# Problem A5 from the 2022 Putnam Exam

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## 1 The Problem

Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize it. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

## 2 The Solution

Let $f(n)$ denote the number of uncovered squares after the game is complete on a board of size $n$ (assuming optimal play from both sides).

$$f(2022) = 290$$

*Proof.* Any tile placement will break the large board into 1 or 2 smaller boards which will then both be solved optimally but individually by being recursively split into smaller board until a base case is hit. Therefore, let us assume that Alice and Bob place tiles to split the board into a base case and a remainder since if instead they placed tiles to split the board into two remainders those remainders would simply reduce to base cases in the future anyways. Let us look at the base cases for Alice's moves first. The cases denote the size of the section $s$ that Alice isolates from the rest of the board, thus it is now Bob's turn to respond.

- *Case 1,* $s = 1$ In this case alice has successfully gotten 1 point since a board of size 1 cannot have any tiles placed on it.
- *Case 2,* $s = 2$ It is impossible for Alice to gain any points in this case since a board of size two must be completely covered.
- *Case 3,* $s = 3$ same as case 1 once 1 more tile is placed (by either party).

- *Case 4,* $s = 4$ Alice has a potential to get two points from a board of size 4, so Bob will reduce it to a board of size 2 on his turn to prevent that and leave alice with 0 points.
- *Case 5,* $s = 5$ No arrangement of tile placements on a board of size 5 allows Alice to get more or less than 1 point.
- *Case 6,* $s = 6$ Bob can place a tile over the middle two squares preventing alice from getting any points
- *Case 7,* $s = 7$ Bob can reduce the board to size 5 or 3 thus allowing Alice 1 point.

Any larger case will simply reduce to one of these cases. For example, suppose we have a board of size 8. Alice can reduce it to a 1 and 5, 2 and 4, or a 3 and 3. Bob will then play on one of those sections leaving Alice with a board that is described by one of the base cases. As we can see if Alice creates a board of even length, she gets no points so she will obviously be creating a board of odd length. In all cases Bob can play such that she gets exactly 1 point so she will choose to break it into a piece of size one because that has the best ratio of uncovered squares to squares.

Now that it is Bob's turn he wants to break the remaining board into the largest section possible such that Alice cannot win any points on it. As we established, Alice will win at least one point on any odd sized board, and any board of size greater than 7. Thus Bob will break it into a piece of size 2, 4, or 6. If Alice is presented with a board of size 4 or 6 she can win 2 points, but she wins no points on a board of size 2, thus Bob will break it into a board of size 2.

Now that we have established optimal play we can find out the number of uncovered tiles at the end of the game. The first round we'll see Alice place a tile on the second left most tile (leaving the leftmost tile uncovered), and then Bob placing a tile 2 spaces to the right of Alice's. One subsequent rounds we'll see Alice place her tile in the same manner she did on round one except treating Bob's last tile as the new left edge of the board, Bob will then play exactly as he did on round one. Thus each round, Alice will get one point and the board will be reduced by 7 (1 square that Alice blocked off, 4 for both tiles placed, and 2 for the gap between the two tiles). Therefore,

$$f(n) = f(n-7) + 1$$

This can be rewritten as $f(n) = \lfloor \frac{n}{7} \rfloor + f(n \mod 7)$. $2022 = 288 \times 7 + 6$ so it follows that $f(2022) = 288 + f(6) = 288 + 2 = 290$ $\mathbb{Q}.\mathbb{E}.\mathbb{D}$