Math 220 A - Lecture 17

November 20, 2020

Office hour next week : Tuesday 2 – 3:30 PM.

Applications of the Residue theorem to real analysis

[a] trigonometric functions

[b] rational functions

[c] Fourier integrals

[d] logarithmic integrals

[e] Mellin transforms

$\boxed{6}$ Rational functions $\quad I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx.$,

Require : (1) Q has no zeros on the real axis

(2) $\deg P - \deg Q \leq -2.$

Claim : The integral converges absolutely

Write $f(x) = \dfrac{P(x)}{Q(x)}$.

By (2) $\Rightarrow \lim\limits_{|x|\to\infty} x^2 f(x) = \alpha \Rightarrow \exists\, R>0$ with

$$|f(x)| < \frac{\alpha+1}{x^2} \quad \text{for } |x|>R. \qquad (*)$$

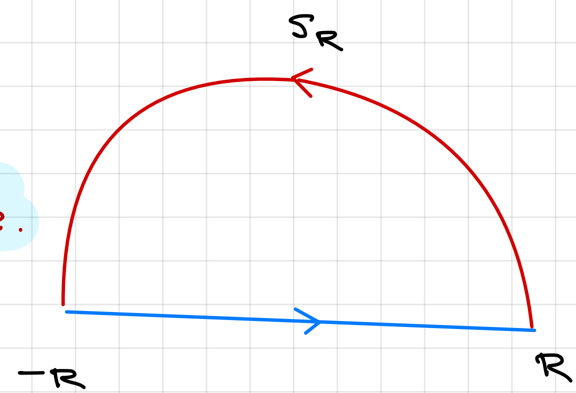
By the comparison test $\Rightarrow \int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$ QED.

Conclusion $\quad I = \lim\limits_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$

Strategy

[i] $f(z) = \dfrac{P(z)}{Q(z)}$

$$\gamma_R = [-R, R] \cup S_R.$$



[ii] Residue theorem

$$\int_{-R}^{R} f(x)\,dx + \int_{S_R} f\,dz = \int_{\gamma_R} f\,dz = 2\pi i \sum_{\substack{a_j \in \mathbb{H}^+ \\ |a_j| < R}} \operatorname{Res}(f, a_j).$$

[iii] Make $R \to \infty$. Show $\displaystyle\lim_{R\to\infty} \int_{S_R} f\,dz = 0$

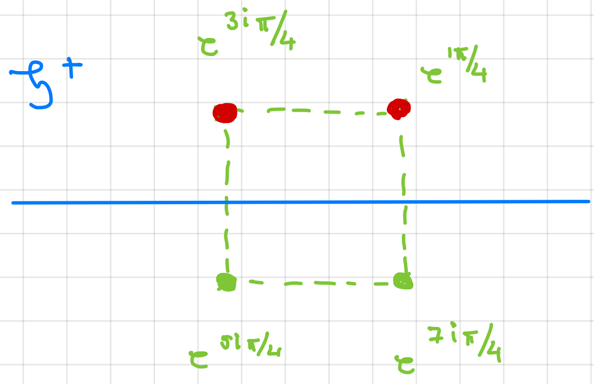
$$\left| \int_{S_R} f\,dz \right| \le \pi R \cdot \frac{\alpha + 1}{R^2} \longrightarrow 0 \text{ as } R \to \infty, \text{ using } (*).$$

From [ii], we obtain

Conclusion

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a_j \in \mathbb{H}^+} \operatorname{Res}\left(\frac{P}{Q}, a_j\right).$$

Example $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$



Poles at $z^4+1=0$

$$\Rightarrow \quad z_k = e^{\frac{\pi i}{4}(2k+1)}, \quad k = 0,1,2,3.$$

Only $e^{\pi i/4}, \ e^{3\pi i/4} \in \mathfrak{H}^+$.

By Method 1,

$$\operatorname*{Res}_{z=z_k} \frac{1}{z^4+1} = \frac{1}{4z^3}\Big/_{z=z_k} = \frac{1}{4z_k^3} = -\frac{z_k}{4}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \left(\operatorname*{Res}_{z=e^{\pi i/4}} \frac{1}{z^4+1} + \operatorname*{Res}_{z=e^{3\pi i/4}} \frac{1}{z^4+1} \right)$$

$$= 2\pi i \left(-\frac{1}{4} e^{\pi i/4} - \frac{1}{4} e^{3\pi i/4} \right)$$

$$= \frac{\pi}{\sqrt{2}}.$$

C Fourier Integrals I

$$I = \int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx \quad \text{(use upper half plane)}$$

$$I = \int_{-\infty}^{\infty} f(x)\, e^{-ix}\, dx \quad \text{(use lower half plane)}$$

Require (1) f extends meromorphically to $\mathfrak{H}^+$

(2) no poles on the real axis.

(3) $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

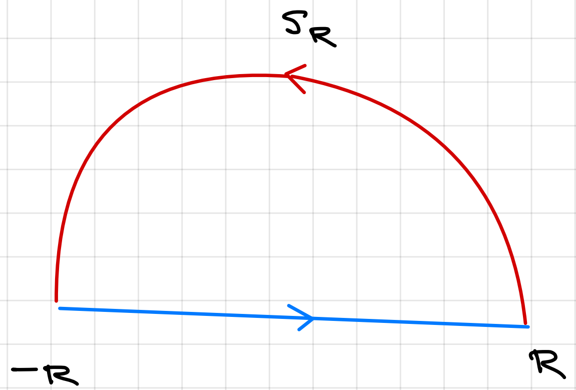
Convergence: By (3), $\int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx$ converges absolutely

Thus

$$I = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, e^{ix}\, dx.$$

Strategy Use the same contour

$\gamma_R = [-R, R] \cup S_R.$



By the residue theorem

$$\int_{-R}^{R} f(x) e^{ix} dx + \int_{S_R} f \, dz = \int_{\gamma_R} f \, dz = 2\pi i \sum_{z = a_j} \operatorname{Res} (f(z) e^{iz})$$

Make $R \to \infty$. Assume moreover

(4) $$\lim_{\substack{z \to \infty \\ z \in \overline{\mathfrak{H}}^+}} f(z) = 0.$$

The next lemma shows $\displaystyle\lim_{R \to \infty} \int_{S_R} f \, dz = 0.$

Conclusion

$$\int_{-\infty}^{\infty} f(x) e^{ix} dx = 2\pi i \sum_{a_j \in \mathfrak{H}^+} \operatorname{Res} (f(z) e^{iz}, a_j).$$

Lemma If $\lim\limits_{\substack{z \to \infty \\ z \in \overline{\mathbb{H}}^+}} |f(z)| = 0$ then

$$\lim_{R \to \infty} \int_{S_R} f(z) e^{iz} \, dz = 0$$

Proof Write $z = Re^{it}$, $0 \leq t \leq \pi$.

$$M_R = \sup_{z \in S_R} |f(z)| \quad , \quad M_R \longrightarrow 0 \text{ as } R \longrightarrow \infty$$

$$\left| \int_{S_R} f(z) e^{iz} dz \right| = \left| \int_0^{\pi} f(Re^{it}) \, e^{iRe^{it}} \cdot iRe^{it} \, dt \right|$$

$$\leq \int_0^{\pi} M_R \cdot \left| e^{iRe^{it}} \right| \cdot R \, dt$$

$$= \int_0^{\pi} M_R \left| e^{iR(\cos t + i \sin t)} \right| R \, dt$$

$$= \int_0^{\pi} R M_R \left| e^{iR\cos t} e^{-R \sin t} \right| dt$$

$$= \int_0^{\pi} R M_R \, e^{-R\sin t} dt$$

$$= 2 \int_0^{\pi/2} R \, M_R \, e^{-R \sin t} dt$$

Claim

$$\leq 2 \int_0^{\pi/2} R M_R \, e^{-R \cdot \frac{2}{\pi} t} = \pi M_R (1 - e^{-R}) \longrightarrow 0$$

as $R \longrightarrow \infty$.

Claim $\frac{2}{\pi} \le \frac{\sin t}{t} \quad \forall\, t \in \left(0, \frac{\pi}{2}\right]$

Proof $f(t) = \frac{\sin t}{t}$.

$$f(\pi/2) = \frac{2}{\pi}$$

We show f is decreasing. Then $f(t) \ge f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \Rightarrow$

$\Rightarrow \frac{\sin t}{t} \ge \frac{2}{\pi}$

To this end, compute $f'(t) = \frac{t \cos t - \sin t}{t^2} \le 0$

$$\Leftrightarrow t \cos t \le \sin t$$

$$\Leftrightarrow \tan t - t \ge 0.$$

Let

$$g(t) = \tan t - t, \quad g(0) = 0$$

We compute $g'(t) = \frac{1}{\cos^2 t} - 1 \ge 0 \Rightarrow g \nearrow \Rightarrow$

$\Rightarrow g(t) \ge g(0) = 0$ as needed. QED

Example $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \operatorname{Re} I. = \frac{\pi}{e}.$

Let $f(z) = \dfrac{1}{1+z^2}$, $z = i$ is the only pole in $\mathbb{H}^+$.

Method 1

$$I = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{1+z^2}\right)\Big/_{z=i} =$$

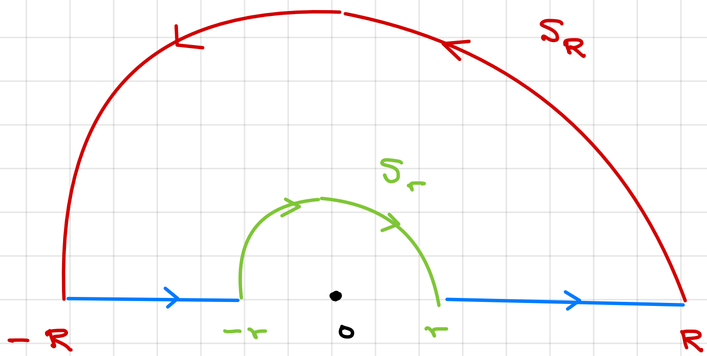
$$= 2\pi i \cdot \frac{e^{iz}}{2z}\Big/_{z=i}$$

$$= \cancel{2}\pi \cancel{i} \cdot \frac{e^{-1}}{\cancel{2}\cancel{i}} = \frac{\pi}{e}.$$

Fourier Integrals – Part II. – Poles on the real axis

Example $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

- issues at 0 & ∞.



- $\displaystyle I = \lim_{\substack{r \to 0 \\ R \to \infty}} \int_r^R \frac{\sin x}{x}\,dx$

Strategy
- $f(z) = \dfrac{e^{iz}}{z}$

- $\gamma = S_R + [-R, -r] + (-S_r) + [r, R]$

$$0 = \int_\gamma f\,dz = \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R \frac{e^{iz}}{z}\,dz + \int_{-R}^{-r} \frac{e^{iz}}{z}\,dz$$

$$= \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R \frac{e^{iz}}{z} - \frac{e^{-iz}}{z}\,dz$$

$$= \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R 2i\,\frac{\sin z}{z}\,dz$$

Make $r \to 0$, $R \to \infty$. By the claim:

$$0 = 0 - i\pi + 2i\int_0^\infty \frac{\sin x}{x}\,dx \Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Claims $\boxed{a}$ $$\lim_{R \to \infty} \int_{S_R} \frac{e^{iz}}{z}\, dz = 0$$

$\boxed{b}$ $$\lim_{r \to 0} \int_{S_r} \frac{e^{iz}}{z}\, dz = i\pi$$

Claim $\boxed{a}$ follows from the previous Lemma applied to

$f(z) = \frac{1}{z}$.

Claim $\boxed{b}$ requires a proof. We will go over the proof next time.