Theory of Algorithms Homework 3

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1 Problem 1

Find the Big Oh bounds of the following recursions using substitution.

(a) $T(n) = T(n-2) + 1$

$$\begin{aligned}
&\text{Guess: } cn\\
&T(n) \leq cn\\
&T(n-2) \leq cn - 2c\\
&T(n) = T(n-2) + 1\\
&\qquad \leq cn - 2c + 1\\
&-2c + 1 = 0\\
&c = \tfrac{1}{2}\\
&cn - 2c + 1 = \tfrac{1}{2}n - \tfrac{1}{2}2 + 1\\
&\qquad = \tfrac{1}{2}n\\
&\qquad = cn
\end{aligned}$$

$$\therefore T(n) = O(n)$$

(b) $T(n) = T(n-2) + n$

$$\begin{aligned}
&\text{Guess: } cn^2 + bn\\
&T(n) \leq cn^2 + bn\\
&T(n-2) \leq cn^2 - 4cn + 4c + bn - 2b\\
&T(n) = T(n-2) + n\\
&\qquad \leq cn^2 - 4cn + 4c + bn - 2b + n\\
&\qquad = cn^2 + (1 + b - 4c)n - 2b + 4c\\
&1 + b - 4c = b\\
&4c = 1\\
&c = \tfrac{1}{4}\\
&-2b + 1 = 0
\end{aligned}$$

$2b = 1$

$b = \frac{1}{2}$

$cn^2 - 4cn + 4c + bn - 2b + n = \frac{1}{4}n^2 - 4\frac{1}{4}n + 4\frac{1}{4} + \frac{1}{2}n - 2\frac{1}{2} + n$

$= \frac{1}{4}n^2 - n + 1 + \frac{1}{2}n - 1 + n$

$= \frac{1}{4}n^2 + \frac{1}{2}n$

$= cn^2 + bn$

$$\therefore T(n) = O(n^2)$$

(c) $T(n) = T(\frac{n}{2}) + 1$

Guess: $c\ lg(n)$

$T(n) \leq c\ lg(n)$

$T(\frac{n}{2}) \leq c\ lg(n) - c$

$T(n) = T(\frac{n}{2}) + 1$

$= c\ lg(n) - c + 1$

$c = 1$

$c\ lg(n) - c + 1$

$= 1 \times lg(n)$

$= c\ lg(n)$

$$\therefore T(n) = O(lg(n))$$

(d) $T(n) = T(\frac{n}{2}) + n + 3$

Guess: $cn + b\ lg(n)$

$T(n) \leq cn + b\ lg(n)$

$T(\frac{n}{2}) \leq \frac{c}{2}n + b\ lg(n) - b$

$T(n) = T(\frac{n}{2}) + n + 3$

$= \frac{c}{2}n + b\ lg(n) - b + n + 3$

$\frac{c}{2} + 1 = c$

$c + 2 = 2c$

$c = 2$

$-b + 3 = 0$

$b = 3$

$\frac{c}{2}n + b\ lg(n) - b + n + 3 = \frac{2}{2}n + 3\ lg(n) - 3 + n + 3$

$= cn + b\ lg(n)$

$$\therefore T(n) = O(n)$$

(e) $T(n) = 8T(\frac{n}{2}) + n^2$

Guess: $n^3 + bn^2$

$T(n) \leq n^3 + bn^2$

$T(\frac{n}{2}) \leq \frac{1}{8}n^3 + \frac{b}{4}n^2$

$T(n) = 8T(\frac{n}{2}) + n^2$

$= 8(\frac{1}{8}n^3 + \frac{b}{4}n^2) + n^2$

$= n^3 + 2bn^2 + n^2$

$2b + 1 = b$

$b + 1 = 0$

$b = -1$

$n^3 + 2bn^2 + n^2 = n^3 - 2n^2 + n^2$

$= n^3 - n^2$

$= n^3 + bn^2$

$$\therefore T(n) = O(n^3)$$

(f) $T(n) = 5T(\frac{n}{4}) + n$

Guess: $n^{\log_4(5)} + bn$

$T(n) \leq n^{\log_4(5)} + bn$

$T(\frac{n}{4}) \leq (\frac{n}{4})^{\log_4(5)} + \frac{b}{4}n$

$T(n) = 5T(\frac{n^{\log_4(5)}}{4^{\log_4(5)}} + \frac{b}{4}n) + n$

$= 5(\frac{n^{\log_4(5)}}{5} + \frac{b}{4}n) + n$

$= n^{\log_4(5)} + \frac{5}{4}bn + n$

$\frac{5}{4}b + 1 = b$

$5b + 4 = 4b$

$b = -4$

$n^{\log_4(5)} + \frac{5}{4}bn + n = n^{\log_4(5)} + -\frac{5}{4}4n + n$

$= n^{\log_4(5)} - 4n$

$= n^{\log_4(5)} + bn$

$$\therefore T(n) = O(n^{\log_4(5)})$$

(g) $T(n) = T(\frac{n}{2}) + lg\ (n)$

Guess: $c\ lg^2(n) + b\ lg(n)$

$T(n) \leq c\ lg^2(n) + b\ lg(n)$

$T(\frac{n}{2}) \leq c\ lg^2(n) - 2c\ lg(n) + c + b\ lg(n) - b$

$$
\begin{aligned}
T(n) &= T(\tfrac{n}{2}) + lg(n) \\
&= c\ lg^2(n) - 2c\ lg(n) + c + b\ lg(n) - b + lg(n) \\
&= c\ lg^2(n) + (1 - 2c + b)\ lg(n) + c - b
\end{aligned}
$$

$c = b$

$1 - 2c + b = b$

$1 = 2c$

$b = c = \frac{1}{2}$

$$
\begin{aligned}
c\ lg^2(n) + (1 - 2c + b)\ lg(n) + c - b &= \tfrac{1}{2}\ lg^2(n) + (1 - 2\tfrac{1}{2} + \tfrac{1}{2})\ lg(n) + \tfrac{1}{2} - \tfrac{1}{2} \\
&= \tfrac{1}{2}\ lg^2(n) + (1 - 1 + \tfrac{1}{2})\ lg(n) \\
&= \tfrac{1}{2}\ lg^2(n) + \tfrac{1}{2}\ lg(n) \\
&= c\ lg^2(n) + b\ lg(n)
\end{aligned}
$$

$$\therefore T(n) = O(lg^2(n))$$

2 Problem 2

Find the Θ bounds for the following recursions using the master method (if possible).

(a) $T(n) = 4T(\frac{n}{2}) + n$

$a = 4$

$b = 2$

$n^{\log_2(4)} = n^2$

$f(n) = n = n^{2-1} = O(n^{2-\epsilon})$ for $\epsilon = 1$

$\therefore$ by MM case 1, $T(n) = \Theta(n^2)$

(b) $T(n) = 4T(\frac{n}{2}) + n^2$

$a = 4$

$b = 2$

$n^{\log_2(4)} = n^2$

$f(n) = n^2 = \Theta(n^2)$

$\therefore$ by MM case 2, $T(n) = \Theta(n^2\ lg(n))$

(c) $T(n) = 3T(n-1) + n$

Not solvable by master method since it doesn't fit the form $a\ T(\frac{n}{b}) + f(n)$

(d) $T(n) = T(\frac{n}{3}) + n$

$a = 1$

$b = 3$

$n^{\log_3(1)} = n^0 = 1$

$f(n) = n = \Omega(n^{0+\epsilon})$ for $\epsilon = 1$.

WTS $a\ f(\frac{n}{b}) \leq c\ f(n)$ for some $0 < c < 1$

$a\ f(\frac{n}{b}) = \frac{n}{3} \leq cn$

$\frac{1}{3} \leq c$

for $\frac{1}{3} \leq c < 1$, $a\ f(\frac{n}{b}) \leq c\ f(n)$

$\therefore$ by MM case 3, $T(n) = \Omega(n)$

(e) $T(n) = T(\frac{n}{4}) + n\ lg(n)$

$a = 1$

$b = 3$

$n^{\log_3(1)} = n^0 = 1$

Unsolvable since $n^{0+\epsilon} \neq n\ lg(n)$ for any $\epsilon > 0$

(f) $T(n) = T(\frac{n}{2}) + T(\frac{n}{3}) + n^2$

Not solvable by master method since it doesn't fit the form $a\ T(\frac{n}{b}) + f(n)$

(g) $T(n) = 4T(\frac{n}{4}) + n + 2$

$a = 4$

$b = 4$

$n^{\log_4(4)} = n^1$

$f(n) = n + 2 = \Theta(n^1)$

$\therefore$ by MM case 2, $T(n) = \Theta n\ lg(n)$

(h) $T(n) = 4T(\frac{n}{2}) + n^2\ lg(n)$

$a = 4$

$b = 2$

$n^{\log_2(4)} = n^2$

Unsolvable since $n^{0+\epsilon} \neq n^2\ lg(n)$ for any $\epsilon > 0$