B1 from the 2009 Putnam Exam

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1 The Problem

Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For examples

$$\frac{10}{9} = \frac{2! \times 5!}{3! \times 3! \times 3!}$$

2 The Solution

Proof. By the definition of rational numbers, $\forall x \in \mathbb{R}$, $\exists p, q \in \mathbb{Z}$ such that $x = \frac{p}{q}$. So it follows that if every integer can be expressed as a quotient of products of prime factorials, then so can every rational number. By the fundamental theorem of arithmetic, every positive integer can be expressed as a unique product of primes. So it follows that if every prime number can be expressed as a quotient of products of prime factorials, then so can every integer. Let P_n denote the nth prime number where $P_1 = 2$, $P_2 = 3$, $P_3 = 5$, and so on and so forth. Let F(n) be the statement "n can be expressed as a quotient of product of non-distinct prime factorials", and P(n) be the statement "The nth prime number can be expressed as a quotient of products of non-distinct prime factorials". We proceed by induction on P_n.

Basis Step: n = 1

$P_1 = 2 = \frac{2! \times 2!}{2!}$ thus P(1) is true.

Inductive Step

Assume there exists some $k \geq 1$ such that $P(1)...P(k)$ are true. We now seek to prove $P(k+1)$.

$$P_{k+1} = \frac{P_{k+1}!}{(P_{k+1} - 1)!}$$

$$(P_{k+1} - 1)! = 2 \times 3 \times 4....P_{k+1} - 1$$

For every integer x from $2...P_{k+1} - 1$, x is either a prime number of a composite number.

Case 1: x is prime

In this case $F(x)$ is true by our assumption since $x \leq P_k$.

Case 2: x is composite

By the fundamental theorem of arithmetic, x can be expressed as a product of prime numbers. Since $x < P_{k+1}$ it follows that each of the prime factors is less than P_{k+1}. Thus, each prime factor of x is in $P_1...P_k$ and can therefore be expressed as a quotient of products of non-distinct primes. The product of several quotients of products of non-distinct primes will itself be one. So in either case, $F(x)$ is true.

Since $F(x)$ is true for all x from 2 to P_k, $F(P_{k+1} - 1!)$ is true. Thus $P(k+1)$ is also true.

Conclusion

Therefore by the principle of mathematical strong induction, $P(n)$ is true for all n.

Since all prime numbers can be expressed as a quotient of products of non-distinct prime numbers, it follows that the same it true for all integers, and therefore all rationals as well which is what we set out to prove. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$