Problem A2 from the 2014 Putnam Exam

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1 The Problem

Let A be an $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{min(i,j)}$$

for $1 \leq i \leq n$. Compute $\det(A)$.

2 The Proof

For any n, A will look like so.

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \frac{1}{2} & \dots & \frac{1}{2} \\ \dots & \dots & \frac{1}{n-1} & \frac{1}{n-1} \\ 1 & \frac{1}{2} & \frac{1}{n-1} & \frac{1}{n} \end{bmatrix}$$

We will compute the determinant by triangularizing the matrix and then taking the product of the main diagonal. To triangularize the matrix we will create a new $n \times n$ matrix called "T" such that the nth row of T equals the nth row of A minus the $n-1$th row of A (the first row of T will simply be equal to the first row of A). Since the first $n-1$ terms of a row are equal to the first $n-1$ terms of the row above it, then performing this operation will zero out the first $n-1$ terms of each row thus making T a triangular matrix. Since the only row operation was adding a scalar multiple of a row to another row, $det(T) = det(A)$.

Proof. Let $T(n)$ be the n-th entry in the main diagonal of A. $T(1) = 1$, and for $n > 1$; $T(n) = \frac{1}{n} - \frac{1}{n-1}$

$$\begin{aligned} T(n) &= \frac{1}{n} - \frac{1}{n-1} \\ &= \frac{n-1}{n(n-1)} - \frac{n}{n(n-1)} \\ &= \frac{n-1-n}{n(n-1)} \\ &= -\frac{1}{n(n-1)} \end{aligned}$$

As mentioned previously the determinant is the product of the main diagonal. Thus,

$$\begin{aligned}
det(A) &= 1 \times \prod_{x=2}^{n} T(n) \\
&= 1 \times \prod_{x=2}^{n} -\frac{1}{n(n-1)} \\
&= \frac{1}{1! \times 0!} \times -\frac{1}{n(n-1)} \times -\frac{1}{(n-1)(n-2)} ... \times -\frac{1}{2 \times 1} \\
&= (-1)^{n-1} \times \frac{1}{n!(n-1)!}
\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$