HOMEWORK 2

DUE APRIL 14, 2021 AT 11:59PM

1. Let R be a commutative ring with unit and M an R-module. Let F_M be the functor from the category of R-modules to itself defined for

$$F_M(X) = \mathrm{Hom}_R(X, M) = \{f : X \longrightarrow M; f \text{ is a homomorphism of } R\text{-modules}\}.$$

(a) Show that F_M is a contravariant functor.

(b) Show that F_M is left exact. Is this still the case if we thing of $F_M : R\textbf{-mod} \longrightarrow \mathbb{Z}\textbf{-mod}$ instead?

(c) Find a nontrivial R-module M such that F_M is exact.

(d) Is F_M always exact? Prove or find a counterexample.

2. Formulate and do the same exercise for the covariant functor $G_M(X) = \mathrm{Hom}_R(M, X)$.

3. And now do the same for the functor $F_M : R\textbf{-mod} \longrightarrow R\textbf{-mod}$ $F_M(X) = M \otimes_R X$.

(a) Show that F_M is a covariant functor.

(b) Show that F_M is right exact. Is this still the case if we thing of $F_M : R\textbf{-mod} \longrightarrow \mathbb{Z}\textbf{-mod}$ instead?

(c) Find a nontrivial abelian group M such that F_M is exact.

(d) Is F_M always exact? Prove or find a counterexample.

4. Let G be a group. Denote $\mathbb{Z}[G]$ the free abelian group (or free $\mathbb{Z}$-module) on the set G. That is,

$$\mathbb{Z}[G] = \left\{ \sum_{\sigma \in G} a_\sigma \sigma \,;\, a_\sigma \in \mathbb{Z} \,\forall \sigma \in G \text{ and all but finitely many } a_\sigma\text{'s are equal to zero} \right\}$$

with the natural addition.

(a) Show that $\mathbb{Z}[G]$ becomes a ring with the multiplication

$$\left(\sum_{\sigma \in G} a_\sigma \sigma \right) \cdot \left(\sum_{\tau \in G} b_\tau \tau \right) = \sum_{\sigma \in G} \left(\sum_{\sigma' \tau = \sigma} a_{\sigma'} b_\tau \right) \sigma.$$

(Do show that the multiplication is well-defined.)

(b) What is the multiplicative identity element in this ring?

(c) Show that the set R of finitely supported functions $f : G \longrightarrow \mathbb{Z}$ becomes a ring under the usual function addition and multiplication given by convolution. That is, the elements of R are maps of sets $f : G \longrightarrow \mathbb{Z}$ with the property that $f(\sigma) = 0$ for all but finitely many $\sigma \in G$; the addition is given by $(f + g)(\sigma) = f(\sigma) + g(\sigma)$ for all $\sigma \in G$; and the multiplication is given by

$$(f * g)(\sigma) = \sum_{\tau \in G} f(\tau) g(\tau^{-1} \sigma).$$

(d) Show that $\mathbb{Z}[G]$ is naturally isomorphic to R (as rings).

5. Let G be a group. A (left) *G-module* is an abelian group M on which there is a G action which satisfies for all $m, m' \in M$ and $\sigma, \tau \in G$,

$$\begin{aligned} 1_G m &= m, \\ \sigma(\tau m) &= (\sigma\tau)m, \\ \sigma(m+m') &= \sigma m + \sigma m'. \end{aligned}$$

That is, there is a group homomorphism $G \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(M) : \sigma \mapsto \sigma(\cdot)$. A morphism of G-modules $f : M \longrightarrow N$ is a group homomorphism which also satisfies $f(\sigma m) = \sigma f(m)$, for $m \in M$ and $\sigma \in G$. For a G-module M, the subgroup of *G-invariant* elements of M is

$$M^G := \{m \in M; \sigma m = m, \forall \sigma \in G\}.$$

Consider the functor $F(M) = M^G$ from the category of G-modules to the category of abelian groups.

(a) Show that the category of left G-modules is the same as the category of left modules over the ring $\mathbb{Z}[G]$. (Nothing fancy is warranted here; just describe the correspondence between the two categories.)

(b) Show that F is a left exact functor.

(c) Let t be a variable and let $G = \{t^n; n \in \mathbb{Z}\}$ be the infinite cyclic group generated by t. Let $N = \mathbb{Z}[G] = \mathbb{Z}[t, t^{-1}]$, and let M be the sub-G-module of N,

$$M = \{n \in N; n = n'(t-1) \text{ for some } n' \in N\} = \mathbb{Z}[t, t^{-1}](t-1).$$

Show that N and M are G-modules under left-multiplication. Show that as abelian groups $N/M \cong \mathbb{Z}$ and that the action of G on $\mathbb{Z}$, induced by this isomorphism, is trivial (i.e., $\sigma a = a$ for all $\sigma \in G, a \in \mathbb{Z}$).

(d) Use the exact sequence of G-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow \mathbb{Z} \longrightarrow 0$$

to show that F is not exact.

6. Write down explicitly the isomorphism $\mathrm{Hom}_R(M \otimes_R N, P) \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ and show that it is functorial, i.e. for each pair of R-module homomorphisms $f : M' \longrightarrow M$ and $g : P \longrightarrow P'$, and for any R-module N the diagram

$$\begin{tikzcd} \mathrm{Hom}_R(M \otimes_R N, P) \arrow[r, "\approx"] \arrow[d, "g\circ(-)\circ(f\otimes 1_N)"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \arrow[d, "g_*\circ(-)\circ f"] \\ \mathrm{Hom}_R(M' \otimes_R N, P') \arrow[r, "\approx"'] & \mathrm{Hom}_R(M', \mathrm{Hom}_R(N, P')) \end{tikzcd}$$

is commutative. Here g_* denotes the pushforward of g.

7. Find the left adjoints (and prove that they are adjoints) for the following forgetful functors.

(a) From the category of commutative rings (with unit) to the category of sets.

(b) From the category of K-vector spaces to the category of sets.

(c) (Bonus) From the category of rings with unit (not necessarily commutative) to the category of abelian groups.