A1 from the 2017 Putnam Exam

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1 The Problem

Let S be the smallest set of positive integers such that

(a) 2 is in S

(b) n is in S whenever n^2 is in S

(c) (n^2+5) is in S whenever n is in S

Which positive integers are not in S? (The set S is "smallest" in the sense that S is contained in any other such set)

2 Solution

2.1 Properties of S

We begin by describing a 4th property of S

(c) $n \in S \to (n+5)^2 \in S$

(b) $(n+5)^2 \in S \to n+5 \in S$

(c) $n+5 \in S \to (n+10)^2 \in S$

(b) $(n+10)^2 \in S \to n+10 \in S$

Thus properties (b) and (c) can be combined to form a new property,

(d) $n \in S \to n+5k \in S$ for all $k \in \mathbb{N}$

By (a) and (b) we know that $2+5k \in S$ for all $k \in \mathbb{N}$. We will computer some other numbers in S to show what number *are* in S before demonstrating which ones *aren't*.

2.2 Elements of S

Proof. We begin with 2 and branch out from there computing other elements of S.

$$
\begin{aligned}
&2 \in S && (a)\\
&2 \in S \implies 49 \in S \implies 54^2 \in S && (c)\\
&\text{Since the last digit of } 54^2 \text{ and } 256^2 \text{ is } 6\ , 54^2 \in S \implies 256^2 \in S && (d)\\
&256^2 \in S \implies 256 \in S \implies 16 \in S && (d)\\
&16 \in S \implies 36 \in S \implies 6 \in S && (d),(b)\\
&16 \in S \implies 4 \in S \implies 9 \in S \implies 3 \in S && (b),(d),(b)
\end{aligned}
$$

By (d) since $2, 3, 4, 6 \in S$, we know that

$$\begin{aligned} &2 + 5k \in S \\ &3 + 5k \in S \\ &4 + 5k \in S \\ &6 + 5k \in S && \forall k \in \mathbb{N} \end{aligned}$$

Thus the only elements not proven to be in S are 1, and all multiples of 5.

2.3 Elements not in S

Any number n is implied to be an element of S if either $n^2 \in S$ or $n - 5k \in S$ for any $k \geq 1$. For example, we know 12 is an elements of S since $12 - 5 * (2) \in S$. 1 would be implied to be in S if either $-4 \in S$ or $1^2 \in S$. The former cannot be true since S is defined to only contain positive integers, and the latter is a self-implication so we can decide it to be false, thus $1 \notin S$.

Now the only numbers not in S are the multiples of 5. Let $x = 5k$ for some $k > 1$.

$$x \in S \iff x^2 \in S \vee x - 5a \in S \text{ for some } 1 \leq a < k$$

$$\begin{aligned} &x = 5k \\ &x^2 = 25k^2 \\ &x^2 = 5(5k^2) \\ &\text{thus } x^2 \text{ is a multiple of 5.} \\ &x = 5k \\ &x - 5a = 5k - 5a \\ &x - 5a = 5(k - a) \\ &\text{thus } x - 5a \text{ is a multiple of 5.} \end{aligned}$$

This means that a multiple of 5 can only be implied to be in S by another multiple of 5 meaning unless one is in S, then none of them are. Since none of them have to be, we can conclude that no multiples of 5 are elements of S.

$$\therefore 1,\ 5k \notin S\ \forall k \geq 1$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}$