Math 220, Problem Set 6.

1. (*Biholomorphic rectangles.*) Recall from lecture the rectangles $R_{a,b}$ with vertices at 0, a, bi and $a+bi$ for $a, b \in \mathbb{R} \setminus \{0\}$. Assume that there exists a biholomorphism

$$f : R_{a,b} \to R_{a',b'}$$

that extends bijectively and continuously over the boundary, sending edges to edges and vertices to vertices. Show that either

$$\frac{a'}{a} = \pm \frac{b'}{b} \text{ or } aa' = \pm bb'.$$

(i) Let S and S' be the horizontal sides of the two rectangles lying on the real axis. We have seen in class that if $f(S) = S'$ and $f(0) = 0$, $f(a) = a'$ then $f(z) = \alpha z$ where

$$\alpha = \frac{a'}{a} = \frac{b'}{b}.$$

We wish to reduce all remaining cases to this situation.

Assume first that $f(S) = S'$, but $f(0) = a'$, $f(a) = 0$. Show that the map

$$\ell(w) = a' - w$$

sends $R_{a',b'}$ to $R_{a',-b'}$ and conclude that

$$\ell \circ f : R_{a,b} \to R_{a',-b'}$$

in such a fashion that 0 is sent to 0 and a is sent to a'. Conclude that

$$\frac{a'}{a} = -\frac{b'}{b}.$$

This settles the case when $f(S) = S'$ without any particular assumption on the images of the vertices.

(ii) If $f(S) \neq S'$, there are three other possible edges of $R_{a',b'}$ which can be candidates for $f(S)$. Use composition with suitable (linear) holomorphic maps

$$\ell : R_{a',b'} \to R_{a'',b''},$$

for a new pair (a'', b''), such that the side S is mapped by the composition $\ell \circ f$ to the corresponding horizontal side of $R_{a'',b''}$ lying on the real axis. Conclude using (i).

Remark: Of course you can solve the problem by using Schwarz reflection to deal directly with the remaining cases. Neither argument seems faster.

You can convince yourselves that all four cases

$$\frac{a'}{a} = \pm \frac{b'}{b} \text{ or } aa' = \pm bb'$$

can in fact be achieved by explicit examples. This follows directly from your solution to (i) and (ii) (but do not hand in).

2. (*Schwarz Reflection across arcs.*) Solve Conway, IX.1.2, page 213. The (slightly) modified statement is as follows.

Let $U \subset \mathbb{C}$ be an open set outside of the unit disc whose boundary shares an arc with the unit circle. Define

$$U^* = \{z : 1/\bar{z} \in U.\}\,.$$

The set U^* is the reflection of U across the unit circle $|z| = 1$.

(i) If $U = \{1 < |z| < R\}$, what is U^*?

(ii) Show that U^* is an open subset of $\Delta \setminus \{0\}$.

(iii) Let $f : U \to \mathbb{C}$ be holomorphic and nowhere zero, and define $f^*(z) = 1/\overline{f\,(1/\bar{z})}$. Show that f^* is holomorphic in U^*.

(iv) What would it mean for an open set V to be symmetric with respect to an arc of the unit circle?

(v) Formulate and prove a version of Schwarz reflection where the unit circle $|z| = 1$ replaces the real axis (both in the domain and the target of your function).

Perhaps the easiest proof is to use the Cayley transform.

3. (*Schwarz Reflection and Conformal Annuli.*) Consider the annuli

$$A_1 = \{z : 1 < |z| < r\}, \quad A_2 = \{z : 1 < z < R\}.$$

Assume that there exists a bijective continuous map

$$f : \overline{A}_1 \to \overline{A}_2$$

which is holomorphic in the interior A_1. Show that $r = R$.

(i) Show that f maps the circle $\{|z| = 1\}$ either to the circle $\{|z| = 1\}$ or to the circle $\{|z| = R\}$.

(ii) Assume first that f maps $\{|z| = 1\}$ to $\{|z| = 1\}$. Use Schwarz reflection to obtain extensions

$$f^+ : \Delta \setminus \{0\} \to \Delta \setminus \{0\}.$$

(iii) Using (ii) show that $r = R$.

(iv) If $\{|z| = 1\}$ is mapped to $\{|z| = R\}$, consider the function $R/f(z)$ and conclude.

Remark: The result is true also without the assumption that f is continuous over the boundary.