MANUAL FOR THE STRUCTURE BINARYTREEWITHROOTANDTOPS

1. Structure BinaryTreeWithRootandTops

1.1. Notation. Given a Pair $p = (p.1, p.2)$, we have

- $\mathsf{Finset}\alpha$ denotes a *finite* subset of α (Lean's `Finset`).
- Given $p : \mathsf{Finset}\alpha \times \mathsf{Finset}\alpha$, $\mathsf{pairToFinset}(p) \;=\; \{p.1, p.2\}$.

```
noncomputable def pairToFinset {α : Type*} [DecidableEq α] : α × α → Finset α
| (p1, p2) => {p1, p2}
```

- $\mathsf{Combinatorial\_Support}(p) = p.1 \cup p.2$ is the "combinatorial support"

```
noncomputable def Combinatorial_Support {α : Type*} [DecidableEq α] (p : Finset α×Finset α) :
Finset α := p.1 ∪ p.2
```

- $\mathbf{Disjoint}(A, B)$ abbreviates $A \cap B = \varnothing$.

1.2. Overview. Data. The record

$$\texttt{BinaryTreeWithRootandTops}\big(\alpha : \mathrm{Type}^*\big) \quad [\mathrm{DecidableEq}\,\alpha]$$

stores four finite objects:

- **Childs** : $\mathsf{Finset}\alpha$ — the ground set (assumed non-empty);
- **Root** : $\mathsf{Finset}\alpha \times \mathsf{Finset}\alpha$ — the first split of **Childs**;
- **Branches** : $\mathsf{Finset}\big(\mathsf{Finset}\alpha \times \mathsf{Finset}\alpha\big)$ — all interior nodes, including **Root**;
- **Tops** : $\mathsf{Finset}\alpha$ — the labels of distinguished leaves.

Laminar family of supports. The supports of the branches satisfy two crucial axioms:

(L1) *Cover.* $\displaystyle\bigcup_{p \in \mathbf{Branches}} \mathsf{Combinatorial\_Support}(p) \;=\; \mathbf{Childs}.$

(L2) *Nesting.* Whenever $p \neq q$ in **Branches**, the supports are either disjoint or properly nested:

$$\mathsf{Combinatorial\_Support}(p) \cap \mathsf{Combinatorial\_Support}(q) = \varnothing \quad \text{or}$$
$$\mathsf{Combinatorial\_Support}(p) \subsetneq \mathsf{Combinatorial\_Support}(q) \quad \text{or}$$
$$\mathsf{Combinatorial\_Support}(q) \subsetneq \mathsf{Combinatorial\_Support}(p).$$

Indeed something even stronger happens, the family

$$\mathcal{L} \;=\; \{p.1, p.2 \mid p \in \mathbf{Branches}\}$$

is a *maximal laminar family* (nested set system) on **Childs**. The record not only remembers this hierarchy of blocks but also keeps track of *how each block is split*: for every non-leaf support $S \in \mathcal{L}$ there exists *at most one* ordered pair $(A, B) \in \mathbf{Branches}$ with $A \cup B = S$, $A \cap B = \varnothing$ and $\mathsf{Combinatorial\_Support}(A, B) = S$. This extra orientation (left vs. right child) upgrades the laminar family into a full binary tree.

Singleton supports $\{t\}$ correspond precisely to the leaf pairs $(\{t\}, \{\,\})$ or $(\{\,\}, \{t\})$; the set of all such labels is stored in the field **Tops**.

1.3. Fields.

`Root` : $\mathsf{Finset}\alpha \times \mathsf{Finset}\alpha$: The distinguished ordered pair that serves as the root of the tree.
`Childs` : $\mathsf{Finset}\alpha$: The finite set of all basic symbols that may appear in either component of a branch pair.
`Branches` : $\mathsf{Finset}(\mathsf{Finset}\alpha \times \mathsf{Finset}\alpha)$: The finite family of ordered pairs that form the vertices of the tree, including `Root`.
`Tops` : $\mathsf{Finset}\alpha$: A specified finite subset of `Childs` whose singletons appear as leaf pairs.

1.4. Structure properties.

All quantifiers range over the corresponding finite sets declared above.

(1) **Non-empty components.**
$\forall y \in \texttt{Branches},\ y.1 \neq \varnothing\ \wedge\ y.2 \neq \varnothing.$
(2) **Disjoint components.**
$\forall p \in \texttt{Branches},\ p.1 \cap p.2 = \varnothing.$
(3) **Distinct pairs are disjoint as sets.**
$\forall p, q \in \texttt{Branches},\ p \neq q\ \rightarrow\ \big(\mathsf{pairToFinset}(p)\big) \cap \big(\mathsf{pairToFinset}(q)\big) = \varnothing.$
(4) **Root belongs to the branch set.**
$\texttt{Root} \in \texttt{Branches}.$
(5) **Every child occurs in some branch.**
$\forall a \in \texttt{Childs},\ \exists p \in \texttt{Branches},\ a \in \mathsf{Combinatorial\_Support}(p).$
(6) **Children are contained in the root pair.**
$\texttt{Childs} \subseteq \texttt{Root}.1 \cup \texttt{Root}.2.$
(7) **Branch components are made of children.**
$\forall p \in \texttt{Branches},\ p.1 \subseteq \texttt{Childs}\ \wedge\ p.2 \subseteq \texttt{Childs}.$
(8) **Recursive tree structure.**
$\forall p \in \texttt{Branches},\ p \neq \texttt{Root}\ \rightarrow\ \exists q \in \texttt{Branches},\ p.1 \cup p.2 \in \mathsf{pairToFinset}(q).$
(9) **Tops are witnessed by singleton pairs.**
$\forall t \in \texttt{Tops},\ \exists q \in \texttt{Branches},\ \{t\} \in \mathsf{pairToFinset}(q).$
(10) **Singletons in any branch are tops.**
$\forall p \in \texttt{Branches},\ \forall x,\ \{x\} \in \mathsf{pairToFinset}(p)\ \rightarrow\ x \in \texttt{Tops}.$
(11) **Non-empty root and child set.**
$\texttt{Root}.1 \neq \varnothing,\ \texttt{Root}.2 \neq \varnothing,\ \texttt{Childs} \neq \varnothing.$
(12) **Root components are disjoint.**
$\texttt{Root}.1 \cap \texttt{Root}.2 = \varnothing.$
(13) **Support property (tree compatibility).**

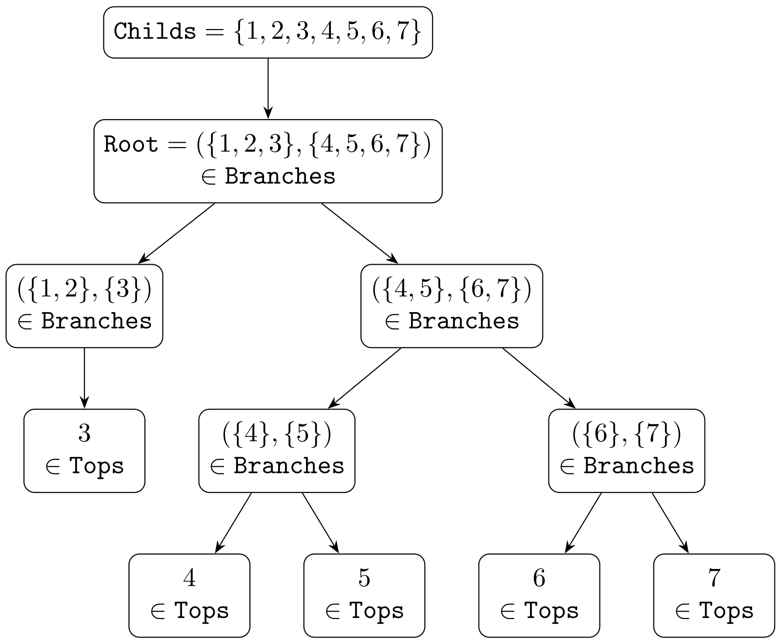
$$\begin{aligned}
&\forall p, q \in \texttt{Branches},\ p \neq q\ \rightarrow \\
&\big(\,\mathsf{Combinatorial_Support}(p) \cap \mathsf{Combinatorial_Support}(q) = \varnothing\,\big) \vee \\
&\mathsf{Combinatorial_Support}(p) \subseteq q.1\ \vee\ \mathsf{Combinatorial_Support}(p) \subseteq q.2 \vee \\
&\mathsf{Combinatorial_Support}(q) \subseteq p.1\ \vee\ \mathsf{Combinatorial_Support}(q) \subseteq p.2.
\end{aligned}$$

```
structure BinaryTreeWithRootandTops (α : Type*)[DecidableEq α] where
  Root                  : Finset α× Finset α
  Childs                : Finset α
  Branches              : Finset (Finset α × Finset α)
  NonemptyPairs:
    ∀ y ∈ Branches, y.1.Nonempty ∧ y.2.Nonempty
  Tops: Finset α
  DisjointComponents:
    ∀ p ∈ Branches, Disjoint p.1 p.2
  DistinctComponentsPairs:
    ∀ p ∈ Branches, ∀ q ∈ Branches, (p ≠ q)->Disjoint (pairToFinset p) (pairToFinset q)
  RootinBranches:
    Root ∈ Branches
  EveryChildinaBranch:
    ∀ a ∈ Childs,  ∃ p ∈ Branches, a ∈ Combinatorial_Support p
  RootcontainsChilds:
   Childs ⊆  Root.1 ∪ Root.2
  TreeStructureChilds:
    ∀ p ∈ Branches, p.1 ⊆ Childs ∧ p.2 ⊆ Childs
  TreeStructure:
    ∀ p ∈ Branches,    p ≠ Root → ∃ q ∈ Branches, p.1 ∪ p.2 ∈ pairToFinset q
  TopsareTops:
    ∀ p ∈ Tops, ∃ q∈ Branches, {p}∈ pairToFinset q
  SingletonsAreTops :
    ∀ p ∈ Branches,
       ∀ x, {x} ∈ pairToFinset p → (x ∈ Tops)
  NonemptyRoot: Root.1.Nonempty ∧ Root.2.Nonempty
  NonemptyChilds: Childs.Nonempty
  DisjointRoot: Disjoint Root.1 Root.2
  SupportProperty :
    ∀ p ∈ Branches,  ∀ q ∈ Branches, p ≠ q →
    (Disjoint (Combinatorial_Support p) (Combinatorial_Support q)∨
    Combinatorial_Support p ⊆ q.1 ∨
    Combinatorial_Support p ⊆ q.2 ∨
    Combinatorial_Support q ⊆ p.1 ∨
```

1.5. **Example.** Let

$$
\begin{aligned}
\texttt{Childs} &= \{1,2,3,4,5,6,7\},\\
\texttt{Root} &= (\{1,2,3\},\ \{4,5,6,7\}),\\
\texttt{Branches} &= \{(\{1,2,3\},\{4,5,6,7\}),\ (\{1,2\},\{3\}),\ (\{4,5\},\{6,7\}),\\
&\quad (\{4\},\{5\}),\ (\{6\},\{7\})\},\\
\text{Tops} &= \{3,4,6,7\}.
\end{aligned}
$$

and we can see the tree structure graphically below



Main Result

Theorem 1.1 (exists_tree_childs_eq_C_and_all_childs_in_Tops_of_card_ge_two). *Let C be a finite set with at least two elements. Then there exists a structure*

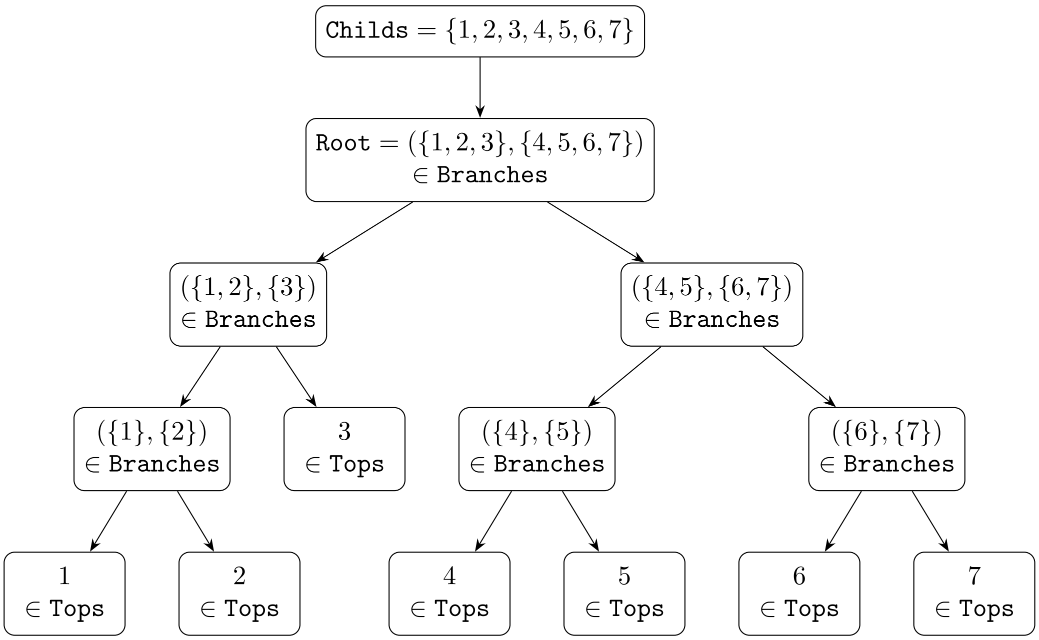
$$T \ : \ \mathtt{BinaryTreeWithRootandTops}$$

such that

$$T.\mathtt{Childs} = C \quad \text{and} \quad T.\mathtt{Tops} = C.$$

```
theorem exists_tree_childs_eq_C_and_all_childs_in_Tops_of_card_ge_two
{α : Type*} [DecidableEq α] {C : Finset α} (hC : C.card ≥ 2) :
∃ T : BinaryTreeWithRootandTops α, T.Childs = C ∧ C = T.Tops := by
```

1.6. **Example.** In the previous example we do not have $T.Tops = T.Childs$, however we can complete it as



1.7. **Connection with Laminar Families of Subsets.** Given a finite set C, a *laminar family* $\mathcal{L}$ is a family of subsets of C satisfying the following condition: for every pair $A, B \in \mathcal{L}$, one of the following holds:

- $A \subset B$,
- $B \subset A$, or
- $A \cap B = \emptyset$.

For more details, see `https://en.wikipedia.org/wiki/Laminar_set_family`.

A laminar family $\mathcal{L}$ is called *maximal* if there does not exist a laminar family of C containing $\mathcal{L}$ properly. Every laminar family of subsets of a finite set C has a rooted tree representation; see Alexander Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, 2004, page 215, Theorem 13.21.

A tree structure named `BinaryTreeWithRootandTops`, with `Tops` = `Childs`, is a *rooted tree representation* of a maximal laminar family of `Childs`. For example, the tree in Section 1.6 represents the maximal laminar family of $\{1, 2, 3, 4, 5, 6, 7\}$ given by

$$\{\{1,2,3,4,5,6,7\},\ \{1,2,3\},\ \{4,5,6,7\},\ \{1,2\},\ \{4,5\},\ \{6,7\},\ \{1\},\ \{2\},\ \{3\},\ \{4\},\ \{5\},\ \{6\},\ \{7\}\}\,.$$