Solutions: Homework 5

Problem 1. *Let f, g be two entire functions of finite order λ. Assume $f(a_n) = g(a_n)$ for a sequence $\{a_n\}_{n\geq 0}$ with*

$$\sum_{n=0}^{\infty} \frac{1}{|a_n|^{\lambda+1}} = \infty.$$

Show that $f = g$.

Proof. Let us suppose that $f \neq g$. Let $h = f - g$. Then

$$\text{order } (h) \leq \max(\text{order } (f), \text{order } (g)) = \lambda.$$

Suppose h has a zero of order m at 0 so that

$$h(z) = z^m H(z).$$

We have seen in class that multiplication by a polynomial does not affect the order. Thus, H has order $\leq \lambda$ as well and $H(a_n) = 0$.

Now, let $\{b_n\}_{n\geq 0}$ be the non-zero zeros of H. Then, $\{a_n\}_{n\geq 0} \subset \{b_n\}_{n\geq 0}$, and hence

$$\sum_{n=0}^{\infty} \frac{1}{|b_n|^{\lambda+1}} = \infty.$$

In particular, if α is the exponent of convergence we must have $\alpha > \lambda + 1$. By part (ii) of Problem 5, HW 4, we have $\alpha \leq \lambda$. This is a contradiction.

Hence, $h = 0$ so that $f = g$. □

Problem 2. *(i) Find all entire functions f of finite order such that $f(\log n) = n$ for all integers $n \geq 1$.*
(ii) Give an example of an entire function f with zeroes only at $\log n$ for integers $n \geq 1$.

Proof. (i) Note that in Problem 1 above, we just used the fact that the order of f and g is $\leq \lambda$, not necessarily equal to λ. Suppose that f is an entire function of finite order such that $f(\log n) = n$ for all integers $n \geq 1$. Let λ denote $\max\{\text{order of } f, 1\}$. Let $N \geq 2$ be such that for all $n \geq N$,

$$\log n \leq n^{\frac{1}{\lambda+1}}$$

Then

$$\infty = \sum_{n=N}^{\infty} \frac{1}{n} \leq \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\lambda+1}}$$

Applying Problem 1 to f and $g(z) = e^z$, we see that $f(z) = e^z$.
(ii) By Theorem 5.12, the function

$$f(z) = z \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{\log(n+1)}\right)$$

is an entire function with zeros only at $\log n$ for integers $n \geq 1$.

□

Problem 3. *If f is an entire function of order λ, show that f' also has order λ.*

Proof. Let $M'(R) = \sup\{|f'(z)| : |z| = R\}$, and let λ' denote the order of f'. Let $|z| = R$. Then, applying Cauchy's estimate to f on $\Delta(z;1) \subset \Delta(0, R+1)$, we have

$$|f'(z)| \leq \sup_{|w-z|=1} |f(w)| \leq M(R+1)$$

and hence

$$M'(R) \leq M(R+1).$$

Thus

$$\limsup_{R\to\infty} \frac{\log\log M'(R)}{\log R} \leq \limsup_{R\to\infty} \frac{\log\log M(R+1)}{\log(R)} = \limsup_{R\to\infty} \frac{\log\log M(R+1)}{\log(R+1)}.$$

This shows that

$$\lambda' \leq \lambda.$$

For the opposite inequality, WLOG, we can assume that $f(0) = 0$ since else we can work with the function $f - f(0)$ which has the same order as shown in class. We then have

$$f(z) = \int_0^1 (f(tz))'\,dt = z\int_0^1 f'(tz)dt.$$

Note that

$$M'(R) = \sup_{|w|=R} |f'(w)| = \sup_{|w|\leq R} |f'(w)|$$

by the maximum modulus principle. Hence for all $|z| = R$, we have

$$|f'(tz)| \leq M'(R)$$

for $0 \leq t \leq 1$, and thus by the above we conclude

$$|f(z)| \leq RM'(R) \implies M(R) \leq RM'(R).$$

Fix $\epsilon > 0$. Then, taking log, we have

$$\log M(R) \leq \log R + \log M'(R) \leq \log R + R^{\lambda'+\epsilon} \leq R^{\lambda'+2\epsilon}$$

for $R >> 0$. Thus

$$\lambda = \lim_{R\to\infty} \frac{\log\log M(R)}{\log R} \leq \lambda' + 2\epsilon.$$

As $\epsilon > 0$ is arbitrary, this shows that

$$\lambda \leq \lambda'.$$

In conclusion, $\lambda = \lambda'$. □

Problem 4. *Let f be entire, $|f'(z)| \leq e^{|z|}$ and*

$$f(\sqrt{n}) = 0 \qquad \textit{for all } n \in \mathbb{Z}_{>0}.$$

Show that $f = 0$.

Proof. Since $|f'(z)| \leq e^{|z|}$ it follows that f' has order $\lambda' \leq 1$. By the previous question, the order of f must satisfy $\lambda \leq 1$. In particular, the rank

$$p \leq h \leq \lambda \leq 1.$$

By definition of the rank, this means that

$$\sum_n \frac{1}{|\sqrt{n}|^{p+1}} = \sum_n \frac{1}{n} < \infty$$

which is clearly a contradiction. Thus $f = 0$. □

Problem 5. *Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z - \sin z$.*
(i) Show that f is an odd entire function of order less or equal to 1.
(ii) Using (i), show that f can be represented as a product

$$f(z) = \frac{z^3}{6} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

where $\{a_n\}$ is a sequence of non-zero complex numbers with

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

Proof. (i) The fact that f is entire and odd is clear. For the order, note that if $|z| = R$, we have

$$|f(z)| \leq |z| + |\sin z| \leq |z| + \left|\frac{1}{2i}(e^{iz} - e^{-iz})\right| \leq |z| + \frac{1}{2}|e^{iz}| + \frac{1}{2}|e^{-iz}|$$

$$\leq |z| + \frac{1}{2}e^{\mathrm{Re}(iz)} + \frac{1}{2}e^{\mathrm{Re}(-iz)} \leq |z| + \frac{1}{2}e^{|iz|} + \frac{1}{2}e^{|-iz|} = |z| + e^{|z|} \leq R + e^R.$$

Thus

$$\lambda \leq \limsup_{R\to\infty} \frac{\log\log(R + e^R)}{\log R} = 1.$$

(ii) Since $\lambda \leq 1$ by Hadamard's theorem, we must have $h \leq \lambda \leq 1$. Thus the rank $p \leq 1$, the and by the definition of the rank, we must have

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty,$$

where $\{a_n\}$ denote the zeroes of f not equal to 0. Since $p + 1 \leq 2$ and $a_n \to \infty$, it follows that $|a_n| > 1$ for n sufficiently large and

$$\frac{1}{|a_n|^2} \leq \frac{1}{|a_n|^{p+1}}.$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

By Weierstraß factorization we have

$$f(z) = z^m e^g \prod_{n=1}^{\infty} E_1\left(-\frac{z}{a_n}\right).$$

Recall that in Weierstraß we can increase the value of p without affecting convergence, so using $p = 1$ is justified. Alternatively, one can split this into two cases $p = 0$ which is simpler, and $p = 1$ which is treated explicitly below.

Since f is odd, the zeroes of f come in pairs $(a_n, -a_n)$. We can combine

$$E_1\left(\frac{z}{a}\right)E_1\left(-\frac{z}{a}\right) = \left(1-\frac{z}{a}\right)e^{\frac{z}{a}}\left(1+\frac{z}{a}\right)e^{-\frac{z}{a}} = 1-\frac{z^2}{a^2}.$$

Thus, we may write

$$f(z) = z^m e^g \prod_{n=1}^{\infty}\left(1-\frac{z^2}{a_n^2}\right),$$

after relabelling/discarding some of the zeroes. Combining terms is justified by the local absolute convergence of the product.

Note that m is order of f at 0. Computing the Taylor expansion, we see that

$$f(z) = z - \sin z = \frac{z^3}{6} + \dots.$$

Thus $m = 3$.

The degree q of g satisfies

$$q \le h \le \lambda \le 1.$$

Thus $g(z) = az + b$ for some a, b. Since f is odd, it follows at once that e^g must be even so

$$e^{g(z)} = e^{g(-z)} \implies e^{az+b} = e^{-az+b} \implies e^{2az} = 1 \implies a = 0.$$

Thus $g = b$ must be a constant

$$f(z) = z^3 e^b \prod_{n=1}^{\infty}\left(1-\frac{z^2}{a_n^2}\right).$$

Therefore,

$$\lim_{z\to 0}\frac{f(z)}{z^3} = e^b$$

using the fact that the product converges to an entire (hence continuous) function. However,

$$\lim_{z\to 0}\frac{f(z)}{z^3} = \frac{1}{6}$$

as we see from the Taylor expansion for instance. Thus $e^b = \frac{1}{6}$ and

$$f(z) = \frac{z^3}{6} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right).$$

$\square$

Problem 6. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function of order λ. Let*

$$\mu = \limsup_{n\to\infty} \frac{n \log n}{-\log |c_n|} > 0.$$

Show that $\lambda = \mu$.
(i) First show that $\lambda \geq \mu$ by showing that for all $\epsilon > 0$ we have $\lambda > \mu - \epsilon$.
(ii) Conversely, show that $\lambda \leq \mu$ by showing that $\lambda < \mu + \epsilon$ for all $\epsilon > 0$.
(iii) Let $a > 0$. Show that the function

$$f(z) = \sum_n \frac{z^n}{n^{an}}$$

is entire and find its order.

Proof. (i) Let $0 < \epsilon < \mu$. By definition,

$$n \log n \geq -(\mu - \epsilon) \log |c_n|$$

for infinitely many n. Using Cauchy's estimate, we have

$$|c_n| \leq \frac{M(R)}{R^n}$$

for all $R > 0$. So we have

$$-\log |c_n| \geq n \log R - \log M(R)$$

and hence,

$$\log M(R) \geq n \log R - \frac{n \log n}{\mu - \epsilon}$$

for infinitely many n, and for all $R > 0$. Putting $R_n = (en)^{\frac{1}{\mu-\epsilon}}$, we have

$$\log M(R_n) \geq \frac{n}{\mu - \epsilon} = \frac{R_n^{\mu-\epsilon}}{\mu - \epsilon}$$

Since $R_n \to \infty$, we have

$$\lambda \geq \mu - \epsilon$$

Since $0 < \epsilon < \mu$ was arbitrary, we have

$$\lambda \geq \mu$$

(ii) Fix $\epsilon > 0$. By definition, there exists $N \geq 1$ such that

$$n \log n \leq -(\mu + \epsilon) \log |c_n|$$

for all $n \geq N$, i.e.

$$|c_n| \leq n^{-\frac{n}{\mu+\epsilon}}$$

for all $n \geq N$. This shows that there exists $C \geq 1$ such that

$$|c_n| \leq C n^{-\frac{n}{\mu+\epsilon}}$$

for all $n \geq 1$. Now, for $|z| = R$, we have

$$\left| \sum_{n=0}^{k} c_n z^n \right| \leq \sum_{n=0}^{k} |c_n||z|^n \leq C \sum_{n=0}^{k} R^n n^{-\frac{n}{\mu+\epsilon}} \leq C \sum_{n=0}^{\infty} R^n n^{-\frac{n}{\mu+\epsilon}}$$

Letting $k \to \infty$, we have

$$|f(z)| \leq C \sum_{n=0}^{\infty} R^n n^{-\frac{n}{\mu+\epsilon}}$$

for all $|z| = R$. So we have

$$M(R) \leq C \sum_{n=0}^{\infty} R^n n^{-\frac{n}{\mu+\epsilon}}$$

Now, let

$$S_1 = \sum_{n \leq (2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{\mu+\epsilon}} \quad \text{and} \quad S_2 = \sum_{n > (2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{\mu+\epsilon}}$$

We have

$$S_1 = \sum_{n \leq (2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{\mu+\epsilon}} \leq R^{(2R)^{\mu+\epsilon}} \sum_{n \leq (2R)^{\mu+\epsilon}} n^{-\frac{n}{\mu+\epsilon}} \leq R^{(2R)^{\mu+\epsilon}} \sum_{n=1}^{\infty} n^{-\frac{n}{\mu+\epsilon}} = AR^{(2R)^{\mu+\epsilon}}$$

where $A = \sum_{n=1}^{\infty} n^{-\frac{n}{\mu+\epsilon}}$. Similarly, we have

$$S_2 = \sum_{n>(2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{\mu+\epsilon}} \leq \sum_{n>(2R)^{\mu+\epsilon}} R^n (2R)^{-n} \leq \sum_{n>(2R)^{\mu+\epsilon}} \left(\frac{1}{2}\right)^n \leq 1$$

Putting this together, we have

$$M(R) \leq C(S_1 + S_2) \leq C(AR^{(2R)^{\mu+\epsilon}} + 1)$$

for all $R > 0$. So we have

$$\lambda \leq \mu + \epsilon$$

Since $\epsilon > 0$ is arbitray, we have $\lambda \leq \mu$, and hence, combining part (i), we have $\lambda = \mu$.

(iii) We have

$$\limsup_{n\to\infty} \left(\frac{1}{n^{an}}\right)^{\frac{1}{n}} = \limsup_{n\to\infty} \frac{1}{n^a} = 0$$

and hence, f is a power series with $R = \infty$, and is therefore entire. We have

$$\mu = \limsup_{n\to\infty} \frac{n \log n}{-\log n^{-an}} = \limsup_{n\to\infty} \frac{n \log n}{an \log n} = \frac{1}{a} > 0$$

Thus the order of f is $\frac{1}{a}$. □