Problem Set 9

1

$$E_F = \frac{\pi \hbar c}{L} n_F$$ — defines n_F corresponding to E_F

(a) in 3D:

$$N = (2\overset{s=1/2}{S}+1)\,\frac{1}{8}\,\frac{4\pi}{3}\,n_F^3 = \frac{\pi}{3}\,n_F^3$$

$$\therefore \quad n_F = \left(\frac{3N}{\pi}\right)^{1/3}$$

$$\therefore \quad E_F = \frac{\pi \hbar c}{L} n_F = \pi \hbar c \left(\frac{3N}{\pi V}\right)^{1/3} \qquad (V = L^3)$$

2

(b) in 1D: $N = (\overset{s=1/2}{2S+1})\, n_F = 2n_F$

$\therefore\ n_F = \frac{N}{2}$

$\therefore\ E_F = \frac{\pi\hbar c}{L} n_F = \frac{\pi\hbar c N}{2L}$

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in 2D: $N = \underbrace{(2S+1)}_{S=1/2} \frac{1}{4}\pi n_F^2 = \frac{\pi n_F^2}{2}$

$\therefore\ n_F = \left(\frac{2N}{\pi}\right)^{1/2}$

$\therefore\ E_F = \frac{\pi\hbar c}{L} n_F = \frac{\pi\hbar c}{L}\sqrt{\frac{2N}{\pi}} = \hbar c\sqrt{\frac{2\pi N}{A}}$ $(A=L^2)$

(c) $U_0 = \int_0^{E_F} dE \cdot \underbrace{D(E)} \cdot E$ $\longrightarrow$ (as in Problem 1(b))

$\hookrightarrow D(E) = \frac{(\overset{s=1/2}{2S+1})\, V}{2\pi^2\hbar^3 c^3} E^2$ $\rightarrow$ from Problem 3, Problem Set 6

(in 3D)

$$U_0 = \int_0^{E_F} dE \left(\frac{V}{\pi^2\hbar^3c^3}\right) E^3 = \frac{V}{4\pi^2\hbar^3c^3} E_F^4$$

Use the result of (a), that $E_F^3 = (\pi\hbar c)^3 \frac{3N}{\pi V}$

$$\therefore\ U_0 = \frac{V}{4\pi^2\hbar^3c^3} (\pi^3\hbar^3c^3) \cdot \frac{3N}{\pi V} \cdot E_F = \frac{3}{4} N E_F$$

Thus: $\boxed{U_0 = \frac{3}{4} N E_F}$ (in 3D)

2

$$p = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$$

For a gas in the ground state $U \to U_0$

(a) For a non-relativistic Fermi gas:

$$U_0 = \frac{3}{5} N E_F = \frac{3\hbar^2}{10m}\left(\frac{6\pi^2}{2S+1}\right)^{2/3} N \left(\frac{N}{V}\right)^{2/3}$$ – from lecture notes (Lecture 13); or can be found similarly to Problem 1, but in 3D

$$\therefore \; P = -\left(\frac{\partial U_0}{\partial V}\right)_{S,N} = -\frac{3\hbar^2}{10m}\left(\frac{6\pi^2}{2S+1}\right)^{2/3} N^{5/2} \frac{\partial}{\partial V}\left(\frac{1}{V^{2/3}}\right)$$

$$= \frac{\hbar^2}{5m}\left(\frac{6\pi^2}{2S+1}\right)^{2/3}\left(\frac{N}{V}\right)^{5/3}$$

For electrons $S = 1/2$: $\boxed{P = \frac{\hbar^2 (3\pi^2)^{2/3}}{5m}\left(\frac{N}{V}\right)^{5/3}}$

(b) For a relativistic electron gas

$$U_0 = \frac{3}{4} N E_F = \frac{3}{4} N \hbar \pi c \left(\frac{3N}{\pi V}\right)^{1/3}$$ – from Problem 2(c) and (a)

$$\therefore \; p = -\left(\frac{\partial U_0}{\partial V}\right)_{S,N} = -\frac{3}{4} N \hbar \pi c \left(\frac{3N}{\pi}\right)^{1/3} \frac{\partial}{\partial V}\left(\frac{1}{V^{1/3}}\right)$$

$$\therefore \; \boxed{p = \frac{(3\pi^2)^{1/3}}{4} \hbar c \left(\frac{N}{V}\right)^{4/3}}$$

3 For a degenerate gas of spin 1/2 particles (non-relativistic), the heat capacity is

$$C_V = \frac{\pi^2 N k_B^2 T}{2 E_F}$$ – from lecture notes, Lecture 14

where $E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$ – nonrelativistic result

$$\therefore \; C_V = \underbrace{\frac{\pi^2 k_B}{2 \cdot \frac{\hbar^2}{2m} \cdot \left(\frac{3\pi^2 N}{V}\right)^{2/3}}}_{\text{numerical coefficient}} N k_B T$$

where, m is the mass of a ^3He atom

$m_{^3He} = 3 \text{ a.m.u.} \approx 3 m_p$

(^3He consists of 2 protons, 1 neutron, and 2 electrons; neglect the mass of electons, and take $m_p \simeq m_n \simeq 1.67 \times 10^{-27}$ kg)

Thus $m \approx 3 m_p \simeq 3 \times 1.67 \times 10^{-27}$ kg.

In addition, we can use the mass-density $\rho = \frac{M}{V}$ – where M is the total mass of the gas, i.e. $M = N \times m_{^3He}$ – to find N

$$\rho = \frac{N m}{V} \implies \underline{\underline{N = \frac{\rho V}{m}}}$$

Thus:

$$C_V = \underbrace{\frac{\pi^2 k_B}{\frac{\hbar^2}{m}\left(\frac{3\pi^2 \rho \cancel{V}}{m \cancel{V}}\right)^{2/3}}}_{\text{Here:}} N k_B T$$

Here: $m \simeq 3 \times 1.67 \times 10^{-27}$ kg
$\rho = 81$ kg/m^3
$k_B = 1.381 \times 10^{-23}$ J/K
$\hbar = 1.0546 \times 10^{-34}$ J.s

This gives:

$C_V \simeq 1 \cdot N k_B T$ – reasonably close to the experimental value of $2.89\, N k_B T$; the difference is due to the interactions between the ^3He atoms, which were neglected in our treatment of the gas as an <u>ideal</u> gas.

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4 (a) $U_G \sim -\frac{GM}{R}$ – follows from the exact result of $U_G = -G \int_0^R \frac{M(r)\, dM(r)}{r} = \ldots$

$$= -\frac{3GM^2}{5R}$$

The mass of the white dwarf star is $M \simeq N m_p$ (as $m_p \gg m_e$); N – is the number of protons equal to the number of electrons. Under extremely high densities in white dwarf stars, the atoms are ionized into thei nuclei and free electrons; we treat the electrons as a degenerate, nonrelativistic ideal Fermi gas. The kinetic energy of the electron gas has to balance the gravitational attraction which is due to protons.

(b) $U_0 = \frac{3}{5} N E_F = \frac{3\hbar^2}{10m}\left(3\pi^2\right)^{2/3} \frac{N^{5/3}}{V^{2/3}}$ – from lecture notes (Lecture 13)

Here m is the mass of an electron m_e,

$V = \frac{4}{3}\pi R^3$ – volume of the star, wher R is the radius

[For an order-of-magnitude estimate, we adapt the result for U_0 – derived for a gas in a cubic box of the same volume V]

$\therefore \quad U_0 \sim \frac{\hbar^2}{m_e} \frac{N^{5/3}}{R^2}$ [keeping track of the numerical coefficient gives $\frac{3\cdot(3\pi^2)^{2/3}}{10\,(4\pi/3)^{2/3}} = 1.105$]

(c) From $\underbrace{U_G + U_0 = 0}_{\text{to balance}} \implies \frac{GM^2}{R} \sim \frac{\hbar^2 N^{5/3}}{m_e R^2}$, where $N \simeq \frac{M}{m_p}$

$\therefore \quad \frac{GM^2}{R} \sim \frac{\hbar^2 M^{5/3}}{m_e\, m_p^{5/3} R^2} \implies \boxed{M^{1/3} R \sim \frac{\hbar^2}{G\, m_e\, m_p^{5/3}}}$

With $m_e \simeq 9.1 \times 10^{-31}$ kg
$m_p \simeq 1.67 \times 10^{-27}$ kg
$G = 6.67 \times 10^{-11}$ $N \cdot m^2 \cdot kg^{-2}$
$\hbar = 1.0546 \times 10^{-34}$ J·s

we obtain $\boxed{M^{1/3} R \sim 10^{17} \text{ kg}^{1/3} \cdot \text{m}}$

(d) If $M = M_\odot = 2\times10^{30}$ kg, then the equilibrium radius of the white dwarf would be:

$$R \sim \frac{10^{17}\ \mathrm{kg}^{1/3}\cdot \mathrm{m}}{M_\odot^{1/3}} = \frac{10^{17}}{(2\times10^{30})^{1/3}}\ \mathrm{m} \simeq 8\times10^{6}\ \mathrm{m}$$

(for comparison, the radius of the Sun is: $R_\odot = 7\times10^{8}$ m)

Mass-density of the dwarf with $M = M_\odot$

$$\rho = \frac{M}{V} = \frac{M_\odot}{\frac{4}{3}\pi R^3} = \frac{2\times10^{30}}{\frac{4}{3}\pi\,(8\times10^{6})^3}\ \frac{\mathrm{kg}}{\mathrm{m}^3} \approx 8\times10^{8}\ \frac{\mathrm{kg}}{\mathrm{m}^3}$$

(for comparison, the density of the sun is: $\rho_\odot = 10^{3}\ \frac{\mathrm{kg}}{\mathrm{m}^3}$)

Are the electrons non-relatovistic?

The average kinetic energy per electron can be estimated from $U_0 = \frac{3}{5} N E_F$, i.e. $\frac{U_0}{N} = \frac{3}{5} E_F \sim E_F$

– it is of the order of E_F.

The nonrelatovistic result for E_F (from the lecture notes) is:

$$E_F = \frac{\hbar^2}{2m_e}\left(\frac{3\pi^2 N}{V}\right)^{2/3} = \frac{\hbar^2}{2m_e}\left(\frac{3\pi^2\ M/m_p}{\frac{4\pi}{3}R^3}\right)^{2/3}$$

With the above estimate of $R \sim 8\times10^{6}$ m (and $M = 2\times10^{30}$ kg) one obtains:

$$E_F \sim 0.5\times10^{-13}\ \mathrm{J} = 3\times10^{5}\ \mathrm{eV}$$

This is comparable withe the value of $m_e c^2 \sim 5\times10^{5}$ eV; i.e. relativistic effects are significant, but not dominant under these densities; at higher densities relativistic effects will be dominant.

(e) For electrons in the extreme relativistic regime

$$U_0 = \frac{3}{4} N E_F ,$$

where $E_F = \hbar \pi c \left(\frac{3N}{\pi V}\right)^{1/3}$ – from problem No. 4

Therefore $U_0 = \frac{3}{4} N \cdot \hbar \pi c \left(\frac{3N}{\pi V}\right)^{1/3}$

i.e $U_0 \sim \dfrac{\hbar c N^{4/3}}{V^{1/3}}$

where $N = \dfrac{M}{m_p}$ and $V = \dfrac{4\pi}{3} R^3$

Therefore $U_0 \sim \dfrac{\hbar c}{R} \left(\dfrac{M}{m_p}\right)^{4/3}$

Now, from $U_G + U_0 = 0$ we get the following equilibrium condition

$$\frac{G M^2}{\cancel{R}} \simeq \frac{\hbar c}{\cancel{R}} \left(\frac{M}{m_p}\right)^{4/3}$$

– radius R drops out from the answer, and we get just an equilibrium condition for the mass

$$M^{2/3} \simeq \frac{\hbar c}{G m_p^{4/3}}$$

or

$$\boxed{M \simeq \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2}} \simeq 3.4 \times 10^{30}\ \text{kg}$$

– this has a meaning of a critical mass M_{cr}, which is the only possible value of the mass that can maintain an equilibrium of a white dwarf star with extreme relativistic electrons.

If the actual mass of the star $M \neq M_{cr}$ then the star can not be in equilibrium. If $M < M_{cr}$ (not enough mass for gravitational pull to ballance the Fermi pressure), then the star will expand untill the particles become non-relativistic (they'll slow down in expansion), and then we are back to the situation of mass-radius relationship of part (c). If, on the other hand, $M > M_{cr}$, the star will contract (collapse) without limit.

(f) There is, however, an intermediate stage in such a collapse, owing to the reaction

$$p + e^- \longrightarrow n + \nu$$

(neutron) (neutrino)

This might form a <u>neutron star</u>.

Repeating the calculation for a neutron star, first for nonrelativistic neutrons, gives:

$$\frac{GM^2}{R} \simeq \frac{\hbar^2 N^{5/3}}{m_n R^2}, \quad \text{where } N = \frac{M}{m_n}$$

$m_n = 1.675 \times 10^{-27}$ kg — mass of the neutron

$$\therefore \quad \frac{GM^2}{R} \simeq \frac{\hbar^2 M^{5/3}}{m_n \, m_n^{5/3} R^2}$$

$$\therefore \quad M^{1/3} R \simeq \frac{\hbar^2}{G m_n^{8/3}} \sim 0.5 \times 10^{14} \text{ kg}^{1/3} \cdot \text{m}$$

If $M = M_\odot = 2 \times 10^{30}$ kg, then $R \sim 4 \times 10^3$ m —

— just 4 km!, and $\rho \sim 7 \times 10^{18}$ kg/m^3

(g) For an extreme relativistic neutron star:

$$U_0 = \frac{3}{4} N E_F \quad , \quad E_F = \hbar \pi c \left(\frac{3N}{\pi V} \right)^{1/3}$$

– again from Problem No. 4 (or as in part (e)). The result for neutrons is the same as for an electron gas as the spin of a neutron is $s = 1/2$.

Thus $U_0 \simeq \dfrac{\hbar c N^{4/3}}{V^{1/3}}$, where

$N = \dfrac{M}{m_n}$ (compare with part (e), where the role of m_n was played by m_p)

$m_n \simeq m_p$ – so the numerical estimates will be the same.

Therefore, we get similar results as in part (e):

$$U_0 \simeq \frac{\hbar c}{R} \left(\frac{M}{m_n} \right)^{4/3}$$

and from the equilibrium condition $U_G + U_0 = 0$, we get the value of the critical mass

$$M_{cr} \sim \frac{1}{m_n^2} \left(\frac{\hbar c}{G} \right)^{3/2} \simeq 3.4 \times 10^{30}\,\text{kg}$$

– with the same conclusions about the fate (collapse) of a neutron star as in part (e).