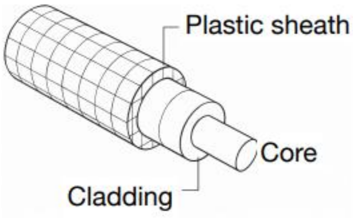
Problem 5.1
Part A
An optical fibre is a waveguide based on total internal reflection. Fibres typically have the structure indicated in the image, where there is a guiding core surrounded by a cladding layer. Light is coupled in from the end of the fibre. Fibres are usually made of fused silica glass, with slight modifications to the formulation to introduce changes in the index of refraction.



(a) What do we require of the index of the core and the index of the cladding so that the fibre will be guiding?

(b) The index contrast is defined as

$$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2}.$$

Assuming one of the materials has an refractive index $n_1 = 1.450$, determine the full acceptance angle of the fibre if it has an index contrast of 1% using Snell's law. Note that the acceptance angle is defined at the air/fibre interface, and assume the fibre has a flat input face.

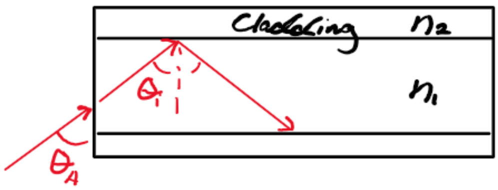
(c) The numerical aperture of the fibre is defined as $\mathrm{NA} = \sin\theta_A$, where θ_A is the half-acceptance angle. Show that $\mathrm{NA} = \sqrt{n_1^2 - n_2^2}$, where n_1 and n_2 are the index of the core and cladding respectively. *Hint:* Define Snell's law at the interface of the fibre, and use the trig identities $1 = \sin^2\theta + \cos^2\theta$ and $\cos(\alpha - \beta) = \cos(\alpha)\cos\beta + \sin(\alpha)\sin\beta$.

a) To induce total internal reflection, we understand there must be no transmission:

Snells law tells us:

$$n_i \sin\theta_i = n_t \sin\theta_t$$

$$\frac{n_i}{n_t} = \frac{\sin\theta_t}{\sin\theta_i}$$



We understand the critical angle occurs at: $\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$
In order for θ_i to be larger than θ_c, meaning total internal reflection has occured.

$$\theta_i > \theta_c : \theta_i > \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

We notice $\sin\theta_c \leq 1$ by trig

thus:

$$\frac{n_2}{n_1} \leq 1$$

And thus $n_2 \leq n_1$. However for criticality, and $\theta_i > \theta_c$ we then understand:

$$n_2 \ll n_1$$

$$n_{cladding} \ll n_{fibers}.$$

b) $$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2}$$

We understand:

$\Delta = 0.01$ (1%)

and $n_1 = 1.450$

$$0.01 = \frac{1.45^2 - n_2^2}{2 \times 1.45^2}$$

$$n_2 = 1.435$$

We understand from the figure above with acceptance angle $\pm\theta_A$, we have a full acceptance angle of $2\theta_A$.

In this case:

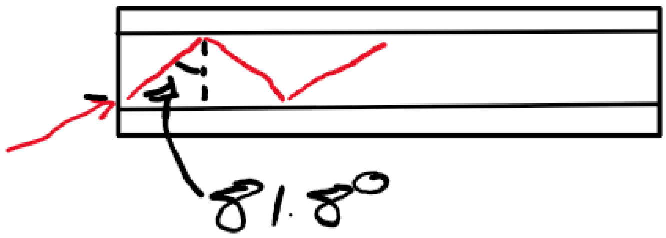
$n_{cladding} = 1.435$

$n_{fiber} = 1.45$

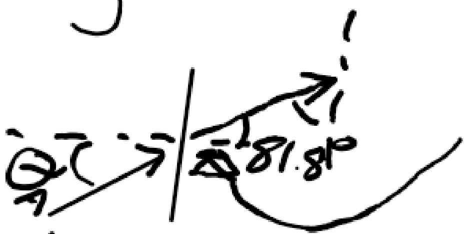
And thus: $\theta_c = \sin^{-1}\left(\frac{1.435}{1.45}\right)$

$= 81.8^\circ$

i.e:



Looking at the air/fibre boundary:



$\theta = 90 - 81.8^\circ$
$= 8.2^\circ$

with air as $n=1$,

$1 \times \sin\theta_A = 1.45 \sin(8.2)$
$\therefore \theta_A = 11.9^\circ$

Thus the full acceptance angle is $11.9 \times 2 = 23.9^\circ$

c) Defining $NA = \sin\theta_A$

Since:

$$\sin\theta_A = n_1 \sin\theta_i$$

n_1 — fibre

$$NA = n_1 \sin\theta_i = n_1 \cos(90 - \theta_c)$$
$$= n_1 \cos(\theta_c) \quad (1)$$

Additionally:

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

$$\therefore \frac{n_2}{n_1} = \sin(\theta_c)$$

using $1 = \cos^2\theta + \sin^2\theta$

$$\cos\theta_c = \sqrt{1 - \left(\frac{n_2}{n_1}\right)^2} = \sqrt{\frac{n_1^2 - n_2^2}{n_1^2}}$$

Subbing into (1):

$$NA = n_1 \sqrt{\frac{n_1^2 - n_2^2}{n_1^2}}$$
$$= \sqrt{n_1^2 - n_2^2}$$

Part B Advanced

When coupling light into the fibre, there will be a loss due to Fresnel reflection due to the index mismatch between the air and fibre core. Numerically determine the average coupling loss due to reflection for unpolarised light (equal probability of $s-$ or p-polarisation) across the acceptance angle.

Fresnel reflection due to air $(n=1)$ fibre (n) core:

$$r_{\parallel} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t} \qquad r_{\perp} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$= \frac{n\cos\theta_A - \cos\theta_t}{n\cos\theta_A + \cos\theta_t} \qquad = \frac{\cos\theta_A - n\cos\theta_t}{\cos\theta_A + n_t \cos\theta_t}$$

$$T = \frac{I_t}{I_i} = 1 - R = 1 - r^2$$

Given $r_{\parallel} = r_{\perp}$, as described in the question, using $r_{\parallel}$:

$$T = 1 - \left(\frac{n\cos\theta_A - \cos\theta_t}{n\cos\theta_A + \cos\theta_t}\right)^2$$

For the conditions observed above

$n = 1.450$, $\theta_A = 11.9$, $\theta_t = 8.2^\circ$

We find:

$T \approx 0.968$ thus we understand there is $(1 - 0.968) \times 100 = 3.17\%$ loss due to reflection.

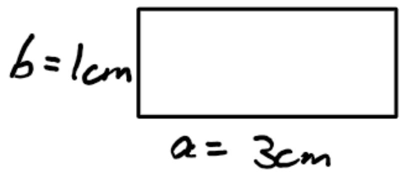
Problem 5.2
Part A
A radar engineer is designing the communication system between a high-frequency oscillator amplifier and an antenna. The engineer uses a rectangular waveguide of dimensions $a = 3$ cm and $b = 1$ cm and wants to operate the system with minimal interference from higher-order modes and minimal attenuation.

(a) What is the lowest frequency at which the waveguide can operate in single-mode TE_{10} transmission?

(b) The engineer selects a frequency 10% above this cut-off, what is the phase velocity in the waveguide ?

(c) Find the guided wavelength and compare it to the free-space wavelength.

a) For a rectangular waveguide of dimensions:



TE_{10} mode refers to a wave with only verticle polarisation:

$$\omega_c = c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

$$= c\sqrt{\left(\frac{\pi}{a}\right)^2 \times 0^2}$$

$$= \frac{c\pi}{a} = \frac{c\pi}{3\times10^{-2}} = 3.1\times10^{10}$$

Finally given $f_c = \frac{\omega_c}{2\pi}$

$$f_c = \frac{c}{2\times3\times10^{-2}} = 5\,GHz$$

Thus 5GHz is the lowest frequency at which it can operate in single mode TE_{10}.

b) $5 \times \frac{110}{100}$ ← 10% higher

$f = 5.5\,GHz$

$\therefore \quad \omega = 2\pi \times 5.5 \times 10^9$

$= 3.45 \times 10^{10}$

Given $V_{phase} = \dfrac{c}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$

$$= \frac{c}{\sqrt{1 - \left(\frac{3.1}{3.45}\right)^2}}$$

$$= 6.8 \times 10^8\, m/s$$

c) For free space:

$\lambda_o = \dfrac{c}{f} = \dfrac{3 \times 10^8}{5.5 \times 10^9}$

$= 0.055\,m$ ← Note this is for the 10% extra case

In the guide:

$$\lambda_g = \frac{6.8 \times 10^8}{5.5 \times 10^9} = 0.124\,m$$

Thus the guided wavelength is 0.069 m longer than in free space.

Part B Advanced

(a) What is the maximum frequency the engineer can use before the TE_{20} mode starts to propagate? What is the frequency range required so that only TE_{10} propagates ?

(b) What is the relation between group velocity and wavenumber ? Using the same frequency as in question b, calculate the group velocity v_g of the TE_{10} mode in the waveguide. Verify that $v_p.v_g = c^2$ and interpret the meaning of the relation in the context of waveguides.

a) TE_2 occurs when:

$$f_c = \frac{\omega_c}{2\pi} = \frac{c}{2\pi}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

$$= \frac{c}{2\pi}\sqrt{\left(\frac{2\pi}{3\times10^{-2}}\right)^2 \times 0^2}$$

$$= 10\ GHz$$

We understand TE_{20} is the second lowest cutoff frequency as

$$TE_{01} = \frac{c}{2\pi}\sqrt{\left(\frac{1\pi}{1\times10^{-2}}\right)^2} = 15\ GHz$$

$$TE_{11} = \frac{c}{2\pi}\sqrt{\left(\frac{2\pi}{3\times10^{-2}}\right)^2 + \left(\frac{\pi}{1\times10^{-2}}\right)^2} = 18\ GHz$$

Thus, we understand only TE_{10} propagates within the frequency range:

$$5GHz < f < 10GHz$$

b)

$$V_g = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$

$$= c\sqrt{1-\left(\frac{3.1}{3.45}\right)^2}$$

$$= 1.3 \times 10^8 \, m/s$$

Given $V_p = 6.8 \times 10^8 \, m/s$

$$V_g \cdot V_p = 8.84 \times 10^{16}$$
$$\approx c^2$$

as $\sqrt{V_g \cdot V_p} = 2.97 \times 10^8 \, m/s \approx c$

Interpreting this, we understand that for a waveguide of no loss, there is a phase-group velocity tradeoff. I.E:

$$V_p \propto \frac{1}{V_g} \quad \text{and} \quad V_p V_g = c^2$$

We will always find $V_p > c$, as the waves phase fronts appear faster, yet the overall group velocity always obeys $V_g < c$.

One interesting result is that as $\omega \to \omega_c$, $V_p \to \infty$ and $V_g \to 0$. This is extremely bad as it is this indicates that the energy is barely flowing.