B1 from the 2019 Putnam Exam

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1 The Problem

Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2 + y^2 = 2^k$ for some $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

2 The Proof

2.1 Elements of P_n

Lemma 2.1.1. *The square of an even integer modulo* 4 *is equal to 0, and the square of an odd integer modulo* 4 *is equal to 1.*

We now seek to prove what ordered pairs a in P_n. We begin by induction on k, proceeding by case.

Case 1, k is even

Let $T(n)$ be the statement

$$\text{“}x^2 + y^2 = 2^k \text{ where } x, y, k \in \mathbb{Z}^+ \text{ , and } k = 2j \text{ for some } j \in \mathbb{Z} \implies x \vee y = 0.\text{”}$$

Basis Step: $n = 0$

$x^2 + y^2 = 1$
The integer solutions for (x, y) are $(1, 0), (-1, 0), (0, 1), (0, -1)$
thus the basis step holds since all solutions have one 0.

Induction Hypothesis

$$\exists n \geq 0 \text{ such that for all even } i \text{ where } 0 \leq i \leq n,\ P(i) \text{ is true}$$

Inductive Step
We now seek to prove that $P(k + 2)$ is true given the induction hypothesis.

$$\begin{aligned} x^2 + y^2 &= 2^{k+2} \\ &= 4(2^k) \\ 2^{k+2} \bmod 4 \ = 0 &\implies x^2 \bmod 4 \ + y^2 \bmod 4 \ = 0 \end{aligned}$$

By lemma 2.1.1 we know that the square of an integer modulo 4 is either 0 or 1, so $x^2 \bmod 4 = 0$ and $y^2 \bmod 4 = 0$ is the only solution that satisfies the equality. Thus x and y are even. So let $x = 2a$ and $y = 2b$

for some $j, k \in \mathbb{Z}$

$$\begin{aligned}
x^2 + y^2 &= 2^{k+2} \\
(2a)^2 + (2b)^2 &= 4(2^k) \\
4a^2 + 4b^2 &= 4(2^k) \\
a^2 + b^2 &= 2^k \\
a^2 + b^2 &= 2^k \implies a \vee b = 0 && \text{I.H.} \\
a \vee b &= 0 \implies x \vee y = 0
\end{aligned}$$

So the Induction Hypothesis holds for $k+2$

Conclusion.
Therefore, by the principles of mathematical strong induction $P(n)$ is true for all even $n \geq 0$.

Case 2, k is odd

Let $T(n)$ be the statement

"$x^2 + y^2 = 2^k$ where $x, y, k \in \mathbb{Z}^+$, and $k = 2j+1$ for some $j \in \mathbb{Z} \implies |x| = |y| = 2^{\frac{n-1}{2}}$."

Basis Step: $n = 1$

$x^2 + y^2 = 2$
The integer solutions for (x, y) are $(1, 1), (-1, 1), (-1, 1), (-1, -1)$
thus the basis step holds since for each solution $|x| = |y| = 2^0$.

Induction Hypothesis

$$\exists n \geq 1 \text{ such that for all odd } i \text{ where } 1 \leq i \leq n,\ P(i) \text{ is true}$$

Inductive Step
We now seek to prove that $P(k+2)$ is true given the induction hypothesis.

$$\begin{aligned}
x^2 + y^2 &= 2^{k+2} \\
&= 4(2^k) \\
2^{k+2} \bmod 4 &= 0 \implies x^2 \bmod 4 + y^2 \bmod 4 = 0
\end{aligned}$$

By lemma 2.1.1 we know that the square of an integer modulo 4 is either 0 or 1, so $x^2 \bmod 4 = 0$ and $y^2 \bmod 4 = 0$ is the only solution that satisfies the equality. Thus x and y are even. So let $x = 2a$ and $y = 2b$ for some $j, k \in \mathbb{Z}$

$$\begin{aligned}
x^2 + y^2 &= 2^{k+2} \\
(2a)^2 + (2b)^2 &= 4(2^k) \\
4a^2 + 4b^2 &= 4(2^k) \\
a^2 + b^2 &= 2^k \\
a^2 + b^2 &= 2^k \implies |a| = |b| = 2^{\frac{k-1}{2}} && \text{I.H.} \\
(2 \times 2^{\frac{k-1}{2}})^2 + (2 \times 2^{\frac{k-1}{2}})^2 &= 2^{k+2} \\
4 \times 2^{k-1} + 4 \times 2^{k-1} &= 2^{k+2} \\
8 \times 2^{k-1} &= 2^{k+2} \\
2^3 \times 2^{k-1} &= 2^{k+2} \\
2^{k+2} &= 2^{k+2}
\end{aligned}$$

So the Induction Hypothesis holds for $k + 2$

Conclusion.
Therefore, by the principles of mathematical strong induction $P(n)$ is true for all odd $n \geq 0$.

We have now proven that $x^2 + y^2 = 2^k \implies x \vee y = 0$ or $|x| = |y| = 2^{\frac{k-1}{2}}$.

$$\therefore f(n) = 5n + 1$$

2.2 The Solution

Proof. We can then say about P_n

$$P_n = P_{n-1} \cup \{(0, \pm 2^{\frac{n}{2}}), (\pm 2^{\frac{n}{2}}, 0\} \qquad \text{when } n \text{ is even}$$
$$P_n = P_{n-1} \cup \{(2^{\frac{n-1}{2}}, \pm 2^{\frac{n-1}{2}}), (-2^{\frac{n-1}{2}}, 2^{\frac{n-1}{2}})\} \qquad \text{when } n \text{ is odd}$$

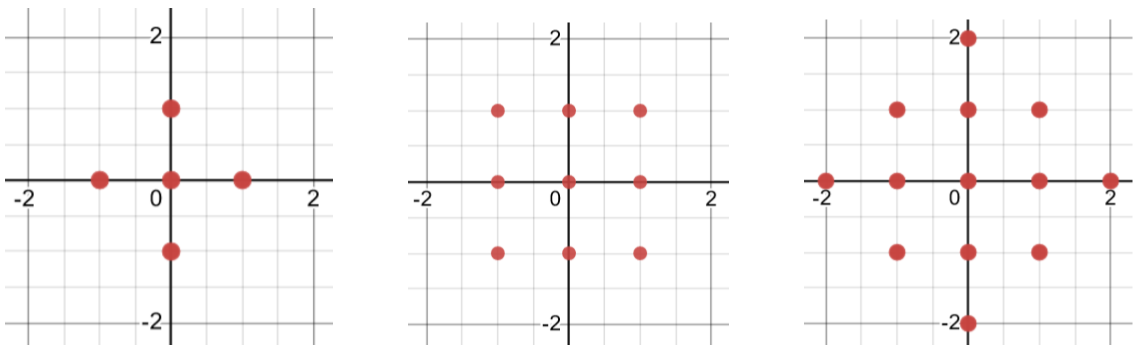

Fig. 2.2.1 P_1, P_2, P_3 (left to right)

Observe that P_1 has 1 four-point subset that represents the four vertices of a square. Now, as n gets larger, P_n continues in this pattern with the four new points being on the axes on one iteration, and the corners on the next, but notice that under a 45^o rotation, the graph is identical and the four new points are always on the corner, so we will only consider that when calculating the number of new squares and apply that rotation to all iterations.

When the four corners are added as in P_2 to form a large perimeter square, the four points in $P_n - P_{n-1}$ are the vertices of a square. Additionally, One points from $P_n - P_{n-1}$, the origin, and the 2 corresponding points in $P_{n-1} - P_{n-2}$ are the vertices of another square, which can be copied once per quadrant for a total of 5 additional squares. The "edge" of our cluster in points will only ever have 3 points on it that can act as vertices to a square. The first square described utilizes the outer points, and 2 of the second squares described utilize the middle point and one outer point. This leaves no additional combinations left so no more additional squares can be drawn giving us 5 new squares each iteration after $n = 0$.

So let f(n) be the function described in the problem we determined that $f(0) = 1$ and f(n) = f(n - 1) + 5.

$$\therefore f(n) = 5n + 1$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$