Problem 1.1
Part A
A simplified model of a tropical cyclone is shown in Fig. 1.

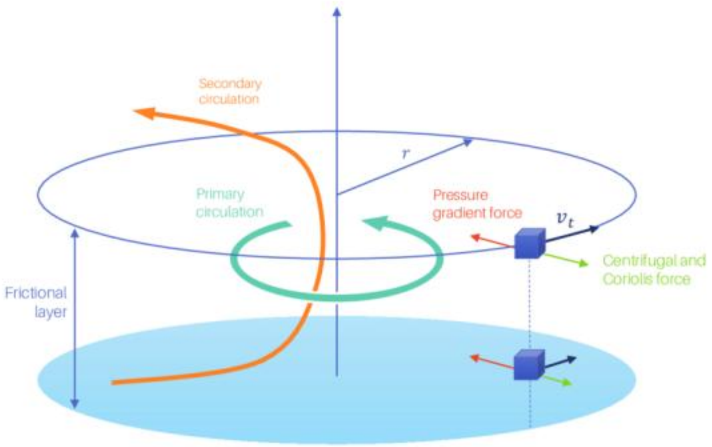


Figure 1: Simplified model of a tropical cyclone. The primary circulation overcomes frictional loss via the driving of the weaker secondary circulation. [S. Zitzmann MS. Thesis, 2022.]

(a) The primary circulation is characterised by its tangential velocity v_t, indicated by the green line. The magnitude of the velocity field is given by

$$v_t = \frac{M}{r} - \frac{fr}{2},$$

where $M = 5 \times 10^7$ m^{2}s^{-1} is the angular momentum per unit mass and $f = 2\Omega\sin\theta$ s^{-1} is the Coriolis parameter, Ω the rotation rate of the earth (use radian units!). Determine f for the latitude of Brisbane.

(b) Using cylindrical coordinates, write down the vector field for v_t.

(c) Find the curl of the tangential velocity field. This is also known as the vorticity.

(d) Show that the tangential velocity field does not obey Stokes theorem. This is because the velocity field has a singular point at the origin.

a) We understand Brisbane is on the latitude $27.5^\circ = 27.5 \times \frac{\pi}{180} = 0.48$ rad

The Earth rotates at 7.3×10^{-5} rad/s

Thus: $f = 2\Omega \sin\theta \quad s^{-1}$

$= 2 \times 7.3 \times 10^{-5} \sin(0.48)$

$= 6.7 \times 10^{-5} \quad s^{-1}$

b) Modelling the object as a velocity vector field using cylindrical coordinates will take the following form:

$$\vec{V}(r,\varphi,z) = A\hat{r} + B\hat{\varphi} + C\hat{z}$$

We can make the following assumptions about the system:

1. As the secondary circulation overcomes the frictional layers, there is no translation up or down and thus:

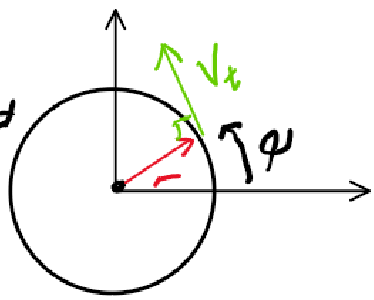
   $C = 0$, i.e no $\hat{z}$ component.

2. Similarly, as no information is given about changes of radius and thus:

   no $\hat{r}$ component.

This leaves us to consider our $\hat{\varphi}$ component. Given $\hat{\varphi}$ represents the rotational component of velocity as shown below:

We understand the velocity in the $\hat{\varphi}$ component to be equal to the tangential velocity.



Top down view of cyclone.

This leaves: $\vec{v}(r,\varphi,z) = \left(\frac{M}{r} - \frac{fr}{2}\right)\hat{\varphi}$

Curl $= \nabla \times \vec{v}$ ← For cylindrical coordinates

$$\text{Curl} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & rv_t & 0 \end{vmatrix} = \frac{1}{r}\left(\left(0 - \frac{\partial v_t}{\partial z}\right)\hat{r} + (0)\hat{\varphi} + \left(\frac{\partial v_t}{\partial r} - 0\right)\hat{z}\right)$$

$$\Rightarrow \frac{\partial}{\partial z}\left(\frac{M}{r} - \frac{fr}{2}\right)r = 0$$

$$\frac{\partial}{\partial r}\left(Mr - \frac{fr^2}{2}\right) = -fr$$

$$\therefore \text{Curl} = \frac{1}{r}\left(-fr\,\hat{z}\right) = -f\hat{z}$$

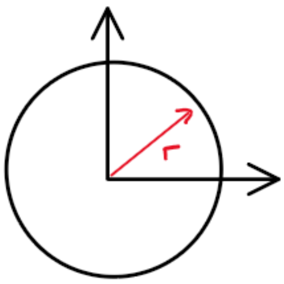
Stokes Theorem:

$$\oint_C F \cdot dr = \iint_S (\nabla \times F)\cdot \hat{n}\, dS$$

To move completely around the cylinder we find by taking the cross section:

$C = 0 \leq \varphi \leq 2\pi$

$S = \left\{ \begin{array}{l} 0 \leq r \leq r \\ 0 \leq \varphi \leq 2\pi \end{array} \right)$ at $z = 0$



$\therefore \oint_0^{2\pi} \left(\frac{M}{r} - \frac{rf}{2}\right) r\hat{\varphi} \cdot d\hat{\varphi}$ Jacobian

$\iint_0^{2\pi}\!\!\int_0^{r} (-f\hat{z})\cdot \hat{n}\, dr\, d\varphi$

$= \left[\left(\frac{M}{r} - \frac{rf}{2}\right)\varphi r\right]_0^{2\pi}$

$= 2\pi\left(M - \frac{r^2 f}{2}\right)$

$= 2\pi M - \pi f r^2$

As $\hat{z}$ and $\hat{n}$ are both normal to the cylinder surface:

$\hat{z}\cdot\hat{n} = 1$

$\int_0^{2\pi}\!\!\int_0^{r} -f r\, dr\, d\varphi$ Jacobian

$= \int_0^{2\pi} -\frac{f r^2}{2}\, d\varphi$

$= -\pi f r^2$

LHS $\neq$ RHS

Reasoning: For Stokes theorem to hold, $\vec{v}$ must be differentiable at all points. However, as:

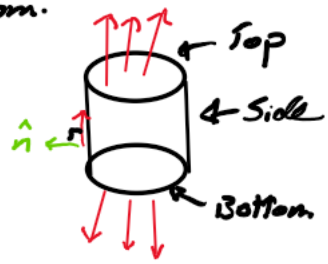
$\hat{\varphi}$ has a $\frac{M}{r}$ term,

We notice that $r = 0$ results in an undefined $\frac{M}{r}$ and

thus it is not differentiable. Therefore, Stokes Theorem does not hold

Part B:

The question of net divergence requires us to consider two parts. The flux through the cylinder sides and its top and bottom.



Given the definition of a tangential vector field, we have previously shown that the tangential vectors are parallel to the surface normal such that:

$$Flux = \int_A \vec{V} \cdot \hat{n} \, dA = 0$$

In the case proposed previously the frictional layer directly cancelled with the secondary circulation, such that there was no $\hat{z}$ velocity. As such:

$$Flux = \int_A (0\hat{z}) \cdot \hat{z} \, dA = 0$$

for both the top and bottom. Thus the vector field had a zero divergence.

In order to attain a non-zero divergence, the velocity effect of the frictional layer and secondary circulation cannot be equal. This inequality would result in a non-zero $\hat{z}$ velocity component and thus:

$$Flux = \int_A B\hat{z} \cdot \hat{z} \, dA \neq 0$$

As such we now have a net divergence from the origin.

Problem 1.2
Part A

Consider the scalar field

$$A(x, y) = (x^2 + y^2)\exp(1 - x^2 - y^2)$$

(a) Using a graphing software of your choice, create a countour plot of the function.

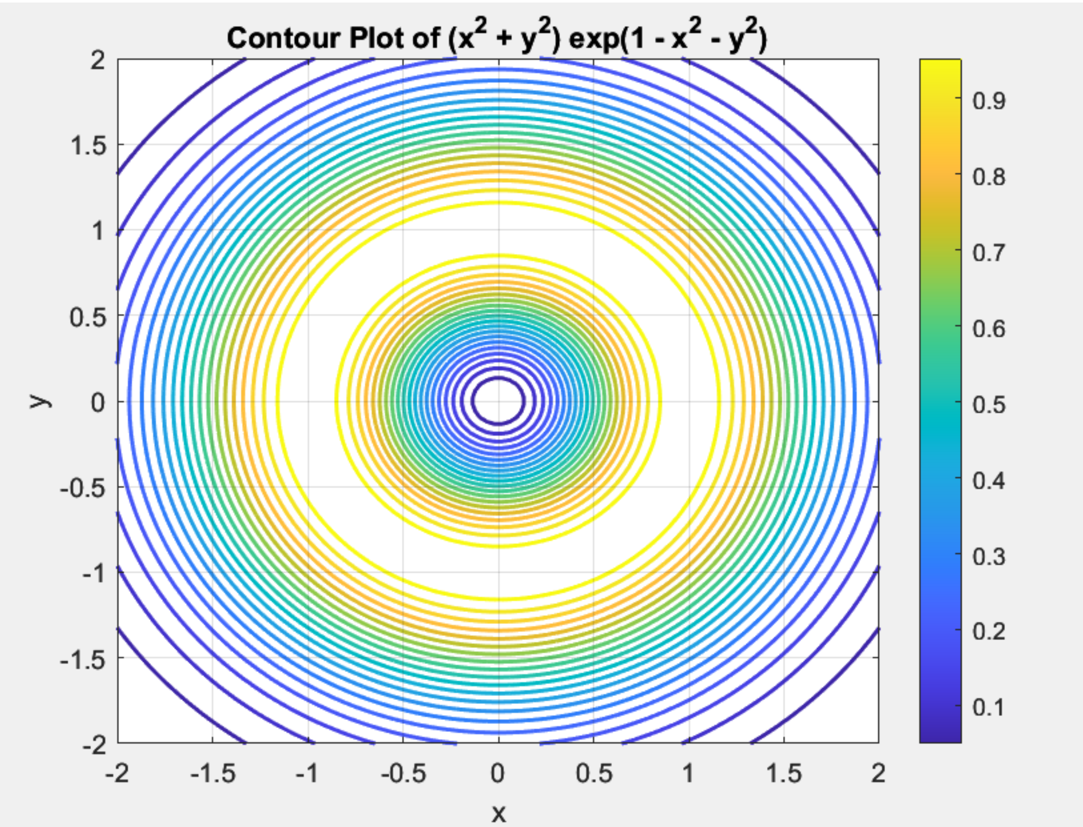
(b) Without calculating the values, discuss whether grad, curl or div exist for this field, and explain your reasoning.

(c) Calculate the gradient of the field and produce a quiver (arrow) plot.

(d) Calculate the divergence of the gradient.

Part B - Advanced
Show that in this case, Stokes theorem holds for the gradient of the field.

a)
Using MATLAB, it was found that the function was bound by $0 < A(x,y) < 1$ and thus the contours were drawn below.



b) Grad: $\nabla A = \begin{bmatrix} \frac{\partial A}{\partial x} \\ \frac{\partial A}{\partial y} \end{bmatrix}$

If we investigate the contour lines above, we notice the symmetric decrease of an inner and outer circle simultaneously. It is the grad vector that quantifies this change, with ∇A for a given point being perpendicular to the given contour, i.e.

For a point $(x, y) = (a, b)$

∇A = A vector perpendicular to the contour $A(a,b)$.

As such, the pure existence of changing contour lines implies the existence of a grad vector.

Curl and Div:

Given curl and divergence are calculated as $\nabla \times A$ and $\nabla \cdot A$ respectively, it becomes clear that the scalar field A lacks the vector components nessasary to conduct the dot and cross operations. As such, it is apparent that no curl or div exist for A.

c)

We understand $\nabla A = \begin{bmatrix} \frac{\partial A}{\partial x} \\ \frac{\partial A}{\partial y} \end{bmatrix}$

$$A(x,y) = (x^2+y^2)e^{(1-x^2-y^2)}$$

$\frac{\partial A}{\partial x}$ => let $(x^2+y^2) = u$ $\quad \frac{\partial u}{\partial x} = 2x$

$e^{(1-x^2-y^2)} = v$ $\quad \frac{\partial v}{\partial x} = -2xe^{1-x^2-y^2}$

Product rule:

$$\frac{\partial A}{\partial x} = (x^2+y^2) \times -2xe^{(1-x^2-y^2)} + 2xe^{(1-x^2-y^2)}$$

$$= e^{(1-x^2-y^2)}(-2x(x^2+y^2) + 2x)$$

$$= -2x(x^2+y^2-1)e^{(1-x^2-y^2)}$$

By the same reasoning:

$$\frac{\partial A}{\partial y} \Rightarrow \quad \text{let} \quad (x^2+y^2) = u \qquad \frac{\partial u}{\partial y} = 2y$$

$$e^{(1-x^2-y^2)} = v \qquad \frac{\partial v}{\partial y} = -2y\, e^{1-x^2-y^2}$$

Product rule:

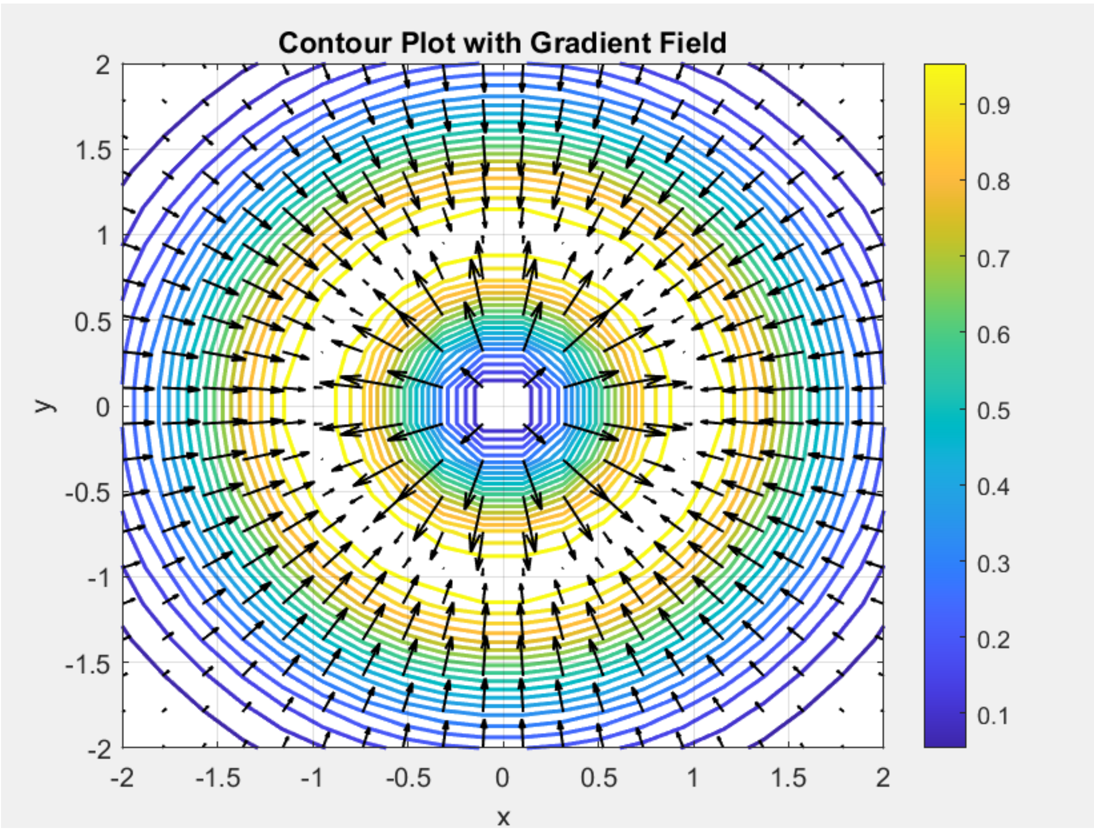
$$\frac{\partial A}{\partial x} = (x^2+y^2) \times -2y\, e^{(1-x^2-y^2)} + 2y\, e^{(1-x^2-y^2)}$$

$$= e^{(1-x^2-y^2)}\left(-2y\,(x^2+y^2) + 2y\right)$$

$$= -2y(x^2+y^2-1)e^{(1-x^2-y^2)}$$

Thus:

$$\nabla A = \begin{bmatrix} -2x\,(x^2+y^2-1)\,e^{(1-x^2-y^2)} \\ -2y\,(x^2+y^2-1)\,e^{(1-x^2-y^2)} \end{bmatrix}$$



d) Divergence = $\nabla \cdot F$

where $F = \nabla A$ so in essence:

$$Div = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} \quad \text{in this case}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{\partial}{\partial x}\left(-2x\,(x^2+y^2-1)e^{(1-x^2-y^2)}\right)$$

let $u = -2x \quad \frac{\partial u}{\partial x} = -2$

$v = (x^2+y^2-1)e^{(1-x^2-y^2)}$

$\hookrightarrow$ let $(x^2+y^2-1) = b \quad \frac{\partial b}{\partial x} = 2x$

$c = e^{1-x^2-y^2}$

Product rule $\rightarrow$ $\frac{\partial c}{\partial x} = -2xe^{1-x^2-y^2}$

$$\therefore \frac{\partial v}{\partial x} = -2xe^{1-x^2-y^2}(x^2+y^2-1) + 2xe^{1-x^2-y^2}$$

$$\therefore \frac{\partial^2 A}{\partial x^2} = 4x^2 e^{(1-x^2-y^2)}(x^2+y^2-1) - 4x^2 e^{(1-x^2-y^2)} - 2(x^2+y^2-1)e^{(1-x^2-y^2)}$$

$$= (x^2+y^2-1)e^{(1-x^2-y^2)}(4x^2-2) - 4x^2 e^{(1-x^2-y^2)}$$

$$= e^{(1-x^2-y^2)}\left[(x^2+y^2-1)(4x^2-2) - 4x^2\right]$$

$$= e^{(1-x^2-y^2)}(4x^4 - 2x^2 + 4x^2y^2 - 2y^2 - 4x^2 + 2 - 4x^2)$$

$$= e^{(1-x^2-y^2)}(4x^4 - 10x^2 + 4x^2y^2 - 2y^2 + 2)$$

$$= 2e^{(1-x^2-y^2)}(2x^4 + x^2(-5+2y^2) - y^2 + 1)$$

Through the same process, we find:

$$\frac{\partial^2 A}{\partial y^2} = 2e^{(1-y^2-x^2)}(2y^4 + y^2(-5+2x^2) - x^2 + 1)$$

Thus: $\text{div} = \dfrac{\partial^2 A}{\partial x^2} + \dfrac{\partial^2 A}{\partial y^2}$

$$= 2e^{(1-y^2-x^2)}\begin{bmatrix} 2x^4 - 5x^2 + 2x^2y^2 - y^2 + 1 \\ + 2y^4 - 5y^2 + 2x^2y^2 - x^2 + 1 \end{bmatrix}$$

$$= 2e^{(1-y^2-x^2)}\left[2(x^4+y^4) - 6(x^2+y^2) + 4x^2y^2 + 2\right]$$

$$= 4e^{(1-y^2-x^2)}\left[(x^4+y^4) - 3(x^2+y^2) + 2x^2y^2 + 1\right]$$

Part B

Stokes Theorem: $\oint_C F \cdot dr = \iint (\nabla \times F) \cdot dS$

Where $F = \nabla A$

We have previously discussed that the grad vector has no curl and thus:

$$\iint (\nabla \times F) \cdot ds = \iint 0 \cdot ds = 0$$

Turning our attention to $\int_C F \cdot dr$ we notice the line integral will take place along a circle. This is relevent as for a vector field (F) created by the grad of a scalar field ($\nabla f = F$), F is conservative, i.e:

$$\oint A \cdot dr = A_{end} - A_{start}$$

Since $A_{end} = A_{start}$ for a closed integral we notice:

$$\oint_C \nabla A \cdot dr = 0$$

Thus Stoke Theorem holds.