Problem 1 from the 2005 Putnam Exam

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1 The Problem

Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

2 The Solution

Let $P(n)$ be the n-tuple with components that sum to n and do not divide any other component in the tuple. We begin be defining bases cases.

- $P(1) = (2^0 3^0)$
- $P(2) = (2^1 3^0)$
- $P(3) = (2^0 3^1)$
- $P(4) = (2^2 3^0)$
- $P(5) = (2^1 3^0, 2^0 3^1)$
- $P(6) = (2^1 3^1)$
- $P(7) = (2^2 3^0, 2^0 3^1)$

For the sake of the proof later on, we will rewrite $P(5)$ and $P(7)$ as follows

- $P(5) = (2^1, 3^1, 6 \times 0)$
- $P(7) = (2^2, 3^1, 6 \times 0)$

Note that a scalar multiple times an n-tuple merely multiplies each component of the tuple by the scalar, i.e $3 \times P(5) = (2^2 3^1, 2^0 3^2)$. Also note that if a number n has a solution (i.e $P(n)$) exists, then the solution for any number that can be expressed as $2^a 3^b \times n$ is $2^a 3^b \times P(n)$.

Lemma 2.1. *Let y be the smallest positive integer that satisfies $3^b|6y$ for some $b \in \mathbb{Z}^+$. By definition of divides, $6y = p \times 3^b$ for some integer p. Since $6y$ is even and 3^b is odd, we know that p must be even, and since y is the smallest number to satisfy the equation, then p must also be the smallest number to satisfy the equation, thus $p = 2$. Furthermore,*

$$6y + 3^b = 3^{b+1}$$

Lemma 2.2. *Let y be the smallest positive integer that satisfies $2^b|6y$ for some $b \in \mathbb{Z}^+$. By definition of divides,*

$$6y = p \times 2^b$$

$$3y = p \times 2^{b-1}$$

Since $3y$ has and odd factor, and 2^b has none, we know that p have at least one odd factor, and since y is the smallest number to satisfy the equation, then p must also be the smallest number to satisfy the equation, thus $p = 3$. Furthermore,

$$6y + 3^b = 2^{b+2}$$

Proof. We proceed by induction on $P(n)$ solving by case.

Basis Step: $1 \leq n \leq 7$
Solutions for $P(1)...P(7)$ have already been found so the basis step holds.

Inductive Step:
Assume there exists some $k \geq 7$ such that solutions for $P(1)...P(k)$ exist.

Case 1: $n \mod 6 = 0$
By definition of modulo, $n = 6x$ for some x.

$$\begin{aligned} n &= 6x \\ n &= 2^1 3^1 \times x \\ P(n) &= 2^1 3^1 \times P(x) \end{aligned}$$

Thus a solution exists for $P(n)$

Case 2: $n \mod 6 = 1$
By definition of modulo, $n = 6x + 1$ for some x. By our assumption, a solution exists for $P(n - 6)$ and be previous work we know it will be of the form $(2^a, 3^b, 6x)$. $6x$ does not divide 2^a or 3^b for any a, b, or x. If 2^a does not divide $6(x + 1)$ and neither does 3^b, then $P(n) = (2^a, 3^b, 6(x + 1))$. If $2^b|6(x + 1)$, then by Lemma 2.2, $P(n) = (2^{a+2}, 3^b, 0)$, otherwise if $3^b|6(x + 1)$ then by Lemma 2.1, $P(n) = (2^a, 3^{b+1}, 0)$. In all cases, a solution exists for $P(n)$.

Case 3: $n \mod 6 = 2$
By definition of modulo, $n = 6x + 2$ for some x.

$$\begin{aligned} n &= 6x + 2 \\ n &= 2^1 \times (3x + 1) \\ P(n) &= 2^1 \times P(3x + 1) \end{aligned}$$

Thus a solution exists for $P(n)$

Case 4: $n \mod 6 = 3$
By definition of modulo, $n = 6x$ for some x.

$$\begin{aligned} n &= 6x \\ n &= 2^1 3^1 \times x \\ P(n) &= 2^1 3^1 \times P(x) \end{aligned}$$

Thus a solution exists for $P(n)$

Case 5: $n \mod 6 = 4$

By definition of modulo, $n = 6x$ for some x.

$$\begin{aligned} n &= 6x + 4 \\ n &= 2^1 \times (3x + 2) \\ P(n) &= 2^1 \times P(3x + 2) \end{aligned}$$

Thus a solution exists for $P(n)$

Case 6: $n \mod 6 = 5$
By definition of modulo, $n = 6x + 5$ for some x. By our assumption, a solution exists for $P(n-6)$ and be previous work we know it will be of the form $(2^a, 3^b, 6x)$. $6x$ does not divide 2^a or 3^b for any a, b, or x. If 2^a does not divide $6(x+1)$ and neither does 3^b, then $P(n) = (2^a, 3^b, 6(x+1))$. If $2^b | 6(x+1)$, then by Lemma 2.2, $P(n) = (2^{a+2}, 3^b, 0)$, otherwise if $3^b | 6(x+1)$ then by Lemma 2.1, $P(n) = (2^a, 3^{b+1}, 0)$. In all cases, a solution exists for $P(n)$.

Therefore, by the principal of strong mathematical induction, $P(n)$ is true for all positive integers. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$