PHYS2941 Problem set 5

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Problem 9.2)

$$\frac{-\hbar^2}{2m}\nabla^2\Psi + V(r)\Psi = E\Psi$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$V(x,y,z) = \begin{cases} 0 & x\in[0,a],\ y\in[0,b],\ z\in[0,c] \\ \infty & \text{elsewhere} \end{cases}$$

For inside the box, $V(r)=0$ Thus:

$$\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi = E\Psi$$

Expanding this we find:

$$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial y^2} + \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial z^2} = E\Psi$$

Alternatively, we can employ separation of variables using $\Psi(x,y,z) = f(x)g(y)h(z)$ to find:

$$\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f(x)g(y)h(z) = E f(x)g(y)h(z)$$

$$= \frac{-\hbar^2}{2m}\left(g(y)h(z)\frac{\partial^2 f(x)}{\partial x^2} + f(x)h(z)\frac{\partial^2 g(y)}{\partial y^2} + f(x)g(y)\frac{\partial^2 h(z)}{\partial z^2}\right)$$

Dividing by $\frac{-\hbar^2}{2m}f(x)g(y)h(z)$

$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} + \frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2} + \frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2} = \frac{-2mE}{\hbar^2}$$

Given we understand that $\frac{-2mE}{\hbar^2}$ must remain constant, so too must the sum on the left for Schrödingers equation to hold true. i.e:

$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} = k_x^2$$

$$\frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2} = k_y^2$$

$$\frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2} = k_z^2$$

where k^2 terms were chosen similar to the 1-D box case.

In similar fashion for the 1-D case, for $k = \frac{2mE}{\hbar^2}$, to satisfy, $V=0$ when $0 \le x \le a$ and ∞ elsewhere:

$$\psi_n = A\sin(k_x x) \quad \text{and thus in our case:}$$

$$f(x) = \sin(k_x x)\sqrt{\frac{2}{a}} \qquad g(y) = \sin(k_y y)\sqrt{\frac{2}{b}}$$

$$h(z) = \sin(k_z z)\sqrt{\frac{2}{c}}$$

And thus:

$$\Psi(x,y,z) = \sqrt{\frac{8}{abc}}\sin(k_x x)\sin(k_y y)\sin(k_z z)$$

Which as in the 1-D case $k = \frac{\pi n}{a}$;

$$\Psi(x,y,z) = \sqrt{\frac{8}{abc}}\sin\left(\frac{\pi n_x x}{a}\right)\sin\left(\frac{\pi n_y y}{b}\right)\sin\left(\frac{\pi n_z z}{c}\right)$$

Similarly, just as for 1-D $E = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 n^2$ the total E will be the sum of each dimension:

$$E = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 n_x^2 + \frac{\hbar^2}{2m}\left(\frac{\pi}{b}\right)^2 n_y^2 + \frac{\hbar^2}{2m}\left(\frac{\pi}{c}\right)^2 n_z^2 = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)$$

Additionally to check that $\Psi(x,y,z)$ is normalised we check:

$$\iiint |\Psi|^2 dx dy dz = 1$$

$$= \int_0^a\int_0^b\int_0^c \frac{8}{abc} \sin^2\left(\frac{n_x \pi x}{a}\right) \sin^2\left(\frac{n_y \pi y}{b}\right) \sin^2\left(\frac{n_z \pi z}{c}\right) dz dy dx$$

$$= \frac{8}{abc} \int_0^a \sin^2\left(\frac{n_x \pi x}{a}\right) dx \int_0^b \sin^2\left(\frac{n_y \pi y}{b}\right) dy \int_0^c \sin^2\left(\frac{n_z \pi z}{c}\right) dz$$

For $\int_0^a \sin^2\left(\frac{n_x \pi x}{a}\right) dx$

$$= \int_0^a \frac{1-\cos\left(\frac{2n_x \pi x}{a}\right)}{2} dx = \left[\frac{1}{2}x\right]_0^a - \left[\frac{-1}{2}\frac{a}{2n_x\pi} \sin\left(\frac{2n_x \pi x}{a}\right)\right]_0^a$$

$$= \frac{a}{2} + \frac{a}{4n_x\pi} \sin(2n_x \pi)$$

Thus the solution follows:

$$= \frac{8}{abc}\left(\frac{a}{2} + \frac{a}{4n_x\pi}\sin(2n_x\pi)\right)\left(\frac{b}{2} + \frac{b}{4n_y\pi}\sin(2n_y\pi)\right)\left(\frac{c}{2} + \frac{c}{4n_z\pi}\sin(2n_z\pi)\right)$$

However, as n for x, y and z must follow the 1-D solutions $n = 1, 2, 3, \ldots$ we find $\sin(2n\pi) = 0$ always

$$= \frac{8}{abc}\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left(\frac{c}{2}\right) = 1$$

Thus $\Psi(x,y,z)$ is normalised.

b) as previously discussed:

$$E = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)$$

which if $a=b=c=L$ for a cubic case:

$$E = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_x^2+n_y^2+n_z^2}{L^2}\right)$$

For $E_{\{1,2,3\}}$ we find : $E = \frac{\hbar^2\pi^2}{2m}\frac{14}{L^2}$

Thus we are interested in all combinations of $n_x^2 + n_y^2 + n_z^2 = 14$

As previously solved in lectures the degeneracy of $E_{\{1,2,3\}}$ is 6:

$\{1,2,3\}$ $\{2,3,1\}$
$\{1,3,2\}$ $\{3,1,2\}$
$\{2,1,3\}$ $\{3,2,1\}$

Problem 9.3)

a)

First to find $\langle r \rangle$: we must first understand that ground state implies $n=1$, $l=0$, $m=0$ i.e.:

First to find the ground state of hydrogens electron we understand ψ_{100}

$$\psi_{nlm} = R_{nl}(r)\, Y_l^m(\theta,\phi)$$

$$= \frac{1}{r}\left(\frac{r}{na_0}\right)^{l+1} e^{-r/na_0}\, v\left(\frac{r}{na_0}\right) \cdot \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, e^{im\phi}\, P_l^m \cos\theta$$

for $n=1$ $m=0$ $l=0$ $\psi_{100} = R_{10}(r)\, Y_0^0(\theta,\phi)$

$$\psi_{100} = \frac{1}{r}\left(\frac{r}{a_0}\right)^1 e^{-r/a_0}\, v\left(\frac{r}{a_0}\right) \cdot \sqrt{\frac{1}{4\pi}\frac{0!}{0!}}\, P_0^0 \cos\theta\, e^0$$

$$= \sqrt{\frac{1}{a_0^3\pi}}\, e^{-r/a_0}$$

As solved in lectures

$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$ where a_0 is the Bohr radius.

Given polar coordinates we understand:

$\langle r \rangle = \int_0^\infty \int_0^{\pi} \int_0^{2\pi} r\, |\psi|^2 r^2 \sin\phi \, d\theta\, d\phi\, dr$ → Note this form is valid as $\hat{r}$ is not an operator, i.e $\hat{r} = r$

Thus $\langle r \rangle = \int \psi^* r \psi \, dr = \int r|\psi|^2 dr$

$= \int_0^\infty \int_0^{\pi} \int_0^{2\pi} \left(\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}\right)^2 r^3 \sin\theta \, d\theta\, d\phi\, dr$

$= \int_0^\infty \frac{r^3}{\pi a_0^3} e^{-2r/a_0} dr \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta$

$= \frac{4\pi}{\pi a_0^3} \int_0^\infty r^3 e^{-2r/a_0} dr$

Let $u = \frac{2r}{a_0}$ $du = \frac{2}{a_0} dr$

$= \frac{4}{a_0^3} \int_0^\infty \left(\frac{u a_0}{2}\right)^3 e^{-u} \frac{a_0\, du}{2} = \frac{a_0}{4} \int_0^\infty u^3 e^{-u} du$

$= \frac{a_0}{4} \times 3! = \frac{3}{2} a_0$

as $\int_0^\infty u^n e^{-u} du = n!$

Thus $\langle r \rangle = \frac{3}{2} a_0$

Following the same process for $\langle r^2 \rangle$

$\langle r^2 \rangle = \int_0^\infty \int_0^{2\pi} \int_0^{\pi} r^2 |\psi|^2 r^2 \sin\phi \, dr\, d\phi\, d\theta = \frac{4}{a_0^3} \int_0^\infty r^4 e^{-2r/a_0} dr$

$= \frac{4}{a_0^3} \int_0^\infty \left(\frac{u a_0}{2}\right)^4 e^{-u} \frac{a_0}{2} du = \frac{4}{a_0^3} \frac{a_0}{2} \left(\frac{a_0}{2}\right)^4 \int_0^\infty u^4 e^{-u} du$

$= \frac{a_0^2}{8} \times 4! = 3a_0^2$

b) Given the symmetry of the ground state wavefunction we understand that given:

<x> =<y> = <z>

$r^2 = x^2+y^2+z^2$, $\langle r^2\rangle = \langle x^2\rangle + \langle y\rangle^2 + \langle z\rangle^2 = 3\langle x^2\rangle$

must also be true.

From the previously calculated $\langle r^2\rangle$ we understand: $\frac{\langle r^2\rangle}{3} = \langle x^2\rangle = \frac{3a_0^2}{3} = a_0^2$

Additionally given the spherical symmetry we understand $\langle x\rangle = 0$

As such:

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{a_0^2} = a_0$$

c) In order to find the most probable value we should solve for the probability density function. Unlike $\Psi(x,y,z)$ with a PDF $|\Psi(x,y,z)|^2$, in polar coordinates we employ the form:

$$\cancel{P(r)dr = |\Psi|^2 dV}$$

and given $\frac{dV}{dr}$:

Radial probability density is given by the mod-squared wavefunction (for n = 1, m = 0, l = 0) times the spherical shell of some radius r.

$$P = |\psi|^2 4\pi r^2 = \frac{1}{a_0^3\pi} e^{-2r/a_0} 4\pi r^2$$

$$P = \frac{4}{a_0^3} e^{-2r/a_0} r^2$$

Thus:

$$\frac{dP(r)}{dr} = 0 = \frac{d}{dr}\left(\frac{4}{a_0^3} e^{-2r/a_0} r^2\right) = \frac{4}{a_0^3}\left(2re^{-2r/a_0} + r^2\left(-\frac{2}{a_0} e^{-2r/a_0}\right)\right)$$

$$= \frac{8r}{a_0^3} e^{-2r/a_0}\left(1 - \frac{r}{a_0}\right)$$

$$0 = 1 - \frac{r}{a_0}$$

$$r = a_0$$

Thus we have shown that the most probable radius is ~~when~~ Bohr's radius.

Problem 10.2)

a)

$$[\hat{L}_x, \hat{L}_y] = (y\hat{p}_z - z\hat{p}_y)(z\hat{p}_x - x\hat{p}_z) - (z\hat{p}_x - x\hat{p}_z)(y\hat{p}_z - z\hat{p}_y)$$

$$= y\hat{p}_z z\hat{p}_x - y\hat{p}_z x\hat{p}_z - z\hat{p}_y z\hat{p}_z + z\hat{p}_y x\hat{p}_z$$

$$- z\hat{p}_x y\hat{p}_z + z\hat{p}_x z\hat{p}_y + x\hat{p}_z y\hat{p}_z - x\hat{p}_z z\hat{p}_y$$

$$= [y\hat{p}_z, z\hat{p}_x] - [y\hat{p}_z, x\hat{p}_z] - [z\hat{p}_y, z\hat{p}_x] + [z\hat{p}_y, x\hat{p}_z]$$

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$$= y\hat{p}_x[\hat{p}_z, z] - yx[\hat{p}_z, \hat{p}_z] - \hat{p}_y\hat{p}_x[z, z] + x\hat{p}_y[z, \hat{p}_z]$$

$$= y\hat{p}_x(-i\hbar) - 0 - 0 + x\hat{p}_y(i\hbar)$$

$$= i\hbar(x\hat{p}_y - y\hat{p}_x) = i\hbar L_z$$

This exact same process is used to show:

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \quad \text{and} \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

$$[\hat L_y, \hat L_z] = [z\hat p_x - x\hat p_z,\ x\hat p_y - y\hat p_x]$$

$$= (z\hat p_x - x\hat p_z)(x\hat p_y - y\hat p_x) - (x\hat p_y - y\hat p_x)(z\hat p_x - x\hat p_z)$$

$$= z\hat p_x x\hat p_y - z\hat p_x y\hat p_x - x\hat p_z x\hat p_y + x\hat p_z y\hat p_x$$

$$- x\hat p_y z\hat p_x + x\hat p_y x\hat p_z + y\hat p_x z\hat p_x - y\hat p_x x\hat p_z$$

$$= [z\hat p_x, x\hat p_y] - [z\hat p_x, y\hat p_y] - [x\hat p_z, x\hat p_y] + [x\hat p_z, y\hat p_x]$$

$$= z\hat p_y[\hat p_x, x] - zy[\hat p_x, \hat p_y] - \hat p_y\hat p_z[x, x] + y\hat p_z[x, \hat p_x]$$

$$= z\hat p_y(i\hbar) - 0 - 0 + y\hat p_z(i\hbar)$$

$$= i\hbar(z\hat p_y - y\hat p_z) = i\hbar\hat L_x$$

$$[\hat L_z, \hat L_x] = [x\hat p_y - y\hat p_x,\ y\hat p_z - z\hat p_y]$$

$$= x\hat p_y y\hat p_z - x\hat p_z z\hat p_y\ y\hat p_x y\hat p_z + y\hat p_x z\hat p_y$$

$$- y\hat p_z x\hat p_y + x\hat p_z z\hat p_y + y\hat p_x y\hat p_z - y\hat p_x z\hat p_y$$

$$= x\hat p_z[\hat p_y, y] - xy[\hat p_z, \hat p_y] - \hat p_x\hat p_z[y, y] + z\hat p_x[y, \hat p_y]$$

$$= x\hat p_z(i\hbar) - 0 - 0 + z\hat p_x(i\hbar)$$

$$= i\hbar\hat L_y$$

Commutators tell us if two variables can be simultaneously measured without an uncertainty principle i.e a commutator of 0 results in no uncertainty principle.

Thus these commutators tell us that each $\hat L_s$ shares an uncertainty principle with ~~the other~~ another angular momentums equal to $i\hbar$ times the other angular momentum.

b)

The scalar operator $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ further:

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_x]$$

$$= [\hat{L}_x^2, L_x] + [\hat{L}_y^2, L_x] + [\hat{L}_z^2, L_x]$$

Commutes with itself

$$= 0 + \hat{L}_y[\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y + \hat{L}_z[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_z$$

Subbing in results from (a)

$$= \hat{L}_y(-i\hbar L_z) + (-i\hbar L_z)\hat{L}_y + \hat{L}_z(i\hbar \hat{L}_y) + (i\hbar \hat{L}_y)\hat{L}_z$$

$$= 0$$

This same process follows to prove $[\hat{L}^2, \hat{L}_y] = 0$
and $[\hat{L}^2, \hat{L}_z] = 0$

$$[\hat{L}^2, \hat{L}_y] = [\hat{L}_x^2, \hat{L}_y] + [\hat{L}_y^2, L_y] + [\hat{L}_z^2, L_y]$$

$$= \hat{L}_x[\hat{L}_x, \hat{L}_y] + [\hat{L}_x, \hat{L}_y]\hat{L}_x \overset{+0}{+} \hat{L}_z[\hat{L}_z, \hat{L}_y] + [\hat{L}_z, \hat{L}_y]\hat{L}_z$$

$$= \hat{L}_x(i\hbar L_z) + (i\hbar L_z)\hat{L}_x + \hat{L}_z(-i\hbar L_x) + (-i\hbar L_x)L_z$$

$$= 0$$

$$[\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2, \hat{L}_z] + [\hat{L}_z^2, L_z]$$

$$= \hat{L}_x(i\hbar \hat{L}_y) + (i\hbar \hat{L}_y)\hat{L}_x + \hat{L}_y(-i\hbar \hat{L}_x) + (-i\hbar \hat{L}_x)\hat{L}_y + 0$$

$$= 0$$

This result tells us that to total angular momentum and one directional component of the angular momentum can be simultaneously measured without an uncertainty principle.

c)

$$[\hat{L}_z, x] = [x\hat{p}_y - y\hat{p}_x, x] = [x\hat{p}_y, x] - [y\hat{p}_x, x]$$

$$= x[\hat{p}_y, x] + [x, x]\hat{p}_y - y[\hat{p}_x, x] - [y, x]\hat{p}_x$$

$$= 0 + 0 - - y i\hbar - 0 = y i\hbar$$

$$[\hat{L}_z, y] = [x\hat{p}_y - y\hat{p}_x, y] = [x\hat{p}_y, y] - [y\hat{p}_x, y]$$

$$= x[\hat{p}_y, y] + [x, y]\hat{p}_y - y[\hat{p}_x, y] - [y, y]\hat{p}_x$$

$$= x(i\hbar) + 0 - \cancel{y(i\hbar)} - 0$$

$$= -x i\hbar$$

$$[\hat{L}_z, z] = [x\hat{p}_y - y\hat{p}_x, z] = [x\hat{p}_y, z] - [y\hat{p}_x, z]$$

$$= x[\hat{p}_y, z] + [x, z]\hat{p}_y - y[\hat{p}_x, z] - [y, z]\hat{p}_x$$

$$= 0 + 0 - 0 - 0$$

$$= 0$$

$$[\hat{L}_z, \hat{p}_x] = [xp_y - y\hat{p}_x, \hat{p}_x] = [x\hat{p}_y, \hat{p}_x] - [y\hat{p}_x, \hat{p}_x]$$

$$= x[\hat{p}_y, \hat{p}_x] + [x, \hat{p}_x]\hat{p}_y - y[\hat{p}_x, \hat{p}_x] - [y, \hat{p}_x]\hat{p}_x$$

$$= 0 + i\hbar\hat{p}_y - 0 - 0$$

$$= i\hbar\hat{p}_y$$

$$[\hat{L}_z, \hat{p}_y] = [x\hat{p}_y - y\hat{p}_x, \hat{p}_y] = [x\hat{p}_y, \hat{p}_y] - [y\hat{p}_x, \hat{p}_y]$$

$$= x[\hat{p}_y, \hat{p}_y] + [x, \hat{p}_y]\hat{p}_y - y[\hat{p}_x, \hat{p}_y] - [y, \hat{p}_y]\hat{p}_x$$

$$= 0 + \overset{0}{\cancel{i\hbar\hat{p}_y}} - 0 - i\hbar\hat{p}_x$$

$$= -i\hbar\hat{p}_x$$

$$[\hat{L}_z, \hat{p}_z] = [xp_y - yp_x, \hat{p}_z] = [x\hat{p}_y, \hat{p}_z] - [y\hat{p}_x, \hat{p}_z]$$

$$= x[\hat{p}_y, \hat{p}_z] + [x, \hat{p}_z]\hat{p}_y - y[\hat{p}_x, \hat{p}_z] - [y, \hat{p}_z]\hat{p}_x$$

$$= 0 + 0 - 0 - 0$$

$$= 0$$

Problem 10.3)

First to show $\hat{L}_\mp$ is the Hermitian conjugate of $\hat{L}_\pm$. Thus: $\langle l,m L_\pm | l,m\rangle = \langle l,m | L_\mp l,m\rangle$ must be satisfied.

$L_\pm = L_x \pm iL_y$

$\langle l,m L_\pm | l,m\rangle = \langle l,m\,(L_x \pm iL_y) | l,m\rangle$

$$= \int_0^{2\pi}\int_0^{\pi} \left(Y_l^m(\theta,\phi)(L_x \pm iL_y)\right)^* Y_l^m(\theta,\phi)\, d\theta\, d\phi$$

we understand that $(L_x + iL_y)^* = L_x - iL_y$

$(L_x - iL_y) = L_x + iL_y$

and thus $(L_x \pm iL_y)^* = (L_x \mp iL_y)$ as L_x and L_y are Hermitian

$$= \int_0^{2\pi}\int_0^{\pi} \left(Y_l^m(\theta,\phi)\right)^* (L_x \mp iL_y)\, Y_l^m(\theta,\phi)\, d\theta\, d\phi$$

$$= \langle l,m | (L_x \mp iL_y)\, l,m\rangle = \langle l,m | \hat{L}_\mp\, l,m\rangle$$

Thus we have shown that $L_\mp$ is the Hermitian conjugate of $L_\pm$.

Now we find $\langle f_l^m | \hat{L}_\mp \hat{L}_\pm f_l^m\rangle$

$= \langle (\hat{L}_\mp)^\dagger f_l^m | \hat{L}_\pm f_l^m\rangle$

$= \langle \hat{L}_\pm f_l^m | \hat{L}_\pm f_l^m\rangle = \langle A_l^m f_l^{m\pm1} | A_l^m f_l^{m\pm1}\rangle$

$= (A_l^m)^2 \langle f_l^{m\pm1} | f_l^{m\pm1}\rangle$

Further: $\langle f_l^m | \hat{L}_\mp \hat{L}_\pm f_l^m\rangle = \langle f_l^m | \hat{L}^2 - \hat{L}_z^2 \mp \hbar \hat{L}_z\, f_l^m\rangle$

$= \hbar^2(l(l+1) - m^2 \mp m)\langle f_l^{m+1} | f_l^{m+1}\rangle$

$= (A_l^m)^2 \langle f_l^{m+1} | f_l^{m+1}\rangle$

Thus we understand that $\hat{L}_+\hat{L}_- \equiv (A_l^m)^2$

If all f_l^m are normalised, we understand A_l^m must always equal 1 i.e:

$$A_l^m = \hbar\sqrt{l(l+1)-m^2 \mp m} = \hbar\sqrt{(l\mp m)(l\pm m+1)} = \sqrt{\hat{L}_\mp \hat{L}_\pm}$$

Thus to normalise all f_l^m, A_l^m must equal $\hbar\sqrt{(l\mp m)(l\pm m-1)}$

Expanding on this, we understand that for f_l^m to be normalised, for $L_\pm f_l^m$, $\langle f_l^{m\pm1} | f_l^{m\pm1}\rangle = 1$ must be satisfied for any m or l. Thus, our derivation of $A^m{}_l$ ensures this as $L_\pm f_l^m = A_l^m f_l^{m\pm1}$ is satisfied given f_l^m was already normalised.

We also note that when m goes outside the bounds of l, A_l^m goes to 0 as expected, i.e: $m=l$ or $m=-l$,

$$A_l^m = \hbar\sqrt{(m\mp m)(m\pm m-1)} = 0$$ assuming correct sign in $\pm$ and $\mp$ is used.

For $m=l$: $A_l^m = \hbar\sqrt{(l-l)(l+l-1)} = 0$

$m=-l$ $A_l^m = \hbar\sqrt{(l+-l)(l-l-1)} = 0$

Fell off the ladder!