A Short Trip to the Wonderful World of Numbers

From Natural to Complex Numbers

Algebraic Equations

Find x that satisfies these equations:

$6 + x = 8$

$4x = 8$

$2x^2 = 8$

$4 + x^2 = 8$

Solution in N: $x = 2$

Algebraic Equations

Similar-looking equations:

$2 + x = 6$	$2 + x = 1$
$2x = 8$	$2x = 1$
$2x^2 = 8$	$2x^2 = 1$
$2 + x^2 = 6$	$2 + x^2 = 1$
Solution in N	No solution in N

Natural Numbers -- N

• Numbers	$m = \|A\| \quad n = \|B\|$	
• Equality	$m = n$ if $\|A\| = \|B\|$	
• Order	$m < n$ if $A \subset B$	
• Sum	$m + n = \|A \cup B\|$	if $A \cap B = \phi$
zero	$0 = \|\phi\|$	$0 + m = m$
negative	X	
• Product	$m\, n = \|A \mathrm{x} B\|$	$0m = 0$
one	$1 = \|\{0\}\|$	$1m = m$
inverse	X	

Integers -- Z

Subtraction fails:

$$6 + x = 5$$

No solution in N

Define –1 and its multiples → "negative" numbers

$-1 + 1 = 0$ $\qquad$ $-m = m\,(-1)$ $\qquad$ $-m + m = 0$

Subtraction: $\qquad n - m = n + (-m)$

Rational Numbers

Division fails:

$6\,x = 5$

No solution in Z

Define a/b as rational → "fractional" numbers

$(a/b)\,(b/a) = 1$

Division: $(c/d) / (a/b) = (c/d)\,(b/a)$

$(5/1) / (6/1) = (5/1)\,(1/6) = 5/6$

Rational Numbers -- Q

• Numbers	$p = a/b \quad q = c/d$	$a,c \in Z$ & $b,d \in N^+$
• Equality	$p = q \leftrightarrow ad = bc$	
• Order	$p < q \leftrightarrow ad < bc$	
• Sum	$p + q = (ad+bc)/(bd)$	
zero	$0 = 0/b$	$0+p = p$
negative	$-p = -a/b$	$p+(-p) = 0$
• Product	$p\,q = (ac)/(bd)$	$0p = 0$
one	$1 = b/b$	$1p = p$
inverse	$p^{-1} = b/a \quad (a\neq 0)$	$p\,p^{-1} = 1$

Real Numbers -- *R*

Square root fails:

$x^2 = 2$

No solution in *Q*

Define an infinite set of theoretical numbers

→ "irrational" numbers

Such a number may require an infinite sequence of digits

Complex Numbers

Square root fails:

$5 + x^2 = 0$

No solution in R

Define just one "imaginary" number $i^2 = -1$

Numbers of type $x+iy$ solve *all* algebraic equations

Complex Numbers -- C

• Numbers	$w = a+ib \quad z = c+id$	$a,b,c,d \in R$
• Equality	$w = z \leftrightarrow a=c \ \& \ b=d$	
• Sum	$w + z = (a+c)+i(b+d)$	
zero	$0 = 0+i0$	$0+w = w$
negative	$-w = (-a)+i(-b)$	$w+(-w) = 0$
• Product	$w\,z = (ac-bd)+i(ad+bc)$	$0w = 0$
one	$1 = 1+i0$	$1w = w$
magnitude	$r^2 = a^2 + b^2$	$r = 0 \leftrightarrow w = 0$
inverse	$w^{-1} = a/r^2 - ib/r^2 \quad (r \neq 0)$	$w\,w^{-1} = 1$

Hierarchy of Numbers

$$N \subset Z \subset Q \subset R \subset C$$

negative, fractional, irrational, and imaginary numbers are added to natural numbers for various needs

"Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is." -- Paul Erdös

Discover the Exponential

- Find a real function $e(x)$, equal to its own derivative at all points:

$$e'(x) = e(x), \qquad e(0) = 1$$

- Suppose $\quad e(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n + \ldots$

$$e'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots$$

Thus, $\quad a_n = a_{n-1}/n$ and $a_0 = 1$ for $n = 1, 2, 3\ldots$

$$e(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- Multiplying $e(a)$ and $e(b)$, we get $e(a+b)$ for all $a,b \in R$

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{j=0}^{\infty} \frac{b^j}{j!} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{a^k b^j}{k!\, j!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} \frac{n!}{(n-i)!\, i!} a^{n-i} b^i = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!}$$

Discover the Number e

- If we define $e(1)=e$, then $e(2) = e(1)\, e(1) = e^2$ and $e(k)=e^k$ for all $k \in N$

- Since $e(k)\, e(-k) = e(0) = 1$, $e(-k) = 1/e(k)$ thus $e(k)=e^k$ for all $k \in Z$

- Since $e(½)\, e(½) = e(1)=e$, we have $e(½) = \sqrt{e}$ ... $e(p)=e^p$ for all $p \in Q$

- And finally $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in R$

Decimal Approximation to e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Only 9 iterations are sufficient for 7-digit accuracy:

n	t	sum
0	1.0	1.0
1	1.0	2.0
2	0.5	2.5
3	0.1666667	2.6666667
4	0.0416667	2.7083334
5	0.0083333	2.7166667
6	0.0013889	2.7180556
7	0.0001984	2.7182540
8	0.0000248	2.7182788
9	0.0000027	2.7182815

Rational Approximation to e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

The same iterations using fractional numbers:

n	t	sum
0	1/1	1/1 = 1.0
1	1/1	2/1 = 2.0
2	1/2	5/2 = 2.5
3	1/6	8/3 = 2.6666667
4	1/24	65/24 = 2.7083333
5	1/120	163/60 = 2.7166667
6	1/720	1957/720 = 2.7180555
7	1/5040	685/252 = 2.7182539
8	1/40320	109601/40320 = 2.7182788
9	1/362880	98641/36288 = 2.7182815

Important Property of i

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

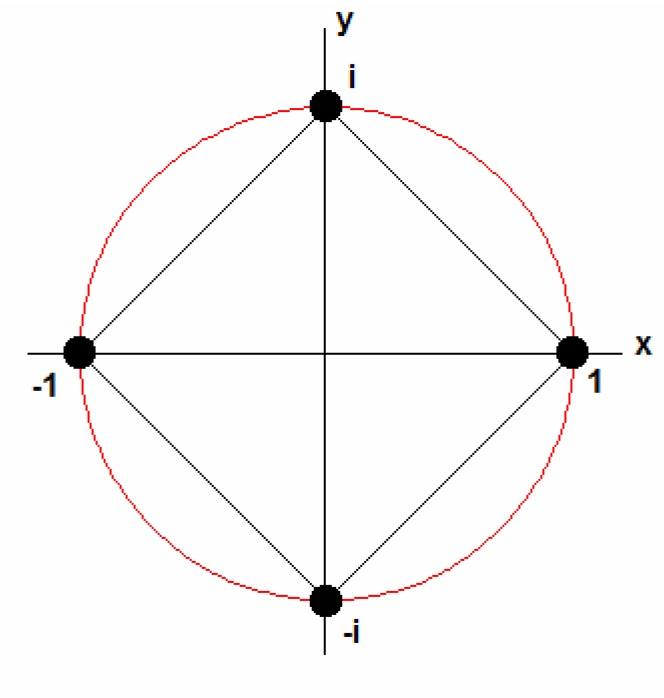
$i^3 = -i$

$i^4 = 1$

…

$i^{2k} = (-1)^k \quad (k \in Z)$

$i^{2k+1} = (-1)^k\, i$



Roots of $z^4 = 1$

http://mathworld.wolfram.com/RootofUnity.html

Complex Exponential

- Consider *e(ix)* for $x \in R$

$$e(ix) = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = c(x) + is(x)$$

$$c(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad \text{and} \quad s(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

- Clearly, $c'(x) = -s(x)$ and $s'(x) = c(x)$ with $c(0) = 1$ and $s(0) = 0$

- $e(ia+ib) = c(a+b) + i\, s(a+b)$

  $= e(ia)\, e(ib) = (c(a) + i\, s(a))\, (c(b) + i\, s(b))$

  $= (c(a)\, c(b) - s(a)\, s(b)) + i\, (c(a)\, s(b) + s(a)\, c(b))$

  → $c(a+b) = c(a)\, c(b) - s(a)\, s(b)$

  and $s(a+b) = c(a)\, s(b) + s(a)\, c(b)$

Discover cosine, sine, & π

- Let $h(x) = c^2(x) + s^2(x)$

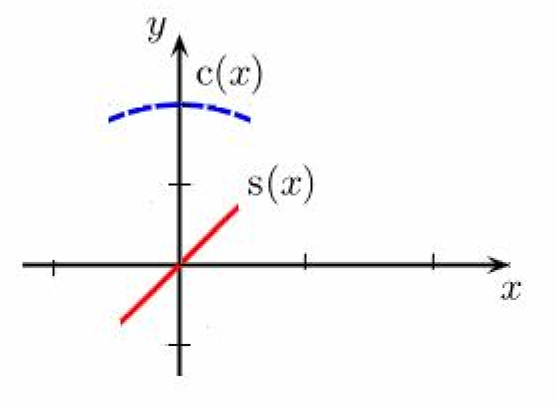
  $h'(x) = -2c(x)s(x) + 2s(x)c(x) = 0 \rightarrow h(x) = \text{constant}$

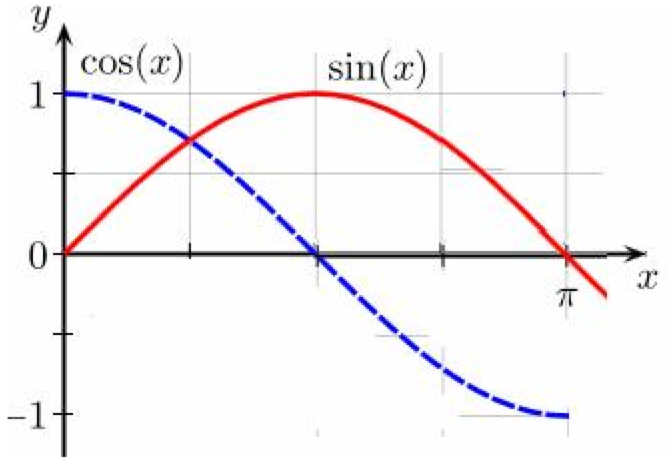
  Since $h(0) = 1$, $c^2(x) + s^2(x) = 1$ for all $x \epsilon R$

- $c(x) = 1 - x^2/2$ and $s(x) = x$ for small $|x|$

  $s(x) = 0$ for some $x>0$, denote this value as π

  $c(\pi) = -1$ and $s(\pi) = 0 \rightarrow e(i\pi) = -1$





Calculation of sin π

$$\sin\pi = \sum_{k=0}^{\infty}(-1)^k \frac{\pi^{2k+1}}{(2k+1)!}$$

k	t	sum
0	π = 3.14159	3.14159
1	$-\pi^3/3!$ =-5.16771	-2.02612
2	$\pi^5/5!$ = 2.55016	0.52404
3	$-\pi^7/7!$ =-0.59926	-0.07522
4	$\pi^9/9!$ = 0.08215	0.00693
5	$-\pi^{11}/11!$ =-0.00737	-0.00045
6	$\pi^{13}/13!$ = 0.00047	0.00002
7	$-\pi^{15}/15!$ =-0.00002	0.00000

sum of positive terms	5.77437
sum of negative terms	-5.77437

Euler's Equation: $e^{i\pi}+1=0$

Numerical demonstration of this wonderful equation using complex numbers:

n	t	sum
0	1	1
1	3.141593i	1 +i 3.141593
2	-4.934802	-3.934802 +i 3.141593
3	-5.167713i	-3.934802 -i 2.026120
4	4.058712	0.123910 -i 2.026120
5	2.550164i	0.123910 +i 0.524044
6	-1.335263	-1.211353 +i 0.524044
7	-0.599265i	-1.211353 -i 0.075221
8	0.235331	-0.976022 -i 0.075221
9	0.082146i	-0.976022 +i 0.006925
10	-0.025807	-1.001829 +i 0.006925
11	-0.007370i	-1.001829 -i 0.000445
12	0.001930	-0.999999 -i 0.000445
13	0.000466i	-0.999999 +i 0.000021

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k+1}}{(2k+1)!}$$

Discover the Logarithm

Consider the inverse of $e(x)$:

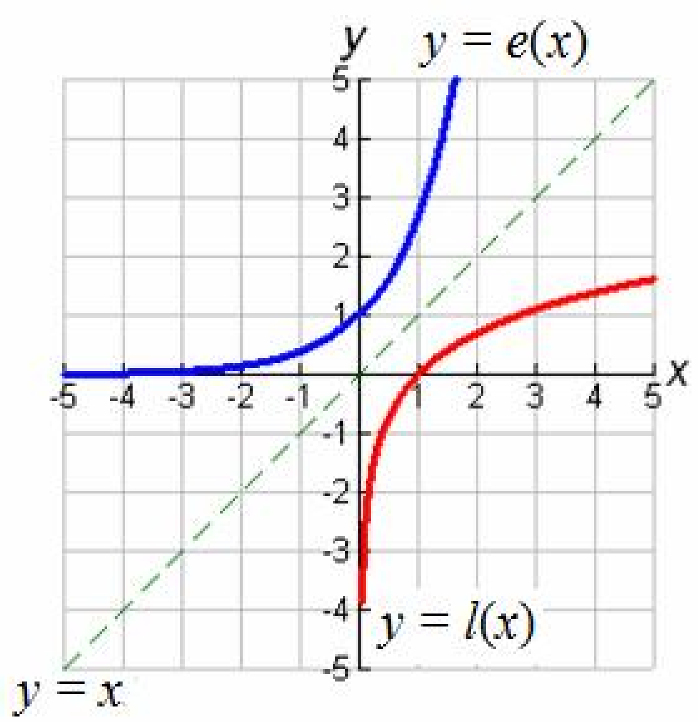
$y = e(x)$, $\quad e(0) = 1$

$l(y) = x$, $\quad 0 = l(1)$

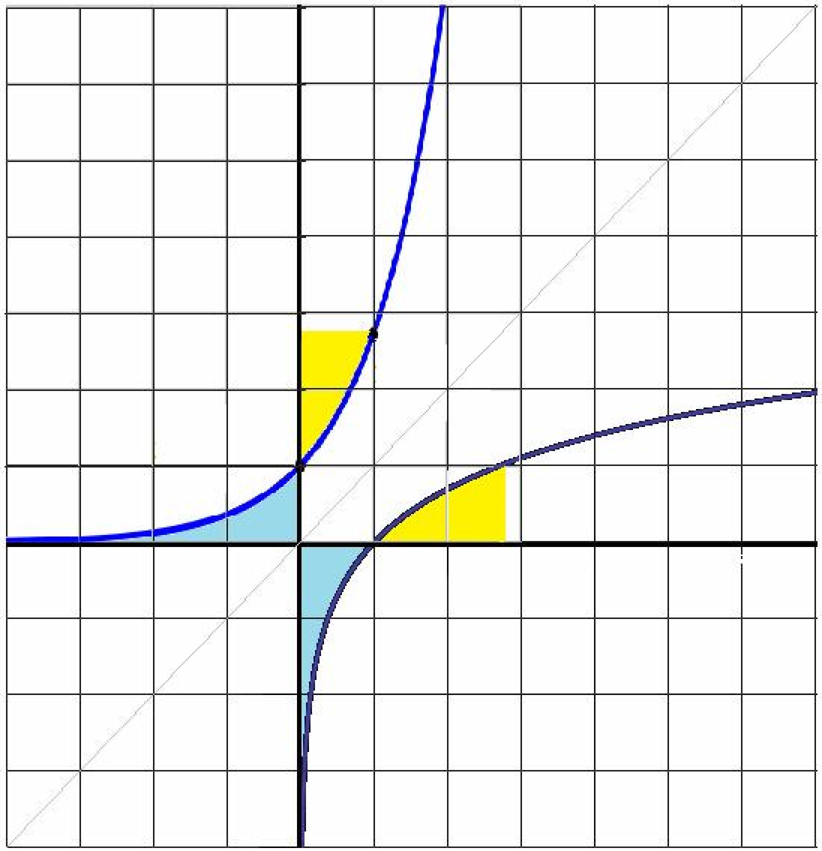
$dy/dx = y \rightarrow dx/dy = 1/y$

$$l(a) = \int_1^a \frac{dy}{y}$$

$$l(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} = l(a) + l(b)$$



Exponential and Logarithm



Blue areas:

$$\int_{-\infty}^{0} e^x dx = 1 \text{ and } \int_{0}^{1} (\log x) dx = -1$$

Yellow areas:

$$\int_{0}^{1} (e - e^x) dx = 1 \text{ and } \int_{1}^{e} (\log x) dx = 1$$

“The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours of the words, must fit together in a harmonious way. Beauty is the first test: There is no permanent place in the world for ugly mathematics.”

G. H. Hardy

References

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