Math 220B — Lecture 24

March 5, 2021

Where are we ?

K compact, $K \subseteq \mathcal{U}$, $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic

Wish $\forall \, \varepsilon \;\; \exists \, R$ rational function with prescribed poles

in a suitable set S.

$$|f - R| < \varepsilon \quad \text{in } K$$

Conway VIII. 1.1.

Step 1 We found segments $\gamma_1, \dots, \gamma_n \subseteq \mathcal{U} \setminus K$

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(w)}{w - z} \, dw \qquad \forall \, z \in K.$$

Step 2 Find rational functions R with

Conway VIII. 1.5

$|f - R| < \varepsilon$ in K, poles of R are on the segments γ_j.

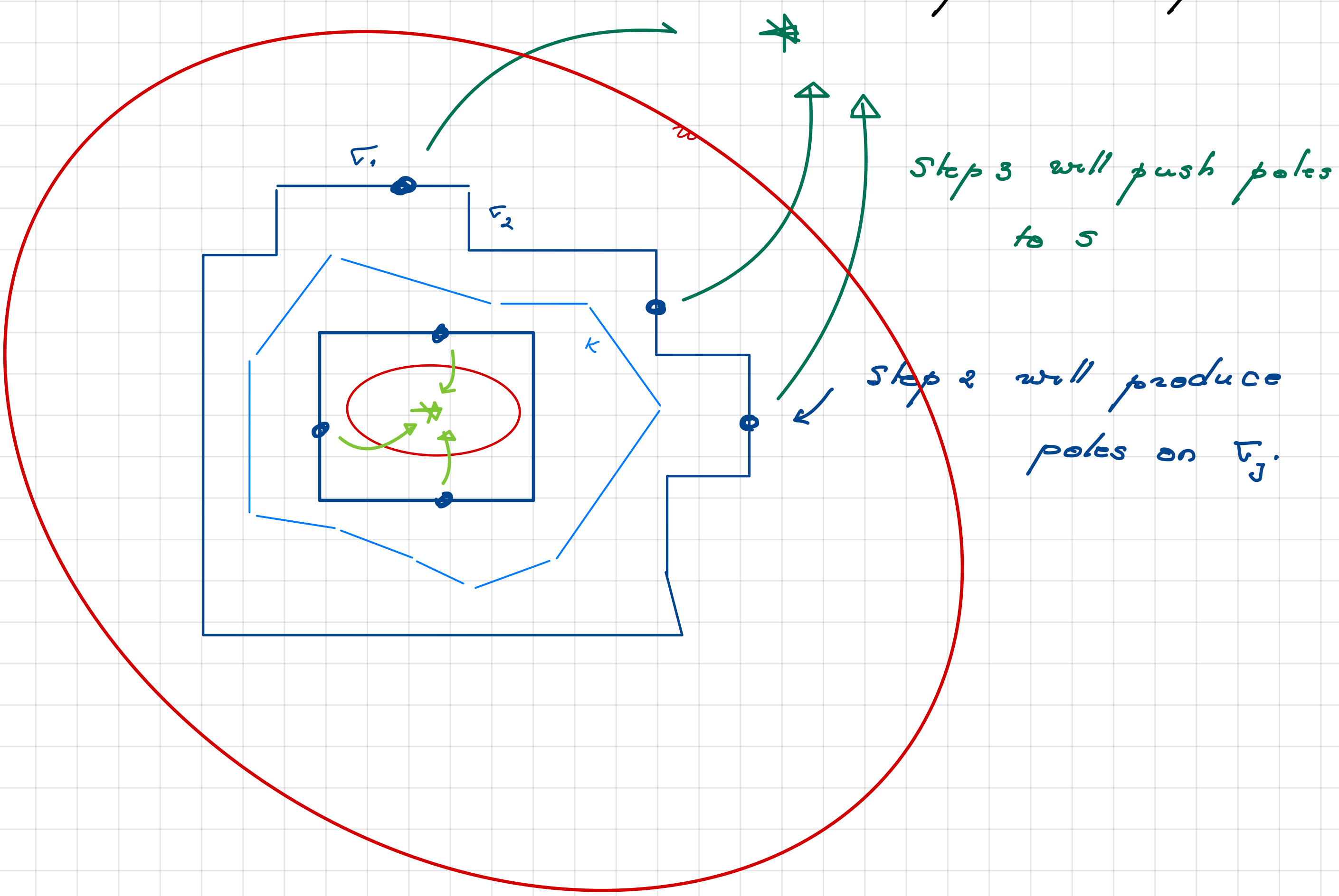
Step 3 Push the poles to prescribed locations.

Conway 1.6 – 1.13.

Visualization of the strategy

$S = \{ *, * \}$

prescribed poles



For step 2 we argue one segment Γ_j at a time showing

$$F_j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f(w)}{w - z}\, dw$$ can be approximated by

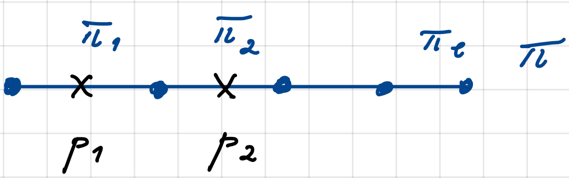
rational functions with poles in Γ_j.

Proof of Step 2

- K compact, π segment (compact), $\pi \cap K = \emptyset$
- f continuous in K

Main Claim (Conway VIII. 1.5)

$$F(z) = \int_{\pi} \frac{f(w)}{w - z}\, dw$$ can be approximated uniformly on K by rational functions with poles in π.



Proof Let $\varphi(w,z) = \dfrac{f(w)}{w-z} : \pi \times K \longrightarrow \mathbb{C}$, $w \in \pi$, $z \in K$.

Since $\pi \cap K = \emptyset \Rightarrow \varphi$ is continuous hence uniformly cont.

$\Rightarrow \forall \varepsilon \ \exists \delta$ such that

$$|w - w'| < \delta \Rightarrow |\varphi(w,z) - \varphi(w',z)| < \varepsilon.$$

- Subdivide π into subsegments $\pi_1, \dots, \pi_\ell$ of length $< \delta$.

- Pick $p_k \in \pi_k$

- Let $c_k = f(p_k) \displaystyle\int_{\pi_k} dw$

- $R = \displaystyle\sum_{k=1}^{\ell} \frac{c_k}{p_k - z}$. rational function with pole at $p_k \in \pi$.

Claim

$$\left| F(z) - R(z) \right| = \left| \int_{\pi} \frac{f(w)}{w-z}\, dw - \sum_{k=1}^{\ell} \frac{f(p_k)}{p_k - z} \int_{\pi_k} dw \right|$$

$$\stackrel{\cdot}{=} \left| \sum_{k=1}^{\ell} \int_{\pi_k} \left(\frac{f(w)}{w-z} - \frac{f(p_k)}{p_k - z} \right) dw \right|$$

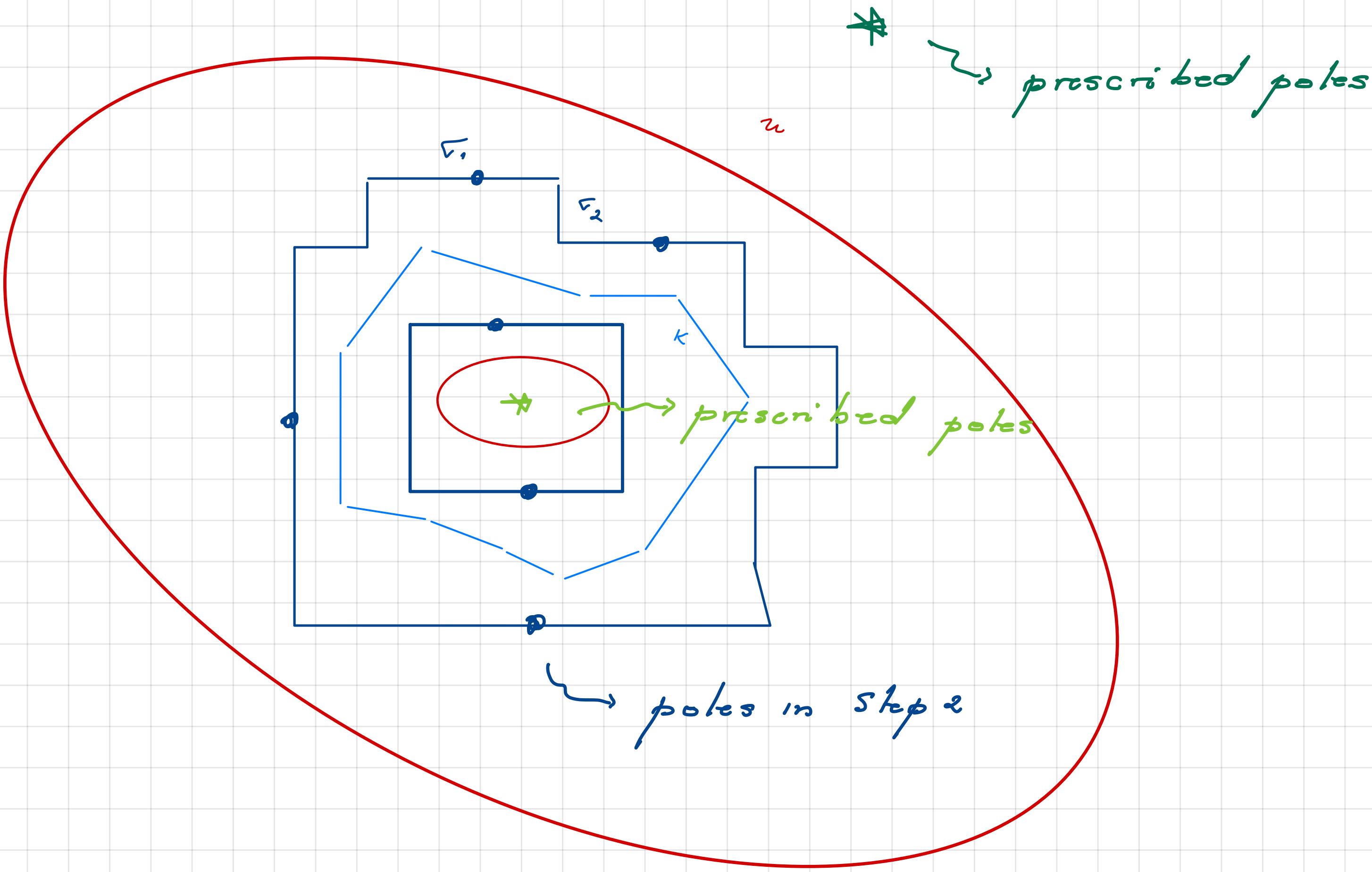
$$\leq \sum_{k=1}^{\ell} \left| \int_{\pi_k} \varphi(w,z) - \varphi(p_k, z)\, dw \right|$$

$$\leq \sum_{k=1}^{\ell} \varepsilon \cdot \operatorname{length}(\pi_k) = \varepsilon \cdot \operatorname{length}(\pi).$$

Here we used $|\varphi(w,z) - \varphi(p_k, z)| < \varepsilon$ since

$|w - p_k| < \delta$ which is true as $p_k, w \in \pi_k$, $\operatorname{length}(\pi_k) < \delta$.

The proof of Step 2 is completed.



Where are we?

- $K \subseteq \mathcal{U}$, f holomorphic
- $\exists\, R$ with poles in Γ_j, $|f - R| < \varepsilon$ on K.

Final Step Fix S a set of poles, one from each component of $\hat{\mathbb{C}} \setminus K$.

Push the poles from Γ_j to the points of S.

Step 3 Pole pushing to prescribed location.

Let $\widehat{\mathbb{C}} \setminus K = \bigcup_i H_i$ = connected components

Let H be a fixed component.

pole produced in Step 2

prescribed location

Lemma $\forall\, a, b \in H$. Then

$\frac{1}{z-a}$ can be approximated uniformly in K by

poles at b

polynomials in $\frac{1}{z-b}$

If H is unbounded & $b = \infty$ then

$\frac{1}{z-a}$ can be approximated uniformly in K by polynomials

Polynomials in z = Rational Functions with poles possibly only at ∞.

Proof of the Lemma

- keep b fixed & vary a. Consider the set

- $W = \{ c \in H : \frac{1}{z-c}$ can be approximated uniformly in K polynomials in $\frac{1}{z-b} \}$

We wish to prove $W = H$.

- $W \neq \Phi$. because $b \in W$.

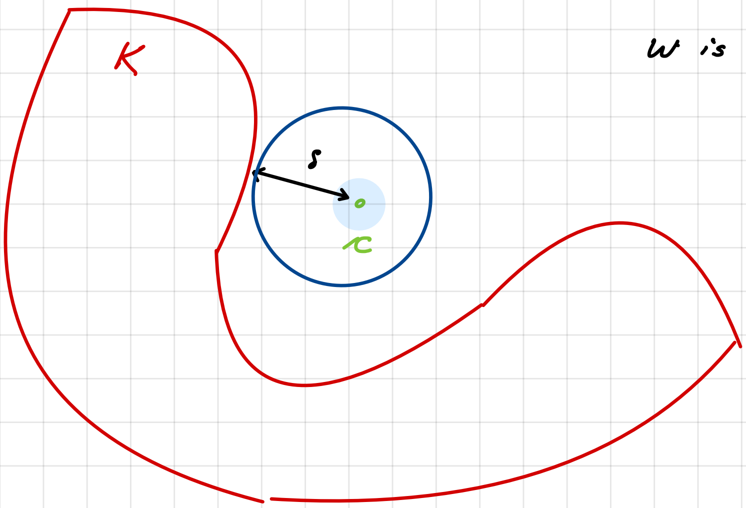
Key Claim

(*) $\forall\ c \in W$, let $\delta = d(c, K)$. Then $\Delta(c, \delta) \subseteq W$.

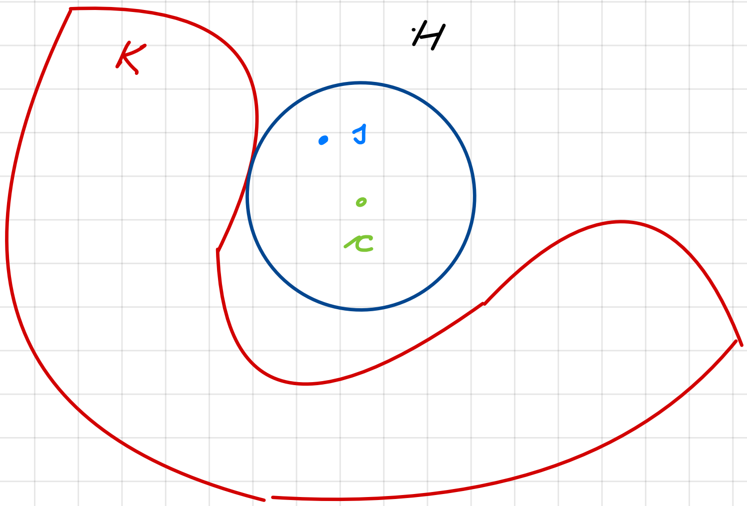
$\Downarrow$

Exercise This implies

W is closed & open hence $W = H$.



Proof of Key Claim Let $s \in \Delta(c, \delta)$. We wish to show that $s \in W$. $\Rightarrow \Delta \subseteq W$ as needed.



Idea $\frac{1}{z-s} \rightsquigarrow$ poly in $\frac{1}{z-c} \rightsquigarrow$ poly in $\frac{1}{z-b}$. $\Rightarrow s \in W$.

Consider the Laurent expansion of $\frac{1}{z-s}$ at c in $\Delta(c; \delta, \infty)$

$$\frac{1}{z-s} = \frac{1}{z-c} \cdot \frac{1}{1 - \frac{s-c}{z-c}} = \frac{1}{z-c} \sum_{k \geq 0} \left(\frac{s-c}{z-c} \right)^k = \sum_{k \geq 0} \frac{(s-c)^k}{(z-c)^{k+1}}$$

Convergence: $|z-c| > \delta > |s-c|$.

Note $z \in K$, $\delta = d(c, K) \Rightarrow K \subseteq \Delta(c; \delta, \infty)$. The Laurent expansion in $\Delta(c; \delta, \infty)$ converges locally uniformly

(Math 220 A, Lecture 12).

Pick T a Laurent polynomial in $\frac{1}{z-c}$ from the Laurent expansion above so that

$$\left|\frac{1}{z-s} - T\right| < \frac{\varepsilon}{2} \quad \text{over } K.$$

Since $c \in W \Rightarrow \frac{1}{z-c}$ can be approximated by polynomials in $\frac{1}{z-b}$. The same is then true about $T =$ polynomial in $\frac{1}{z-c}$. Then $\exists\, P$ polynomial in $\frac{1}{z-b}$ so that

$$|T - P| < \frac{\varepsilon}{2} \quad \text{in } K$$

Then $\left|\frac{1}{z-s} - \underbrace{P}_{\text{polynomial in } \frac{1}{z-b}.}\right| \le \left|\frac{1}{z-s} - T\right| + |T - P| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ in K. This shows $s \in W$.

If H is unbounded Let $K \subseteq \Delta(0,r)$

- first move the poles to $|c| > r$.

- Taylor expand $\frac{1}{z-c}$ near $z=0$ in

$$\Delta(0,|c|) \supseteq \Delta(0,r) \supseteq K$$

The Taylor series converges locally uniformly. Hence we can approximate $\frac{1}{z-c}$ by polynomials uniformly on K.

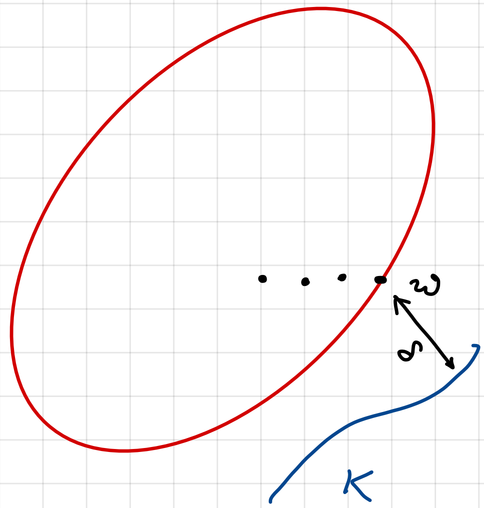
Proof of the Exercise

- W open. Indeed $\forall\, c \in W$ $\exists\, \Delta(c,\delta) \subseteq W$

by $(*)$ showing W open

- We show W closed in H.

Assume $w_n \longrightarrow w$, $w_n \in W$, $w \in H$.



Let $d(w, K) = \delta$

Fix n with $d(w, w_n) < \frac{\delta}{2}$.

$$\Rightarrow d(w_n, K) \geq d(w, K) - d(w, w_n) > \frac{\delta}{2}$$

$\Rightarrow \Delta\left(w_n, \frac{\delta}{2}\right) \subseteq W$ since $w_n \in W$ and $(*)$

$\Rightarrow w \in W.$ since $w \in \Delta\left(w_n, \frac{\delta}{2}\right)$. This proves the Exercise.

Remark This completes the proof of Runge.

Summary: start with f $\overset{\text{Step 1}}{\rightsquigarrow}$ Cauchy for compact sets

$\overset{\text{Step 2}}{\rightsquigarrow}$ rational approximation with poles in Γ_j.

$\overset{\text{Step 3}}{\rightsquigarrow}$ further approximation with prescribed poles