Math 220A - Lecture 11

October 30, 2020

Midterm - Friday Nov 6

- closed book, closed notes
- honor code - no zoom proctoring
- available Friday 3PM, due Friday 4 PM.
- upload answers in Grade Scope
- time zone issues - email me
- buffer : 10 minutes to upload solutions, 4:10 PM.
- if questions arise, please email.

$\boxed{1.}$ Last time

We looked at zeros of holomorphic functions.

The following result guarantees existence.

Lemma $f: \mathcal{U} \to \mathbb{C}$ holomorphic, $\bar{\Delta}(a, R) \subseteq \mathcal{U}$

Assume $\min\limits_{z \in \partial\Delta} |f(z)| > |f(a)|$. Then f has a zero in $\mathcal{U}$.

Proof Assume $f \neq 0$, let $g = \frac{1}{f}$.

Note $|g(a)| = \frac{1}{|f(a)|} > \frac{1}{\min |f(z)|} = \max\limits_{z \in \partial\Delta} |g(z)|$.

This contradicts the $k=0$ case of Cauchy's estimate

$$|g(a)| \leq \max_{z \in \partial\Delta} |g(z)| \quad \text{(last time)}$$

Thus f has a zero in $\mathcal{U}$.

Main theorems

[i] Identity Principle

[ii] Open Mapping Theorem

[iii] Maximum Modulus Principle

[2] Open Mapping Theorem.

Recall: $f: X \longrightarrow Y$ is open map if $\forall\, U \subseteq X$ open, $f(U)$ is open.

$f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longrightarrow x^2$ is not open, $U = (-1,1)$, $f(U) = [0,1)$

$f: \mathbb{C} \longrightarrow \mathbb{C}$, $z \longrightarrow z^2$ is open. This is because:

Theorem $f: U \longrightarrow \mathbb{C}$ not constant holomorphic $\Longrightarrow$

$\Longrightarrow$ f is open.

Proof Suffices to show $f(U)$ is open. Else if $V \subseteq U$, work with $f|_V : V \longrightarrow \mathbb{C}$. This is not constant because of identity principle.

Let $a \in U$. We may assume $f(a) = 0$.

Claim $\exists\, r$ such that $\Delta(0,r) \subseteq f(U)$. This would show $f(U)$ contains a neighborhood of $f(a)=0$. $\Rightarrow f(U)$ open.

Proof Since U open $\Rightarrow \exists\, \bar{\Delta}(a) \subseteq U$. We may assume $f/_{\partial\bar{\Delta}(a)}$ has no zeros. (Argue by contradiction. This would give a sequence of accumulating zeros for f. contradicting identity principle).

$$\text{Let } r = \frac{1}{2} \min_{z \in \partial\bar{\Delta}(a)} |f(z)| > 0.$$

Let $w \in \Delta(0,r)$. We need to show $\exists\, z \in U$, $f(z) = w$.

Apply the Lemma to $f-w$. to guarantee $\exists$ zero z for $f-w$.

We need

$$\min_{z \in \partial\bar{\Delta}(a)} |f(z)-w| > |f(a)-w| = |0-w| = |w|$$

Indeed,

$$|f(z)-w| \geq |f(z)| - |w| \geq 2r - |w| \overset{?}{>} |w|$$

since $|w| < r$. This completes the proof.

Example $f: U \longrightarrow \mathbb{C}$, $P \in \mathbb{R}[X, Y]$ not constant

$$P(\operatorname{Re} f, \operatorname{Im} f) = 0 \implies f \text{ constant.}$$

$P = aX + bY - c$.

$$a \operatorname{Re} f + b \operatorname{Im} f = c \implies f \text{ constant.}$$

$P = X^{2020} + Y^{2020} - 1$.

$$(\operatorname{Re} f)^{2020} + (\operatorname{Im} f)^{2020} = 1. \implies f \text{ constant.}$$

Proof By OMT, $f(U)$ is open so it contains a disc Δ.

Since $P(\operatorname{Re} f, \operatorname{Im} f) = 0 \implies f(U) \subseteq \{(x,y) : P(x,y) = 0\}$.

$\implies \Delta \subseteq \{(x,y) : P(x,y) = 0\}$ This cannot happen.

Indeed, write $P(x,y) = \sum_{k=0}^{N} a_k(x)\, y^k$.

Fix x such that $a_N(x) \neq 0$. (finitely many roots). For such x, y takes on at most N values for which $P(x,y) = 0$.

But if $\Delta \subseteq \{(x,y) : P(x,y) = 0\}$, for each x there would be ∞-many y's. contradiction.

Example $f: U \longrightarrow V$ bijective, holomorphic & $f'(a) \neq 0$ $\forall a \in U$. Then f^{-1} holomorphic.

Proof We show f^{-1} continuous. This is the OMT.

$$(f^{-1})^{-1}(W) = f(W) = \text{open.} \quad \forall\, W \subseteq V \text{ open.}$$

We show f^{-1} is differentiable. Use the definition.

$$1 = \frac{f(f^{-1}(z+h)) - f(f^{-1}(z))}{h} =$$

$$= \lim_{h \to 0} \frac{f(f^{-1}(z+h)) - f(f^{-1}(z))}{f^{-1}(z+h) - f^{-1}(z)} \lim_{h \to 0} \frac{f^{-1}(z+h) - f^{-1}(z)}{h}.$$

The first limit exists since f is holomorphic & f^{-1} is continuous. It equals $f'(f^{-1}(z)) \neq 0$. The second limit must exist as well, giving the derivative $(f^{-1})'(z) = \dfrac{1}{f'(f^{-1}(z))}$.

Remark We assumed $f'(a) \neq 0$ $\forall a$. This is automatic (see later).