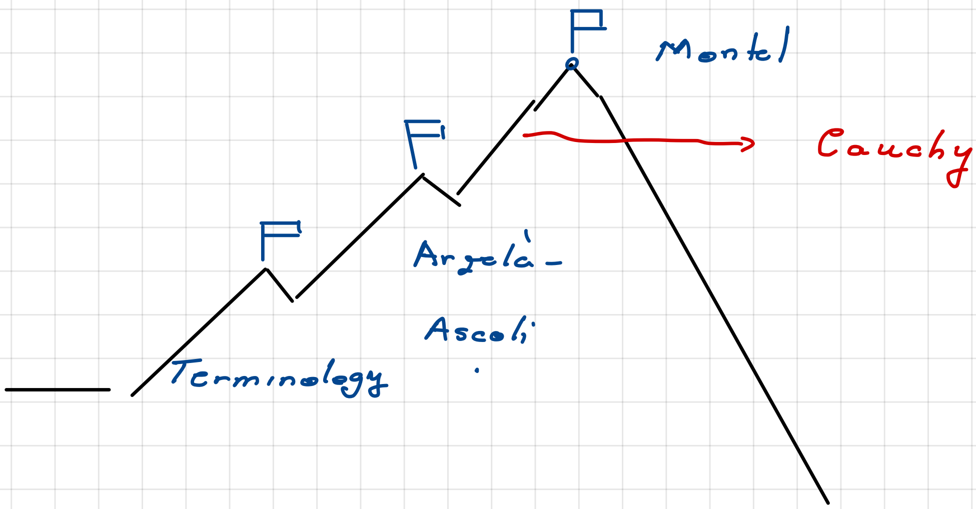
Math 220B — Lecture 12

February 1, 2021



I. Last time

Point #1 All notions we use are local e.g.

local boundedness, local uniform convergence, local equicontinuity (today)

Point #2 Work with families

$\mathcal{F}$ family of continuous or holomorphic functions in $\mathcal{U}$

Definitions

locally uniformly

$\mathcal{F}$ normal $\Longleftrightarrow$ every sequence in $\mathcal{F}$ has convergent subsequence

$\mathcal{F}$ locally bounded (LB) $\Longleftrightarrow$ $\forall\, x \;\exists\, \Delta_x \subseteq \mathcal{U}$, $\mathcal{F}/_{\Delta_x}$ uniformly bounded

i.e. $\exists\, M > 0 \;\forall f \in \mathcal{F}$, $|f| \leq M$ in Δ_x.

Montel's Theorem $\mathcal{F}$ family of holomorphic functions in $\mathcal{U}$.

$\mathcal{F}$ normal $\Longleftrightarrow$ $\mathcal{F}$ locally bounded.

This fails in real analysis,

$\mathcal{F} = \{\sin nx\}_n$ locally bounded in $\mathbb{R}$ & not normal.

(we can't even arrange pointwise convergence)

Question C.2. What is the correct statement in real analysis i.e. continuous functions?

Remark

This requires the notion of equicontinuity.

There will be several versions.

II. Notions of Equicontinuity

strongest

$\boxed{1}$ $\mathcal{F}$ equicontinuous on $\mathcal{U}$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall \ |x-y| < \delta \quad \forall f \in \mathcal{F}: \quad |f(x) - f(y)| < \varepsilon.$$

Main Point If $\mathcal{F} = \{f\}$ this says f uniformly continuous.

In general, this says

all $f \in \mathcal{F}$ are uniformly continuous, "uniformly".

that is, the same δ in the definition of uniform continuity works for all $f \in \mathcal{F}$, uniformly.

I. Fix $M > 0$. The family

$$\mathcal{F} = \{ f : (0,1) \longrightarrow \mathbb{R},\ |f(x) - f(y)| \leq M\,|x-y| \}$$ equicontinuous.

Suffices to take $\delta = \frac{\varepsilon}{M}$ and note

$$|x-y| < \delta \implies |f(x) - f(y)| \leq M\,|x-y| < \varepsilon \quad \forall f \in \mathcal{F}.$$

II. $$\mathcal{F} = \left\{ f = \sum_{k=0}^{2021} a_k x^k,\ |a_k| \leq 1 \right\}$$ equicontinuous on $[-1,1]$.

$$\left| \frac{f(x) - f(y)}{x-y} \right| = \left| \sum_{k=0}^{2021} a_k \left(x^{k-1} + \dots + y^{k-1} \right) \right|$$

$$\leq \sum_{k=0}^{202} |a_k| \left(|x|^{k-1} + \dots + |y|^{k-1} \right)$$

$$\leq \sum_{k=0}^{2021} 1 \cdot \underbrace{(1 + \dots + 1)}_{k} = \sum_{k=0}^{2021} k = M$$ & use part I

III. $\mathcal{F} = \{ f_n \}$; $f_n(x) = nx$ not equicontinuous in $[0,1]$.

IV. See also the Proposition at the end of lecture.

Variations

[i] equicontinuous

[ii] equicontinuous at each point (Conway).

$\forall x \in \mathcal{U} \quad \forall \varepsilon > 0 \quad \exists \Delta(x,\delta) \text{ s.t. } \forall y \in \Delta(x,\delta) \Rightarrow |f(y) - f(x)| < \varepsilon.$

$\forall f \in \mathcal{F}$

When $\mathcal{F} = \{f\}$ this says f is continuous at each point.

[iii] locally equicontinuous

$\forall x \quad \exists \Delta_x \subseteq \mathcal{U}, \quad \mathcal{F}/_{\Delta_x}$ is equicontinuous

[iv] equicontinuous on all compacts (Rudin, Ahlfors, us)

$\forall K \subseteq \mathcal{U}$ compact, $\mathcal{F}/_K$ equicontinuous

[ii] – [iii] – [iv] are equivalent.

[iv] ⟹ [iii] Just use $K = \overline{\Delta}_*$ where Δ_* is a bounded neighborhood of $*$ in $\mathcal{U}$.

[iii] ⟹ [ii] clear from definitions

[ii] ⟹ [iv] requires a compactness argument

(HWK 4, #6 or Conway VII.1).

III. Question C Characterization of normality?

Theorem (Arzelà - Ascoli)

$\mathcal{F}$ family of continuous functions

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ is locally equicontinuous & locally bounded.

Theorem (Montel) $\mathcal{F}$ family of holomorphic functions.

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ locally bounded.

Question D Why is local equicontinuity needed in real analysis?

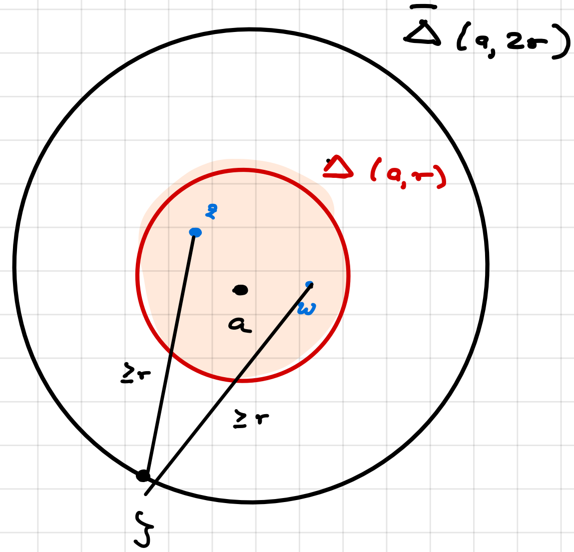
Question E Why is local equicontinuity NOT needed in complex analysis?

Answer to E

Proposition $\mathcal{F}$ family of holomorphic functions.

$\mathcal{F}$ is locally bounded $\Longrightarrow$ $\mathcal{F}$ is locally equicontinuous.

Proof Fix $a \in \mathcal{U}$.



$\Rightarrow \exists\ \bar{\Delta}(a, 2r)$ such that

$\mathcal{F}|_{\bar{\Delta}(a,2r)}$ is bounded by M.

Claim $\mathcal{F}|_{\Delta(a,r)}$ is equicontinuous.

Fix $\varepsilon > 0$. Let $z, w \in \Delta(a, r)$. Take $f \in \mathcal{F}$.

$$|f(z) - f(w)| = \left| \frac{1}{2\pi i} \int_{|\zeta - a| = 2r} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{|\zeta - a| = 2r} \frac{f(\zeta)}{\zeta - w}\, d\zeta \right|$$

Cauchy's formula

$$= \frac{1}{2\pi} \left| \int_{|\zeta - a| = 2r} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta \right|$$

$$= \frac{1}{2\pi} \left| \int_{|\zeta - a| = 2r} f(\zeta) \cdot \frac{z - w}{\underbrace{(\zeta - z)}_{\geq r}\ \underbrace{(\zeta - w)}_{\geq r}}\, d\zeta \right|$$

$$\leq \frac{1}{2\pi} \cdot M \cdot |z-w| \cdot \frac{1}{r^2} \cdot 2\pi \cdot (2r)$$

$$= \frac{2M}{r} \cdot |z-w| = K|z-w| \quad \text{for} \quad K = \frac{2M}{r}.$$

The Claim follows by Example $\boxed{c}$ above. or directly,

let $\delta = \frac{\varepsilon}{K}$. If $|z-w| < \delta \Rightarrow |f(z) - f(w)| \leq K|z-w| < \varepsilon$.

QED

Conclusion

Proposition + Arzelà - Ascoli $\Rightarrow$ Montel

We only prove Arzelà - Ascoli (next time)