# Problem B5 from the 1995 Putnam Exam

Evan Dreher

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## 1 The Problem

A game starts with four heaps of beans, containing 3, 4, 5, and 6 beans. The two players move alternately. A move consists of taking **either**

1. one bean form a heap, provided at least two beans are left behind in that heap
2. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

## 2 The Solution

Let $p_1$ and $p_2$ denote the two players of the game where $p_1$ goes first.

First consider a game played by the same rules but with only 1 heap of $n$ beans where $n \geq 3$. If $n$ is odd, $p_1$ will win because they will reduce it to an even number, and player 2 will reduce it back to odd. This will repeat until player 1 is left with 3 beans in the heap and they will take the heap and win the game. Likewise if $n$ is even $p_2$ will win.

Now consider the same game but with only two piles of size $n$ and $k$ where both are greater than 3.

If $n$ and $k$ are even, $p_1$ will lose. Regardless of which heap they choose from they will reduce a pile to an odd quantity. If they reduce a pile to size 3, $p_2$ can simply take the whole pile and leave an even pile on $p_1$'s turn which wins the game for $p_2$. If instead the odd pile is greater than size 3, $p_2$ will simply reduce it back to an even number and repeat until victory.

If $n$ and $k$ are odd, $p_1$ will also lose. $p_1$ will reduce a pile to an even quantity.$p_2$ will then take one from the other pile reducing it to an even quantity as well thus leaving $p_1$ with $n$ and $k$ even which has been previously established to be a win for $p_2$

Thus if only one of $n$and $k$ is even, it will be a win for $p_1$ since they can simply reduce the odd pile to an even one and give $p_2$ the losing position.

Now consider the same game but with three piles the three piles greater than size 3 denoted $n$, $k$, and $x$. I assert that if the game starts like this and $n+k+x \mod 2 = 1$, then $p_1$ will win.

*Proof.* Since each pile is larger than three, then eventually a move will be made the reduce a pile from size 3 to size 4 and the other boards will both be greater than 3. The current player, denoted $p_c$ will then consider the quantities of the other two boards, denoted $n$ and $k$. If $n \mod 2 = k \mod 2$, then $p_c$ will simply take the entirety of the pile with 3 beans leaving the other player with a lost position. If $n \mod 2 \neq k \mod 2$, then $p_c$ will reduce the pile of size 3 to size two. If the other player takes the pile of size 2, then it is $p_c$'s turn when $n \mod 2 \neq k \mod 2$ which was previously established to be a win for $p_c$. If they do not, then $p_c$ takes the pile of two leaving the other player with a board where $n \mod 2 = k \mod 2$ which was previously established to be a win for $p_c$.

thus in all cases, once one of the boards is reduced to size 3 for the first time, $p_c$ wins the game. Now let the board state be $n$, $k$, $x$ while it is $p_1$'s turn. $p_1$ (knowing that they cannot reduce a pile to size 3 without losing the game) will avoid drawing from the pile of size 4 and draw from any pile larger than that. But since $p_2$ also knows this, they will do the same. With both sides avoiding piles of size 4, there will be $n-4+k-4+x-4$ moves before a player is forced to draw from a pile of size 4. If $n-4+k-4+x-4$ is even, then an even number of moves before a player is forced to reduce a board to size 3. After an even number of moves it is $p_1$'s turn thus $p_1$ will be forced to make a board of size 3 and lose. If instead $n-4+k-4+x-4$ then $p_2$ will be the one forced to make a board of size 3 and lose. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$

Now consider the actual game. The one with board state 3, 4, 5, 6. I assert that $p_1$ wins.

*Proof.* $p_1$ begins by reducing the pile of size 3 to size 2. If $p_2$ responds by taking the remainder of the pile then it is $p_1$'s turn on the board state 4,5,6 which is a win for $p_1$ since $0+1+2 \mod 2 = 1$. If $p_2$ response by taking from the pile of size 5 or 6, $p_1$ takes the remainder of the pile of size two leaving it to be $p_2$'s turn on either 4,4,6 or 4,5,5 which is a loss for $p_2$ since the sum of the values mod 2 is 0. If $p_2$ responds by reducing the pile of size 4 to size 3, $p_1$ will respond by taking the remainder of the pile. This leaves the board state 2,5,6 when it is $p_2$'s turn. If $p_2$ takes the pile of size 2, then it is $p_1$'s turn when $n \mod 2 \neq k \mod 2$ which is a win for $p_1$. If instead $p_2$ takes from one of the other piles, $p_2$ will take the remainder of the pile of size 2 leaving it to be $p_2$'s turn when $n \mod 2 = k \mod 2$ which is a loss for $p_2$. In all cases, $p_1$ wins by this strategy, therefore, in order to win the game you want to move first. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$