PHYS2941 problem set 4

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Problem 7.3)

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a) $\hat{Q} = \sum_n q_n |e_n\rangle\langle e_n|$ thus: $\hat{H} = \sum_n E_n |\psi_n\rangle\langle\psi_n|$

Given ~~the~~ the state vector form of the Schrödinger equation:

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = \hat{H}|\Psi(t)\rangle \quad \text{we see:}$$

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = \sum_n E_n |\psi_n\rangle\langle\psi_n| \, |\Psi(t)\rangle$$

Note $\langle\psi_n|\Psi(t)\rangle = C_n = \langle\psi_n||\Psi(t)\rangle$

We notice E_n is a scalar and thus its position is irrelevent:

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = \sum_n E_n|\psi_n\rangle C_n = \sum_n C_n E_n |\psi_n\rangle$$

Now investigating the left side we find:

$$i\hbar\frac{d|\Psi(t)\rangle}{dt} = i\hbar \frac{d}{dt}\sum_n C_n|\psi_n\rangle \quad \text{thus:}$$

$$i\hbar\frac{d}{dt}\sum_n C_n|\psi_n\rangle = \sum_n E_n C_n|\psi_n\rangle$$

Thus for any n we can solve the first order differential:

y' + f(x) y = 0 has the general solution y = $e^{\int f(x)dx}$

$$i\hbar \frac{d}{dt} C_n|\psi_n\rangle = E_n C_n|\psi_n\rangle$$

$$i\hbar\frac{d\overset{C_n}{|\psi_n\rangle}}{dt} = E_n\overset{C_n}{|\psi_n\rangle}$$

~~This is~~ Thus the general solution follows:

$$C_n|\psi_n\rangle = Ce^{-i\frac{E_n}{\hbar}t}$$

which given $|\Psi(t)\rangle = \sum_n C_n|\psi_n\rangle$

$$|\Psi(t)\rangle = \sum_n C e^{-i\frac{E_n}{\hbar}t}$$

given at $t=0$, $|\Psi(0)\rangle = \sum_n C_n|\psi_n\rangle$

we know:

$$|\Psi(t)\rangle = \sum_n C_n|\psi_n\rangle e^{-i\frac{E_n}{\hbar}t}$$

which proves for a general n:

$$\Rightarrow \sum_n C_n|\psi_n\rangle e^{-i\frac{E_n}{\hbar}t} = |\Psi(t)\rangle$$

b) From the question we understand:

$\langle \psi_6 | \Psi(0) \rangle = \frac{1}{\sqrt{6}}$ $\quad \langle \psi_{17} | \Psi(0) \rangle = \frac{-i}{\sqrt{2}}$

$\langle \psi_{271} | \Psi(0) \rangle = \frac{1}{\sqrt{3}}$ and any other n, $\langle \psi_n | \Psi(0) \rangle = 0$

we also see $\langle \psi_n | \Psi(0) \rangle = C_n$.

Given the previously derived: $|\Psi(t)\rangle = \sum_n C_n |\psi_n\rangle e^{-\frac{iE_n}{\hbar}t}$

we know for $n = 6$, 17 and 271.

We finally conclude that: $|\Psi(t)\rangle = \sum_{n=1}^{\infty} \langle \psi_n | \Psi(0) \rangle \, |\psi_n\rangle \, \mathrm{e}^{-iE_n t/\hbar}$ thus:

$$|\Psi(t)\rangle = C_6 |\psi_6\rangle e^{-iE_6 t/\hbar} + C_{17} |\psi_{17}\rangle e^{-iE_{17} t/\hbar} + C_{271} |\psi_{271}\rangle e^{-iE_{271} t/\hbar}$$

$$= \frac{1}{\sqrt{6}} |\psi_6\rangle e^{-iE_6 t/\hbar} + \frac{-i}{\sqrt{2}} |\psi_{17}\rangle e^{-iE_{17} t/\hbar} + \frac{1}{\sqrt{3}} |\psi_{271}\rangle e^{-iE_{271} t/\hbar}$$

Problem 7.4)

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Given some arbitrary operator $\hat{A}$ we see:

$\hat{A} = \hat{I}\hat{A}\hat{I}$ where $\hat{I} = \sum_n |\psi_n\rangle\langle\psi_n|$ as $|\psi_n\rangle$ form a complete orthonormal basis.

With the left $\hat{I}$ being in terms of m and the right n:

$$\hat{A} = \sum_m |\psi_m\rangle\langle\psi_m| A \sum_n |\psi_n\rangle\langle\psi_n|$$

$$= \sum_m \sum_n |\psi_m\rangle\langle\psi_m| A |\psi_n\rangle\langle\psi_n|$$

As $\langle\psi_m|A|\psi_n\rangle = A_{m,n}$ we see

$$\hat{A} = \sum_m \sum_n |\psi_m\rangle A_{m,n} \langle\psi_n| \qquad \text{QED}$$

Note that $A_{m,n}$ is a scalar and can therefore be moved around. As such, we have shown that any arbitrary operator A-hat can be written as $\hat{A} = \sum_m \sum_n A_{m,n} |\psi_m\rangle\langle\psi_n|$,

Problem 8.4)

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a) $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

Focusing on LHS:

$[\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$

Focusing on RHS:

$[\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

$= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$

$= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A}$

$= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$

LHS = RHS

QED: $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

b)

To solve for the Hermitian Conjugate of $\hat{a}_+$ we must first assert that from calculations the following is true:

$$(i)^\dagger = -i \quad \text{and} \quad \left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$$

we also know that ~~for~~ to find the Hermitian conjugate,

$$\langle \hat{a}_+ \psi \} | \psi \rangle = \langle \psi | \hat{a}_+^\dagger \psi \rangle$$

Additionally for the next two questions, let me prove a few extremely helpful identities:

$(i)^\dagger = -i$: $\langle i\psi | \psi \rangle = \langle \psi | -i\psi \rangle$

$$\langle i\psi|\psi\rangle = \int_{-\infty}^{\infty} (i\psi)^* \psi \, dx = \int_{-\infty}^{\infty} -i\psi^*\psi \, dx = \int_{-\infty}^{\infty} \psi^*(-i\psi) \, dx = \langle \psi | -i\psi \rangle$$

$\therefore (i)^\dagger = -i$ QED

$\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$: $\langle \frac{d}{dx}\psi | \psi \rangle = \langle \cancel{d}\psi | \frac{d}{dx}\psi \rangle$

$$\int_{-\infty}^{\infty} \left(\frac{d}{dx}\psi^*\right)^* \psi \, dx = \int_{-\infty}^{\infty} \frac{d}{dx}\psi^* \psi \, dx = \langle \frac{d}{dx}\psi | \psi \rangle$$

Let $\frac{du}{dx} = \frac{d}{dx}\psi^*$ $\quad u = \psi^*$

$v = \psi$ $\quad \frac{dv}{dx} = \frac{d\psi}{dx}$ $\quad$ using Integration by parts:

$$\int_{-\infty}^{\infty} \frac{d}{dx}\psi^*\psi \, dx = \left[\psi\psi^*\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi^* \frac{d}{dx}\psi \, dx$$

$$= \langle \psi | -\frac{d}{dx}\psi \rangle \quad \text{QED}$$

$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$: $\langle \psi | \hat{A}\hat{B}\phi \rangle = \langle \hat{A}^\dagger \psi | \hat{B}\phi \rangle = \langle \hat{B}^\dagger \hat{A}^\dagger \psi | \phi \rangle$

Thus $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$ QED

$(\hat{a} + \hat{b})^\dagger = \hat{a}^\dagger + \hat{b}^\dagger$:

$\langle \psi | (\hat{a} + \hat{b})^\dagger | \phi \rangle = \langle \psi | \hat{a} + \hat{b} | \phi \rangle^*$ (Definition of Hermitian Conjugate)

$= \langle \psi | \hat{a} | \phi \rangle^* \langle \psi | \hat{b} | \phi \rangle^*$

$= \langle \psi | \hat{a}^\dagger | \phi \rangle \langle \psi | \hat{b}^\dagger | \phi \rangle$

$= \langle \psi | \hat{a}^\dagger + \hat{b}^\dagger | \phi \rangle$

$\therefore (\hat{a} + \hat{b})^\dagger = \hat{a}^\dagger + \hat{b}^\dagger$ QED

$$(\hat{a}_+)^\dagger = \left(\frac{1}{\sqrt{2\hbar m\omega}}\right)^\dagger(-i\hat{p} + m\omega x)^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left((-i\hat{p})^\dagger + m\omega x\right)$$

$$(-i\hat{p})^\dagger = (-i)^\dagger(\hat{p})^\dagger = (i)(\hat{p})$$

$$\therefore \hat{a}_+^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{p} + m\omega x\right) = \hat{a}_-$$

c)

$$\hat{X} = \hat{a}_+ + \hat{a}_-$$

$$: \quad \hat{a}_+^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(\hat{p}i + m\omega x\right) = \hat{a}_-$$

$$\hat{a}_-^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left((\hat{p}i)^\dagger + m\omega x\right) = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega x\right) = \hat{a}_+$$

$$\therefore \hat{X}^\dagger = (\hat{a}_+ + \hat{a}_-)^\dagger = \hat{a}_+^\dagger + \hat{a}_-^\dagger = \hat{a}_- + \hat{a}_+ = \hat{X}$$

Thus $\hat{X}$ is Hermitian.

Similarly;

for $\hat{n} = \hat{a}_+\hat{a}_-$: $\hat{n}^\dagger = (\hat{a}_-)^\dagger(\hat{a}_+)^\dagger = \hat{a}_+\hat{a}_-$

Thus, we have shown explicitly that both the sum and product of ladder operators are Hermitian.

Problem 8.5)

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a) $$\frac{d\hat{Q}(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{Q}(t)\right]$$ For $\hat{a}_+(t)$ we find:

$$\frac{d\hat{a}_+(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{a}_+(t)\right]$$

$$\frac{d\hat{a}_+(t)}{dt} = \frac{i}{\hbar}\left(\hat{H}\hat{a}_+(t) - \overset{\hat{a}(t)\hat{H}}{\cancel{\hat{H}\hat{a}_+(t)}}\right) = \frac{i}{\hbar}\left(\hbar\omega\left(\hat{a}_+\hat{a}_- + \tfrac{1}{2}\right)\hat{a}_+ - \hat{a}_+\hbar\omega\left(\hat{a}_+\hat{a}_- + \tfrac{1}{2}\right)\right)$$

$$= i\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ + \tfrac{1}{2}\hat{a}_+ - \hat{a}_+\hat{a}_+\hat{a}_- - \tfrac{1}{2}\hat{a}_+\right) = i\omega\hat{a}_+\left(\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_-\right)$$

Evidently: $$\frac{d\hat{a}_+(t)}{dt} = i\omega\hat{a}_+\left[\hat{a}_-, \hat{a}_+\right] = i\omega\hat{a}_+(t)$$

This follows a linear homogeneous first order differential:

$y' + f(x)y = 0$ with general solution $y = Ce^{\int f(x)dx}$

Thus:

$$\hat{a}_+(t) = Ce^{-\int i\omega dt} = Ce^{-i\omega t}$$

Similarly for $\hat{a}_-(t)$

$$\frac{d\hat{a}_-(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{a}_-(t)\right] = i\omega\hat{a}_-\left[\hat{a}_-, \hat{a}_+\right]$$

and thus: $\hat{a}_-(t) = Ce^{-i\omega t}$

To find C we understand that at $t=0$:

$\hat{a}_+(t) = \hat{a}_+(0)$: $\hat{a}_-(t) = \hat{a}_-(t)$ Thus:

$\hat{a}_+(t) = a_+(0)e^{-i\omega t}$ $\quad \hat{a}_-(t) = \hat{a}_-(0)e^{-i\omega t}$

b) $\hat{n} = \hat{a}_+\hat{a}_-$ To show $\hat{n}$ and $\hat{H}$ commute,

$[\hat{n}, \hat{H}] = 0$. must be true.

$[\hat{H}, \hat{n}] = [\hat{H}, \hat{a}_+\hat{a}_-] = \hat{H}\hat{a}_+\hat{a}_- - \hat{a}_+\hat{a}_-\hat{H}$

$\Rightarrow$ Given $\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$,

it has previously been shown that substituting

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$ and $\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$

we find: $\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$

Thus:

$[\hat{H}, \hat{n}] = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})\hat{a}_+\hat{a}_- - \hat{a}_+\hat{a}_-\hbar\omega(a_+a_- + \frac{1}{2})$

$= \hbar\omega\hat{a}_+\hat{a}_-\hat{a}_+\hat{a}_- + \frac{\hbar\omega}{2}\hat{a}_+\hat{a}_- - \hbar\omega\hat{a}_+\hat{a}_-\hat{a}_+\hat{a}_- - \frac{\hbar\omega}{2}a_+a_-$

$= 0$

Thus they commute.

Additionally we understand:

$\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$

In the same way that in quantum mechanics

$\hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$ and $\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$

Thus:

$\hat{n}|n\rangle = \hat{a}_+\hat{a}_-|n\rangle = \hat{a}_+\sqrt{n}|n-1\rangle$

$= \sqrt{n}\sqrt{n}|n\rangle$

$= n|n\rangle$

Thus we have shown $\hat{n}|n\rangle = n|n\rangle$

Thus for $\hat{n}|n\rangle$ ~~eigenstate~~, the corresponding eigenvalue is n.

Additionally, as $\hat{H}$ and $\hat{n}$ commute, it can be concluded that there are a common set of states $|n\rangle$ that satisfy both, ~~$\hat{H}$ and $\hat{n}$~~ such that they are simultaneous eigenstates.