HW 7 - SOLUTIONS

Q1. Let $f(z) = \frac{1}{z}$. This function is holomorphic in K since it extends holomorphically to Δ^*, a neighborhood of K. However, f cannot be approximated uniformly in K by functions which are holomorphic in Δ:

$$f_n \rightrightarrows f.$$

Indeed, if this was possible, we could integrate over $|z| = \frac{1}{2}$ to get

$$\int_{|z|=\frac{1}{2}} f_n \, dz \to \int_{|z|=\frac{1}{2}} f \, dz.$$

By Cauchy's theorem,

$$\int_{|z|=\frac{1}{2}} f_n \, dz = 0$$

whereas by the residue theorem

$$\int_{|z|=\frac{1}{2}} f \, dz = 2\pi i.$$

This is a contradiction.

Q2.

(i) This is an application of Runge's theorem, compact form. The set

$$K = \{4 \leq |z| \leq 5\}$$

is compact. The function

$$f(z) = \frac{1}{(z-2)(z-6)}$$

extends to the neighborhood

$$U = \left\{ z : \frac{7}{2} < |z| < \frac{11}{2} \right\}$$

of K. The set $\widehat{\mathbb{C}} - K$ has two components $\{z : |z| < 4\}$ and $\{z : |z| > 5\} \cup \{\infty\}$. Let $S = \{3, 7\}$ which is a set of points from each component. We can invoke Runge's theorem, thus approximating f by rational functions with poles at 3 and 7.

(ii) The answer is no and the argument is very similar to that of **Q1**. If we could find rational functions with poles only at 7 such that

$$R_n \rightrightarrows f$$

in K, then we could integrate over $|z| = \frac{9}{2}$ to get

$$\int_{|z|=\frac{9}{2}} R_n \, dz \to \int_{|z|=\frac{9}{2}} f \, dz.$$

By the residue theorem,

$$\int_{|z|=\frac{9}{2}} R_n \, dz = 0$$

since the poles of R_n are exterior to $|z| = \frac{9}{2}$. By the residue theorem

$$\int_{|z|=\frac{9}{2}} f \, dz = 2\pi i \operatorname{Res}(f, 2) = 2\pi i \frac{1}{z-6}|_{z=2} = -\frac{\pi i}{2}.$$

This is a contradiction.

Q3.

(i) The set K_n is clearly closed and bounded hence compact. The function f_n extends to a neighborhood U_n of K_n where

$$U_n = \left\{ z = x + iy : \frac{1}{2n} < |y| < n+1, |x| < n+1 \right\} \cup \left\{ z = x + iy : |x| < n+1, |y| < \frac{1}{4n} \right\}.$$

We can write as before

$$U_n = U_n^+ \cup U_n^- \cup U_n^0,$$

with the obvious meanings. Note that U_n^+, U_n^-, U_n^0 are disjoint open sets. The extension is given by

$$f_n = \begin{cases} 1 & \text{if } z \in U_n^+ \\ -1 & \text{if } z \in U_n^- \\ 0 & \text{if } z \in U_n^0. \end{cases}.$$

The function f_n is thus holomorphic in K_n. Since

$$\mathbb{C} \setminus K_n$$

is connected, it follows from Little Runge's theorem that f_n can be approximated uniformly on K_n by polynomials. In particular, we can find polynomials p_n with

$$|p_n(z) - f_n(z)| < \frac{1}{2^n} \text{ for all } z \in K_n.$$

(ii) Let z be arbitrary. If $z = x + iy \in \mathfrak{h}^+$, we can find n sufficiently large so that

$$\frac{1}{n} < y < n \text{ and } |x| < n.$$

This uses the fact that $y > 0$ so that we can pick $n > \max(y, \frac{1}{y}, |x|)$. Thus $z \in K_n$. Note that

$$K_n \subset K_m \text{ for all } m \geq n.$$

Thus $z \in K_m$ for all $m \geq n$, and in particular by (i) we have

$$|p_m(z) - f_m(z)| < \frac{1}{2^m} \implies |p_m(z) - 1| < \frac{1}{2^m}.$$

Thus $p_m(z) \to 1$, pointwise, as $m \to \infty$. The argument when $z \in \mathfrak{h}^-$ or $z \in \mathbb{R}$ is entirely similar, the only changes that are needed concern the values of the function f.

Q4.

(i) This statement is true. Indeed, $\widehat{\mathbb{C}} \setminus U$ is a union of two closed sets $\overline{\Delta}\left(\frac{1}{4}, \frac{3}{4}\right)$ and the set $\{z : |z| \geq 1\} \cup \{\infty\}$. These two closed sets have a point in common, namely $z = 1$. This means that $\widehat{\mathbb{C}} \setminus U$ is connected, hence by Little Runge for open sets, every holomorphic function on U can be approximated locally uniformly in U by polynomials.

(ii) This statement is false. Take the function $f(z) = \frac{1}{z}$, which is holomorphic in U and continuous in K. Assume

$$p_n \rightrightarrows f$$

in K, for polynomials p_n. Then there exists N such that for all $n \geq N$ we have for $z \in K$ that

$$|p_n(z) - f(z)| < \frac{1}{2} \implies \left|p_n(z) - \frac{1}{z}\right| < \frac{1}{2} \implies |zp_n(z) - 1| < \frac{|z|}{2} \leq \frac{1}{2}.$$

Let $g(z) = zp_n(z) - 1$ for fixed $n \geq N$. We have $g(0) = 1$ and $|g(z)| < \frac{1}{2}$ in U. The function g is in fact entire so in particular it extends continuously over the circle $|z| = 1$. By the maximum modulus principle applied to the disc $\Delta(0,1)$ and the holomorphic function $g(z)$, we obtain

$$1 = |g(0)| \leq \sup_{z \in \overline{\Delta}} |g(z)| = \sup_{z \in \partial\Delta} |g(z)| \leq \frac{1}{2}.$$

This is a contradiction.

(iii) This statement is true. The complement $\widehat{\mathbb{C}} \setminus K$ consists of two components $\Delta\left(\frac{1}{4}, \frac{3}{4}\right)$ and $\{z : |z| > 1\} \cup \{\infty\}$. Let $S = \{0, \infty\}$. By Runge's theorem for compact sets, any function f holomorphic in K can be approximated uniformly in K by rational functions with poles only at S. Such a rational function

$$R = \frac{P}{Q}$$

must be a Laurent polynomial. Indeed, since there are no poles in $\mathbb{C}$ other than 0, Q must be a power of z hence $Q = z^m$. Writing $P = \sum_{j=0}^{p} a_j z^j$ we see that

$$R = \sum_{j=0}^{p} a_j z^{j-m}$$

is a Laurent polynomial.