Problem B4 on the 1995 Putnam Exam

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1 The Problem

Evaluate

$$\sqrt[8]{2207-\frac{1}{2207-\frac{1}{2207-\frac{1}{\cdots}}}}$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$ where a, b, c, d are integers.

2 The Solution

$$\sqrt[8]{2207-\frac{1}{2207-\frac{1}{2207-\frac{1}{\cdots}}}}=\phi^2=\frac{3+\sqrt{5}}{2}$$

Proof.

$$\begin{aligned}
x &= \sqrt[8]{2207-\frac{1}{2207-\frac{1}{2207-\frac{1}{\cdots}}}} \\
x^8 &= 2207-\frac{1}{2207-\frac{1}{2207-\frac{1}{\cdots}}} \\
x^8 &= 2207-\frac{1}{x^8} \\
x^{16} &= 2207x^8-1 \\
y^2-2207y+1 &= 0 \qquad \text{with a variable change } y=x^8 \\
y &= \frac{2207+\sqrt{4870845}}{2} \\
&= \frac{2207+987\sqrt{5}}{2} \\
x &= \sqrt[8]{\frac{2207+987\sqrt{5}}{2}}
\end{aligned}$$

We will now find x, and y such that $x^2 + 5y^2 = 4414$ and $xy = 987$. The factors of 987 are 3, 7, and 47. The ordered pair $(47, 21)$ satisifies both equations, so we will proceed with those.

$$
\begin{aligned}
x &= \sqrt[8]{\frac{2207 + 987\sqrt{5}}{2}} \\
&= \sqrt[8]{\frac{4414 + 1976\sqrt{5}}{4}} \\
&= \sqrt[8]{\frac{47^2 + 2(47)(21)(\sqrt{5}) + 5(21^2)}{4}} \\
&= \sqrt[4]{\frac{47 + 21\sqrt{5}}{2}} \\
&= \sqrt[4]{\frac{96 + 42\sqrt{5}}{4}} \\
&= \sqrt[4]{\frac{7^2 + 2(7)(3)\sqrt{5} + 45}{4}} && \text{by the same process seen before} \\
&= \sqrt{\frac{7 + 3\sqrt{5}}{2}} \\
&= \sqrt{\frac{14 + 6\sqrt{5}}{4}} \\
&= \sqrt{\frac{3^2 + 2(1)(3)\sqrt{5}}{4}} && \text{by the same process seen before} \\
&= \frac{3 + \sqrt{5}}{2}
\end{aligned}
$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}$