Math 281a – Problem Set # 1
Module 2

Professor Jelena Bradic

Due @ October 9th, 2021

Problem 1

Consider a sequence of $i.i.d.$ random variables, X_n, in $\mathbb{R}$, where $n = 1, 2, 3, \ldots$ and $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$. Show that $X_n \xrightarrow{d} X$ with $X \sim \mathcal{N}(\mu, \sigma^2)$, if and only if $\lim_{n\to\infty} \mu_n \to \mu$ and $\lim_{n\to\infty} \sigma_n^2 \to \sigma^2$. If additional conditions are needed for this result to hold, identify those carefully.

Problem 2

Establish the following claim. If random variables X_n, X, in $\mathbb{R}$, where $n = 1, 2, 3, \ldots$, are such that $\|X_n - X\|_2 \xrightarrow{p} 0$, then $X_n \xrightarrow{d} X$.

Problem 3

Let $X_1, \cdots, X_n$ be i.i.d. with density $f_{\lambda,a}(x) = \lambda e^{-\lambda(x-a)} \mathbb{1}\{x \geq a\}$. Calculate the maximum likelihood estimator of λ, a and show that it converges in probability to the correct target λ, a.

Problem 4

If $E(X_n) \to \mu$ and var$(X_n) \to 0$ then $X_n \xrightarrow{p} \mu$. Show this.

Problem 5

Suppose that X_n are independent and such that for any $0 < \alpha < 1$,

$$P(X_n = n) = P(X_n = -n) = n^{-\alpha}/4$$
$$P(X_n = 0) = 1 - n^{-\alpha}/2.$$

Let $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Let $S_n = X_1 + \cdots + X_n$ denote the partial sum. Use LFT to show that $S_n/s_n \xrightarrow{d} \mathcal{N}(0,1)$.

Problem 6

A condition stronger than Lindebergs that is often easier to check is the Lyapounov condition:

$$\exists \delta > 0 \text{ such that } \lim_{n\to\infty} \sum_j E[|X_{ij}|^{2+\delta}] = 0.$$

Show that Lyapounovs condition implied Lindebergs condition.

Problem 7

Let X_n be a sequence of independent random variables with mean zero and variance σ_n^2. Show that the following condition

$$1/B \leq \sigma_n^2 \leq B, \qquad 0 < B$$

is not sufficient for

$$\sum_i X_i / \sqrt{\sum_i \sigma_i^2} \xrightarrow{d} \mathcal{N}(0,1).$$

[Hint: find a counterexample]

Problem 8 [Optional]

Let $X_1, X_2, X_3, \ldots$ be independent random variables. Suppose that as $n \to \infty$

$$\sum_{i=1}^n P(|X_i| > n) \to 0$$

and

$$n^{-2} \sum_{i=1}^n E[X_i^2 \mathbb{1}\{|X_i| \leq n\}] \to 0.$$

Show that

$$\frac{T_n - b_n}{n} \xrightarrow{p} 0, \qquad T_n = \sum_{i=1}^n X_i, \qquad b_n = \sum_{i=1}^n E[X_i \mathbb{1}\{|X_i| \leq n\}]$$