Q1)

a) For signals with odd symmetry we understand:

$$a_0 = \frac{1}{T}\int_{t_0}^{T+t_0} f(t)\, dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{T+t_0} f(t)\sin(n\omega_0 t)\, dt$$

we understand:

$$f(t) = \begin{cases} V_m & -\frac{T}{4} \leq t \leq \frac{T}{4} \\ -V_m & \frac{T}{4} < t < \frac{3T}{4} \end{cases}$$

We understand: $f = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{N} a_b \sin\left(\frac{2\pi n t}{T}\right)$

where $T =$ the period of the square wave

Given we start at $t=0$: $t_0 = 0$ Thus

$a_0 = \frac{1}{T}\int_0^T f(t)\, dt$ within $0 < t < T$ we find both V_m and $-V_m$ thus separate the integral:

$$a_0 = \frac{1}{T}\int_0^{T/2} V_m\, dt + \frac{1}{T}\int_{T/2}^{\frac{3T}{4}} -V_m\, dt$$

$$= \frac{1}{T}\left(\left[V_m t\right]_0^{\frac{T}{2}} + \left[-V_m t\right]_{T/2}^{\frac{3T}{4}}\right) = \frac{1}{T}\left(V_m \frac{T}{2} + \left(-V_m \frac{3T}{4}\right) - \left(-V_m \frac{T}{2}\right)\right)$$

$$= \frac{1}{T}(0) = 0$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi n t}{T}\right) dt$$

$$= \frac{2}{T}\int_0^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right) dt + \frac{2}{T}\int_{T/2}^{T} f(t)\cos\left(\frac{2\pi n t}{T}\right) dt$$

$$= \frac{2}{T}\left(\left[V_m \frac{T}{2\pi n} \sin\left(\frac{2\pi n t}{T}\right)\right]_0^{T/2} + \left[-V_m \frac{T}{2\pi n} \sin\left(\frac{2\pi n t}{T}\right)\right]_{T/2}^{T} \right)$$

$$= \frac{2}{T}\left(\frac{V_m T}{2\pi n}\sin(\pi n) - \frac{V_m T}{2\pi n}\sin(0) - \frac{V_m T}{2\pi n}\sin(2\pi n) + \frac{V_m T}{2\pi n}\sin(\pi n) \right)$$

$$= \frac{2}{T}(0) = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right) dt$$

$$= \frac{2}{T}\int_0^{T/2} V_m \sin\left(\frac{2\pi n t}{T}\right) dt + \frac{2}{T}\int_{T/2}^{T} -V_m \sin\left(\frac{2\pi n t}{T}\right) dt$$

$$= \frac{2}{T}\left(\left[V_m \frac{-T}{2\pi n} \cos\left(\frac{2\pi n t}{T}\right)\right]_0^{T/2} + \left[+V_m \frac{T}{2\pi n} \cos\left(\frac{2\pi n t}{T}\right)\right]_{T/2}^{T} \right)$$

$$= \frac{2}{T} V_m \frac{T}{2\pi n}\left(-\cos(\pi n) + \cos(0) + \cos(2\pi n) - \cos(\pi n) \right)$$

$$= \frac{V_m}{\pi n}\left(1 + \cos(2\pi n) - 2\cos(\pi n)\right)$$

Notice for any $n = 1, 2, 3 \ldots$ $\cos(2\pi n) = 1$

$$\therefore\ b_n = \frac{V_m}{\pi n}\left(2 + 2\cos(\pi n)\right) = \frac{2V_m}{\pi n}\left(1 + \cos(\pi n)\right)$$

Notice that if n is even, $b_n = \frac{2V_m}{\pi n}(1+1) = \frac{4V_m}{\pi n}$

odd, $b_n = \frac{2V_m}{\pi n}(1-1) = 0$

Now we sub into f:

$$f = \sum_{n=1}^{N} \left(\frac{2V_m}{n\pi}\right)\left(1 + \cos(\pi n)\right)\sin\left(\frac{2\pi n t}{T}\right)$$ Given only ~~odd~~ even n:

$$f = \sum_{\substack{n=2 \\ (\text{Even } n)}}^{N} \frac{4V_m}{n\pi}\sin\left(\frac{2\pi n t}{T}\right) =$$

b) Given the Symmetric function, we understand:

$$f(t) = f(-t) \quad ①$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt = \frac{1}{T}\left(\int_{-T/2}^{0} f(t)\,dt + \int_{0}^{T/2} f(t)\,dt\right)$$

Given ①: if $t = -x$, $dt = -dx$ ②

$$a_0 = \frac{1}{T}\left(\int_{-T/2}^{0} f(-x)(-dx) + \int_{0}^{T/2} f(t)\,dt\right)$$

$$= \frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{2}{T}\int_0^{T/2} \frac{Rise}{Run}t + y\text{-int}\,dt = \frac{2}{T}\int_0^{T/2} \frac{4V_p}{T}t - V_p\,dt$$

$$= \frac{2}{T}\left[\frac{2V_p}{T}t^2 - V_p t\right]_0^{T/2} = \frac{2}{T}\left(\frac{TV_p}{2} - \frac{V_pT}{2}\right) = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n t}{T}\right)dt = \frac{2}{T}\left(\int_{-T/2}^{0} f(t)\sin\left(\frac{2\pi n t}{T}\right)dt + \int_0^{T/2} f(t)\sin\left(\frac{2\pi n t}{T}\right)dt\right)$$

Given the same ② reasoning:

$$= \frac{2}{T}\left(\int_{T/2}^{0} f(-x)\sin\left(\frac{-2\pi n x}{T}\right)(-dx) + \int_0^{T/2} f(t)\sin\left(\frac{2\pi n t}{T}\right)dt\right)$$

$$= \frac{2}{T}\left(-\int_0^{T/2} f(-x)\sin\left(\frac{-2\pi n x}{T}\right)(-dx) + \int_0^{T/2} f(t)\sin\left(\frac{2\pi n t}{T}\right)dt\right)$$

$$= \frac{2}{T}(0) = 0$$

$$a_n = \frac{2}{T}\left(\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt\right)$$

$$= \frac{2}{T}\left(\int_{-T/2}^{0} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt + \int_0^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt\right)$$

which using ② and proved in lectures:

$$= \frac{4}{T}\int_0^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt$$

$$= \frac{4}{T}\int_0^{T/2}\left(\frac{4V_p}{T}t - V_p\right)\cos\left(\frac{2\pi n t}{T}\right)dt \quad \text{or} \quad \frac{T}{4}a_n = \int_0^{T/2}\left(\frac{4V_p}{T}t - V_p\right)\cos\left(\frac{2\pi n t}{T}\right)dt$$

Let $u = \frac{4V_p}{T}t - V_p$: $du = \frac{4V_p}{T}dt$

$dv = \cos\left(\frac{2\pi n t}{T}\right)$ $v = \frac{T}{2\pi n}\sin\left(\frac{2\pi n t}{T}\right)$

Using integration by parts,

$$a_n = \left[\left(\frac{4V_p t}{T} - V_p\right)\left(\frac{T}{2\pi n}\sin\left(\frac{2\pi n t}{T}\right)\right)\right]_0^{T/2} - \int_0^{T/2} \frac{T}{2\pi n}\sin\left(\frac{2\pi n t}{T}\right)\frac{4V_p}{T}\,dt$$

$$= \left(\cancel{3}\,2V_p - V_p\right)\left(\frac{T}{2\pi n}\sin(\pi n)\right) - \left(-V_p\right)\left(\frac{T}{2\pi n}\sin(0)\right)$$

$$- \frac{4V_p}{T}\frac{T}{2\pi n}\left[\frac{-T}{2\pi n}\cos\left(\frac{2\pi n t}{T}\right)\right]_0^{T/2}$$

$$= \frac{TV_p}{2\pi n}\sin(\pi n) - \frac{2V_p}{\pi n}\left(\frac{-T}{2\pi n}\cos(\pi n) - \frac{T}{2\pi n}\right)$$

$$= \frac{TV_p}{2\pi n}\sin(\pi n) - \frac{TV_p}{\pi^2 n^2}\left(\cos(\pi n) - 1\right)$$

Given $n = 1, 2, 3\ldots$ $\sin(\pi n) = 0$ always:

$$\frac{T}{4}a_n = \frac{TV_p}{\pi^2 n^2}\left(1 - \cos(\pi n)\right) \quad \text{Thus } a_n = \frac{4V_p}{\pi^2 n^2}\left(1 - \cos(\pi n)\right)$$

Notice that if n is even: $a_n = \frac{4V_p}{n^2\pi^2}(1-1) = 0$

odd: $= \frac{4V_p}{n^2\pi^2}(1+1) = \frac{8V_p}{n^2\pi^2}$

Thus:

$$f(t) = \sum_{n=1}^{N} \frac{8V_p}{n^2\pi^2}\cos\left(\frac{2\pi n t}{T}\right) \quad \text{for odd } n$$

Question 2)

We understand the current circuit to follow:

$$V_o = V_i\left(\frac{Z_I}{Z_I + R}\right)$$

$$Z_I = j\omega L = j(10\times10^{-3})(2\pi f) \qquad f = \frac{1}{T} = \frac{1}{200\pi\times10^{-6}}$$

$$\therefore\ V_o = V_i\left(\frac{100}{100+300j}\right)$$

Now we solve for fourier series of V_o

$$V_i(t) = \begin{cases} V_m & -T/4 < t < T/4 \\ -V_m & T/4 < t < 3T/4 \end{cases}$$

where $V_m = 30\pi$ V

$$a_0 = \int_{-T/4}^{3T/4} V_i(t)\,dt = \int_{-T/4}^{T/4} 30\pi\,dt + \int_{T/4}^{3T/4} -30\pi\,dt$$

$$= \left[30\pi t\right]_{-T/4}^{T/4} + \left[-30\pi t\right]_{T/4}^{\frac{3T}{4}}$$

$$= \frac{30\pi T}{4} + \frac{30\pi T}{4} - \frac{90\pi T}{4} + \frac{30\pi t}{4} = 0$$

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} V_i(t)\cos\left(\frac{2\pi n t}{T}\right)dt = \frac{2}{T}\int_{-T/4}^{T/4} 30\pi\cos\left(\frac{2\pi n t}{T}\right)dt + \frac{2}{T}\int_{T/4}^{3T/4} -30\pi\cos\left(\frac{2\pi n t}{T}\right)dt$$

$$= \frac{2}{T}\left[\frac{15T}{n}\sin\left(\frac{2\pi n t}{T}\right)\right]_{-T/4}^{T/4} + \frac{2}{T}\left[-\frac{15T}{n}\sin\left(\frac{2\pi n t}{T}\right)\right]_{T/4}^{\frac{3T}{4}}$$

$$= \frac{2}{T}\left(\frac{15T}{n}\sin\left(\frac{\pi n}{2}\right) - \frac{15T}{n}\sin\left(\frac{-\pi n}{2}\right) - \frac{15T}{n}\sin\left(\frac{3\pi n}{2}\right) + \frac{15T}{n}\sin\left(\frac{\pi n}{2}\right)\right)$$

$$= \frac{2}{T}\left(\frac{45T}{n}\sin\left(\frac{\pi n}{2}\right) - \frac{15T}{n}\sin\left(\frac{3\pi n}{2}\right)\right) = \frac{2}{T}\left(\frac{45T}{n}\sin\left(\frac{\pi n}{2}\right) + \right.$$

$$= 0$$

$$b_n = \frac{2}{T}\int_{-T/4}^{3T/4} V_i(t)\sin\left(\frac{2\pi nt}{T}\right)dt = \frac{2}{T}\int_{-T/4}^{T/4} 30\pi\sin\left(\frac{2\pi nt}{T}\right)dt + \frac{2}{T}\int_{T/4}^{3T/4} -30\pi\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$= \frac{2}{T}\left[\frac{15T}{n}\cos\left(\frac{2\pi nt}{T}\right)\right]_{-T/4}^{T/4} + \frac{2}{T}\left[\frac{15T}{n}\cos\left(\frac{2\pi nt}{T}\right)\right]_{T/4}^{3T/4}$$

$$= \frac{2}{T}\frac{15T}{n}\left(-\cos\left(\frac{\pi n}{2}\right) + \cos\left(\frac{-\pi n}{2}\right) + \cos\left(\frac{3\pi n}{2}\right) - \cos\left(\frac{\pi n}{2}\right)\right)$$

Now observe for even n: $b_n = \frac{30}{n} \times (4) = \frac{120}{n}$

odd n: $b_n = \frac{30}{n} \times (0) = 0$

Thus $b_n = \frac{120}{n}$ for even n:

$$V_i(t) = \sum_{n=2}^{N} \frac{120}{n}\sin\left(\frac{2\pi nt}{T}\right) \quad \text{even } n$$

Thus the first 3 non-zero terms: $n = 2, 4, 6$

$$= \frac{120}{2}\sin\left(\frac{4\pi t}{T}\right) + \frac{120}{4}\sin\left(\frac{8\pi t}{T}\right) + \frac{120}{6}\sin\left(\frac{12\pi t}{T}\right)$$

where $T = 200\pi\ \mu s$

Since $V_i(t) = \left(\frac{100j}{300+100j}\right)V_o(t)$

$$V_o = V_i(t)\left(\frac{300+100j}{100j}\right) = V_i(t)(1-3j)$$

$$= (1-3j)60\sin\left(\frac{4\pi t}{T}\right) + (1-3j)30\sin\left(\frac{8\pi t}{T}\right) + (1-3j)20\sin\left(\frac{12\pi t}{T}\right)$$

$$= (1-3j)60\sin(20\,000t) + (1-3j)30\sin(40\,000t) + (1-3j)20\sin(60\,000t)$$

Thus the first three non-zero terms are:

$n=2$: $60\sin(20\,000t) - 180j\sin(20\,000t)$

$n=4$: $30\sin(40\,000t) - 90j\sin(40\,000t)$

$n=6$: $20\sin(60\,000t) - 60j\sin(60\,000t)$