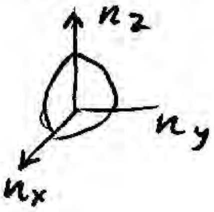
Problem Set 11

1 Number of thermal photons

$$N = \underbrace{2}_{\text{2 polarizations}} \sum_{\underline{n}} f_P(E_{\underline{n}}) \longrightarrow 2 \cdot \underbrace{\frac{1}{8}}_{\substack{\text{positive}\\ \text{octant}\\ \text{in } \underline{n}\text{-space}}} 4\pi \int_0^\infty dn\, n^2 \frac{1}{e^{\hbar\omega_n/k_B T} - 1}$$

$\underline{n} = (n_x, n_y, n_z)$, $n_{x,y,z} = 0, 1, 2, \ldots$



where $\omega_n = \frac{\pi c}{L} n$

$$= \pi \int_0^\infty dn \frac{n^2}{e^{\frac{\hbar \pi c}{L k_B T} n} - 1} = \pi \frac{L^3 (k_B T)^3}{\pi^3 \hbar^3 c^3} \underbrace{\int_0^\infty dx \frac{x^2}{e^x - 1}}_{\simeq 2.404}$$

def. $x \equiv \frac{\hbar \pi c}{L k_B T} n$

$$= 2.404 \cdot \frac{V}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 \qquad \text{where } V = L^3$$

Thus:

$$\boxed{N = 2.404 \cdot \frac{V}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3}$$

2 Heat capacity of intergalactic space

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N}$$

For a gas of H atoms (in the classical regime):

$$U_H = \frac{3}{2} N k_B T = \frac{3}{2} (n V) k_B T$$

↳ concentration

$$\therefore \quad C_V^{(H)} = \left(\frac{\partial U_H}{\partial T}\right)_{V,N} = \frac{3}{2} n V k_B$$

For radiation:

$$U_R = \frac{\pi^2 V}{15 \hbar^3 c^3} (k_B T)^4$$

[see Lecture 18, Stefan-Boltzmann law]

$$\therefore \quad C_V^{(R)} = \left(\frac{\partial U_R}{\partial T}\right)_{V,N} = \frac{4\pi^2 V}{15 \hbar^3 c^3} k_B^4 T^3$$

Therefore: $$\frac{C_V^{(H)}}{C_V^{(R)}} = \frac{45 n}{8\pi^2} \left(\frac{\hbar c}{k_B T}\right)^3 \simeq \mathbf{2.8 \times 10^{-10}}$$

3 Pressure of thermal radiation

(a). $P = -\left(\frac{\partial U}{\partial V}\right)_S$

↳ at constant entropy!

Because $U = 2\sum_{\underline{n}} \hbar\omega_{\underline{n}} \underbrace{f_P(E_n)}_{\langle s_{\underline{n}}\rangle}$

$\langle s_{\underline{n}} \rangle$ thermal average occupancy of a mode of frequency $\omega_{\underline{n}}$ and definite polarization [Planck distribution function, $f_P(E_{\underline{n}}) = f_P(\hbar\omega_{\underline{n}})$ where $\omega_{\underline{n}} = \frac{\pi c}{L} n = \frac{\pi c}{V^{1/3}} n$

we obtain

$$P = -\frac{\partial}{\partial V}\left(2\sum_{\underline{n}} \hbar\omega_n f_P(\hbar\omega_{\underline{n}})\right)_S$$

$$= -\frac{\partial}{\partial V}\left(2\sum_{\underline{n}} \frac{\hbar\omega_{\underline{n}}}{e^{\hbar\omega_{\underline{n}}/k_BT} - 1}\right)_S$$

— need to ensure that S remains constant

Here $\frac{\hbar\omega_{\underline{n}}}{k_BT} = \frac{\hbar\pi c}{k_B T V^{1/3}} n$

On the other hand, the entropy of a photon gas is [see lecture 18]

$$S = \frac{4\pi^2}{45}\frac{k_B^4 V T^3}{(\hbar c)^3}, \text{ i.e. } S \propto T^3 V$$

If S to stay constant, then T^3V should also stay constant, and hence $TV^{1/3}$ – const.

Thus, in the expression for P

$$P = -\frac{\partial}{\partial V}\left(2\sum_n \frac{\hbar\omega_n}{e^{\hbar\omega_n/k_BT}-1}\right)_S$$

the ratio $\frac{\hbar\omega_n}{k_BT} = \frac{\hbar\pi c n}{k_BTV^{1/3}}$ remains constant under constant S; accordingly, the derivative $\frac{\partial}{\partial V}$ doesn't affect the term $\frac{1}{e^{\hbar\omega_n/k_BT}-1}$ under const. S, and therefor

$\frac{\partial}{\partial V}$ acts only on ω_n

(b) Now.

$$\frac{\partial\omega_n}{\partial V} = \frac{\partial}{\partial V}\left(\frac{\pi c n}{V^{1/3}}\right) = -\frac{1}{3}\pi c n\, V^{-\frac{1}{3}-1}$$

$$= -\frac{1}{3V}\underbrace{\frac{\pi c n}{V^{1/3}}}_{=\,\omega_n} = -\frac{\omega_n}{3V}$$

(c) Contonue with P

$$P = -2\sum_n \frac{\hbar}{e^{\hbar\omega_n/k_BT}-1}\frac{\partial}{\partial V}(\omega_n)$$

$$= \frac{1}{3V}\, 2\sum_n \frac{\hbar\omega_n}{e^{\hbar\omega_n/k_BT}-1}$$

$\rightarrow$ but this is the definition of U [Lecture 18]

Thus:

$$\boxed{P = \frac{U}{3V}} = \frac{1}{3\cancel{V}} \frac{\pi^2 \cancel{V}}{15 \hbar^3 c^3} (k_B T)^4$$

(d) Radiation pressure:

$$P = \frac{\pi^2}{45 \hbar^3 c^3} (k_B T)^4$$

Kinetic pressure of an ideal gas of H atoms (in the classical regime)

$$PV = N k_B T \quad \text{— ideal gas law}$$

$$\therefore \quad P = \frac{N}{V} k_B T = n k_B T$$

↳ concentration

$$n = \frac{N}{V} = 6.02 \times 10^{29} \ m^{-3}$$

From

$$\frac{\pi^2 (k_B T)^4}{45 \hbar^3 c^3} = n k_B T$$

→ solve for T

$$T_0 \simeq \frac{\hbar c}{k_B} \left(\frac{45 n}{\pi^2} \right)^{1/3} \sim 3 \times 10^7 \ K$$

4. Photon gas in one dimension

Heat capacity: $c_V = \left(\frac{\partial U}{\partial T}\right)_V$

$\therefore$ Need to find U first.

$$U = \sum_n E_n f_P(E_n)$$

where $E_n = \hbar\omega_n$ – energy associated with 1 mode of frequency ω_n (energy of an "orbital" that can be occupied by a certain number of photons)

$$f_P(E_n) = \frac{1}{e^{E_n/k_BT}-1} = \frac{1}{e^{\hbar\omega_n/k_BT}-1}$$

– Planck distribution function (average number of photons in the mode of frequency ω_n) – average occupancy at temperature T.

This is the same as

$$U = \underbrace{\sum_n}_{\text{sum over all modes}} \underbrace{\langle E_n \rangle}$$

$\hookrightarrow$ average energy in the mode of frequency ω_n; $E_n = \hbar\omega_n$, and if the thermal average occupancy is $\langle s_n \rangle \equiv f_P(E_n) = \frac{1}{e^{\hbar\omega_n/k_BT}-1}$

then $\langle E_n \rangle = \langle s_n \rangle \cdot \hbar\omega_n = \dfrac{\hbar\omega_n}{e^{\hbar\omega_n/k_BT}-1}$

In 1D: $\omega_n = \frac{\pi c}{L} n$, with $n = 1, 2, 3, \ldots$

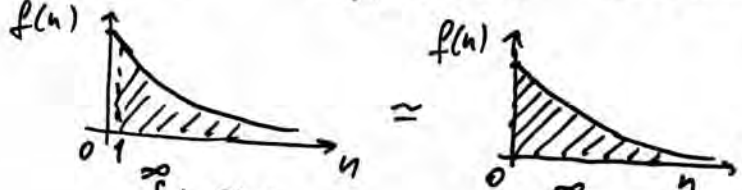
$\Rightarrow$ the spectrum of energies is $E_n = \hbar\omega_n = \frac{\hbar\pi c}{L} n$

Thus :

$$U = \sum_{n=1}^{\infty} \frac{\hbar\omega_n}{e^{\hbar\omega_n/k_BT}-1} \simeq \int_1^{\infty} dn \frac{\hbar\omega_n}{e^{\hbar\omega_n/k_BT}-1}$$

$$= \int_1^{\infty} dn \frac{\frac{\hbar\pi c}{L}n}{e^{\frac{\hbar\pi c n}{Lk_BT}}-1} \simeq \int_0^{\infty} dn \frac{\frac{\hbar\pi c}{L}n}{e^{\frac{\hbar\pi c n}{Lk_BT}}-1}$$

can extend the lower limit to 0 becaus the integrand $\left(\frac{n}{e^{\frac{\hbar\pi c n}{Lk_BT}}-1}\right)$ is finite at $n=0$ and the difference between $\int_1^{\infty} dn\, f(n)$ and $\int_0^{\infty} dn\, f(n)$ is very small



$\int_1^{\infty} dn\, f(n)$ $\simeq$ $\int_0^{\infty} dn\, f(n)$

introduce $x \equiv \frac{\hbar\pi c}{Lk_BT}n$

$$= \frac{\hbar\pi c}{L}\left(\frac{Lk_BT}{\hbar\pi c}\right)^2 \underbrace{\int_0^{\infty} dx \frac{x}{e^x-1}}_{=\frac{\pi^2}{6}}$$

$$= \frac{L(k_BT)^2}{\hbar\pi c}\cdot\frac{\pi^2}{6} = \frac{\pi L (k_BT)^2}{6\hbar c}$$

Thus: $U = \frac{\pi L k_B^2 T^2}{6\hbar c} \Rightarrow \boxed{C_V = \left(\frac{\partial U}{\partial T}\right)_L = \frac{\pi L k_B^2 T}{3\hbar c}}$

5

Photon condensation (hypotetic)

Definin T_c as the temperature at which the total number of photons in all excited orbitals $N_e(T_c)$ approximately equals to N, and using the result for N found in the previous problem (Problem 1), we find

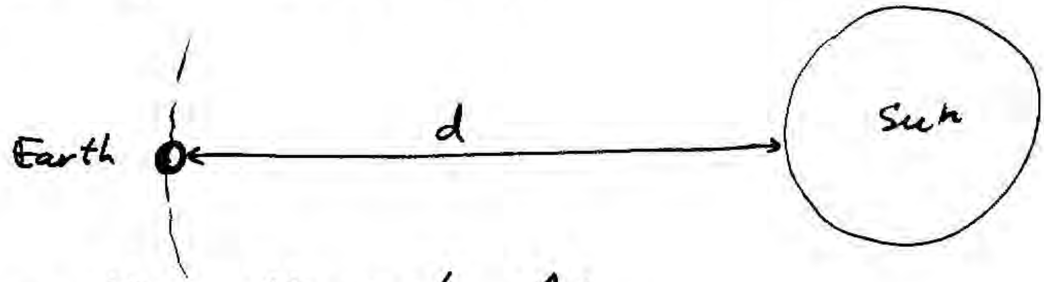
$$N_e(T_c) \simeq N \implies \frac{2.404\, V (k_B T_c)^3}{\pi^2 \hbar^3 c^3} = N$$

$$\therefore \quad T_c = \frac{\hbar c}{k_B} \left(\frac{\pi^2 N}{2.404\, V} \right)^{1/3} \simeq 1.7 \times 10^6 \text{ K}$$

6 Surface temperature and age of the Sun.

(a) Def. $j_{sol.const.}$ — solar constant

From its physical meaning, multiplying $j_{sol.const.}$ by an area A will give the energy flux through that area (on the Earth, since $j_{sol.const.}$ is defined with respect to the distance d from the sun to the Earth):



Energy flux through A:

$$I = j_{sol.const.} \times A$$

If A is replaced by the area of the entire sphere of radius d, $S = 4\pi d^2$, then $j_{sol.const.} \times S$ would give the total rate of energy generation:

$$I_{tot.} = j_{sol.const.} \times 4\pi d^2 \sim 4 \times 10^{26}\ \mathrm{J\ s^{-1}}$$

(b) On the other hand, the same $I_{tot.}$ can be found from the Stefan's law $[J = \sigma_B T^4]$:

$$I_{tot} = J \times 4\pi R_\odot^2$$

where J is the intensity [or energy rate per unit area] of thermal radiation from a black-body at temperature T.

Thus:

$$I_{tot.} = 4\pi \sigma_B \cdot R_\odot^2 \, T^4$$

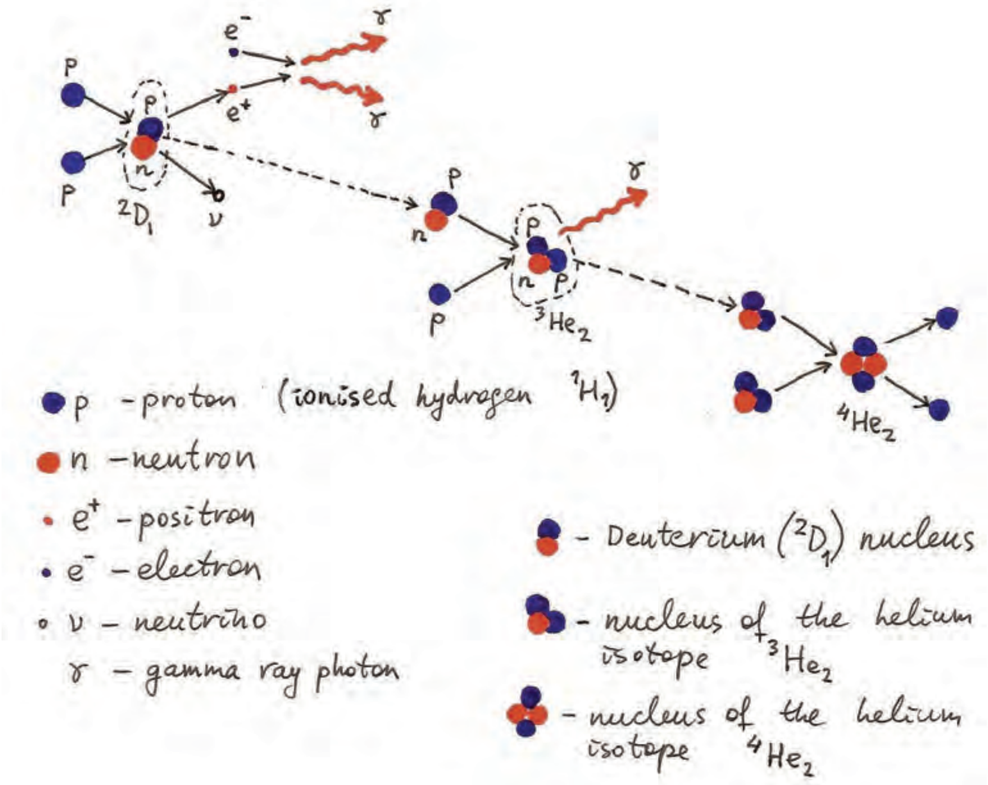
$$\therefore \quad T = \left(\frac{I_{tot}}{4\pi \sigma_B \, R_\odot^2} \right)^{1/4}$$

Using I_{tot} from (a), $\sigma_B = 5.7 \times 10^{-8} \frac{J}{s \cdot m \cdot K^4}$, and $R_\odot = 7 \times 10^8$ m, we obtain

$$\boxed{T \simeq 6000 \text{ K.}}$$

(c) Under the extreme conditions in the thermonuclear core of the Sun (high densities, high temperatures, $T_{\text{core}} \simeq 1.5 \times 10^7$ K), hydrogen atoms are completely dissociated (ionised) into bare protons and free electrons (the plasma state). The same extremely high temperatures provide the protons with enough kinetic energy to overcome their Coulomb repulsion and get close enough that the strong nuclear force takes over and pulls the protons together into one larger nucleus. If two light nuclei fuse, they will generally form a single nucleus with a slightly smaller mass than the sum of their original masses. The difference in mass is released as energy according to Einstein's mass-energy equivalence formula $E = \Delta m\, c^2$.

The simplest fusion chain (the so called proton-proton chain) is the fusion of four hydrogen nuclei to make one helium nucleus. This "hydrogen-burning" phase supplies energy to stars similar to the Sun. Several individual reactions are involved in the proton-proton chain and are shown in the figure below.



1. The first step involves the fusion of two hydrogen nuclei $^1H_1$ (protons) into deuterium $^2D_1$ nucleus, releasing a positron and a neutrino as one proton changes into a neutron:

$$^1\mathrm{H}_1 \;+\; ^1\mathrm{H}_1 \longrightarrow \;^2\mathrm{D}_1 \;+ \mathrm{e}^+ + \nu + 0.42 \text{ MeV}$$

The positron immediately annihilates with an electron, and their mass energy is carried off by two gamma ray photons:

$$\mathrm{e}^+ + \mathrm{e}^- \longrightarrow 2\gamma + 1.02 \text{ MeV}$$

2. After this, the deuterium produced in the first stage can fuse with another hydrogen nucleus to produce a light isotope of helium, $^3He_1$:

$$^2\mathrm{D}_1 \;+\; ^1\mathrm{H}_1 \longrightarrow \;^3\mathrm{He}_2 \;+ \gamma + 5.49 \text{ MeV}$$

3. From here, pairs of nuclei of $^3_2$He atoms produced in steps 1 and 2 fuse into a helium isotope $^4_2$He and release two protons:

$$^3\mathrm{He}_2 \;+\; ^3\mathrm{He}_2 \;\longrightarrow\; ^4\mathrm{He}_2 \;+ 2\;^1\mathrm{H}_1 + 12.86 \text{ MeV}$$

The complete proton-proton chain releases a net energy of

$$(\Delta m)c^2 = (4m_H - m_{He})c^2 \simeq 4.27 \times 10^{-12} \text{ J } \simeq 26.7 \text{ MeV}$$

Assuming for simplicity that the Sun consist of only hydrogen atoms, one can use the mass of the Sun and the mass of hydrogen atoms to estimate the total number of hydrogen atoms: $N_H = M_\odot / m_H \simeq 1.2 \times 10^{57}$. Taking into account that four hydrogen atoms are required to give one helium atom (and therefore to release the above amount of energy per proton-proton chain, 4.27×10^{-12} J), and taking next 10% of these, we estimate the total amount of energy of the Sun available for radiation as

$$U_{tot} = (\Delta m)c^2 \times \frac{N_H}{4} \times 0.1 \simeq 1.3 \times 10^{44} \text{ J}$$

(d) Since

$$U_{tot} = I_{tot} \times t,$$

where I_{tot} is from part (a), the life expectancy can be estimated as

$$t = \frac{U_{tot}}{I_{tot}} \simeq 3 \times 10^{17} \text{ s } \sim 10^{10} \text{ years.}$$