Math 220 B – Lecture 2

January 6, 2021

Last time – Infinite products   Conway VII. 5

Given $p_k \in \mathbb{C}$, define $P = \prod_{k=1}^{\infty} p_k$ convergent product

if $\exists N$ with

$$\lim_{n\to\infty} \prod_{k=N}^{n} p_k = \hat{P} \neq 0 \quad \text{and set}$$

$$P = p_1 \cdots p_{N-1} \hat{P} = \text{value independent of } N.$$

Remarks [i] $\exists$ finitely many zero terms

[ii] $P = 0 \iff \exists\, k$ with $p_k = 0$

[iii] Note

$$p_n = \frac{\prod_{k=N}^{n} p_k}{\prod_{k=N}^{n-1} p_k} \longrightarrow \frac{\hat{P}}{\hat{P}} = 1. \quad \text{as } n \to \infty.$$

Henceforth, we will assume $p_k = 1 + a_k$, $a_k \to 0$.

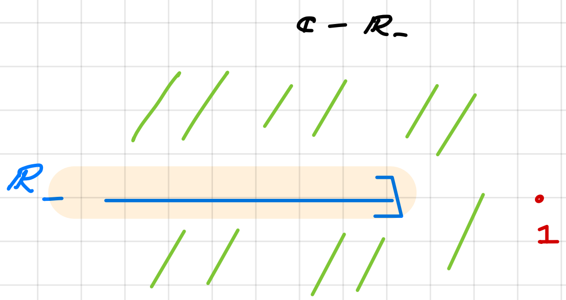
$$\prod_{k=1}^{\infty} (1 + a_k)$$

We seek to connect infinite products to infinite series.

Recall Principal branch of logarithm $z \neq 0$, $z \in \mathbb{C} \setminus \mathbb{R}_-$

$$\mathrm{Log}\,(z) = \log r + i\theta$$

$$\theta \in (-\pi, \pi)$$



$\mathrm{Log}\,(1+z)$. makes sense if z small since $1+z \notin \mathbb{R}_-$.

Lemma $\prod_{k=1}^{\infty} (1+a_k)$ converges $\Longleftrightarrow$ $\exists\, N > 0$ such that $\sum_{k=N}^{\infty} \mathrm{Log}\,(1+a_k)$ converges

Proof Write

$$S_n = \sum_{k=N}^{n} \mathrm{Log}\,(1+a_k)$$

$$P_n = \prod_{k=N}^{n} (1+a_k)$$

$$\Longrightarrow \quad e^{S_n} = P_n.$$

Proof "$\Leftarrow$": If $s_n \to s$, $p_n = e^{s_n} \to e^{s} = \hat{p} \neq 0$.

"$\Rightarrow$" Assume $p_n \to \hat{p}$. We wish to show $s_n \to s$.

Pick α such that $\hat{p} \notin \mathbb{R}_{\geq 0} e^{i\alpha}$. We use the branch Log_α

$$\mathrm{Log}_\alpha z = \log r + i\theta, \quad \theta \in (\alpha, \alpha + 2\pi)$$

$$e^{s_n} = p_n \implies s_n = \mathrm{Log}_\alpha p_n + 2\pi i \ell_n, \quad \ell_n \in \mathbb{Z}.$$

$$\underset{e^{\mathrm{Log}_\alpha p_n}}{\overset{\parallel}{}}$$

We claim $\ell_n = \ell_{n-1}$ if $n \gg 0$ $\implies \exists\, \ell$, $\ell_n = \ell$.

$$\implies s_n = \mathrm{Log}_\alpha p_n + 2\pi i \ell_n \longrightarrow \mathrm{Log}_\alpha \hat{p} + 2\pi i \ell := s$$

To prove the claim, consider

$$\underbrace{s_n - s_{n-1}}_{\downarrow \atop 0} = \underbrace{\mathrm{Log}_\alpha p_n - \mathrm{Log}_\alpha p_{n-1}}_{\downarrow \atop 0} + 2\pi i (\ell_n - \ell_{n-1}) \quad \text{as } n \to \infty$$

Note $s_n - s_{n-1} = \mathrm{Log}(1 + a_n) \longrightarrow \mathrm{Log}\, 1 = 0$

$$\mathrm{Log}_\alpha p_n - \mathrm{Log}_\alpha p_{n-1} \longrightarrow \mathrm{Log}_\alpha \hat{p} - \mathrm{Log}_\alpha \hat{p} = 0$$

This shows
$$\left.\begin{array}{l} \ell_n - \ell_{n-1} \longrightarrow 0 \ \text{ as } n \to \infty \\ \ell_n - \ell_{n-1} \in \mathbb{Z} \end{array}\right\} \implies \ell_n = \ell_{n-1} \text{ if } n \gg 0.$$

Absolute convergence

Question How do we define absolutely convergent products $\prod_{k=1}^{\infty} p_k$

Wrong Answer. $\prod_{k=1}^{\infty} |p_k|$ converges

But then for $p_k = (-1)^k$, $\prod_{k=1}^{\infty} (-1)^k$ converges absolutely which is absurd.

Def $\prod_{k=1}^{\infty} (1+a_k)$ converges absolutely iff $\exists N$ such that

$\sum_{k=N}^{\infty} \operatorname{Log}(1+a_k)$ converges absolutely.

Lemma TFAE

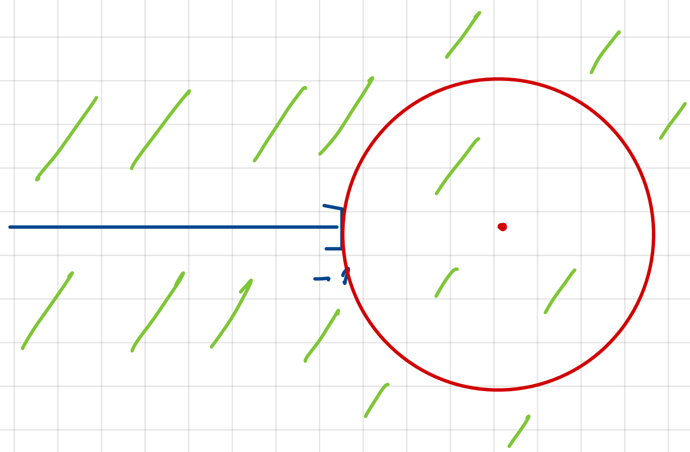
[i] $\prod_{k=1}^{\infty}(1+a_k)$ converges absolutely

[ii] $\sum_{k=1}^{\infty} a_k$ converges absolutely

[iii] $\prod_{k=1}^{\infty}(1+|a_k|)$ converges

Proof Consider Taylor expansion in $\Delta(0,1) \subseteq \mathbb{C}\setminus(-\infty,-1]$

$$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$



$\Downarrow$

$$\frac{\mathrm{Log}(1+z)}{z} = 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots$$

$\Rightarrow \lim_{z\to 0} \frac{\mathrm{Log}(1+z)}{z} = 1 \Rightarrow \exists\, \rho > 0$ such that if $|z|<\rho,\ z\neq 0$.

$$\frac{1}{2} \leq \left|\frac{\mathrm{Log}(1+z)}{z}\right| \leq \frac{3}{2}$$

Important inequality $\exists\, \rho$ s.t. if $|z|<\rho$

$$\frac{1}{2}|z| \leq |\mathrm{Log}(1+z)| \leq \frac{3}{2}|z|.$$

$\boxed{i} \iff \boxed{ii}$ By defn, $\prod_{k=1}^{\infty}(1+a_k)$ converges absolutely

$\iff \sum_{k=N}^{\infty} \operatorname{Log}(1+a_k)$ converges absolutely

$\iff \sum_{k=N}^{\infty} a_k$ converges absolutely (comparison test + important inequality)

Finally,

$\boxed{ii} \iff \sum_{k=N}^{\infty} |a_k|$ converges absolutely

$\iff \prod_{k=1}^{\infty}(1+|a_k|)$ converges absolutely by $\boxed{i} \iff \boxed{ii}$ for $\tilde{a}_k = |a_k|$

$\iff \prod_{k=1}^{\infty}(1+|a_k|)$ converges $\iff \boxed{iii}$

indeed, absolute convergence of the product is superfluous

$$\sum_{k=N}^{\infty} |\operatorname{Log}(1+|a_k|)| = \sum_{k=N}^{\infty} \operatorname{Log}(1+|a_k|)$$

Remark (Rearrangements).

Math 140A we learned that if $\sum_{k=1}^{\infty} b_k$ is absolutely convergent then $\forall\ \sigma : \mathbb{N} \longrightarrow \mathbb{N}$ bijection then $\sum_{k=1}^{\infty} b_{\sigma(k)}$ converges to the same sum.

The same happens for absolutely convergent products $\prod_{k=1}^{\infty} p_k$ can be rearranged, $b_k = \operatorname{Log}(1+a_k)$, $p_k = 1+a_k$.

2. Infinite Products of Holomorphic Functions

$f_k : U \longrightarrow \mathbb{C}$ holomorphic, $U \subseteq \mathbb{C}$

Assumption $\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly

Terminology $\sum_{k=1}^{\infty} f_k$ converges absolutely locally uniformly.

Define

$$(*) \quad F(z) = \prod_{k=1}^{\infty} (1 + f_k(z)).$$

Remark $(*)$ converges absolutely $\forall z \in U \Rightarrow$ can rearrange the product.

Proposition Under the above **Assumption**

$\boxed{i}$ the partial products of $(*)$ converge locally uniformly.

$\boxed{ii}$ F is holomorphic

$\boxed{iii}$ $F(z_0) = 0 \iff \exists\, k$ with $1 + f_k(z_0) = 0$

Proof will be given next time.

Examples $\boxed{i}$ $z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$ defines an entire function with zeroes only at the integers. & nowhere else.

Indeed, apply the Proposition to $f_k(z) = \frac{z^2}{k^2}$.

$\boxed{ii}$ $\prod_{k=1}^{\infty} (1 + q^k z)$ is an entire function if $|q| < 1$ with zeroes only at $z = -q^{-k}$.

Apply the Proposition to $f_k(z) = q^k z$.