A Constructive Lower Bound on Exponential Approximations (LBF Framework)

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Abstract

We present a recursive and geometric approach to bounding the difference between exponential terms a^x and b^y, leading to a constructive lower bound of the form $|a^x - b^y| \gtrsim a^{x/2}$. This has implications for the irrationality of $\log a/\log b$, Diophantine equations, and exponential dynamics such as Collatz-type behaviors. This summary introduces the core structure of the framework and is intended for early feedback and discussion.

Conjecture

Let $a, b \in \mathbb{N}$ with $\log a, \log b \notin \mathbb{Q}$. Then for any integers $x, y > N$, we have

$$|a^x - b^y| > \min(a^x, b^y)^{1/2}$$

This implies a lower bound on the associated linear form in logarithms:

$$|x \log a - y \log b| > \frac{1}{a^{x/2}}$$

WLOG we examine this in the context of the gap between powers of 2 and powers of 3 where the power of 3 is a very close under-approximation of a power of 2, proving that:

$$|2^x - 3^y| \gtrsim 3^{\frac{y}{2}}$$

for all $x, y \in \mathbb{N}$.

Definitions

The proof relies on the fact that powers of 3 that are very close either above or below a power of 2 can be constructed recursively, where any close under-approximation $3^x = 2^m - a$, with $a \ll 2^m$, must be multiplied by a very close over-approximation $3^y = 2^n + b$ where $\frac{a}{2^m} \approx \frac{b}{2^n}$ to generate a new, proportionally better under-approximation.

This is equivalent to saying: if we have a new best proportional under-approximant $3^x \approx 2^m$, the previous best under-approximant must be a factor of it, as all powers of 3 divide one another. As both powers are close to powers of 2 and 3, the complementary factor 3^{y-x} must also be a good approximant. Therefore we have:

$$(2^m - a)(2^n + b) \approx 2^{m+n}$$

Throughout this proof we refer to these powers as "Current/New Best Over/Under Approximants" (CBU/CBO/NBU/NBO). We prove that the numerical error of the NBU is bounded by a lower bounding function (LBF) such that:

$$\text{LBF} \gtrsim 3^{(x+y)/2}$$

Error Factorization

We choose any factor pair of powers of 3 that multiply to our NBU:
Given $3^{x+y} = (2^m - a)(2^n + b)$, we have

$$\text{error} = b \cdot 2^m - a \cdot 2^n - ab$$

We will define the error as comprising the Major Error: $b \cdot 2^m - a \cdot 2^n$ and Minor Error ab.

Lemma 1: Dominance of Major Error

Lemma: the total error is dominated by the major error for all such factor pairs, not only

$$CBU \times CBO$$

Consider an impossible power of 3 pair such that

$$3^x \times 3^y = (2^m - a)(2^n + b) = 2^{m+n}, a, b \neq 0$$

In this scenario we find that the major and minor errors must perfectly cancel in order to leave only the dominant term

$$2^{m+n} \rightarrow 2^m b - 2^n a = ab$$

Any increase of $\frac{b}{a}$ must result in an over-approximation so for all pairs resulting in an NBU $\frac{b}{a}$ must decrease to allow integer values of a and b.

Consider some such decrease in b, new error becomes

$$\epsilon = 2^m b - 2^n a - 2^m \delta - ab + a\delta$$

now

$$2^m \times b - 2^n a = ab$$

so magnitude of major error becomes

$$2^n a - 2^m b + 2^m \delta$$

and magnitude of minor error becomes

$$ab - a\delta$$

Therefore any perturbation from an impossible perfect factor pair that still results in an underapproximation must have minor error ab smaller than major error

$$2^m b - 2^n a$$

and for small a,b the minor error decays quadratically as a proportion of

$$2^{m+n}$$

which allows:

$$2^m b - 2^n a \gg ab$$

Lemma 2: Asymptotic Lower Bound

For $2^{m+n} \approx 3^{x+y}$, the difference satisfies

$$|2^{m+n} - 3^{x+y}| \gtrsim 3^{\frac{x+y}{2}}$$

based on residue-scaled error analysis.

This can be proved taking the major error as always being of greater magnitude than the minor error, meaning that the major error always determines whether or not a product is an over or under-estimate, ab being unable to flip an NBO to an NBU. As a result we can say for any NBU

$$\epsilon > 2^m b - 2^n a$$

for all product pairs $CBU \times CBO$ or even non-approximant pairs that multiply to our NBU.

Factoring the error shows that whichever is the smaller power of 2 is a factor of the entire major error, WLOG assuming this is m:

$$\epsilon > 2^m (b - 2^{n-m} a) \rightarrow \epsilon > 2^m$$

.

$$b - 2^{n-m} a \neq 0$$

as b,a are the difference between powers of 2 and 3 implies b is odd and $2^{n-m}a$ is even.

This rigorously proves that $|2^{m+n} - 3^{x+y}| > 2^{min(m,n)}$

Now, consider a high power of 3 that closely under-approximates a power of 2. We have shown that for all possible factor pairs of previous powers of 3 that multiply to this NBU $(2^m - a)(2^n + b)$, the error is dominated by $2^m b - 2^n a$ and therefore $\epsilon \gg 2^{min(m,n)}$ as the minor error only amplifies any underapproximation.

For low powers of 3 this still allows error to be relatively numerically low, for example in the case of $3^5 = 243$ we could write $243 = (2^5 - 5)(2^3 + 1)$ and extract $\epsilon = 32 \times 1 - 8 \times 5 - 5 \times 1 = -13$ which in this case is smaller than our asymptotic bound $LBF : \epsilon > 3^{\frac{x+y}{2}} = 3^{2.5} = 15.6$. Our bound in this case is determined by $2^{min(m,n)} = 8$, which 13 comfortably exceeds.

However, as powers of 3 get larger more factor pairs appear and factor pairs in the center converge on the square root of our NBU, meaning that quickly with the NBU $3^{65} \approx 2^{111}$ we can say $3^{33} \times 3^{32}$ has $\frac{ab}{2^m b - 2^n a} = 0.72$, not representing an insignificant minor error but still allowing the major error to dominate and proving $\epsilon > 3^{32} \approx 3^{\frac{3^{65}}{2}}$.

As our power of 3 increases significantly the density of good approximations around the square root of the NBU will increase exponentially by Weyl's Distribution Theorem and the Erdos Turan inequality, deomnstrating that the LBF is asymptotic to $3^{\frac{x+y}{2}}$ in the long term and very quickly converges close to this line. Thus

$$|2^x - 3^y| \gtrsim 3^{\frac{y}{2}}$$

For all $x, y \in \mathbb{N}$

QED

Conclusion

This elementary, geometric construction provides a recursive bound that complements Baker's theorem for bounding logarithmic linear forms in the case of exponential approximations. It avoids transcendental methods and has implications for Diophantine problems like Catalan's and Collatz. This initial proof will be fully written up with numerical testing code, graphical demonstrations, and discussion of the application of the LBF to higher Collatz loops, Mihailescu's Theorem, and wider problems in number theory.

Visuals

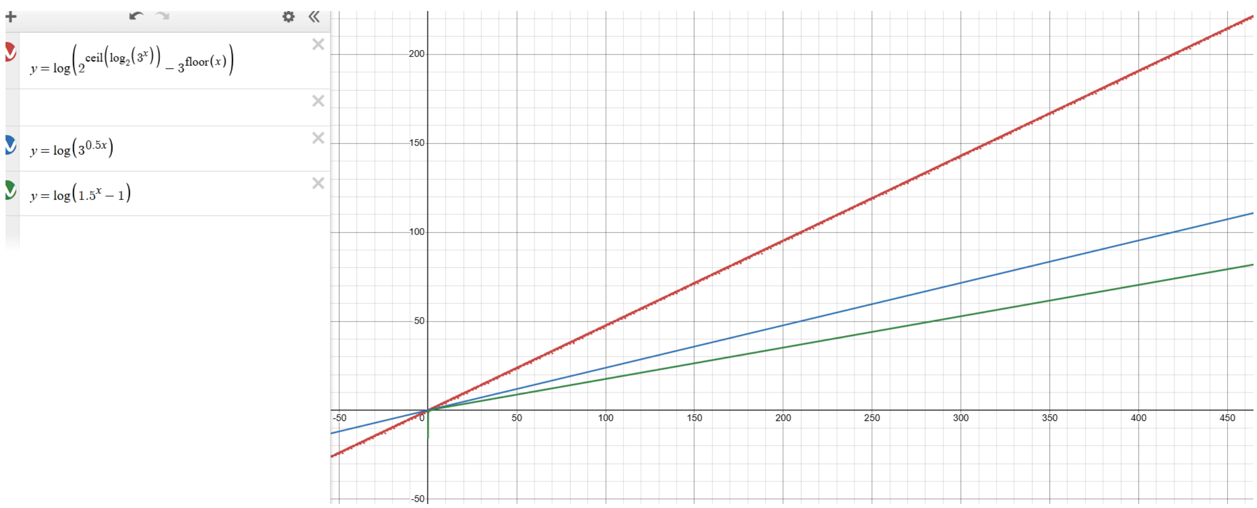

Figure 1: Log-scale graph: Red = error gaps between powers of 3 and nearest powers of 2 above; Blue = lower bound function (LBF), conservative beyond $3^5 = 243$; Green = Collatz upper error from +1 terms, bounded by y=$(3/2)^x wherexisthenumberofoddsteps$.