Statistical Models for Bursty Events

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Motivation

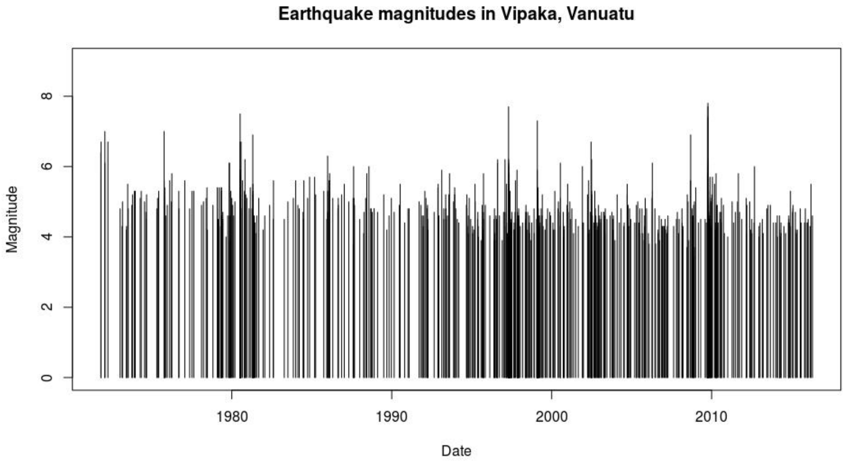
Classical extreme value theory assumes that events happen uniformly.

However this is not always the case, in many systems the events occur in bursts.

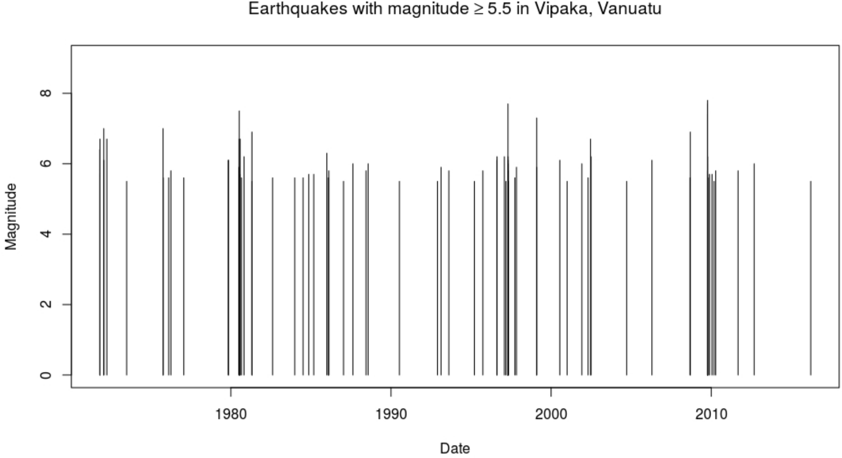
Examples include both human-created events and physical phenomena:

- Communication
- Financial Trades
- Network Traffic
- Neuron Firing Sequences
- Seismic Activity

Example Process



Example Process After Thresholding



Notation

Let $J_1, J_2, \ldots$ be a sequence of i.i.d. random variables that model the jump sizes (event magnitudes).

Let $W_1, W_2, \ldots$ be a sequence of i.i.d positive random variables that model the waiting times between the jumps.

We can then define $(W_1, J_1), (W_2, J_2), \ldots$ to be a sequence of i.i.d $\mathbb{R} \times \mathbb{R}^+$ random variables.

Notation Contd.

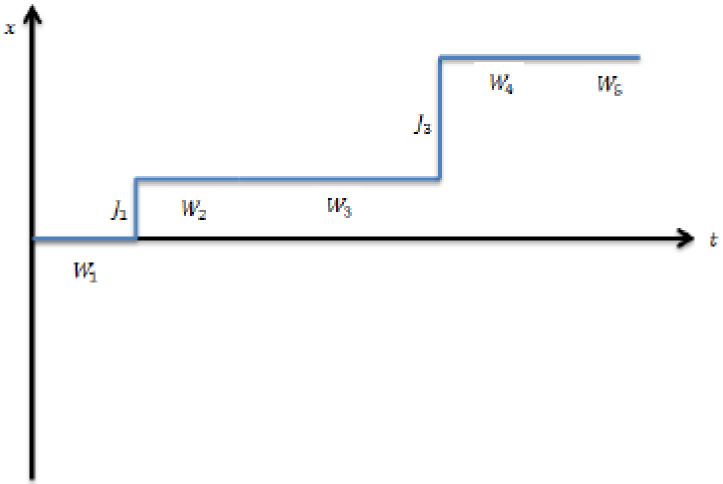
Now define the sum of the first n waiting times to be $S(n) := \sum_{i=1}^{n} W_i$

Define the maximum of the first n jumps to be $M(n) := \bigvee_{i=1}^{n} J_i$

Define a renewal process $N(t) := \max\{n \geq 0 : S(n) \leq t\}$

Finally we define the Continuous Time Random Maxima (CTRM) to be $V(t) := M(N(t)) = \bigvee_{i=1}^{N(t)} J_i$

CTRM Example



A Possible Model

We first assume that the waiting times and jump sizes are independent.

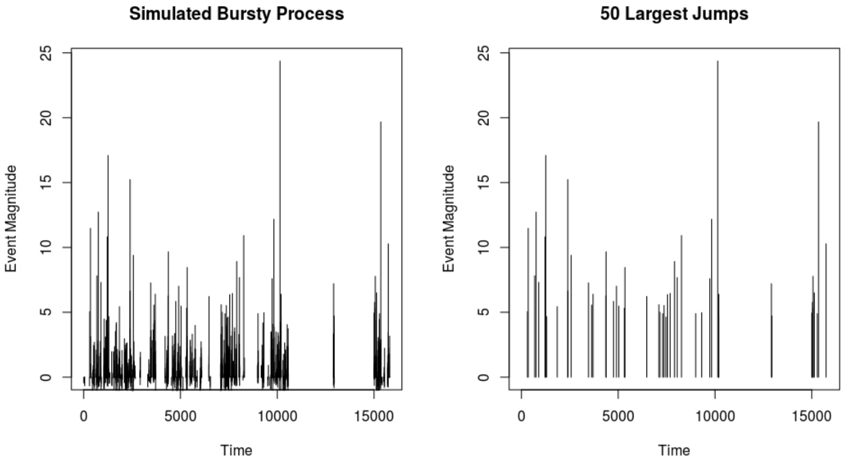
Waiting times W_i are modelled according to a stable distribution with stability parameter $\beta \in (0, 1)$, skewness parameter 1, location parameter 0, and scale parameter $=1$.

Jump sizes J_i are modelled according to Generalized Extreme Value (GEV) distribution with location parameter μ, scale parameter σ and shape parameter ψ.

The primary goal is to design methodology that fits models to data sets of bursty events.

Simulated data

$$\mu = 0, \sigma = 1, \psi = 0.3, \beta = 0.7$$



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Distribution of Durations

Proposition (Meerschaert and Stoev (2007))

Let F be the cdf of a GEV random variable. Let a be the threshold level, then define T_a as the duration between jumps that have been thresholded at the a-level. Then

$$T_a \sim (-\log F(a))^{\frac{-1}{\beta}} X^{\frac{1}{\beta}} D(1).$$

Where $D(1)$ is a random variable of the stable distribution with stability parameter β and skewness parameter 1 and where X is a standard exponential random variable.

Distribution of Durations Contd.

After taking the logarithms of both sides we arrive at

$$\log T_a \sim \frac{1}{\beta}\log X + \log D(1) - \frac{1}{\beta}\log(-\log F(a)).$$

Now we have that,

$$\begin{aligned} f_{\frac{1}{\beta}\log X}(x) &= \frac{d}{dx}\mathbb{P}\left(\frac{1}{\beta}\log X \leq x\right) \\ &= \frac{d}{dx}\mathbb{P}(X \leq e^{x\beta}) \\ &= f_X(e^{x\beta})\beta e^{x\beta}. \end{aligned}$$

Distribution of Durations Contd.

We also have,

$$\begin{aligned} f_{\log D}(x) &= \frac{d}{dx}\mathbb{P}(\log D \leq x) \\ &= \frac{d}{dx}\mathbb{P}(D \leq e^x) \\ &= f_D(e^x)e^x. \end{aligned}$$

Using convolution we arrive at

$$f_{\frac{1}{\beta}\log X+\log D}(x) = \int_{-\infty}^{\infty} f_{\frac{1}{\beta}\log X}(x-y) f_{\log D}(y)dy.$$

Shifting the above expression to the right by $\frac{1}{\beta}\log(-F(a))$ gives us the density of $\log T_a$. That is,

$$f_{\log T_a}(x) = \int_{-\infty}^{\infty} f_{\frac{1}{\beta}\log X}\left(x - y + \frac{1}{\beta}\log(-\log F(a))\right) f_{\log D}(y)dy.$$

Distribution of Durations Contd.

Substituting our expressions for $f_{\log D}(x)$ and $f_{\frac{1}{\beta}\log X}(x)$ we get

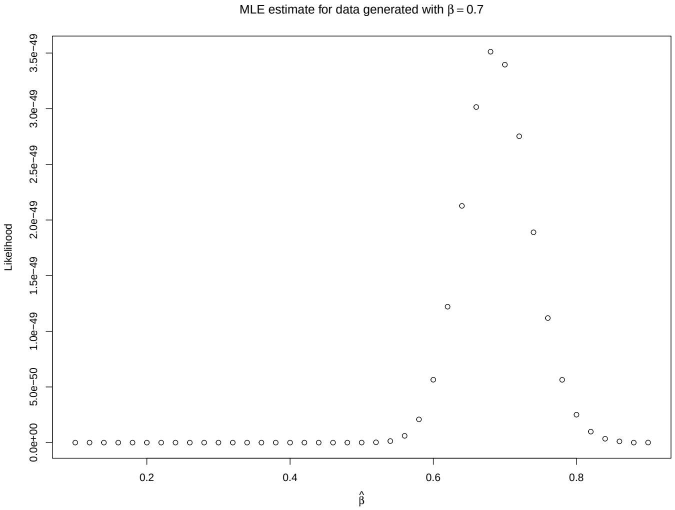
$$f_{\log T_a}(x) = \int_{-\infty}^{\infty} f_X(e^{\beta(x-y+\frac{1}{\beta}\log(-\log F(a)))})\beta e^{\beta(x-y+\frac{1}{\beta}\log(-\log F(a)))} f_D(e^y)e^y\,dy$$

which simplifies to

$$f_{\log T_a}(x) = \int_{-\infty}^{\infty} -\log F(a) f_X(-\log F(a)e^{\beta(x-y)})\beta e^{\beta(x-y)} f_D(e^y)e^y\,dy.$$

Likelihood Profile

The above density was used in order to calculate an MLE for the simulated data and the following likelihood profile was generated.



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Convergence of Waiting Times

Theorem (Meerschaert and Sikorskii (2011))

Suppose that W_i are i.i.d. and positive with $\mathbb{P}(W_n > t) = ct^{-\beta}$ for all $t > c^{1/\beta}$, some $c > 0$ and $0 < \beta < 1$. Then

$$n^{-1/\beta}(W_1 + ... + W_n) \to D \text{ as } n \to \infty.$$

Where D is distributed according to a one-sided stable distribution.

This theorem can be generalised for any waiting times that follow a heavy tailed distribution (i.e mean is not finite) as follows

$$a_n S(n) \Rightarrow D.$$

Convergence of Waiting Times Contd.

Since we will be working in continuous time, we need to define partial processes.

Define the partial sum-process as $S(t) := \sum_{i=1}^{\lfloor t \rfloor} W_i$

Similarly let the partial max-process be $M(t) := \bigvee_{i=1}^{\lfloor t \rfloor} J_i$

Theorem (Meerschaert and Sikorskii (2011))

Given $a_n S(n) \Rightarrow D$, where the sequence of positive constants $\{a_n\}$ is regular varying with index $-1/\beta, (0 < \beta < 1)$, then writing $a(t) := a_t$,

$$\{a(c)S(ct)\}_{t\geq 0} \xrightarrow[c\to\infty]{J_1} \{D(t)\}_{t\geq 0},$$

where $\{D(t)\}_{t\geq 0}$ is β-stable subordinator.

Convergence of Maxima

Theorem (Lamperti (1964))

Recall $M(n) = \bigvee_{i=1}^{n} J_i$. Suppose there exists constants $b_n > 0$ and d_n such that,

$$\mathbb{P}(M_n \leq b_n x + d_n) = F^n(b_n x + d + n) \Rightarrow G(x).$$

Now set

$$A^c(t) = \begin{cases} b_c(M_{\lfloor ct \rfloor} - d_c), & t \geq 1/c \\ b_c(J_1 - d_c), & 0 < t < 1/c. \end{cases}$$

Then $A^c \xrightarrow[c\to\infty]{J_1} A$, where $\{A(t)\}_{t\geq 0}$ is an extremal process generated by G. That is

$$\{b(c)(M(ct) - d(c))\}_{t\geq 0} \xrightarrow[c\to\infty]{J_1} \{A(t)\}_{t\geq 0}.$$

Notation for Scaled W_i and J_i

Define $W_i^c := a(c)W_i$ and $J_i^c = b(c)(J_i - d(c))$.

Define the scaled partial sum-process as $S^c(t) := \sum_{i=1}^{\lfloor ct \rfloor} W_i^c$

Define the scaled partial max-process as $M^c(t) := \bigvee_{i=1}^{\lfloor ct \rfloor} J_i^c$

Define a renewal process $N(t) := \max\{n \geq 0 : S^c(n) \leq t\}$

Finally we define the scaled Continuous Time Random Maxima (CTRM) to be $V^c(t) := \bigvee_{i=1}^{N^c(t)} J_i^c$

Convergence of Joint Process

Theorem

Let (W_i^c, J_i^c) be a sequence of i.i.d $\mathbb{R}^+ \times \mathbb{R}$ random vectors such that

$$\{S^c(t), M^c(t)\}_{t\geq 0} \xrightarrow[c\to\infty]{J_1} \{(D(t), A(t))\}_{t\geq 0}$$

where the paths of $\{D(t))\}_{t\geq 0}$ are non-decreasing almost surely. Then,

$$\{V^c(t)\}_{t\geq 0} \xrightarrow[c\to\infty]{J_1} \{(A_- \circ E)_+(t)\}_{t\geq 0},$$

where $E := \inf\{u > 0 : D(u) > t\}$ is the inverse of D.