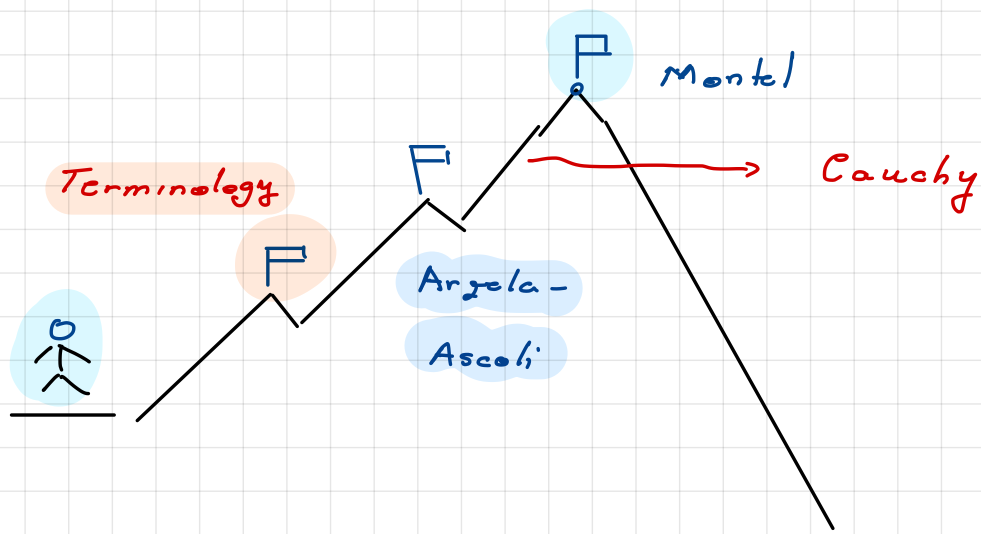
Math 220B — Lecture 11

January 29, 2021

Next few lectures – Normal Families   Conway VII. 1 & 2.



Why climb the mountain – Motivation

Sequences of complex numbers

$\{a_n\}$ bounded $\Longrightarrow$ $\exists$ convergent subsequence

Indeed, if $|a_n| \leq M$ $\Rightarrow$ $a_n \in \overline{\Delta}(0, M)$. The closed disc $\overline{\Delta}(0, M)$ is compact.

We wish to make similar statements for sequences of functions (continuous or holomorphic).

Dream Statement

Given a "bounded" sequence of functions, there exists a "convergent" subsequence.

Question A What could "—" mean?

Question B Is this connected to compactness?

Answer is "yes" but it has no consequences for the current lecture.

Remark Dream statement makes sense in

[i] real analysis (continuous functions)

Arzela - Ascoli

[ii] complex analysis (holomorphic functions).

Montel.

We will investigate both.

Question A $f_n : U \longrightarrow \mathbb{C}$ "convergent" could mean

[i] pointwise weak

[ii] uniform strong

[iii] local uniform OK for us

$\Updownarrow$

[iv] uniform convergence on compact sets OK for us

"bounded" could mean

[i] pointwise bounded — weak

$\forall x \in \mathcal{U} \quad \exists M(x)$ with $|f_n(x)| < M(x) \quad \forall n$

[ii] uniformly bounded — strong

$\exists M \quad \forall x \in \mathcal{U} \quad |f_n(x)| < M \quad \forall n$

[iii] locally uniformly bounded. — OK for us

$\forall x \ \exists \Delta_x \subseteq \mathcal{U}$ neighborhood of x, such that the restrictions $f_n/_{\Delta_x}$ are uniformly bounded.

[iv] uniformly bounded on compact sets — OK for us

$\forall K \ \exists M(K), \quad |f_n(x)| \leq M(K) \quad \forall x \in K \ \forall n$

Remark We have $\boxed{III} \Leftrightarrow \boxed{IV}$ that is,

locally uniformly bounded $\Longleftrightarrow$

uniformly bounded on each compact

Why? $\Leftarrow$ If $x \in \mathcal{U}$, let $K = \bar{\Delta}_x$ be a compact neighborhood of x.

$\Rightarrow$ For all $x \in \mathcal{U}$, $\exists\, \Delta_x$ where $f_n|_{\Delta_x}$ are bounded by M_x.

Then $K \subseteq \bigcup_{x \in K} \Delta_x \Rightarrow K \subseteq \bigcup_i \Delta_{x_i}$ and let

$$M = \max(M_{x_1}, \dots, M_{x_n}) > 0.$$

This is a bound for all f_n's over K.

Example

$\boxed{\text{i}}$ $f_n(x) = \sin nx$ uniformly bounded by 1 in $\mathbb{R}$.

$\boxed{\text{ii}}$ $f_n(z) = z^n$ in $\Delta(0,1)$ uniformly bounded by 1

$\boxed{\text{iii}}$ $f_n(z) = nz^n$ locally uniformly bound in $\Delta(0,1)$

but not uniformly bounded.

Proof

For $0 \le r < 1$: $|f_n(z)| \le nr^n$. in $\Delta(0,r)$. Since

$$\lim_{n\to\infty} nr^n = 0 \implies \{nr^n\} \text{ is bounded by } M \implies$$

$$\implies |f_n(z)| \le M \quad \text{in} \quad \Delta(0,r).$$

Each $K \subseteq \Delta(0,1)$ compact, $K \subseteq \overline{\Delta}(0,r)$ for $r<1 \implies$

$\implies \{f_n\}$ uniformly bounded (locally/on compacts).

Since $f_n\left(\frac{1}{\sqrt[n]{2}}\right) = \frac{n}{2} \longrightarrow \infty$.

$\implies \{f_n\}$ not uniformly bounded.

Dream Statement Revisited

$f_n : \mathcal{U} \longrightarrow \mathbb{C}$ locally uniformly bounded

$\Longrightarrow$ f_n admits a locally convergent subsequence

---

Question Could this be true?

Example No.

Let $\mathcal{U} = \mathbb{R}$. The sequence

$$f_n(x) = \sin nx$$

is uniformly bounded, but we can't get a convergent subsequence, not even pointwise.

Question C1 Could this be true in complex analysis i.e. holomorphic functions? YES

Question C2 What is the correct statement in real analysis i.e. continuous functions?

Answer to C1

Main Theorem (Montel)

$f_n : U \longrightarrow \mathbb{C}$ holomorphic & locally uniformly bounded

$\Rightarrow$ f_n admits a locally uniformly convergent subsequence.

More generally – Families

$\mathcal{F}$ family of continuous or holomorphic functions.

Required for applications (Riemann-mapping & Picard's theorems)

---

[i] Any sequence determines $\mathcal{F} = \{f_1, f_2 \dots f_n \dots\}$ = family.

[ii] $\mathcal{F} = \{ f : \Delta(0,1) \longrightarrow \mathbb{C}$ holomorphic

$$f(z) = \sum_{k=1}^{\infty} a_k z^k, \quad |a_k| \leq k \}$$

[iii] $\mathcal{F} = \{ f : \Delta(0,1) \longrightarrow \mathbb{C}$ holomorphic, $f(0) = 1$, $\mathrm{Re} f > 0 \}$

Def $\mathcal{F}$ is normal if all sequences in $\mathcal{F}$ admit a locally uniformly convergent subsequence.

Remark The limit does not have to be in $\mathcal{F}$.

Example

$\boxed{1}$ $\mathcal{F}$ normal family of holomorphic functions

$\Longrightarrow$ $\mathcal{F}'$ is normal where $\mathcal{F}' = \{ f' : f \in \mathcal{F} \}$

Proof Definition + Weierstraß Convergence

Let $\{ f_n' \} \subseteq \mathcal{F}'$ be a sequence with $f_n \in \mathcal{F}$.

Pick a subsequence $f_{n_k} \overset{l.u.}{\rightrightarrows} f$. By Weierstraß, $f_{n_k}' \overset{l.u.}{\rightrightarrows} f'$ showing $\mathcal{F}'$ is normal.

Remark

We can define $\mathcal{F}$ uniformly bounded, locally uniformly bounded etc just as before.

Examples

$\boxed{1}$ $\mathcal{F} = \left\{ f : \Delta(0,1) \longrightarrow \mathbb{C} \text{ holomorphic}, \ f = \sum_{k=1}^{\infty} a_k z^k, \ |a_k| \leq k \right\}$

locally uniformly bounded.

Indeed, since all compacts $K \subseteq \overline{\Delta}(0,r)$ suffices to work over $\overline{\Delta}(0,r)$. Then

$$|f(z)| \leq \sum_{k=1}^{\infty} |a_k| \, |z|^k \leq \sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2} \quad \forall \, |z| \leq r$$

$$\forall \, f \in \mathcal{F}$$

$\Rightarrow \mathcal{F}$ locally uniformly bounded.

[ii] $\mathcal{F}$ family of holomorphic functions in $\mathcal{U}$

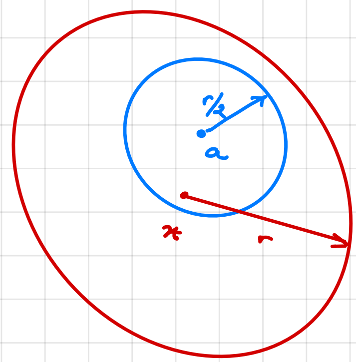
$\mathcal{F}$ locally uniformly bounded. $\Rightarrow$

$\mathcal{F}'$ locally uniformly bounded.

Proof Cauchy's estimates.

Take $x \in \mathcal{U}$. $\Rightarrow \exists\, \Delta(x,r) \subseteq \mathcal{U}$ such that $\forall\, f \in \mathcal{F}$:

$$|f| \leq M \quad \text{over} \quad \Delta(x,r).$$



We bound $|f'|$ over $\Delta(x, r/2)$.

Let $a \in \Delta(x, r/2)$. By Cauchy's estimate

$$|f'(a)| \leq \frac{\sup |f| \text{ over } \bar{\Delta}(a, r/2)}{r/2} \leq \frac{M}{r/2}.$$

where we used $\bar{\Delta}(a, r/2) \subseteq \Delta(x,r)$.

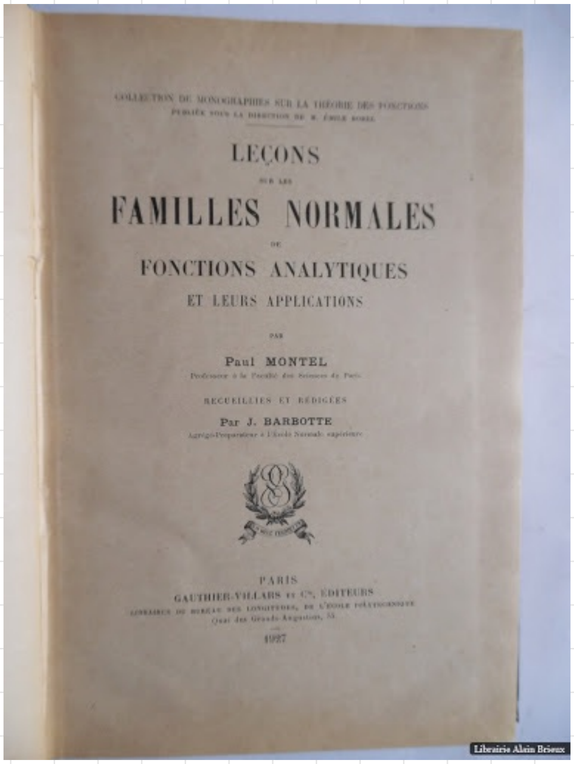
[iii] We have seen that $\mathcal{F} = \{z^n\}_n$ is uniformly bounded but $\mathcal{F}' = \{n z^{n-1}\}_n$ is not uniformly bounded

Montel Rephrased (Dream statement)

$\mathcal{F}$ family of holomorphic functions in $\mathcal{U} \subseteq \mathbb{C}$

$\mathcal{F}$ locally uniformly bounded $\iff$ $\mathcal{F}$ normal.

Remark Both sides are well behaved under taking derivatives as we noted.



LEÇONS

FAMILLES NORMALES

FONCTIONS ANALYTIQUES

ET LEURS APPLICATIONS

Paul MONTEL

Par J. BARBOTTE

PARIS

GAUTHIER-VILLARS

1927

Paul Montel (1876 – 1975).

Students:

Henri Cartan
Jean Dieudonné

Montel introduced and developed the notion of normal family.

He published the theorem named after him in his thesis in 1907. In 1927 he published a monograph on normal families.

Une suite infinie de fonctions analytiques et bornees a l'interieur d'un domaine simplement connexe, admet au moins une fonction limite a l'interieur de ce domaine.

(An infinite sequence of functions that are analytic and bounded in the interior of a simply connected domain admits at least one limit function in the interior of this domain.)

P. Montel, 1907