Problem A2 from the 2009 Putnam Exam

Evan Dreher

September 2023

1 The Problem

Functions f, g, and h are differentiable on some open interval around 0 and satisfy the equations and initial conditions.

$$f' = 2f^2gh + \frac{1}{gh} \tag{1}$$

$$g' = fg^2h + \frac{4}{fh} \tag{2}$$

$$f' = 3fgh^2 + \frac{1}{fg} \tag{3}$$

$$f(0) = g(0) = h(0) = 1 \tag{4}$$

2 The Proof

$$f = 2^{-\frac{1}{12}}(\frac{\sin(6x + \frac{\pi}{4})}{\cos^2(6x + \frac{\pi}{4})})^{\frac{1}{6}}$$

Proof. Rewriting equations 1,2, and 3 to eliminate fractions yields the following equations.

$$fgh' = 2f^2g^2h^2 + 1 = 2(fgh)^2 + 1 \tag{5}$$

$$fg'h = f^2g^2h^2 + 4 = (fgh)^2 + 4 \tag{6}$$

$$fgh' = 3f^2g^2h^2 + 1 = 3(fgh)^2 + 1 \tag{7}$$

$$\begin{aligned}
f'gh + fg'h + fgh' &= 6(fgh)^2 + 6 && \text{by summing equations 5,6,7} \\
(fgh)' &= 6((fgh)^2 + 1) && \text{by reversing the product rule} \\
\frac{d(fgh)}{dx} &= 6((fgh)^2 + 1) \\
\frac{d(fgh)}{(fgh)^2 + 1} &= 6dx \\
\arctan(fgh) &= 6x + c && \text{by integration of both sides} \\
fgh &= \tan(6x + C) \\
1 &= \tan(C) && \text{when } x = 0 \\
C &= \frac{\pi}{2}
\end{aligned}$$

$$fgh = \tan(6x + \frac{\pi}{4}) \tag{8}$$

With equation 8, we will now do substitutions into previous equations to derive an explicit formula for f.

$$
\begin{aligned}
f' &= 2f\tan(6x+\frac{\pi}{4}) + \frac{1}{gh} && \text{by plugging eq 8 into eq 1}\\
f' &= 2f\tan(6x+\frac{\pi}{4}) + \frac{f}{\tan(6x+\frac{\pi}{4})}\\
\frac{f'}{f} &= 2\tan(6x+\frac{\pi}{4}) + \cot(6x+\frac{\pi}{4})\\
\ln(f)+C &= 2\int\tan(6x+\frac{\pi}{4})dx + \int\cot(6x+\frac{\pi}{4})dx && \text{by integrating both sides}\\
\ln(f)+C &= \frac{1}{3}\int\tan(u)du + \frac{1}{6}\int\cot(u)du && \text{with } u = 6x+\frac{\pi}{4}\\
&= \frac{1}{6}\ln(\sin u) - \frac{1}{3}\ln(\cos u)\\
6\ln(f)+C &= ln(\sin u) - ln(\cos^2 u)\\
&= ln(\frac{\sin u}{\cos^2 u})\\
\ln(f) &= \frac{1}{6}ln(\frac{\sin u}{\cos^2 u}) + C\\
f &= e^C(\frac{\sin u}{\cos^2 u})^{\frac{1}{6}}\\
&= e^C(\frac{\sin(6x+\frac{\pi}{4})}{\cos^2(6x+\frac{\pi}{4})})^{\frac{1}{6}}\\
1 &= e^C(\frac{\sin(\frac{\pi}{4})}{\cos^2(\frac{\pi}{4})})^{\frac{1}{6}} && \text{when } x = 0\\
&= e^C(\frac{\frac{\sqrt{2}}{2}}{\frac{2}{4}})^{\frac{1}{6}}\\
&= \sqrt[12]{2}e^C\\
C &= -\frac{1}{12}\ln(2)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f &= e^C(\frac{\sin u}{\cos^2 u})^{\frac{1}{6}}\\
&= e^{-\frac{1}{12}\ln(2)}(\frac{\sin u}{\cos^2 u})^{\frac{1}{6}}\\
&= 2^{-\frac{1}{12}}(\frac{\sin(6x+\frac{\pi}{4})}{\cos^2(6x+\frac{\pi}{4})})^{\frac{1}{6}}
\end{aligned}
$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}$