Associativity on π_1
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This animation was created to explain why

$$\alpha \star (\beta \star \gamma) \simeq (\alpha \star \beta) \star \gamma$$

in the fundamental group π_1, motivated by an answer we gave to a question on Mathematics Stack Exchange.

The path $\alpha \star (\beta \star \gamma)$ evolves as follows:

1. During the first half of the interval $[0, 1/2]$, it travels along α at twice the original speed.
2. During the second half $[1/2, 1]$, it moves along $\beta \star \gamma$. More precisely:
   2.1. On $[1/2, 3/4]$, it moves along β at four times the original speed.
   2.2. On $[3/4, 1]$, it moves along γ also at four times the original speed.

On the other hand, the path $(\alpha \star \beta) \star \gamma$ evolves as follows:

1. During the first half $[0, 1/2]$, it travels along $\alpha \star \beta$. In this interval:
   1.1. On $[0, 1/4]$, it moves along α at four times the original speed.
   1.2. On $[1/4, 1/2]$, it moves along β also at four times the original speed.
2. During the second half $[1/2, 1]$, it moves along γ at twice the original speed.

Thus, in both cases the path moves through α, β, and γ in the same order, but with different speeds. The homotopy is constructed by continuously changing, with a parameter $s \in [0, 1]$, the speeds at which we traverse each segment:

- At $s = 0$, we move through α, β, and γ at speeds $2x$, $4x$, $4x$ respectively.
- At $s = 1$, we move through them at speeds $4x$, $4x$, $2x$ respectively.

To be more precise:

- For fixed s, the map

$$\frac{4t}{2-s}$$

sends the interval $\left[0, \frac{2-s}{4}\right]$ to $[0, 1]$, with derivative $\frac{4}{2-s}$. So at $s = 0$ the derivative is 2, and at $s = 1$ it is 4.

- For fixed s, the map

$$4t + s - 2$$

sends $\left[\frac{2-s}{4}, \frac{3-s}{4}\right]$ to $[0, 1]$, with constant derivative 4.

- For fixed s, the map

$$\frac{4t + s - 3}{s+1}$$

sends $\left[\frac{3-s}{4}, 1\right]$ to $[0, 1]$, with derivative $\frac{4}{s+1}$. At $s = 0$ this is 4, and at $s = 1$ it is 2.

In other words, for every fixed s, the path $t \mapsto H(t, s)$ is just a reparametrization of the initial path $\omega = \alpha \star (\beta \star \gamma)$, moving at different "speeds". That is,

$$H(t, s) = \omega(\theta_s(t)),$$

where $\theta_s \colon [0, 1] \to [0, 1]$ is the homeomorphism defined by

$$\theta_s(t) = \begin{cases} \dfrac{2t}{2-s}, & t \in \left[0, \frac{2-s}{4}\right], \\ t + \dfrac{s}{4}, & t \in \left[\frac{2-s}{4}, \frac{3-s}{4}\right], \\ \dfrac{3}{4} + \dfrac{t + \frac{s}{4} - \frac{3}{4}}{s+1}, & t \in \left[\frac{3-s}{4}, 1\right]. \end{cases}$$

The Python script generates an animation that shows the deformation given by θ_s as s varies from 0 to 1.

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