Problem A2 from the 2005 Putnam Exam

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1 The Problem

Let $S_n = \{(a, b)|1 \leq a \leq n, 1 \leq b \leq 3\}$. A *rook tour* of S_n is a polygonal path made up of line segments connecting points $p_1, p_2 ... p_{3n}$ in sequence such that

- $p_i \in S_n$
- p_i and p_{i+1} are a unit distance apart for $1 \leq i < 3n$
- for each $p \in S_n$ there is a unique i such that $p_i = p$.

How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$

2 The Solution

Let $P(n)$ denote the number of rook tours which can be formed on S_n. S_n will have n columns in it. The rook will always enter a column c for the first time from the left. Furthermore it will enter on the first or third row as entering on the second row would cause $(1, c)$ and/or $(3, c)$ to become unreachable, from here I will use the term "breaking move" to refer to a move the rook makes in which it enters a new column. The rook will make $n-1$ breaking moves on S_n, however the breaking move for the last column must always be on the third row since entering on the first row would put the rook on the square (1,n) too early. So the rook gets to choose $n-2$ breaking moves with two options per chioce. This means that $P(n)$ has an upper bound of 2^{n-2}. I assert however that for every possible combination of breaking moves has 1 valid rook tour associated with it, thus

$$P(n) = \lfloor 2^{n-2} \rfloor$$

Note that the floor operator is used so that $P(1) = 0$, but for $n > 1$, $P(n)$ is simply 2^{n-2}

Proof. Consider a rook that has made a breaking moves along the third row. This sequence of moves must be contiguous in the path since each vertex connects to the next one. If $a = n$, then the path the rook took is $(1, 1)...(3, 1)...(3, n)$. The only way for it to complete the path is to continue like so $(3, n)...(2, n)...(2, 2)$

...$(1,2)$...$(1,n)$. If instead $a < n$, then at somepoint the breaking moves will switch from the third row to the first row. To connect the two moves, the rook need simply move up two spaces, but since a contiguous breaking moves were made along the third row, the previous $(a-1)$ columns have not had their 1st and 2nd row vertices visited yet and traveling up two spaces would split the graph in two making it unsolvable. Thus the only way to connect the two moves is go up one row, to the left (a-1) rows, up 1 row, and to the right (a-1) rows in order to make the next breaking move. Note that if instead of switching from the third row to the first the rook switches vice versa the same applies but the moves are simply reflected vertically. If multiple switches occer then there still only exists 1 path to connect the breaking moves. In all cases, there is exaclty 1 path that can be constructed given a set of predetermined breaking moves, so it follows that $P(n)$ is equal to the number of possible combinations of breaking moves. Therefore $P(n) = 2^{n-2}$ $\mathbb{Q}.\mathbb{E}.\mathbb{D}$