PHYS2901 Problem set 2

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Problem 3.4)

3.4)

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^*(x,t)\,\Psi_2(x,t)\,dx = 0$$

$$= \int_{-\infty}^{\infty} \frac{d}{dt}\Psi_1^*(x,t)\Psi_2(x,t)\,dx$$

$$\frac{d}{dt}\Psi_1^*(x,t)\Psi_2(x,t) = \frac{d\Psi_1^*}{dt}\Psi_2(x,t) + \frac{d\Psi_2}{dt}\Psi_1^*(x,t)$$

$$= \int_{-\infty}^{\infty} \frac{d\Psi_1^*}{dt}\Psi_2(x,t) + \frac{d\Psi_2}{dt}\Psi_1^*(x,t)\,dx$$

$$\frac{d\Psi_1^*}{dt} = \frac{i\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2} + \frac{i}{\hbar}V\Psi_1^*$$

$$\frac{d\Psi_2^*}{dt} = \frac{i\hbar}{2m}\frac{\partial^2\Psi_2^*}{\partial x^2} - \frac{i}{\hbar}V\Psi_2^*$$

$$\cancel{= \int_{-\infty}^{\infty} \frac{i\hbar}{2m}}$$

cancel

$$= \int_{-\infty}^{\infty} \left(\frac{i\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2} - \frac{i}{\hbar}V\Psi_1^*\right)\Psi_2 + \left(\frac{i\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2} - \frac{i}{\hbar}V\Psi_2\right)\Psi_1^*\,dx$$

$$= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2 - \frac{\partial^2\Psi_2}{\partial x^2}\Psi_1^*\right)dx$$

Integrate by parts

$$= \frac{i\hbar}{2m}\left(\left[\frac{\partial\Psi_1^*}{\partial x}\Psi_2\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx - \left[\frac{\partial\Psi_2}{\partial x}\Psi_1^*\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\partial\Psi_2}{\partial x}\frac{\partial\Psi_1^*}{\partial x}dx\right)$$

$$= 0 \qquad \text{QED}$$

Problem 3.5)

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By first rearranging Schrödingers equation, we find:

$$i\hbar \frac{\partial \Psi(x)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x)}{\partial x^2} + V\Psi(x)$$

As $\Psi(x)$ is time independant:

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + V\Psi(x) = E\Psi(x)$$

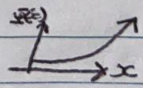
$$\frac{d^2\Psi(x)}{dx^2} = \frac{-2m}{\hbar^2}\left(E - V(x)\right)\Psi(x)$$

For the case $E < V_{min}$, we find:

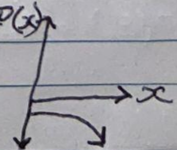
$\frac{d^2\Psi(x)}{dx^2} =$ ~~negative~~ positive as $\frac{-2m}{\hbar}(E - V(x))$ is ~~negative~~ positive.

As $\frac{d^2\Psi(x)}{dx^2}$ and $\Psi(x)$ now share the same sign, regardless of whether it is negative or positive, $\Psi(x)$ will never converge to 0 as $x \to \infty$. To show this:

($\Psi(x)$ is positive): The function will be positive with a concave up motion:



($\Psi(x)$ is negative): The function will be negative with a concave down motion:



Quantum vs Classical:

We have already investigated the case where a particles energy is less than the minimum potential of V(x), in which case, we found the wave function is not normalisable. This indicates that the energy of a particle must be greater than the minimum potential in order for its wavefunction to be normalisable. However, I now want you to consider the case where the energy of a particle is directly equal to the minimum potential. In such a case, there is the possibility for the energy of the particle to equal the minimum potential of V(x). This indicates that somewhere in the square well, the particle may be stationary. However, we know this to violate Heisenberg's Uncertainty Principle, and as such, similarly proves that E must be greater than Vmin. However, in classical mechanics, the kinetic energy of an object can indeed be 0, thus we observe a major difference between classical and quantum mechanics.

Problem 4.3)

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a) The probability that the particle is in the left half $(0 \leq x \leq \frac{a}{2})$ is given by:

$$p = \int_0^{a/2} |\Psi(x,0)|^2 \, dx$$

$$|\Psi(x,0)|^2 = \frac{8}{5a}\left(1+\cos\left(\frac{\pi x}{a}\right)\right)^2 \sin^2\left(\frac{\pi x}{a}\right)$$

$$\therefore \; p(0 \leq x \leq a/2) = \int_0^{a/2} \frac{8}{5a}\left(1+\cos\left(\frac{\pi x}{a}\right)\right)^2 \sin^2\left(\frac{\pi x}{a}\right) dx$$

First, we further simplify the expression under the integral.

$$= \frac{8}{5a}\left(1+\cos\left(\frac{x\pi}{a}\right)\right)^2\left(1-\cos^2\left(\frac{x\pi}{a}\right)\right)$$

$$= \frac{8}{5a}\left(1+2\cos\left(\frac{x\pi}{a}\right)+\cos^2\left(\frac{x\pi}{a}\right)\right)\left(1-\cos^2\left(\frac{x\pi}{a}\right)\right)$$

$$= \frac{8}{5a}\left(1-\cos^2\left(\frac{x\pi}{a}\right)+2\cos\left(\frac{x\pi}{a}\right)-2\cos^3\left(\frac{x\pi}{a}\right)+\cos^2\left(\frac{x\pi}{a}\right)-\cos^4\left(\frac{x\pi}{a}\right)\right)$$

$$= \frac{8}{5a}\left(1+2\cos\left(\frac{x\pi}{a}\right)-2\cos^3\left(\frac{x\pi}{a}\right)-\cos^4\left(\frac{x\pi}{a}\right)\right)$$

This leaves:

$$= \frac{8}{5a}\left(\int_0^{a/2} 1\,dx + 2\int_0^{a/2}\cos\left(\frac{\pi x}{a}\right)dx - 2\int_0^{a/2}\cos^3\left(\frac{\pi x}{2}\right)dx - \int_0^{a/2}\cos^4\left(\frac{\pi x}{a}\right)dx\right)$$

$$= 0.8395$$

Thus there is a 83.95% chance of finding it in the first half of the box.

b)

$$\Psi(x,0) = \sqrt{\frac{8}{5a}}\left(1+\cos\left(\frac{\pi x}{a}\right)\right)\sin\left(\frac{\pi x}{a}\right)$$

$$= \sqrt{\frac{8}{5a}}\left(\sin\left(\frac{\pi x}{a}\right) + \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\right)$$

$$= \sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{8}{5a}}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)$$

Notice that $\sin(2A) = 2\sin(A)\cos(A)$ Thus:

$$\Psi(x,0) = \sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right) + \frac{1}{2}\sqrt{\frac{8}{5a}}\sin\left(\frac{2\pi x}{a}\right)$$

Given the two facts:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \quad \text{where} \quad n = 1,2,3\ldots$$

$$E = \frac{n^2\pi\hbar^2}{2ma^2}$$

We know:

$\sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right)$:

$n=1$

$E_1 = \frac{\pi\hbar^3}{2ma^2}$

$P(E_1) = |c|^2 = \frac{8}{5a}$

$\frac{1}{2}\sqrt{\frac{8}{5a}}\sin\left(\frac{2\pi x}{a}\right)$:

$n=2$

$E_2 = \frac{4\pi\hbar^2}{2ma^2}$

$P(E_2) = |c|^2 = \frac{8}{20a} = \frac{2}{5a}$

Since $P(E_1) + P(E_2) = 1$ $\cancel{= \frac{8}{5a} + \frac{8}{20a}}$

$\therefore$ $P(E_1) = 4P(E_2)$ Thus

$E_1 = \frac{\pi\hbar^2}{2ma^2}$

$P(E_1) = 0.8$

$E_2 = \frac{4\pi\hbar^2}{2ma^2}$

$P(E_2) = 0.2$

c)

To solve for $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \psi_n(x) \Psi(x,0)\, dx$ we must remember the following:

$$\Psi(x,0) = \sum_{n=1}^{\infty} C_n \psi_n(x)$$

$$C_n = \int_{-\infty}^{\infty} \psi_n(x)\, \Psi(x,0)\, dx$$

Given $|C_n|^2$ is the probability that the wave function ~~Ψ~~ depicts some given $\psi_n(x)$, we know over a complete basis of eigenfunctions that:

$$\sum_{n=1}^{\infty} |C_n|^2 = 1$$

Thus we ~~[illegible]~~ notice that.

$$\sum_{n=1}^{\infty} C_n = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \psi_n(x) \Psi(x,0)\, dx$$

In the same way that the sum of C_n squared ~~is~~ equalling 1 indicates that the particle must be somewhere, the sum of all projections of $\psi_n(x)$ onto $\Psi(x,0)$ must be 1 as by definition $\psi_n(x)$ forms a complete basis and $\Psi(x,0)$ is normalised.

Problem 4.4)

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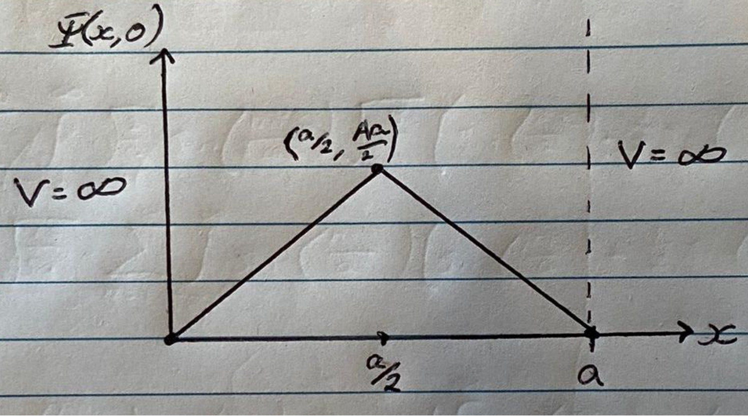
a)

$$\Psi(x,0) = \begin{cases} Ax & \text{if } 0 \le x \le a/2 \\ A(a-x) & \text{if } a/2 \le x \le a \end{cases}$$



To determine A, we recall:

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx$$

$$= \int_0^{a/2} A^2 x^2 \, dx + \int_{a/2}^{a} A^2 (a-x)^2 \, dx$$

$$\frac{1}{A^2} = \left[\frac{x^3}{3}\right]_0^{a/2} + \int_{a/2}^{a} x^2 - 2ax + a^2 \, dx$$

$$= \frac{1}{3}\left(\frac{a}{2}\right)^3 - 0 + \left[\frac{x^3}{3} - ax^2 + a^2 x\right]_{a/2}^{a}$$

$$= \frac{a^3}{24} + \frac{a^3}{3} - a^3 + a^3 - \frac{1}{3}\left(\frac{a}{2}\right)^3 + a\left(\frac{a}{2}\right)^2 - a^2\left(\frac{a}{2}\right)$$

$$= \frac{a^3}{24} + \frac{a^3}{3} - \frac{a^3}{24} + \frac{a^3}{4} - \frac{a^3}{2}$$

$$= \frac{a^3}{12} = \frac{1}{A^2} \implies A = \frac{2\sqrt{3}}{a^{3/2}}$$

b) Given: $\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$ and

$$c_n = \int_{-\infty}^{\infty} \psi_n^* \, \Psi(x,0) \, dx$$

$$\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

Thus for $0 \leq x \leq \frac{a}{2}$ we find:

$$c_n = \int_0^{a/2} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) Ax \, dx$$

Integrate by parts: $u = Ax$ $\quad du = A$

$\frac{dv}{dx} = \sin\left(\frac{n\pi x}{a}\right)$ $\quad v = \frac{-a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)$

$$= \left[\frac{-a}{n\pi} \cos\left(\frac{n\pi x}{a}\right) Ax\right]_0^{a/2} - \int_0^{a/2} A \frac{-a}{n\pi} \cos\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{-a}{n\pi} \cos\left(\frac{n\pi}{2}\right) \frac{Aa}{2} + \left[\frac{Aa^2}{n\pi^2} \sin\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a}$$

$$= -\frac{Aa^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{Aa^2}{n^2\pi^2} \sin\left(\frac{\cancel{}}{n\pi}\right) - \frac{Aa^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$C_n = \left(\frac{-Aa^2}{2\pi n}\cos\left(\frac{n\pi}{2}\right) + \frac{Aa^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\sqrt{\frac{2}{a}}$$

$$= A\sqrt{\frac{2}{a}}\left(\frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{a^2}{2\pi n}\cos\left(\frac{n\pi}{2}\right)\right)$$

Now for $a/2 \leq x \leq a$:

$$C_n = \int_{a/2}^{a} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)A(a-x)\,dx$$

Integrate by parts: $u = A(a-x)$ $\quad \frac{du}{dx} = -A$

$\frac{dv}{dx} = \sin\left(\frac{n\pi x}{a}\right)$ $\quad v = \frac{-a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)$

$$\frac{C_n}{\sqrt{\frac{2}{a}}} = \left[A(a-x)\frac{-a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a} - \int_{a/2}^{a} -A\frac{-a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)dx$$

$$= 0 + \frac{Aa^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \left[\frac{Aa^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a}$$

$$= \frac{Aa^2}{2\pi n}\cos\left(\frac{n\pi}{2}\right) - \frac{Aa^2}{n^2\pi^2}\sin(n\pi) + \frac{Aa^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$= A\sqrt{\frac{2}{a}}\left(\frac{a^2}{2\pi n}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

With this we can now conclude:

$$C_n = C_n(0 \leq x \leq a/2) + C_n(a/2 \leq x \leq a)$$

$$= A\sqrt{\frac{2}{a}}\left(\frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{a^2}{2\pi n}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{2\pi n}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$= A\sqrt{\frac{2}{a}}\left(\frac{2a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

C)

Given these series of coefficient Cn, we understand that the probability P(E) will be largest for a number that maximises the $\frac{2a^2}{n^2\pi^2}$ and $\sin\left(\frac{n\pi}{2}\right)$ terms given $P(E_n) = |C_n|^2$. This, by observation, indicates that the most probable energy level will be n = 1 which means:

$$E_1 = \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{\pi^2\hbar^2}{2ma^2}$$