MATH2901 Assignment 1

Q1) $\frac{dy}{dx} = -x + x^2 y \qquad y(0)=0$

By definition $\phi_0 = 0$ and $f(x,y) = -x + x^2 y$

$\phi_n(x) = \int_0^x f(x,y)\, ds$ where $x = s$ and $y = \phi_{n-1}(x)$

$\therefore \phi_1(x) = \int_0^x f(s,0)\, ds = \int_0^x -s + s^2 \cdot 0\, ds = \left[-\frac{1}{2}s^2\right]_0^x = \frac{-1}{2}x^2$

$\phi_2(x) = \int_0^x f(s, -\frac{1}{2}s^2)\, ds = \int_0^x -s + s^2\left(-\frac{1}{2}s^2\right) ds = \left[-\frac{1}{10}s^5 - \frac{1}{2}s^2\right]_0^x$

$= -\frac{1}{10}x^5 - \frac{1}{2}x^2$

$\phi_3(x) = \int_0^x f(s, -\frac{1}{10}s^5 - \frac{1}{2}s^2)\, ds = \int_0^x -s + s^2\left(-\frac{1}{10}s^5 - \frac{1}{2}x^2\right) ds$

$= \left[-\frac{1}{2}s^2 - \frac{1}{10}s^5 - \frac{1}{80}s^8\right]_0^x$

$= -\frac{1}{2}x^2 - \frac{1}{10}x^5 - \frac{1}{80}x^8$

Pattern: $\phi_k(x) = \sum_{n=1}^{k} -\frac{1}{\prod_{i=1}^{n}(3i-1)} x^{(3n-1)}$

Prove $k=1$ is true:

$\phi_1(x) = \sum_{n=1}^{1} -\frac{1}{\prod_{i=1}^{1}(3i-1)} x^{(3n-1)} = \frac{-1}{(3-1)} x^{(3-1)} = \frac{-1}{2}x^2$

Now prove $k+1$: Since k is true.

Apply iterative rule to $\phi_k(x) = \sum_{n=1}^{k} \frac{1}{\prod_{i=1}^{n}(3i-1)} x^{(3n-1)}$

$$\phi_{k+1} = \int_0^x f(s, \phi_k)\, ds = \int_0^x -s + s^2\left(\sum_{n=1}^{k} -\frac{1}{\prod_{i=1}^{n}(3i-1)} s^{(3n-1)}\right) ds$$

$$= \int_0^x -s\, ds + \int_0^x \sum_{n=1}^{k} -\frac{1}{\prod_{i=1}^{n}(3i-1)} s^{(3n+1)}\, ds \quad \text{(Index laws)}$$

$$= -\frac{1}{2}x^2 + \left[\sum_{n=1}^{k} -\frac{1}{\prod_{i=1}^{n}(3i-1)} s^{(3n+2)} \times \frac{1}{(3n+2)}\right]_0^x$$

$$= -\frac{1}{2}x^2 + \sum_{n=1}^{k} -\frac{1}{\prod_{i=1}^{n}(3i-1)} x^{(3n+2)} \times \frac{1}{3n+2}$$

$$= -\frac{1}{2}x^2 + \sum_{n=1}^{k} \frac{-1}{\prod_{i=1}^{n}(3i-1)} x^{3(n+1)-1} \times \frac{1}{3(n+1)-1}$$

$$= -\frac{1}{2}x^2 + \sum_{n=2}^{k+1} \frac{-1}{\prod_{i=1}^{n+1}(3i-1)} x^{3n-1}$$

$$= \sum_{n=1}^{k+1} \frac{-1}{\prod_{i=1}^{n+1}(3i-1)} x^{(3n-1)}$$

QED : This formula is ~~po~~ able to generate a sequence of functions for some n proven by induction.

Q2) $y-3+(x+2y)\frac{dy}{dx}=0 \qquad y(0)=-3$

Since ODE is of form:

$P(x,y)+Q(x,y)\frac{dy}{dx}=0$

We see that: $\frac{\partial P}{\partial y}=1$ and $\frac{\partial Q}{\partial x}=1 \quad \therefore \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

thus there is some $f(x,y)$ where:

$\frac{\partial}{\partial x}f(x,y)=P(x,y) \qquad \frac{\partial}{\partial y}f(x,y)=Q(x,y)$

$\therefore \int y-3\,dx = xy-3x+g(y)$

$\int x+2y\,dy = xy+y^2+m(x)$

$\therefore g(y)=y^2$ and $m(x)=-3x$

$\therefore f(x,y)=c=xy-3x+y^2 \qquad y(0)=-3$

$c=0-\cancel{3x}\,0+9 \qquad \therefore c=9$

$9=xy-3x+y^2$

Q3)

$$y(x) = Ae^{x} + Be^{-x} + Cxe^{x}$$

given a guessed characteristic equation:

$$0 = (\lambda-1)^2(\lambda+1)$$
$$0 = (\lambda^2-2\lambda+1)(\lambda+1)$$
$$0 = \lambda^3+\lambda^2-2\lambda^2-2\lambda+\lambda+1$$
$$0 = \lambda^3-\lambda^2-\lambda+1$$
$$\therefore\ y'''-y''-y'+y=0$$

Sub in $y(x)$:

$$y'(x) = Ae^{x} - Be^{-x} + Cxe^{x} + Ce^{x}$$
$$y''(x) = Ae^{x} + Be^{-x} + Cxe^{x} + 2Ce^{x}$$
$$y'''(x) = Ae^{x} - Be^{-x} + Cxe^{x} + 3Ce^{x}$$

$$\therefore\ 0 = y'''-y''-y'+y$$
$$= Ae^{x} - Be^{-x} + Cxe^{x} + 3Ce^{x} - Ae^{x} - Be^{-x} - Cxe^{x} - 2Ce^{x}$$
$$- Ae^{x} + Be^{-x} - Cxe^{x} - Ce^{x} + Ae^{x} + Be^{-x} + Cxe^{x}$$
$$=0$$

QED: $y'''-y''-y'+y=0$ is a general solution ODE to

$$y(x) = Ae^{x} + Be^{-x} + Cxe^{x}$$

Q4)

a) If $f(x)$ is a solution to to the ODE
$xy'' - (1+x)y' + y = 0$ where $f(x) = A(1+x) + Be^{x}$,
Then subbing in $f(x)$ and $f''(x)$ should equal 0

$f'(x) = A + Be^{x}$ $\qquad$ $f''(x) = Be^{x}$

$0 = x(Be^{x}) - (1+x)(A+Be^{x}) + A(1+x) + Be^{x}$

$0 = xBe^{x} - A - Be^{x} - Ax - xBe^{x} + A + Ax + Be^{x}$

$0 = 0$

LHS = RHS $\qquad$ QED: $f(x)$ is a solution to the ODE $xy'' - (1+x)y' + y = 0$

b) The general solution to the ODE ① will be of the form $y(x) = y_H(x) + y_P(x)$ where:
$y_H(x) = A(1+x) + Be^{x}$

① $xy'' - (1+x)y' + y = x^2 e^{2x} \rightarrow y'' - \frac{(1+x)}{x}y' + \frac{1}{x}y = xe^{2x}$

To find y_P, we can use variation of parameters, where $y_1 = (1+x)$ and $y_2 = e^{x}$ can be a solution to $y_P = uy_1 + vy_2$

Solving for the Wronskian,

$$w = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = f_1(x)f_2'(x) - f_1'(x)f_2(x)$$

$= (1+x) \times e^{x} - e^{x}$ $\qquad$ and $r = xe^{2x}$

$\therefore u = -\int \frac{y_2 r}{w} dx$ $\qquad$ and $\qquad$ $v = -\int \frac{y_1 r}{w}$

$$u = -\int \frac{y_2 r}{W}\,dx$$

$$u = -\int \frac{e^{x} x e^{2x}}{x e^{x}}$$

$$= -\int e^{2x}\,dx$$

$$= \frac{-1}{2} e^{2x} + C$$

where $c = 0$

Since any solution is valid.

$$V = +\int \frac{y_1 r}{W}\,dx$$

$$V = +\int \frac{(1+x) x e^{2x}}{x e^{x}}\,dx$$

$$= +\int (1+x) e^{x}\,dx$$

$$= +\int e^{x} + x e^{x}\,dx$$

$$= +e^{x} + \int x e^{x}\,dx$$

Integration by Parts:

let $u = x \quad V = e^{x}$

$$\therefore \int x e^{x}\,dx = x e^{x} - \int 1 \times e^{x}\,dx$$

$$= x e^{x} - e^{x} + C$$

$$V = e^{x} + x e^{x} - e^{x}$$

$$\therefore y_p = u y_1 + v y_2$$

$$= \frac{-1}{2} e^{2x}(1+x) + (e^{x} + x e^{x} - e^{x}) e^{x}$$

$$= e^{2x}\left(\frac{-1}{2}(1+x) + x\right)$$

$$= e^{2x}\left(-\frac{1}{2} + \frac{1}{2}x\right)$$

$\therefore$ The general solution to $xy'' + (x+1)y' + y = x^2 e^{2x}$ is

$$y = A(1+x) + Be^{x} + \frac{1}{2}e^{2x}(x-1)$$

Q5) $\begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{pmatrix} = A$

The ~~basis~~ basis of an orthogonal complement to A will be equal to the null space of A transposed by definition. i.e $Col(A^T)$ with respect to the Euclidean inner product!

$$A^T = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{pmatrix} \qquad Null(A^T) = \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 4 & -2 & 0 & 3 & 0 \\ 5 & 1 & -1 & 5 & 0 \\ 6 & 4 & -2 & 7 & 0 \\ 9 & -1 & -1 & 8 & 0 \end{array}\right)$$

By following the following transformations: $R_5 \Rightarrow -R_2 - R_3$
we receive:
$R_4 \Rightarrow -R_3 - R_1$
$R_3 \Rightarrow -R_1 - R_2$

$$Null(A^T) = \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 4 & -2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad R_2 \Rightarrow -4R_1$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & -14 & 4 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This results in the simultaneous equations:

$$x_1 + 3x_2 - x_3 \quad 2x_4 = 0 \quad (1)$$
$$-14x_2 + 4x_3 - 5x_4 = 0 \quad (2)$$

$$(1) + \frac{3}{14} \times (2) = x_1 + \left(-1 + \frac{12}{14}\right)x_3 + \left(2 - \frac{15}{14}\right)x_4 = 0$$
$$\Rightarrow x_1 = +\frac{1}{7}x_3 - \frac{13}{14}x_4$$
$$x_2 = -\frac{4}{14}x_3 - \frac{5}{14}x_4$$
$$x_3 = x_3$$
$$x_4 = x_4$$

$\therefore$ The ~~total~~ vector becomes = $\begin{bmatrix} \frac{1}{7}x_3 - \frac{13}{14}x_4 \\ \frac{-4}{14}x_3 - \frac{5}{14}x_4 \\ x_3 \\ x_4 \end{bmatrix}$

$\therefore$ Taking ~~out~~ x_3 ~~and~~ x_4 as linear combinations:

$$\begin{bmatrix} \frac{1}{7} \\ -4/14 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -\frac{13}{14} \\ -5/14 \\ 0 \\ 1 \end{bmatrix} x_4$$

Thus the basis of $Null(A^T)$ is $\left\{ \begin{pmatrix} \frac{1}{7} \\ -\frac{2}{7} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -13 \\ -5 \\ 0 \\ 14 \end{pmatrix} \right\}$.

$\therefore$ This is also the basis for the orthogonal component of A, with respect to the Euclidean inner product of the column space A.

Q6)

$$\langle \underline{u}, \underline{v} \rangle = \int_0^1 x\, u(x)\, v(x)\, dx$$

a) For $\{\sqrt{2}, 6x-4\}$ to be an orthonormal basis, it must satisfy the following:

① They must be orthogonal to each other,

② They must be unit vectors

③ They must form a basis.

① To be orthogonal, their inner product must be equal to zero.

$$0 = \langle \sqrt{2}, 6x-4 \rangle = \int_0^1 x\sqrt{2}(6x-4)\, dx$$
$$= \int_0^1 6\sqrt{2}x^2 - 4\sqrt{2}x\, dx$$
$$= \left[2\sqrt{2}x^3 - 2\sqrt{2}x^2\right]_0^1$$
$$= (2\sqrt{2} - 2\sqrt{2}) - 0$$
$$= 0$$

LHS = RHS QED, they are orthogonal

② To be unit vectors, $\|\sqrt{2}\|$ and $\|6x-4\|$ must equal ①.

$$\|\sqrt{2}\| = \langle \sqrt{2}, \sqrt{2} \rangle = \int_0^1 x(\sqrt{2})^2\, dx = \left[x^2\right]_0^1 = 1 - 0 = 1$$

$$\|6x-4\| = \langle 6x-4, 6x-4 \rangle = \int_0^1 x(6x-4)^2\, dx = \int_0^1 x(36x^2 - 48x + 16)\, dx$$
$$= \int_0^1 36x^3 + 48x^2 + 16x\, dx = \left[9x^4 - 16x^3 + 8x^2\right]_0^1$$

$$= 9 - 16 + 8 = 1$$

$\therefore$ $\sqrt{2}$ and $6x-4$ are both unit vectors

③ For $\{\sqrt{2}, 6x-4\}$ to be a basis, they must be linearly independent and have a span equal to the span of $\{1, x\}$.

$\sqrt{2}$ and $6x-4$ are clearly linearly independent ~~as~~ no A could ever make the following statement true:

$$A\sqrt{2} = 6x-4 \quad \text{where } A \in \mathbb{R}$$

To prove $\{\sqrt{2}, 6x-4\}$ spans $\{1, x\}$ we observe:

$$\underline{v} = \alpha_1 \sqrt{2} + \alpha_2(6x-4) = \alpha_2 6x + (\alpha_1\sqrt{2} - 4\alpha_2)$$

for some vector within space $\{1, x\}$: $r_1 x + r_2$

$$\alpha_2 = \frac{r_1}{6} \quad \text{and} \quad \alpha_1 = \frac{r_1 + 4\alpha_2}{\sqrt{2}}$$

As observed, any linear combination of $\{1, x\}$ can be constructed from linear combinations of $\{\sqrt{2}, 6x-4\}$. Thus, with it also being linearly independent, it is a basis.

b) For $\| x^2 - a\sqrt{2} - b(6x-4) \|$ to be minimised, ~~when~~ we can use Best Approximation Theorem where:

$$V = x^2 \quad \text{and} \quad u = +a\sqrt{2} + b(6x-4)$$

$\| V - \mathrm{Proj}_u(\underline{v}) \|$ where $V = x^2$

$$\mathrm{Proj}_{u[0,1]}(\underline{v}) = a\sqrt{2} + b(6x-4)$$

$$\mathrm{Proj}_u(\underline{v}) = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2$$

$$\therefore a\sqrt{2} + b(6x-4) = \langle x^2, \sqrt{2} \rangle \sqrt{2} + \langle x^2, 6x-4 \rangle (6x-4)$$

$$a = \langle x^2, \sqrt{2} \rangle = \int_0^1 x \times x^2 \times \sqrt{2}\, dx = \sqrt{2}\left[\frac{1}{3}x^3\right]_0^1 = \frac{\sqrt{2}}{4}$$

$$b = \langle x^2, 6x-4 \rangle = \int_0^1 x \times x^3 \times (6x-4)\, dx = \int_0^1 6x^5 - 4x^4\, dx$$

$$= \left[x^6 - \frac{4}{5}x^5\right]_0^1 = 1 - \frac{4}{5} = \frac{1}{5}$$

$\therefore$ For $\| x^2 - a\sqrt{2} - b(6x-4) \|$ to be minimised, a ~~and~~ b must equal $\frac{\sqrt{2}}{4}$ and $\frac{1}{5}$ respectively as given by the Best Approximation Theorem.