CTRM STATISTICS

Abstract. We fit a max-renewal process, (aka 'CTRM', Continuous Time Random Maxima) to data.

1. Introduction

The stochastic model $M(t)$ below has been studied under the various names "Shock process" [EM73, SS83, SS84, SS85, And87, Gut99], "max-renewal process", [Sil04, ST04, BŠ14] and "CTRM (continuous time random maxima) [BSM07, MS09, **?**, HS15]: Assume i.i.d. pairs of random variables (W_k, J_k), $k = 1, 2, \dots$ where $W_k > 0$ represents an inter-arrival time of certain events and $J_k \in \mathbb{R}$ the corresponding event magnitude. Write $N(t) = \max\{n \in \mathbb{N} : W_1 + \dots + W_n \le t\}$ for the renewal process associated with the W_k (where the maximum of the empty set is set to 0), and define

$$M(t) = \bigvee_{k=1}^{N(t)} J_k = \max\{J_k : k = 1, \dots, N(t)\}. \tag{1.1}$$

If W_k is interpreted as the time leading up to the event with magnitude J_k, then $M(t)$ is the largest magnitude up to time t. The alternative case where W_k is the inter-arrival time *following* J_k is termed "second type" (in the shock model literature) or OCTRM (overshooting CTRM), and the largest magnitude up to time t is then given by

$$\tilde{M}(t) = \bigvee_{k=1}^{N(t)+1} J_k. \tag{1.2}$$

Finally, the model is called coupled when W_k and J_k are not independent; in this article we focus on the uncoupled case. For this case, $M(t)$ and $\tilde{M}(t)$ have the same limiting behaviour for large times and magnitudes [HS15] and hence we focus on the CTRM $M(t)$.

If J_k is constant or light-tailed (e.g. exponentially distributed), then, by the strong law of large numbers, on large time scales the number $N(t)$ of events by time t is well approximated by $t/\mathbf{E}[W_k]$. Magnitudes in excess of a given (large) threshold ℓ then arrive at a constant rate (which depends on ℓ), and the inter-arrival times are exponentially distributed. The dynamics of the record process are well known.

In this article, we are concerned with heavy-tailed inter-arrival times W_k: We assume that $\mathbf{P}[W_k > t] \sim Ct^{-\beta}$ as $t \to \infty$, for some $\beta \in (0, 1)$ and some constant C. Then $N(t)$ has temporal scaling exponent β, i.e. $Cn^{-\beta} N(nt) \to E(t)$, where $E(t)$ is the inverse of a stable subordinator (increasing Lévy process) $D(t)$, in the sense of

$$E(t) = \inf\{u : D(u) > t\}. \tag{1.3}$$

The process $E(t)$ is self-similar, and its points of increase model 'bursts', meaning that $E(t)$ is starkly increasing in some short time intervals and constant in others [KKBK12].

[MS09] have shown that for large times t, the process $M(t)$ converges to a (rescaled) time-changed extreme-value process $A(E(t))$. Here, $A(t)$ is an extreme-value process in the sense of [Lam64] and [Res13], and

The goal of our statistical method is to make inferences about the dynamics of $M(t)$, given data (W_k, J_k). Since the waiting times J_k are not assumed exponential, $M(t)$ is not Markov, and the same applies to $A(E(t))$.

Any probability distribution with cumulative distribution function (CDF) F on $\mathbb{R}$ lies in the max-domain of attraction of a generalized extreme value (GEV) distribution with CDF Λ [LLR12]. That is, there exist functions $a : \mathbb{R}^+ \to \mathbb{R}^+$ and $d : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$F(a(n)x + b(n))^n \to \Lambda(x), \quad n \to \infty, \quad x \in (x_L, x_R), \tag{1.4}$$

where x_L and x_R denote the left and right endpoint of the GEV distribution.

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