$$\boxed{g(N,s) = \frac{N!}{(\frac{1}{2}N + s)!\,(\frac{1}{2}N - s)!} = \frac{N!}{N_\uparrow!\,N_\downarrow!}.} \tag{15}$$

We take the logarithm of both sides of (15) to obtain

$$\log g(N,s) = \log N! - \log(\tfrac{1}{2}N + s)! - \log(\tfrac{1}{2}N - s)!\ , \tag{22}$$

by virtue of the characteristic property of the logarithm of a product:

$$\log xy = \log x + \log y; \qquad \log(x/y) = \log x - \log y. \tag{23}$$

With the notation

$$N_\uparrow = \tfrac{1}{2}N + s; \qquad N_\downarrow = \tfrac{1}{2}N - s \tag{24}$$

for the number of magnets up and down, (22) appears as

$$\log g(N,s) = \log N! - \log N_\uparrow! - \log N_\downarrow!. \tag{25}$$

We evaluate the logarithm of $N!$ in (25) by use of the **Stirling approximation**, according to which

$$N! \simeq (2\pi N)^{1/2} N^N \exp[-N + 1/(12N) + \cdots]\ , \tag{26}$$

for $N \gg 1$. This result is derived in Appendix A. For sufficiently large N, the terms $1/(12N) + \cdots$ in the argument may be neglected in comparison with N. We take the logarithm of both sides of (26) to obtain

$$\log N! \cong \tfrac{1}{2}\log 2\pi + (N + \tfrac{1}{2})\log N - N. \tag{27}$$

Similarly

$$\log N_\uparrow! \cong \tfrac{1}{2}\log 2\pi + (N_\uparrow + \tfrac{1}{2})\log N_\uparrow - N_\uparrow; \tag{28}$$

$$\log N_\downarrow! \cong \tfrac{1}{2}\log 2\pi + (N_\downarrow + \tfrac{1}{2})\log N_\downarrow - N_\downarrow. \tag{29}$$

After rearrangement of (27),

$$\log N! \cong \tfrac{1}{2}\log(2\pi/N) + (N_\uparrow + \tfrac{1}{2} + N_\downarrow + \tfrac{1}{2})\log N - (N_\uparrow + N_\downarrow)\ , \tag{30}$$

where we have used $N = N_\uparrow + N_\downarrow$. We subtract (28) and (29) from (30) to obtain for (25):

$$\log g \cong \tfrac{1}{2}\log(1/2\pi N) - (N_\uparrow + \tfrac{1}{2})\log(N_\uparrow/N) - (N_\downarrow + \tfrac{1}{2})\log(N_\downarrow/N). \tag{31}$$

This may be simplified because

$$\log(N_\uparrow/N) = \log\tfrac{1}{2}(1 + 2s/N) = -\log 2 + \log(1 + 2s/N)$$
$$\cong -\log 2 + (2s/N) - (2s^2/N^2) \tag{32}$$

by virtue of the expansion $\log(1 + x) = x - \frac{1}{2}x^2 + \cdots$, valid for $x \ll 1$. Similarly,

$$\log(N_\downarrow/N) = \log\tfrac{1}{2}(1 - 2s/N) \simeq -\log 2 - (2s/N) - (2s^2/N^2). \tag{33}$$

On substitution in (31) we obtain

$$\log g \cong \tfrac{1}{2}\log(2/\pi N) + N\log 2 - 2s^2/N. \tag{34}$$

We write this result as

$$\boxed{g(N,s) \cong g(N,0)\exp(-2s^2/N)\ ,} \tag{35}$$

where

$$g(N,0) \simeq (2/\pi N)^{1/2}2^N. \tag{36}$$

Such a distribution of values of s is called a **Gaussian distribution**. The integral* of (35) over the range $-\infty$ to $+\infty$ for s gives the correct value 2^N for the total number of states. Several useful integrals are treated in Appendix A.

The exact value of $g(N,0)$ is given by (15) with $s = 0$:

$$g(N,0) = \frac{N!}{(\frac{1}{2}N)!\,(\frac{1}{2}N)!}. \tag{37}$$

---

* The replacement of a sum by an integral, such as $\sum_s(\ldots)$ by $\int(\ldots)ds$, usually does not introduce significant errors. For example, the ratio of

$$\sum_{s=0}^{N} s = \tfrac{1}{2}(N^2 + N) \qquad \text{to} \qquad \int_0^N s\,ds = \tfrac{1}{2}N^2$$

is equal to $1 + (1/N)$, which approaches 1 as N approaches ∞.

2 Flipping 10 coins

(a) There are 2 possible outcomes for the first coin, and for each of these, two for the second coin, and for each of these, two for the third coin, and so on. So the total number of microstates is

$$2^{10} = 1024$$

(b) The sequence in question is just one possible outcome out of 2^{10}. So its probability is

$$\frac{1}{1024}$$

(c) The <u>probability</u> of getting 6 heads (and therefore 4 tails for the remaining coins out of total 10) is given by the fraction of all states that have 6 heads, i.e. by the multiplicity of a macrostate with 6 heads

$$g(6) = \binom{10}{6} = \frac{10!}{6!\,\underbrace{(10-6)!}_{4!}} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{1\cdot2\cdot3\cdot4\cdot5\cdot6 \times 1\cdot2\cdot3\cdot4}$$

$$= \frac{7\cdot8\cdot9\cdot\cancel{10}^{5}}{\cancel{2}\cdot3\cdot4} = 7\cdot2\cdot3\cdot5 = 210$$

relative to the total number of possible outcomes, i.e.

$$P(6) = \frac{g(6)}{2^{10}} = \frac{210}{1024} = 0.205 \quad \text{or} \quad \underline{\underline{20.5\%}}$$

3.

The number of ways of choosing five specific cards from 52 (in no particular order) is simply

$$\binom{52}{5} = \frac{52!}{(5!)(47!)} = 2.6 \times 10^6,$$

or 2.6 million. Of all of these possible hands, only four are royal flushes, so the probability of getting a royal flush on the first deal is

$$\frac{4}{2.6 \times 10^6} = 1.54 \times 10^{-6},$$

that is, somewhat better than one in a million.

n here is the same thing as q_tot

4. (a) $U = \sum_{i=1}^{N} j_i \hbar\omega = n\hbar\omega$

$$\underbrace{j_1 + j_2 + j_3 + \cdots + j_N}_{N \text{ terms}} = n$$

N terms, N harmonic oscillators

The number of excitation quanta on each oscillator can vary $j_i = 0, 1, 2, \ldots n$ (up to the maximum value of n, defined by the given total energy $U = n\hbar\omega$).

Examples of microstates:

$j_1 = n$, j_1

j_1	j_2	j_3	$\cdots$	j_N
n	0	0	0	0
0	n	0	0	0
$n-1$	1	0	0	0
1	$n-1$	0	0	0
3	0	10	0	1
$\vdots$				

$\rightarrow$ such that $3 + 10 + 1 = 14$ (if n was $n = 14$)

or in general, such that

$j_1 + j_2 + \cdots j_N = n$

The problem of finding out the multiplicity $g(N,n)$ is equivalent to the number of ways of distributing n balls among N boxes
(n excitation quanta among N harmonic oscillators)

"oscillator 1" $\equiv$ "box 1"

"n quanta" $\equiv$ "n balls"

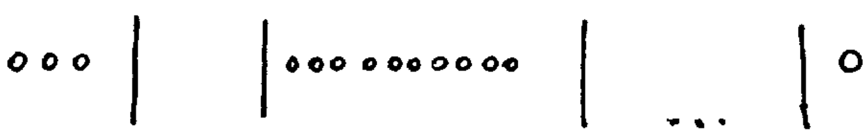
An example of a microstate:

$j_1 = 3$ (3 quanta) $\quad$ $j_2 = 0$ (0 quanta) $\quad$ $j_3 = 10$ (10 quanta) $\quad$ $j_N =$ [illegible]

box No. 1 $\quad$ No. 2 $\quad$ No. N

A box may be empty since $j_i = 0$ is possible.

The numbe of ways to distribute n balls among N boxes can be obtained by finding the number of permutations of placing in a row all the balls together with $(N-1)$ matchsticks that designate the dividing walls [For N boxes there is $N-1$ dividing walls]



If one labels all the balls and matches with the running numbers $1, 2, \dots, n+N-1$ (n balls, $N-1$ matches), then

$$
\begin{array}{cccccccccccccccccc}
\circ & \circ & \circ & | & & | & \circ & \circ & \circ\circ\circ\circ\circ\circ\circ\circ\circ & | & \cdots & | & \circ \\
1 & 2 & 3 & 4 & & & 5 & 6 & 7\,8\,9\,10\,11\,12\,13\,\cdots & & & & n+N-1
\end{array}
$$

then the number of permutations of these is

$$(n+N-1)!$$

This number $(n+N-1)!$, however, overcounts the required number of ways of distributing n balls among N boxes, since the n balls and $(N-1)$ matches are indistinguishable.

One could not tell the difference if we rearrange n balls ($n!$ permutations) and $N-1$ matches ($(N-1)!$ permutations) within each configuration, since the total energy $U=n\hbar\omega$ would be the same.

Thus we need to divide $(n+N-1)!$ by $n!$ and by $(N-1)!$

Therefore the number of ways of distributing n balls among N boxes (the number of microstates corresponding to the total energy $U=n\hbar\omega$) is:

$$g(N,n) = \frac{(n+N-1)!}{n!\,(N-1)!}$$

(b) $S = k_B \ln(g) = k_B \ln \dfrac{(n+N-1)!}{n!\,(N-1)!}$

$$= k_B \left[\ln (n+N-1)! \quad - \ln n! \quad - \ln (N-1)! \right]$$

since $\ln \frac{a}{bc} = \ln a - \ln(bc)$
$= \ln a - \ln b - \ln c$

stirling approximation:

$$\simeq k_B \Big[(n+N-1) \ln(n+N-1) - (n+N-1)$$
$$- n \ln n \quad + n$$
$$- (N-1) \ln (N-1) \quad + (N-1) \Big]$$

next, use $N \gg 1$ to replace $N-1$ by N

$$\simeq k_B \Big[(n+N) \ln (n+N) \ \cancel{- n} \ \cancel{- N} \quad - n \ln n \quad \cancel{+ n}$$
$$- N \ln N \quad \cancel{+ N} \Big]$$

$$= k_B \left[(n+N) \ln (n+N) \quad - n \ln n \quad - N \ln N \right]$$

Thus $\boxed{\dfrac{S}{k_B} = (n+N) \ln (n+N) \quad - n \ln n \quad - N \ln N}$

(c) From $U = n\hbar\omega$, $n = \frac{U}{\hbar\omega}$.

Substitute $n = \frac{U}{\hbar\omega}$ onto the expression for S :

$$\frac{S}{k_B} = \left(\frac{U}{\hbar\omega} + N\right) \ln\left(\frac{U}{\hbar\omega} + N\right) - \frac{U}{\hbar\omega} \ln \frac{U}{\hbar\omega} - N \ln N$$

From the definition of temperature

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N$$

we find (by differentiating $\frac{S}{k_B}$ with respect to U)

$$\frac{1}{T} = k_B \left[\frac{1}{\hbar\omega} \ln\left(\frac{U}{\hbar\omega} + N\right) + \left(\frac{U}{\hbar\omega} + N\right) \frac{1}{\frac{U}{\hbar\omega} + N} \cdot \frac{1}{\hbar\omega} \right.$$

$$\left. - \frac{1}{\hbar\omega} \ln \frac{U}{\hbar\omega} - \frac{U}{\hbar\omega} \cdot \frac{1}{\frac{U}{\hbar\omega}} \cdot \frac{1}{\hbar\omega} \right]$$

$$= k_B \left[\frac{1}{\hbar\omega} \ln\left(\frac{U}{\hbar\omega} + N\right) - \frac{1}{\hbar\omega} \ln \frac{U}{\hbar\omega} \right]$$

$$= \frac{k_B}{\hbar\omega} \ln \frac{\frac{U}{\hbar\omega} + N}{\frac{U}{\hbar\omega}}$$

Thus

$$\ln \frac{\frac{U}{\hbar\omega} + N}{\frac{U}{\hbar\omega}} = \frac{\hbar\omega}{k_B T}$$

$$\Longrightarrow \quad \frac{\frac{U}{\hbar\omega} + N}{\frac{U}{\hbar\omega}} = e^{\frac{\hbar\omega}{k_B T}}$$

or

$$1 + \frac{N\hbar\omega}{U} = e^{\hbar\omega/k_B T}$$

or

$$\frac{N\hbar\omega}{U} = e^{\frac{\hbar\omega}{k_B T}} - 1$$

or

$$\boxed{U = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1}} \qquad !!!$$

From counting the number of ways of putting n "balls" into N "boxes" we have derived the famous result by

Max Plank,

which explained the full spectrum of black-body radiation and led to the birth of quantum mechanics!