HW 4.

Problem 1. Let $Y_i = X_i^2 \in [0,1]$.

By Example 2.4 of Wainwright, Y_i's are sub-Gaussian with parameter 1. (By Exercise 2.4, the parameter can be $\frac{1}{2}$).

By Hoeffding bound,

$$\mathbb{P}\left(\sum_{i=1}^{n}\{Y_i - E(Y_i)\} \geq t\right) \leq \exp\left(-\frac{t^2}{2n}\right), \quad \text{where} \quad E(Y_i) = E(X_i^2) = \int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}.$$

That is,

$$\mathbb{P}\left(\|X\|_2^2 - \frac{n}{3} \geq t\right) \leq \exp\left(-\frac{t^2}{2n}\right), \quad \forall t \geq 0$$

Similarly, since $-Y_i$'s are also sub-Gaussian with parameter 1,

$$\mathbb{P}\left(-\|X\|_2^2 + \frac{n}{3} \geq t\right) \leq \exp\left(-\frac{t^2}{2n}\right), \quad \forall t \geq 0.$$

Hence,

$$\mathbb{P}\left(\left|\|X\|_2^2 - \frac{n}{3}\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2n}\right), \quad \forall t \geq 0.$$

Problem 2. $E(|X|^k) = \int_0^\infty P(|X|^k > t)\,dt = \int_0^\infty P(|X| > t^{\frac{1}{k}})\,dt$ $\left(\text{Let } u = \frac{2t^{\frac{1}{k}}}{\lambda}, \text{ then } t = (\frac{\lambda u}{2})^k, \ dt = (\frac{\lambda}{2})^k k u^{k-1} du\right)$

$$\begin{aligned}
&\leq 2\int_0^\infty \exp\left(\frac{-2t^{\frac{1}{k}}}{\lambda}\right) dt \\
&= 2\int_0^\infty \left(\frac{\lambda}{2}\right)^k k u^{k-1} e^{-u}\, du \\
&= k \cdot 2^{1-k} \cdot \lambda^k (k-1)! \qquad \left(\text{since } \int_0^\infty e^{-u} u^{k-1} du = \Gamma(k) = (k-1)!\right) \\
&\leq \lambda^k k!
\end{aligned}$$

Problem 3. $\mathbb{E}(e^{tZ}) = 1+\sum_{k=2}^{\infty} \frac{t^k \mathbb{E}(Z^k)}{k!} = 1+\sum_{k=2}^{\infty} \frac{t^k \mathbb{E}\{X^2-\mathbb{E}(X^2)\}^k}{k!} \leq 1+\sum_{k=2}^{\infty} \frac{|t|^k \mathbb{E}|X^2+\mathbb{E}(X^2)|^k}{k!}$

$\leq 1+\sum_{k=2}^{\infty} \frac{|t|^k \cdot 2^{k-1}[\mathbb{E}(X^{2k})+\{\mathbb{E}(X^2)\}^k]}{k!}$ since $\mathbb{E}(X+Y)^k \leq 2^{k-1}\{\mathbb{E}(X^k)+\mathbb{E}(Y^k)\}$ $\forall X,Y\geq 0$

$\leq 1+\sum_{k=2}^{\infty} \frac{|t|^k 2^k \mathbb{E}(X^{2k})}{k!}$ Jensen Inequality.

$\leq 1+\sum_{k=2}^{\infty} \frac{|t|^k 2^k (2\sigma^2)^k \cdot 2\cdot k!}{k!}$ $\mathbb{E}(X^{2k}) \leq (2\sigma^2)^k \cdot 2 \cdot k!$ if X is sub-Gaussian (σ^2) and $\mathbb{E}(X)=0$.

$= 1+2(4t\sigma^2)^2 \sum_{k=0}^{\infty} (4|t|\sigma^2)^k$

$\leq 1+32t^2\sigma^4 \sum_{k=0}^{\infty} 4^{-k}$ when $|t| \leq \frac{1}{16\sigma^2}$

$= 1+\frac{128}{3}t^2\sigma^4 \leq e^{\frac{128}{3}t^2\sigma^4} \leq e^{\frac{(16\sigma^2)^2 t^2}{2}}$, $\forall |t| \leq \frac{1}{16\sigma^2}$.

That is, Z is sub-exponential $(16\sigma^2, 16\sigma^2)$.

Problem 4. (a) $\forall t>0$.

$\mathbb{E}\{X_{(n)}\} = t^{-1}\mathbb{E}[\log\{e^{tX_{(n)}}\}]$

$\leq t^{-1}\log[\mathbb{E}\{e^{tX_{(n)}}\}]$ Jensen Inequality

$= t^{-1}\log\{\mathbb{E}(\max e^{tX_i})\}$

$\leq t^{-1}\log\{\sum_{i=1}^{n}\mathbb{E}(e^{tX_i})\}$

$\leq t^{-1}\log(\sum_{i=1}^{n} e^{t^2\sigma^2/2}) = \frac{\log n}{t}+\frac{t\sigma^2}{2}$.

Set $t=\sqrt{\frac{2\log(n)}{\sigma^2}}$, then $\mathbb{E}\{X_{(n)}\} \leq \sigma\sqrt{2\log(n)}$

(b) $\mathbb{P}(X_{(n)}>t) = \mathbb{P}(\bigcup_{i=1}^{n}[X_i>t])$

$\leq \sum_{i=1}^{n}\mathbb{P}(X_i>t)$

$\leq n\exp(-\frac{t^2}{2\sigma^2})$