# Math 220 A — Lecture 19

November 25, 2020

# Applications of the Residue theorem to real analysis

- [a] trigonometric functions
- [b] rational functions
- [c] Fourier integrals
- [d] logarithmic integrals
- [e] Mellin transforms

Mellin transforms: $\int_0^\infty \frac{R(x)}{x^\alpha}\,dx$

$R$ = rational function, no poles on positive real axis

**Example** $R(x) = \dfrac{1}{x+1} \Rightarrow \boxed{\displaystyle\int_0^\infty \frac{dx}{x^\alpha(x+1)} = \frac{\pi}{\sin \pi\alpha}}$

for $0 < \alpha < 1$

**Homework** $R(x) = \dfrac{1}{x^n+1} \Rightarrow \displaystyle\int_0^\infty \frac{dx}{x^\alpha(x^n+1)}$

**Convergence** uses $0 < \alpha < 1$.

- $0 < x < 1$ : $\displaystyle\int_0^1 \frac{dx}{x^\alpha(x+1)} < \int_0^1 \frac{dx}{x^\alpha} = \frac{x^{-\alpha+1}}{-\alpha+1}\Bigg|_{x=0}^{x=1} < \infty$

- $1 < x < \infty$ : $\displaystyle\int_1^\infty \frac{dx}{x^\alpha(x+1)} < \int_1^\infty \frac{dx}{x^{\alpha+1}} = \frac{x^{-\alpha}}{-\alpha}\Bigg|_{x=1}^{x=\infty} < \infty$

$$I = \int_0^\infty \frac{dx}{x^\alpha(x+1)} = \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{dx}{x^\alpha(x+1)} .$$

$$I = \int_0^\infty \frac{dx}{x^\alpha (x+1)}$$

Question : [a] What function?

[b] What contour?

Issues

[i] extend $x^\alpha$ holomorphically

$$z^\alpha = \exp(\alpha\, \ell(z)) \rightsquigarrow \text{branch cut along } [0,\infty).$$

For $z = re^{it}$, $\ell(z) = \log r + it$, $0 < t < 2\pi$

[ii] pole at 0 — use $C_r$ to isolate the pole

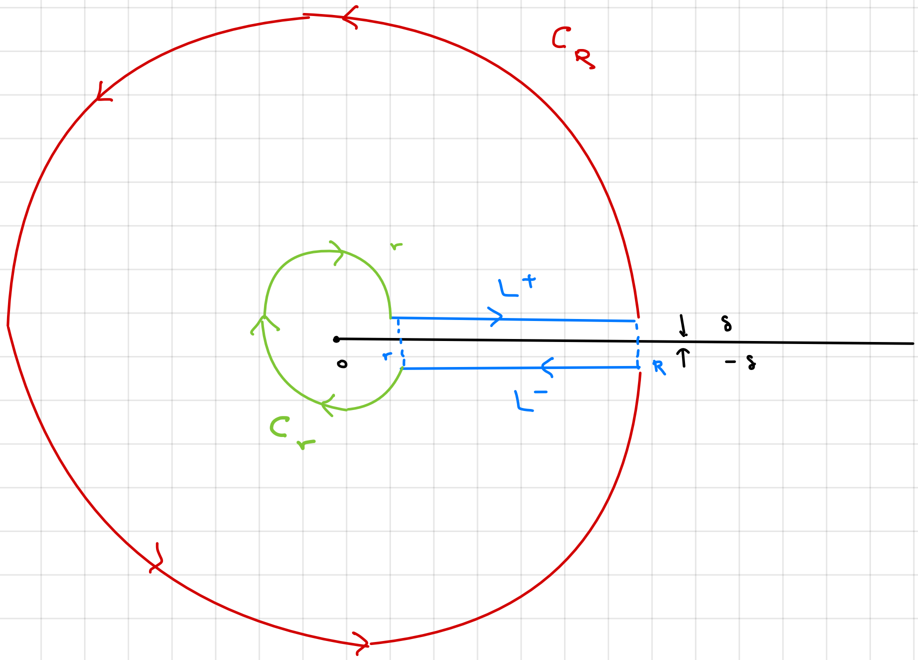
Remark It is precisely the fact that we cut along $[0,\infty)$

(= domain of integration) that allows us to calculate $I$.

Before, we were cutting away from domain of integration

Solutions (a) $f(z) = \dfrac{1}{z^{\alpha}(z+1)}$

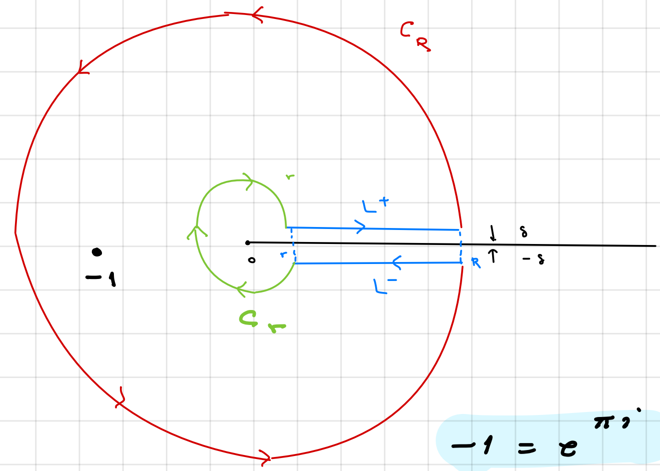
(b) $\gamma$ = key-hole contour




$$\gamma = C_R + (-L^-) + (-C_r) + L^+$$

## Residue theorem

$$f(z) = \frac{1}{z^{\alpha}(z+1)} \quad . \text{ pole at } -1.$$




$$-1 = e^{\pi i}$$

Method 1

$$\operatorname{Res}(f, -1) = \operatorname*{Res}_{z=-1} \frac{\frac{1}{z^{\alpha}}}{z+1} = \frac{1}{(-1)^{\alpha}} = \frac{1}{e^{\pi i \alpha}} = e^{-\pi i \alpha}$$

$$\int_{\gamma} f\, dz = 2\pi i \operatorname{Res}(f, -1) = 2\pi i \exp(-\alpha \pi i).$$

$$\int_{C_R} f\, dz - \int_{C_r} f\, dz + \int_{L^+} f\, dz - \int_{L^-} f\, dz \qquad (R).$$

Make $r \to 0$, $R \to \infty$, $\delta \to 0$.

Claim $\boxed{a}$ $\lim_{\substack{\rho \to 0 \\ \rho \to \infty}} \int_{C_\rho} \frac{dz}{z^\alpha (z+1)} = 0$

Claim $\boxed{b}$ $\lim_{\substack{\delta \to 0 \\ r \to 0 \\ R \to \infty}} \int_{L^+} \frac{dz}{z^\alpha (z+1)} = I$

$\boxed{c}$ $\lim_{\substack{\delta \to 0 \\ r \to 0 \\ R \to \infty}} \int_{L^-} \frac{dz}{z^\alpha (z+1)} = e^{-2\pi i \alpha} I$

Conclude In $(R)$ make $\delta \to 0$, $r \to 0$, $R \to \infty$:

$$0 - 0 + I - e^{-2\pi i \alpha} I = e^{-\pi i \alpha} \cdot 2\pi i$$

$$I = \frac{2\pi i \, e^{-\pi i \alpha}}{1 - e^{-2\pi i \alpha}} = \frac{2\pi i}{e^{\pi i \alpha} - e^{-\pi i \alpha}} = \frac{\pi}{\sin \pi \alpha}.$$

## Proof of $\boxed{a}$

$$\left| \int_{C_\rho} \frac{dz}{z^{\alpha}(z+1)} \right| \le 2\pi\rho \cdot \frac{1}{\rho^{\alpha}|\rho-1|} \longrightarrow 0$$

as $\rho \to 0$ or $\rho \to \infty$. because $0 < \alpha < 1$.

## Proof of $\boxed{b}$

$g(z) = \frac{1}{{}^{-\alpha}}$, $L^{+} = \{ t + i\delta : r \le t \le R \}$.

$$\lim_{\delta \to 0} \int_{L^+} \frac{g(z)}{z+1}\, dz \overset{(+)}{=} \int_r^R \frac{t^{-\alpha}}{t+1}\, dt. \longrightarrow I.$$

as $r \to 0$
$R \to \infty$.

Why $(+)$?

$$\int_{L^+} \frac{g(z)}{z+1}\, dz = \int_r^R \frac{g(t+i\delta)}{1+t+i\delta}\, dt \xrightarrow{\delta \to 0} \int_r^R \frac{t^{-\alpha}}{1+t}\, dt.$$

Define

$$G(t,\delta) = \begin{cases} \dfrac{g(t+i\delta)}{1+t+i\delta} - \dfrac{t^{-\alpha}}{1+t}, & \delta \ne 0. \\ 0, & \delta = 0. \end{cases}$$

$r \le t \le R$, $0 \le \delta \le 1$.

$G$ continuous (uniformly). Given any $\varepsilon$, $\exists\, \tau > 0$ such that if

$$|\delta - 0| < \tau, \quad |t - t| < \tau \implies |G(t,\delta) - \underbrace{G(t,0)}_{0}| < \varepsilon.$$

$$\implies |G(t,\delta)| < \varepsilon. \implies \left| \int_r^R G(t,\delta)\, dt \right| \leq (R-r)\cdot \varepsilon \quad \text{as } |\delta| < \tau.$$

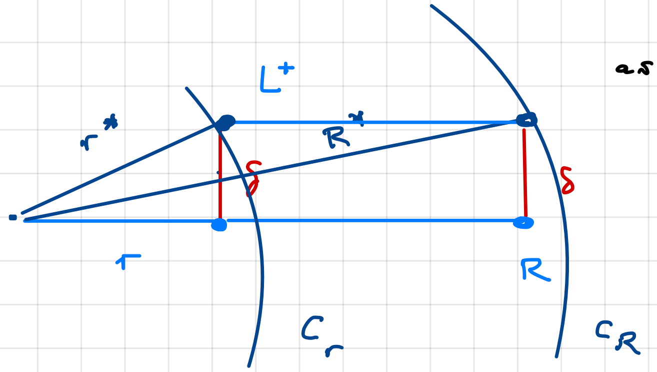
**Remark** If we wish to parametrize $L^+$ by $r \leq t \leq R$,

we'd need to use circles $C_R$, $C_r$ of radii

$R^* = \sqrt{R^2 + \delta^2}$, $r^* = \sqrt{r^2+\delta^2}$. The argument in $\boxed{1}$

still applies since $r^* \to 0$, $R^* \to \infty$.

as $r \to 0$, $\delta \to 0$, $R \to \infty$.



## Proof of $\boxed{c}$

**Difference** $\quad g(t-i\delta) \longrightarrow t^{-\alpha}\, e^{-2\pi i\alpha}$.

The rest of the proof is the same as $\boxed{b}$.

Indeed $\quad g(t-i\delta) = (t-i\delta)^{-\alpha} = \exp(-\alpha\, \ell(t-i\delta))$

$$\xrightarrow{\delta\to 0} \exp(-\alpha\log t - 2\pi i\alpha) = t^{-\alpha}\, e^{-2\pi i\alpha}.$$

This explains the extra factor $e^{-2\pi i\alpha}$ in the answer to $\boxed{c}$.

# II. Residues at $\infty$ & Shadows of Riemann Surfaces

## [A]. Topology on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

- basic neighborhood of $\infty$

$$\mathcal{U} = \{\infty\} \cup \{|z| > R\} \text{ for some } R.$$

- $\hat{\mathbb{C}}$ is a topological space

- $\hat{\mathbb{C}}$ compact

Remark $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $z \to \frac{1}{z}$, $f(0) = \infty$

$f(\infty) = 0$

(punctured) neighborhoods of $0$ $|z| < r$

$z \to \frac{1}{z}$.

$\left|\frac{1}{z}\right| > \frac{1}{r}$

(punctured) neighborhoods of $\infty$

## $\boxed{B.}$ Singularities & Residues at $\infty$

Recall Conway $\overline{V}$. 1. 13 — Pset 5 / Problem 6

If $f : \{|z| > R\} \longrightarrow \mathbb{C}$ holomorphic

$\Longrightarrow$ $\infty$ is isolated singularity

Types

$\boxed{i}$ removable

$\boxed{ii}$ pole $\quad \Longrightarrow \quad g(z) = f\left(\frac{1}{z}\right).$

$\boxed{iii}$ essential $\quad$ Inspect singularity at 0.

Example $\quad f(z) = \dfrac{z^5 + 2}{z - 1}. \rightarrow$ poles at $1, \infty \in \hat{\mathbb{C}}$.

has a pole at $z = 1$. Inspect $\infty$.

$$g(z) = f\left(\frac{1}{z}\right) = \frac{\frac{1}{z^5} + 2}{\frac{1}{z} - 1} = \frac{1 + 2z^5}{1 - z} \cdot \frac{1}{z^4} \quad \text{pole at } z = 0$$

$\Longrightarrow$ $f$ has a pole at $\infty$.

## Residue at $\infty$

$\mathrm{Res}(f, \infty) = ?$

**Beware**

$$\mathrm{Res}(f, \infty) \neq \mathrm{Res}(g, 0).$$

**Instead**

$$\mathrm{Res}(f, \infty) := -\frac{1}{2\pi i} \int_{|z|=\rho} f \, dz \qquad \text{where } \rho > R.$$

By Homotopy Cauchy this does not depend on $\rho$.

**Question**

Why do we care about the residue at $\infty$?

**Homework Example**

$$\int_{|z|=5} \frac{z^3}{(1-z)(2-z)(3-z)(4-z)} \, dz = -2\pi i \, \mathrm{Res}(\_\_, \infty).$$

This is better than computing 4 different residues.

Next time we will answer the following:

<u>Question</u> How do we calculate the residue at $\infty$?