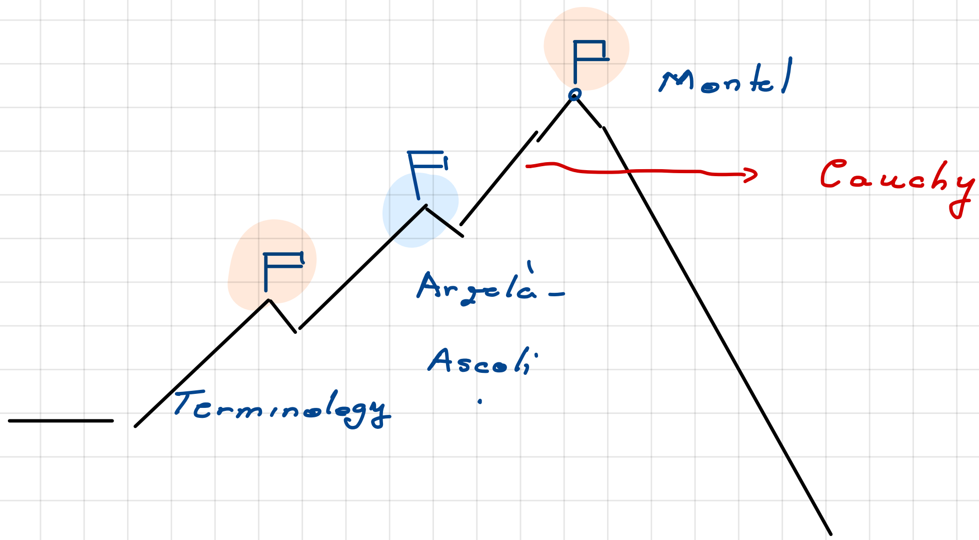
Math 220B — Lecture 13

February 3, 2021

Last time



Arzelà - Ascoli $\mathcal{F}$ family of continuous functions in $\mathcal{U}$

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ locally equicontinuous and locally bounded.

Today - we give the proof.

All functions today are continuous.

Notation & Preliminaries

$f : U \longrightarrow \mathbb{C}$ continuous, $K \subseteq U$ compact

$$\|f\|_K = \sup_{z \in K} |f(z)|$$

Note

$\boxed{\text{i}}$ $\|f+g\|_K \leq \|f\|_K + \|g\|_K$

$\boxed{\text{ii}}$ $f_n \overset{K}{\rightrightarrows} f \iff \|f_n - f\|_K \longrightarrow 0$ as $n \longrightarrow \infty$.

Def f_n is uniformly Cauchy in K if

$\forall \varepsilon > 0 \;\; \exists N \;\; \forall n, m \geq N, \quad \|f_n - f_m\|_K < \varepsilon.$

Lemma f_n converges uniformly in K

$\iff$ f_n uniformly Cauchy in K.

Proof We will only use „$\Leftarrow$" so we only give its proof.

Fix $\varepsilon > 0 \Rightarrow \exists N$ with $|f_n(z) - f_m(z)| < \varepsilon$ $\forall\, n, m \geq N$.

$\forall\, z \in K$. $(*)$

Thus $\{f_n(z)\}$ is Cauchy for fixed z. Then $\{f_n(z)\}$ converges pointwise to $f(z)$. Make $m \to \infty$ in $(*)$ to conclude that

$\forall\, \varepsilon \quad \exists\, N$ with $|f_n(z) - f(z)| \leq \varepsilon$ $\forall\, n \geq N$, $z \in K$.

Thus $f_n \rightrightarrows f$ in K.

Proof of Arzelà – Ascoli

„$\Longrightarrow$" Let $\mathcal{F}$ be normal.

(1) $\mathcal{F}$ locally bounded

Let $K \subseteq \mathcal{U}$ compact. We show $\mathcal{F}|_K$ bounded. i.e.

$$\exists M > 0 \quad \forall f \in \mathcal{F} \implies \|f\|_K < M.$$

Assume not for a contradiction. Then

$$\forall M > 0 \quad \exists f_M \in \mathcal{F} \text{ with } \|f_M\|_K \geq M$$

Letting $M = n.$, we obtain a sequence f_n with $\|f_n\|_K \geq n.$

Since $\mathcal{F}$ normal, we can find a subsequence $f_{n_k} \overset{K}{\rightrightarrows} f$

Thus $\|f_{n_k} - f\|_K < 1$ if k sufficiently large.

Note f_{n_k} continuous $\Rightarrow$ f continuous. so $\|f\|_K < M$. Then

$$M. > \|f\|_K \geq \|f_{n_k}\|_K - \|f_{n_k} - f\|_K \geq n_k - 1 \longrightarrow \infty \text{ as } k \to \infty$$

This gives a contradiction.

(2) $\mathcal{F}$ locally equicontinuous

Let $K \subseteq \mathcal{U}$ compact. We show $\mathcal{F}/_K$ equicontinuous.

that is $\forall\, \varepsilon\ \exists\, \delta : \forall\, x, y \in K,\ |x-y| < \delta\ \ \forall f \in \mathcal{F}$ then

$$|f(x) - f(y)| < \varepsilon.$$

Assume not, then

$$\exists\, \varepsilon\quad \forall\, \delta\quad \exists\ x_\delta, y_\delta \in K \text{ with } |x_\delta - y_\delta| < \delta\ \ \exists f_\delta \in \mathcal{F} \text{ but}$$

$$|f_\delta(x_\delta) - f_\delta(y_\delta)| \geq \varepsilon.$$

Take $\delta = \frac{1}{n}$. Then

$$\exists\ x_n, y_n \in K,\quad |x_n - y_n| < \frac{1}{n}\quad \exists f_n \in \mathcal{F} \quad \text{with}$$

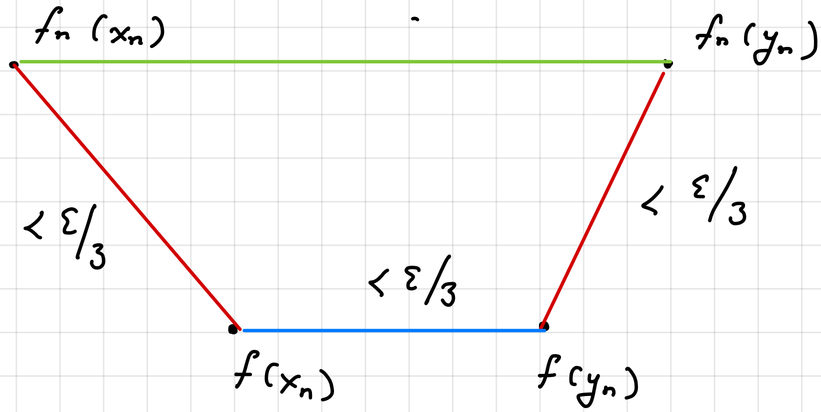
$$|f_n(x_n) - f_n(y_n)| \geq \varepsilon.$$

After passing to a subsequence & relabelling, we arrange

[i] $f_n \overset{K}{\rightrightarrows} f$ because $\mathcal{F}$ normal

[ii] $|x_n - y_n| < \frac{1}{n}$

[iii] $|f_n(x_n) - f_n(y_n)| \geq \varepsilon.$



Using f_n continuous, $f_n \rightrightarrows f$ we get f continuous.

Since K compact $\Rightarrow f|_K$ uniformly continuous.

Then $\exists\ \tau > 0$ with

$$|x-y| < \tau, \quad x,y \in K \Rightarrow |f(x) - f(y)| < \varepsilon/3. \quad (1)$$

Let N be so that $\forall\ n \geq N$, we have $\frac{1}{n} < \tau$ and

$$\|f_n - f\|_K < \varepsilon/3 \quad . \quad (2)$$

Then $|x_n - y_n| < \frac{1}{n} < \tau \Rightarrow |f(x_n) - f(y_n)| < \varepsilon/3$ by (1).

$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{3}$ & $|f_n(y_n) - f(y_n)| < \varepsilon/3$ by (2).

By triangle inequality (see picture)

$$|f_n(x_n) - f_n(y_n)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

contradicting $\boxed{\text{**}}$

The Converse

Assume $\mathcal{F}$ is locally equicontinuous & locally bounded.

$$\overset{?}{\Longrightarrow} \quad \mathcal{F} \text{ normal}$$

Let $f_n \in \mathcal{F}$. We wish to find a subsequence converging locally uniformly?

How do we find such a subsequence?

Plan $\boxed{I}$ arrange pointwise convergence of f_n

$\boxed{II}$ show local uniform convergence

Better Plan $\boxed{I}$ arrange pointwise convergence of f_n only at a countable dense set

$\boxed{II}$ show local uniform convergence

Let $\{a_k\}$ be the set of points in $\mathcal{U}$ with rational coordinates enumerated in any order. Dense!

Claim $\boxed{i}$ After passing to a subsequence of f_n & relabelling, we may assume

(*) $\forall k$, the sequence $f_n(a_k)$ converges as $n \to \infty$.

Claim $\boxed{ii}$ If $\{f_n\}$ equicontinuous & (*) $\Rightarrow$ f_n converges locally uniformly.

We win!

Proof of Claim $\boxed{2}$ Cantor diagonalization

We only use pointwise boundedness of $\{f_n\}$.

Consider $f_1(a_1)\quad f_2(a_1)\quad \cdots\quad f_n(a_1)\quad \cdots$ bounded

Find a subsequence

$(S_1)\qquad f_{11},\ f_{12},\ \cdots\ f_{1n},\ \cdots$ ← converges at a_1

Look at the values of (S_1) at a_2 & repeat. We find

$\underline{\underline{(S_2)}}\qquad f_{21},\ f_{22},\ \cdots\ f_{2n}\ \cdots$ ← converges at a_2 and a_1

Look at the values of (S_2) at a_3 & repeat.

We obtain an array:

$$
\begin{array}{cccccccl}
 & f_1 & f_2 & \cdots & f_n & \cdots & & \text{converges} \\
(S_1) & \boxed{f_{11}} & f_{12} & f_{13} & \cdots & f_{1n} & \cdots & \text{at } a_1 \\
(S_2) & f_{21} & \boxed{f_{22}} & f_{23} & \cdots & f_{2n} & \cdots & \text{at } a_1, a_2 \\
(S_3) & f_{31} & f_{32} & \boxed{f_{33}} & \cdots & f_{3n} & \cdots & \text{at } a_1, a_2, a_3 \\
 & \cdots & & & \ddots & & &
\end{array}
$$

Each row is a subsequence of the previous one.

Consider the diagonal subsequence

$$f_{11},\ f_{22},\ f_{33},\ \cdots\ f_{nn}\ \cdots$$

It is a subsequence of the original sequence. & converges at each a_k. Indeed

$$f_{11},\ f_{22},\ \cdots\ f_{k-1,k-1},\ \big|\ f_{kk},\ f_{k+1,k+1},\ \cdots$$

initial terms

part of (S_k) so we have convergence at a_k.

Proof of Claim [ii]

Know [a] $\{a_k\}$ dense in $\mathcal{U}$ and

$\forall k$, the sequence $\{f_n(a_k)\}_n$ converges

[b] f_n locally equicontinuous

Wish $\forall\, \alpha \in \mathcal{U}$, $\exists$ Δ = bounded open ball in $\mathcal{U}$, $\alpha \in \mathcal{U}$

$f_n/_{\overline{\Delta}}$ converges uniformly.

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(1) $\forall\, \alpha$ $\exists\, \alpha \in \overline{\Delta}$, $\mathcal{F}/_{\overline{\Delta}}$ equicontinuous.

Thus $\forall\, \varepsilon$ $\exists\, \delta$: $\forall\, |x-y| < \delta$, $x, y \in \overline{\Delta}$, $\forall\, f \in \mathcal{F}$

$$|f(x) - f(y)| < \varepsilon/3$$

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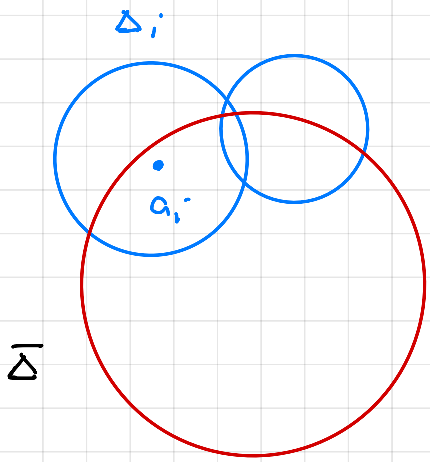
(2) $\overline{\Delta}$ can be covered by $\Delta_i = \Delta(a_i; \delta)$ for $a_i \in \overline{\Delta}$.



This because $\{a_i\}_i \cap \overline{\Delta}$ is dense in $\overline{\Delta}$.

By compactness, we may assume

$$\overline{\Delta} \subseteq \bigcup_{i=1}^{\ell} \Delta(a_i, \delta).$$

(3) Since $\{f_n(a_i)\}_{i=\overline{1,\ell}}$ is convergent, it is Cauchy. Hence

$$\forall \varepsilon \quad \exists N \quad \forall n,m \geq N \quad \forall 1 \leq i \leq \ell$$

$$|f_n(a_i) - f_m(a_i)| < \varepsilon/3$$

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(4) Let $z \in \overline{\Delta}$. By (2), $\exists i$ with $|z - a_i| < \delta$. Let $n,m \geq N$ as in (3). Then

$$|f_n(z) - f_m(z)| \leq \underbrace{|f_n(z) - f_n(a_i)|}_{\text{use (1)}} + \underbrace{|f_n(a_i) - f_m(a_i)|}_{\text{use (3)}} + \underbrace{|f_m(a_i) - f_m(z)|}_{\text{use (1)}}$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

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(5) Conclusion $\|f_n - f_m\|_{\overline{\Delta}} < \varepsilon \quad \forall n,m \geq N$.

$\Longrightarrow \{f_n\}$ uniformly Cauchy in $\overline{\Delta}$

$\overset{\text{Lemma}}{\Longrightarrow} \{f_n\}$ converges uniformly in $\overline{\Delta}$.

This completes the proof.

Remark the converse only used pointwise boundedness

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ pointwise bounded + locally equicont

$\iff$ $\mathcal{F}$ locally bounded + locally equicont.

The second version bears connections with Montel & it is more uniform.