HW 1 Solutions

Prob 1. $\mathbb{E} e^{itX_n} = e^{it\mu_n - \frac{1}{2}\sigma_n^2 t^2}$, $\mathbb{E} e^{itX} = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$.

$X_n \xrightarrow{d} X \iff \mathbb{E} e^{itX_n} \to \mathbb{E} e^{itX}$, $\forall t \in \mathbb{R}$ (Levy's continuity theorem)

$\iff e^{it\mu_n - \frac{1}{2}\sigma_n^2 t^2} \to e^{it\mu - \frac{1}{2}\sigma^2 t^2}$, $\forall t \in \mathbb{R}$

$\iff \mu_n \to \mu$, $\sigma_n^2 \to \sigma^2$. $\blacksquare$ ("$\Rightarrow$" can be shown by choosing $t = \pm 1$).

Prob 2. $\forall a \in \mathbb{R}, \varepsilon > 0$, $\mathbb{P}(X_n \le a) \le \mathbb{P}(X \le a+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon)$, since $\{X_n \le a\} \subseteq \{X \le a+\varepsilon\} \cup \{|X_n - X| > \varepsilon\}$.

Similarly, $\mathbb{P}(X \le a-\varepsilon) \le \mathbb{P}(X_n \le a) + \mathbb{P}(|X_n - X| > \varepsilon)$.

Hence, $\mathbb{P}(X \le a-\varepsilon) - \mathbb{P}(|X_n - X| > \varepsilon) \le \mathbb{P}(X_n \le a) \le \mathbb{P}(X \le a+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon)$.

where $\mathbb{P}(|X_n - X| > \varepsilon) \to 0$.

Let $\varepsilon \to 0$, then for every a s.t. the CDF $F_X(a)$ is continuous,

$P(X \le a-\varepsilon) \to P(X \le a)$ and $P(X \le a+\varepsilon) \to P(X \le a)$

Hence, $\mathbb{P}(X_n \le a) \to \mathbb{P}(X \le a)$. $\blacksquare$

Prob 3. Let $X := (X_1, \cdots X_n)$. Then. $f_{\lambda,a}(X) = \lambda^n e^{-\lambda\sum_{i=1}^n (X_i - a)} \mathbb{1}\{\min_i X_i \geq a\} = \begin{cases} \lambda^n e^{-\lambda\sum_{i=1}^n (X_i - a)}, & \text{if } \min_i X_i \geq a \\ 0, & \text{o/w}. \end{cases}$

Since $\lambda > 0$, we know that $0 < \lambda^n e^{-\lambda\sum_{i=1}^n (X_i - a)}$ and hence the MLE satisfies $\hat{a} \leq \min_i X_i$.

For any $a \leq \min_i X_i$, we have the log-likelihood

$$\ell(\lambda, a; X) := \log f_{\lambda,a}(X) = n \log \lambda - \lambda \sum_{i=1}^n (X_i - a) = n \log \lambda - \lambda \sum_{i=1}^n X_i + n\lambda a$$

is monotone increasing w.r.t. a. Hence when $a \leq \min_i X_i$, $\ell(\lambda, a; X)$ is optimized at $a = \min_i X_i$.

To sum up, the MLE for a is $\hat{a} = \min_i X_i$ with a corresponding log-likelihood $\ell(\lambda, \hat{a}; X) = n \log \lambda - \lambda \sum_{i=1}^n (X_i - \min_j X_j)$

Now, we try to find the MLE for λ. Define $g(\lambda, X) := \ell(\lambda, \hat{a}; X)$.

Then, $\frac{\partial}{\partial\lambda} g(\lambda, X) = n\lambda^{-1} - \sum_{i=1}^n (X_i - \min_j X_j)$.

Set $\frac{\partial}{\partial\lambda} g(\lambda, X)\big|_{\lambda = \hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^n (X_i - \hat{a})}$

Now, it remains to show $\hat{a} \xrightarrow{P} a$ and $\hat{\lambda} \xrightarrow{P} \lambda$.

$\forall \varepsilon > 0$, $\mathbb{P}(|\hat{a} - a| > \varepsilon) = \mathbb{P}(\hat{a} > a + \varepsilon) + \mathbb{P}(\hat{a} < a - \varepsilon)$

$= \mathbb{P}(\hat{a} > a + \varepsilon)$ since $\mathbb{P}(\hat{a} < a - \varepsilon) \leq \mathbb{P}(\hat{a} < a) = 1 - \mathbb{P}(\min_i X_i \geq a) = 1 - \{\mathbb{P}(X_1 \geq a)\}^n = 0$.

$= \mathbb{P}(\bigcap_{i=1}^n [X_i > a + \varepsilon])$

$= \prod_{i=1}^n \mathbb{P}(X_i > a + \varepsilon) = \{\mathbb{P}(X_1 > a + \varepsilon)\}^n$

$= e^{-n\lambda\varepsilon} \to 0$ since $\mathbb{P}(X_1 > a + \varepsilon) = \int_{a+\varepsilon}^\infty \lambda e^{-\lambda(x-a)} dx = \int_\varepsilon^\infty \lambda e^{-\lambda x} dx = \lim_{u \to \infty} -e^{-\lambda x}\Big|_\varepsilon^u = e^{-\lambda\varepsilon}$.

Hence. $\hat{a} \xrightarrow{P} a$. By Slutsky's Lemma, to show $\hat{\lambda} \xrightarrow{P} \lambda$, it remains to show $n^{-1}\sum_{i=1}^n (X_i - a) \xrightarrow{P} \lambda^{-1}$.

$X_i - a \overset{iid}{\sim} \text{Exp}(\lambda)$. $\Rightarrow \mathbb{E}(X_i - a) = \lambda^{-1}$, $\text{Var}(X_i - a) = \lambda^{-2}$.

Hence. $\mathbb{E}\{n^{-1}\sum_{i=1}^n (X_i - a)\} = \lambda^{-1}$, $\text{Var}\{n^{-1}\sum_{i=1}^n (X_i - a)\} = n^{-1}\lambda^{-2} \to 0$.

By Prob. 4, $n^{-1}\sum_{i=1}^n (X_i - a) \xrightarrow{P} \lambda^{-1}$ $\boxed{B}$

Prob 4. By Chebyshev's inequality,

$$\mathbb{P}(|X_n-\mu_n| \geq k\sigma_n) \leq k^{-2}, \quad \forall k>0. \quad \mu_n = \mathbb{E}(X_n), \ \sigma_n^2 = \mathrm{Var}(X_n).$$

Hence, $\forall \varepsilon > 0$,

$$\mathbb{P}(|X_n-\mu_n| > \tfrac{\varepsilon}{2}) \leq 4\sigma_n^2 \varepsilon^{-2}.$$

$$\Rightarrow \mathbb{P}(|X_n-\mu| > \varepsilon) \leq \mathbb{P}(|X_n-\mu_n| > \tfrac{\varepsilon}{2}) + \mathbb{P}(|\mu_n-\mu| > \tfrac{\varepsilon}{2}) \leq 4\sigma_n^2\varepsilon^{-2} + \mathbb{I}\{|\mu_n-\mu| > \tfrac{\varepsilon}{2}\} \to 0. \quad \blacksquare$$

Another approach: $E(X_n-\mu)^2 = \mathrm{Var}(X_n) + \{E(X_n)-\mu\}^2 \to 0 + (\mu-\mu)^2 = 0.$

By Markov's inequality,

$$\mathbb{P}(|X_n-\mu| > \varepsilon) = \mathbb{P}((X_n-\mu)^2 > \varepsilon^2) \leq \frac{E(X_n-\mu)^2}{\varepsilon^2} \to 0 \text{ as } n\to\infty. \quad \blacksquare$$

Prob J. For any $k = 1, \cdots, n$,

$\mathbb{E}(X_k) = 0$, $Var(X_k) = k^2 \cdot \frac{k^{-\alpha}}{2} = \frac{k^{2-\alpha}}{2}$.

$S_n^2 = \sum_{k=1}^n \sigma_k^2 = \frac{1}{2}\sum_{k=1}^n k^{2-\alpha} \simeq n^{3-\alpha}$ as $n \to \infty$.

Remark: $S_n^2 = O(n^{3-\alpha})$ is not enough. We need S_n^2 to be (or at least be) of a rate $n^{3-\alpha}$, rather that at most of a rate $n^{3-\alpha}$.

Need to show the Lindeberg's condition:

$S_n^{-2} \sum_{k=1}^n \mathbb{E}(X_k^2 \mathbb{1}\{|X_k| > \varepsilon S_n\}) \to 0$ as $n \to \infty$.

For $0 < \alpha < 1$, $k \le n$, $k^{-1} S_n \geq n^{-1} S_n \simeq n^{\frac{1-\alpha}{2}} \to \infty$

Hence, $\mathbb{1}\{k > \varepsilon S_n\} = 0$ for large enough n.

Remark: This statement is stronger than $\mathbb{1}\{k > \varepsilon S_n\} \to 0$ as $n \to \infty$.

$\Rightarrow S_n^{-2} \sum_{k=1}^n \mathbb{E}(X_k^2 \mathbb{1}\{|X_k| > \varepsilon S_n\}) = S_n^{-2} \sum_{k=1}^n \mathbb{E}(k^2 \mathbb{1}\{|k| > \varepsilon S_n\}) = 0$

By LFT, $S_n / s_n \xrightarrow{d} N(0,1)$.

Prob 6. $\sum_{j=1}^{n} \mathbb{E}(X_{nj}^2 \mathbb{1}\{|X_{nj}|>\varepsilon\}) \le \sum_{j=1}^{n} \mathbb{E}(|X_{nj}|^{2+\delta} \cdot |X_{nj}|^{-\delta} \mathbb{1}\{|X_{nj}|>\varepsilon\})$

$\le \sum_{j=1}^{n} \mathbb{E}(|X_{nj}|^{2+\delta}) \cdot \varepsilon^{-\delta} \to 0$ as $n \to \infty$. $\blacksquare$

Prob 7. Let $\mathbb{P}(X_k = k) = \mathbb{P}(X_k = -k) = k^{-2}/2$, $\mathbb{P}(X_k = 0) = 1 - k^{-2}$

Then, $\sigma_k^2 = k^2 \cdot k^{-2} = 1$ for all k.

$\mathbb{P}(|n^{-\frac{1}{2}} \sum_{k=1}^{n} X_k| > \varepsilon) = \mathbb{P}(|\sum_{j=1}^{n} X_k| > \sqrt{n} \cdot \varepsilon)$

$\le \mathbb{E}(|\sum_{k=1}^{n} X_k|)/(\sqrt{n}\varepsilon)$ (Markov's inequality).

$\le \sum_{k=1}^{n} \mathbb{E}|X_k| / (\sqrt{n}\varepsilon)$

$= \sum_{k=1}^{n} k^{-1} / (\sqrt{n}\varepsilon)$ since $\mathbb{E}|X_k| = k \cdot k^{-2} = k^{-1}$.

$\le (\log n + 1) / (\sqrt{n}\varepsilon)$ fact: $\sum_{k=1}^{n} k^{-1} \le \log n + 1$.

$\to 0$ as $n \to \infty$.

That is, $n^{-\frac{1}{2}} \sum_{k=1}^{n} X_k \xrightarrow{\mathbb{P}} 0$.

Prob 8. Let $Y_i = X_i \mathbb{1}\{|X_i| \le n\}$.

Step 1: Show $\frac{\sum Y_i - b_n}{n} \xrightarrow{P} 0$.

$\mathbb{P}\left(\left|\frac{\sum Y_i - b_n}{n}\right| > \varepsilon\right) \le \varepsilon^{-2} \mathbb{E}\left|\frac{\sum Y_i - b_n}{n}\right|^2$ (Markov's inequality)

$= n^{-2} \varepsilon^{-2} \sum_i \mathbb{E}(Y_i^2) \to 0$ by assumption.

Step 2: Show $\frac{\sum X_i - \sum Y_i}{n} \xrightarrow{P} 0$.

$\mathbb{P}\left(\left|\frac{\sum X_i - \sum Y_i}{n}\right| > \varepsilon\right) \le \mathbb{P}\left(\bigcup_{i=1}^{n} [X_i \neq Y_i]\right)$

$\le \sum_{i=1}^{n} \mathbb{P}(X_i \neq Y_i) = \sum_{i=1}^{n} \mathbb{P}(|X_i| > n) \to 0$ by assumption.

Hence, $\frac{T_n - b_n}{n} = \frac{\sum X_i - \sum Y_i}{n} + \frac{\sum Y_i - b_n}{n} \xrightarrow{P} 0$ by Slusky's Lemma.