Math 220B - Winter 2021 - Midterm

Problem 1.

(i) Find an entire function with simple zeroes only at $z = \sqrt{n}$ for each $n \in \mathbb{Z}_{\geq 0}$, and no other zeroes.

(ii) Give an example of a meromorphic function function with poles only at $z = -\sqrt{n}$ and principal parts $\frac{1}{z+\sqrt{n}}$, for $n \in \mathbb{Z}_{\geq 0}$.

(iii) Assume $\{a_n\}, \{b_n\}$ be sequences with no common terms, such that $\sum_n |a_n - b_n| < \infty$ and $b_n \to \infty$ as $n \to \infty$. Show that

$$f(z) = \prod_{n=1}^{\infty} \frac{z - a_n}{z - b_n}$$

defines a holomorphic function in the open set $\mathbb{C} \setminus \{b_1, b_2, \ldots\}$. What are the zeros of f?

`Solution:`

(i) *Let* $p_n = 2$ *for all* n. *Note that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{\sqrt{n}}\right)^{p_n+1} = \sum_{n=1}^{\infty} \frac{r^{3/2}}{n^{3/2}} < \infty,$$

for all r. *By the Weierstraß factorization theorem we obtain that*

$$f(z) = z \prod_{n=1}^{\infty} E_2\left(\frac{z}{\sqrt{n}}\right)$$

solves the Weierstraß problem in (i).

(ii) *We Taylor expand around the Laurent tail* q_n *at the origin for* $n \neq 0$*:*

$$q_n(z) = \frac{1}{z + \sqrt{n}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{1 + \frac{z}{\sqrt{n}}} = \frac{1}{\sqrt{n}} \left(1 - \frac{z}{\sqrt{n}} + \frac{z^2}{n} - \ldots\right).$$

Let

$$q_n = \frac{1}{\sqrt{n}} \left(1 - \frac{z}{\sqrt{n}}\right).$$

We compute

$$|q_n - h_n| = \left|\frac{z^2}{n(z + \sqrt{n})}\right|.$$

Letting $r_n = \frac{n^{1/8}}{2} < \sqrt{n}$. *We have*

$$|q_n - h_n| = \left|\frac{z^2}{n(z + \sqrt{n})}\right| \leq \frac{r_n^2}{n(\sqrt{n} - r_n)} := c_n.$$

Note that

$$\lim_{n\to\infty} \frac{c_n}{n^{5/4}} < \infty.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}} < \infty$ *it follows by the limit comparison test that*

$$\sum_{n=1}^{\infty} c_n < \infty.$$

Thus the choices q_n, c_n, r_n *verify the requirements in the proof of Mittag-Leffler, and*

$$f = \frac{1}{z} + \sum_{n=1}^{\infty}(q_n - h_n) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+\sqrt{n}} - \frac{1}{\sqrt{n}}\left(1 - \frac{z}{\sqrt{n}}\right)\right)$$

is the Mittag-Leffler solution.

(iii) *Let*

$$f_n(z) = \frac{z-a_n}{z-b_n} - 1 = \frac{b_n - a_n}{z - b_n}.$$

We show that $\prod_{n=1}^{\infty}(1+f_n)$ *converges absolutely and locally uniformly in*

$$U = \mathbb{C} \setminus \{b_1, b_2, \ldots\}.$$

The set U *is indeed open in* $\mathbb{C}$ *since the complement is closed as* $b_n \to \infty$. *The limit is a holomorphic function in* U *with zeros whenever*

$$1 + f_n(z) = 0 \text{ for some } n \iff \frac{z-a_n}{z-b_n} = 0 \iff z = a_n \text{ for some } n.$$

To show the product converges absolutely and locally uniformly, it suffices to show that the series

$$\sum_{n=1}^{\infty} f_n$$

converges absolutely and locally uniformly in U. *Let* K *be a compact set in* U, *and assume* $K \subset \Delta(0,R)$ *for some* R. *Since* $b_n \to \infty$, *we can find* N *such that for all* $n \geq N$:

$$|b_n| \geq R + 1.$$

Then

$$|z - b_n| \geq |b_n| - |z| \geq |b_n| - R \geq 1$$

if $n \geq N$ *and* $z \in K$. *Thus*

$$|f_n(z)| = \frac{|a_n - b_n|}{|z - b_n|} \leq |a_n - b_n| = M_n$$

for $n \geq N$ *and* $z \in K$. *Note that*

$$\sum M_n < \infty$$

by assumption. Thus by the Weierstraß M-test, we obtain that

$$\sum_{n=1}^{\infty} f_n$$

converges absolutely and uniformly on K, *as needed.*

Problem 2.

Let $f : \Delta(0,1) \setminus \{0\} \to \mathbb{C}$ be a holomorphic function on the punctured unit disc. Let

$$f_n : \Delta(0,1) \setminus \{0\} \to \mathbb{C}, \quad f_n(z) = f\left(\frac{z}{n}\right).$$

Show that if the family $\mathcal{F} = \{f_n : n \geq 1\}$ is normal iff f has a removable singularity at the origin.

`Solution:` *Assume that f has a removable singularity at the origin. We show that $\{f_n\}$ is a locally bounded family. Let $K \subset \Delta$ be compact. Without loss of generality we may assume*

$$K \subset \overline{\Delta}(0,r)$$

for some $r < 1$. Since f is continuous in the compact set $\overline{\Delta}(0,r)$, we find a bound

$$|f(z)| \leq M \text{ for all } |z| \leq r.$$

Then

$$z \in K \implies |z| \leq r \implies \left|\frac{z}{n}\right| \leq \frac{r}{n} \leq r \implies \left|f\left(\frac{z}{n}\right)\right| \leq M \implies |f_n(z)| \leq M.$$

Thus $\mathcal{F}$ is bounded on K, hence locally bounded.

For the converse, assume $\mathcal{F}$ is normal. In particular, $\mathcal{F}$ is locally bounded, hence bounded on each compact set e.g. the circle $|z| = \frac{1}{2}$. Thus

$$|f_n(z)| \leq M$$

for all n and $|z| = \frac{1}{2}$, so

$$|f(z)| \leq M$$

whenever $|z| = \frac{1}{2n}$. Write C_n for the circle $|z| = \frac{1}{2n}$ and consider the annulus $D_n = \overline{\Delta}\left(0; \frac{1}{2n+2}; \frac{1}{2n}\right)$ with boundary

$$\partial D_n = C_n \cup C_{n+1}.$$

By the maximum modulus principle, using that f is bounded by M on ∂D_n, it follows that

$$|f(z)| \leq M \quad \forall z \in D_n.$$

Since

$$\cup_{n=1}^{\infty} D_n = \Delta\left(0, \frac{1}{2}\right) \setminus \{0\},$$

it follows that f is bounded in $\Delta\left(0, \frac{1}{2}\right) \setminus \{0\}$. By the removable singularity theorem, it follows that the singularity at 0 is removable.

Problem 3.

Let $f : \Delta(0,1) \to \mathbb{C}$ be such that Re $f(z) > 0$ for all $z \in \Delta$, and assume that $f(0) = 1$.

(i) Show that for all $z \in \Delta(0,1)$ we have
$$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}.$$

(ii) Find the minimum and maximum value of $\left|f\left(\frac{1}{2}\right)\right|$.

`Solution:`

(i) *Consider*
$$\phi : \{z : Re\ z > 0\} \to \Delta(0,1)$$
the Cayley-like transform
$$\phi(z) = \frac{z-1}{z+1}.$$
Note that
$$\phi(z) = C(iz) \quad where \quad C(w) = \frac{w-i}{w+i}$$
is the usual Cayley transform taking us from $\Delta(0,1) \to \mathfrak{h}^+$. The multiplication by i in the argument is needed so that we map from
$$\{z : Re\ z > 0\} \to \mathfrak{h}^+.$$
Let
$$g = \phi \circ f : \Delta(0,1) \to \Delta(0,1), \quad g = \frac{f-1}{f+1}.$$
Note that $g(0) = 0$. By Schwarz's lemma, we obtain
$$|g(z)| \leq |z| \iff \left|\frac{1-f(z)}{1+f(z)}\right| \leq |z| \iff |1-f(z)| \leq |z||1+f(z)|.$$
Using the triangle inequality
$$1-|f(z)| \leq |1-f(z)|, \quad |1+f(z)| \leq 1+|f(z)|,$$
we obtain
$$1-|f(z)| \leq |z|(1+|f(z)|) \iff (1-|z|) \leq |f(z)|(1+|z|) \iff \frac{1-|z|}{1+|z|} \leq |f(z)|.$$
Similarly, using
$$|f(z)|-1 \leq |f(z)-1|, \quad |1+f(z)| \leq 1+|f(z)|,$$
we obtain
$$|f(z)|-1 \leq |z|(1+|f(z)|) \iff |f(z)|(1-|z|) \leq 1+|z| \iff |f(z)| \leq \frac{1+|z|}{1-|z|}.$$

(ii) *For* $z = \frac{1}{2}$, *we have*

$$\frac{1}{3} \leq \left| f\left(\frac{1}{2}\right) \right| \leq 3.$$

The minimum value equals $\frac{1}{3}$ *and the maximum value is* 3.

The maximum value is indeed achieved. We can consider the inverse

$$\phi^{-1} : \Delta \to \{z : Re\ z > 0\}, \quad \phi^{-1}(z) = \frac{1+z}{1-z}.$$

Letting

$$f(z) = \phi^{-1}(z) = \frac{1+z}{1-z}$$

we see that $f(1/2) = 3$. *Using*

$$f(z) = \phi^{-1}(-z) = \frac{1-z}{1+z}$$

we achieve the minimum value $f(1/2) = 1/3$.

Problem 4.

Recall the function
$$G(z) = \prod_{n=1}^{\infty} E_1\left(-\frac{z}{n}\right).$$

(i) Show that
$$\left(z+\frac{1}{2}\right) G(z) G\left(z+\frac{1}{2}\right) = e^{h(z)} G(2z),$$
for some entire function h.

(ii) Show furthermore that $h(z) = az + b$.

`Solution:`

(i) *The product $G(z)$ converges absolutely and locally uniformly by Weierstraß factorization theorem since indeed setting all $p_n = 1$, we have that for all $r > 0$,*
$$\sum_n \left(\frac{r}{n}\right)^{p_n+1} = r^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$
The function $G(z)$ has zeros at all negative integers
$$-1, -2, \ldots.$$
The function $G\left(z+\frac{1}{2}\right)$ has zeros at
$$-\frac{3}{2}, -\frac{5}{2}, \ldots.$$
Thus $\left(z+\frac{1}{2}\right) G(z) G\left(z+\frac{1}{2}\right)$ has zeros at
$$-\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots.$$
Similarly, $G(2z)$ has zeros the negative half integers
$$-\frac{1}{2}, -1, -\frac{3}{2}, \ldots.$$
The two sides are both entire and have exactly the same zeroes hence
$$\left(z+\frac{1}{2}\right) G(z) G\left(z+\frac{1}{2}\right) = e^{h(z)} G(2z),$$
for some entire function $h(z)$.

(ii) *We compute the logarithmic derivatives of both sides. This yields*
$$\frac{1}{z+\frac{1}{2}} + \frac{G'(z)}{G(z)} + \frac{G'\left(z+\frac{1}{2}\right)}{G\left(z+\frac{1}{2}\right)} = h'(z) + 2\frac{G'(2z)}{G(2z)}.$$
Since
$$G(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}.$$

We have

$$\frac{G'(z)}{G(z)} = \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right).$$

Furthermore,

$$\frac{G'\left(z+\frac{1}{2}\right)}{G\left(z+\frac{1}{2}\right)} = \sum_{n=1}^{\infty} \left(\frac{1}{z+\frac{1}{2}+n} - \frac{1}{n} \right).$$

Finally,

$$\frac{G'(2z)}{G(2z)} = \sum_{n=1}^{\infty} \left(\frac{1}{2z+n} - \frac{1}{n} \right).$$

This yields

$$h'(z) = \frac{1}{z+\frac{1}{2}} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+\frac{1}{2}+n} - \frac{1}{n} \right) - 2 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{2z+n} - \frac{1}{n} \right).$$

In the last sum (doubled), the terms with $n \to 2n$ become $\sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right)$ so they cancel the first sum. For $n \to 2n+1$, the terms involving z match with the terms in the second sum, but the free terms differ by

$$-\frac{1}{n} + \frac{2}{2n+1}.$$

Rearrangements into even and odd summands are allowed since we have seen in class that the logarithmic derivative preserves the absolute and local uniform convergence. Thus $h'(z)$ does not depend on z, showing $h' = a$ for a constant a. In fact,

$$a = \sum_{n=1}^{\infty} \left(-\frac{1}{n} + \frac{2}{2n+1} \right).$$

This gives $h(z) = az + b$ for a constant b.

Problem 5.

Let $a, b \neq 0$ and $a, b \in \Delta(0,1)$. Consider the twice punctured discs

$$D_1 = \Delta(0,1) \setminus \{0, a\}, \quad D_2 = \Delta(0,1) \setminus \{0, b\}.$$

Find a necessary and sufficient condition for D_1, D_2 to be biholomorphic, and determine all biholomorphic maps

$$f : D_1 \to D_2.$$

`Solution:` *We claim that the condition is $|a| = |b|$. Indeed, if $|a| = |b|$ we can use the rotation*

$$f(z) = \alpha z$$

for $\alpha = \frac{b}{a}$ with $|\alpha| = 1$. We immediately verify that

$$f(0) = 0, \quad f(a) = b, \quad f : D_1 \to D_2 \textit{ is biholomorphic.}$$

Conversely, assume that $f : D_1 \to D_2$ is a biholomorphism. Write $\Delta = \Delta(0,1)$ and note $D_2 \subset \Delta$. Since

$$|f(z)| < 1$$

for all z in a neighborhood of the punctures 0 and a, it follows by the removable singularity theorem that f extends across 0 and a to a holomorphic map

$$F : \Delta \to \overline{\Delta}.$$

We claim furthermore that

$$F : \Delta \to \Delta.$$

Indeed, if there existed $z_0 \in \Delta$ such that $F(z_0) \in \partial\Delta$, then we would contradict the open mapping theorem applied to the nonconstant holomorphic function F.

We claim next that

$$F(0) = 0, F(a) = b \textit{ or } F(0) = b, F(a) = 0.$$

Indeed, let $F(0) = \alpha$. If $\alpha \in D_2$, then since f is bijective, we must have $f(z) = \alpha$ for some $z \in D_1$. Pick

$$\Delta_0, \Delta_z, \Delta_\alpha$$

small discs around $0, z, \alpha$ such that

$$\Delta_\alpha \subset F(\Delta_0), \quad \Delta_\alpha \subset f(\Delta_z), \quad \Delta_0 \cap \Delta_z = \emptyset, \quad a \notin \Delta_0, \quad a \notin \Delta_z.$$

This is possible by the open mapping theorem. Let $w \in \Delta_\alpha \setminus \{\alpha\}$. Then the above inclusions show

$$w = F(u) = f(v)$$

for $u \in \Delta_0$ *and* $v \in \Delta_z$*. Since* $w \neq \alpha$*, we must have* $u \neq 0$ *and also* $u \neq a$ *since* $a \notin \Delta_0$*. Thus* $F(u) = f(u)$ *and*

$$w = f(u) = f(v).$$

This contradicts f *injective since* $u \neq v$ *as* $\Delta_0 \cap \Delta_z = \emptyset$*. Thus* $F(0) \notin D_2$*. Similarly,* $F(a) \notin D_2$*. In fact, applying the same argument yet one more time, we can also see that* $F(0) \neq F(a)$*. For if* $F(0) = F(a) = z$*, then picking small neighborhoods around* $0, a, \alpha$*, we would contradict injectivity of* f *in those neighborhoods.*

Thus

$$\{F(0), F(a)\} = \{0, b\}.$$

In particular, $F : \Delta \to \Delta$ *is bijective using that* $f : D_1 \to D_2$ *bijective. Thus* F *is a biholomorphism of* Δ*.*

- *If* $F(0) = 0$ *and* $F(a) = b$*, then* F *must be a rotation* $F(z) = \alpha z$ *with* $|\alpha| = 1$*. We must also have* $b = \alpha a$ *which implies*

  $$|a| = |b|$$

  since $|\alpha| = 1$*. Furthermore,*

  $$f(z) = \frac{b}{a} z.$$

- *If* $F(0) = b$ *and* $F(a) = 0$*, then we can use* ϕ_b *to recenter* F*. Let*

  $$\widetilde{F} = \phi_b \circ F.$$

  We obtain

  $$\tilde{F}(0) = 0, \quad \widetilde{F}(a) = \phi_b(0) = -b.$$

  In this case, $\widetilde{F}$ *is also a rotation*

  $$\widetilde{F}(z) = \alpha z$$

  for $|\alpha| = 1$*. Since* $\widetilde{F}(a) = -b$ *it follows* $\alpha = -\frac{b}{a}$ *which gives again*

  $$|a| = |b|.$$

  In this case

  $$\widetilde{F} = \phi_b \circ f \implies f = \phi_{-b} \circ \widetilde{F} \implies f(z) = \frac{\widetilde{F}(z) + b}{1 + \bar{b}\widetilde{F}(z)} = \frac{ab - bz}{a - b\bar{b}z}.$$

The necessary and sufficient condition is then $|a| = |b|$ *and there are 2 biholomorphisms between* D_1, D_2*.*