STATISTICS OF BURSTY EVENTS

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Abstract

The prediction of extreme events resulting from human behaviour is becoming significant in areas as far reaching as the control of disease spread, resource allocation, and emergency response. Classical methods in extreme value theory fail to recognise that most human created events do not occur uniformly, but in bursts. In this thesis, we aim to design statistical methods for the inference and prediction of Continuous Time Random Maxima (CTRM).

Contents

Chapter 1

Introduction

Time series displaying inhomogeneous behaviour have received strong interest in the recent statistical physics literature, [Bar05, OB05, VOD$^+$06, VRLB07, OS11, MGV11, KKP$^+$11, BB13], and have been observed in the context of earthquakes, sunspots, neuronal activity, human communication etc., see [KKBK12, VTK13] for a list of references. Such time series exhibit high activity in some 'bursty' intervals, which alternate with other, quiet intervals. Although several mechanisms are plausible explanations for bursty behaviour (most prominently self-exciting point processes [Haw71]), there seems to be one salient feature which very typically indicates the departure from temporal homogeneity: A heavy-tailed distribution of waiting times [VOD$^+$06, KKBK12, VTK13]. A simple renewal process with heavy-tailed waiting times captures these dynamics. For many systems, the renewal property is appropriate, as can be checked by a simple test: the dynamics do not change significantly if the waiting times are randomly reshuffled [KKBK12].

When a magnitude can be assigned to each event in the renewal process, such as for earthquakes, sun flares, neuron voltages or the impact of an email, two natural and basic questions to ask are: What is the distribution of the largest event up to a given time t? What is the probability that an event exceeds a given level ℓ within the next t units of time? A probabilistic extreme value model which assumes that the events form a renewal process is available in the literature. This model has been studied under the names "Continuous Time Random Maxima process" (CTRM) [BSM07, **?**, HS15, **?**], "Max-Renewal process" [Sil02, ST04, BŠ14], and "Shock process" [EM73, SS83, SS84, SS85, And87, Gut99]. This article aims to develop concrete statistical inference methods for this model, a problem which has seemingly received little attention by the statistical community.

Chapter 2

Preliminaries

In this chapter, we introduce the notation and review the background theory of probability and stochastic processes that are necessary in developing the statistical methods outlined in this thesis.

2.1 Probability and Measure Theory

Definition 2.1.1. *Let Ω be a nonempty set and let $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a* ***σ-algebra*** *if it satisfies the following conditions*

1. *The empty set $\emptyset \in \mathcal{F}$,*
2. *If $A \in \mathcal{F}$ then the complement $A^c \in \mathcal{F}$, and*
3. *If $A_1, A_2, \ldots \in \mathcal{F}$ then their union $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

It follows that a σ-algebra is also closed under intersection.

Definition 2.1.2. *Let Ω be a nonempty set and let $\mathcal{F}$ be a σ-algebra on Ω. Then the pair $(\Omega, \mathcal{F})$ is called a* ***measurable space****.*

Definition 2.1.3. *Let $(\Omega, \mathcal{F})$ be a measurable space. A* ***measure*** *μ is a real-valued set function on $\mathcal{F}$ satisfying the following conditions*

1. *For every set $A \in \mathcal{F}$, $\mu(A) \geq 0$, and*
2. *If $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$ then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Now that the concept of a measure has been defined we can interpret a σ-algebra as the set of all events that can be measured from the underlying set Ω.

Definition 2.1.4. *Let $(\Omega, \mathcal{F})$ be a measurable space. A **probability measure** $\mathbb{P}$ is a measure such that $\mathbb{P}(\Omega) = 1$.*

Definition 2.1.5. *Let μ be a measure defined on the measurable space $(\Omega, \mathcal{F})$. We call the triplet $(\Omega, \mathcal{F}, \mu)$ a **measure space**. In particular, if $\mathbb{P}$ is a probability measure then we call $(\Omega, \mathcal{F}, \mathbb{P})$ a **probability space**.*

A particular example of a probability measure that we will use later on is the Dirac measure.

Definition 2.1.6. *Let $(\Omega, \mathcal{F})$ be some measurable space. A **Dirac measure** δ_x is a measure defined on a given point $x \in \Omega$ and a set $A \in \mathcal{F}$*

$$\delta_x(A) = \begin{cases} 0, & x \notin A; \\ 1, & x \in A. \end{cases}$$

A useful result of Dirac measures is the identity

$$\int \delta_x(dy) f(y) = f(x).$$

The Dirac measure can be thought of as the indicator function. Another example of a measure is the Lebesgue measure, which is the standard measure on the Euclidean space $\mathbb{R}^d$. The rigorous definition of a Lebesgue measure is fairly analytical and is omitted. The Lebesgue measure is essentially an extension of the standard notions of length, area and volume to d-dimensional Euclidean spaces.

Definition 2.1.7. *Let* $(\Omega, \mathcal{F})$ *and* $(E, \mathcal{E})$ *be measurable spaces. A function* $f : \Omega \to E$ *is* $(\mathcal{F}, \mathcal{E})$*-**measurable** if* $f^{-1}(S) \in \mathcal{F}$ *for every* $S \in \mathcal{E}$*.*

Definition 2.1.8. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $(E, \mathcal{E})$ *a measurable space. A **random variable** is a function* $X : \Omega \to E$ *that is* $(\mathcal{F}, \mathcal{E})$*-measurable. We call* $(E, \mathcal{E})$ *the **state space**.*

2.2 Stochastic Processes

Generally speaking, a stochastic process $\{X_t\}_{t\geq 0}$ is a collection of random variables that represent the evolution of some object over time. As this thesis focuses heavily on stochastic processes it is important that we rigorously define what a stochastic process is.

Definition 2.2.1. *A **stochastic process** is a collection of random variables* $\{X_t\}_{t\in\mathcal{I}}$ *that is defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and is indexed by an ordered set* $\mathcal{I}$*.*

If we fix a $t_0 \in \mathcal{I}$ then $X(t_0, \omega)$ is simply a random variable. If we fix a $\omega_0 \in \Omega$ then $X(t, \omega_0)$ is a function with respect to t and is the trajectory of a single realisation of the stochastic process $X(t)$. From now on we will only consider stochastic processes where our ordered set $\mathcal{I} = \mathbb{R}^+$ represents an interval in continuous time. We now write $\{X(t)\}_{t\geq 0} : \mathbb{R}^+ \times \Omega \to \mathbb{R}$.

Definition 2.2.2. *A stochastic process* $\{X(t)\}_{t\geq 0}$ *has a **jump** at time* t *if the process is discontinuous at that point. i.e.* $X(t) \neq X_-(t)$*.*

Definition 2.2.3. *A stochastic process* $\{X(t)\}_{t\geq 0}$ *is cádlág (derived from the French term "continue droite, limite gauche" if it has right continuous paths with left hand limits. i.e* $X_-(t)$ *exists and* $X_-(t)$ *exists and is equal to* $X(t)$*.*

Definition 2.2.4. *A* ***filtration*** $\{\mathcal{F}_t\}_{t\geq 0}$ *is a collection of* σ*-algebras such that for any* $0 \leq s \leq t$*,* $\mathcal{F}_s \subseteq \mathcal{F}_t$*.*

Note that with each successive σ-algebra the amount of sets that can be measured is non-decreasing. Thus a filtration can be interpreted as the accumulation of information about our stochastic process with respect to time. We write $\sigma(X_s, s \leq t)$ as the smallest σ-algebra generated by $\{X_s : s \leq t\}$.

Definition 2.2.5. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with the filtration* $\mathcal{F} = \{\mathcal{F}\}_{t\geq 0}$ *and let* $(E, \mathcal{E})$ *a measurable space. We say the stochastic process* $\{X_t\}_{t\geq 0}$ *is* $\boldsymbol{\mathcal{F}}$***-adapted*** *if for every* $t \geq 0$*,* $X_t : \Omega \to E$ *is* $(\mathcal{F}_t, \mathcal{E})$ *- measurable.*

2.3 Weak Convergence

The notion of weak convergence or convergence in distribution is also required to understand results utilised in this thesis.

Definition 2.3.1. *A* ***metric*** *d on a set E is a non-negative real-valued function on the product space $E \times E$ such that*

- $d(x, y) = 0$ *iff* $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) + d(y, z)$.

A metric is a function that defines the distance between two points in a set.

Definition 2.3.2. *Let E be a set and let d be a metric on E. Then the ordered pair (E,d) is a* ***metric space****.*

Definition 2.3.3. *A sequence* $\{x_n\}_{n\geq 1}$ *in a metric space (E,d)* ***converges to a limit*** *$x \in E$ if, for all $\epsilon > 0$, there exists an integer n_0 such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$.*

Definition 2.3.4. *A sequence of probability measures* $\{\mathbb{P}_n\}_{n\geq 1}$ *on the measurable space* $(\Omega, \mathcal{F})$ ***converges weakly*** *to a probability measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$ *if*

$$\lim_{n\to\infty} \int_E f d\mathbb{P}_n = \int_E f d\mathbb{P}$$

for all continuous and bounded real-valued functions f *on* Ω*. We then write*

$$\mathbb{P}_n \to \mathbb{P}.$$

Note that this definition is essentially 2.3.3 with E as the set of all probability measures on the measurable space $(\Omega, \mathcal{F})$ and $d(\mathbb{P}, \mathbb{Q}) = |\int_E f d\mathbb{P} - \int_E f d\mathbb{Q}|$.

Definition 2.3.5. *Let* $X_n : \Omega \to E$ *be a sequence of random variables with a corresponding sequence of probability measures* $\{\mathbb{P}_{X_n}\}_{n\geq 1}$ *on the measurable space* $(\Omega, \mathcal{F})$*. Then* $\{X_n\}_{n\geq 1}$ ***converges in distribution*** *to a random variable* $X : \Omega \to E$ *if*

$$\mathbb{P}_{X_n} \to \mathbb{P}_X$$

where $\mathbb{P}_X$ *is the probability measure of* X*. We then write* $X_n \xrightarrow{d} X$*.*

Chapter 3

Scaling Limit Theorems

3.1 Problem Formulation

The Continuous Time Random *Walk* (CTRW) has been a highly successful model for anomalous diffusion in the past two decades [MK00, HTSL10], likely due to its tractable and flexible scaling properties. The stochastic process we study in this article is conceptually very close to the CTRW, since essentially the jumps J_k are reinterpreted as magnitudes, and instead of the cumulative sum, one tracks the cumulative maximum. Similarly tractable scaling properties apply to the CTRM, see below. For the above reasons, we use the name CTRM in this article.

Definition 3.1.1. *Assume i.i.d. pairs of random variables (J_k, W_k), $k = 1, 2, \ldots$ where $W_k > 0$ represents the inter-arrival times of certain events and $J_k \in \mathbb{R}$ the corresponding event magnitudes. Write*

$$N(t) = \max\{n \in \mathbb{N} : W_1 + \ldots + W_n \leq t\} \tag{3.1.1}$$

for the renewal process associated with the W_k (where the maximum of the empty set is set to 0). Then the process

$$M(t) = \bigvee_{k=1}^{N(t)} J_k = \max\{J_k : k = 1, \ldots, N(t)\}, \quad t \geq 0. \tag{3.1.2}$$

is called a CTRM (Continuous Time Random Maxima process).

It is clear that the sample paths of $N(t)$ and $M(t)$ are right-continuous with left-hand limits. If W_k is interpreted as the time *leading up* to the event with magnitude J_k, then $M(t)$ is indeed the largest magnitude up to time t. The alternative case where W_k represents the inter-arrival time *following* J_k is termed “second type” (in the shock model literature) or OCTRM (overshooting CTRM), and the largest magnitude up to time t is then given by

$$\tilde{M}(t) = \bigvee_{k=1}^{N(t)+1} J_k, \quad t \geq 0. \tag{3.1.3}$$

Finally, the model is called *coupled* when W_k and J_k are not independent. In this article we focus on the uncoupled case, for which it can be shown that the processes $M(t)$ and $\tilde{M}(t)$ have the same limiting distributions at large times [**?**], and hence we focus on the CTRM $M(t)$.

We aim to make inference on the distribution of the following quantities:

Definition 3.1.2. *Let $M(t)$ be an uncoupled CTRM whose magnitudes J_k are supported on the interval $[x_0, x_F]$. Then the exceedance time of level $\ell \in [x_0, x_F]$ is the random variable*

$$T_\ell = \inf\{t : M(t) > \ell\}$$

and the exceedance is

$$X_\ell = M(T_\ell) - \ell.$$

Lemma 3.1.3. *Given a level $\ell \in [x_0, x_F]$, exceedance X_ℓ and exceedance time T_ℓ are independent. Moreover,*

$$\mathbf{P}[X_\ell > x] = \frac{\overline{F}_J(\ell + x)}{\overline{F}_J(\ell)}, \quad x > 0,$$
$$\mathbf{P}[T_\ell > t] = \overline{F}_J(\ell) \sum_{n=1}^{\infty} \int_0^\infty \overline{F}_W(t - t')\mathbf{P}[L_{n-1} \leq \ell]\mathbf{P}[S_{n-1} \in dt'], \quad t > 0$$

where $L_n = \bigvee_{k=1}^n J_k$, $L_0 = x_0$, and $S_n = \sum_{k=1}^n W_k$, $S_0 = 0$.

Proof. Let $\tau_\ell = \min\{k : J_k > x\}$. Then

$$T_\ell > t, X_\ell > x \iff S_{\tau_\ell} > t, L_{\tau_\ell} > \ell + x \iff \exists n : L_{n-1} \leq \ell, J_n > \ell + x, S_n > t$$
$$\iff \exists n : L_{n-1} \leq \ell, J_n > \ell + x, W_n > t - S_{n-1}$$

and such n must be unique, since $x > 0$. Thus

$$\mathbf{P}[T_\ell > t, X_\ell > x] = \sum_{n=1}^{\infty} \mathbf{P}[J_n > \ell + x, W_n > t - S_{n-1}, L_{n-1} \leq \ell]$$
$$= \sum_{n=1}^{\infty} \int_0^\ell \int_0^\infty \mathbf{P}[J_n > \ell + x, W_n > t - t' | L_{n-1} = m', S_{n-1} = t'] \mathbf{P}[L_{n-1} \in dm', S_{n-1} \in dt']$$
$$= \sum_{n=1}^{\infty} \int_0^\infty \overline{F}_J(\ell + x) \overline{F}_W(t - t') \mathbf{P}[L_{n-1} \leq \ell] \mathbf{P}[S_{n-1} \in dt']$$

since the sequences J_k and W_k are i.i.d. and independent of each other. Letting $t \downarrow 0$, we see

$$\mathbf{P}[X_\ell > x] = \sum_{n=1}^{\infty} \overline{F}_J(\ell + x) \mathbf{P}[L_{n-1} \leq \ell] = \overline{F}_J(\ell + x) \sum_{n=1}^{\infty} F_J(\ell)^{n-1}$$
$$= \frac{\overline{F}_J(\ell + x)}{\overline{F}_J(\ell)},$$

and letting $x \downarrow 0$, one gets $\mathbf{P}[T_\ell > t]$. One checks that $\mathbf{P}[T_\ell > t, X_\ell > x] = \mathbf{P}[T_\ell > t] \mathbf{P}[X_\ell > x]$, implying independence. $\square$

The distribution of the exceedance is hence, as expected, simply the conditional distribution of a magnitude J_k given $J_k > \ell$. We will look at the distributions of exceedances and exceedance times in further detail in chapter 4.

3.2 Scaling Limit of the Sum of Waiting Times

As mentioned in the introduction, we assume a heavy-tailed distribution for the waiting times, more precisely that the tail function $\overline{F}_W := 1 - F_W$ of the CDF of W is regularly varying:

$$t \mapsto \overline{F}_W(t) \in RV(-\beta), \quad \beta \in (0, 1) \tag{3.2.1}$$

meaning that [Sen76, MS01]

$$\lim_{t\to\infty} \frac{\overline{F}_W(\lambda t)}{\overline{F}_W(t)} = \lambda^{-\beta}.$$

By [MS01, Cor. 8.2.19] (also compare [Whi01, Th. 4.5.1]), this is equivalent to W_1 being in the (sum-) domain of attraction of a positively skewed stable law D with stability parameter β, which means the weak convergence

$$b(n)(W_1 + \ldots + W_n) \Rightarrow D, \quad n \to \infty. \tag{3.2.2}$$

The i.i.d. nature of the W_k allows for a more general, functional limit theorem [MS01, Ex. 11.2.18], [Whi01, Th. 4.5.3]:

$$b(c)(W_1 + \ldots + W_{\lfloor cr \rfloor}) \Rightarrow D(cr), \quad c \to \infty, \tag{3.2.3}$$

where $\{D(r)\}_{t\geq 0}$ is a stable subordinator, defined via

$$\mathbf{E}[e^{-\lambda D(r)}] = e^{-r\lambda^\beta}, \tag{3.2.4}$$

where $\lfloor r \rfloor$ denotes the largest integer not larger than r, and where convergence holds on the stochastic process level, i.e. in the sense of weak convergence in Skorokhod space endowed with the J_1 topology [Whi01, Sec. 3.3].

The renewal process paths $N(t)$ result, up to a constant, from inverting the non-decreasing sample paths of the sum $r \mapsto W_1 + \ldots + W_{\lfloor r \rfloor}$. Accordingly, the scaling limit of the renewal process is [MS04]

$$N(ct)/\tilde{b}(c) \Rightarrow E(t), \quad c \to \infty \tag{3.2.5}$$

where $E(t)$ denotes the inverse stable subordinator [MS13]

$$E(t) = \inf\{r : D(r) > t\}, \quad t \geq 0, \tag{3.2.6}$$

and where $\tilde{b}(c)$ is asymptotically inverse to $1/b(c)$, in the sense of [Sen76, p.20]:

$$1/b(\tilde{b}(c)) \sim c \sim \tilde{b}(1/b(c)) \tag{3.2.7}$$

where a $\sim$ symbol indicates that the quotient of both sides converges to 1 as $c \to \infty$. Note that $\tilde{b}(c) \in RV(\beta)$.

The inverse stable subordinator $E(t)$ governs the temporal dynamics of the scaling limit of the CTRM process $M(t)$, see Theorem **??**. It is self-similar with exponent β [MS04], non-decreasing, and the (regenerative, random) set $\mathcal{R}$ of its points of increase is a fractal with dimension β [Ber99]. $E(t)$ is a model for time series with intermittent, 'bursty' behaviour, for the following reasons:

i) Conditional on $t \geq 0$ being a point of increase, any interval $(t, t + \epsilon)$ almost surely contains uncountably many other points of $\mathcal{R}$;

ii) $\mathcal{R}$ has Lebesgue measure 0, and hence $E(t)$ is constant at any "randomly" chosen time t.

Having only two parameters ($\beta \in (0, 1)$ and a scale parameter) the inverse stable subordinator hence models scaling limits of heavy-tailed waiting times parsimoniously.

3.3 Scaling Limit of the Maximum of Event Magnitudes

3.4 Scaling Limit of CTRM

Chapter 4

Distribution of CTRM

4.1 Distribution of Exceedances

4.2 Distribution of Exceedance Durations

Chapter 5

Statistical Inference

5.1 Log Moment Estimator for Mittag-Leffler Random Variables

5.2 Maximum Likelihood Estimator for Generalised Pareto Random Variables

5.3 Stability of Parameter Estimates

5.4 Goodness of Fit

Chapter 6

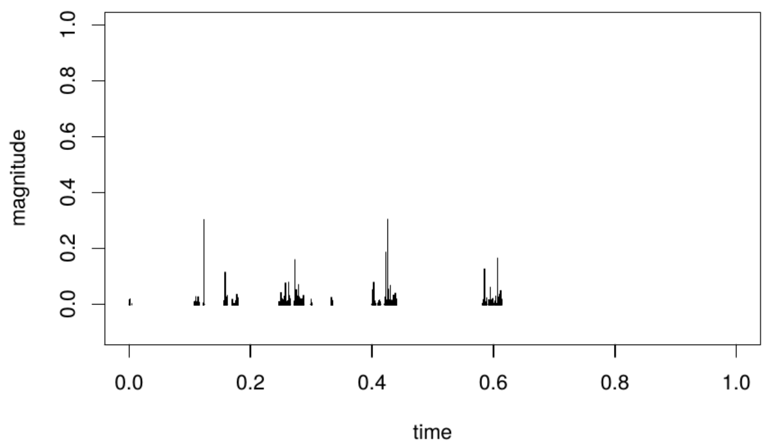
Model Fitting

The accuracy of the Mittag-Leffler log moment estimator and the Generalised Pareto MLE has been established in [Cah13] and [?] respectively. However it is yet to be confirmed that these estimation methods work for exceedances and exceedance durations. Thus before we apply our inference methods to actual data, we will check that we can correctly estimate the theoretical Mittag-Leffler and Generalised Pareto parameters derived from a simulated bursty process. If we are able to accurately estimate these parameters we can then proceed to apply the estimation methods to actual data, confident in knowing that the actual parameters will be estimated.

6.1 Model Fit of Simulated Data

We assume n i.i.d. waiting times are drawn from the positively skewed stable distribution with stability parameter $\beta = 0.8$, scaled with $n^{-1/\beta}$ to ensure convergence to a stable subordinator. This defines a renewal process, at whose renewal times we assume n i.i.d. magnitudes are drawn from a Generalized Extreme Value Distribution with shape parameter $\xi = 0.7$.

Figure 6.1: A simulated bursty process



Recall that we have defined the exceedance duration of level $\ell \in [x_0, x_F]$ as the random variable

$$T_\ell = \inf\{t : M(t) > \ell\}$$

and the exceedance as

$$X_\ell = M(T_\ell) - \ell.$$

As these random variables are the quantities of interest it is natural for us to then consider the (simulated) sample exceedances and their corresponding durations over the threshold level ℓ.

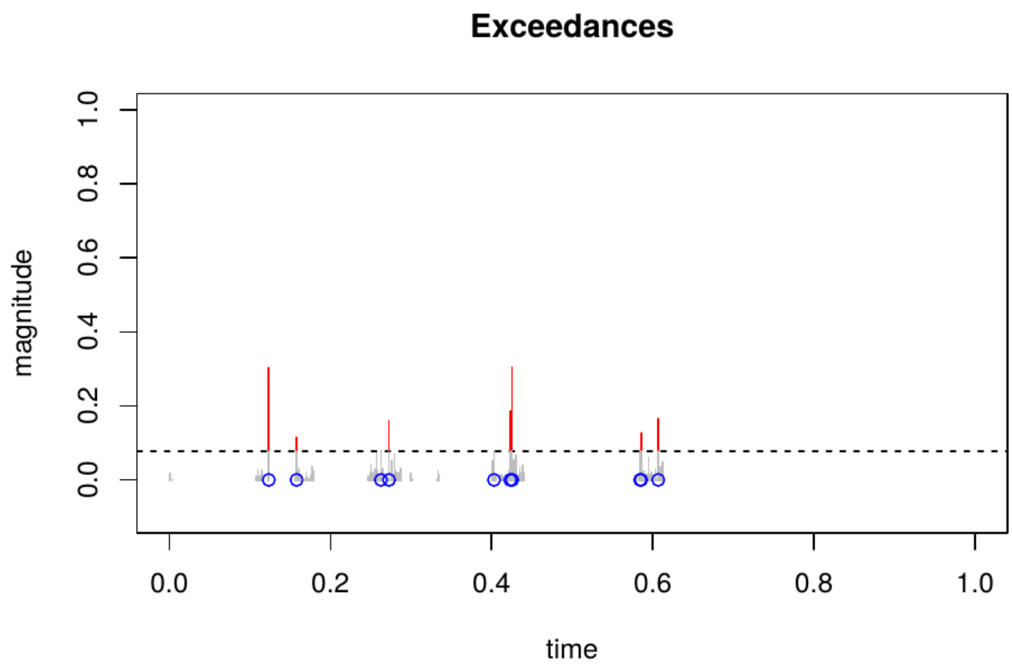


Figure 6.2: Exceedance durations (blue circles) and Exceedance sizes (red lines).

Assume now a time series of magnitudes, and that interest lies in the estimation of the timings of the large magnitudes. Consider a minimum threshold ℓ_0, e.g. at the 95% quantile. Vary the threshold ℓ on the interval $[\ell_0, x_F]$, and consider the resulting sequences of exceedance sizes and exceedance times $\{(X_{\ell,i}, T_{\ell,i})\}$. Due to the renewal property each sequence is i.i.d.. Now $T_{\ell,1}, T_{\ell,2}, \ldots$ can be modelled by a Mittag-Leffler distribution and $J_{\ell,1}, J_{\ell,2}, \ldots$ can be modelled by a Generalised Pareto distribution as shown in chapter 4.

6.2 Model Fit of Real Data

Chapter 7

Conclusion

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