Problem A2 from the 2021 Putnam Exam

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1 The Problem

For every real number x, let
$$g(x) = \lim_{r \to 0}((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}$$

Find $\lim_{x\to\infty} \frac{g(x)}{x}$

2 The Solution

$$\lim_{x\to\infty} \frac{g(x)}{x} = e$$

Proof.

$$\begin{aligned}
g(x) &= \lim_{r\to 0}((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}} \\
&= e^{\lim_{r\to 0} \ln(((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}})} \\
\lim_{r\to 0} \frac{\ln((x+1)^{r+1} - x^{r+1})}{r} &= \frac{0}{0} \\
&= \lim_{r\to 0} \frac{((x+1)^{r+1}\ln(x+1) - x^{r+1}\ln(x))}{(x+1)^{r+1} - x^{r+1}} && \text{by L'Hopital's rule} \\
&= (x+1)\ln(x+1) - x\ln(x) \\
g(x) &= e^{(x+1)\ln(x+1) - x\ln(x)} \\
&= \frac{(x+1)^{x+1}}{x^x} \\
\lim_{x\to\infty} \frac{g(x)}{x} &= \lim_{x\to\infty} \frac{(x+1)^{x+1}}{x^x} \times \frac{1}{x} \\
&= \lim_{x\to\infty} \frac{(x+1)^{x+1}}{x^{x+1}} \\
&= \lim_{x\to\infty} (1 + \frac{1}{x})^{x+1} \\
&= e
\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}$