Problem Set 8

(1.) Quantum densities for a non-relativistic
ideal gas

The single-particle energies (in the non-relativistic
limit) are quantized according to

$$E_{\underline{n}} = \frac{\hbar^2 \pi^2}{2m L^2} n^2 \qquad (1)$$

where :   in 1D : $\underline{n} \rightarrow n = 1, 2, 3, \ldots$

in 2D : $\underline{n} \equiv (n_x, n_y)$, with $n = |\underline{n}| = \sqrt{n_x^2 + n_y^2}$

and $n_{x,y} = 1, 2, 3, \ldots$

in 3D : $\underline{n} \equiv (n_x, n_y, n_z)$, with $n = |\underline{n}| = \sqrt{n_x^2 + n_y^2 + n_z^2}$

and $n_{x,y,z} = 1, 2, 3, \ldots$
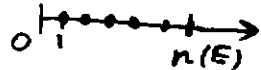
From Eq.(1)   , in the   continuous limit,

$$n(E) = \left(\frac{2m}{\hbar^2} \right) \frac{L}{\pi} \sqrt{E} \qquad \left(\begin{array}{c} \text{will use} \\ \text{later} \end{array} \right) \qquad (2)$$

The density of single-particle orbitals is
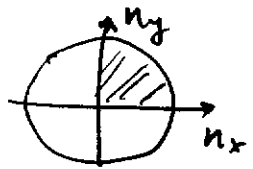defined   as

$$D(E) = \frac{\partial \mathcal{N}(E)}{\partial E}$$

where $\mathcal{N}(E)$ is the total number of states
with energies less than E (including
the spin states, i.e. $(2S+1)$ different
values of the spin projection quantum number S_z
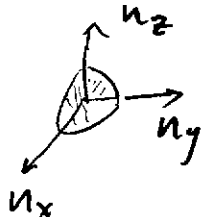for a given spin S)

In 1D : $\mathcal{N}(E) = (2S+1)\, n(E) = (2S+1)\left(\dfrac{2m}{\hbar^2}\right)^{1/2} \dfrac{L}{\pi}\sqrt{E}$

$\underbrace{}_{\text{from Eq.(2)}}$

$0 \quad n(E)$

In 2D : $\mathcal{N}(E) = \underbrace{(2S+1)\dfrac{1}{4}\pi\, n(E)^2}_{\substack{\text{area} \\ \text{on the} \\ \text{positive} \\ \text{quadrant}}} = (2S+1)\dfrac{1}{4}\pi\,\dfrac{2m}{\hbar^2}\left(\dfrac{L}{\pi}\right)^2 E$

$n_y \quad n_x$

In 3D : $\mathcal{N}(E) = \underbrace{(2S+1)\dfrac{1}{8}\dfrac{4\pi}{3}\, n(E)^3}_{\substack{\text{volume} \\ \text{of the} \\ \text{positive} \\ \text{octant}}} = (2S+1)\dfrac{1}{8}\dfrac{4\pi}{3}\left(\dfrac{2m}{\hbar^2}\right)^{3/2}\left(\dfrac{L}{\pi}\right)^3 E^{3/2}$

$n_z \quad n_y \quad n_x$

Differentiating each of these $\mathcal{N}(E)$ with respect to E, we obtain

$$D(E) = \dfrac{d\mathcal{N}(E)}{dE} = \begin{cases} (2S+1)\dfrac{L}{2\pi}\left(\dfrac{2m}{\hbar^2}\right)^{1/2}\dfrac{1}{\sqrt{E}} & \text{— in 1D} \\[4mm] (2S+1)\dfrac{m\,A}{2\pi\,\hbar^2} & \text{— in 2D (where } A=L^2) \\[4mm] (2S+1)\dfrac{V}{4\pi^2}\left(\dfrac{2m}{\hbar^2}\right)^{3/2}\sqrt{E} & \text{— in 3D (where } V=L^3) \end{cases}$$

② (a)

$\rightarrow$ S = 1/2 for electrons

(i) in 1D : $N = (2S+1)\, n_F = 2\, n_F$

$$\therefore \quad n_F = \frac{N}{2}$$

$$E_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n_F^2 = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 \left(\frac{N}{2}\right)^2 = \frac{\pi^2 \hbar^2 N^2}{8 m L^2}$$

(ii) in 2D: $N = (2S+1)\cdot \frac{1}{4}\pi\, n_F^2 = \frac{1}{2}\pi\, n_F^2$

$\quad\quad \rightarrow S = \frac{1}{2}$

$$\therefore \quad n_F = \sqrt{2N/\pi}$$

$$E_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n_F^2 = \frac{\hbar^2}{2m}\,\frac{\pi^2}{A}\cdot\frac{2N}{\pi} = \frac{\pi \hbar^2 N}{m A}$$

$\quad\quad \rightarrow L^2 = A$

(b) Total average energy of the gas, in general, is given by

$\rightarrow$ Fermi-Dirac distr. fn.

$$U = \sum_{S_z}\sum_{n} E_n\, f_F(E_n) = (2S+1)\sum_{n} E_n\, f(E_n)$$

$\underbrace{\quad}$ spin projection (from $-S, -S+1, \cdots , S-1, S$)
here $S = 1/2$, so $S_z = \pm 1/2$

$$\rightarrow \begin{cases} (2S+1)\displaystyle\int_0^\infty dn\; E_n\, f_F(E_n), & \text{in 1D} \\[2em] (2S+1)\,\frac{1}{4}\,2\pi\displaystyle\int_0^\infty dn\; n\, E_n\, f_F(E_n), & \text{in 2D} \\[2em] (2S+1)\,\frac{1}{8}\,4\pi\displaystyle\int_0^\infty dn\; n^2\, E_n\, f_F(E_n), & \text{in 3D} \end{cases}$$

or $\longrightarrow \displaystyle\int_0^\infty dE\, D(E)\, E\, f_F(E)$

$\quad\quad \rightarrow$ density of states in 1D, 2D or 3D

$$U = \int_0^\infty dE \, D(E) \, E \, f_F(E) \qquad \text{in} \quad 1D, 2D, \text{ or } 3D$$

For a gas of electrons $(s = 1/2)$ in the ground state $(T = 0)$

$$f_F(E) = \begin{cases} 1, & \text{if } E \leq E_F \\ 0, & \text{if } E > E_F \end{cases}$$

∴ The ground state energy of the gas is:

$$U_0 = \int_0^{E_F} dE \, D(E) \, E \qquad (\text{in } 1D, 2D, \text{ or } 3D)$$

(i) in 1D:
$$D(E) = \frac{L}{\pi} \left(\frac{2m}{\hbar^2} \right)^{1/2} \frac{1}{\sqrt{E}} \qquad \rightarrow \text{from Problem 1,}$$

Problem Set 6 with $s = 1/2$

$$\therefore U_0 = \int_0^{E_F} dE \cdot E \cdot \frac{L}{\pi} \left(\frac{2m}{\hbar^2} \right)^{1/2} \frac{1}{\sqrt{E}}$$

$$= \frac{L}{\pi} \left(\frac{2m}{\hbar^2} \right)^{1/2} \int_0^{E_F} dE \sqrt{E} = \frac{L}{\pi} \left(\frac{2m}{\hbar^2} \right)^{1/2} \frac{2}{3} E_F^{3/2}$$

Use the result of (a)-(i) that $E_F^{1/2} = \dfrac{\pi \hbar N}{\sqrt{8m} \, L}$

$$\therefore U_0 = \frac{L}{\pi} \left(\frac{2m}{\hbar^2} \right)^{1/2} \frac{2}{3} \cdot E_F \cdot \underbrace{\frac{\pi \hbar N}{\sqrt{8m} \, L}}_{E_F^{3/2}} = \frac{1}{3} N E_F$$

Thus
$$\boxed{U_0 = \frac{1}{3} N E_F} \qquad \text{in } 1D$$

(ii) in 2D : $\quad D(E) = (2S+1) \dfrac{mA}{2\pi\hbar^2} \;\overset{\to S=1/2}{=}\; \dfrac{mA}{\pi\hbar^2}$ (from problem 1, Problem set 6)

$$\therefore \; U_0 = \int_0^{E_F} dE \cdot E \cdot D(E) = \frac{mA}{\pi\hbar^2} \int_0^{E_F} dE\, E = \frac{mA}{2\pi\hbar^2} E_F^{\,2}$$

Use the result of (a)-(ii) that $E_F = \dfrac{\pi\hbar^2}{mA} N$

$$\therefore \; U_0 = \frac{mA}{2\pi\hbar^2} E_F \cdot \frac{\pi\hbar^2}{mA} N = \frac{1}{2} N E_F$$

Thus $\boxed{U_0 = \dfrac{1}{2} N E_F}$ in 2D

③

The density of copper (which I looked up in an introductory physics textbook) is 8.93 g/cm³, and the atomic mass (which I got off a periodic table) is 63.5 g/mol. Let's consider a chunk of copper containing one mole of atoms. Then the mass is 63.5 g, and the volume is

$$V = \frac{\text{mass}}{\text{density}} = \frac{63.5 \text{ g}}{8.93 \text{ g/cm}^3} = 7.11 \times 10^{-6} \text{ m}^3.$$

Assuming that each atom contributes one conduction electron, the number of conduction electrons per unit volume is

$$\frac{N}{V} = \frac{6.02 \times 10^{23}}{7.11 \times 10^{-6} \text{ m}^3} = 8.47 \times 10^{28} \text{ m}^{-3},$$

and therefore the Fermi energy is

$$\epsilon_F = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3} = \frac{(6.63 \times 10^{-34} \text{ J}\cdot\text{s})^2}{8(9.11 \times 10^{-31} \text{ kg})}\left(\frac{3}{\pi}\cdot 8.47 \times 10^{28} \text{ m}^{-3}\right)^{2/3}$$
$$= 1.13 \times 10^{-18} \text{ J} = 7.05 \text{ eV}.$$

The Fermi temperature is just this divided by Boltzmann's constant,

$$T_F = \frac{\epsilon_F}{k} = \frac{7.05 \text{ eV}}{8.62 \times 10^{-5} \text{ eV/K}} = 82,000 \text{ K}.$$

This is 270 times greater than room temperature, so yes, room temperature is sufficiently low for this electron gas to be considered degenerate. The degeneracy pressure for this system is

$$P = \frac{2}{5}\frac{N}{V}\epsilon_F = (.4)(8.47 \times 10^{28} \text{ m}^{-3})(1.13 \times 10^{-18} \text{ J}) = 3.8 \times 10^{10} \text{ N/m}^2 = 3.8 \times 10^5 \text{ atm},$$

④

To determine whether either approximation is valid, let's calculate the Fermi temperature for the electron gas at the center of the sun:

$$T_F = \frac{\epsilon_F}{k} = \frac{h^2}{8mk}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{(6.63 \times 10^{-34} \text{ J·s})^2}{8(9.1 \times 10^{-31} \text{ kg})(1.38 \times 10^{-23} \text{ J/K})}\left(\frac{3(10^{32} \text{ m}^{-3})}{\pi}\right)^{2/3}$$
$$= 9.1 \times 10^6 \text{ K}.$$

This is quite close to the actual temperature, of order 10^7 K. Since T is neither much greater than nor much less than T_F, neither approximation would be very accurate: We can't treat the gas as degenerate ($T \approx 0$), nor can we treat it as an ordinary "classical" ideal gas ($T \gg T_F$).