Problem A2 from the 2001 Putnam Exam

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1 The Problem

You have coins $C_1, C_2, ..., C_n$. For each k, C_k is biased so that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

2 The Solution

Let $P(n)$ denote the probability that the nth coin falls heads such that $P(n) = \frac{1}{2n+1}$. Let $T(n)$ denote the probability that when n coins are tossed, the number of heads is odd.

$$T(n) = \frac{n}{2n+1}$$

Proof. The probability that n coins produce an odd number of heads is the probability that $n-1$ coins produces even heads and the nth coin falls heads or that $n-1$ coins produce odd heads and then nth coins falls tails. As an equation, this is expressed as follows

$$\begin{aligned} T(n) &= T(n-1) \times (1 - P(n-1)) + (1 - T(n-1)) \times P(n) \\ &= T(n-1) - P(n)T(n-1) + P(n) - P(n)T(n-1) \\ &= T(n-1) + P(n) - 2P(n)T(n-1) \end{aligned}$$

We proceed by induction on $T(n)$

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Basis Step: n-1
$T(1) = P(1) = \frac{1}{3}$, thus the basis step holds.

Inductive Step
Assume $\exists k; T(k) = \frac{k}{2k+1}$

$$\begin{aligned}
T(k+1) &= P(k+1) + T(k) - 2(P(k+1)T(k)) \\
&= \frac{1}{2(k+1)+1} + \frac{k}{2k+1} - 2(\frac{1}{2(k+1)+1}\frac{k}{2k+1}) \\
&= \frac{2k+1}{(2(k+1)+1)(2k+1)} + \frac{(k)(2k+3)}{(2(k+1)+1)(2k+1)} - \frac{2k}{(2(k+1)+1)(2k+1)} \\
&= \frac{2k+1+2k^2+3k-2k}{(2(k+1)+1)(2k+1)} \\
&= \frac{2k^2+3k+1}{(2(k+1)+1)(2k+1)} \\
&= \frac{(2k+1)(k+1)}{(2(k+1)+1)(2k+1)} \\
&= \frac{k+1}{2(k+1)+1}
\end{aligned}$$

Therefore, by the principle of weak mathematical induction, $T(n) = \frac{n}{2n+1}$ for all $n \geq 1$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}$