Math 280A Fall '21

Lecture 3

§ 2.1

Probability Space

$$(\Omega, \mathcal{B}, P)$$

non-empty set

σ-field of subsets of Ω

"probability measure"

$$P : \mathcal{B} \longrightarrow [0, 1]$$

- $P(\Omega) = 1$
- $P\left(\biguplus_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} P(B_n)$

$\biguplus$ indicates disjoint union

"countable additivity"

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Consequences

1. $P(\emptyset) = 0$.

Use c.a. with $B_1 = B_2 = \cdots = B_n = \cdots = \emptyset$

$$P(\emptyset) = P\left(\dot{\bigcup_{n=1}^{\infty}} \emptyset\right) = \sum_{n=1}^{\infty} P(\emptyset)$$

which forces $P(\emptyset) = 0$.

③

2. Finite Additivity: $P\left(\dot{\bigcup}_{1}^{n} B_k\right) = \sum_{k=1}^{n} P(B_k)$

Proof

Given disjoint $B_1, B_2, \dots, B_n$ in $\mathcal{B}$

take $B_k = \emptyset$ for $k > n$

Then $\dot{\bigcup}_{1}^{\infty} B_k = \dot{\bigcup}_{1}^{n} B_k$

$\therefore$ By c.a.

$$P\left(\dot{\bigcup}_{1}^{n} B_k\right) = P\left(\dot{\bigcup}_{1}^{\infty} B_k\right) = \sum_{1}^{\infty} P(B_k) = \sum_{1}^{n} P(B_k) \quad //$$

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3. **Complements** $P(B^c) = 1 - P(B)$

If $B \in \mathcal{B}$ then also $B^c \in \mathcal{B}$ and $B \,\dot\cup\, B^c = \Omega$

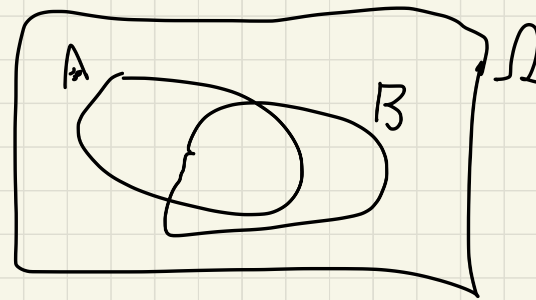
By 2.

$$1 = P(\Omega) = P(B \,\dot\cup\, B^c) = P(B) + P(B^c)$$

$$\therefore \; P(B^c) = 1 - P(B)$$

4. Inclusion-Exclusion

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$$P(A \cup B) = P(A \setminus B) + P(B \setminus A) + P(AB)$$

$$= P(A) + P(B \setminus A)$$

$$= P(A) + P(B) - P(AB)$$

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup AB$$

(disjoint)

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See the text (Ex. 4, p. 30, formula (2.2)) for

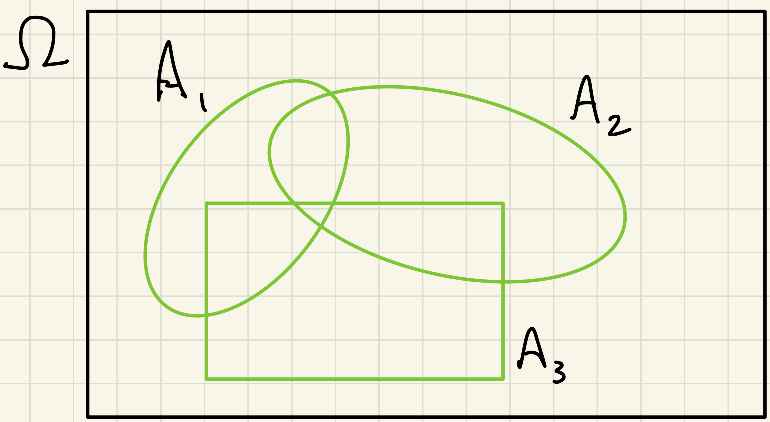
$$P\left(\bigcup_{1}^{n} A_k\right) = \sum_{1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i A_j)$$

$$+ \sum_{1\le i<j<k\le n} P(A_i A_j A_k)$$

$$\vdots$$

$$\pm P(A_1 \cdots A_n)$$

$n = 3$



Cut $A_1 \cup A_2 \cup A_3$ into 7 disjoint pieces; use finite additivity; re-assemble

5. **Monotonicity**: $A \subset B \implies P(A) \le P(B)$

proof: $B = A \,\dot\cup\, (B \setminus A)$

$$\therefore \quad P(B) = P(A) + \underbrace{P(B \setminus A)}_{\ge 0}$$

$$\ge P(A).$$

Bonus:

$P(B \setminus A) = P(B) - P(A)$ provided $A \subset B$.

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6. Subadditivity

$$(*) \quad P\left(\bigcup_{1}^{n} A_k\right) \le \sum_{1}^{n} P(A_k)$$

case $n=2$

proof Induction on n. Case $n=2$ is Inc.-Exc.

Suppose $(*)$ holds. Then

$$P\left(\bigcup_{1}^{n+1} A_k\right) = P\left(\left(\bigcup_{1}^{n} A_k\right) \cup A_{n+1}\right) \le P\left(\bigcup_{1}^{n} A_k\right) + P(A_{n+1})$$

$$\le \sum_{1}^{n} P(A_k) + P(A_{n+1}) \quad \text{(induction hypo.)}$$

$$= \sum_{1}^{n+1} P(A_k)$$

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7. Continuity

If $A_n \uparrow A$ (i.e. $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ and $\bigcup_1^\infty A_n = A$)

then $\uparrow \lim_n P(A_n) = P(A)$

proof "disjointify"

Define
$$B_1 := A_1$$
$$B_2 := A_2 \setminus A_1$$
$$\vdots$$
$$B_k := A_k \setminus A_{k-1}$$
$$\vdots$$

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$$\therefore \quad A = \bigcup_{1}^{\infty} A_k = \dot{\bigcup_{1}^{\infty}} B_k \qquad \text{(check this!)}$$

$$\therefore \text{(c.a.)} \quad P(A) = \sum_{1}^{\infty} P(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$$

$$\text{But : } \sum_{1}^{n} P(B_k) = P(B_1) + P(B_2) + \cdots + P(B_n)$$

$$= P(A_1) + P(A_2 \setminus A_1) + \cdots + P(A_n \setminus A_{n-1})$$

$$= P(A_1) + [P(A_2) - P(A_1)] + \cdots + [P(A_n) - P(A_{n-1})]$$

(telescopes)

$$= P(A_n)$$

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Because $A_n \uparrow$, $P(A_n) \uparrow$ and is bdd above by $P(A)$.

$\therefore \uparrow \lim_n P(A_n)$ exists, and glancing back

$$P(A) = \lim_n \sum_1^n P(B_k) = \lim_n P(A_n).$$

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Corollary 1: $A_n \downarrow A$ (i.e. $A_1 \supset A_2 \supset \cdots$ and $\bigcap_n A_n = A$)

$$\Rightarrow \downarrow \lim_n P(A_n) = P(A)$$

Corollary 2: Countable Subadditivity

$$P\left(\bigcup_1^\infty A_k\right) \leq \sum_1^\infty P(A_k)$$

Just combine 6 and 7.