

MANUAL FOR THE STRUCTURE BINARYTREEWITHROOTANDTOPS

1. STRUCTURE BINARYTREEWITHROOTANDTOPS

1.1. **Notation.** Given a Pair $p = (p.1, p.2)$, we have

- $\text{Finset } \alpha$ denotes a *finite* subset of α (Lean’s `Finset`).
- Given $p : \text{Finset } \alpha \times \text{Finset } \alpha$, $\text{pairToFinset}(p) = \{p.1, p.2\}$.

```
noncomputable def pairToFinset {α : Type*} [DecidableEq α] : α × α → Finset α
| (p1, p2) => {p1, p2}
```

- $\text{Combinatorial_Support}(p) = p.1 \cup p.2$ is the “combinatorial support”

```
noncomputable def Combinatorial_Support {α : Type*} [DecidableEq α] (p : Finset α × Finset α) :
Finset α := p.1 ∪ p.2
```

- $\text{Disjoint}(A, B)$ abbreviates $A \cap B = \emptyset$.

1.2. **Overview. Data.** The record

$\text{BinaryTreeWithRootandTops}(\alpha : \text{Type}^*) \quad [\text{DecidableEq } \alpha]$

stores four finite objects:

- **Childs** : $\text{Finset } \alpha$ — the ground set (assumed non-empty);
- **Root** : $\text{Finset } \alpha \times \text{Finset } \alpha$ — the first split of **Childs**;
- **Branches** : $\text{Finset}(\text{Finset } \alpha \times \text{Finset } \alpha)$ — all interior nodes, including **Root**;
- **Tops** : $\text{Finset } \alpha$ — the labels of distinguished leaves.

Laminar family of supports. The supports of the branches satisfy two crucial axioms:

(L1) *Cover.* $\bigcup_{p \in \text{Branches}} \text{Combinatorial_Support}(p) = \text{Childs}.$

(L2) *Nesting.* Whenever $p \neq q$ in **Branches**, the supports are either disjoint or properly nested:

$$\begin{aligned} \text{Combinatorial_Support}(p) \cap \text{Combinatorial_Support}(q) &= \emptyset \quad \text{or} \\ \text{Combinatorial_Support}(p) &\subseteq \text{Combinatorial_Support}(q) \quad \text{or} \\ \text{Combinatorial_Support}(q) &\subseteq \text{Combinatorial_Support}(p). \end{aligned}$$

Indeed something even stronger happens, the family

$$\mathcal{L} = \{p.1, p.2 \mid p \in \text{Branches}\}$$

is a *maximal laminar family* (nested set system) on **Childs**. The record not only remembers this hierarchy of blocks but also keeps track of *how each block is split*: for every non-leaf support $S \in \mathcal{L}$ there exists *at most one* ordered pair $(A, B) \in \text{Branches}$ with $A \cup B = S$, $A \cap B = \emptyset$ and $\text{Combinatorial_Support}(A, B) = S$. This extra orientation (left vs. right child) upgrades the laminar family into a full binary tree.

Singleton supports $\{t\}$ correspond precisely to the leaf pairs $(\{t\}, \{\})$ or $(\{\}, \{t\})$; the set of all such labels is stored in the field **Tops**.

1.3. Fields.

Root : $\text{Finset}\alpha \times \text{Finset}\alpha$: The distinguished ordered pair that serves as the root of the tree.

Childs : $\text{Finset}\alpha$: The finite set of all basic symbols that may appear in either component of a branch pair.

Branches : $\text{Finset}(\text{Finset}\alpha \times \text{Finset}\alpha)$: The finite family of ordered pairs that form the vertices of the tree, including **Root**.

Tops : $\text{Finset}\alpha$: A specified finite subset of **Childs** whose singletons appear as leaf pairs.

1.4. Structure properties. All quantifiers range over the corresponding finite sets declared above.

(1) Non-empty components.

$$\forall y \in \text{Branches}, y.1 \neq \emptyset \wedge y.2 \neq \emptyset.$$

(2) Disjoint components.

$$\forall p \in \text{Branches}, p.1 \cap p.2 = \emptyset.$$

(3) Distinct pairs are disjoint as sets.

$$\forall p, q \in \text{Branches}, p \neq q \rightarrow (\text{pairToFinset}(p)) \cap (\text{pairToFinset}(q)) = \emptyset.$$

(4) Root belongs to the branch set.

$$\text{Root} \in \text{Branches}.$$

(5) Every child occurs in some branch.

$$\forall a \in \text{Childs}, \exists p \in \text{Branches}, a \in \text{Combinatorial_Support}(p).$$

(6) Children are contained in the root pair.

$$\text{Childs} \subseteq \text{Root}.1 \cup \text{Root}.2.$$

(7) Branch components are made of children.

$$\forall p \in \text{Branches}, p.1 \subseteq \text{Childs} \wedge p.2 \subseteq \text{Childs}.$$

(8) Recursive tree structure.

$$\forall p \in \text{Branches}, p \neq \text{Root} \rightarrow \exists q \in \text{Branches}, p.1 \cup p.2 \in \text{pairToFinset}(q).$$

(9) Tops are witnessed by singleton pairs.

$$\forall t \in \text{Tops}, \exists q \in \text{Branches}, \{t\} \in \text{pairToFinset}(q).$$

(10) Singletons in any branch are tops.

$$\forall p \in \text{Branches}, \forall x, \{x\} \in \text{pairToFinset}(p) \rightarrow x \in \text{Tops}.$$

(11) Non-empty root and child set.

$$\text{Root}.1 \neq \emptyset, \text{Root}.2 \neq \emptyset, \text{Childs} \neq \emptyset.$$

(12) Root components are disjoint.

$$\text{Root}.1 \cap \text{Root}.2 = \emptyset.$$

(13) Support property (tree compatibility).

$$\begin{aligned} & \forall p, q \in \text{Branches}, p \neq q \rightarrow \\ & (\text{Combinatorial_Support}(p) \cap \text{Combinatorial_Support}(q) = \emptyset) \vee \\ & \text{Combinatorial_Support}(p) \subseteq q.1 \vee \text{Combinatorial_Support}(p) \subseteq q.2 \vee \\ & \text{Combinatorial_Support}(q) \subseteq p.1 \vee \text{Combinatorial_Support}(q) \subseteq p.2. \end{aligned}$$

```

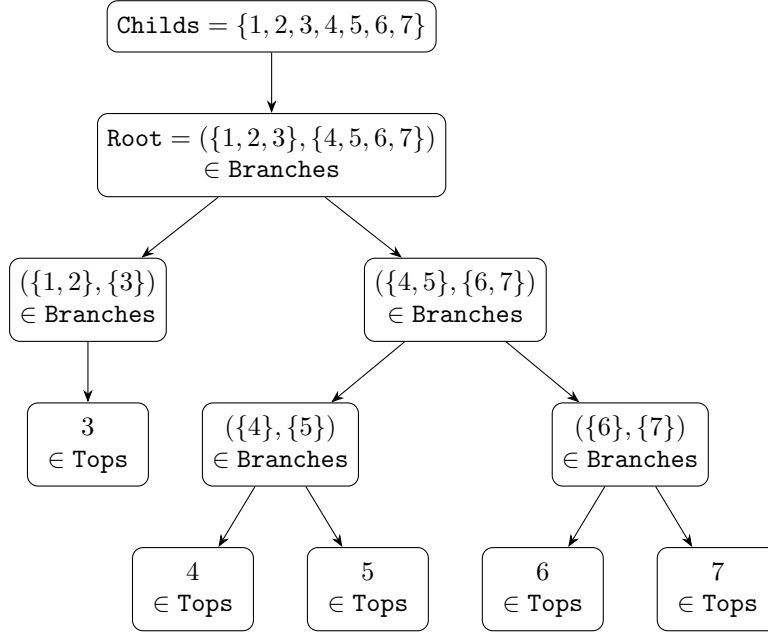
structure BinaryTreeWithRootandTops (α : Type*) [DecidableEq α] where
  Root      : Finset α × Finset α
  Childs    : Finset α
  Branches  : Finset (Finset α × Finset α)
  NonemptyPairs:
    | ∀ y ∈ Branches, y.1.Nonempty ∧ y.2.Nonempty
  Tops: Finset α
  DisjointComponents:
    | ∀ p ∈ Branches, Disjoint p.1 p.2
  DistinctComponentsPairs:
    | ∀ p ∈ Branches, ∀ q ∈ Branches, (p ≠ q) → Disjoint (pairToFinset p) (pairToFinset q)
  RootinBranches:
    | Root ∈ Branches
  EveryChildinaBranch:
    | ∀ a ∈ Childs, ∃ p ∈ Branches, a ∈ Combinatorial_Support p
  RootcontainsChilds:
    | Childs ⊆ Root.1 ∪ Root.2
  TreeStructureChilds:
    | ∀ p ∈ Branches, p.1 ⊆ Childs ∧ p.2 ⊆ Childs
  TreeStructure:
    | ∀ p ∈ Branches, p ≠ Root → ∃ q ∈ Branches, p.1 ∪ p.2 ∈ pairToFinset q
  TopsareTops:
    | ∀ p ∈ Tops, ∃ q ∈ Branches, {p} ∈ pairToFinset q
  SingletonsAreTops :
    | ∀ p ∈ Branches,
      | | ∀ x, {x} ∈ pairToFinset p → (x ∈ Tops)
  NonemptyRoot: Root.1.Nonempty ∧ Root.2.Nonempty
  NonemptyChilds: Childs.Nonempty
  DisjointRoot: Disjoint Root.1 Root.2
  SupportProperty :
    | ∀ p ∈ Branches, ∀ q ∈ Branches, p ≠ q →
      | (Disjoint (Combinatorial_Support p) (Combinatorial_Support q)) ∧
      | Combinatorial_Support p ⊆ q.1 ∨
      | Combinatorial_Support p ⊆ q.2 ∨
      | Combinatorial_Support q ⊆ p.1 ∨

```

1.5. **Example.** Let

$$\begin{aligned}
 \text{Childs} &= \{1, 2, 3, 4, 5, 6, 7\}, \\
 \text{Root} &= (\{1, 2, 3\}, \{4, 5, 6, 7\}), \\
 \text{Branches} &= \{(\{1, 2, 3\}, \{4, 5, 6, 7\}), (\{1, 2\}, \{3\}), (\{4, 5\}, \{6, 7\}), \\
 &\quad (\{4\}, \{5\}), (\{6\}, \{7\})\}, \\
 \text{Tops} &= \{3, 4, 6, 7\}.
 \end{aligned}$$

and we can see the tree structure graphically below



MAIN RESULT

Theorem 1.1 (`exists_tree_childs_eq_C_and_all_childs_in_Tops_of_card_ge_two`). *Let C be a finite set with at least two elements. Then there exists a structure*

$$T : \text{BinaryTreeWithRootandTops}$$

such that

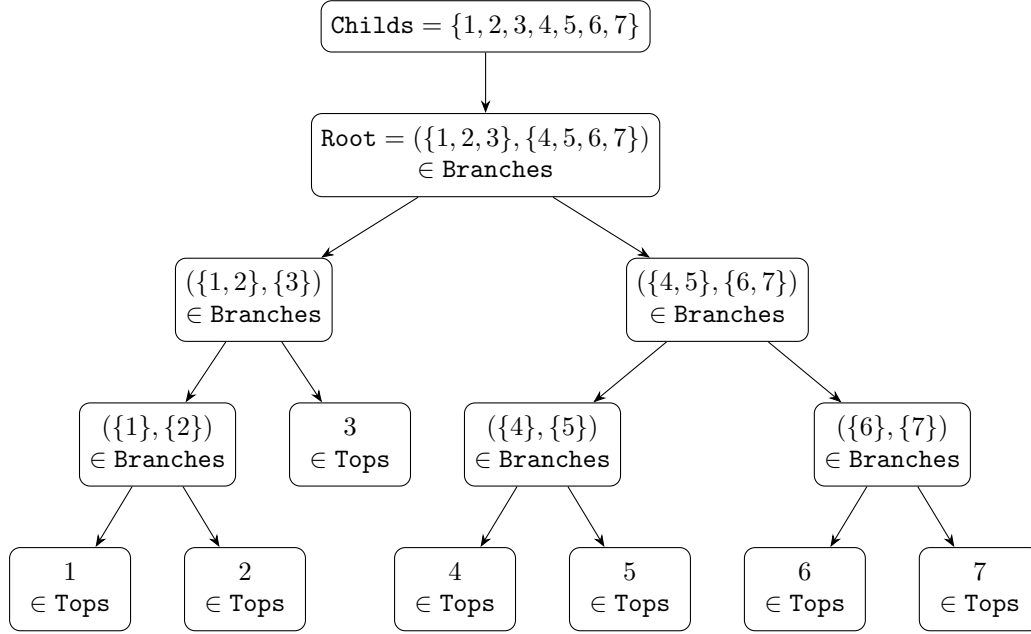
$$T.\text{Childs} = C \quad \text{and} \quad T.\text{Tops} = C.$$

```

theorem exists_tree_childs_eq_C_and_all_childs_in_Tops_of_card_ge_two
{α : Type*} [DecidableEq α] {C : Finset α} (hC : C.card ≥ 2) :
  ∃ T : BinaryTreeWithRootandTops α, T.Childs = C ∧ C = T.Tops := by

```

1.6. Example. In the previous example we do not have $T.\text{Tops} = T.\text{Childs}$, however we can complete it as



1.7. Connection with Laminar Families of Subsets. Given a finite set C , a *laminar family* \mathcal{L} is a family of subsets of C satisfying the following condition: for every pair $A, B \in \mathcal{L}$, one of the following holds:

- $A \subset B$,
- $B \subset A$, or
- $A \cap B = \emptyset$.

For more details, see https://en.wikipedia.org/wiki/Laminar_set_family.

A laminar family \mathcal{L} is called *maximal* if there does not exist a laminar family of C containing \mathcal{L} properly. Every laminar family of subsets of a finite set C has a rooted tree representation; see Alexander Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, 2004, page 215, Theorem 13.21.

A tree structure named `BinaryTreeWithRootandTops`, with `Tops = Childs`, is a *rooted tree representation* of a maximal laminar family of `Childs`. For example, the tree in Section 1.6 represents the maximal laminar family of $\{1, 2, 3, 4, 5, 6, 7\}$ given by

$\{\{1, 2, 3, 4, 5, 6, 7\}, \{1, 2, 3\}, \{4, 5, 6, 7\}, \{1, 2\}, \{4, 5\}, \{6, 7\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$.