Problem 3.1
Part A
Consider an electromagnetic wave with electric field given by:

$$\vec{\mathbf{E}}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}[\cos(kr-\omega t) - (1/kr)\sin(kr-\omega t)]\hat{\boldsymbol{\phi}}.$$

As usual, k is the wave number and ω is the angular frequency. A is a constant.

(a) Show that $\vec{\mathbf{E}}$ obeys all four of Maxwell's equations in vacuum.

(b) What is the magnetic field associated with this electric field?

(c) Calculate the Poynting vector.

a) Notice we are dealing with spherical coordinates:
Maxwells equations in free space:

① $\nabla\cdot\vec{E} = 0$

First notice that $\vec{E}(r,\theta,\phi,t)$ only has a $\hat{\phi}$ component. Thus:

$$\nabla\cdot\vec{E} = \frac{1}{r\sin\theta}\frac{\partial\vec{E}_\phi}{\partial\phi}$$

Spherical coords

$\vec{E}_\phi$ has no ϕ dependent terms thus:

$$\frac{\partial\vec{E}_\phi}{\partial\phi} = 0 \quad \therefore \quad \nabla\cdot\vec{E} = 0$$

QED

② For $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ we can set true by purely solving for $\vec{B}$ from $\vec{E}$.
If the resulting $\vec{B}$ then satisfies

$$\nabla\cdot\vec{B} = 0 \quad \text{and} \quad c^2\nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t}$$

then $\vec{E}$ must satisfy all of Maxwells equations.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\therefore \quad \vec{B} = -\int \nabla \times \vec{E}\, dt$$

$$(\nabla \times \vec{E})_\theta = \frac{1}{r}\left(\frac{1}{\sin\theta}\underbrace{\frac{\partial \vec{E}_r}{\partial \phi}}_{\hookrightarrow 0} - \frac{\partial}{\partial r} r E_\phi\right)\hat{\theta}$$

$$= \frac{-1}{r}\frac{\partial}{\partial r} A\sin\theta\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\hat{\theta}$$

$$\frac{-1}{r}\cos(kr-\omega t) + \frac{1}{kr^2}\sin(kr-\omega t)$$

$$= \frac{-1}{r} A\sin\theta\left(-k\sin(kr-\omega t) - \frac{1}{r}\cos(kr-\omega t) + \frac{1}{kr^2}\sin(kr-\omega t)\right)\hat{\theta}$$

$$-\int (\nabla \times E_\theta)\, dt = \vec{B}_\theta$$

$$\vec{B}_\theta = \frac{-1}{r} A\sin\theta\left(\frac{-k}{\omega}\cos(kr-\omega t) + \frac{1}{r\omega}\sin(kr-\omega t) + \frac{1}{kr^2\omega}\cos(kr-\omega t)\right)\hat{\theta}$$

$$(\nabla \times E)_r = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(E_\phi \sin\theta) - \underbrace{\frac{\partial E_\theta}{\partial \phi}}_{\hookrightarrow 0}\right)\hat{r}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{A\sin^2\theta}{r}\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\hat{r}$$

$$\Rightarrow \frac{\partial}{\partial\theta}\sin^2\theta = 2\cos\theta\sin\theta$$

$$= \frac{2A\cos\theta}{r^2}\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\hat{r}$$

$$\int (\nabla\times E)_r\,dt = \frac{2A\cos\theta}{r^2}\int \cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\,dt\ \hat{r}$$

$$= \frac{2A\cos\theta}{r^2}\left(-\frac{1}{\omega}\sin(kr-\omega t) - \frac{1}{kr\omega}\cos(kr-\omega t)\right)\hat{r}$$

Thus:

$$\vec{B} = \frac{2A\cos\theta}{r^2}\left(-\frac{1}{\omega}\sin(kr-\omega t) - \frac{1}{kr\omega}\cos(kr-\omega t)\right)\hat{r}$$

$$-\frac{1}{r}A\sin\theta\left(\frac{-k}{\omega}\cos(kr-\omega t) + \frac{1}{r\omega}\sin(kr-\omega t) + \frac{1}{kr^2\omega}\cos(kr-\omega t)\right)\hat{\theta}$$

We now show:

$$\nabla\cdot\vec{B} = 0$$

$$= \frac{1}{r^2}\frac{\partial\, r^2\vec{B}_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\vec{B}_\theta\sin\theta$$

$$\frac{1}{r^2}\frac{\partial r^2\vec{B}_r}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\, r^2\times\frac{-2A\cos\theta}{r^2}\left(\frac{1}{\omega}\sin(kr-\omega t) - \frac{1}{kr\omega}\cos(kr-\omega t)\right)$$

$$= \frac{-2A\cos\theta}{r^2}\left(\frac{k}{\omega}\cos(kr-\omega t) - \frac{1}{r\omega}\sin(kr-\omega t) - \frac{1}{kr^2\omega}\cos(kr-\omega t)\right)$$

$$\frac{1}{r\sin\theta}\frac{\partial \vec{B}_\theta\sin\theta}{\partial\theta}$$

$$= -\frac{1}{r}A\sin^2\theta\left(\frac{-k}{\omega}\cos(kr-\omega t) + \frac{1}{r\omega}\sin(kr-\omega t) + \frac{1}{kr^2\omega}\cos(kr-\omega t)\right)$$

$$= \frac{1}{r\sin\theta}\times\frac{-A}{r}\times 2\sin\theta\cos\theta\ \times$$

$$= \frac{-2A\cos\theta}{r^2}\left(\frac{-k}{\omega}\cos(kr-\omega t) + \frac{1}{r\omega}\sin(kr-\omega t) + \frac{1}{kr^2\omega}\cos(kr-\omega t)\right)$$

Note the sign difference

$$\therefore\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\vec{B}_\theta\sin\theta = \frac{-1}{r^2}\frac{\partial r^2\vec{B}_r}{\partial r}$$

And Thus $\nabla\cdot\vec{B} = 0$

Now finally ④

$$c^2 \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r} r\vec{B}_\theta - \frac{\partial \vec{B}_r}{\partial \theta}\right)\hat{\phi}$$

purely $\hat{\phi}$

as no component of $\vec{B}$ depends on ϕ.

$$\frac{\partial \vec{B}_r}{\partial \theta} = \frac{\partial}{\partial \theta} \frac{-2A\cos\theta}{r^2}\left(-\frac{1}{\omega}\sin(kr-\omega t) - \frac{1}{kr\omega}\cos(kr-\omega t)\right)$$

$$= -\frac{2A\sin\theta}{r^2}\left(-\frac{1}{\omega}\sin(kr-\omega t) - \frac{1}{kr\omega}\cos(kr-\omega t)\right)$$

$$= A\sin\theta\left(\frac{-2}{r^2\omega}\sin(kr-\omega t) - \frac{2}{kr^3\omega}\cos(kr-\omega t)\right)$$

$$-\frac{\partial}{\partial r}\left(r\vec{B}_\theta\right):$$

$$= -\frac{\partial}{\partial r}\, r\, \frac{-1}{r} A\sin\theta\left(\frac{-k}{\omega}\cos(kr-\omega t) + \frac{1}{r\omega}\sin(kr-\omega t) + \frac{1}{kr^2\omega}\cos(kr-\omega t)\right)$$

$$= -A\sin\theta\left(\frac{k^2}{\omega}\sin(kr-\omega t) - \frac{1}{r^2\omega}\sin(kr-\omega t) + \frac{k}{r\omega}\cos(kr-\omega t)\right.$$
$$\left. + \frac{\cancel{k}}{\cancel{k}r^2\omega}\sin(kr-\omega t) - \frac{1}{kr^3\omega}\cos(kr-\omega t)\right)$$

$$= A\sin\theta\left(\left(-\frac{k^2}{\omega} + \frac{2}{r^2\omega}\right)\sin(kr-\omega t) + \left(\frac{-1}{r^2\omega} - \frac{2}{kr^3\omega} + \frac{k}{\omega r}\right)\cos(kr-\omega t)\right)$$

$\therefore$ The sum of these parts:

$$\nabla\times\vec{B} = \frac{1}{r}A\sin\theta\left(\left(\frac{k^2}{\omega}+\frac{1}{r^2\omega}-\frac{1}{r^2\omega}\right)\sin(kr-\omega t)+\left(\frac{2}{kr^3\omega}-\frac{2}{kr^3\omega}+\frac{k}{\omega r}\right)\cos(kr-\omega t)\right)$$

$$= \frac{A\sin\theta}{r}\left(\left(\frac{k^2}{\omega}\right)\sin(kr-\omega t)+\left(\frac{k}{\omega r}\right)\cos(kr-\omega t)\right)$$

Given:

$$\frac{\partial\vec{E}}{\partial t} = \frac{A\sin\theta}{r}\left(\omega\sin(kr-\omega t)+\frac{\omega}{kr}\cos(kr-\omega t)\right)\hat{\phi}$$

We notice:

$$c^2\nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t}$$

thus:

$$\frac{k^2c^2}{\omega}=\omega \qquad \frac{k}{\omega r}c^2 = \frac{\omega}{kr}$$

$$\frac{k^2\omega^2}{\omega k^2}=\omega \qquad \frac{k\,\omega^2}{\omega k^2 r} = \frac{\omega}{kr}$$

$$\omega=\omega \qquad \frac{\omega}{kr}=\frac{\omega}{kr}$$

True $\qquad$ True.

Equating Sin and Cos coeff of $\frac{\partial\vec{E}}{\partial t}$ and $c^2\nabla\times\vec{B}$ $\longrightarrow$

Thus we have shown that

$$c^2\nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t}$$

b) We have shown in a) that:

$$\vec{B} = \frac{2A\cos\theta}{r^2}\left(-\frac{1}{\omega}\sin(kr-\omega t)-\frac{1}{kr\omega}\cos(kr-\omega t)\right)\hat{r}$$

$$-\frac{1}{r}A\sin\theta\left(-\frac{k}{\omega}\cos(kr-\omega t)+\frac{1}{r\omega}\sin(kr-\omega t)+\frac{1}{kr^2\omega}\cos(kr-\omega t)\right)\hat{\theta}$$

c) The poynting vector is denoted by:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$= \frac{1}{\mu_0} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \vec{E}_r & \vec{E}_\theta & \vec{E}_\phi \\ \vec{B}_r & \vec{B}_\theta & \vec{B}_\phi \end{vmatrix}$$

$$= \frac{1}{\mu_0} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 0 & 0 & \vec{E}_\phi \\ \vec{B}_r & \vec{B}_\theta & 0 \end{vmatrix}$$

$$= \frac{1}{\mu_0} \left(-\vec{E}_\phi \vec{B}_\theta \hat{r} - \vec{E}_\phi \vec{B}_r \hat{\theta} \right)$$

$$= -\frac{1}{\mu_0} \vec{E}_\phi \left(\vec{B}_\theta \hat{r} + \vec{B}_r \hat{\theta} \right)$$

$$= -\frac{1}{\mu_0} \frac{A \sin\theta}{r} \left(\cos(kr-\omega t) - \frac{1}{kr} \sin(kr-\omega t) \right)$$

$$\times \left[\frac{2A\cos\theta}{r^2} \left(-\frac{1}{\omega} \sin(kr-\omega t) - \frac{1}{kr\omega} \cos(kr-\omega t) \right) \hat{\theta} \right.$$
$$\left. - \frac{1}{r} A \sin\theta \left(-\frac{k}{\omega} \cos(kr-\omega t) + \frac{1}{r\omega} \sin(kr-\omega t) + \frac{1}{kr^2\omega} \cos(kr-\omega t) \right) \hat{r} \right]$$

Part B - Advanced

Consider a linearly polarised electromagnetic field of wavenumber k travelling in the $+z$ direction described by

$$\mathbf{B} = ik\left[u\hat{\mathbf{y}} + \frac{i}{k}\frac{\partial u}{\partial y}\hat{\mathbf{z}}\right]e^{i(kz-\omega t)}$$

$$\mathbf{E} = i\omega\left[u\hat{\mathbf{x}} + \frac{i}{k}\frac{\partial u}{\partial x}\hat{\mathbf{z}}\right]e^{i(kz-\omega t)},$$

where $u = u(r,\phi,z) = u_0(r,z)e^{+i\ell\phi}$ is a function with cylindrically symmetric amplitude about the propagation axis (ℓ is an integer). The surfaces of constant phase for a field like this are helical, as shown in Fig. ??.

Figure 1: Helical wavefronts

(a) How is the Poynting vector of this electromagnetic field different from that of a plane wave?

(b) How is the wavevector of this electromagnetic field different from that of a plane wave?

a) Quantitatively, we can observe a difference in the poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$= \frac{1}{\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ i\omega u & 0 & -\frac{\omega}{k}\frac{\partial u}{\partial x} \\ 0 & iku & -\frac{\partial u}{\partial y} \end{vmatrix} e^{i(kz-\omega t)\times 2}$$

$$= \frac{e^{2i(kz-\omega t)}}{\mu_0}\left(-\omega k u^2\hat{x} - i\omega u\frac{\partial u}{\partial y}\hat{y} + \frac{\omega iku}{k}\frac{\partial u}{\partial x}\hat{z}\right)$$

$$= \frac{e^{2i(kz-\omega t)}}{\mu_0}\,\omega u\left(-ku\hat{x} - i\frac{\partial u}{\partial y}\hat{y} + i\frac{\partial u}{\partial x}\hat{z}\right)$$

For a standard plane wave traveling in the z direction, we find as an example:

$$\vec{E}_x = \varepsilon_x e^{i(kz-\omega t)} \qquad \vec{E}_y = \varepsilon_y e^{i(kz-\omega t)}$$

$$\vec{B}_x = \mathcal{B}_x e^{ikz-\omega t} \qquad \vec{B}_y = \mathcal{B}_y e^{i(kz-\omega t)}$$

$$\vec{E}_z = \vec{B}_z = 0$$

which results in a poynting vector of purely $\hat{z}$ component.

$$S = \frac{1}{\mu_0} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \mathcal{E}_x & \mathcal{E}_y & 0 \\ \beta_x & \beta_y & 0 \end{vmatrix} e^{2i(kz-\omega t)}$$

$$= \frac{1}{\mu_0} \left(\mathcal{E}_x \beta_y - \mathcal{E}_y \beta_x \right) e^{2i(kz-\omega t)} \hat{z}$$

Thus we notice an obvious difference, such that the helical waveforms poynting vector rotates about the z axis. In other words, despite both traveling along the z axis, the energy flux is spread about in a helix, whilst for a plane wave it is concentrated on the direction of movement.

This poynting vector rotation about the $\hat{z}$ axis is a direct result of the $e^{il\phi}$ dependence in u, which propagates into $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in $\vec{E}$ and $\vec{B}$ given the Poynting vector is always perpendicular to these fields.

b) The wavevector of an EM wave outlines how the phase of the wave changes in space. The following is defined:

$$\vec{k} = (k_x, k_y, k_z)$$

$$|\vec{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda}$$

wavelength

For the plane wave traveling in z, we understand $k_x = k_y = 0$ as all phase is in $\hat{z}$ and thus:

$$\vec{k} = k\hat{z} = \frac{2\pi}{\lambda}\hat{z}$$

However for a helix, the $e^{il\phi}$ dependence adds complexity. The helical structure is caused by:

$$u = u_0(r,\theta)\underbrace{e^{il\phi}}_{\text{cause for helix}}$$

Thus the phase is created by both the phase in z and ϕ:

$$\text{phase} = kz + l\phi$$

Still true as $k_x = k_y = 0$

And thus:

$$\vec{k}_{eff} = \nabla(kz + l\phi)$$

$$= k\hat{z} + \frac{1}{r}l\hat{\phi}$$

$$\nabla l\phi = \frac{1}{r}l\frac{\partial}{\partial\phi}\phi\,\hat{\phi}$$

$$= \frac{l}{r}\hat{\phi}$$

$$= k\hat{z} + \frac{l}{r}\hat{\phi}$$

Thus we clearly notice that the wave vector for the planar wave travels purely down z, whilst the helix wavevector also twists around z.