Solutions: Homework 3

Problem 1. *Show that if $u_n : G \to \mathbb{R}$ are subharmonic/superharmonic converging locally uniformly to $u : G \to \mathbb{R}$, then u is also subharmonic/superharmonic.*

Proof. We prove the statement for subharmonic functions. The same proof works for superharmonic functions, with just the opoosite inequality.

Note first that u is continuous, since continuity is a local property, and $u_n \to u$ locally uniformly.

Let $\overline{\Delta}(a;r) \subset G$. Since the u_n's are subharmonic, we have

$$u_n(a) \leq \frac{1}{2\pi} \int_o^{2\pi} u_n(a + re^{i\theta}) d\theta.$$

Since the u_n's coverge locally uniformly to u, and $\partial\Delta(a;r)$ is a compact set, it follows that u_n converges to u uniformly on $\partial\Delta(a;r)$. So, we have

$$\frac{1}{2\pi} \int_o^{2\pi} u_n(a + re^{i\theta}) d\theta \to \frac{1}{2\pi} \int_o^{2\pi} u(a + re^{i\theta}) d\theta$$

as $n \to \infty$. Since $u_n(a) \to u(a)$ as $n \to \infty$, we have

$$u(a) \leq \frac{1}{2\pi} \int_o^{2\pi} u(a + re^{i\theta}) d\theta$$

and hence u is subharmonic. □

Problem 2. *Let $\phi : G \to \mathbb{R}$ be subharmonic, and let $\overline{\Delta} \subset G$ be a closed disc in G. Consider the Poisson modification $\widetilde{\phi} : G \to \mathbb{R}$ of ϕ along Δ. Show that $\widetilde{\phi}$ is also subharmonic.*

Proof. Let $a \in G$. If $a \in G \setminus \Delta$, let R be chosen so that $\overline{\Delta}(a, R) \subset G$ and furthermore the mean value inequality is satisfied for ϕ in $\overline{\Delta}(a, R)$. Then, for all $0 \leq r < R$, we have

$$\widetilde{\phi}(a) = \phi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(a + re^{it})\, dt \leq \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\phi}(a + re^{it})\, dt$$

using that $\phi \leq \widetilde{\phi}$.

For $a \in \Delta$, let $\overline{\Delta}(a, R) \subset \Delta$. For $r < R$, we have

$$\widetilde{\phi}(a) = h(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a + re^{it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\phi}(a + re^{it})\, dt$$

where h is the harmonic function solving Dirichlet Problem, so that $\phi = h$ in Δ.

The analysis above shows that $\widetilde{\phi}$ satisfies the mean value inequality in sufficiently small discs, so $\widetilde{\phi}$ is subharmonic. $\square$

Problem 3. *Show that the Dirichlet Problem cannot be solved in the punctured disc $G = \Delta(0, 1) \setminus \{0\}$.*

Exhibit a continuous function $f : \partial G \to \mathbb{R}$ which cannot be obtained as the boundary value of a harmonic function in G, continuous in $\overline{G}$.

Proof. Note that $\partial G = \partial\Delta \cap \{0\}$. Define $f : \partial G \to \mathbb{R}$ setting

$$f(0) = 1, \quad f(z) = 0 \text{ for } z \in \partial\Delta.$$

Assume that the Dirichlet problem can be solved for (G, f), and let u be the solution. Then u is continuous at 0, so by Problem Set 2, Problem 2, we conclude that u extends to a harmonic function across 0. Thus u is harmonic in Δ. We can then apply the mean value property conclude

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})\, dt$$

for all $r < 1$. Since u is continuous in $\overline{G}$, it is uniformly continuous there and thus $u(re^{it}) \to u(e^{it})$ uniformly as $r \to 1$. (This can be seen from the definition. Fix $\epsilon > 0$. We can find δ such that $|x - y| < \delta$ implies $|u(x) - u(y)| < \epsilon$. Use this for $x = re^{it}, y = e^{it}$.) Making $r \to 1$ in the above, we obtain

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\, dt = 0.$$

This contradicts $u(0) = 1$. Thus the Dirichlet problem cannot be solved for (G, f).

$\square$

Problem 4. *Let G be open connected, $\zeta_0 \in \partial G$. We say G admits a barrier at ζ_0 provided there exists $\omega : G \to \mathbb{R}$ harmonic, continuous in $\overline{G}$, such that $\omega(\zeta_0) = 0$ and $\omega > 0$ on $\partial G \setminus \{\zeta_0\}$.*

Show that if the Dirichlet Problem is solvable in G, then G admits a barrier at each $\zeta_0 \in \partial G$.

Proof. Let $f(z) = |z - \zeta_0|$. Since the Dirichlet Problem is solvable in G, we can find ω continuous in $\overline{G}$, harmonic in G such that

$$\omega|_{\partial G} = f.$$

In particular $\omega(\zeta_0) = f(\zeta_0) = 0$ and for $z \in \partial G \setminus \{\zeta_0\}$, we have $\omega(z) = |z - \zeta_0| > 0$.

$\square$

Problem 5. *Let G be a region, $\zeta_0 \in \partial G$, and let l be a half-line starting at ζ_0 that intersects $\overline{G}$ only at ζ_0. Let $\zeta_1 \neq \zeta_0$ be a point on the half-line ℓ. Show that ζ_0 is a barrier for ∂G.*

Proof. Let

$$\ell = \{t\zeta_0 + (1-t)\zeta_1 : 0 \leq t \leq 1\}.$$

Then $\ell \cap \overline{G} = \{\zeta_0\}$. Let $\omega : \overline{G} \to \mathbb{R}$ be defined by

$$\omega(z) = \mathrm{Re}\left(\sqrt{\frac{z - \zeta_0}{z - \zeta_1}}\right)$$

where $\sqrt{a} = e^{\frac{1}{2}\mathrm{Log}(a)}$ for any $a \in \mathbb{C} \setminus (-\infty, 0]$ and Log is the principal logarithm, and $\sqrt{0} = 0$.

Note that

$$f(z) = \frac{z - \zeta_0}{z - \zeta_1}$$

is injective and $f(\ell) = (-\infty, 0]$ since

$$f(t\zeta_0 + (1-t)\zeta_1) = -\frac{1-t}{t}.$$

Therefore,

$$f(G) \subset \mathbb{C} \setminus (-\infty, 0].$$

In consequence, $\sqrt{f}$ can be defined as a holomorphic function on G. Then $\omega = \mathrm{Re}(\sqrt{f})$ is harmonic on G. Also, $\sqrt{z}$ is continuous on $\mathbb{C} \setminus (-\infty, 0)$ and hence ω is continuous on $\overline{G}$.

Now, $\omega(\zeta_0) = 0$. Since

$$\sqrt{z} : \mathbb{C} \setminus (-\infty, 0] \to \{z : \mathrm{Re}\ z > 0\}$$

is a surjective map, we have $\omega(z) > 0$ for all $z \in \partial G \setminus \{\zeta_0\}$. Hence, ω is a barrier for ∂G at ζ_0.

□

Problem 6. *Let $\mathcal{H}$ be the family of harmonic functions $h : \Delta \to \mathbb{R}$ with $h(0) = 1$ and $h(z) > 0$ for $z \in \Delta$. Show that every sequence in $\mathcal{H}$ admits a subsequence that converges locally uniformly to a function in $\mathcal{H}$.*

Proof. Let $\{h_n\}$ be a sequence in $\mathcal{H}$. Write $h_n = \mathrm{Re}\ f_n$. Thus $\mathrm{Re}\ f_n > 0$. We know $\mathrm{Re}\ f_n(0) = 1$. By possibly changing f_n by an imaginary constant, we may furthermore assume

$$f_n(0) = 1.$$

We already have seen in Math 220B, Homework 4, Problem 2 that $\{f_n\}$ is a normal family. (On the Qualifying Exam, this statement would require a proof since it is based on a homework question.) Passing to a subsequence, we may assume $f_n \to f$ with $f(0) = 1$. Letting $h = \mathrm{Re}\ f$, we see

$$h_n = \mathrm{Re}\ f_n \to \mathrm{Re}\ f = h.$$

Furthermore, h is harmonic, $h(0) = 1$.

We show $h(z) > 0$ for $z \in \Delta$. Note that

$$h(z) = \lim h_n(z) \geq 0$$

for $z \in \Delta$. If $h(z_0) = 0$ for some $z_0 \in \Delta$, then z_0 would be a maximum for h, violating the maximum principle unless $h \equiv 0$. However, the latter situation is impossible since $h(0) = 1$. Thus $h > 0$ in Δ, proving that $h \in \mathcal{H}$. □