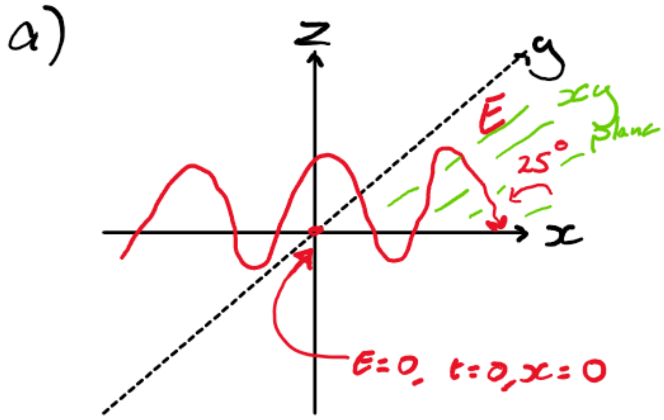
Problem 4.1
Part A
For the following questions, consider an electromagnetic wave of frequency ω and wave number k.

(a) Write an expression for a linearly polarised light of electric field amplitude E_0, propagating along the x-axis with its plane of oscillation at 25^o to the xy-plane. The electric field is zero at $t = 0$ and $x = 0$. *Hint: You can have a nonzero phase constant.*

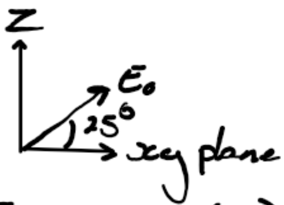
(b) Write an expression for a linearly polarised light of electric field amplitude E_0, propagating along a line in the xy-plane at 45^o to the x-axis and having its plane of oscillation corresponding to the xy-plane. The electric field is zero at $t = 0$, $x = 0$, and $y = 0$. *Hint: What is the wave vector?*

(c) Write an expression for righ-circularly-polarised light propagating in the x-direction such that at $t = 0$ and $x = 0$, the electric field vector points in the $-z$-direction.

a)



xy angle corresponds to:



$$E_z = E_0 \sin(25)$$
$$E_y = E_0 \cos(25)$$

$$\vec{E} = \vec{E_0} \cos(kx - \omega t + \phi) \quad (1)$$

x axis propagation.

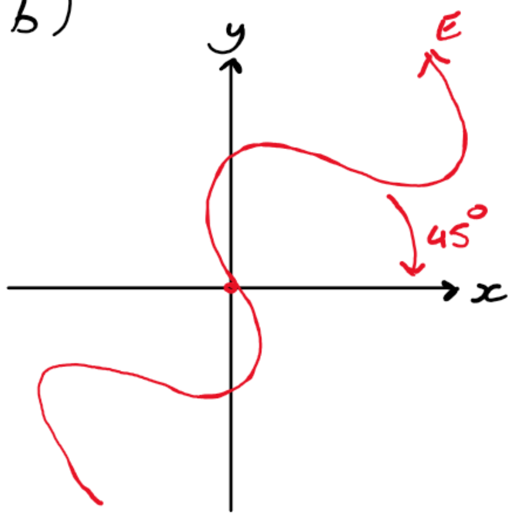
In cylindrical coordinates:

$$\vec{E_0} = E_0\left(\cos\left(25 \times \frac{\pi}{180}\right)\hat{y} + \sin\left(25 \times \frac{\pi}{180}\right)\hat{z}\right)$$

Sub into (1) Thus:

$$\vec{E} = E_0\left(\cos\left(\frac{5\pi}{36}\right)\hat{y} + \sin\left(\frac{5\pi}{36}\right)\hat{z}\right)\cos(kx - \omega t + \phi)$$

b)



We understand the wavevector to be:

$$\vec{k} = k_x \hat{i} + k_y \hat{j}$$

$$= k\cos(45)\hat{i} + k\sin(45)\hat{j}$$

$$= \frac{k}{\sqrt{2}}(\hat{i}+\hat{j})$$

Since we know $\vec{E}_0$ and $\vec{k}$ must be perpendicular but both in the xy plane:

$$\vec{E}_0 \propto (\hat{i}-\hat{j}) \quad \text{as} \quad \vec{k}\cdot\vec{E}_0 = \frac{k}{\sqrt{2}}(\hat{i}+\hat{j})\cdot(\hat{i}-\hat{j})$$

$$= \frac{k}{\sqrt{2}}(1\times 1 + 1\times -1)$$

$$= 0 \quad \text{(Perpendicular)}$$

Since: $\vec{E}(r,t) = E_0\cos(\vec{k}\cdot\hat{r} - \omega t + \phi)$

$\vec{E}=0,\ t=0,\ r=0 \rightarrow \sqrt{x^2+y^2} = |\vec{r}|$

$$0 = E_0\cos(0-0+\phi)$$

$$\phi = n\frac{\pi}{2} \qquad n = 1, 3, 5\ldots$$

Choosing $\phi = \frac{\pi}{2}$ and noticing $\cos(x+\frac{\pi}{2}) = \sin(x)$:

$$\vec{E}_0(r,t) = E_0\sin(\vec{k}\cdot\vec{r} - \omega t)$$

Given $\vec{r} = x\hat{i} + y\hat{j}$: $\vec{k} \cdot \vec{r} = k_x x + k_y y$

$$= k\cos(45)x + k\cos(45)y$$
$$= \frac{k}{\sqrt{2}}(x+y)$$

And $\vec{E_0} \propto (\hat{i} - \hat{j})$ implies:

$$\vec{E_0} = \vec{E_0} \times \frac{(\hat{i}-\hat{j})}{\sqrt{2}}$$

← Direction of $\vec{E_0}$

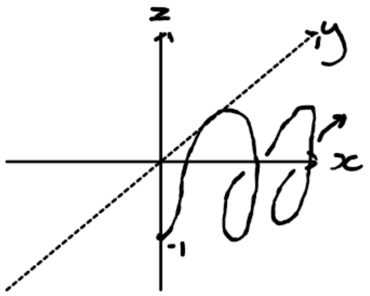
ensures unit vector

Thus finally:

$$\vec{E_0} = E_0 \frac{(\hat{i}-\hat{j})}{\sqrt{2}} \sin\left(\frac{k}{\sqrt{2}}(x+y) - \omega t\right)$$

c) Right circularly polarised (From source)
x propagating : $t=0$, $x=0$, $\vec{E} = -\hat{z}$

Assuming the x and y components are $\frac{\pi}{2}$ out of phase



We understand:

$$\vec{k} = k\hat{i}$$
$$\vec{E_0}(x=t=0) = 0\hat{j} - \hat{E_0}\hat{k}$$

The general form of x propagating right handed:

$$\vec{E}(x,t) = E_0\left(\hat{j}\cos(kx - \omega t + \phi) + \hat{k}\sin(kx - \omega t + \phi)\right)$$

We notice

$$E_j(0,0) = E_0\cos(\phi) = 0$$
$$E_k(0,0) = E_0\sin(\phi) = -E_0$$

Thus $\phi = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$

$$\vec{E}(x,t) = E_0\left(\hat{j}\cos\left(kx - \omega t - \frac{\pi}{2}\right) + \hat{k}\sin\left(kx - \omega t - \frac{\pi}{2}\right)\right)$$

Since $\cos(x - \frac{\pi}{2}) = \sin(x)$: $\sin(x - \frac{\pi}{2}) = -\cos(x)$

$$\vec{E}(x,y) = E_0\left(\hat{j}\sin(kx - \omega t) - \hat{k}\cos(kx - \omega t)\right)$$

Part B - Advanced

Consider Fig. 1. The crystal on top of the word OPTICS is calcite, the two polaroids have their transmission axes parallel to their short edges. The principal section of this particular calcite crystal is approximately 45^o counter-clockwise (looking head-on towards the picture) from the vertical axis.

Figure 1: Calcite

(a) Why are the letters P and T obviously shifted downwards?

(b) Why does the letter C appear unaffected?

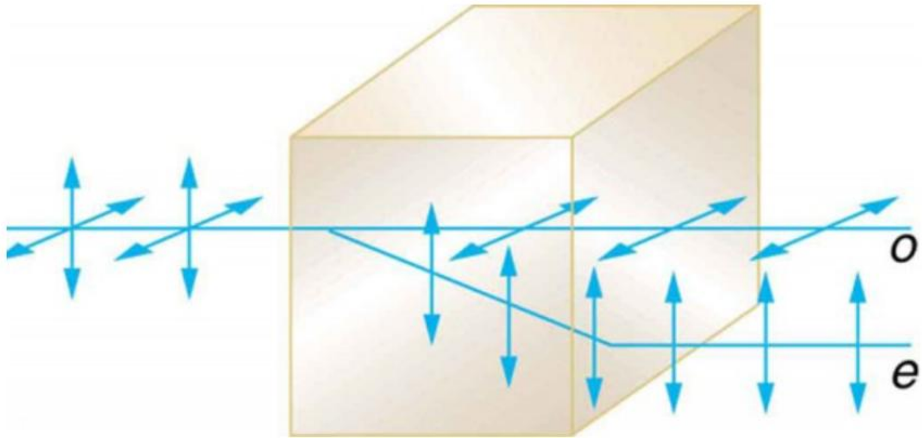
Hint. What does principal section mean in relation to the ordinary and extraordinary waves?

A)

We first need to understand the geometry of the crystal. The principal section of the calcite crystal is approximately 45degrees counterclockwise for our point of view. The principal section is a plane within the crystal, through which the optical axis passes, i.e. the observed light is in the principal section plane.

The letters T and I are both observed to be in the original position, and slightly downshifted. These two images refer to the ordinary wave, where the light passes the medium unrefracted (The darker T and I), and the extraordinary waves, where the light has been refracted by the medium and thus shifted downwards. However, it is relevant to point out that both the ordinary and extraordinary waves pass along the principal section, which given it is rotated 45 degrees anticlockwise from vertical, the downward shifted extraordinary I and T are justified.

It is also relevant to point out that the reason calcite creates these two images at all is due to the birefringence nature of the material, meaning the refraction of light depends on the polarisation of the light itself. As shown in the figure below, birefringence materials split the unpolarised light into its linear components, with the extraordinary wave not obeying Snell's law.



B)

As for the apparent unaffected C character, we understand this implies that there were no extraordinary waves. This further indicates that the incident light was already polarised prior to entering the calcite. This is a direct result of the letter being positioned above a polaroid. The polaroid is placed at 45 degrees, such that the light from the C character is linearly polarised and transmitted through the principal section plane, and thus undergoes no separation into extraordinary waves. As such, C appears to be unchanged.

Problem 4.2
Part A

Fig. 2 shows two nearly overlapped intensity peaks of the sort you might produce with a diffraction grating.

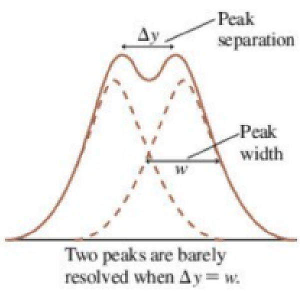


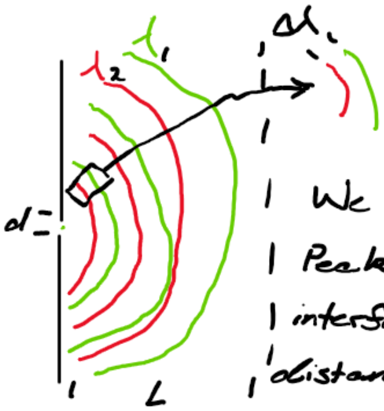
Figure 2: Resolution

As a practical matter, two peaks can just barely be resolved if their spacing equals the width w of each peak, where w is measured at half of the peak's height. Two peaks closer together than w will merge into a single peak. We can use this idea to understand the resolution of a diffraction grating.

(a) In the small-angle approximation, the position of the $m = 1$ peak of a diffraction grating falls at the same location as the $m = 1$ fringe of a double slit: $y_1 = \lambda L/d$ (L is the distance from the observation screen to the grating, d is the width of the slits). Suppose two wavelengths differing by $\Delta\lambda$ pass through a grating at the same time. Find an expression for Δy, the separation of their first-order peaks.

(b) The widths of the bright fringes are proportional to $1/N$, where N is the number of slits in the grating. Let's hypothesise that the fringe width is $w = y_1/N$. Show that this is true for the double-slit pattern.

First visualising this problem we assume their peaks pass the gratting at the same time:



We can note: $\lambda_2 = \lambda_1 - \Delta\lambda$
Peaks occur during constructive peak interference, such that the distance is $n\lambda$ long, where n is an integer.

We are told the first order peaks occur at: $y_1 = \frac{\lambda L}{d}$ thus:

$$y_1(\lambda) = \frac{\lambda L}{d} \qquad y_1(\lambda + \Delta\lambda) = \frac{(\lambda + \Delta\lambda)L}{d}$$

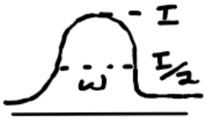
$$\therefore \Delta y = y_1(\lambda + \Delta\lambda) - y_1(\lambda) = \frac{\Delta\lambda L}{d}$$

Separation between first order peak of each λ.

b) $w \propto \frac{1}{N}$ such that $w = \frac{y_1}{N}$
For the double slit experiment $N = 2$
thus:

$$w = \frac{y_1}{2} = \frac{\Delta\lambda L}{2d}$$

By definition w is the width at half maximum:



So to show $\omega = \frac{y_1}{N}$ is true for the double slit, we must show I is the same for both:

Double Slit I:

$$I_{Slit}(\theta) = I_0 \frac{\sin^2\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\left(\frac{\pi b \sin\theta}{\lambda}\right)^2} \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

Diffraction grating I:

$$I_{Diff}(\theta) = \frac{I(0)}{N^2} \frac{\sin^2(\beta)}{\beta^2} \frac{\sin^2(N\alpha)}{\sin^2(\alpha)}$$

$$= \frac{I_0}{4} \frac{\sin^2\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\left(\frac{\pi b \sin\theta}{\lambda}\right)^2} \frac{\sin^2\left(\frac{2\pi d \sin\theta}{\lambda}\right)}{\sin^2\left(\frac{\pi d \sin\theta}{\lambda}\right)}$$

Notice: $\sin\left(\frac{2\pi d \sin\theta}{\lambda}\right) = 2\sin\left(\frac{\pi d \sin\theta}{\lambda}\right)\cos\left(\frac{\pi d \sin\theta}{\lambda}\right)$

$$\therefore = \frac{I_0}{4} \frac{\sin^2\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\left(\frac{\pi b \sin\theta}{\lambda}\right)^2} \left(2\cos\left(\frac{\pi d \sin\theta}{\lambda}\right) \frac{\sin\left(\frac{\pi d \sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi d \sin\theta}{\lambda}\right)}\right)^2$$

$$= \frac{I_0}{4} \frac{\sin^2\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\left(\frac{\pi b \sin\theta}{\lambda}\right)^2} \times 4\cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

$$= I_0 \frac{\sin^2\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\left(\frac{\pi b \sin\theta}{\lambda}\right)^2} \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

Thus: $I_{Diff}(\theta) = I_{Slit}(\theta)$

Since they have the same intensity form, $\omega = \frac{y_1}{2}$ must satisfy the double slit pattern.

Part B - Advanced
Use your results from parts (a) and (b) together with the idea that $\Delta y_{min} = w$ to find an expression for $\Delta\lambda_{min}$, the minimum wavelength separation (in first order) for which the diffraction fringes can barely be resolved.

From a: $\Delta y = \frac{\Delta\lambda L}{d}$

From b: $w = \frac{\lambda L}{dN}$

Given barely resolved when:

$\Delta y = w$: $\frac{\Delta\lambda_{min} L}{d} = \frac{\lambda L}{dN}$

$$\therefore \Delta\lambda_{min} = \frac{\lambda}{N}$$

Thus we have shown the minimum wavelength at which the diffraction fringes can barely be resolved the ratio of the wavelength and the number of slits.