PHYS2955 Problem Set 2 Samuel Allpass

Problem 2.1
Part A
We consider a 1 m total length, 50 mm diameter metal rod that has a heating section in its middle. The heating section has a length of 10 cm and is maintained at a temperature of 65°C. The rod is insulated along its length, meaning that heat can only flow out of its ends. The ends of the rod are maintained at a temperature of 20°C. The thermal conductivity of the rod is $\kappa = 273\ \mathrm{Wm^{-1}K^{-1}}$.

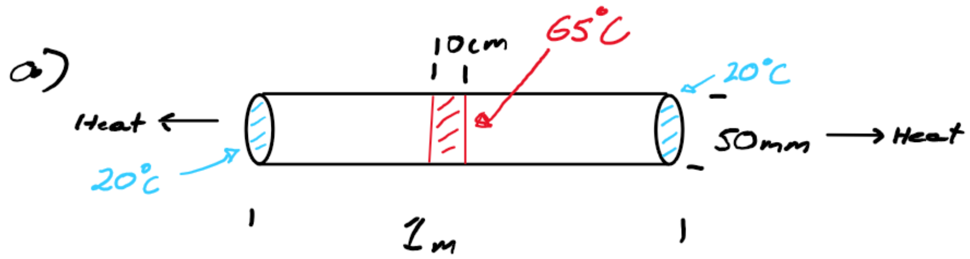
(a) Determine what the steady-state temperature is throughout the rod.

(b) Describe in words how the temperature would evolve in the following situations, and provide a sketch of the temperature profile of the rod in each of the following cases for (i) short (ii) intermediate, and (iii) long times:

- The heating element is suddenly turned off.
- The temperature of the heating element is instantly increased.

Part B Advanced
Consider the original situation, where the heating element is fixed to a temperature of 65°C. How much power must be supplied to the heating element to maintain this situation?

a)



$\kappa = 273\ \mathrm{W/mK}$

However, for two parts, i.e each side of the heat source.
↳ Given the 65°C middle and 20°C ends are kept constant, we can treat each side individually.

We understand that: $\frac{\partial T}{\partial t} = D\nabla^2 T$

and that for the steady state $\frac{\partial T}{\partial t} = 0$
$\therefore D\nabla^2 T = 0$

Using cylindrical coordinates
↳ Note that the temperature is constant over the cross sectional area, thus only changing with z:

$$0 = D\nabla^2 T \Rightarrow \nabla^2 T = 0$$

$$\frac{\partial^2 T}{\partial z^2} = 0$$

$$\frac{\partial T}{\partial z} = C_1 \Rightarrow T = C_1 z + C_2$$

For the left side $T_1(z): 0 \leq z \leq 0.45\,m$

$$T(0) = 20°C = 293K$$

$$T(0.45) = 65°C = 338K$$

$$T(0) \Rightarrow \quad 293 = C_1 \times 0 + C_2$$

$$C_2 = 293$$

$$T(0.45) \Rightarrow \quad 338 = C_1 \times 0.45 + 293$$

$$C_1 = 100$$

$$T_1(z) = 100z + 293$$

For middle $T_2(z): 0.45 \leq z \leq 0.55$

$$T(0.45) = 338K$$

$$T(0.55) = 338K$$

$$\therefore T_2(z) = 338$$

For right side $T_3(z)$: $0.55 \leq z \leq 1$

$$T(0.55) = 338\,K$$

$$T(1) = 293\,K$$

$$T(0.55) \Rightarrow 338 = 0.55C_1 + C_2$$

$$T(1) \Rightarrow 293 = C_1 + C_2$$

$$\hookrightarrow C_1 = 293 - C_2$$

$$338 = 0.55(293 - C_2) + C_2$$

$$C_2 = 393$$

$$C_1 = 293 - 393 = -100$$

$$T_3(z) = -100z + 393$$

Thus we have the final function for the steady state temperature.

$$T(z) = \begin{cases} 100z + 293 & z \leq 0.45 \\ 338 & 0.45 \leq z \leq 0.55 \\ -100z + 293 & z \geq 0.55 \end{cases}$$

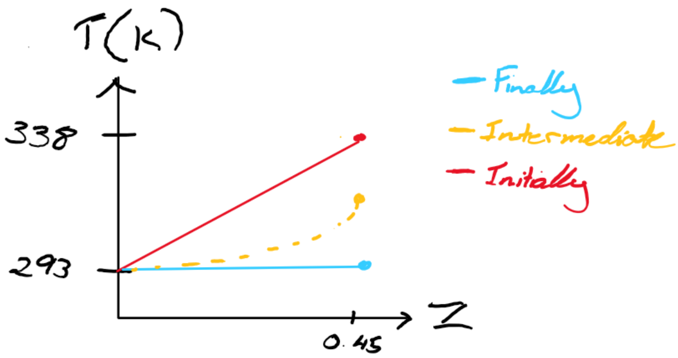
Kelvin

b)

If the heating element is suddenly turned off, such that like the rest of the rod, the thermal conductivity is 273W/mK, we would find that the steady state is broken. In such a case, the temperature in the rods centre begins to decrease from the initial 338K linear equation as the constant 293K ends cool the rod. This cooling continues until thermal equilibrium, at which the whole rod is 293K and in the linear steady state again.

In terms of heat, this correlates to the cooling elements on the end acting as infinite cooling reservoirs, such that heat is being taken from the system into the reservoir.

This process would follow the temperature profiles as seen below for one side but would be identical for the other:

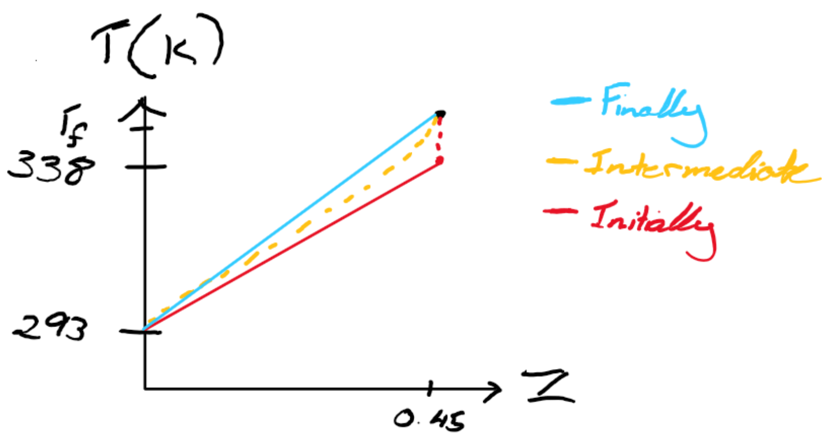


We notice that in the steady state the temperature is linear with distance, and that as it leaves the steady state, the curve follows the 1D heat equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \; where \; \alpha = \frac{k}{pc}$$

What is important to note, is that the gradient of the temperature profile is steeper at the hot ends due to the 1D heat equation.

In the case that the temperature of the heating element is instantly increased, we find that the steady state is again broken. This time, initially the central temperature is “instantly increased” such that it is no longer a linear temperature distribution, nor is it continuous. Assuming the heating element and cool ends are maintained on, such that they act as infinitely large reservoirs, the heat will begin move from the heating element towards the ends. This continues until to a linear temperature distribution is formed between the heating element and the cool ends, one far steeper than before. This would produce the following temperature profile for the first half:



Part B Advanced

Consider the original situation, where the heating element is fixed to a temperature of 65°C. How much power must be supplied to the heating element to maintain this situation?

We understand that the heating element must output the amount of heat leaving through the cool ends in order to maintain this steady state. It is known that:

Power

Fourier heat transfer: $Q = -kA\frac{dT}{dx}$

where: $\frac{dT}{dx} = \frac{T_{heat} - T_{end}}{L_{\text{Half of rod}}}$

First we find $-kA = -273 \times \left(\frac{50}{2} \times 10^{-3}\right)^2 \pi$

$= -0.54$

for half

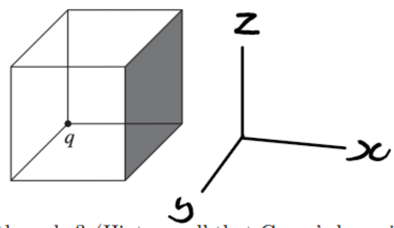
$\therefore Q = -0.54 \times \frac{338 - 293}{0.45}$

$= -53.6\ W$

Thus, to overcome the power loss on both sides, the heating element must output 107.2 W of power.

Problem 2.2
Part A

A positive charge is placed at the corner of the unit cube.

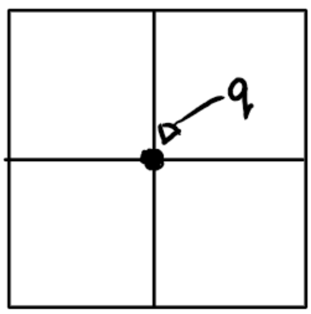


What is the flux through the shaded face of the cube? (Hint: recall that Gauss's law will hold for any arbitrary surface).

We understand that Gauss' law for electrostatics tells us:

$$\oint_S E \cdot ds = \frac{q_{in}}{\epsilon_0} = \Phi_E$$

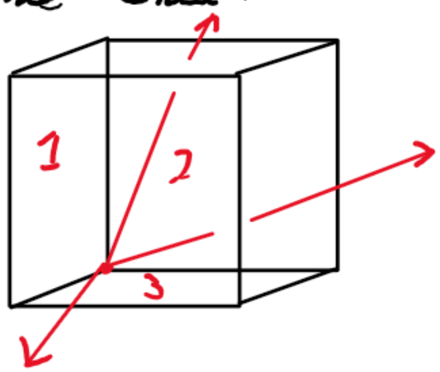
In other words, the electric flux of the surface is proportional to the enclosed charge.
Given the particle sits on the corner, it must be equally shared with 8 other cubes. In 2D:



As such, it becomes obvious that only $\frac{1}{8}$ of the particles charge is within our cube. Thus, as the cube is a Gaussian surface:

$$\Phi_{cube} = \frac{q}{8\epsilon_0}$$

Again following this geometric analysis, we must look at which of the cubes surfaces experiences the flux:



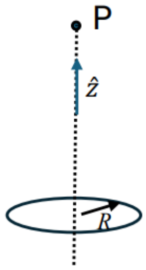
Due to the particle being situated at the corner, we understand the flux of the adjacent sides to be 0 as $\theta = 90°$.

$$\Phi_E = EA\cos\theta = 0$$

Further, due to the symmetry, the remaining 3 surfaces will equally share the flux, such that:

$$\Phi_{shaded} = \frac{1}{3} \times \frac{q}{8\epsilon_0} = \frac{q}{24\epsilon_0}$$

Part B Advanced

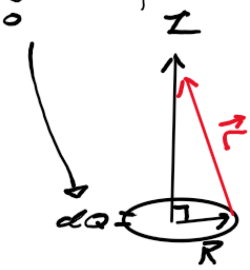


Consider now that a charge Q is evenly distributed on a ring of radius R, as shown in the figure. Find the electric potential at the point P above the centre of the ring. Determine the electric field at the point P from the potential.

We can solve this problem by considering the ring as an infinite series of charged points in a circle.
For a charged surface, we find
$\vec{E} = -\nabla \phi$ with potential function:

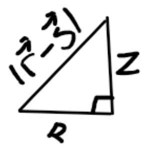
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_0^Q \frac{dQ}{|\vec{r}-\vec{s}|} = \frac{1}{4\pi\epsilon_0}\frac{Q}{|\vec{r}-\vec{s}|}$$

Redrawing the figure:
We find each ring creates some V.



$$\phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{R^2+z^2}}$$

note $|\vec{r}-\vec{s}| \Rightarrow$



$\therefore = \sqrt{R^2+z^2}$

Given the symmetry of the ring, for any point on the z axis: $\bar{E}_x = \bar{E}_y = \bar{0}$
$\therefore$ $\vec{E}$ on $P = \vec{E}_z = -\nabla\phi = -\frac{\partial\phi}{\partial z}$

$$\vec{E}_z = -\frac{\partial}{\partial z} \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{R^2+z^2}} = \frac{-Q}{4\pi\epsilon_0} \frac{\partial}{\partial z} \frac{1}{\sqrt{R^2+z^2}}$$

let $R^2+z^2 = u$ $\qquad \frac{du}{dz} = 2z$

$$F(u) = \frac{1}{\sqrt{u}} = u^{-\frac{1}{2}} \qquad \frac{dF}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$$

Chain rule: $$\frac{dF}{dx} = -\frac{2z}{2} \frac{1}{(R^2+z^2)^{\frac{3}{2}}}$$

$$\therefore \vec{E}_z = \frac{-Q}{4\pi\epsilon_0} \times -\frac{2z}{2} \frac{1}{(R^2+z^2)^{\frac{3}{2}}}$$

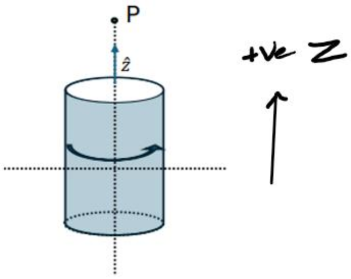
$$= \frac{Qz}{4\pi\epsilon_0} \frac{1}{(R^2+z^2)^{3/2}}$$

Thus, at point P: $(z = P)$

$$\vec{E}(P) = \frac{QP}{4\pi\epsilon_0} \frac{1}{(R^2+P^2)^{3/2}}$$

Problem 2.3
Part A

We consider now a section of tube of length ℓ and radius b, this time centred at the origin. The shaded surface of the tube carries a surface current distributed over the entire surface $\vec{\mathbf{j}}$ anticlockwise around the z-axis as shown. Integrating the current the total current around the surface is $\oint_S \vec{\mathbf{j}} \cdot d\vec{\mathbf{S}} = I_0$. Using Ampere's law determine the magnetic field along the z-axis at the origin, assuming the field is zero radially outside the tube (no field along x or y).



$$\oint \vec{j} \cdot d\vec{S} = I_0$$

In order to find the magnetic field at the origin, we must first recognise that:

$$\vec{B} = \nabla \times \vec{A}(\vec{r}) \quad \text{and} \quad \vec{A}(\vec{r}) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{j(\vec{r})}{|\vec{r}-\vec{3}|} d^3S$$

First translating into cylindrical coordinates: r, φ, z

We find $r = b$ and $-\frac{1}{2}l \leq z \leq \frac{1}{2}l$
$0 \leq \varphi \leq 2\pi$

Representing $|\vec{r} - \vec{3}|$ in this form we find:

$$\vec{r}^2 = x^2 + y^2 + z^2$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$\therefore$ At origin:

$$|\vec{r} - \vec{3}| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + z^2} = \sqrt{b^2 + z^2}$$

$$\therefore \quad \vec{A} = \frac{1}{4\pi\epsilon_0 c^2}\int_0^{2\pi}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} \frac{j(\vec{r})}{\sqrt{b^2+z^2}} dz\, d\theta$$

let $b^2+z^2=u \qquad \frac{du}{dz}=2z$

$$F(u)=\frac{1}{\sqrt{u}} \qquad \frac{dF}{du}=-\frac{1}{2}u^{-\frac{3}{2}}$$

$$\therefore \vec{A}=\frac{I_0\hat{z}}{4\pi\epsilon_0 c^2}\int_0^{2\pi}\frac{-2z}{2(b^2+z^2)^{3/2}}\,d\theta=\frac{-I_0}{2\epsilon_0 c^2}\frac{z}{(b^2+z^2)^{3/2}}\hat{z}$$

We now aim to find $\vec{B}_z$. We will ignore B_x and B_y in the calculation for simplification

$$\vec{B}=\nabla\times\vec{A}=\begin{vmatrix}\frac{1}{r}\hat{r} & \hat{\varphi} & \frac{1}{r}\hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ A_r & rA_\varphi & A_z\end{vmatrix}$$

$$=\left(\frac{\partial}{\partial r}rA_\varphi-\frac{\partial}{\partial\varphi}A_r\right)\hat{z}$$

Note we found only A_z is non-zero

$\therefore \vec{B}_z=0$

Part B Advanced

Determine the magnetic field at the point P on the z-axis outside the tube.

Hint: Use the Biot-Savart law (Feynman 15.25) and split the box up into loops of height dz and then integrate over the tube. Consider how the symmetry of the problem will aid you in finding the on-axis field.

Biot-Savarts law tells us the magnetic for a solarnoid:

Turns

$$B = \frac{\mu_0 N I}{2R} \qquad (1)$$

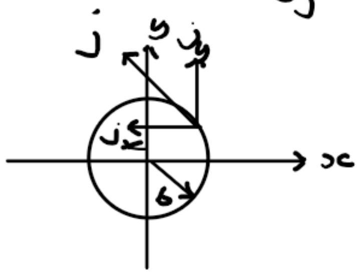
current

radius

Considering the rod as a solarnoid split into dz width wire turns. Given $\oint \vec{j}\, ds = I_0$ we notice for a small section of wire dz:

$\vec{j} = NI$ taking the components:

$j_x = -j\sin(\theta)$ $\quad j_y = j\cos(\theta)$ $\quad j_z = 0$

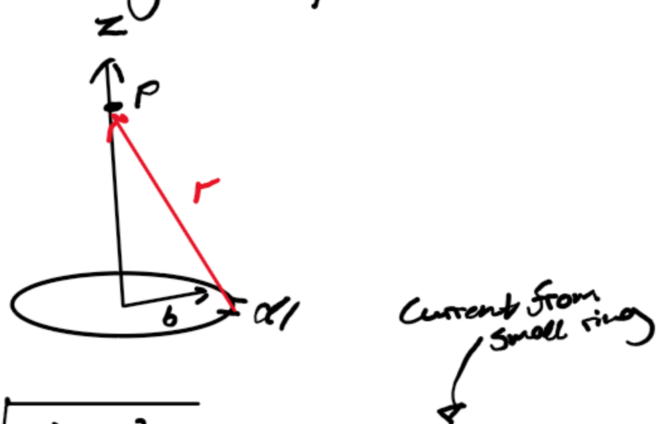


Top down view

Biot-Savarts law tells us:

$$B(\vec{r}) = \frac{\mu_0 I_0}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2}$$

For a single loop:



$$\therefore \quad r = \sqrt{b^2 + z^2} \quad \text{and} \quad dI = \frac{I_0}{2\pi b l} dz$$

We also notice that:

$$I_0 \int d\vec{s} \times \hat{r} = \int dI \times \hat{r} \quad \text{as} \quad \oint I \times d\vec{s} = I_0$$

$$dI \times \hat{r} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ -b\sin\theta & b\cos\theta & 0 \\ -\frac{b\cos\theta}{r} & -\frac{b\sin\theta}{r} & \frac{z}{r} \end{vmatrix} = \frac{b^2}{r}\hat{z}$$

Given by symmetry: $\hat{r} = \hat{\theta} = 0$

For some $d\theta$

$$\therefore \quad \vec{B}_{ring}(\vec{r}) = \frac{\mu_0 I_0 N}{2R} \quad dI = I_0 N \quad \text{and} \quad dI = \frac{I_0}{2\pi b l} dz$$

$$= \frac{\mu_0}{2R} \times \frac{I_0 dz}{2\pi b l} \times \frac{b^2}{r}\hat{z} = \frac{\mu_0 I_0 b \, dz}{4\pi l (b^2 + z^2)^{3/2}} \hat{z}$$

This is for one ring thus in the cylinder case we have infinite rings from $-\frac{1}{2}l \leq z \leq \frac{1}{2}l$

$$d\vec{B}_{cyl} = \frac{\mu_0 I_0 b}{4\pi l (b^2+z^2)^{3/2}} dz\ \hat{z}$$

$$\vec{B}_{cyl} = \frac{\mu_0 I_0 b}{4\pi l} \int_{-\frac{1}{2}l}^{\frac{1}{2}l} \frac{1}{(b^2+z^2)^{3/2}} dz\ \hat{z}$$

We understand :

$$\int \frac{1}{(b^2+z^2)^{3/2}} dz = \frac{z}{b^2(b^2+z^2)^{3/2}}$$

$$\therefore \quad \vec{B} = \frac{\mu_0 I_0}{2} \left[\frac{z}{b^2(b^2+z^2)^{3/2}} \right]_{-\frac{1}{2}l}^{\frac{1}{2}l}$$

$$= b\frac{\mu_0 I_0}{12}\left(\frac{\frac{1}{2}l}{b^2\left(b^2+\left(\frac{l}{2}\right)^2\right)^{3/2}} - \frac{-\frac{1}{2}l}{b^2\left(b^2+\left(\frac{l}{2}\right)^2\right)^{3/2}}\right)$$

$$= \frac{\mu_0 I_0}{4\pi l} b \frac{l}{b^2\left(b^2+\left(\frac{l}{2}\right)^2\right)^{3/2}}$$

$$= \frac{\mu_0 I_0}{4\pi}\frac{1}{b\left(b^2+\left(\frac{l}{2}\right)^2\right)^{3/2}}$$

Thus, we have shown that for any point on the z axis, including point P, a magnetic field strength of

$$\frac{\mu_0 I_0}{4\pi}\frac{1}{b\left(b^2+\left(\frac{l}{2}\right)^2\right)^{3/2}}$$ is experienced.