2024 秋《机器学习概论》作业 4 解答

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题目来源：
P1 P2 P3: https://github.com/maxim5/cs229-2018-autumn/blob/main/problem-sets-solutions/PS4/ps4sol.pdf P2 P3 P5
P4: https://github.com/tuan-ld/cs229-machine-learning-standford/blob/main/midterm/midterm-solutions.pdf P4

P1

(a)

If $\hat{\pi}_0 = \pi_0$, then

$$\begin{aligned}
\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a) &= \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} \frac{\pi_1(s,a)}{\pi_0(s,a)} R(s,a) \\
&= \sum_{(s,a)} \frac{\pi_1(s,a)}{\pi_0(s,a)} R(s,a) p(s) \pi_0(s,a) \\
&= \sum_{(s,a)} R(s,a) p(s) \pi_1(s,a) \\
&= \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}} R(s,a)
\end{aligned}$$

(b)

If $\hat{\pi}_0 = \pi_0$, then

$$\begin{aligned}
\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} &= \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} \frac{\pi_1(s,a)}{\pi_0(s,a)} \\
&= \sum_{(s,a)} p(s,a) \frac{\pi_1(s,a)}{\pi_0(s,a)} \\
&= \sum_{(s,a)} p(s) \pi_0(s,a) \frac{\pi_1(s,a)}{\pi_0(s,a)} \\
&= \sum_{(s,a)} p(s) \pi_1(s,a) \\
&= 1
\end{aligned}$$

$$\frac{\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a)}{\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}} = \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}} R(s,a)$$

(c)

$$\frac{\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a)}{\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}} = \frac{\sum_{(s,a)} p(s,a) \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a)}{\sum_{(s,a)} p(s,a) \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}}$$

If there is only a single data element in the dataset, then

$$\frac{\sum_{(s,a)} p(s,a) \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a)}{\sum_{(s,a)} p(s,a) \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}} = \frac{p(s,a) \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a)}{p(s,a) \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}} = R(s,a)$$

So if $\pi_0 \neq \pi_1$, then

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} R(s,a) \neq \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}} R(s,a)$$

(d)

i.

If $\hat{\pi}_0 = \pi_0$, then

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} ((\mathbb{E}_{a\sim\pi_1(s,a)} \hat{R}(s,a)) = \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}} \hat{R}(s,a)$$

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} (\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}(R(s,a) - \hat{R}(s,a))) = \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}} (R(s,a) - \hat{R}(s,a))$$

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} ((\mathbb{E}_{a\sim\pi_1(s,a)} \hat{R}(s,a)) + \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}(R(s,a) - \hat{R}(s,a))) = \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}} R(s,a)$$

ii.

If $\hat{R}(s,a) = R(s,a)$, then

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} ((\mathbb{E}_{a\sim\pi_1(s,a)} \hat{R}(s,a)) = \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}} R(s,a)$$

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} (\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}(R(s,a) - \hat{R}(s,a))) = 0$$

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}} ((\mathbb{E}_{a\sim\pi_1(s,a)} \hat{R}(s,a)) + \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}(R(s,a) - \hat{R}(s,a))) = \mathbb{E}_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}} R(s,a)$$

(e)

i.

Importance sampling estimator. Estimating $\hat{\pi}_0$ is easier in this situation.

ii.

Regression estimator. Estimating $\hat{R}(s,a)$ is easier in this situation.

P2

$$f_u(x) = \arg\min_{v\in\mathcal{V}} ||x - v||^2 = \frac{uu^T x}{u^T u} = uu^T x$$

$$\begin{aligned}
\arg\min_{u:u^Tu=1} \sum_{i=1}^{m} \|x^{(i)} - f_u(x^{(i)})\|_2^2 &= \arg\min_{u:u^Tu=1} \sum_{i=1}^{m} \|x^{(i)} - uu^T x^{(i)}\|_2^2 \\
&= \arg\min_{u:u^Tu=1} \sum_{i=1}^{m} (x^{(i)} - uu^T x^{(i)})^T (x^{(i)} - uu^T x^{(i)}) \\
&= \arg\min_{u:u^Tu=1} \sum_{i=1}^{m} {x^{(i)}}^T x^{(i)} - {x^{(i)}}^T uu^T x^{(i)} \\
&= \arg\max_{u:u^Tu=1} \sum_{i=1}^{m} {x^{(i)}}^T uu^T x^{(i)} \\
&= \arg\max_{u:u^Tu=1} \sum_{i=1}^{m} u^T x^{(i)} {x^{(i)}}^T u \\
&= \arg\max_{u:u^Tu=1} u^T \Big(\sum_{i=1}^{m} x^{(i)} {x^{(i)}}^T \Big) u
\end{aligned}$$

P3

(a)

$$\begin{aligned}
\|B(V_1) - B(V_2)\|_\infty &= \gamma \left\| \max_{a\in A} \sum_{s'\in S} P_{sa}(s')\left[V_1(s') - V_2(s')\right] \right\|_\infty \\
&= \gamma \max_{s'\in S} \left| \max_{a\in A} \sum_{s'\in S} P_{sa}(s')\left[V_1(s') - V_2(s')\right] \right| \\
&\le \gamma \|V_1 - V_2\|_\infty
\end{aligned}$$

The inequality holds because for any $\alpha, x \in \mathbb{R}^n$, if $\sum_i \alpha_i = 1$ and $\alpha_i \ge 0$, then $\sum_i \alpha_i x_i \le \max_i x_i$

(b)

Assume that V_1 and V_2 are both fixed points, i.e., $B(V_1) = V_1, B(V_2) = V_2$.

$$\|V_1 - V_2\|_\infty = \|B(V_1) - B(V_2)\|_\infty \le \gamma \|V_1 - V_2\|_\infty$$

$$\|V_1 - V_2\|_\infty = 0$$

$$V_1 = V_2$$

So B has at most one fixed point.

P4

(a) [**9 points**] The primal optimization problem for the one-class SVM was given above. Write down the corresponding dual optimization problem. Simplify your answer as much as possible. In particular, w should not appear in your answer. **Answer:** The Lagrangian is given by

$$L(w,\alpha) = \frac{1}{2}w^\top w + \sum_{i=1}^{m} \alpha_i (1 - w^\top x^{(i)}). \tag{4}$$

Setting the gradient of the Lagrangian with respect to w to zero, we obtain $w = \sum_{i=1}^{m} \alpha_i x^{(i)}$. It follows that

$$\max_{\alpha \geq 0} \min_{w} \left(\frac{1}{2}w^\top w + \sum_{i=1}^{m} \alpha_i (1 - w^\top x^{(i)}) \right) \tag{5}$$

$$= \max_{\alpha \geq 0} \frac{1}{2} \left(\sum_{i=1}^{m} \alpha_i x^{(i)} \right)^\top \left(\sum_{i=1}^{m} \alpha_i x^{(i)} \right) + \sum_{i=1}^{m} \alpha_i \left(1 - \left(\sum_{i=1}^{m} \alpha_i x^{(i)} \right)^\top x^{(i)} \right) \tag{6}$$

$$= \max_{\alpha \geq 0} \left(\sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j x^{(i)\top} x^{(j)} \right) \tag{7}$$

The first equality follows from setting the gradient w.r.t. w equal to zero, and solving for w, which gives $w = \sum_{i=1}^{m} \alpha_i x^{(i)}$ and substituting this expression for w. The second equality follows from simplifying the expression.

(b) [**4 points**] Can the one-class SVM be kernelized (both in training and testing)? Justify your answer.

Answer: Yes. For training we can use the dual formulation, in which only inner products of the data appear. For testing at a point z we just need to evaluate $w^\top z = \left(\sum_{i=1}^{m} \alpha_i x^{(i)}\right)^\top z = \sum_{i=1}^{m} \alpha_i x^{(i)\top} z$ in which the training data and the test point z only appear in inner products.

(c) [**5 points**] Give an SMO-like algorithm to optimize the dual. I.e., give an algorithm that in every optimization step optimizes over the smallest possible subset of variables. Also give in closed-form the update equation for this subset of variables. You should also justify why it is sufficient to consider this many variables at a time in each step.

Answer: Since we have convex optimization problem with only independent coordinate wise constraints ($\alpha_i \geq 0$), we can optimize iteratively over 1 variable at a time. Optimizing w.r.t. α_i is done by setting

$$\alpha_i = \max \left\{ 0, \frac{1}{K_{i,i}} (1 - \sum_{j \neq i} \alpha_j K_{i,j}) \right\}$$

(Set the derivative w.r.t. α_i equal to zero and solve for α_i. And take into account the constraint. Here, we defined $K_{i,j} = x^{(i)\top} x^{(j)}$.)