Statistical Models for Bursty Events

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Motivation

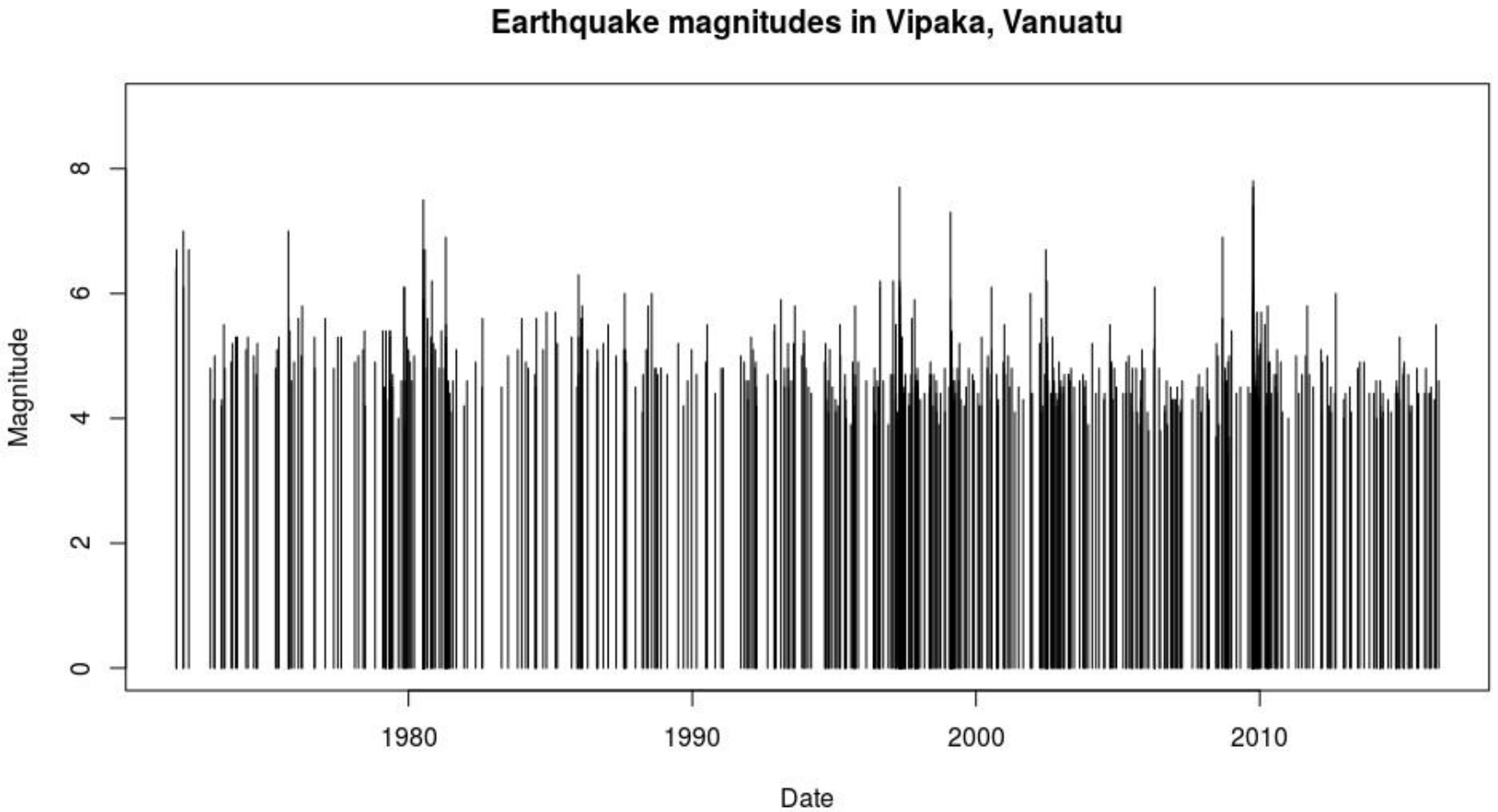
Classical extreme value theory assumes that events happen uniformly.

However this is not always the case, in many systems the events occur in bursts.

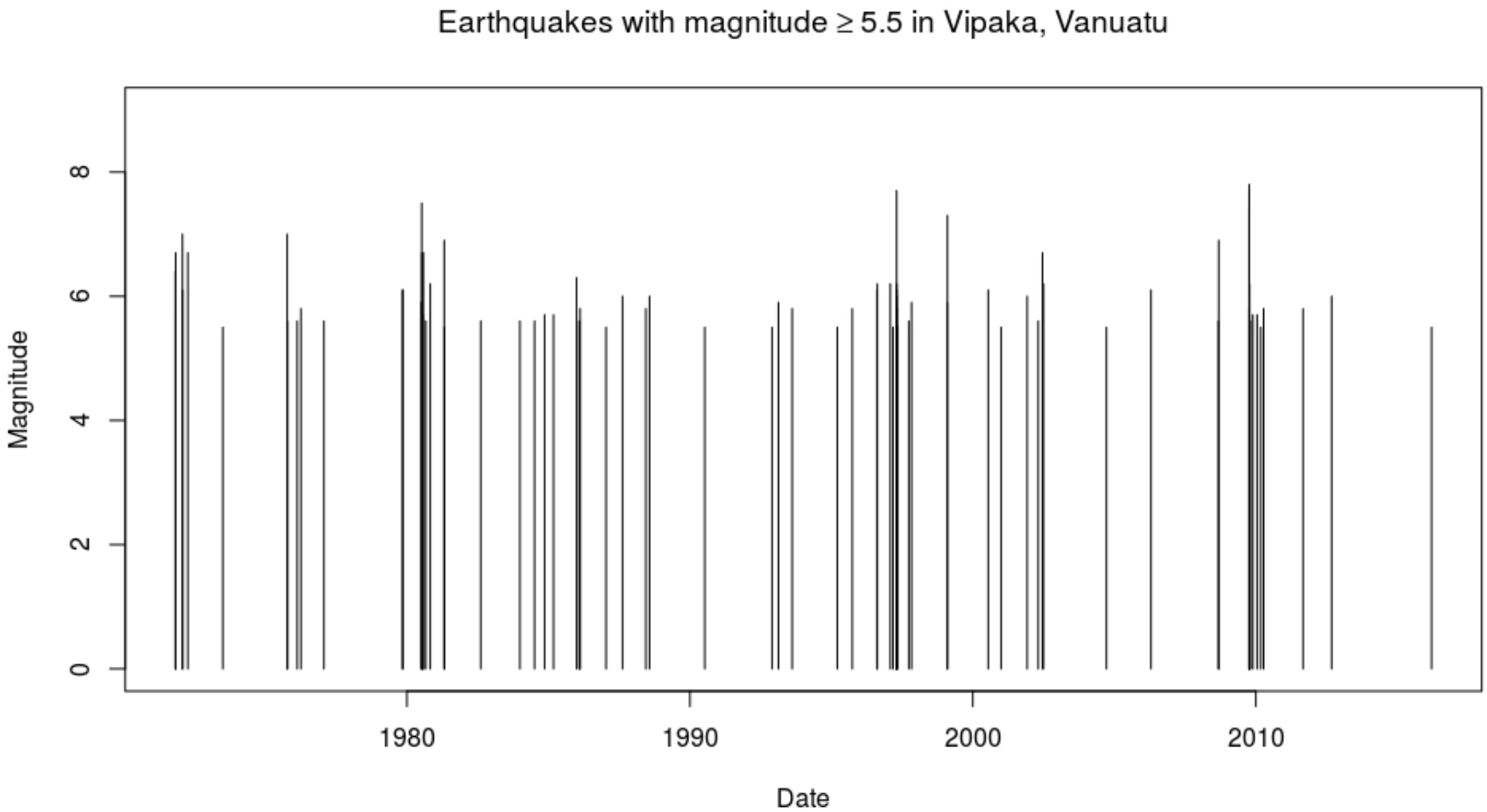
Examples include both human-created events and physical phenomena:

- Communication
- Financial Trades
- Network Traffic
- Neuron Firing Sequences
- Seismic Activity

Example Process



Example Process After Thresholding



Notation

Let $J_1, J_2, \ldots$ be a sequence of i.i.d. random variables that model the jump sizes (event magnitudes).

Let $W_1, W_2, \ldots$ be a sequence of i.i.d positive random variables that model the waiting times between the jumps.

We can then define $(W_1, J_1), (W_2, J_2), \ldots$ to be a sequence of i.i.d $\mathbb{R} \times \mathbb{R}^+$ random variables.

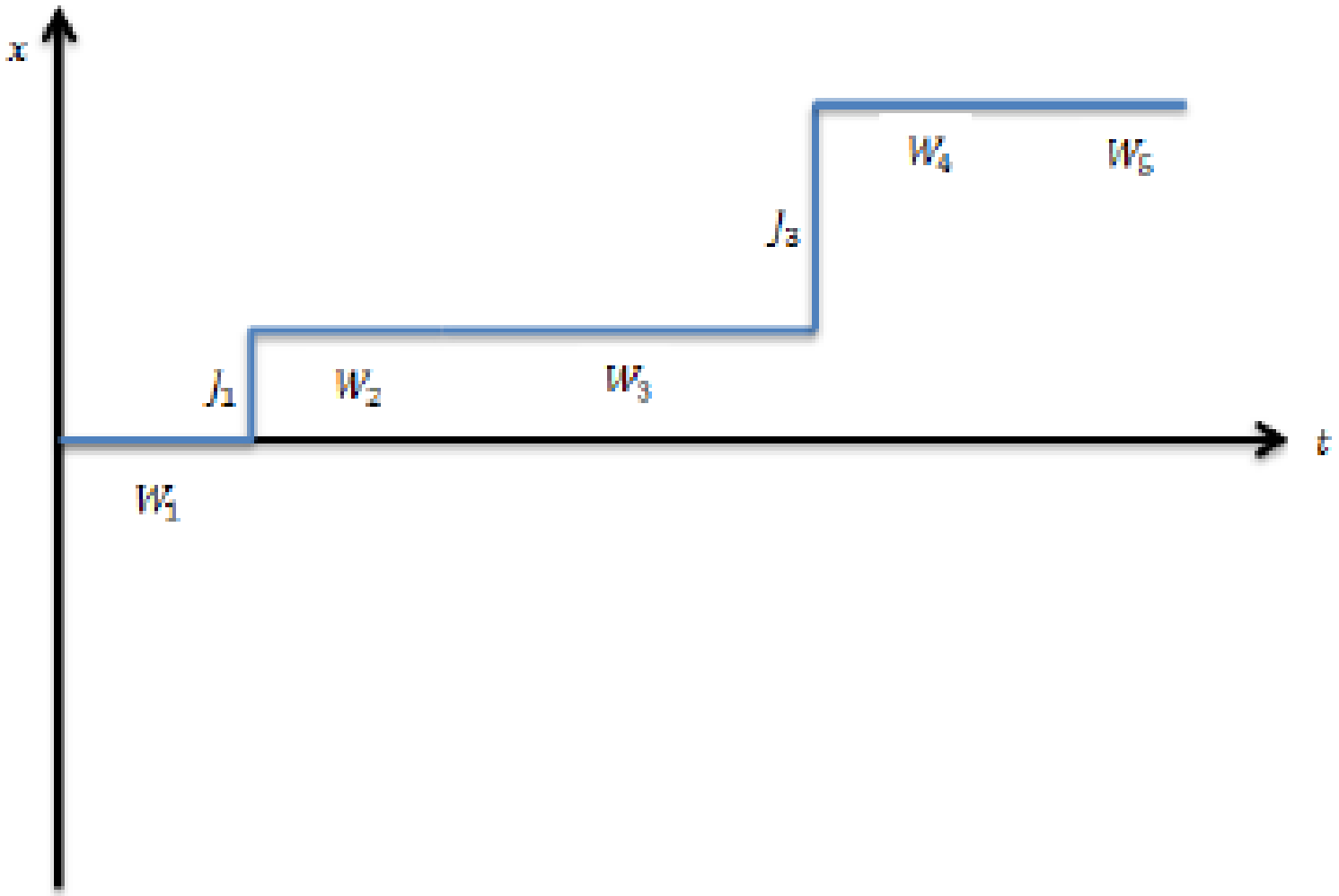
Notation Contd.

Now define the sum of the first n waiting times to be $S(n) := \sum_{i=1}^{n} W_i$

Define a renewal process $N(t) := \max\{n \geq 0 : S(n) \leq t\}$

Finally we define the Continuous Time Random Maxima (CTRM) to be $M(t) := \bigvee_{k=1}^{N(t)} J_k = \max\{J_k : k = 1, \ldots, N(t)\}, \quad t \geq 0.$

CTRM Example



A Possible Simulation

We assume that the waiting times and jump sizes are independent.

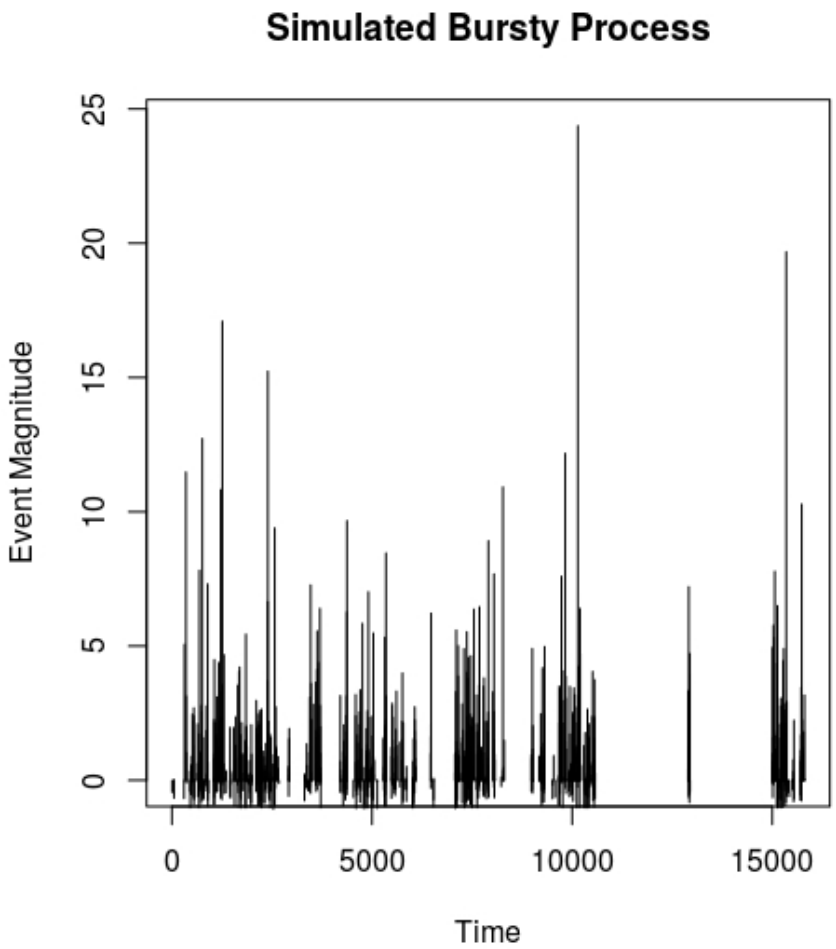
Waiting times W_i are simulated according to a stable distribution with stability parameter $\beta \in (0, 1)$, skewness parameter 1, location parameter 0, and scale parameter $=1$.

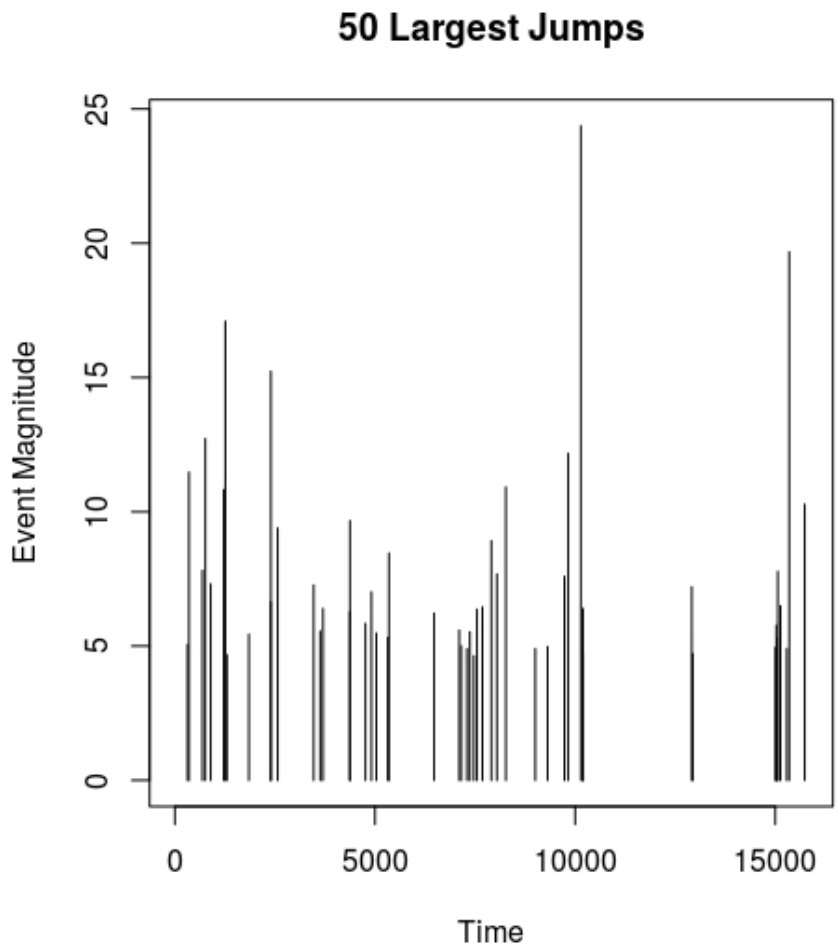
Jump sizes J_i are simulated according to Generalized Extreme Value (GEV) distribution with location parameter μ, scale parameter σ and shape parameter ξ.

The primary goal is to design methodology that fits models to the CTRM of the simulated data.

Simulated data

$$\beta = 0.7, \mu = 0, \sigma = 1, \xi = 0.3$$





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Scaling Limit of Waiting Times

Since we will be working with limits, we need to define partial processes.

Define the partial sum-process as $S_p(t) := \sum_{i=1}^{\lfloor t \rfloor} W_i$

Theorem (Meerschaert and Sikorskii (2011))

Suppose that W_i are i.i.d. and positive with $\mathbb{P}(W_n > t) = ct^{-\beta}$ for all $t > c^{1/\beta}$, some $c > 0$ and $0 < \beta < 1$, then

$$\{b(c)^{-1} S_p(ct)\}_{t \geq 0} \xrightarrow[c \to \infty]{J_1} \{D(t)\}_{t \geq 0},$$

where $\{D(t)\}_{t \geq 0}$ is β-stable subordinator and $b(c) = c^{1/\beta}$.

Scaling Limit of Maxima

Theorem (Lamperti (1964))

Let $F(x)$ be the CDF of J_i. Now suppose there exists constants $a(n) > 0$ and $d(n)$ such that,

$$\mathbb{P}\left(\bigvee_{i=1}^{n} J_i \leq a(n)^{-1}(x - d(n))\right) = F^n(a(n)^{-1}(x - d(n))) \xrightarrow[n\to\infty]{d} G(x).$$

Then $G(x)$ must be a member of the GEV family. Now define the partial max-process as

$$M_p(t) := \begin{cases} \bigvee_{i=1}^{\lfloor t \rfloor} J_i, & t \geq 1 \\ J_1, & 0 < t < 1. \end{cases}$$

Then $\{a(c)^{-1}(M_p(ct) - d(c))\}_{t\geq 0} \xrightarrow[c\to\infty]{J_1} \{A(t)\}_{t\geq 0}$, where $\{A(t)\}_{t\geq 0}$ is an extremal process generated by G.

Scaling Limit of the Joint Process

Theorem

Let (W_i, J_i) be a sequence of i.i.d $\mathbb{R}^+ \times \mathbb{R}$ random vectors such that

$$\{b(c)^{-1}S_p(ct), a(c)^{-1}(M_p(ct) - d(c))\}_{t\geq 0} \xrightarrow[c\to\infty]{J_1} \{(D(t), A(t))\}_{t\geq 0}$$

where the paths of $\{D(t))\}_{t\geq 0}$ are non-decreasing almost surely. Then,

$$\{a(n)^{-1}(M(ct) - d(n))\}_{t\geq 0} \xrightarrow[c\to\infty]{J_1} \{(A(E(t))\}_{t\geq 0},$$

where $E := \inf\{u > 0 : D(u) > t\}$ is the inverse of D and $n = \tilde{b}(c)$ where $\tilde{b}(c)$ is the asymptotic inverse of $b(c)$.

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Variables of Interest

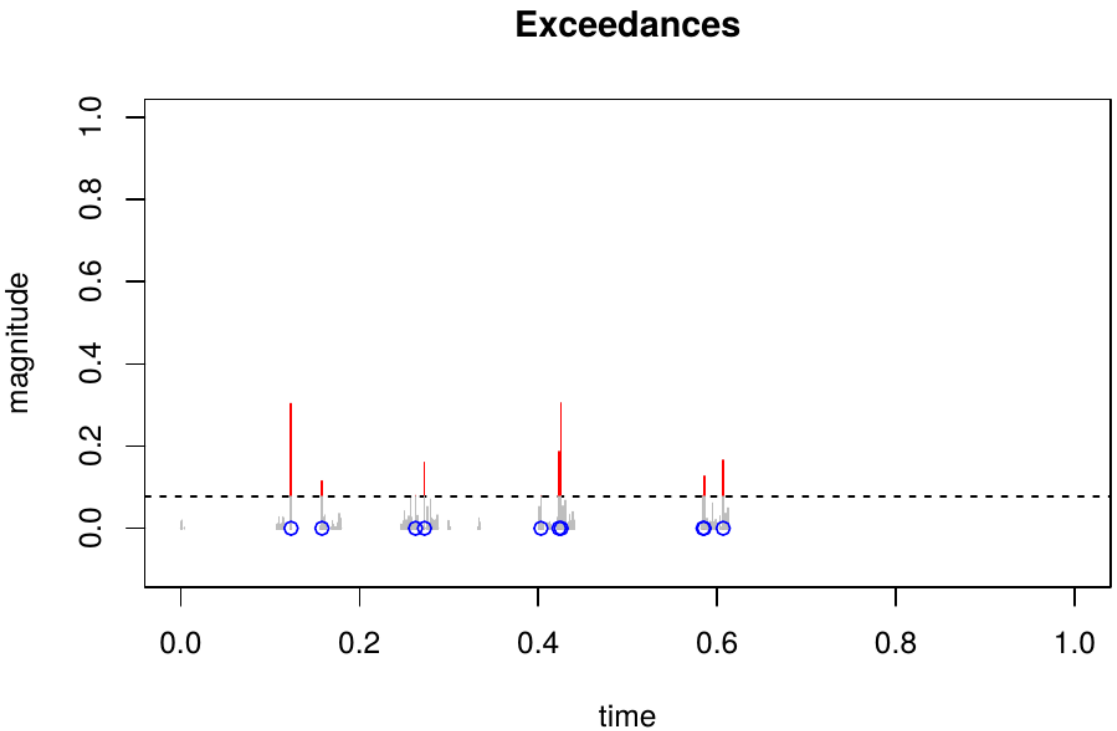

Figure 1: Exceedance times (blue circles) and Exceedance sizes (red lines).

We are interested in modelling the durations
$T_\ell := \inf\{t : M(t) > \ell\}$ and the exceedances $X_\ell = M(T_\ell) - \ell$

Distribution of Durations

Proposition (Meerschaert and Stoev (2007))

Let ℓ be the threshold level, then define $\xi_\ell := \inf\{t : A(E(t)) > \ell\}$ as the hitting time of level ℓ by the process $A(E(t))$. Then

$$\xi_\ell \sim (-\log F(\ell))^{\frac{-1}{\beta}} V_{\beta,1}.$$

Where F is the cdf of a GEV random variable and $V_{\beta,1}$ is a Mittag-Leffler RV with tail parameter β and scale parameter 1.

Using this result it can be shown that

$$T_\ell \sim ML(\beta, \delta),$$

with $\delta = b(n)(-\log F(\ell))^{-1/\beta}$.

Distribution of Exceedances

Theorem (Coles (2001))

If $J_1, J_2, \ldots$ are a sequence of i.i.d. random variables such that

$$\mathbb{P}(\max(J_1, \ldots, J_n) \leq z) \xrightarrow[n\to\infty]{} G(z).$$

Where G is the cdf of a GEV$(\xi, \mu.\sigma)$ random variable, then we have

$$X_\ell \sim GP(\xi, \tilde{\sigma}),$$

with $\tilde{\sigma} = \sigma + \xi(\ell - \mu)$.

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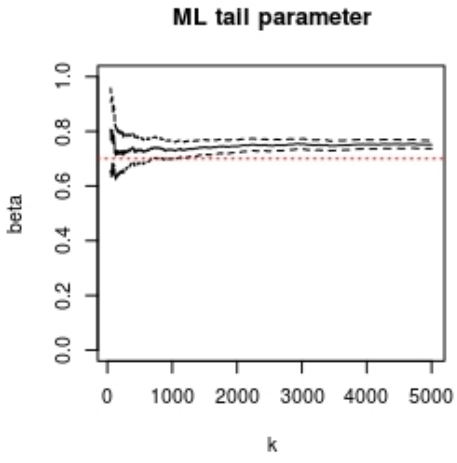
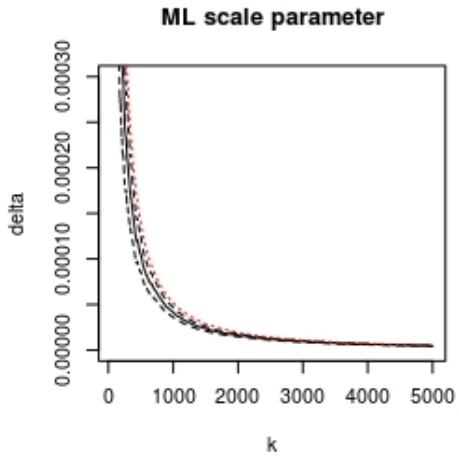
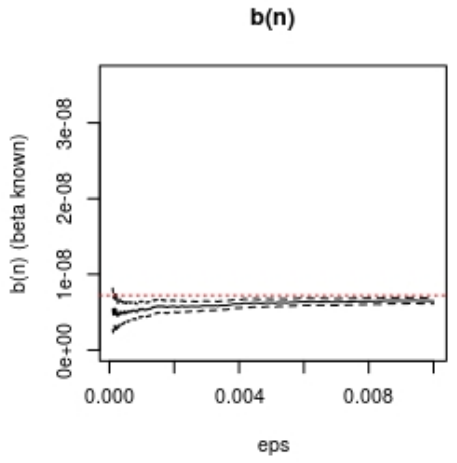
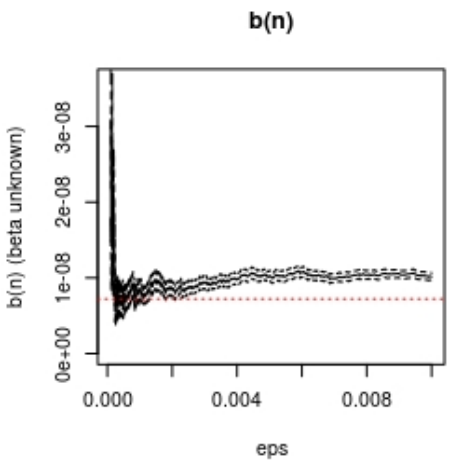
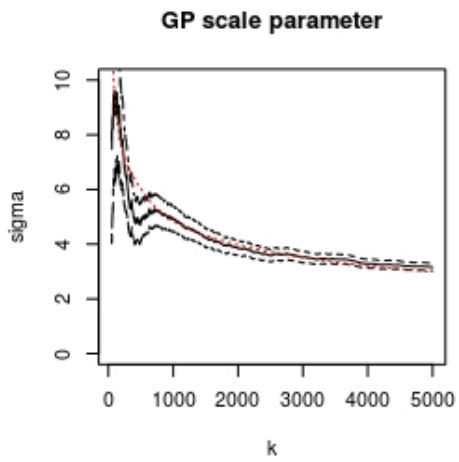
Stability Plots

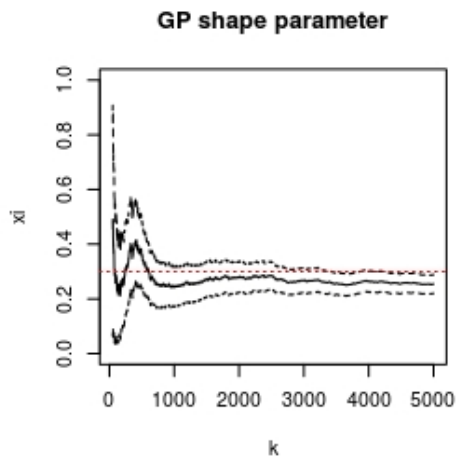












Threshold Selection

If the model fits the data well we should have $\beta, b(n), \xi$ constant and σ linear with respect to the threshold level ℓ.

At high thresholds we have a small amount of data points and thus a high variance.

Since exceedances and durations are GP and ML distributed asymptotically low thresholds introduce bias.

When picking a threshold we should attempt to minimise the variance whilst keeping the threshold high.

Further Research

Apply the model to datasets with heavy tail waiting times such as bond futures trades, seismic activity and network transmissions.

Extend the model to include the case where there is a dependence structure between W_i and J_i.