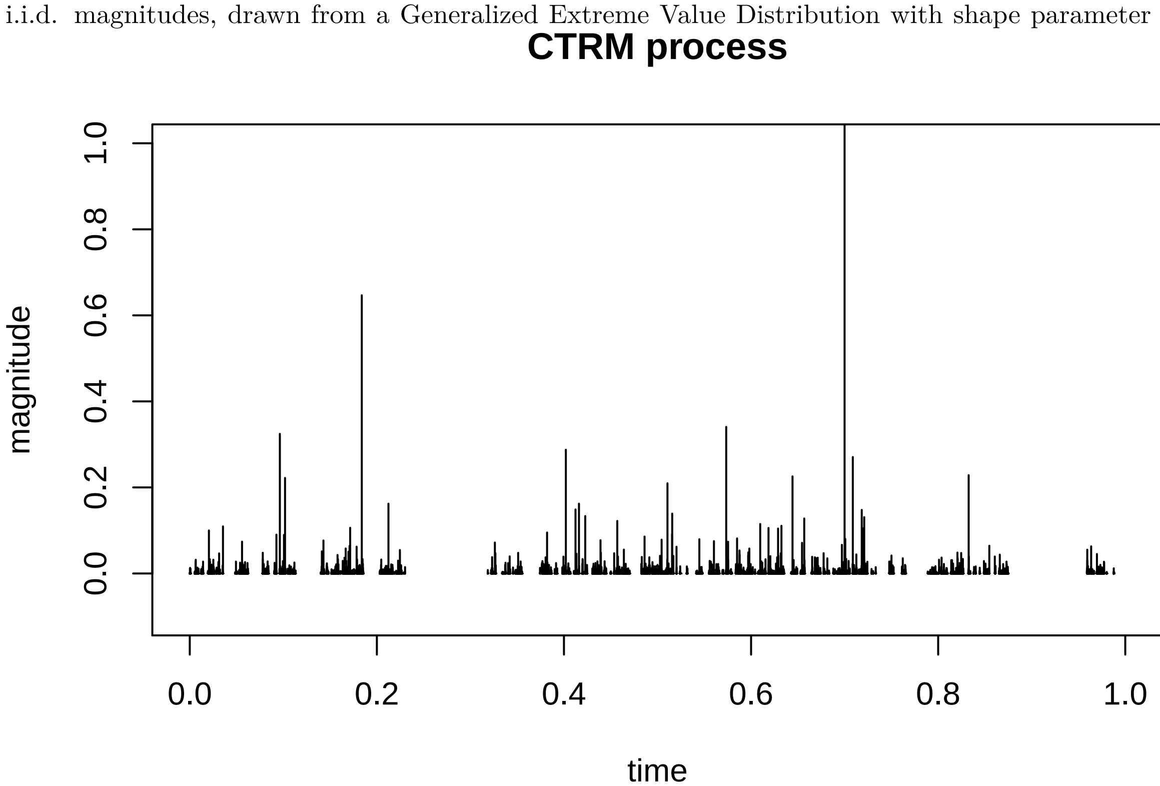
# Simulating and Estimating CTRM processes

### Simulation

We assume i.i.d. waiting times drawn from the positively skewed stable distribution with stability parameter $\beta = 0.8$, scaled with $n^{-1/\beta}$. This defines a renewal process, at whose renewal times we assume i.i.d. magnitudes, drawn from a Generalized Extreme Value Distribution with shape parameter $\xi = 0.7$:




Besides the magnitude of large events, their *timing* is of interest to us. We define the exceedance time of level $\ell \in [x_0, x_F]$ as the random variable

$$T_\ell = \inf\{t : M(t) > \ell\}$$

and the exceedance as

$$X_\ell = M(T_\ell) - \ell.$$

### Probability distribution of Exceedance Times

### Result by Anderson (1987)

Anderson (1987) has shown that as $\ell \uparrow x_F$ (the right end-point of the distribution of magnitudes) the following weak convergence holds:

$$\frac{T_\ell}{n(1/(1 - F_J(\ell)))} \Rightarrow W_\beta. \tag{1}$$

Here,

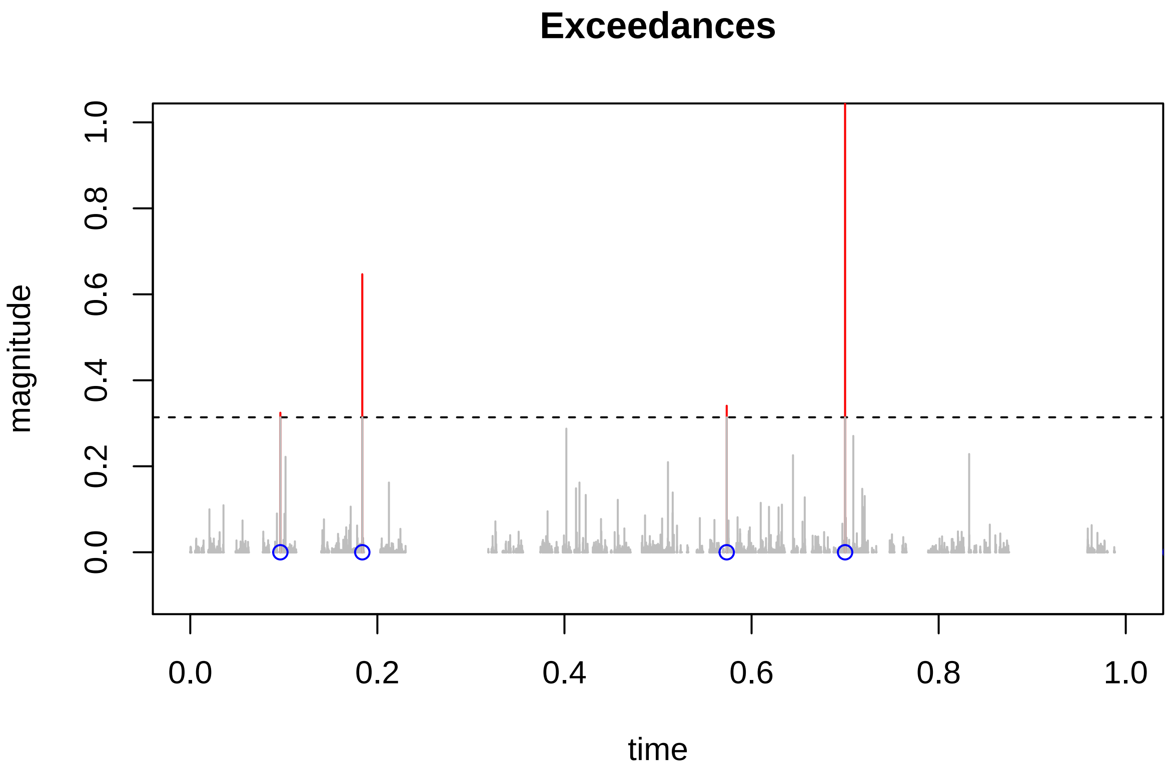



Figure 1: Exceedance times (blue circles) and Exceedance sizes (red lines).

- $F_J(\cdot)$ is the CDF of the magnitudes $J_1, J_2, \ldots$
- $n(\cdot)$ is a norming function which varies regularly at $\infty$ with parameter $1/\beta$
- $W_\beta$ is a Mittag-Leffler random variable, defined via its Laplace transform

$$\mathbf{E}[\exp(-\lambda W_\beta)] = \frac{1}{1+\lambda^\beta}. \tag{2}$$

More precisely, $n(t)$ is an asymptotic inverse to the function

$$g(t) := \frac{t}{\Gamma(2-\beta)\int_0^t (1-F_W(u))\,du} \in RV_\infty(\beta) \tag{3}$$

**Approach in this paper**

Meerschaert and Stoev (2008) have derived a scaling limit theorem for the Continuous Time Random Maxima (CTRM) process

$$M(t) = \bigvee_{k=1}^{N(t)} J_k \tag{4}$$

where $N(t)$ is the renewal process as above: Assume that

- $\left[\bigvee_{k=1}^{\lfloor ct \rfloor} J_k - d(c)\right] / a(c) \Rightarrow A(t)$
- $b(c)^{-1} \sum_{k=1}^{\lfloor ct \rfloor} W_k \Rightarrow D(t)$

converge (weakly in Skorokhod space) for some norming sequences $a(c)$, $d(c)$ and $b(c)$. Then

$$\frac{M(ct) - d(\tilde{b}(c))}{a(\tilde{b}(c))} \Rightarrow A(E(t)) \tag{5}$$

where $\tilde{b}(c)$ is asymptotically inverse to $b(c)$ and $E(t) = \inf\{r : D(r) > t\}$ is the stochastic process inverse to $D(t)$.

For large $c$ and $t$ comparable to $c$, we may hence approximate

$$M(t) \approx a(\tilde{b}(c))A(E(t/c)) + d(\tilde{b}(c)). \tag{6}$$

Recalling that $c \sim b(\tilde{b}(c))$, may substitute $n = \tilde{b}(c)$ to get

$$M(t) \approx a(n)A(E(t/b(n))) + d(n). \tag{7}$$

Then we have

$$T_\ell > t \Longleftrightarrow M(t) \le \ell \overset{\approx}{\Longleftrightarrow} A(E(t/b(n))) \le \frac{\ell - d(n)}{a(n)} =: \ell^* \tag{8}$$

$$\Longleftrightarrow \xi_{\ell^*} > t/b(n) \Longleftrightarrow b(n)\xi_{\ell^*} > t \tag{9}$$

where $\xi_a := \inf\{t : A(E(t)) > a\}$ is the hitting time of level $a$ by the process $A(E(t))$. Hence we may approximate the distribution of $T_\ell$ by the distribution of the random variable $\xi_{\ell^*}$, rescaled with $b(n)$. It was shown by Meerschaert & Stoev (2008) that

$$\xi_a \overset{d}{=} (-\log F_A(a))^{-1/\beta} W_\beta$$

where $F_A(\cdot)$ is the CDF of $A := A(1)$. Summing up, the exceedance time $T_\ell$ is asymptotically Mittag-Leffler distributed:

$$T_\ell \overset{a}{\sim} \mathrm{ML}\left(\beta, b(n)[-\log F_A(\ell^*)]^{-1/\beta}\right) \tag{10}$$

## Estimation

Since

$$\bigvee_{k=1}^{n} J_k \le \ell \overset{\approx}{\Longleftrightarrow} A \le \ell^*, \tag{11}$$

we may estimate $F_A(\ell^*)$ empirically by

$$\varepsilon(\ell^*) := 1 - F_A(\ell^*) \approx \#\{k : J_k \leq \ell\}/n. \tag{12}$$

Assume now a time series of magnitudes, and that interest lies in the estimation of the timings of the large magnitudes. Consider a minimum threshold $\ell_0$, e.g. at the 0.9 quantile. Vary the threshold $\ell$ on the interval $[\ell_0, x_F]$, and consider the resulting sequences of exceedance sizes and exceedance times $\{(X_{\ell,i}, T_{\ell,i})\}$. Due to the renewal property, this sequence is i.i.d., and $T_{\ell,1}, T_{\ell,2}, \ldots$ can be modelled by a Mittag-Leffler distribution. We use the method of log-moments estimator, taken from Cahoy (2012), which provides a point-estimate and an asymptotically normal confidence interval, at default confidence level 95%.

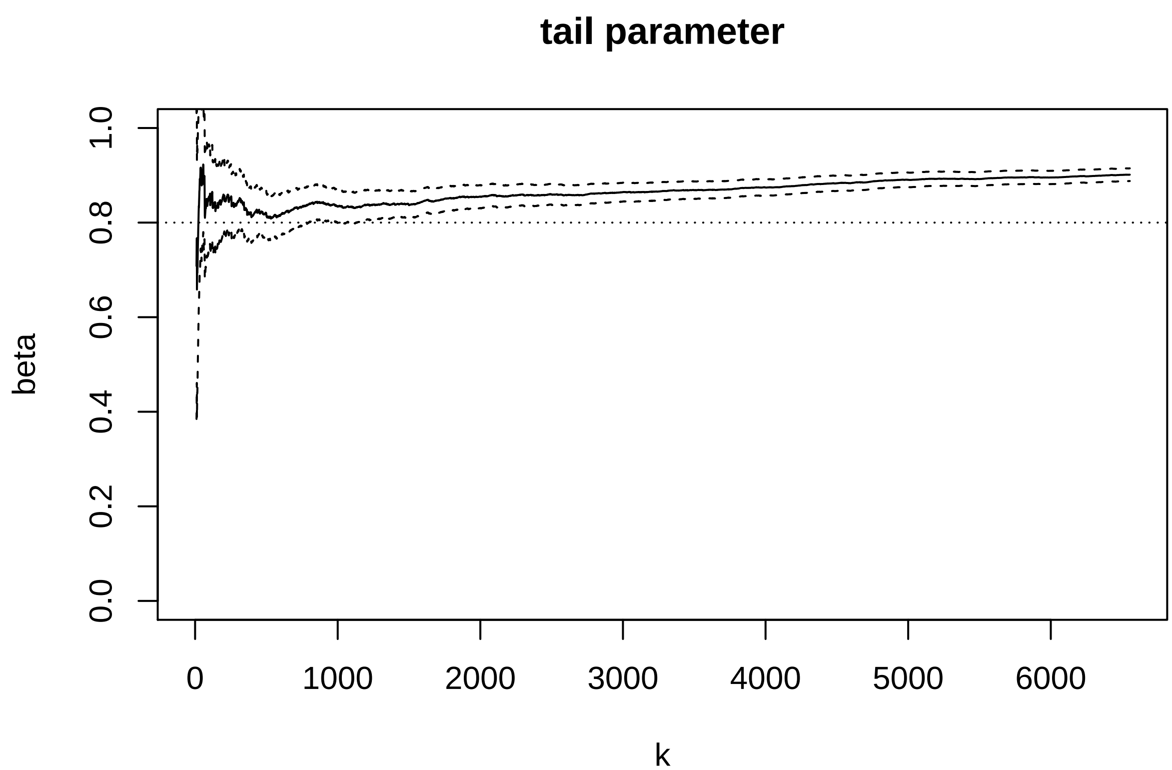



Figure 2: Estimate of tail parameter of Mittag Leffler distribution (y-axis), for observations thresholded at the top k observations (x-axis). The dotted line represents the correct value.

Since the scale parameter $\delta := -b(n)[\log(1 - \varepsilon(\ell^*))]^{-1/\beta}$ depends on the threshold $\ell^*$ as well as on the tail parameter $\beta$, we plot the estimate of $b(n)$

$$b(n) = \delta \times (-\log(1 - \varepsilon(\ell^*)))^{1/\beta} \tag{13}$$

with an estimate for $\beta$ plugged in from the previous step.

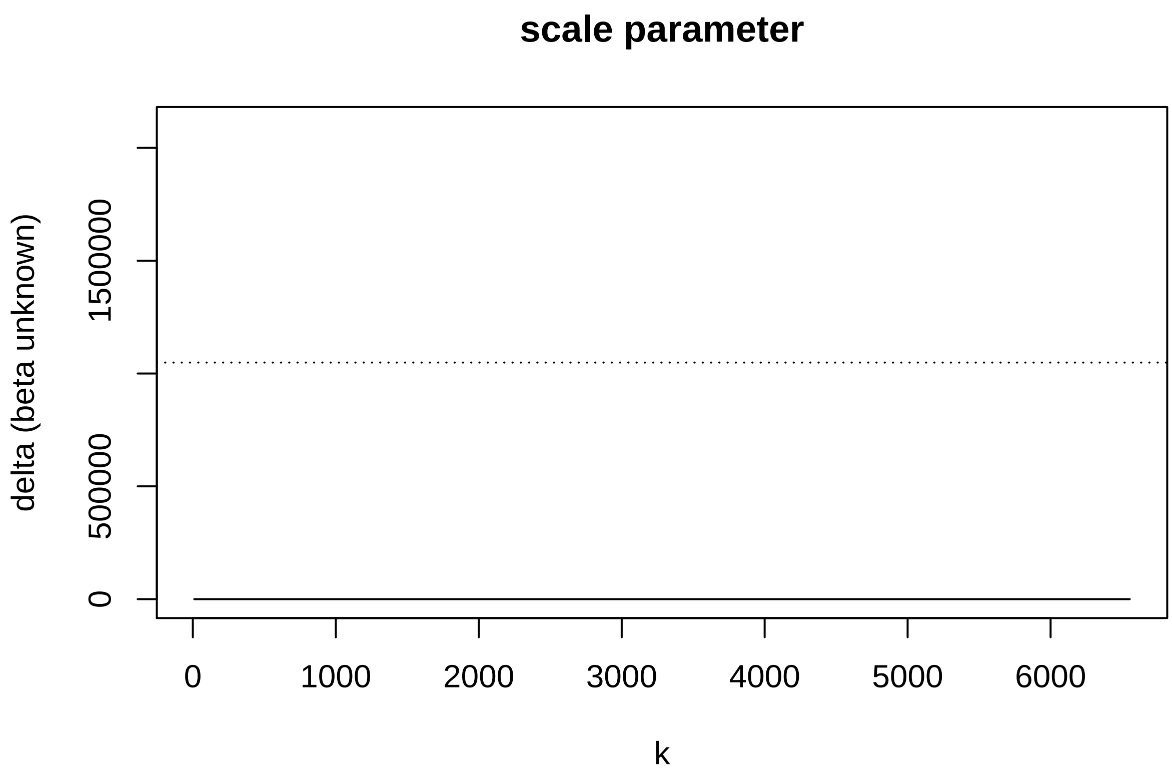



Figure 3: Estimate of scale parameter of Mittag Leffler distribution (y-axis), for observations thresholded at the top k observations (x-axis). The dotted line represents the true scaling parameter.