Modelling Extremes of Bursty Events

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Motivation

Classical extreme value theory assumes that events happen uniformly.

However this is not always the case, in many systems the events occur in bursts.

Examples include both human-created events and physical phenomena:

- Communication
- Financial Trades
- Network Traffic
- Neuron Firing Sequences
- Seismic Activity

Example Processes

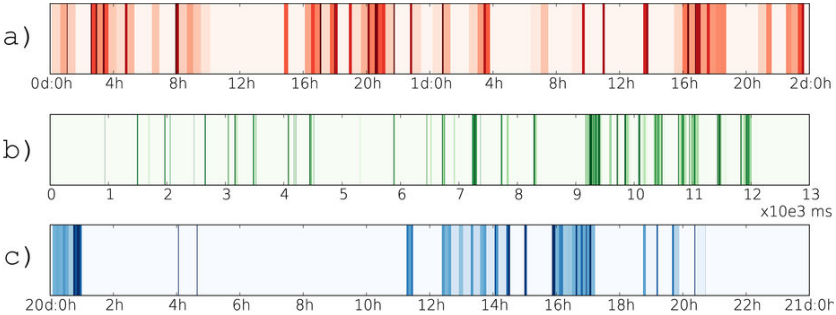


Figure 1 : (a): Sequence of earthquakes with magnitude larger than two at a single location (South of Chishima Island, 8th–9th October 1994) (b): Firing sequence of a single neuron (from rat's hippocampal) (c): Outgoing mobile phone call sequence of an individual. The darker the colour the shorter the time between consecutive events. (Karsai, Kaski, Barabási & Kertész, 2012)

Notation

Let $J_1, J_2, \ldots$ be a sequence of i.i.d. random variables that model the jump sizes (event magnitudes).

Let $W_1, W_2, \ldots$ be a sequence of i.i.d positive random variables that model the waiting times between the jumps.

We can then define $(W_1, J_1), (W_2, J_2), \ldots$ to be a sequence of i.i.d $\mathbb{R} \times \mathbb{R}^+$ random variables.

Notation Contd.

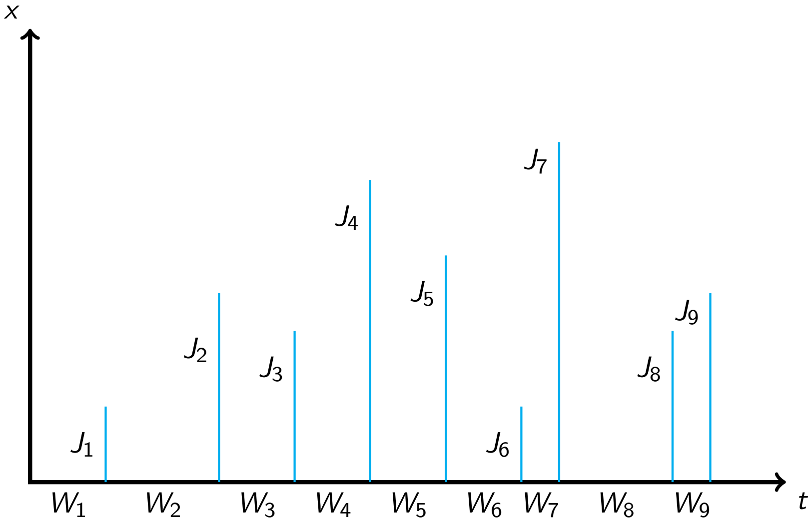
Now define $S(n) := \sum_{i=1}^{n} W_i$ as the partial sum of the first n waiting times.

Define a renewal process $N(t) := \max\{n \geq 0 : S(n) \leq t\}$.

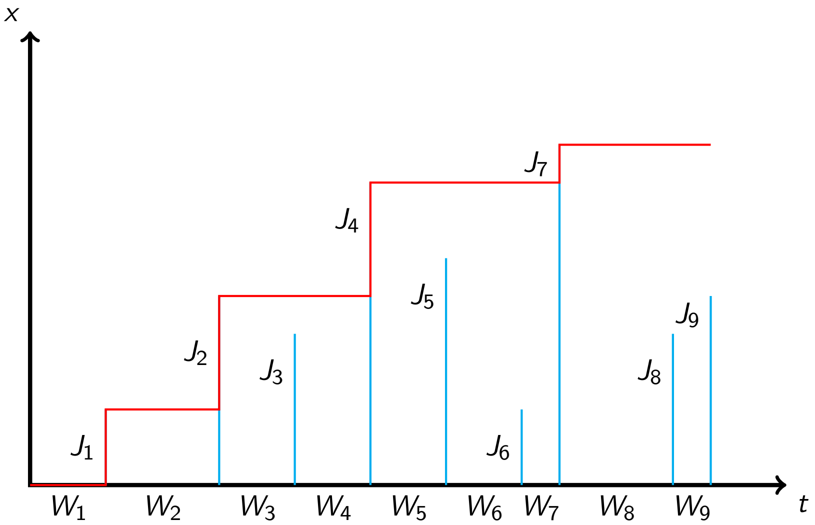
Define $M(n) := \sum_{i=1}^{n} W_i$ as the partial maxima of the first n jumps.

Finally we define the Continuous Time Random Maxima (CTRM) to be $M(N(t)) := \bigvee_{k=1}^{N(t)} J_k = \max\{J_k : k = 1, \ldots, N(t)\}$.

CTRM Example



CTRM Example



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Discrete Random Walk

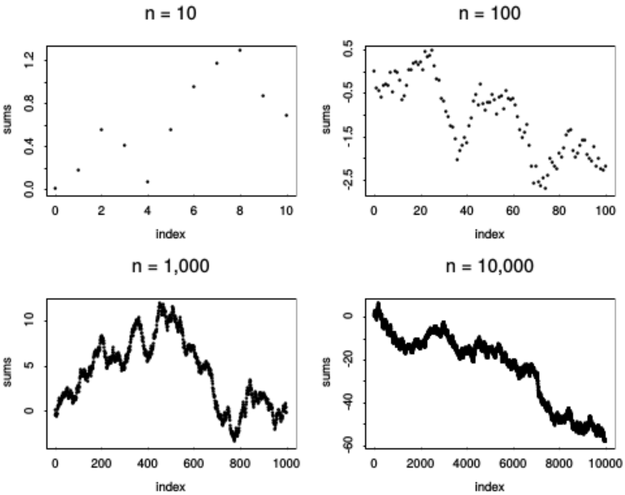

Figure 2 : Possible realisations of a random walk with steps, U_i, uniformly distributed in $[1/2, 1/2]$

A Continuous Analogue

We seek a continuous function that coincides with the Discrete Random Walk at integer arguments.

There are two obvious choices:

- A linear interpolation between every point
- The step function $S_{\lfloor t \rfloor} = U_1 + \cdots + U_{\lfloor t \rfloor}$

Both are equivalent for large t so we consider the step function as it's simpler.

Now add a time scaling constant c such that $S_{\lfloor ct \rfloor} = U_1 + \cdots + U_{\lfloor ct \rfloor}$.

The Classical Functional Central Limit Theorem

Recall that the classical Central Limit Theorem states

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} Z \sim N(0,1) \text{ as } n \to \infty.$$

Then the classical Functional Central Limit Theorem states

$$\frac{S_{\lfloor ct \rfloor} - \lfloor ct \rfloor \mu}{\sqrt{c\sigma^2}} \xrightarrow{d} B_t \sim N(0,t) \text{ as } c \to \infty.$$

Scaling Limit of Waiting Times

The classical central limit theorem can be generalised for random variable with undefined moments.

(Whitt 2001)
If W_k obey a central limit theorem such that

$$\frac{W_1 + \ldots + W_n}{b(n)} \xrightarrow{d} Y, \quad n \to \infty,$$

where Y is a stable random variable. Then we have the following functional central limit theorem

$$\frac{S_{\lfloor ct \rfloor}}{b(c)} \xrightarrow{d} D(t), \quad c \to \infty,$$

where $D(t)$ is a stable subordinator, i.e. its increments are stably distributed.

(Meerschaert & Scheffler 2004)
The scaling limit of the renewal process is then

$$\tilde{b}(c)^{-1}N(ct) \xrightarrow{d} E(t), \quad c \to \infty$$

where $E(t)$ denotes the inverse stable subordinator

$$E(t) = \inf\{r : D(r) > t\}, \quad t \geq 0,$$

and where $\tilde{b}(c)$ is asymptotically inverse to $b(c)$.

Scaling Limit of Maxima

(Lamperti 1964)
If there exists constants $a(n) > 0$ and $d(n)$ such that,

$$\mathbb{P}\left(\frac{M_n - d_n}{a_n} \leq x\right) \xrightarrow{d} G(x).$$

Then $G(x)$ must be a Generalised Extreme Value distribution. We also have the functional extremal limit theorem

$$\frac{M_{\lfloor ct \rfloor} - d(c)}{a(c)} \xrightarrow[c \to \infty]{J_1} A(t),$$

where $\{A(t)\}_{t \geq 0}$ is an extremal process generated by G. That is $\mathbb{P}(A(t) \leq x) = G(x)^t$.

Recap

We have a limit theorem for our partial sum-process,

$$b(c)^{-1}S_{\lfloor ct \rfloor} \xrightarrow{d} D(t), \quad c \to \infty, \tag{1}$$

a limit theorem for our renewal process,

$$N(ct)/\tilde{b}(c) \xrightarrow{d} E(t), \quad c \to \infty, \tag{2}$$

and a limit theorem for our partial maxima-process,

$$\frac{M_{\lfloor ct \rfloor} - d(c)}{a(c)} \xrightarrow{d} A(t), \quad c \to \infty. \tag{3}$$

Scaling Limit of CTRM

Theorem

Let (W_i, J_i) be a sequence of i.i.d $\mathbb{R}^+ \times \mathbb{R}$ random vectors such that the limits in equations (1),(2) and (3) hold. Then,

$$\frac{M(N(ct)) - d(\tilde{b}(c))}{a(\tilde{b}(c))} \xrightarrow{d} A(E(t)), \quad c \to \infty,$$

where $A(E(t))$ is a subordination of the extremal process $A(t)$ by the inverse stable subordinator process $E(t)$.

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Variables of Interest

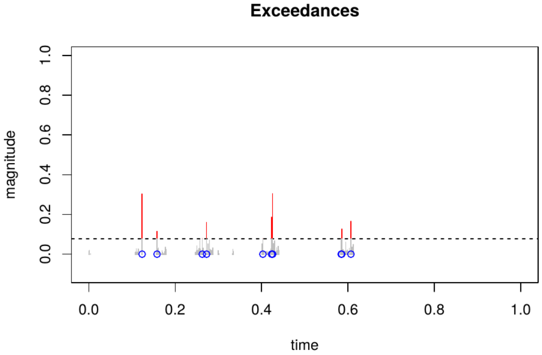

Figure 1: Exceedance times (blue circles) and Exceedance sizes (red lines).

We are interested in modelling the durations $T_\ell := \inf\{t : M(N(t)) > \ell\}$ and the exceedances $X_\ell = M(N(T_\ell)) - \ell$

Distribution of Exceedances and Exceedance Durations

Using the scaling limit theorem for our CTRM it can be shown that for large ℓ

$$T_\ell \sim ML(\beta, \delta),$$

where ML refers to the Mittag-Leffler distribution. It can also be shown that

$$X_\ell \sim GP(\xi, \tilde{\sigma}),$$

where GP refers to the Generalised Pareto distribution.

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Threshold Selection

At high thresholds we have a small amount of data points and thus a high variance.

Low thresholds introduce bias since exceedances and durations are only asymptotically GP and ML distributed.

We can transform all four parameters such that they're constant with respect to the threshold if the asymptotic approximations are accurate.

Thus we should pick the lowest threshold such that the transformed parameters remain constant.

A Possible Simulation

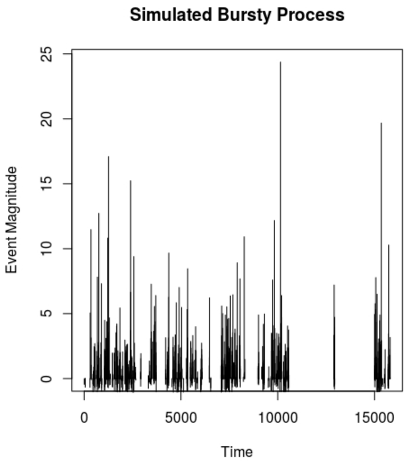
Waiting times W_i are simulated according to a stable distribution with stability parameter $\alpha \in (0, 1)$.

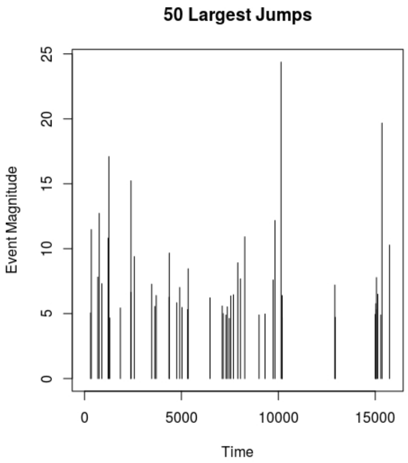
Jump sizes J_i are simulated according to Generalized Extreme Value (GEV) distribution with location parameter μ, scale parameter σ and shape parameter ξ.

We can now test if our theoretical results hold for simulated data.

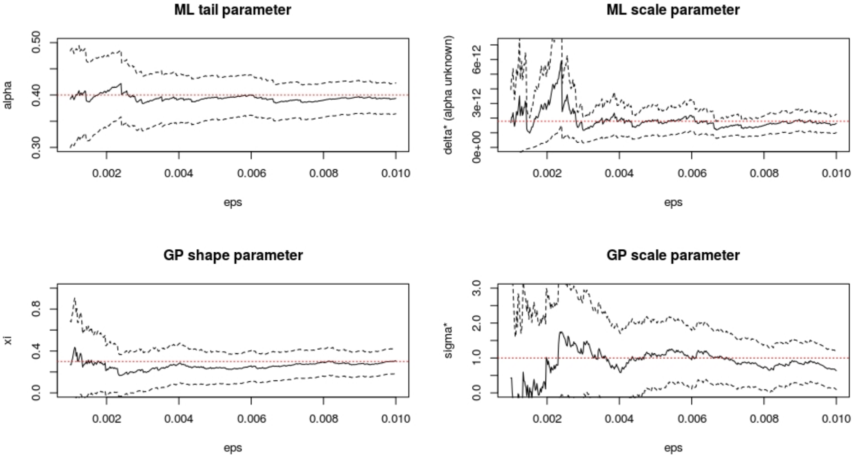
Simulated data

$$\alpha = 0.4, \sigma = 1, \xi = 0.4$$





Stability Plots



Further Research

Apply the model to datasets with heavy tail waiting times such as bond futures trades, seismic activity and network transmissions.

Extend the model to include the case where there is a dependence structure between W_i and J_i.