Math 220A - Fall 2016 - Final Exam

Name: ______________________________

Student ID: ___________________________

Instructions:

Please print your name and student ID (if you know it).

There are 8 questions which are worth 80 points. You have 180 minutes to complete the test. .

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
Total		80

Problem 1. [*10 points.*]

Consider the function $f(z) = ze^{3-z} - 1$. Show that f has exact one zero inside the disc $\Delta(0, 1)$.

Problem 2. [*10 points.*]

Calculate the integral

$$\int_0^\infty \frac{dx}{x^{2n}+1}, \text{ for } n \geq 2.$$

Make sure you explain all the necessary estimates.

Problem 3. [*10 points.*]

Consider

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_n.$$

Show that there exists c with $|c| = 1$ such that

$$|f(c)| \geq 1.$$

Problem 4. [*10 points.*]

Assume that f is entire and $f(z) = f(z+1)$ such that $|f(z)| \leq e^{|z|}$. Show that f is constant.

(i) Consider
$$g(z) = \frac{f(z) - f(0)}{\sin \pi z}.$$
Show that g is periodic and that g can be extended to an entire function.

(ii) By direct calculation, show that g is bounded in the strip $0 \leq \text{Re } z \leq 1$.

(iii) Conclude from (ii) that $g = 0$ hence f is constant.

Problem 5. [*10 points.*]

Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function with $\deg P \leq \deg Q - 2$ such that Q has no zeros along the non-negative real axis. Show that

$$\int_0^\infty f(x)\,dx = -\sum_{a \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}} \operatorname{Res}_{z=a}(f(z) \log z),$$

where for $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ we set $\log z = \log r + i\theta$ and $\theta \in (0, 2\pi)$.

You may wish to integrate along a "keyhole" contour, consisting of two portions of two circles and two line segments, and avoiding the non-negative real axis.

Problem 6. [*10 points.*]

Let $a, b \neq 0$ be real numbers and let U be a connected open set. Let $f : U \to \mathbb{C}$ be a holomorphic function. Show that if $a \operatorname{Re} f + b \operatorname{Im} f$ is constant, then f is constant.

Problem 7. [*10 points.*]

Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Problem 8. [*10 points.*]

Assume that f is continuous in the closed unit disc $\overline{\Delta}$ and holomorphic inside the unit disc Δ. Assume that

$$|f(z)| = 1 \text{ for all } |z| = 1.$$

(i) If f is nonconstant, show that f must have a zero inside Δ.

(ii) Show that if f has a unique simple zero at $z = 0$ then $f(z) = \alpha z$.