Solutions: Homework 4

Problem 1. *Let $f : G \to \mathbb{C}$ be holomorphic, and let $\overline{\Delta}(0, r) \subset G$. Assume that $f(0) \neq 0$. Let $z_1, \dots, z_k$ be zeros of f in the open disc $\Delta(0, r)$. Show that*

$$|f(0)| \leq |z_1 ... z_k| \cdot \frac{M(r)}{r^k}.$$

Proof. Without loss of generality, we may assume $z_1, \dots, z_k$ are all zeros of f in the disc $\Delta(0, r)$, listed with multiplicities. Indeed, if we missed one of the zeroes, say at w, the inequality would only be weaker since

$$\frac{|z_1 \dots z_k w|}{r^{k+1}} \leq \frac{|z_1 \cdots z_k|}{r^k}.$$

By Jensen's formula, we have

$$\log|f(0)| + \sum_{n=1}^{k} \log\left(\frac{r}{|z_n|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| d\theta.$$

Then

$$\log\left|\frac{f(0)r^k}{z_1 ... z_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| d\theta \leq \log M(r)$$

This implies that

$$|f(0)| \leq |z_1 ... z_k| \frac{M(r)}{r^k}.$$

□

Problem 2. *Assume f is a bounded holomorphic function*

$$f : \Delta(0, 1) \to \mathbb{C}$$

with zeroes $a_1, a_2, ...$ listed with multiplicity. Show that

$$\sum_n (1 - |a_n|) < \infty.$$

Proof. There is nothing to prove if f has finitely many zeroes, so let us assume that there are infinitely many zeroes in the unit disc.

Let us first consider the case $f(0) \neq 0$. Let M be such that $|f(z)| \leq M$ for all $z \in \Delta(0,1)$. Let $k > 0$ be fixed, and let r be chosen such that $N(r) > k$. Using Problem 1, we have

$$\prod_{n=1}^{k} |a_n| \geq \frac{|f(0)|}{M} r^k.$$

Now r is arbitrary such that $N(r) > k$, so we can take $r \to 1$, keeping k fixed. We obtain

$$\prod_{n=1}^{k} |a_n| \geq \frac{|f(0)|}{M}.$$

In turn,

$$\sum_{n=1}^{k} (-\log|a_n|) \leq C$$

for some constant

$$C = -\log|f(0)| + \log M.$$

Since $-\log|a_n| > 0$, we conclude that the series

$$\sum_n (-\log|a_n|)$$

converges. Note that

$$-\log|a| \geq 1 - |a|, \text{ for all } |a| < 1.$$

(Write $|a| = e^{-x}$ with $x > 0$ and the inequality to be proven is $1 - x - e^{-x} \geq 0$ which is immediate by examining the critical points.) By the comparison test, we see that

$$\sum_n (1 - |a_n|)$$

converges as well.

Now suppose that $f(0) = 0$. Then $f(z) = z^m h(z)$ for some $m \geq 1$ and h holomorphic on $\Delta(0,1)$ with $h(0) \neq 0$. So, if we prove that h is bounded, then the previous case proves the result for h, and hence f as the non-zero zeros of h and f are the same. Note that for any $z \in \Delta(0,1)$,

$$h(z) \leq \max\left\{ \sup_{|z|=1/2} |h(z)|, \sup_{1/2<|z|<1} |h(z)| \right\}$$

But note that for $1/2 < |z| < 1$,

$$|h(z)| = \frac{|f(z)|}{|z|^m} \leq 2^m |f(z)| \leq 2^m M$$

This proves that h is bounded. $\square$

Problem 3. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.*
(i) If $f(0) = 1$ show that

$$N(r) \log 2 \leq \log M(2r).$$

(ii) Assume that

$$|f(z)| \leq \exp(A|z|^k)$$

for $A > 0$ and k natural number. Show that

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} \leq k.$$

Proof. (i) Fix $r > 0$. Let the zeros of f in $\Delta(0, r)$ be $a_1, ..., a_{N(r)}$ repeated according to multiplicity, and let $a_{N(r)+1}, ..., a_{N(2r)}$ denote the zeros of f in $\Delta(0, 2r) \setminus \Delta(0, r)$ repeated according to multiplicity. Then, by Jensen's formula, we have

$$\log |f(0)| + \sum_{n=1}^{N(2r)} \log\left(\frac{2r}{|a_n|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log |f(2re^{i\theta})| d\theta$$

Rewriting this, we have

$$\sum_{n=1}^{N(r)} \log\left(\frac{2r}{|a_n|}\right) + \sum_{n=N(r)+1}^{N(2r)} \log\left(\frac{2r}{|a_n|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log |f(2re^{i\theta})| d\theta$$

Since the second summation on the LHS above is non-negative, we have

$$\sum_{n=1}^{N(r)} \log\left(\frac{2r}{|a_n|}\right) \leq \frac{1}{2\pi}\int_0^{2\pi} \log |f(2re^{i\theta})| d\theta \leq M(2r)$$

We have, for $1 \leq n \leq N(r)$

$$\log 2 \leq \log\left(\frac{2r}{|a_n|}\right)$$

and so

$$N(r)\log 2 \leq M(2r).$$

(ii) Now suppose that

$$|f(z)| \leq \exp(A|z|^k)$$

for some $A > 0$ and $k \in \mathbb{N}$. Then we have

$$M(2r) \leq \exp(2^k A r^k)$$

and hence, by part (i), we have

$$N(r)\log 2 \leq \log M(2r) \leq 2^k A r^k$$

Taking log, we have

$$\log N(r) + \log\log 2 \leq k\log 2 + \log A + k\log r$$

So, for $r > 1$, we have

$$\frac{\log N(r)}{\log r} \leq \frac{k\log 2 + \log A - \log\log 2}{\log r} + k$$

Taking lim sup, we have

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} \leq k.$$

□

Problem 4. *If f, g are entire functions of order λ_1, λ_2, show that fg has order $\leq \lambda = \max(\lambda_1, \lambda_2)$.*

Proof. We have

$$\lambda_1 \leq \lambda, \quad \lambda_2 \leq \lambda.$$

Let $\epsilon > 0$. Then there exists $R > 1$ such that for all $|z| > R$, we have

$$|f(z)| < \exp(|z|^{\lambda_1 + \frac{\epsilon}{2}}) \leq \exp(|z|^{\lambda + \frac{\epsilon}{2}})$$

and

$$|g(z)| < \exp(|z|^{\lambda_2 + \frac{\epsilon}{2}}) \leq \exp(|z|^{\lambda + \frac{\epsilon}{2}}).$$

Multiplying

$$|fg(z)| < \exp(2|z|^{\lambda+\frac{\epsilon}{2}})$$

Let $R' = \max\{2^{\frac{2}{\epsilon}}, R\}$. Then, for $|z| > R'$,

$$2 \leq |z|^{\frac{\epsilon}{2}}.$$

and so, for $|z| > R'$, we have

$$|fg(z)| < e^{|z|^{\lambda+\epsilon}}.$$

Thus letting $M(R)$ denote the corresponding function for fg we have $M(R) < e^{R^{\lambda+\epsilon}}$ hence

$$\lambda(fg) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \leq \limsup_{R\to\infty} \frac{\log\log e^{R^{\lambda+\epsilon}}}{\log R} = \lambda + \epsilon.$$

Since ϵ was arbitrary, it follows fg has order $\leq \lambda = \max(\lambda_1, \lambda_2)$. □

Problem 5. *Let*

$$|a_1| \leq |a_2| \leq ...$$

be a sequence of non-zero complex numbers and let

$$\alpha = \inf\{t : \sum_n \frac{1}{|a_n|^t} < \infty\}.$$

Assume f is an entire function with zeros only at $a_1, a_2, ...$ Let λ be the order of f.
(i) Show that for any $\epsilon > 0$,

$$\sum_n \frac{1}{|a_n|^{\alpha+\epsilon}} < \infty, \sum_n \frac{1}{|a_n|^{\alpha-\epsilon}} = \infty$$

(ii) Show that $\alpha \leq \lambda$.

Proof. (i) By definition, for any $\epsilon > 0$,

$$\sum_n \frac{1}{|a_n|^{\alpha-\epsilon}} = \infty.$$

For any $\epsilon > 0$, there exists $0 \leq \epsilon' < \epsilon$ such that

$$\sum_n \frac{1}{|a_n|^{\alpha+\epsilon'}} < \infty.$$

Now, since $|a_n| \to \infty$, for $n >> 0$, we have $|a_n| > 1$ and thus

$$\frac{1}{|a_n|^{\alpha+\epsilon}} < \frac{1}{|a_n|^{\alpha+\epsilon'}}.$$

So,

$$\sum_n \frac{1}{|a_n|^{\alpha+\epsilon}} < \infty.$$

(ii) WLOG we may assume $f(0) = 1$. Let $\epsilon > 0$. We have

$$n - 1 \leq N(|a_n|) \leq \frac{\log M(2|a_n|)}{\log 2}$$

Since f is of order λ, there exists $N \geq 1$ such that for $n \geq N$, we have

$$\log M(2|a_n|) \leq (2|a_n|)^{\lambda+\epsilon}$$

So, for $n \geq N$, we have

$$n - 1 \leq \frac{2^{\lambda+\epsilon}}{\log 2}|a_n|^{\lambda+\epsilon}$$

So, for $n \geq N$, we have

$$\frac{1}{|a_n|} \leq \frac{2}{((n-1)\log 2)^{1/(\lambda+\epsilon)}}$$

Then, for $n \geq N$

$$\frac{1}{|a_n|^{\lambda+2\epsilon}} \leq \left(\frac{2^{\lambda+2\epsilon}}{(\log 2)^{(\lambda+2\epsilon)/(\lambda+\epsilon)}}\right) \frac{1}{(n-1)^{(\lambda+2\epsilon)/(\lambda+\epsilon)}}$$

Since $\frac{\lambda+2\epsilon}{\lambda+\epsilon} > 1$, the RHS, considered as a series in $2 \leq n < \infty$ above converges, and hence

$$\frac{1}{|a_n|^{\lambda+2\epsilon}} < \infty$$

This shows that $\lambda + 2\epsilon > \alpha$. As $\epsilon > 0$ was arbitrary, this shows that $\lambda \geq \alpha$. $\square$

Problem 6. *Show that* $\cos\sqrt{z}$ *is an entire function of order* $\frac{1}{2}$ *and genus* 0.

Proof. We know that for all $z \in \mathbb{C}$, we have

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

The square root is initially defined for $z \in \mathbb{C} \setminus (-\infty, 0]$, and we have

$$\cos \sqrt{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$$

Note that the series on the RHS above has radius of convergence $R = \infty$ by the root test

$$R^{-1} = \limsup_{n \to infty} \frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n)!}} = \limsup_{n \to \infty} \frac{1}{(2n+1)(2n+2)} = 0 \implies R = \infty.$$

Thus, RHS defines an entire function. This shows that $\cos \sqrt{z}$ is/can to be extended to an entire function.

To find the order, note that for $|z| = R$, we have

$$\left|\cos \sqrt{z}\right| = \left| \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!} \right| \leq \sum_{n=0}^{\infty} \frac{R^n}{(2n)!} = \frac{e^{R^{1/2}} + e^{-R^{1/2}}}{2} \leq e^{R^{1/2}}.$$

Thus

$$M(R) \leq e^{R^{1/2}} \implies \log \log M(R) \leq \frac{1}{2} \log R \implies \lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R} \leq \frac{1}{2}.$$

Thus $\lambda \leq \frac{1}{2}$. For the opposite inequality, let $\epsilon > 0$. Now,

$$|\cos \sqrt{-n^2}| = |\cos(in)| = \frac{e^n + e^{-n}}{2} \geq \frac{e^n}{2} > e^{n^{1-2\epsilon}} = e^{|-n^2|^{1/2-\epsilon}}$$

for all n large enough. This shows $\lambda \leq \frac{1}{2}$, and so $\lambda = \frac{1}{2}$.

To find the genus, recall from Math 220B, Lecture 4 that

$$\cos z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{\pi^2 (2n-1)^2}\right).$$

Thus

$$\cos \sqrt{z} = \prod_{n=1}^{\infty} \left(1 - \frac{4z}{\pi^2 (2n-1)^2}\right) = \prod_{n=1}^{\infty} E_0\left(\frac{z}{a_n}\right),$$

for $a_n = \frac{\pi^2}{4}(2n-1)^2$. Furthermore by the comparison test with $\lim_{n\to\infty} \frac{a_n}{n^2} < \infty$ and the harmonic series test, we have

$$\sum \frac{1}{|a_n|^{p+1}} < \infty \iff \sum \frac{1}{n^{2(p+1)}} < \infty \iff 2(p+1) > 1 \iff p \geq 0.$$

This shows that the rank is $p = 0$, hence genus $h = 0$.

$\square$