Math 220A - Lecture 1

October 2, 2020

Weierstraß

Let $U \subseteq \mathbb{C}$ open & connected.

Definition $f : U \longrightarrow \mathbb{C}$ is complex differentiable (CD) if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} := f'(z) \text{ exists and is finite.}$$

Examples

$\boxed{I}$ f, g complex differentiable $\Rightarrow f+g,\ fg$ are also

$\boxed{II}$ $1, z, z^2, \dots, z^n, \dots$ complex differentiable

$\bar{z}$ is not

CD = complex differentiable

RD = real differentiable

Remark

We have seen the same definition for $f : U \longrightarrow \mathbb{R}$, $U \subseteq \mathbb{R}$ open.

The two definitions have very different consequences.

$\boxed{i}$ If f is CD $\Rightarrow f'$ is CD $\Rightarrow f''$ is CD $\Rightarrow \dots$

If f is RD this fails. Indeed,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Then

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & , x \neq 0 \\ 0 & \end{cases}$$

is not even continuous.

[II] If f is CD, we will show

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n \quad \text{in some } \Delta(a, r) \subseteq U.$$

If f is RD, this fails. Take

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We have f is C^∞, $f^{(n)}(0) = 0$ $\forall n$ so the Taylor series at 0 is 0. Thus f does not equal its Taylor series in any interval $(-r, r)$, $r > 0$.

[III] If f is CD for $U = \mathbb{C}$ & f bounded $\Rightarrow$ f constant.

If f is RD, $f(x) = \sin x$ is bounded.

[IV] If f is CD and $f = 0$ for $V \subseteq U$ open $\Rightarrow$

$\Rightarrow f \equiv 0$.

This fails if f is RD.

A more appropriate comparison is with functions of two real variables.

Identify $\mathbb{C} \cong \mathbb{R}^2$; $z = x + iy \longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

$$|z| = \sqrt{x^2 + y^2}$$

Definition $f : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is real differentiable (RD) if

$\forall z \in U \quad \exists \ A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\mathbb{R}$ - linear,

$$\lim_{h \to 0} \frac{|f(z+h) - f(z) - Ah|}{|h|} = 0.$$

We write $A = Df(z)$.

Remark f is CD $\Rightarrow$ f is RD.

Indeed, $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is multiplication by $f'(z)$

Remark If $f = u + iv$ is RD then

u_x, u_y, v_x, v_y exist and $A = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ = Jacobian.

Indeed, by definition

$$\lim_{h \to 0} \frac{|f(x+h, y) - f(x,y) - h A \begin{bmatrix} 1 \\ 0 \end{bmatrix}|}{|h|} = 0$$

$$\Rightarrow A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = u_x + i v_x \rightsquigarrow \begin{bmatrix} u_x \\ v_x \end{bmatrix}$$

Similarly $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_y \\ v_y \end{bmatrix}$, as needed.

Conversely If

u_x, u_y, v_x, v_y exist & are continuous $\Rightarrow f$ is RD.

See Math 140 C or Rudin 9.21.

Lemma $A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. $\mathbb{R}$-linear. TFAE

$\boxed{a}$ A is $\mathbb{C}$-linear

$\boxed{b}$ $A(z) = \alpha z$ for $\alpha \in \mathbb{C}$

$\boxed{c}$ $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $\alpha = a + bi$

Proof $\boxed{a} \Longrightarrow \boxed{b}$ Take $\alpha = A(1) \Longrightarrow A(z) = z\,A(1) = \alpha z$.

$\boxed{b} \Longrightarrow \boxed{c}$ $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A(1) = \alpha = \begin{bmatrix} a \\ b \end{bmatrix}$

$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A(i) = \alpha i = ai - b \longrightarrow \begin{bmatrix} -b \\ a \end{bmatrix}$

$\boxed{c} \Longrightarrow \boxed{a}$ Let $\alpha = a + bi$. Then $A(z) = \alpha z$ by the argument above. Thus A is $\mathbb{C}$-linear.

Remark By the lemma, TFAE

[i] f is CD

[ii] f is RD & $Df(z)$ is $\mathbb{C}$-linear $\forall z \in U$.

Remark (Cauchy-Riemann equations).

If f is CD, $Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ is $\mathbb{C}$-linear

so by the lemma

$$u_x = v_y$$
$$u_y = -v_x$$

(CR equations)

Conversely if CR-equations hold & u, v are of class C^1

then $f = u + iv$ is CD.

Indeed, f is RD in this case and

$$Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$ is $\mathbb{C}$ - linear by the Lemma

part $\boxed{c}$ & CR equations. Thus $Df(z)$ is multiplication

by $\alpha = f'(z)$ & f is CD.

Harmonic functions

If u, v satisfy CR & are of class C^2 then

$$u_x = v_y \implies u_{xx} = v_{yx}$$
$$u_y = -v_x \implies u_{yy} = -v_{xy}$$
$$\implies u_{xx} + u_{yy} = 0.$$

Similarly $v_{xx} + v_{yy} = 0$

A function h of class C^2 with

$h_{xx} + h_{yy} = 0$ is said to be harmonic.

Conclusion

Thus if f is CD & f class C^2 then

$u = \operatorname{Re} f$, $v = \operatorname{Im} f$ are harmonic.

Pairs (u, v) arising this way are called harmonic conjugates.

Notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Remark

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned} \implies \begin{aligned} x &= \frac{1}{2}(z + \bar{z}). \\ y &= \frac{1}{2i}(z - \bar{z}). \end{aligned}$$

Think of $z, \bar{z}$ as independent variables. then

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x} \cdot \frac{1}{2} + \frac{\partial}{\partial y} \cdot \frac{1}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y}\right)$$

Lemma

"f depends on z but not on $\bar{z}$"

$$f \text{ is CD} \implies \frac{\partial f}{\partial \bar{z}} = 0$$

Proof

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(f_x + i f_y \right) \text{ by definition}$$

$$= \frac{1}{2}\left(u_x + i v_x + i \left(u_y + i v_y \right)\right)$$

$$= \frac{1}{2}\left(u_x - v_y \right) + \frac{i}{2}\left(v_x + u_y \right) \stackrel{?}{=} 0$$

$$\iff \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}$$

. These are the CR equations.

Math 220 A — Lecture 2

Oct 7, 2020

Last time

[i] $f : U \longrightarrow \mathbb{C}$ is holomorphic provided $\forall z \in U$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

[ii] $f = u + iv$ holomorphic $\Longrightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

(Cauchy Riemann)

$$\Longleftrightarrow \frac{\partial f}{\partial \bar{z}} = 0.$$

[iii] u, v are harmonic conjugates

$$Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Today : [i] geometric consequences

[ii] analytic functions & power series

[iii] logarithm

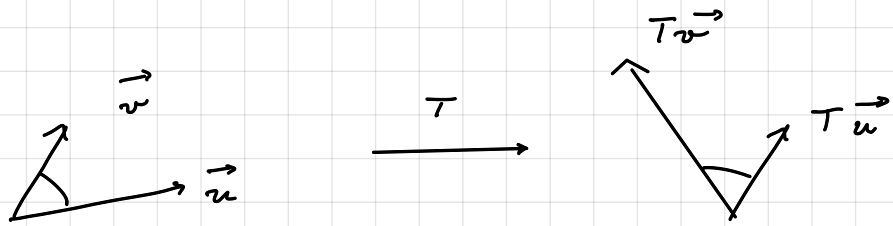
1. Geometric consequences / Conformal maps

Def $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\mathbb{R}$-linear, invertible

[i] T is orientation preserving if $\det T > 0$.

[ii] T is angle preserving if for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$

$$\measuredangle(\vec{u}, \vec{v}) = \measuredangle(T\vec{u}, T\vec{v}).$$



Remark $T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is both orientation & angle preserving (unless $a = b = 0$).

- $\det T = a^2 + b^2 > 0$ if $(a, b) \neq (0, 0)$
- ${}^tT\,T = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = |\alpha|^2 \cdot \mathbb{1}$, $\alpha = a + bi$

$$\Rightarrow \quad Tu \cdot Tv = {}^tT\,Tu \cdot v = |\alpha|^2\, u \cdot v$$

$$\text{If } u = v \Rightarrow Tv \cdot Tv = |\alpha|^2\, v \cdot v \Rightarrow \|Tv\| = |\alpha| \cdot \|v\|$$

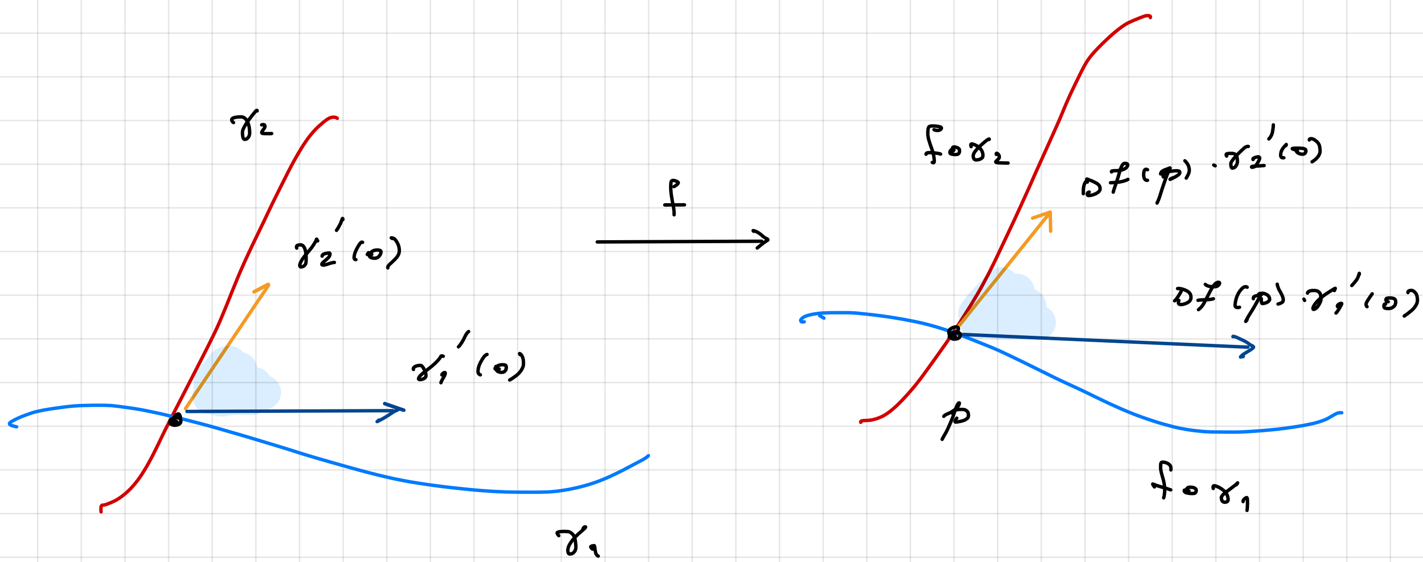
$$\cos \measuredangle(u,v) = \frac{u \cdot v}{\|u\| \cdot \|v\|}$$

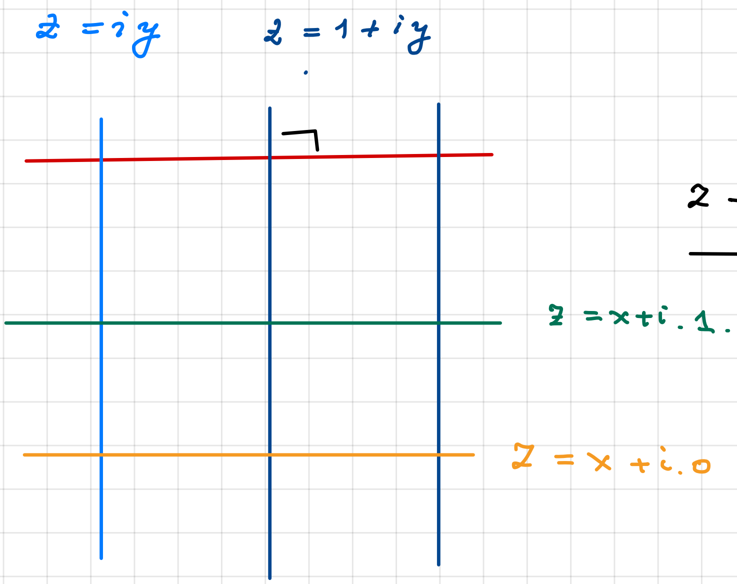
$$\cos \measuredangle(Tu, Tv) = \frac{Tu \cdot Tv}{\|Tu\| \cdot \|Tv\|} = \frac{|\alpha|^2\, u \cdot v}{|\alpha| \cdot \|u\| \cdot |\alpha| \cdot \|v\|}$$

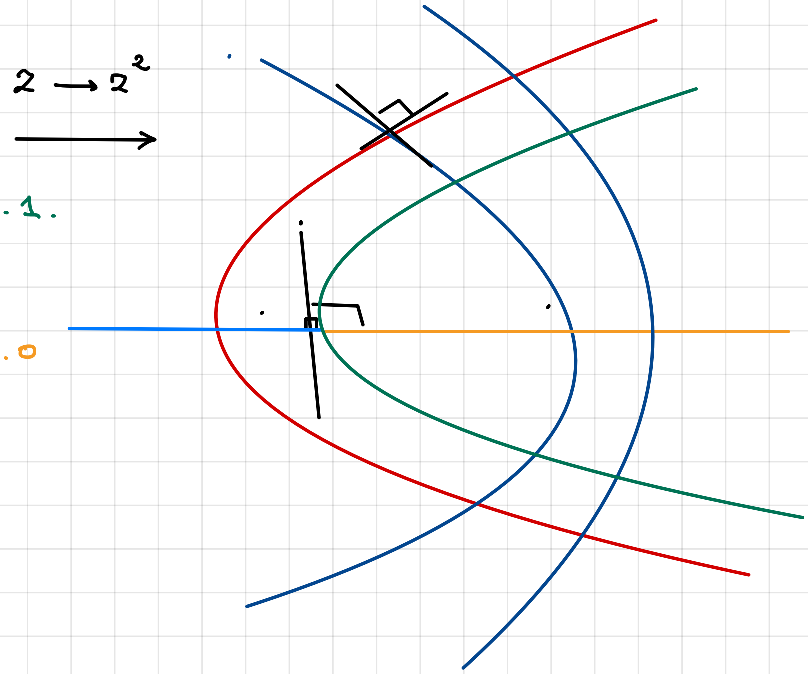
$$\Rightarrow \measuredangle(u,v) = \measuredangle(Tu, Tv)$$

Remark f holomorphic $\Rightarrow$ either $f'(z) = 0$ or else $Df(z)$ is orientation & angle preserving.

$\Rightarrow$ "f preserves angles" / f "conformal"







$f(z) = z^2$

$z = x + i.1 \implies z^2 = \underbrace{x^2 - 1}_{u} + \underbrace{2x}_{v} i$

$\implies u = \dfrac{v^2}{4} - 1$ parabola

$z = iy \implies$ half line

$z = 1 + iy \implies z^2 = \underbrace{1 - y^2}_{u} + \underbrace{2y}_{v} i$

$\implies u = 1 - \dfrac{v^2}{4}$ parabola

<u>Check</u> : Angles are preserved.

2 Power series & Analytic functions

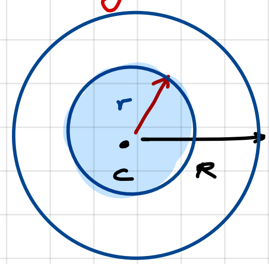
$c \in \mathbb{C}$, $a_n \in \mathbb{C}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n \qquad (*)$$

Def / Theorem (Abel) $\exists R$ $0 \le R \le \infty$ such that

$\boxed{i}$ if $|z-c| < R \Rightarrow (*)$ converges.

If $0 \le r < R \Rightarrow (*)$ converges absolutely & uniformly in $\Delta(c, r)$.



$\boxed{ii}$ if $|z-c| > R \Rightarrow (*)$ diverges.

Furthermore $R^{-1} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$. R = radius of convergence.

Definition $f : U \longrightarrow \mathbb{C}$ is analytic if $\forall z_0 \in U$ $\exists R > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{in } \Delta(z_0, R) \subseteq U.$$

Proof WLOG $c = 0$, else work $z^{new} = z - c$.

$\sum_{n=0}^{\infty} a_n z^n$. We set $R^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Let $|z| < r$

$\boxed{i}$. Let $r < \rho < R$. $\Rightarrow \limsup \sqrt[n]{|a_n|} = \frac{1}{R} < \frac{1}{\rho} \Rightarrow$

$$\Rightarrow \sqrt[n]{|a_n|} < \frac{1}{\rho} \text{ if } n \geq N.$$

$$\Rightarrow |a_n| < \frac{1}{\rho^n} \quad \text{if } n \geq N$$

$$\Rightarrow \underbrace{|a_n z^n|}_{f_n(z).} < \underbrace{\left(\frac{r}{\rho}\right)^n}_{M_n} \quad \text{if } n \geq N.$$

By Weierstraß M-test

$$|f_n| \leq M_n, \ \sum_{n=0}^{\infty} M_n < \infty \Rightarrow \sum_{n=0}^{\infty} f_n \text{ converges absolutely \& uniformly.}$$

$$\Rightarrow \sum_n a_n z^n \text{ converges absolutely \& uniformly in } \Delta(0, r).$$

$\boxed{ii}$ If $|z| > \rho > R \Rightarrow \limsup \sqrt[n]{|a_n|} = \frac{1}{R} > \frac{1}{\rho}$

$$\Rightarrow \sqrt[n]{|a_n|} > \frac{1}{\rho} \text{ for infinitely many } n\text{'s.}$$

$$\Rightarrow |a_n| > \frac{1}{\rho^n} \quad \text{for infinitely many } n\text{'s.}$$

$$\Rightarrow |a_n z^n| > \Big(\underbrace{\frac{|z|}{\rho}}_{>1}\Big)^n \text{ for infinitely many } n\text{'s.}$$

$$\Rightarrow a_n z^n \not\to 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n \text{ diverges}$$

Differentiation

Recall that if $f_n \longrightarrow f$ it doesn't follow $f_n' \longrightarrow f'$ in general. However, for power series we have

Theorem (Rudin 8.1).

If $\sum_{n \geq 0} a_n (z-c)^n$ has radius of convergence R, then $\sum_{n \geq 1} n a_n (z-c)^{n-1}$ has radius of convergence R as well.

Furthermore, if

$$f(z) = \sum_{n \geq 0} a_n (z-c)^n \quad \text{in} \quad \Delta(c, R)$$

$$\Longrightarrow f'(z) = \sum_{n \geq 1} n a_n (z-c)^{n-1} \quad \text{in} \quad \Delta(c, R).$$

Corollary $f^{(k)}(z) = \sum_{n \geq k} a_n n(n-1) \dots (n-k+1) (z-c)^{n-k}$

$$\overset{z=c}{\Longrightarrow} f^{(k)}(c) = a_k k! \Rightarrow a_k = \frac{f^{(k)}(c)}{k!}$$

$$\Rightarrow f(z) = \sum_{n \geq 0} \frac{f^{(n)}(c)}{n!} (z-c)^n \quad \text{if } f \text{ is analytic in } \Delta(c, R).$$

Remark If f is analytic $\overset{\text{Thm}}{\Longrightarrow}$ f is holomorphic.

Examples : exp, cos, sin

$\boxed{i}$ $e^z := 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots$, $R = \infty$.   ↙ $f(z)$.

$$\limsup_{n\to\infty} \sqrt[n]{n!} \geq \limsup_{n\to\infty} \sqrt[n]{\left(\frac{n}{2}\right)^{n/2}} = \limsup_{n\to\infty} \sqrt{\frac{n}{2}} = \infty.$$

$$f'(z) = 0 + 1 + z + \cdots + \frac{z^{n-1}}{(n-1)!} + \cdots = f(z)$$

$\Longrightarrow$ $(e^z)' = e^z$.

$\Longrightarrow$ $e^{z+c} = e^z \cdot e^c$. (Both sides satisfy $y' = y$, $y(0) = e^c$ so they are equal)

$\boxed{ii}$ $\cos z := \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

$\sin z := \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$(\sin z)' = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \cos z$.

$\sin^2 z + \cos^2 z = 1$.

Beware! $\sin z$, $\cos z$ are not bounded functions as $z \in \mathbb{C}$

$$\cos in\pi = \frac{e^{-n\pi} + e^{n\pi}}{2} \longrightarrow \infty \quad \text{as } n \longrightarrow \pm\infty.$$

$\boxed{iii}$ z^n can be defined for all $n \in \mathbb{Z}$, if $z \neq 0$.

[3] Logarithm

Remark

$e^{2\pi i} = 1 \implies$ exponential is not invertible.

$$\log 1 = 0, \pm 2\pi i, \pm 4\pi i, \dots, \pm 2n\pi i$$

Question Define $\log z$. ?

Remark Issues also arise with $\sqrt[n]{z}$ and z^α.

These are related to the logarithm.

$$\sqrt[n]{z} \longleftrightarrow z^\alpha \quad \text{for } \alpha = \frac{1}{n}$$

$$z^\alpha := \exp(\alpha \log z)$$

Def A logarithm function $\ell : \mathcal{U} \longrightarrow \mathbb{C}$ is a continuous function such that

$$e^{\ell(z)} = z. \quad \forall z \in \mathcal{U}.$$

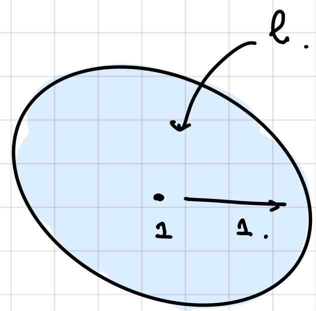
Naturally, for this to make sense, we need $\mathcal{U} \subseteq \mathbb{C} \setminus 0$.

Any two logarithms ℓ', ℓ on $\mathcal{U}$ differ by $2\pi i n$, $n \in \mathbb{Z}$.

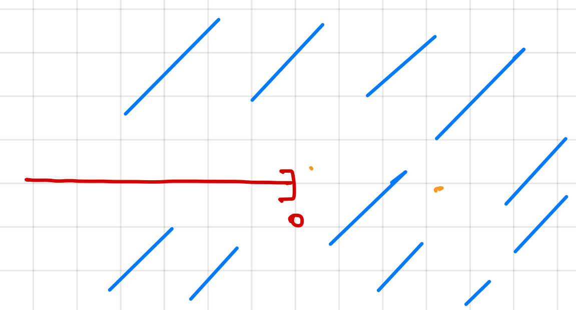
Example A $\mathcal{U} = \Delta(1,1)$,

$$\ell(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

HWK : ℓ is a logarithm in $\mathcal{U}$.



Example B $\mathcal{U} = \mathbb{C} \setminus \mathbb{R}_{\le 0}$



$z \in \mathcal{U}$

$z = r e^{i\theta}$, $\theta \in (-\pi, \pi)$.

$r \neq 0$

$$\operatorname{Log} z = \log r + i\theta$$

$$\Rightarrow e^{\operatorname{Log} z} = e^{\log r + i\theta} = r e^{i\theta} = z \Rightarrow \operatorname{Log} \text{ is a logarithm in } \mathbb{C} \setminus \mathbb{R}_{\le 0}.$$

Remark The two examples above give the same answer in $\Delta(1,1)$.

Indeed the two logarithms $\ell(z)$ and $\operatorname{Log} z$ differ by $2\pi i n$

$$\Rightarrow \quad \operatorname{Log} z - \ell(z) = 2\pi i n. \quad \text{Set } z = 1$$

$$\Rightarrow \quad \underbrace{\operatorname{Log} 1}_{0} - \underbrace{\ell(1)}_{0} = 2\pi i n \quad \Rightarrow n = 0$$

$$\Rightarrow \quad \operatorname{Log} z = \ell(z) \quad \text{in } \Delta(1,1).$$

Math 220 A - Lecture 3

October 9, 2020

Logistics

- 13 votes for MWF 3–3:50
- 5 votes for WF 3–4:15
- 4 votes indifferent

New time: MWF 3–3:50

No lecture: Monday, Oct 26.

Today: Loose ends

i. power series

ii. logarithm

iii. Mobius transformations

Conway $\overline{\underline{III}}$.

I Loose ends from last time

Theorem Assume that $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence R. Then $\sum_{k=1}^{\infty} k a_k z^{k-1}$ has radius of convergence R as well.

Furthermore, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ then

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}.$$

Proof Radius of convergence for 2nd power series

$$R^{-1} = \limsup_{k \to \infty} \sqrt[k]{k |a_k|} = \limsup_{k \to \infty} \sqrt[k]{|a_k|}. \quad \text{since } \sqrt[k]{k} \to 1.$$

Fix $\alpha \in \Delta(0, R)$. We show $f'(\alpha) = g(\alpha)$.

where $g = \sum_{k=1}^{\infty} k a_k z^{k-1}$.

Let $S_N = \sum_{k=0}^{N} a_k z^k$, $R_N = \sum_{k=N+1}^{\infty} a_k z^k$.

Know $S_N \to f$, $S_N' \to g$.

Fix $\varepsilon > 0$. We wish to find $\delta_\varepsilon > 0$

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} - g(\alpha) \right| < \varepsilon \quad \text{if} \quad z \in \Delta(\alpha, \delta)$$

Let $|\alpha| < \rho < R$. For $z \in \Delta(0, \rho)$ we have

$$(*) = \left| \frac{f(z) - f(\alpha)}{z - \alpha} - g(\alpha) \right| \leq \underbrace{\left| \frac{S_N(z) - S_N(\alpha)}{z - \alpha} - S_N'(\alpha) \right|}_{\overline{I}.}$$

$$+ \underbrace{\left| S_N'(\alpha) - g(\alpha) \right|}_{\overline{II}}$$

$$+ \underbrace{\left| \frac{R_N(z) - R_N(\alpha)}{z - \alpha.} \right|}_{\overline{III}.} \overset{?}{<} \varepsilon.$$

We estimate each of these terms. Term $\overline{III}$.

$$\left| \frac{R_N(z) - R_N(\alpha)}{z - \alpha.} \right| \leq \sum_{k=N+1}^{\infty} |a_k| \left| \frac{z^k - \alpha^k}{z - \alpha} \right|$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| \left(|z|^{k-1} + \ldots + |\alpha|^{k-1} \right)$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| \, k \rho^{k-1} < \varepsilon/3$$

if $N \geq N_1$.

Term $\overline{\underline{II}}$: $\left| s_N'(\alpha) - g(\alpha) \right| < \varepsilon/3$ for $N \geq N_2$.

Fix $N \geq N_1$ & N_2. For this N, find δ such that

Term $\overline{\underline{I}}$: $$\left| \frac{s_N(z) - s_N(\alpha)}{z - \alpha} - s_N'(\alpha) \right| < \varepsilon/3 \quad \text{if } z \in \Delta(\alpha, \delta).$$

Then

$$(*) \leq \overline{\underline{I}} + \overline{\underline{II}} + \overline{\underline{III}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ for}$$

$z \in \Delta(\alpha, \delta) \cap \Delta(0, \rho)$. QED.

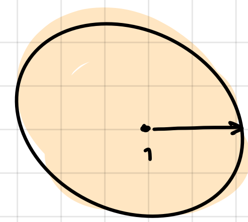
II. Logarithm

$U \subseteq \mathbb{C} \setminus \{0\}$ open & connected

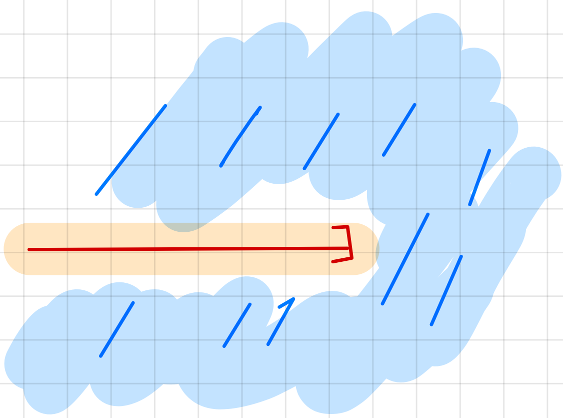
$\ell : U \longrightarrow \mathbb{C}$ continuous & $e^{\ell(z)} = z$.

Example A $\quad U = \Delta(1,1)$

$$\ell(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z-1)$$



Example B $\quad U = \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ slit plane



$z = r e^{i\theta}$

$\operatorname{Log} z = \log r + i\theta$

$\theta \in (-\pi, \pi) \Rightarrow e^{\operatorname{Log} z} = z.$

(Principal branch of logarithm).

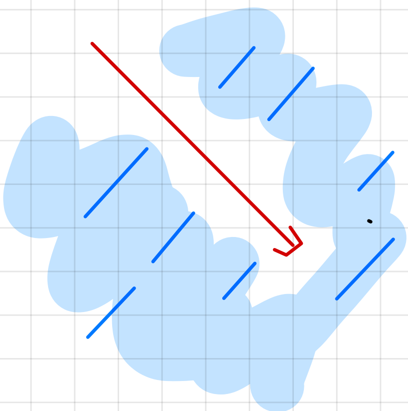
BEWARE $\quad \operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w$

This holds if $\operatorname{Re} z > 0$, $\operatorname{Re} w > 0$.

Example $\operatorname{Log}(1-i) = \log\sqrt{2} + i\left(-\frac{\pi}{4}\right).$

principal branch

Example C Other branches



$$U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}\, e^{i\alpha}.$$

$$z = re^{i\theta}, \quad \theta \in (\alpha, \alpha + 2\pi).$$

$$\operatorname{Log}_\alpha z = \log r + i\theta.$$

Remark [a] $U = \mathbb{C} \setminus \{0\} \Rightarrow$ impossible to define logarithm

[b] $U \subseteq \mathbb{C} \setminus \{0\}$ simply connected

$\Rightarrow$ we can define logarithm (later).

Examples A – C are simply connected.

Remark $z^{\alpha} = \exp(\alpha\, \ell(z))$ is multi-valued

– differ by $\exp(\alpha \cdot 2\pi i \cdot n)$.

Example Principal value of $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

$$(1-i)^{i} = \exp(i \cdot \operatorname{Log}(1-i))$$

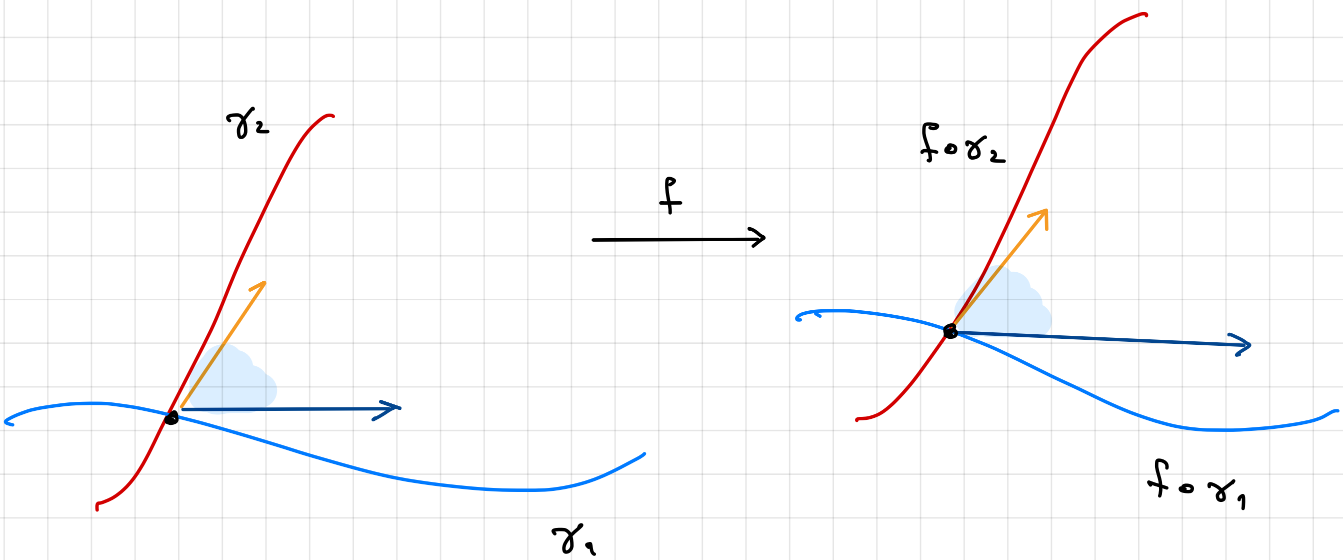
$$= \exp\left(i \cdot \left(\log\sqrt{2} - i\frac{\pi}{4}\right)\right)$$

$$= \exp\left(i \log\sqrt{2} + \frac{\pi}{4}\right)$$

$$= e^{\pi/4}\left(\cos\log\sqrt{2} + i\sin\log\sqrt{2}\right).$$

III. Geometry of holomorphic maps

We have seen holomorphic maps with $f'(z) \neq 0$ preserve angles.



Remark Given $U, V \subseteq \mathbb{C}$, a biholomorphic map $f : U \longrightarrow V$ is

[i] f bijective, holomorphic

[ii] $g = f^{-1} : V \longrightarrow U$ holomorphic.

If $f(p) = q \implies f \circ g(z) = z$

$$\implies g'(q) = \frac{1}{f'(p)}, \quad f'(p) \neq 0.$$

Important Question

Given $U, V \subseteq \mathbb{C}$, are they biholomorphic?

Today we study a class of transformations which are important for geometric arguments.

Möbius transformations (MT)

Fractional linear transformations (FLT)

Linear fractional transformations (LFT)

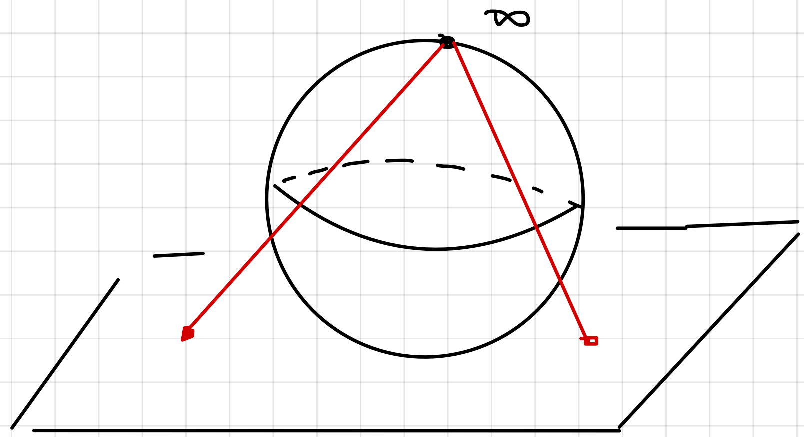
August Ferdinand Möbius (1790–1868)

Möbius strip, Möbius inversion, Möbius transform

Möbius published important work in astronomy.

Definition $\mathbb{C}_\infty = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Riemann sphere



Definition Möbius transformations MT.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow h_A : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}} \qquad \frac{1}{0} = \infty.$$

$A \in GL_2.$

$$\begin{cases} z \longrightarrow \dfrac{az+b}{cz+d} \\ \infty \longrightarrow \lim\limits_{z \to \infty} \dfrac{az+b}{cz+d} = \dfrac{a}{c}. \end{cases}$$

biholomorphism $h_A : \mathbb{C} \setminus \{-\frac{d}{c}\} \longrightarrow \mathbb{C} \setminus \{\frac{a}{c}\}.$

Remark

i. $A = I \Longrightarrow h_A = 1.$

ii. $A = \lambda B \Longleftrightarrow h_A = h_B.$ for $\lambda \neq 0.$

iii. $h_{AB} = h_A \circ h_B$ If $B = A^{-1}. \Longrightarrow h_{A^{-1}} = h_A^{-1}.$

Most famous example: Cayley transform

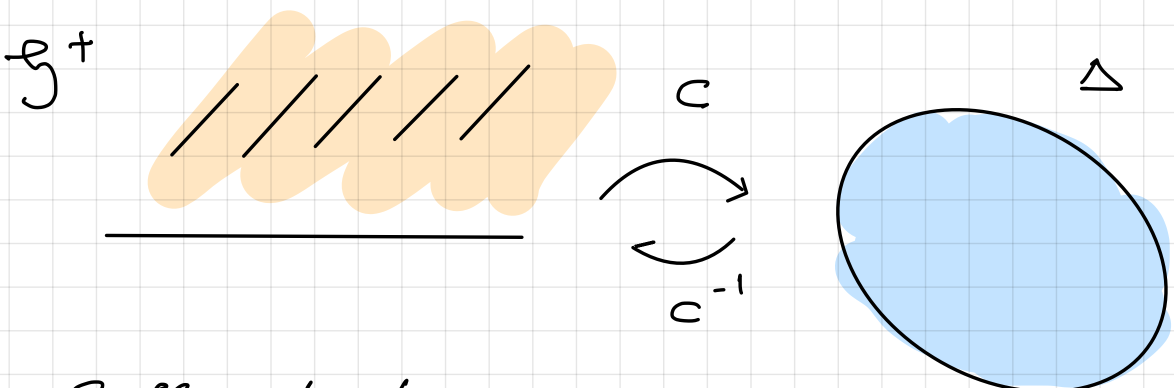
$$C(z) = \frac{z-i}{z+i}, \qquad C^{-1}(w) = i \cdot \frac{1-w}{1+w}.$$

Notation: $\Delta = \Delta(0,1)$

$$\mathfrak{H}^+ = \{ z : \operatorname{Im} z > 0 \}.$$

Claim: C is a biholomorphism

$$C : \mathfrak{H}^+ \longrightarrow \Delta.$$



Suffices to show

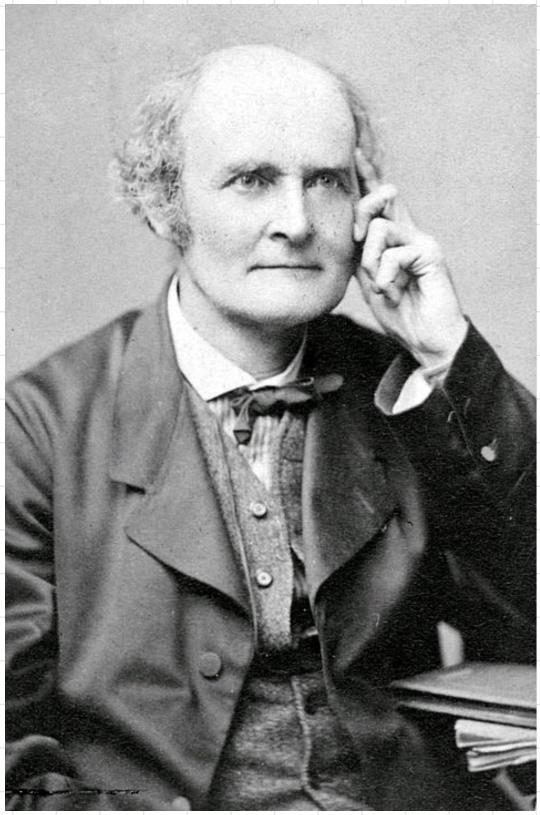
$$z \in \mathfrak{H}^+ \iff C(z) \in \Delta. \qquad \text{Write } z = x + iy$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$y > 0 \qquad\qquad |z - i| < |z+i|$$

$$\Updownarrow$$

$$x^2 + (y-1)^2 < x^2 + (y+1)^2$$



Arthur Cayley (1821 - 1895)

- worked in algebraic geometry, Group theory
- Cayley - Hamilton theorem
- modern definition of a group

Remark

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2} \frac{1}{z+\frac{d}{c}} + \frac{a}{c}$$

$c=0$: $$\frac{az+b}{d} = \frac{a}{d}\cdot z + \frac{b}{d}.$$

Types of Mobius transforms

$\boxed{I}$ translation $\quad Tz = z+\lambda$

$\boxed{II}$ rotations $\quad Rz = e^{i\theta}\cdot z$

$\boxed{III}$ dilations $\quad Dz = mz, \; m \in \mathbb{R}.$

$\boxed{IV}$ inversion $\quad Sz = \frac{1}{z}.$

Lemma All Möbius transforms are compositions of $\boxed{I} - \boxed{IV}$.

Generalized circles in $\hat{\mathbb{C}}$

$\boxed{\text{I}}$ circles in $\mathbb{C}$

$\boxed{\text{II}}$ line $L \cup \{\infty\}$ = circle in $\hat{\mathbb{C}}$ through ∞.

Main theorems about Möbius transforms

Theorem A Any Möbius transform maps generalized circles to generalized circles.

Theorem B Given two triples (z_1, z_2, z_3) and (z_1', z_2', z_3') of distinct points in $\hat{\mathbb{C}}$, $\exists!$ Möbius transformation h with

$$h(z_i) = z_i'.$$

Math 220 A - Lecture 4

October 12, 2020

Last time

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad h_A : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}, \quad z \longrightarrow \frac{az+b}{cz+d}$$

Generalized circles in $\hat{\mathbb{C}}$

$\boxed{\text{I}}$ circles in $\mathbb{C}$

$\boxed{\text{II}}$ line $L \cup \{\infty\}$

Theorem A Any Möbius transform maps generalized circles to generalized circles.

Theorem B PGL_2 acts triply transitively on $\hat{\mathbb{C}}$.

Given (z_1, z_2, z_3), (z_1', z_2', z_3') triples of distinct elts in $\hat{\mathbb{C}}$, $\exists!\ h$ with $h(z_i) = z_i'$

Proof of Thm A Suffices to consider the cases

[i] translation $z \longrightarrow z + \lambda$ clear

[ii] rotation $z \longrightarrow e^{i\alpha} z$ clear

[iii] dilation $z \longrightarrow m z$ clear

[IV] inversion $z \longrightarrow 1/z$.

Claim A generalized circle is given by

$$(*) \quad A z\bar{z} + B z + C\bar{z} + D = 0 . \quad \text{where } A, D \in \mathbb{R}.$$

and B, C are conjugates.

Proof A circle in $\mathbb{C}$ is given by

$$|z - z_0| = r \iff (z - z_0)\cdot(\bar{z} - \bar{z}_0) = r^2$$

$$\iff z\bar{z} - \bar{z}_0 z - z_0 \bar{z} + (z_0\bar{z}_0 - r^2) = 0$$

$\implies$ (*) for $A = 1$, $D = z_0\bar{z}_0 - r^2$, $B = -\bar{z}_0$, $C = -z_0$.

Conversely, if $A \neq 0$, (*) can be brought into this form.

When $A = 0$: $\underbrace{Bz + C\bar{z}}_{\text{linear}} + D = 0 \iff$ line.

Proof $\boxed{IV}$ preserves generalized circles.

$$A z\bar{z} + Bz + C\bar{z} + D = 0.$$

Let $w = \dfrac{1}{z} \Rightarrow A \cdot \dfrac{1}{w\bar{w}} + \dfrac{B}{w} + \dfrac{C}{\bar{w}} + D = 0$

$$\Rightarrow A + B\bar{w} + Cw + Dw\bar{w} = 0.$$

$\Rightarrow$ generalized circle. $\Rightarrow$ Thm A.

In the case of lines $L \cup \infty$, 0 and ∞ correspond under $\boxed{IV}$.

Proof of thm B   **Uniqueness**   Assume $\exists\, h, h'$

$$z_1 \underset{h'}{\overset{h}{\longrightarrow}} z_1' \qquad z_2 \underset{h'}{\overset{h}{\longrightarrow}} z_2' \qquad z_3 \underset{h'}{\overset{h}{\longrightarrow}} z_3'$$

Let $T = h'^{-1} \circ h \implies T(z_i) = z_i$,

$\iff \dfrac{az+b}{cz+d} = z$ has 3 roots z_1, z_2, z_3

$\iff az + b = cz^2 + dz$ has 3 roots

$\implies a = d,\ b = c \implies T = \mathbb{1} \implies h = h'!$

Existence   Suffices: $\exists\, h$ with

$$\begin{tikzcd}
z_1 \arrow[r, "h"] \arrow[rr, bend right, "T"'] & 0 & z_1' \arrow[l, "h'"'] \\
z_2 \arrow[r, "h"] \arrow[rr, bend right, "T"'] & 1 & z_2' \arrow[l, "h'"'] \\
z_3 \arrow[r, "h"] \arrow[rr, bend right, "T"'] & \infty. & z_3' \arrow[l, "h'"']
\end{tikzcd}$$

$h(z_1) = 0$

$h(z_2) = 1$

$h(z_3) = \infty.$

If (z_1', z_2', z_3') is another triple, find h' with

$$h'(z_1') = 0, \quad h'(z_2) = 1, \quad h'(z_3) = \infty.$$

Define $T = h'^{-1} \circ h \implies T(z_i) = z_i'$ as needed.

To deal with (z_1, z_2, z_3) and $(0, 1, \infty)$.

Cross ratio If $z_1, z_2, z_3 \neq \infty$,

$$h(z) = \frac{z - z_1}{z - z_3} \Big/ \frac{z_2 - z_1}{z_3 - z_2}$$

This is sometimes denoted $[z : z_1 : z_2 : z_3]$.

Check $h(z_1) = 0$

$h(z_2) = 1$

$h(z_3) = \infty$.

There are 3 remaining case $z_1 = \infty$, $z_2 = \infty$ or $z_3 = \infty$.

For example, when $z_1 = \infty$. the above expression is

$$h(z) = \frac{z_2 - z_3}{z - z_3}, \quad h(z_1) = 0, \quad h(z_2) = 1, \quad h(z_3) = \infty.$$

II. Cauchy theory & Integration (Conway IV)

The theory of integration is crucial to complex analysis. Many important results have as starting point Cauchy's integral formula.

§1. Complex integration

[a] $U \subseteq \mathbb{C}$ open & connected

$\gamma : [a,b] \longrightarrow U$ C^1-path

[i] length $(\gamma) = \int_a^b |\gamma'(t)|\, dt.$

[ii] C^1-reparametrization $\hat{\gamma} : [\hat{a}, \hat{b}] \longrightarrow U$

$$\hat{\gamma} = \gamma \circ \Phi, \quad \Phi : [\hat{a}, \hat{b}] \longrightarrow [a,b]$$

Orientation preserving: $\phi' > 0.$

[b] A piecewise C^1-path

$$\gamma = \gamma_1 + \dots + \gamma_n, \quad \gamma_i \text{ of class } C^1.$$

if $\exists\ a = a_0 < a_1 < \dots < a_n = b$

$$\gamma\big/_{[a_{i-1}, a_i]} = \gamma_i$$

[c] $f : U \longrightarrow \mathbb{C}$ continuous, Define

$$\int_\gamma f\, dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

substitute

$z = \gamma(t)$

$dz = \gamma'(t)\, dt$

This is independent of orientation preserving reparametrization

$$\int_a^b f(\gamma(t))\, \gamma'(t)\, dt = \int_{\hat{a}}^{\hat{b}} f(\hat{\gamma}(s)) \cdot \hat{\gamma}'(s)\, ds$$

$t = \phi(s)$.

This is change of variables: $f(\gamma(t)) = f(\hat{\gamma}(s))$

$$\gamma'(t)\, dt = \hat{\gamma}'(s)\, ds.$$

Remark $\displaystyle\int_{-\gamma} f\,dz = -\int_{\gamma} f\,dz$ after changing orientation

Remark The definition extends to piecewise C^1 paths

$$\int_{\gamma} f\,dz = \int_{\gamma_1} f\,dz + \dots + \int_{\gamma_n} f\,dz.$$

In particular, we can define $\displaystyle\int_{\partial R} f\,dz$, R retangle.

Remark Conway works with rectifiable paths.

In the elementary theory of analytic functions it is seldom necessary to consider arcs which are rectifiable, but not piecewise differentiable. However, the notion of rectifiable arc is one that every mathematician should know.

Ahlfors – Complex Analysis, 3rd edition

page 105

Fundamental estimate Assume $|f| \leq M$ along γ

$$\Longrightarrow \left| \int_\gamma f \, dz \right| \leq \text{length}(\gamma) \cdot M.$$

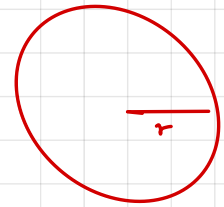
Proof
$$\left| \int_\gamma f \, dz \right| = \left| \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt \right|$$
$$\leq M \int_a^b |\gamma'(t)| \, dt$$
$$= M \cdot \text{length}(\gamma)$$

Augustin-Louis Cauchy (1789 - 1857) was a French mathematician who made contributions to several branches of mathematics including complex analysis.

Cauchy was a prolific writer: 800 research articles and 5 textbooks.

Example A γ = circle of radius r, $\gamma(t) = re^{it}$

$$\int_\gamma z^n \, dz \overset{z = re^{it}}{=} \int_0^{2\pi} r^n e^{int} \cdot re^{it} i \, dt$$



$$= \int_0^{2\pi} r^{n+1} e^{i(n+1)t} i \, dt$$

$$= r^{n+1} \frac{e^{i(n+1)t}}{i(n+1)} \cdot i \Big/_{t=0}^{t=2\pi} = 0. \quad , n \neq -1.$$

When $n = -1$

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{re^{it} i}{re^{it}} dt = \int_0^{2\pi} i \, dt = 2\pi i$$

Example B f admits primitive F, $f = F'$.

$$\int_\gamma f \, dz = \int_a^b F'(\gamma(t)) \cdot \gamma'(t) \, dt$$

$$= \int_a^b \left(F(\gamma(t))\right)' dt = F(\gamma(b)) - F(\gamma(a)).$$

Path independence!

III. Existence of primitives

$U \subseteq \mathbb{C}$ open connected, f continuous. We show three results.

Proposition A TFAE

$\boxed{i}$ f admits a primitive

$\boxed{ii}$ $\displaystyle\int_\gamma f\,dz = 0$ $\forall\ \gamma$ piecewise C^1 loop.

Remark $\boxed{i} \Rightarrow \boxed{ii}$ is clear by Example B.

Remark $\dfrac{1}{z}$ doesn't admit a primitive in $U = \mathbb{C}^\times$.

since $\displaystyle\int_\gamma \frac{dz}{z} = 2\pi i$ by Example A.

$\Rightarrow \nexists$ no logarithm in $U = \mathbb{C}^\times$.

Proposition B If $U = \Delta =$ disc. TFAE

$\boxed{i}$ f admits primitive

$\boxed{ii}$ $\displaystyle\int_{\partial R} f\,dz = 0$ for all rectangles $\overline{R} \subseteq U$.

Compare :

Prop. A	Prop B.
$U \subseteq \mathbb{C}$	$U = \Delta$
γ piecewise C^1	$\gamma = \partial R$

Proposition C If $f : U \to \mathbb{C}$ holomorphic $\Rightarrow$ $\displaystyle\int_{\partial R} f \, dz = 0$

for all rectangles $\bar{R} \subseteq U$.

Corollary $f : \Delta \to \mathbb{C}$ holomorphic $\overset{B+C.}{\Rightarrow}$ f admits a primitive.

Math 220A - Lecture 5

October 14, 2020

Last time — existence of primitives

$U \subseteq \mathbb{C}$ open & connected, $f : U \longrightarrow \mathbb{C}$ continuous

Proposition A TFAE

[i] f admits a primitive

[ii] $\displaystyle\int_\gamma f \, dz = 0$ $\forall \gamma$ piecewise C^1 loop.

Corollary $\not\exists$ no logarithm in $U = \mathbb{C}^\times$.

Proposition B If $U = \Delta =$ disc. TFAE

[i] f admits primitive

[ii] $\displaystyle\int_{\partial R} f \, dz = 0$ for all rectangles $\overline{R} \subseteq U$.

Compare:

Prop. A	Prop B.
$U \subseteq \mathbb{C}$	$U = \Delta$
γ piecewise C^1	$\gamma = \partial R$

Proposition C If $f : U \to \mathbb{C}$ holomorphic $\Longrightarrow \int_{\partial R} f\, dz = 0$ for all rectangles $R \subseteq U$. (Goursat's lemma).

Remark f' is NOT assumed to be continuous.

If f' is continuous an easier proof is possible.

Corollary $f : \Delta \to \mathbb{C}$ holomorphic

$\overset{B+C.}{\Longrightarrow}$ f admits a primitive.

$\overset{A}{\Longrightarrow}$ $\int_\gamma f\, dz = 0 \quad \forall \gamma$ piecewise C^1 loops.

This is a form of Cauchy's theorem.

Edouard Goursat

1858 – 1936

J'ai reconnu depuis longtemps que la demonstration du theoreme de Cauchy, que j'ai donnee en 1883, ne supposait pas la continuite de la derivee.

(I have recognized for a long time that the demonstration of Cauchy's theorem which I gave in 1883 didn't really presuppose the continuity of the derivative.)

Sur la definition generale des functions analytiques, d'apres Cauchy. Trans. AMS, 1900, 14-46.

Proposition A TFAE

$\boxed{i}$ f admits a primitive

$\boxed{ii}$ $\int_\gamma f\,dz = 0$ $\forall\, \gamma$ piecewise C^1 loop.

Proof $\boxed{i} \Rightarrow \boxed{ii}$ follows by path independence

$\boxed{ii} \Rightarrow \boxed{i}$. Fix $p \in U$. Let

$$F(q) = \int_\gamma f\,dz$$ where γ is a piecewise C^1 path in U joining p to q.

This is well-defined $\Longleftrightarrow \int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$.

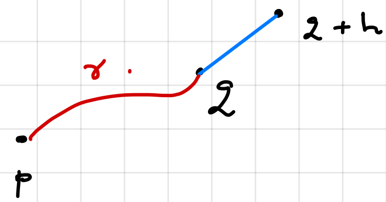
$\Longleftrightarrow \int_\eta f\,dz = 0$ where $\eta = \gamma_1 + (-\gamma_2)$.

which is true by assumption $\boxed{ii}$.

Claim $F' = f$.

Proof Fix $q \in U$. Let $\varepsilon > 0$. Let $\delta > 0$ with

$$(*) \quad |f(z) - f(q)| < \varepsilon \quad \text{if} \quad z \in \Delta(q, \delta).$$



We compute

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_z^{z+h} f(z)\,dz - f(z)\right|$$

$$= \frac{1}{|h|}\left|\int_z^{z+h}\big(f(z)-f(z)\big)\,dz\right|$$

$< \varepsilon$ by $(*)$ if $|h|<\delta$.

$$\leq \frac{1}{|h|}\cdot \text{length}\big([z,z+h]\big)\cdot\varepsilon.$$

$$= \frac{1}{|h|}\cdot |h|\cdot \varepsilon = \varepsilon \implies F' = f$$

Question: Why can we always find a piecewise C^1 path?

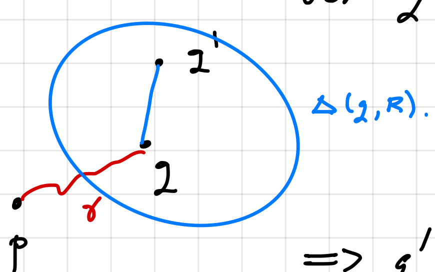
Let

$$\mathcal{X} = \{z \in U : \exists \text{ piecewise } C^1 \text{ path from } p \text{ to } z\}.$$

$\mathcal{X} \neq \emptyset$ since $p \in \mathcal{X}$.

$\mathcal{X}$ open. Let $z \in \mathcal{X} \implies \exists\, R>0$ with $\Delta(z,R) \subseteq U$.

For $z' \in \Delta(z,R)$, join p to z (since $z \in \mathcal{X}$)



join z to z' (via line segment).

$\implies z' \in \mathcal{X} \implies \Delta(z,R) \subseteq \mathcal{X} \implies \mathcal{X}$ open.

$\mathcal{X}$ closed. Let $q \in \partial \mathcal{X}$. We show $q \in \mathcal{X}$.

Let $q' \in \mathcal{X}$, $q' \in \Delta(q, \varepsilon) \subseteq \mathcal{U}$.

Join p to q' by a piecewise C^1 path. &

q' to q by line segment thus joining p to q

$\Rightarrow q \in \mathcal{X}$.

Since $\mathcal{U}$ connected $\Rightarrow \mathcal{X} = \mathcal{U}$.

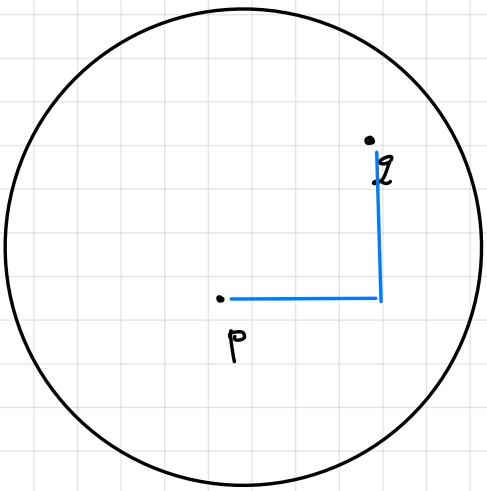
Proposition B If $U = \Delta = \text{disc.}$ TFAE

$\boxed{i}$ f admits primitive

$\boxed{ii}$ $\displaystyle\int_{\partial R} f\, dz = 0$ for all rectangles $\overline{R} \subseteq U$.

Proof We only need $\boxed{ii} \Rightarrow \boxed{i}$. Let $p \in U$.

Δ



Define $F(q) = \displaystyle\int_\gamma f\, dz$ where

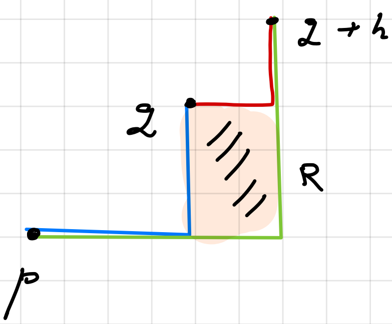
γ is a path from p to q.

consisting of 2 segments parallel to the axes. Such a path exists since

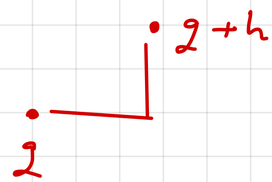
$U = \Delta = \text{disc.}$

The proof $F' = f$ is similar. We have

$$F(q+h) - F(q) = \int_p^{q+h} f\, dz - \int_p^{q} f\, dz = \int_q^{q+h} f\, dz$$

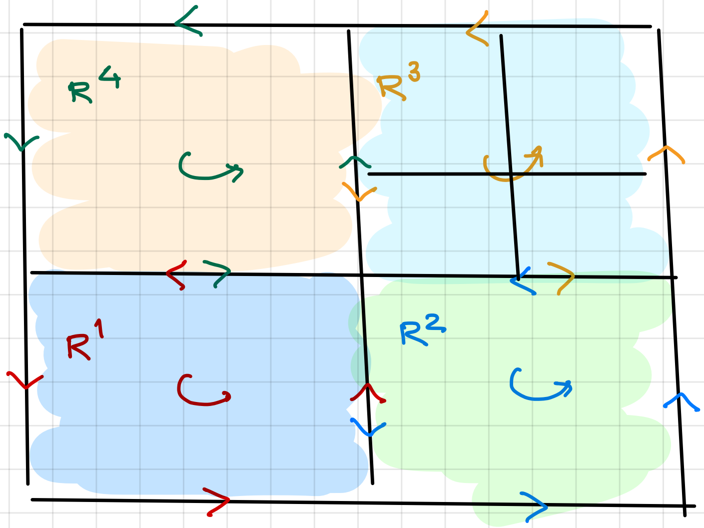


because $\displaystyle\int_{\partial R} f\, dz = 0$.



For the red path from q to $q + h$, the same argument applies, the length of the path $\leq 2\,|h|$.

Proposition C If $f : U \to \mathbb{C}$ holomorphic $\Rightarrow$ $\int_{\partial R} f\, dz = 0$ for all rectangles $\bar{R} \subseteq U$. (Goursat's lemma).



Proof Let $A = \left| \int_{\partial R} f\, dz \right|$.

Let $\varepsilon > 0$ arbitrary. Wish $A = 0$. We will show

$$A < K \varepsilon \quad \forall \varepsilon > 0.$$

for some $K > 0$.

Subdivide rectangle R into 4 equal rectangles R^1, R^2, R^3, R^4.

$$\Rightarrow A = \left| \int_{\partial R} f\, dz \right| = \left| \sum_{j=1}^{4} \int_{\partial R^j} f\, dz \right| \leq \sum_{j=1}^{4} \left| \int_{\partial R^j} f\, dz \right|.$$

$\Rightarrow \exists$ rectangle (out of R^1, R^2, R^3, R^4), call it $R^{(1)}$, with

$$\frac{A}{4} \leq \left| \int_{\partial R^{(1)}} f\, dz \right|$$

Continue inductively. We obtain a sequence of rectangles

$$R \supseteq R^{(1)} \supseteq R^{(2)} \supseteq \dots \quad , \ \operatorname{diam} R^{(n)} \longrightarrow 0.$$

such that
$$\frac{A}{4^n} \leq \left| \int_{\partial R^{(n)}} f \, dz \right| .$$

By compactness, $\bigcap_{n=0}^{\infty} R^{(n)} = \{c\}$. Since f is holomorphic

$$\Bigg| \underbrace{\frac{f(z) - f(c)}{z - c} - f'(c)}_{\chi(z)} \Bigg| < \varepsilon \quad \text{if } z \in \Delta(c, \delta), \text{ for some } \delta > 0.$$

$$\Rightarrow \quad |\chi(z)| < \varepsilon \quad \& \quad f(z) = f(c) + (z-c) f'(c) + (z-c)\chi(z).$$

$$\Rightarrow \frac{A}{4^n} \leq \left| \int_{\partial R^{(n)}} f \, dz \right| = \Bigg| \int_{\partial R^{(n)}} \underbrace{f(c) + (z-c) f'(c)}_{\substack{0 \text{ admits} \\ \text{primitive}}} + (z-c)\chi(z) \, dz . \Bigg|$$

$$= \left| \int_{\partial R^{(n)}} (z-c) \, \chi(z) \, dz \right| . \qquad R^{(n)} \subseteq \Delta(c, \delta) \text{ if } n \gg 0.$$

$$\leq \operatorname{diam}(R^{(n)}) . \, \varepsilon \, \operatorname{length}(\partial R^{(n)}).$$

$$= \varepsilon . \frac{\operatorname{diam}(R)}{2^n} . \frac{\operatorname{length}(\partial R)}{2^n} = \frac{\varepsilon}{4^n} K.$$

$$\Rightarrow \quad A < K\varepsilon \quad \forall \varepsilon > 0 \quad \Rightarrow \quad A = 0.$$

Math 220 A – Lecture 6

October 16, 2020

Last time Δ = disc

Proposition C. If $f : U \to \mathbb{C}$ holomorphic $\Rightarrow \int_{\partial R} f\, dz = 0$

for all rectangles $\overline{R} \subseteq U$. (Goursat's lemma).

Corollary $f : \Delta \to \mathbb{C}$ holomorphic

$\overset{B+C.}{\Longrightarrow}$ f admits a primitive.

$\overset{A}{\Longrightarrow}$ $\int_\gamma f\, dz = 0 \quad \forall \gamma$ piecewise C^1 loop

We seek improvements

New assumption.

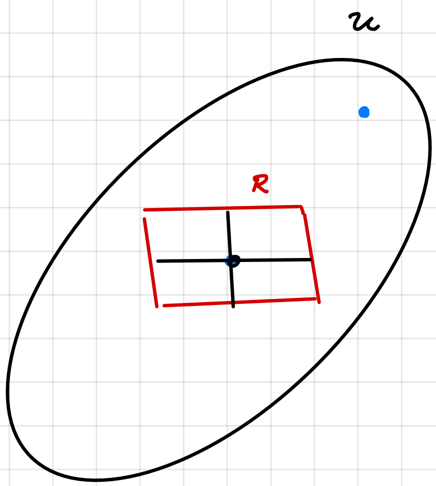
$(*)$ $f : U \to \mathbb{C}$ continuous, holomorphic on $U \setminus \{a\}$.

Proposition C$^+$ f satisfies $(*)$ then $\displaystyle\int_{\partial R} f\,dz = 0$.

for all $\overline{R} \subseteq U$.

Proof $\boxed{i}$ If a is outside $\overline{R}$, let $U^{new} = U \setminus \{a\}$

& apply Proposition C to (f, U^{new})



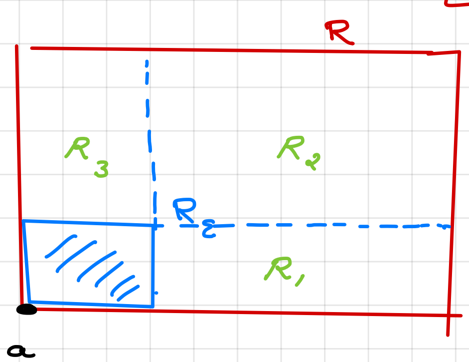
$$\Longrightarrow \int_{\partial R} f\,dz = 0$$

$\boxed{ii}$ If $a \in \overline{R}$, after subdividing $\overline{R}$

we may assume a is a vertex.

$\boxed{iii}$ If a is a vertex, let R_ε be a

square of side ε with vertex a.



$$\int_{\partial R_j} f\,dz = 0 \Longrightarrow \int_{\partial R} f\,dz = \int_{\partial R_\varepsilon} f\,dz.$$

Suffices $\displaystyle\int_{\partial R_\varepsilon} f\,dz \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Since f cont. at $a \Longrightarrow |f(z)| < |f(a)| + 1$ if $z \in \partial R_\varepsilon$

for small ε $\displaystyle\Longrightarrow \left|\int_{\partial R_\varepsilon} f\,dz\right| \leq (|f(a)|+1)\,\underbrace{\text{length}(\partial R_\varepsilon)}_{4\varepsilon} \longrightarrow 0$

Corollary$^+$ $f : \Delta \longrightarrow \mathbb{C}$ continuous, holomorphic in $\Delta \setminus \{a\}$.

$$\overset{\text{Prop } B+C^+}{\Longrightarrow} \quad f \text{ admits a primitive.}$$

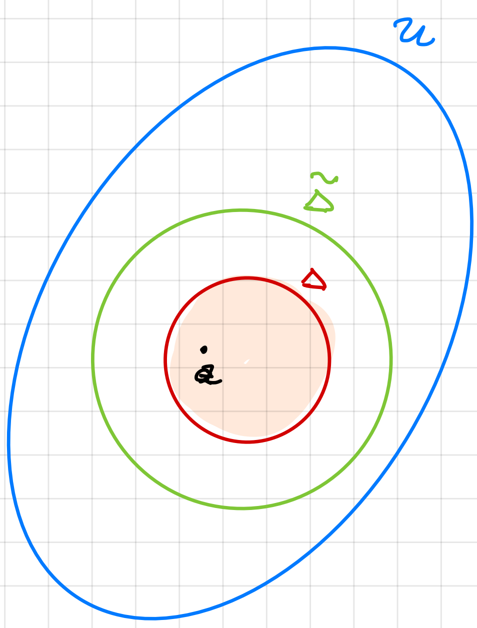
$$\overset{\text{Prop } A}{\Longrightarrow} \quad \int_\gamma f \, dz = 0 \quad \forall \, \gamma \text{ piecewise } C^1 \text{ loop}$$

Local Cauchy Integral Formula

$f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $\overline{\Delta} \subseteq \mathcal{U}$ $\forall \, a \in \Delta$

$$f(a) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(z)}{z-a} \, dz$$

Remark $f|_{\partial \Delta}$ determines f in Δ.



Proof Let

$$F(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases}$$

$\Rightarrow$ F continuous on $\mathcal{U}$. & holomorphic in $\mathcal{U} \setminus \{a\}$.

Let $\tilde{\Delta}$ s.t. $\overline{\Delta} \subseteq \tilde{\Delta} \subseteq \overline{\tilde{\Delta}} \subseteq \mathcal{U}$. Apply Corollary^{+} to $\tilde{\Delta}$ with $\gamma = \partial\Delta$:

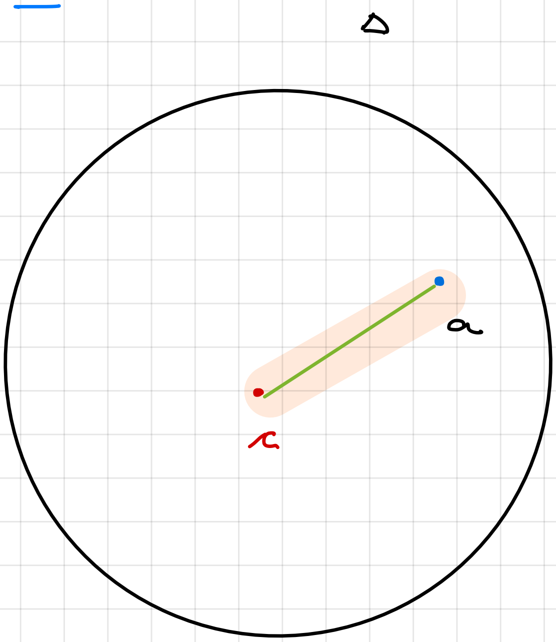
$$\Rightarrow \int_{\partial\Delta} F \, dz = 0 \Rightarrow \int_{\partial\Delta} \frac{f(z) - f(a)}{z-a} \, dz = 0.$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(z)}{z-a} \, dz = f(a) \cdot \underbrace{\frac{1}{2\pi i} \int_{\partial\Delta} \frac{dz}{z-a}}_{1.\ \text{(next lemma)}}.$$

$\Rightarrow$ Local Cauchy follows.

Lemma If $a \in \Delta \implies \displaystyle\int_{\partial\Delta} \frac{dz}{z-a} = 2\pi i$

Proof



Let c be the center of Δ.

$$\implies \int_{\partial\Delta} \frac{dz}{\underbrace{z-c}_{w}} = 2\pi i$$

$$\iff \int_{\partial\Delta(0,R)} \frac{dw}{w} = 2\pi i$$

which we have seen before.

It suffices to show $\displaystyle\int_{\partial\Delta}\left(\frac{dz}{z-a} - \frac{dz}{z-c}\right) = 0 \iff \int_{\partial\Delta} h\, dz = 0$.

Let $h(z) = \dfrac{1}{z-a} - \dfrac{1}{z-c}$. We show that h admits a primitive in $\mathbb{C} \setminus [a,c]$. Let

principal branch

$$\operatorname{Log} \frac{z-a}{z-c} = g(z)$$

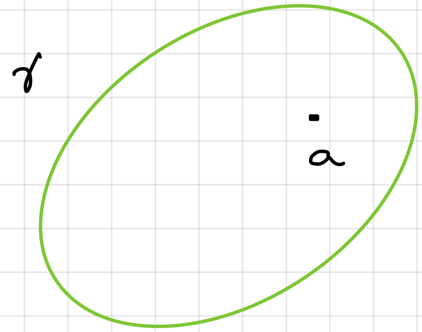
$\implies g' = h$.

Issue We need to show $\dfrac{z-a}{z-c} \in \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

$$\frac{z-a}{z-c} = -u,\ u \in \mathbb{R}_{\geq 0} \iff z = a \cdot \frac{1}{u+1} + c \cdot \frac{u}{u+1} \in$$

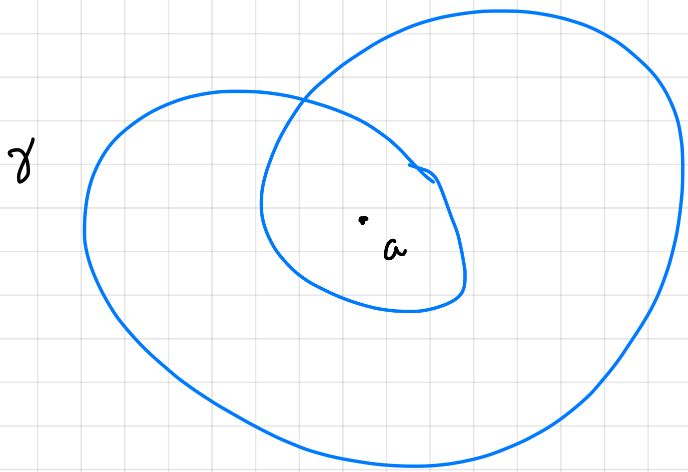
segment from a to c. (false)

Index (winding number) $a \notin \{\gamma\}$. Define



$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}$$

Example A γ circle



$n(\gamma, a) = 1$ if $a \in \operatorname{Int} \gamma$.

by the Lemma.

Example B $\gamma_k(t) = e^{2\pi i t k}$, $0 \le t \le 1$.

$\Longrightarrow n(\gamma_k, 0) = k$.

$$n(\gamma_k, 0) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{dz}{z} =$$

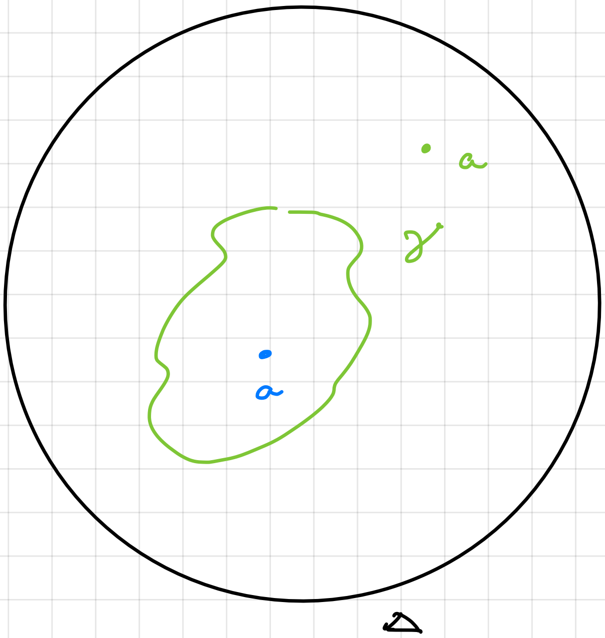
$$= \frac{1}{\cancel{2\pi i}} \int_0^1 \frac{\cancel{e^{2\pi i t k}} \cdot \cancel{2\pi i} k}{\cancel{e^{2\pi i t k}}} \, dt$$

$$= k.$$

Cauchy (revisited) $f : \Delta \longrightarrow \mathbb{C}$ holomorphic,

γ closed C^1 loop in Δ, $a \in U \setminus \{\gamma\}$.

$$f(a) \cdot n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz$$



The proof is identical to the previous proof.

Lemma $n(\gamma, a) \in \mathbb{Z}$, $a \notin \{\gamma\}$.

Proof $n(\gamma, a) = \frac{1}{2\pi i} \int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s)-a}\, ds$ where

$\gamma : [\alpha, \beta] \to U$ is a piecewise C^1 loop $\gamma(\alpha) = \gamma(\beta)$.

Consider

$$h(t) = \int_\alpha^t \frac{\gamma'(s)}{\gamma(s)-a}\, ds, \quad h(\alpha) = 0.$$

Want $h(\beta) \in 2\pi i \mathbb{Z}$.

Compute

$$h'(t) = \frac{\gamma'(t)}{\gamma(t)-a}.$$

$$\Rightarrow \left(e^{-h(t)} \cdot (\gamma(t)-a)\right)' = e^{-h(t)} \big(\underbrace{-h'(t)(\gamma(t)-a) + \gamma'(t)}_{0}\big).$$

$\Rightarrow e^{-h(t)}(\gamma(t)-a)$ constant. Let $t = \alpha$, $t = \beta$:

$$\underbrace{e^{-h(\alpha)}}_{1} (\gamma(\alpha) - a) = e^{-h(\beta)}(\gamma(\beta) - a).$$

$\Rightarrow e^{-h(\beta)} = 1 \Rightarrow h(\beta) \in 2\pi i \mathbb{Z}$. QED.

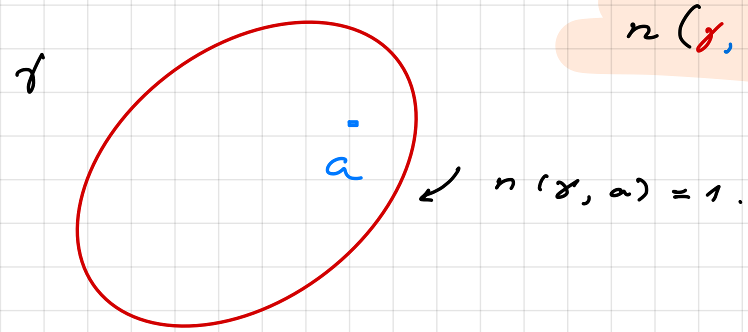
Math 220 A — Lecture 7

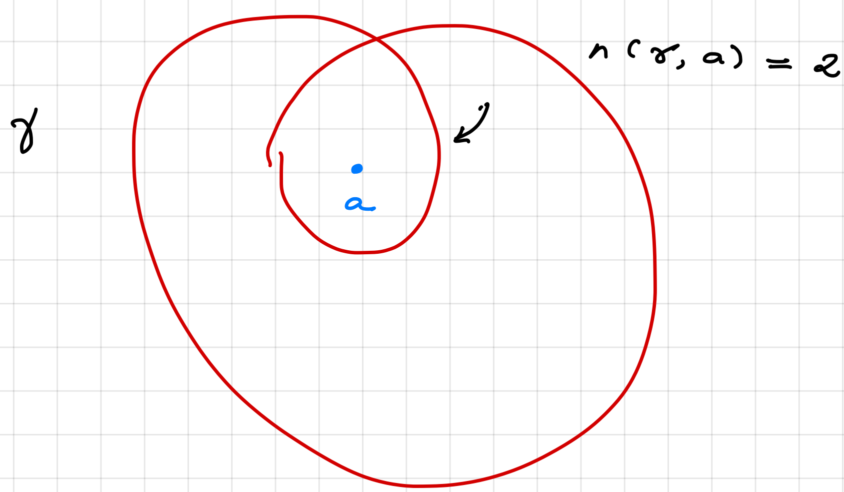
October 19, 2020

Last time : Winding number (index)

- γ piecewise C^1 loop, $a \notin \{\gamma\}$.

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$$





Properties

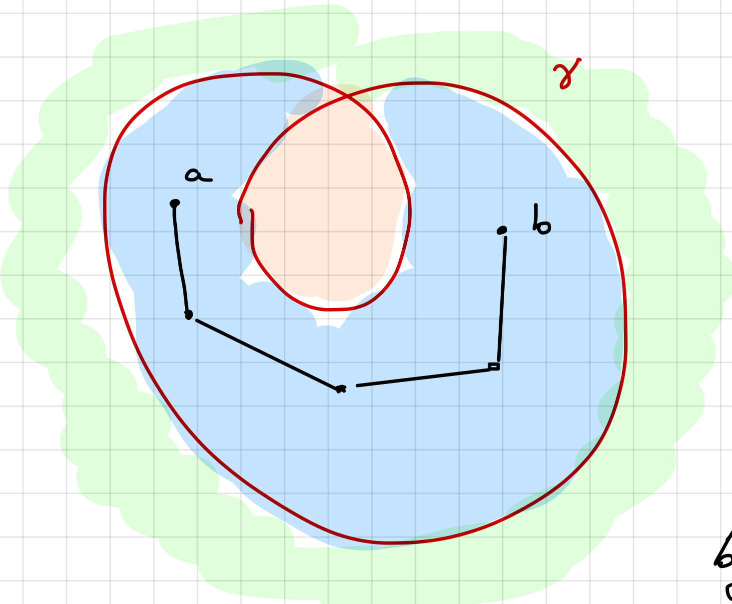
[1] $n(-\gamma, a) = -n(\gamma, a)$ (change of orientation)

Proof:

$$\int_{-\gamma} \frac{dz}{z-a} = -\int_\gamma \frac{dz}{z-a}$$

$\boxed{ii}$ $n(\gamma, -) : \mathbb{C} \setminus \{\gamma\} \longrightarrow \mathbb{Z}$ is locally constant

$n(\gamma, a) = 0$ for a in the unbounded component of $\mathbb{C} \setminus \{\gamma\}$.



Proof.

Let R be a component of $\mathbb{C} \setminus \{\gamma\}$. If $a, b \in \bar{R}$

$\Rightarrow$ a, b can be joined by a polygonal path in R.

This is the same argument used in the past to show we can join by piecewise C^1 path. Suffices to show

if $\overline{ab} \subseteq R \Rightarrow n(\gamma, a) = n(\gamma, b)$

$$\Longleftrightarrow \int_\gamma dz \left(\frac{1}{z-a} - \frac{1}{z-b} \right) = 0 .$$

This is true since $\operatorname{Log} \dfrac{z-a}{z-b}$ is a primitive of the integrand. We showed last time $\operatorname{Log}\dfrac{z-a}{z-b}$ is well defined in $\mathbb{C} \setminus \overline{ab}$.

If U is the unbounded component, let $R \gg 0$ such that $\{\gamma\} \subseteq \Delta(0,R)$. Let m be the value of $n(\gamma, \_)$ on U. Pick $|a| \geq 2R$.

$a \in U$. $\Rightarrow$ $|z-a| \geq |a| - |z| \geq 2R - R = R$ if $z \in \{\gamma\}$ $\Rightarrow$

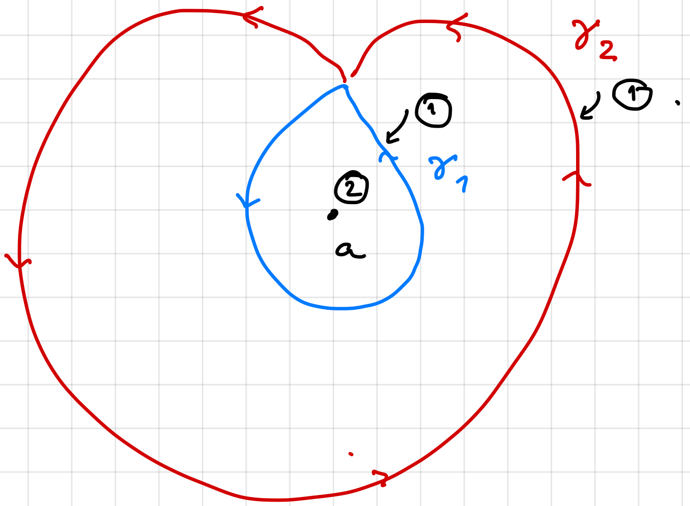
$$|m| = |n(\gamma, a)| = \frac{1}{2\pi}\left|\int_\gamma \frac{dz}{z-a}\right| \leq$$

$$\leq \frac{1}{2\pi} \cdot \frac{1}{R} \cdot \operatorname{length}(\gamma).$$

Make $R \longrightarrow \infty$ $\Rightarrow$ $n(\gamma, a) = m = 0$.

$\boxed{iii}$ $\gamma = \gamma_1 + \gamma_2$

$\Rightarrow n(\gamma, a) = n(\gamma_1, a) + n(\gamma_2, a)$



Proof:

$$\int_{\gamma} \frac{dz}{z-a} = \int_{\gamma_1} \frac{dz}{z-a} + \int_{\gamma_2} \frac{dz}{z-a}.$$

Rudiments of algebraic topology

$$\pi_1(X) = \text{(based) loops in } X \big/ \underset{\text{homotopy}}{\sim}$$

$$\pi_1(\mathbb{C} \setminus \{a\}) \cong \mathbb{Z} \quad \text{isomorphism}$$

$$\gamma \longrightarrow n(\gamma, a).$$

Two questions arise

$\boxed{a}$ Can we define integrals over γ continuous?

$\boxed{b}$ $\gamma_1 \sim \gamma_2 \overset{?}{\Longrightarrow} n(\gamma_1, a) = n(\gamma_2, a).$

Answer to $\boxed{a}$ YES. If f holomorphic, γ continuous we define $\int_\gamma f\,dz$. For instance by analytic continuation

We will not pursue this here.

Answer to $\boxed{b}$ YES. Cauchy's Theorem (Homotopy)

Conway $\overline{\underline{IV}}$. 6.

We reparametrize so that the domain is $I = [0,1]$.

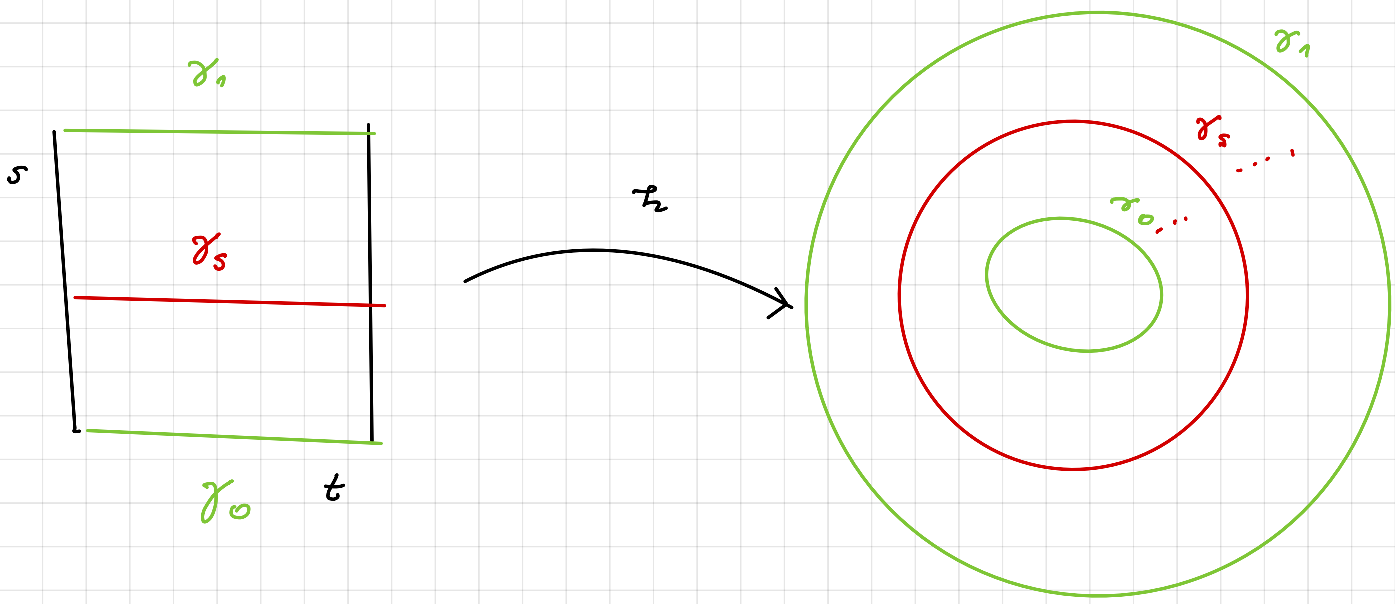
Homotopy $\gamma_0, \gamma_1 : I \longrightarrow U$ continuous loops

$\gamma_0 \overset{U}{\sim} \gamma_1$ if $\exists\, h : I \times I \longrightarrow U$ continuous

$$h(t,0) = \gamma_0(t), \quad h(t,1) = \gamma_1(t).$$

$$h(0,s) = h(1,s).$$

$\Longrightarrow \quad \gamma_s(t) = h(t,s)$. continuous loop.

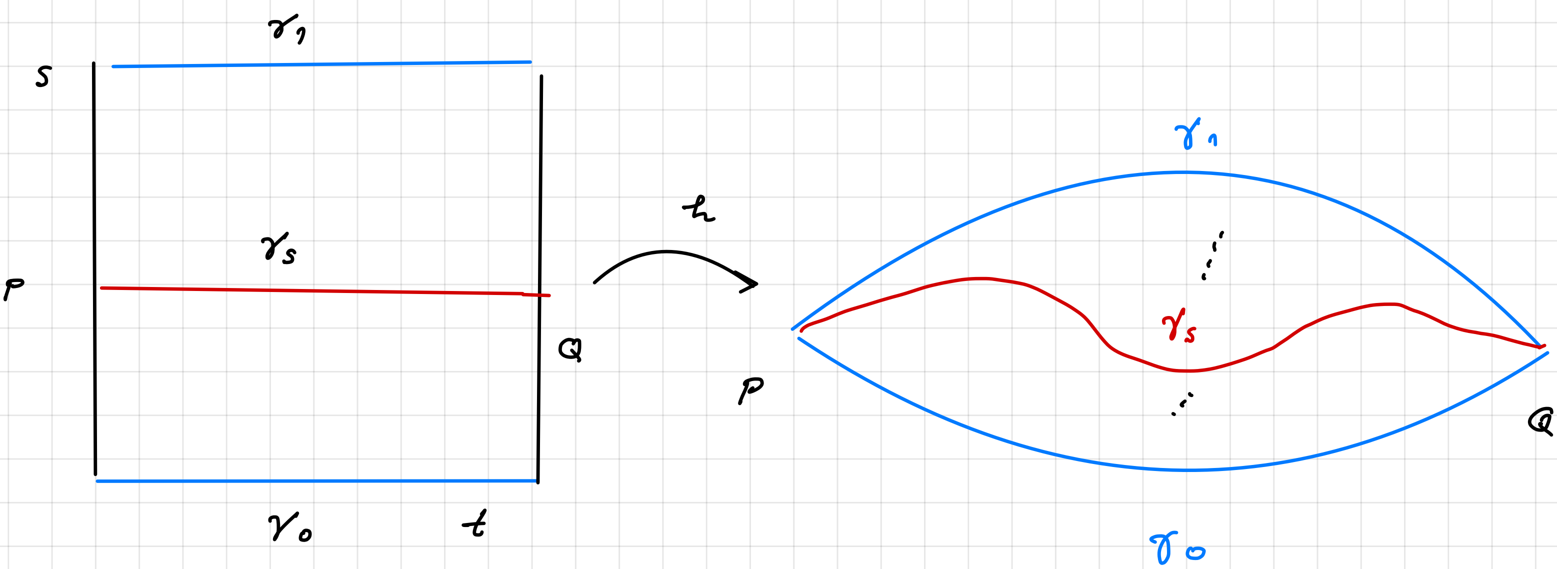


Def $\gamma_0, \gamma_1 : I \longrightarrow \mathcal{U}$ continuous paths from P to Q

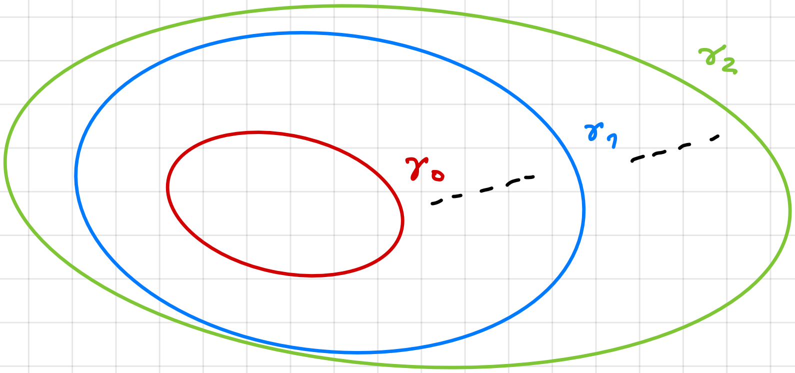
$\gamma_0 \underset{FEP}{\overset{\mathcal{U}}{\sim}} \gamma_1$ if $\exists\, h : I \times I \longrightarrow \mathcal{U}$ continuous

$$h(t,0) = \gamma_0(t) \quad , \quad h(t,1) = \gamma_1(t)$$

$$h(0,s) = P \quad , \quad h(1,s) = Q .$$



Remark [a] $\sim$ is an equivalence relation.



$$\gamma_0 \overset{U}{\sim} \gamma_1,\ \gamma_1 \overset{U}{\sim} \gamma_2 \Rightarrow$$

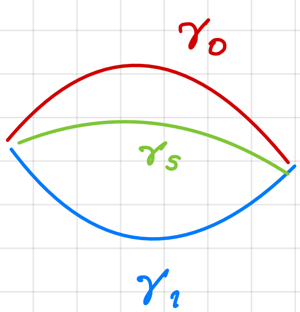
$$\Rightarrow \gamma_0 \overset{U}{\sim} \gamma_2$$

constant loop

[b] Check $\gamma + (-\gamma) \overset{U}{\sim} 0$. $\forall \gamma$ path in U

[c] If $\gamma_0 \overset{FEP}{\sim} \gamma_1$, let $\gamma = \gamma_0 + (-\gamma_1)$ loop

$\Rightarrow \gamma \overset{U}{\sim} 0$. as loops. Indeed let



$$\Gamma_s = \gamma_s + (-\gamma_1).$$

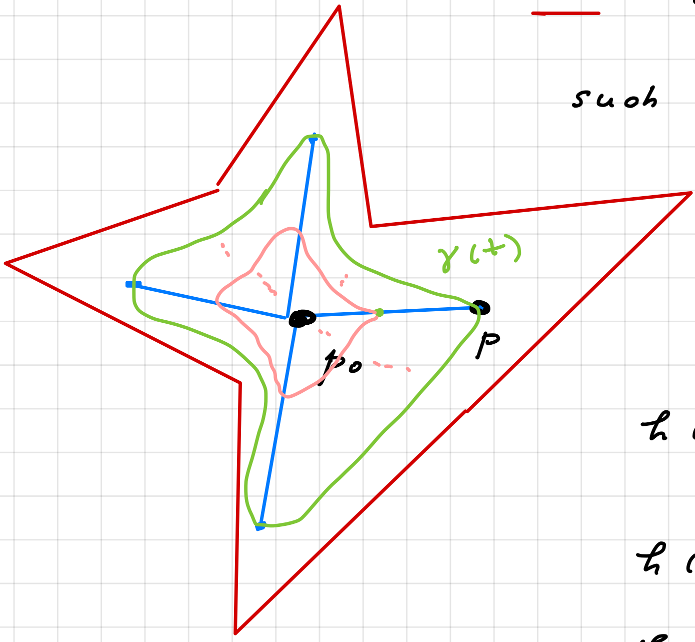
$\Gamma_0 = \gamma$. By [b], $\Gamma_1 \sim 0$.

By [a] $\Rightarrow \gamma \overset{U}{\sim} 0$.

Def U is simply connected if $\forall \gamma$ loop in U,

$\gamma \overset{U}{\sim} 0 \Longleftrightarrow \pi_1(U) = 0$.

Example $\mathcal{U}$ is star convex $\Rightarrow$ $\mathcal{U}$ simply connected



Def $\mathcal{U}$ star convex if $\exists\, p_0 \in \mathcal{U}$ such that $\forall\, p \in \mathcal{U} \Rightarrow \overline{p_0 p} \subseteq \mathcal{U}$.

Let γ be a loop in $\mathcal{U}$.

$$h(t,s) = s\,p_0 + (1-s)\,\gamma(t) \subseteq \mathcal{U}$$

$$h(t,0) = \gamma(t)$$

$$h(t,1) = p_0 \qquad \Rightarrow \gamma \sim 0.$$

Cauchy's Theorem (Homotopy version)

$f : U \to \mathbb{C}$ holomorphic, $\gamma_0 \overset{U}{\sim} \gamma_1$ piecewise C' loops in U $\Rightarrow$ $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

Remarks $\boxed{i}$ $\gamma \overset{U}{\sim} 0 \Rightarrow \int_\gamma f\,dz = \int_0 f\,dz = 0.$

If U simply connected $\Rightarrow \int_\gamma f\,dz = 0 \quad \forall \gamma$ C' loop in U.

$\boxed{ii}$ γ_1, γ_2 piecewise C' paths, $\gamma_1 \overset{FEP}{\sim} \gamma_2$

$\Rightarrow \int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$. Indeed let $\gamma = \gamma_1 + (-\gamma_2)$.

By $\boxed{i}$ $\Rightarrow \int_\gamma f\,dz = 0 \Rightarrow \int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$.

$\boxed{iii}$ $\gamma_0 \overset{U}{\sim} \gamma_1$, $U \subseteq \mathbb{C} \setminus \{a\}$ piecewise C' loops in $U \subseteq \mathbb{C} \setminus \{a\}$ $\Rightarrow \int_{\gamma_0} \frac{dz}{z-a} = \int_{\gamma_1} \frac{dz}{z-a}$

$\Rightarrow n(\gamma_0, a) = n(\gamma_1, a)$.

This proves a previous assertion.

Remark The homotopy in Cauchy's theorem is not assumed to be C^1.

Existence of primitives in simply connected sets

If U simply connected, $f : U \longrightarrow \mathbb{C}$ holomorphic

$$\Rightarrow \int_\gamma f\,dz = 0 \text{ by Remark } \boxed{1}$$

$\Rightarrow$ Prop A, f has a primitive

Corollary Any holomorphic function in a simply connected set admits a primitive.

Take $f(z) = \frac{1}{z}$. A primitive is a branch of logarithm.

Corollary Let $U \subseteq \mathbb{C} \setminus \{0\}$ simply connected. We can define a branch of logarithm in U.

Math 220A – Lecture 8

October 21, 2020

Last time

Cauchy's Theorem (Homotopy version).

$f : U \to \mathbb{C}$ holomorphic, γ_0, γ_1 piecewise C^1 loops in U, $\gamma_0 \overset{U}{\sim} \gamma_1$. Then

$$\int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz$$

Remark We prove a seemingly stronger result

Cauchy's Theorem^{+} (Homotopy version).

$(+)$ $f : U \to \mathbb{C}$ continuous, holomorphic in $U \setminus \{a\}$

$$\implies \int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz \quad \text{if } \gamma_0 \overset{U}{\sim} \gamma_1 \text{ are piecewise } C^1 \text{ loops.}$$

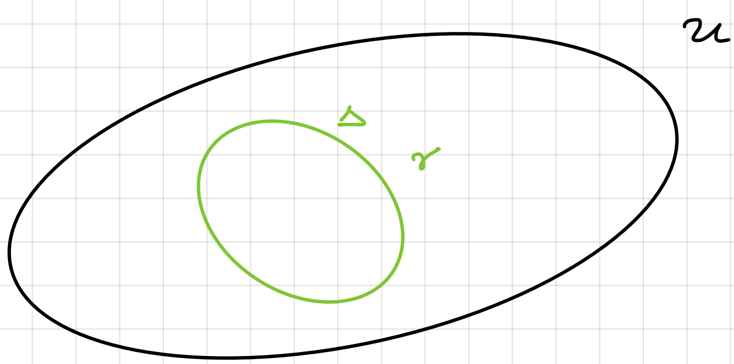
We need this stronger form to prove:

Cauchy's Integral Formula (CIF)

$f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $\gamma \overset{\mathcal{U}}{\sim} 0$, $a \in \mathcal{U} \setminus \{\gamma\}$

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz$$

Remark This generalizes Local Cauchy's Integral Formula we proved before. In that case, $\gamma = \partial\Delta$ where $\overline{\Delta} \subseteq \mathcal{U}$.



Proof of CIF

Let
$$F(z) = \begin{cases} \dfrac{f(z) - f(a)}{z-a} & , z \neq a \\ f'(a) & , z = a \end{cases}$$

$\Longrightarrow$ F continuous in U, holomorphic in $U \setminus \{a\}$.

$$\Longrightarrow \int_\gamma F \, dz = 0 \quad \text{by Cauchy}^{+}$$

$$\Longrightarrow \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(a)}{z-a} \, dz = 0$$

$$\Longrightarrow \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} = f(a) \cdot \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = f(a) \cdot n(\gamma, a).$$

QED.

Remark

$$\begin{array}{ccc} \text{Homotopy Cauchy}^{+} & \Longrightarrow & \text{CIF} \\ \Downarrow & \swarrow & \\ \text{Homotopy Cauchy} & & \end{array}$$

In fact CIF $\Longrightarrow$ Homotopy Cauchy by using CIF

for $\gamma = \gamma_0 + (-\gamma_1)$ & the function $(z-a) f(z)$

Proof of Cauchy$^+$

Recall the assumption

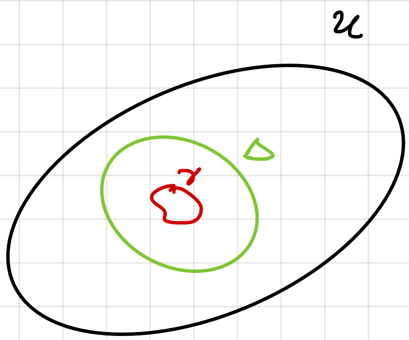
$$(+)\ f \text{ cont } \& \ f \text{ hol. in } \mathcal{U}\setminus\{a\}.$$

For the proof we only use

$\boxed{i}$ f continuous

$\boxed{ii}$ $\forall\ \Delta \subseteq \mathcal{U}$, $\{\gamma\} \subseteq \Delta$ piecewise C^1 loop

$$\Longrightarrow \int_\gamma f\,dz = 0 \qquad (*).$$

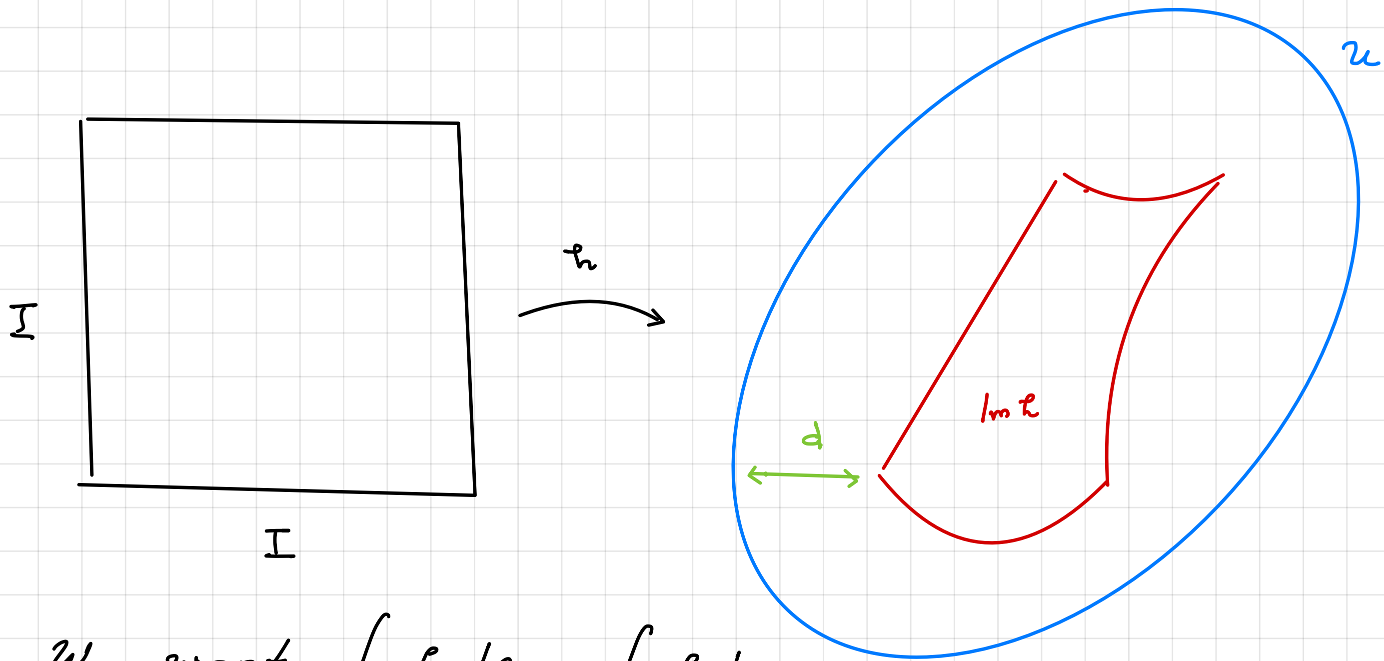


Under assumption $(+)$, item $\boxed{ii}$ follows from a previous

Corollary$^+$ $f : \Delta \longrightarrow \mathbb{C}$ continuous, holomorphic in $\Delta \setminus \{a\}$

$$\Longrightarrow \int_\gamma f\,dz = 0 \quad \forall\ \gamma \text{ piecewise } C^1 \text{ loop}$$

(see Lecture 6).



We want $\displaystyle\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

Let $h : I \times I \longrightarrow \mathcal{U}$ be the homotopy from γ_0 to γ_1

Im h compact, $\mathbb{C} \setminus \mathcal{U}$ closed $\overset{\text{Rudin}}{\Longrightarrow} \exists\, d > 0$.

$$d = \operatorname{dist}(\operatorname{Im} h,\ \mathbb{C} \setminus \mathcal{U})$$

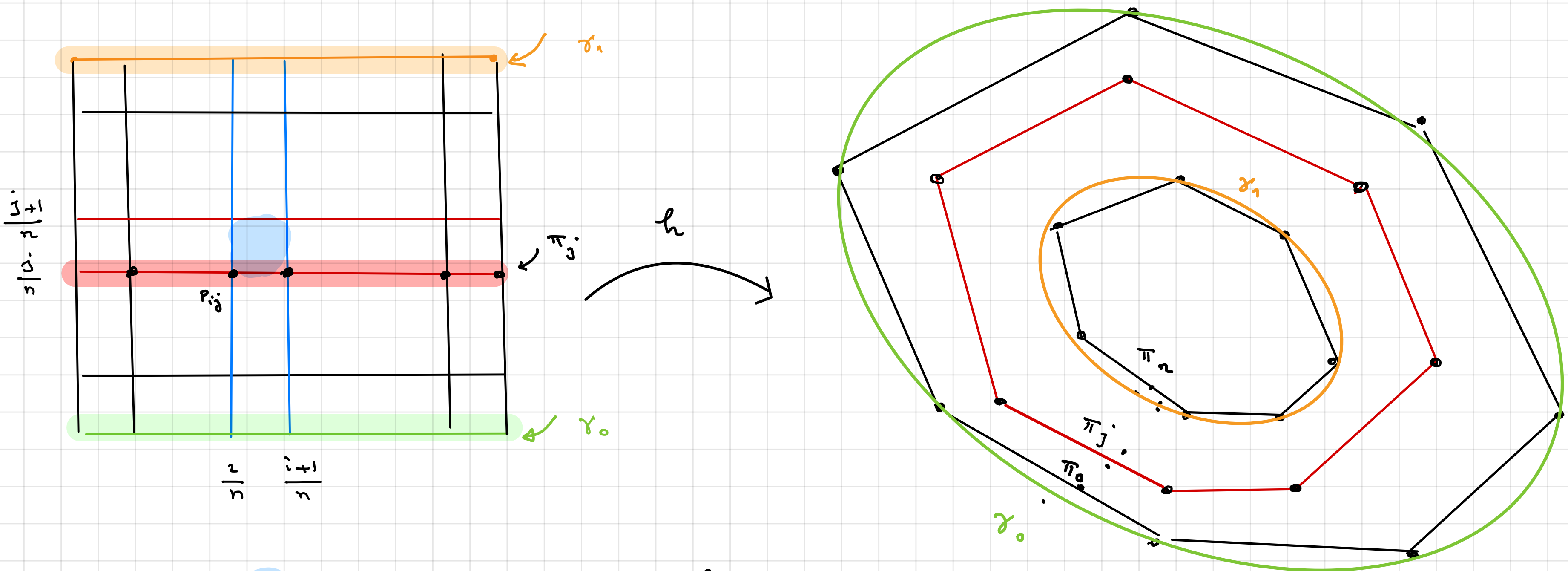
h continuous, $I \times I$ compact $\Longrightarrow$ h uniformly cont.

$\Longrightarrow \exists\, \delta > 0$ such that

$$|t - t'| < \delta,\ |s - s'| < \delta \implies |h(t,s) - h(t',s')| < d.$$

Let $n \in \mathbb{Z}_+$ with $\frac{1}{n} < \delta$. Subdivide I into equal

intervals $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ of length $< \delta$.



Let P_{ij} have coordinates $\left(\frac{i}{n}, \frac{j}{n}\right)$. Let $Q_{ij} = h(P_{ij})$.

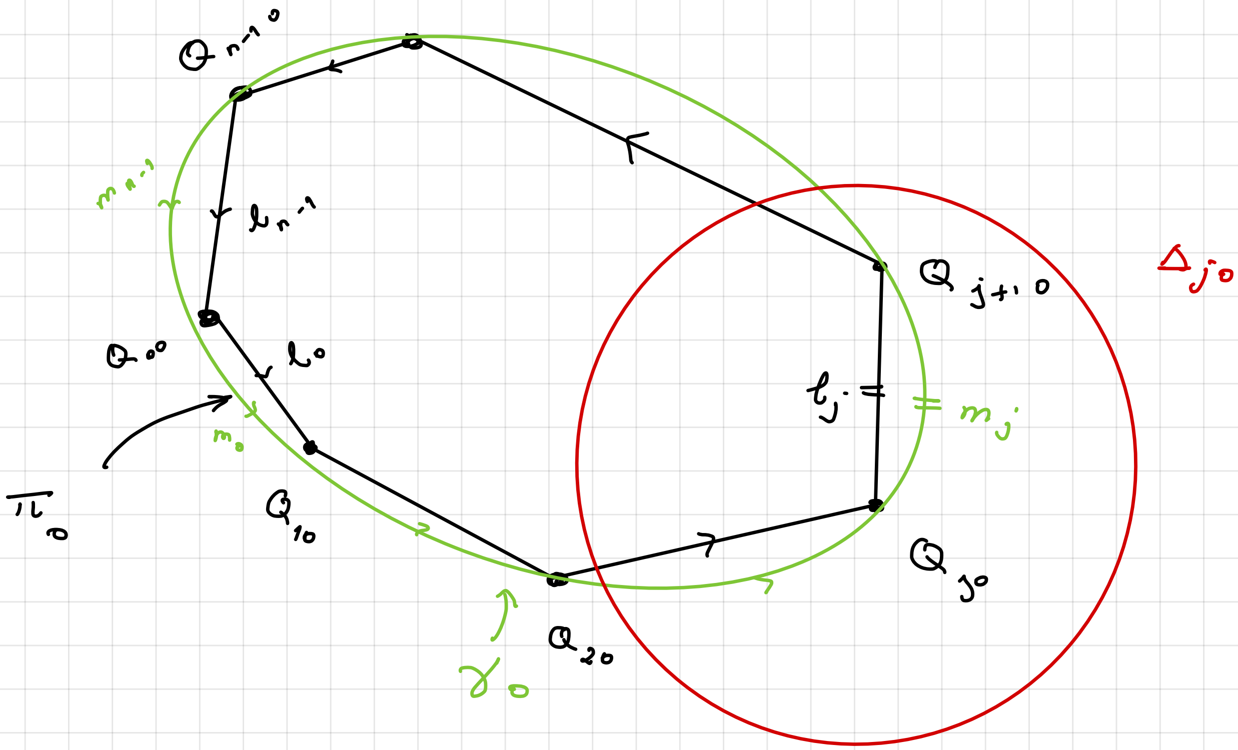
Let $R_{ij} = \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$. Let $\Delta_{ij} = \Delta(Q_{ij}, d)$

Note $\Delta_{ij} \subseteq U$. by the choice of d.

Since sides of R_{ij} have length $< \delta$ $\Longrightarrow$ $h(R_{ij}) \subseteq \Delta_{ij}$ by uniform continuity.

Let π_j be the polygon through $Q_{0j}, Q_{1j}, \dots, Q_{nj} = Q_{0j}$

Claim [a] $\displaystyle\int_{\gamma_0} f\,dz = \int_{\pi_0} f\,dz$ & $\displaystyle\int_{\pi_n} f\,dz = \int_{\gamma_1} f\,dz.$



Let $l_0, l_1, \dots l_{n-1}$ be the edges of the polygon π_0

$m_0, m_1, \dots m_{n-1}$ be the arcs of the curve γ_0.

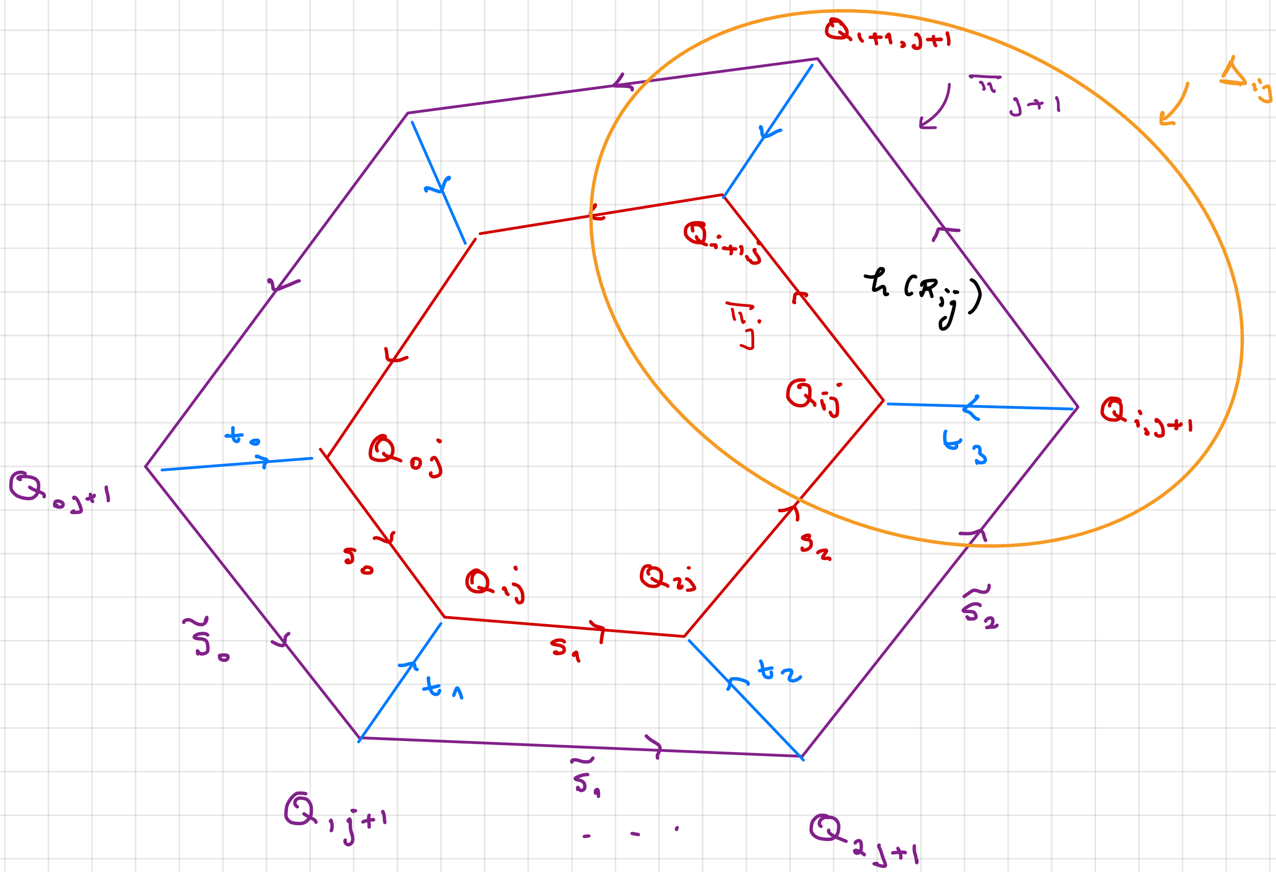
$$m_j = \gamma_0 \big/_{[\frac{j}{n}, \frac{j+1}{n}]}$$

By construction both l_j, m_j are contained in $\Delta_{j0} \subseteq \mathcal{U}$.

By $(*)$ we have $\displaystyle\int_{l_j + (-m_j)} f\,dz = 0 \implies \int_{l_j} f\,dz = \int_{m_j} f\,dz$

Adding for all j, we find $\displaystyle\int_{\pi_0} f\,dz = \int_{\gamma_0} f\,dz.$

Claim [b] $\displaystyle\int_{\pi_j} f\,dz = \int_{\pi_{j+1}} f\,dz$



Let $s_0, \dots s_{n-1}$ be the edges of π_j.

$\tilde{s}_0, \dots \tilde{s}_{n-1}$ the edges of π_{j+1}

$t_0, \dots, t_{n-1}$ the segments joining Q_{ij} to $Q_{i,j+1}$.

Since $h(R_{ij}) \subseteq \Delta_{ij} \Rightarrow \tilde{s}_i + t_{i+1} + (-s_i) + (-t_i)$ is

a loop in Δ_{ij}. By $(*)$

$$\Longrightarrow \int_{\tilde{s}_i + t_{i+1} + (-s_i) + (-t_i)} f \, dz = 0$$

$$\Longrightarrow \int_{\tilde{s}_i} f \, dz - \int_{s_i} f \, dz = \int_{t_i} f \, dz - \int_{t_{i+1}} f \, dz.$$

Add these for all i: We find

$$\int_{\pi_{j+1}} f \, dz - \int_{\pi_j} f \, dz = 0 \Longrightarrow \text{Claim } \boxed{b}.$$

From Claims $\boxed{a}$ & $\boxed{b}$,

$$\int_{\gamma_0} f \, dz = \int_{\pi_0} f \, dz = \dots = \int_{\pi_n} f \, dz = \int_{\gamma_1} f \, dz.$$

QED.

Math 220 A — Lecture 9

October 23, 2020

No lecture on Monday, Oct 26.

Last time

Cauchy's Integral Formula (CIF)

$f : U \to \mathbb{C}$ holomorphic, $\gamma \overset{U}{\sim} 0$, $a \in U \setminus \{\gamma\}$

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

Example $|a| < |b|$. We compute

$$\int_{|z|=r} \frac{e^z}{(z-a)(z-b)}\, dz$$

$\boxed{i}$ $r < |a|$, the integrand is holomorphic so answer $= 0$.

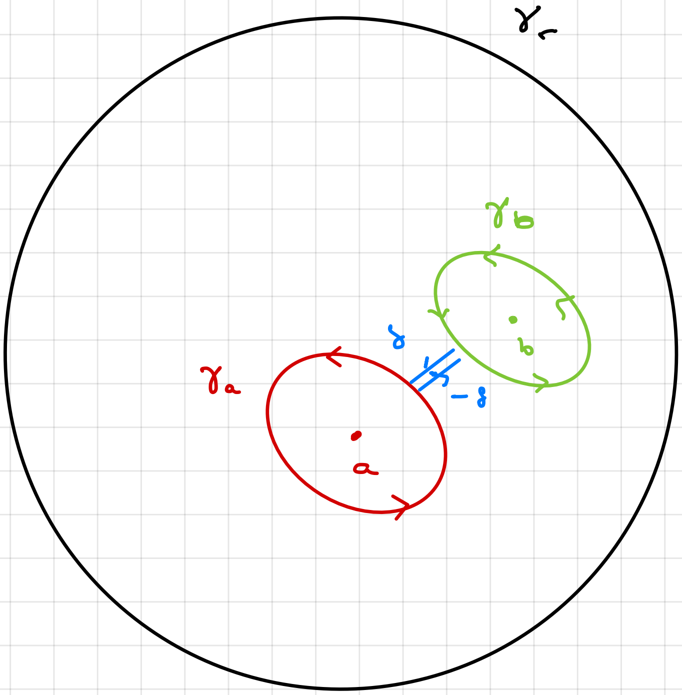
$\boxed{ii}$ $|a| < r < |b|$. Write

holomorphic in $|z| \le r$.

$$\int_{|z|=r} \frac{e^z/(z-b)}{z-a} = 2\pi i \cdot \frac{e^z}{z-b}\Bigg|_{z=a} = 2\pi i \cdot \frac{e^a}{a-b}.$$

$\boxed{iii}$ $|a| < |b| < r$

Let $\gamma_r = \{|z| = r\}$.



Let $f(z) = \dfrac{e^z}{(z-a)(z-b)}$.

Let γ_a, γ_b be two circles centered at a, b and δ a segment joining them.

Let $\gamma = \gamma_a + \delta + \gamma_b + (-\delta)$.

Note $\gamma \sim \gamma_r$ in $\mathbb{C} \setminus \{a, b\}$.

By homotopy Cauchy

$$\int_{\gamma_r} f\,dz = \int_{\gamma} f\,dz = \int_{\gamma_a} f\,dz + \int_{\gamma_b} f\,dz + \cancel{\int_{\delta} f\,dz} + \cancel{\int_{-\delta} f\,dz}$$

$$= \int_{\gamma_a} \frac{e^z/z-b}{z-a}\,dz + \int_{\gamma_b} \frac{e^z/z-a}{z-b}\,dz$$

$$= 2\pi i \cdot \frac{e^z}{z-b}\Big|_{z=a} + 2\pi i \frac{e^z}{z-a}\Big|_{z=b}$$

$$= 2\pi i \cdot \frac{e^a - e^b}{a-b}.$$

Taylor Expansion

Theorem $f: U \to \mathbb{C}$ holomorphic, $a \in U$, $\Delta(a,R) \subseteq U$.

Then in $\Delta(a,R)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad (*).$$

$\Rightarrow$ f analytic $\Rightarrow$ f is ∞ - many times differentiable.

ANALYTIC = HOLOMORPHIC = DIFFERENTIABLE

Proof Let $\Delta(a,R) \subseteq U$. We pick $0 < r < R$. Let $z \in \Delta(a,r)$. By CIF

$$f(z) = \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{t-z}\, dt$$

Key: $$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \frac{1}{t-a} \cdot \frac{1}{1 - \frac{z-a}{t-a}}$$

$$= \frac{1}{t-a} \sum_{k=0}^{\infty} \frac{(z-a)^k}{(t-a)^k}$$ converges since

$$\left|\frac{z-a}{t-a}\right| = \frac{|z-a|}{r} < 1.$$

$$\Rightarrow \frac{f(t)}{t-z} = \sum_{k=0}^{\infty} f(t) \cdot \frac{(z-a)^k}{(t-a)^{k+1}}. \quad (+)$$

Claim This converges uniformly in t over $|t-a|=r$.

Indeed, let $f_k(t) = f(t) \cdot \frac{(z-a)^k}{(t-a)^{k+1}}$.

$$\Rightarrow |f_k(t)| \le M \cdot \frac{|z-a|^k}{r^{k+1}} = M_k, \quad |f(t)| \le M \text{ for } |t-a|=r$$

Note $\sum M_k < \infty$ since $|z-a|<r$. Thus the claim follows

by Weierstraß M-test

Since the convergence is uniform, we can integrate (Rudin)

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{t-z}\, dt \overset{(+)}{=}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}}\, dt \cdot (z-a)^k$$

$$= \sum_{k=0}^{\infty} a_k (z-a)^k.$$

Def A holomorphic $f : \mathbb{C} \longrightarrow \mathbb{C}$ is said to be entire.

Remark f entire $\overset{\text{Taylor}}{\Longrightarrow}$ $f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in \mathbb{C}$

Remark $f : U \longrightarrow \mathbb{C}$, $\overline{\Delta}(a, r) \subseteq U$.

$$a_k = \frac{f^{(k)}(a)}{k!} = \underbrace{\frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}}\, dt}_{\text{from the proof of the Theorem.}}$$

Lecture 2/3

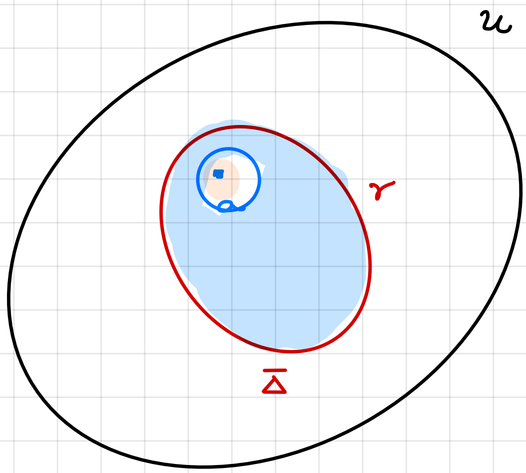
Thus $$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

This is local CIF for derivatives.

Cauchy's Integral Formula (for derivatives)

If $\bar{\Delta} \subseteq \mathcal{U}$, $a \in \Delta$, $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\partial\Delta} \frac{f(t)}{(t-a)^{k+1}}\, dt.$$



Proof

If a is the center of Δ we showed this on the previous page.

If a is not the center then

let γ_a be a small circle centered at a. Then $\gamma_a \sim \gamma$ where $\gamma = \partial\Delta$. We have

$$\underbrace{f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\gamma_a} \frac{f(t)}{(t-a)^{k+1}}\, dt}_{\gamma_a \text{ has center } a} \underbrace{=}_{\substack{\text{homotopy} \\ \text{Cauchy}}} \frac{k!}{2\pi i} \int_{\gamma} \frac{f(t)}{(t-a)^{k+1}}\, dt$$

Remark (Homotopy version)

$$f: U \longrightarrow \mathbb{C}, \quad \gamma \overset{U}{\sim} 0, \quad a \in \mathbb{C} \setminus \{\gamma\}$$

$$n(\gamma, a)\, f^{(k)}(a) = \frac{k!}{2\pi i} \int_\gamma \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

The case $\gamma = \partial \Delta$, $\overline{\Delta} \subseteq U$ is considered above

A possible proof is via Conway IV. 2.2 / HWK 3

Exercise 7. Another proof is via the residue theorem to be stated later.

Example

$$\int_{|z|=r} \frac{e^z}{(z-a)^k}\, dz \quad , \ r \neq |a|$$

If $|a| > r$ the answer is 0 because the integrand is holomorphic

If $r > |a|$, apply CIF for derivatives:

$$\frac{1}{(k-1)!} \cdot 2\pi i \cdot \partial^{(k-1)} e^z \Big/_{z=a} = \frac{e^a}{(k-1)!} \cdot 2\pi i$$

Cauchy's Estimate

Let $f: U \to \mathbb{C}$ holomorphic, $\overline{\Delta}(a, R) \subseteq U$. Let

$$M_R = \sup_{|z-a|=R} |f(z)|.$$

Then

$$|f^{(k)}(a)| \leq k! \, \frac{M_R}{R^k}.$$

Proof By CIF for derivatives

$$|f^{(k)}(a)| = \left| \frac{k!}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^{k+1}} \, dz \right|$$

$$\leq \frac{k!}{2\pi} \cdot \frac{M_R}{R^{k+1}} \cdot \text{length } |z-a|=R$$

$$= \frac{k!}{2\pi} \cdot \frac{M_R}{R^{k+1}} \cdot 2\pi R = k! \, \frac{M_R}{R^k}.$$

Liouville's theorem

If $f: \mathbb{C} \longrightarrow \mathbb{C}$ entire & bounded $\Rightarrow$ f constant.

We prove this next time.

Math 220A - Lecture 10

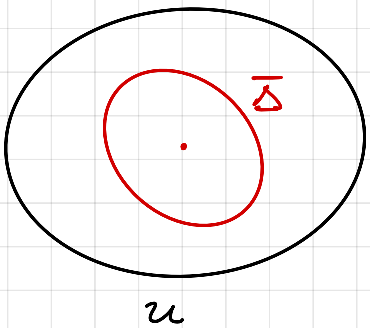
October 28, 2020

Recall — Midterm next Friday

[0] Last time (Cauchy's Estimate)

$$f : \mathcal{U} \longrightarrow \mathbb{C} \text{ holomorphic}, \quad \overline{\Delta}(a, R) \subseteq \mathcal{U}$$

$$|f^{(k)}(a)| \leq k! \, \frac{M_R}{R^k}, \qquad M_R = \sup_{|z-a|=R} |f(z)|.$$

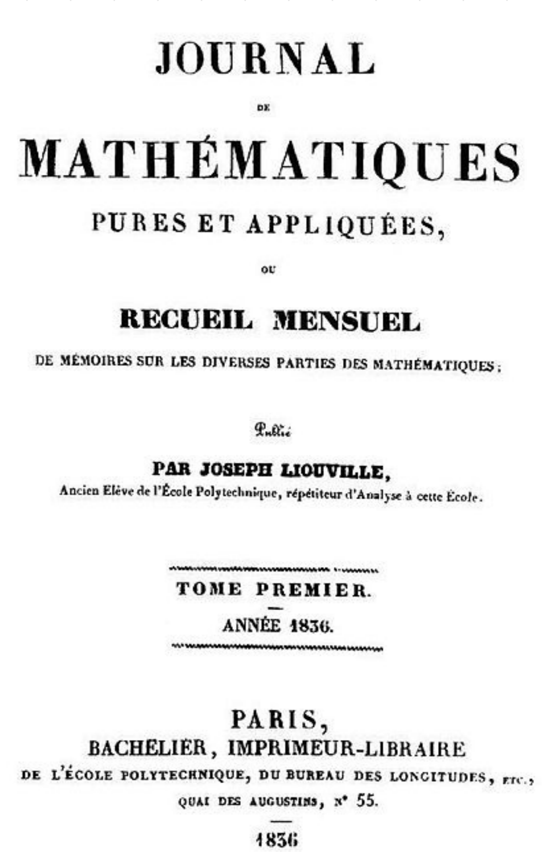


Remark $k = 0$:

$$|f(a)| \leq \sup_{z \in \partial\Delta} |f(z)|$$

[1.] Liouville's theorem

If $f : \mathbb{C} \longrightarrow \mathbb{C}$ entire & bounded $\Rightarrow$ f constant.



JOURNAL

DE

MATHÉMATIQUES

PURES ET APPLIQUÉES,

OU

RECUEIL MENSUEL

DE MÉMOIRES SUR LES DIVERSES PARTIES DES MATHÉMATIQUES;

Publié

PAR JOSEPH LIOUVILLE,

Ancien Elève de l'École Polytechnique, répétiteur d'Analyse à cette École.

TOME PREMIER.

ANNÉE 1836.

PARIS,

BACHELIER, IMPRIMEUR-LIBRAIRE

DE L'ÉCOLE POLYTECHNIQUE, DU BUREAU DES LONGITUDES, ETC.,

QUAI DES AUGUSTINS, N° 55.

1836

Joseph Liouville

1809 – 1882

Journal de Liouville

Known for: Liouville's theorem

Sturm-Liouville theory

Liouville numbers

Liouville function

...

Proof: f is bounded by M, $|f(z)| \leq M \quad \forall z \in \mathbb{C}$.

Cauchy's estimate for $k=1$. Take $\overline{\Delta}(a, R) \subseteq \mathbb{C}$.

$$|f'(a)| \leq \frac{M_R}{R} \leq \frac{M}{R}.$$

Take $R \longrightarrow \infty$.

Thus $f'(a) = 0. \quad \forall a \implies f$ constant.

Fundamental Theorem of Algebra

Any nonconstant polynomial $f \in \mathbb{C}[z]$ has at least one complex root.

Proof: WLOG f monic

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n.$$

Assume

f has no roots $\implies f(z) \neq 0 \quad \forall z$.

Let $g = \frac{1}{f}$. $\implies g$ is entire. We show g bounded $\implies$

$\implies g$ constant. $\implies f$ constant. This is a contradiction.

We show g bounded. If $|z| = R$

$$|f(z)| = \left| z^n + a_1 z^{n-1} + \dots + a_n \right| \geq |z|^n - \sum_{k=1}^{n-1} |a_k| \, |z|^{n-k}$$

$$= R^n - \sum_{k=1}^{n-1} |a_k| R^{n-k} \longrightarrow \infty \quad \text{as} \quad R \longrightarrow \infty.$$

If $R \geq R_0 \implies |f(z)| \geq 1. \implies |g(z)| \leq 1.$

If $R \leq R_0 \implies$ by continuity of g: $|g(z)| \leq K.$ $\Bigg\} \implies$

$$\implies |g(z)| \leq M = \max(1, K). \quad \forall z$$

$\boxed{2}$ Zeros of holomorphic functions — Conway IV. 3.

$f : U \longrightarrow \mathbb{C}$ holomorphic, $f \not\equiv 0$, U connected.

$a \in U$ is a zero of order N if

$$f(a) = 0, \quad f'(a) = 0, \quad \dots, \quad f^{(N-1)}(a) = 0, \quad f^{(N)}(a) \neq 0.$$

$\implies$ Taylor expansion in $\Delta(a, R) \subseteq U$

$$f(z) = \sum_{k \geq N} \frac{f^{(k)}(a)}{k!} \cdot (z-a)^k = (z-a)^N g(z) \qquad (*)$$

where g is a power series in $\Delta(a, R)$.

$$g(a) = \frac{f^{(N)}(a)}{N!} \neq 0.$$

We need to rule out the case $N = \infty$.

Lemma $f : U \longrightarrow \mathbb{C}$, U connected. TFAE

$\boxed{i}$ $f \equiv 0$

$\boxed{ii}$ $\exists\, a \in U$, $f^{(k)}(a) = 0 \quad \forall k$

$\boxed{iii}$ $S = \{ z : f(z) = 0 \}$ has a limit point in U.

Proof $\boxed{i} \Rightarrow \boxed{ii}$, $\boxed{i} \Rightarrow \boxed{iii}$

$\boxed{iii} \Rightarrow \boxed{ii}$ Let a be a limit point for S, $a \in U$.

Clearly $f(a) = 0$. Let us assume a has finite order N.

By $(*)$, $f(z) = (z-a)^N g(z)$ in $\Delta(a, R)$. with

g power series, $g(a) \neq 0$. By continuity of g, $g(z) \neq 0$ in

some $\Delta(a, r) \subseteq \Delta(a, R)$. Then

$$S \cap \Delta(a, r) = \{ z : (z-a)^N g(z) = 0 \} = \{ a \}.$$

contradiction with a being a limit point.

Thus $N = \infty$. $\Rightarrow \boxed{ii}$.

$\boxed{iii} \Rightarrow \boxed{ii}$. Let $A = \{a : f^{(k)}(a) = 0 \ \forall k\} \subseteq U$.

By assumption $A \neq \emptyset$. We show A is closed & open.

Thus $A = U$. $\Rightarrow f \equiv 0$.

- A closed. Indeed $A = \bigcap_{k=0}^{\infty} \left(f^{(k)}\right)^{-1}(0) =$ closed.

Since $f^{(k)}$ is continuous $\Rightarrow f^{(k)^{-1}}(0)$ is closed $\Rightarrow A$ closed

- A open. Let $a \in A$. By Taylor if $\Delta(a, R) \subseteq U$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k = 0 \quad \text{since } f^{(k)}(a) = 0.$$

Since $f = 0$ in $\Delta(a, R) \Rightarrow f^{(k)} = 0$ in $\Delta(a, R) \Rightarrow$

$\Rightarrow \Delta(a, R) \subseteq A \Rightarrow A$ open.

Corollary (Identity principle) Let $f, g : U \longrightarrow \mathbb{C}$. holomorphic.

If

$$S = \{z : f(z) = g(z)\} \text{ has a limit point} \Rightarrow f = g.$$

Proof Work with $h = f - g$. Apply Lemma above.

Remarks

$\boxed{i}$ The zeros of $f : U \longrightarrow \mathbb{C}$ holomorphic cannot have a limit point in U.

$\boxed{ii}$ $f(z) = \sin \dfrac{1+z}{1-z}$ holomorphic in $\underbrace{\mathbb{C} \setminus \{1\}}_{U}$.

Zeros $\quad \dfrac{1+z}{1-z} = n\pi \iff z = \dfrac{-1+n\pi}{1+n\pi} \longrightarrow 1.$

Thus the zeros can accumulate to ∂U.

$\boxed{iii}$ This fails for C^∞ - functions

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-\frac{1}{x^2}} \sin \frac{1}{x}, & x \neq 0. \end{cases}$$

Check f is C^∞. Also f has zeros at $\frac{1}{n\pi} \longrightarrow 0.$ which has a limit point.

$\boxed{\text{IV}}$ $f \not\equiv 0$ has at most countably many zeros.

Let $U = \bigcup_{n=1}^{\infty} K_n$ where K_n compact. In each compact set K_n, f can only have finitely many zeros.

(indeed this is because $\mathrm{zero}(f)$ can't accumulate in K_n)

$$\Rightarrow \quad \mathrm{Zero}(f) = \bigcup_n \underbrace{\mathrm{Zero}(f) \cap K_n}_{\text{finite}} = \text{countable}.$$

Aufgaben und Lehrsätze,

erstere aufzulösen, letztere zu beweisen.

1.

(Von Herrn *N. H. Abel.*)

49. **Theorème.** Si la somme de la série infinie

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_m x^m + \ldots\ldots$$

est égale à zéro pour toutes les valeurs de x entre deux limites réelles α et β; on aura nécessairement

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0 \ldots\ldots a_m = 0 \ldots\ldots$$

en vertu de ce que la somme de la série s'évanouira pour une valeur quelconque de x.

Identity theorem: Crelle's Journal 1827, page 286

Math 220A – Lecture 11

October 30, 2020

Midterm - Friday Nov 6

- closed book, closed notes

- honor code - no zoom proctoring

- available Friday 3PM, due Friday 4 PM.

- upload answers in Grade Scope

- time zone issues - email me

- buffer : 10 minutes to upload solutions, 4:10 PM.

- if questions arise, please email.

[1.] Last time

We looked at zeros of holomorphic functions.

The following result guarantees existence.

Lemma $f : U \longrightarrow \mathbb{C}$ holomorphic, $\overline{\Delta}(a, R) \subseteq U$

Assume $\min\limits_{z \in \partial\Delta} |f(z)| > |f(a)|$. Then f has a zero in U.

Proof Assume $f \neq 0$, let $g = \frac{1}{f}$.

Note $|g(a)| = \frac{1}{|f(a)|} > \frac{1}{\min |f(z)|} = \max\limits_{z \in \partial\Delta} |g(z)|$.

This contradicts the $k = 0$ case of Cauchy's estimate

$$|g(a)| \leq \max_{z \in \partial\Delta} |g(z)| \quad \text{(last time)}$$

Thus f has a zero in U.

Main theorems

i) Identity Principle

ii) Open Mapping Theorem

iii) Maximum Modulus Principle

2 Open Mapping Theorem.

Recall: $f : X \longrightarrow Y$ is open map if $\forall\, U \subseteq X$ open, $f(U)$ is open.

$f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longrightarrow x^2$ is not open, $U = (-1,1)$, $f(U) = [0,1)$

$f : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longrightarrow z^2$ is open. This is because:

Theorem $f : U \longrightarrow \mathbb{C}$ not constant holomorphic $\Longrightarrow$ $\Longrightarrow$ f is open.

Proof Suffices to show $f(U)$ is open. Else if $V \subseteq U$, work with $f|_V : V \longrightarrow \mathbb{C}$. This is not constant because of identity principle.

Let $a \in U$. We may assume $f(a) = 0$.

Claim $\exists\, r$ such that $\Delta(0,r) \subseteq f(U)$. This would show $f(U)$ contains a neighborhood of $f(a)=0$. $\Rightarrow f(U)$ open.

Proof Since U open $\Rightarrow \exists\, \bar{\Delta}(a) \subseteq U$. We may assume $f\big|_{\partial\bar{\Delta}(a)}$ has no zeros. (Argue by contradiction. This would give a sequence of accumulating zeros for f. contradicting identity principle).

$$\text{Let } r = \frac{1}{2} \min_{z \in \partial\bar{\Delta}(a)} |f(z)| > 0.$$

Let $w \in \Delta(0,r)$. We need to show $\exists\, z \in U$, $f(z) = w$.

Apply the Lemma to $f-w$. to guarantee $\exists$ zero z for $f-w$.

We need

$$\min_{z \in \partial\bar{\Delta}(a)} |f(z)-w| > |f(a)-w| = |0-w| = |w|$$

Indeed,

$$|f(z)-w| \geq |f(z)| - |w| \geq 2r - |w| \overset{?}{>} |w|$$

since $|w| < r$. This completes the proof.

Example $f: U \longrightarrow \mathbb{C}$, $P \in \mathbb{R}[X,Y]$ not constant

$$P(\operatorname{Re} f, \operatorname{Im} f) = 0 \implies f \text{ constant}.$$

$P = aX + bY - c$.

$$a \operatorname{Re} f + b \operatorname{Im} f = c \implies f \text{ constant}.$$

$P = X^{2020} + Y^{2020} - 1$.

$$(\operatorname{Re} f)^{2020} + (\operatorname{Im} f)^{2020} = 1. \implies f \text{ constant}.$$

Proof By OMT, $f(U)$ is open so it contains a disc Δ.

Since $P(\operatorname{Re} f, \operatorname{Im} f) = 0 \implies f(U) \subseteq \{(x,y) : P(x,y) = 0\}$.

$\implies \Delta \subseteq \{(x,y) : P(x,y) = 0\}$ This cannot happen.

Indeed, write $P(x,y) = \sum_{k=0}^{N} a_k(x) \, y^k$.

Fix x such that $a_N(x) \neq 0$. (finitely many roots). For such

x, y takes on at most N values for which $P(x,y) = 0$.

But if $\Delta \subseteq \{(x,y) : P(x,y) = 0\}$, for each x there would be

∞-many y's. contradiction.

Example $f: U \longrightarrow V$ bijective, holomorphic & $f'(a) \neq 0$ $\forall a \in U$. Then f^{-1} holomorphic.

Proof We show f^{-1} continuous. This is the OMT.

$$(f^{-1})^{-1}(W) = f(W) = \text{open}. \ \forall\ W \subseteq V \text{ open}.$$

We show f^{-1} is differentiable. Use the definition.

$$1 = \frac{f(f^{-1}(z+h)) - f(f^{-1}(z))}{h} =$$

$$\underset{h\to 0}{\overset{=}{=}} \lim \frac{f(f^{-1}(z+h)) - f(f^{-1}(z))}{f^{-1}(z+h) - f^{-1}(z)} \lim_{h\to 0} \frac{f^{-1}(z+h) - f^{-1}(z)}{h}.$$

The first limit exists since f is holomorphic & f^{-1} is continuous. It equals $f'(f^{-1}(z)) \neq 0$. The second limit must exist as well, giving the derivative $(f^{-1})'(z) = \dfrac{1}{f'(f^{-1}(z))}$.

Remark We assumed $f'(a) \neq 0$ $\forall a$. This is automatic (see later).

Math 220A - Lecture 12

November 2, 2020

[I] Main theorems

[i] Identity Principle

[ii] Open Mapping Theorem

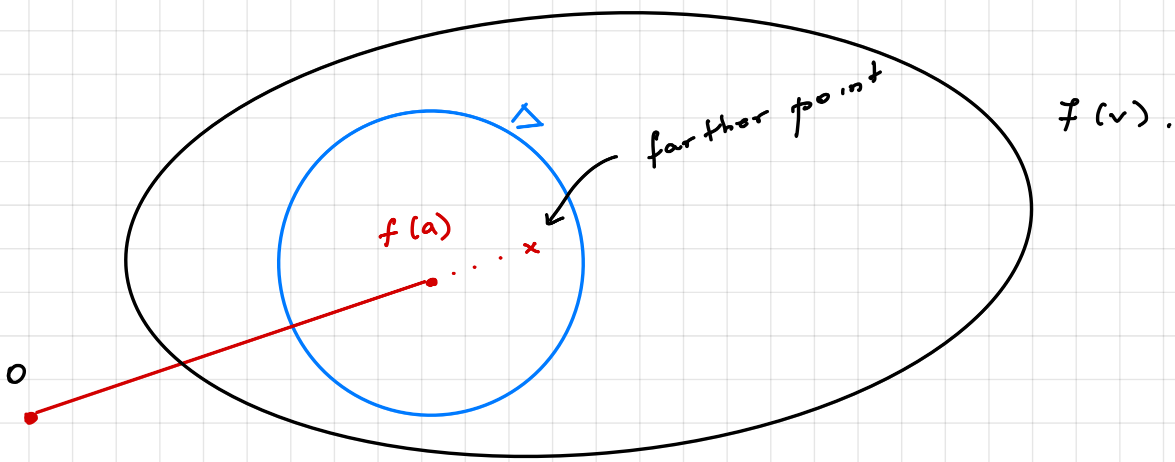
[iii] Maximum Modulus Principle

Theorem $f: U \to \mathbb{C}$ holomorphic, non constant $\Rightarrow$ $|f|$ cannot have local maxima.

Proof Assume that $|f|$ achieves a local maximum at a.

$\Rightarrow \exists\, V \ni a,\ V \subseteq U$, $|f|$ has a maximum at a.

By OMT, $f(V)$ is open. $\Rightarrow$ $\exists$ disc Δ centered at $f(a)$ $\Delta \subseteq f(V)$. Note that $|f|$ measures distance from the origin. The disc Δ has points farther from 0 than $f(a)$ contradicting the assumption $|f|$ has maximum at a. (in V).



Remarks $\boxed{i}$ Minimum modulus principle

$f: U \longrightarrow \mathbb{C}$ holomorphic, not constant, f has no zeros in U.

$\Longrightarrow |f|$ has no local minimum

Proof Let $g = \frac{1}{f} : U \longrightarrow \mathbb{C}$. holomorphic. Apply the maximum modulus to the function g & conclude.

$\boxed{ii}$ U bounded, $f: \overline{U} \longrightarrow \mathbb{C}$ continuous, holomorphic in U

$$\Longrightarrow \max_{\overline{U}} |f| = \max_{\partial U} |f| \quad (*).$$

Proof Since U bounded $\Rightarrow \overline{U}, \partial U$ compact so f achieves maxima on these sets. Let f achieve maximum in $\overline{U}$ at $a \in \overline{U}$.

If $a \in U \Rightarrow f|_U$ has a maximum at $a \xrightarrow{\text{MMP}}$

$\Rightarrow f =$ constant & there's nothing to prove.

Otherwise $a \in \partial U$ proving $(*)$.

2. Laurent Series & Functions in annular regions (Conway V.1)

We have seen $f: \Delta(a,r) \longrightarrow \mathbb{C}$ then

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k \quad \text{— Taylor series}$$

We consider Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

Convergence of Laurent series

$$f^+(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

$$f^-(z) = \sum_{k=-\infty}^{-1} a_k (z-a)^k = \sum_{k=1}^{\infty} a_{-k} (z-a)^{-k}$$

$$f(z) = f^+(z) + f^-(z).$$

Def f converges absolutely & uniformly provided f^+, f^- do so.

Remark

radius of convergence

f^+ converges if $|z-a| < R$.

f^- converges if $|z-a|^{-1} < r^{-1} \iff |z-a| > r$.

For power series, convergence is absolute & uniform on compact subsets.

Define $\Delta(a; r, R) = \{ z : r < |z-a| < R \}$, $0 \le r < R \le \infty$.

Theorem Let $f : \Delta(a; r, R) \longrightarrow \mathbb{C}$ holomorphic. Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$ can be expanded into

Laurent series, converging absolutely & uniformly on compact sets in $\Delta(a; r, R)$. Furthermore,

$$a_k = \frac{1}{2\pi i} \int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{k+1}} \, dw. \quad \forall \; r < \rho < R.$$

Remark An important case is $r = 0$. Then

$$\Delta^*(a, R) = \Delta(a, R) \setminus \{a\} = \text{punctured disc.}$$

$$f : \Delta^*(a, R) \longrightarrow \mathbb{C} \text{ holomorphic} \Rightarrow f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

Compare this to Taylor expansion.

Pierre Alphonse Laurent

1813 – 1854

(engineer in the army).

The original work on Laurent series was not published.

Cauchy writes (C.R. Acad. Sci. Paris, 1843, page 938)

L'extension donnée par M. Laurent nous parait digne de remarque

(After Remmert, Complex Analysis, page 350)

Proof (of Laurent expansion) $A = \Delta(a; r, R)$.

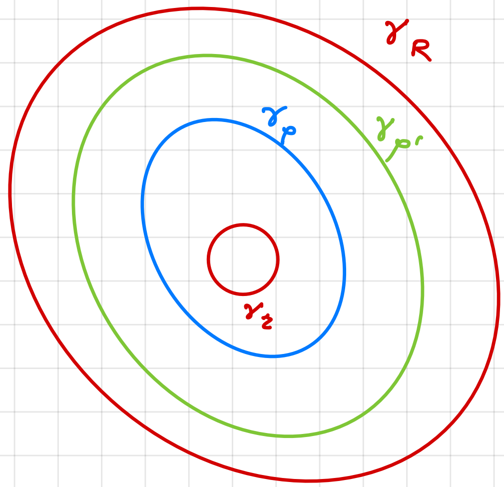
$\boxed{a}$ WLOG $a = 0$; else work with $f(z+a)$.

$\boxed{b}$ the expression $a_k = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(w)}{w^{k+1}}\, dw$

is independent of ρ. Indeed

$\gamma_\rho \overset{A}{\sim} \gamma_{\rho'}$ and use

Cauchy Homotopy Theorem.

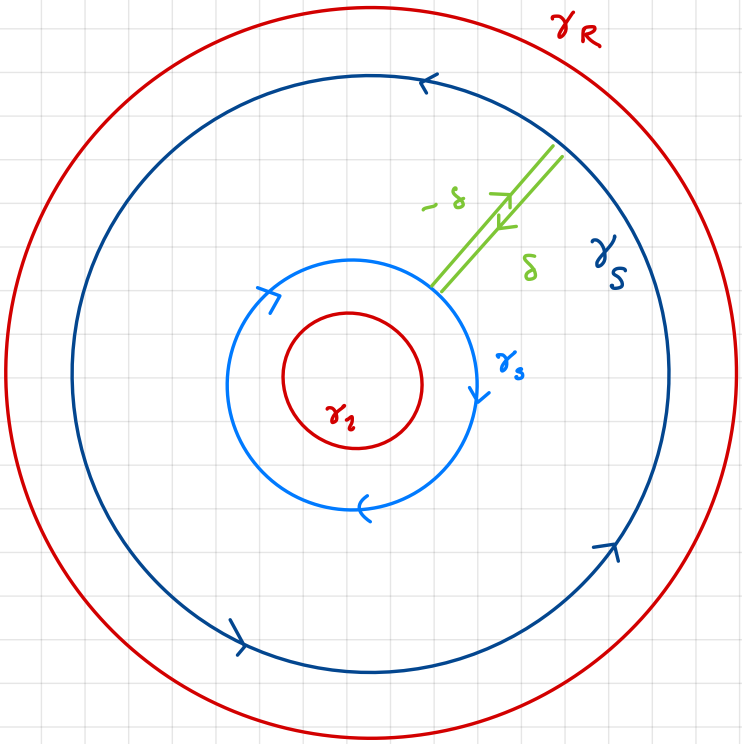


$\boxed{c}$ suffices to prove pointwise convergence in z.

Indeed, convergence of f $\iff$ convergence of f^+ & f^- in $r < |z| < R$.

But then f^+ converges in $|z| < R$ (power series have discs of converge) & we remarked convergence is absolute & uniform on compacts. Same for f^-.

Pointwise convergence Let $r < s < |z| < S < R$



Let δ be a segment joining γ_s, γ_S avoiding z.

Let

$$\gamma = \gamma_S + \delta + \gamma_s + (-\delta).$$

Note $\gamma \overset{A}{\sim} 0$. This can be seen by continuously shrinking $\delta \to 0$.

Also $n(\gamma, z) = 1$ since $n(\gamma_s, z) = 0$ as z is outside and $n(\gamma_S, z) = 1$ as z is interior to γ_S. $\Rightarrow n(\gamma, z) = 1$.

CIF:

$$(+) \quad f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, dw.$$

$$= \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w - z} \, dw - \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w - z} \, dw$$

(cancelling the contribution of $\delta, -\delta$).

The two terms will give the positive / negative parts of Laurent series.

Key expansions (Remember them) $s < |z| < S$.

$\boxed{i}$ over γ_S : $\dfrac{1}{w - z} = \dfrac{1}{w} \cdot \dfrac{1}{1 - \frac{z}{w}} = \sum_{k=0}^{\infty} \frac{1}{w}\left(\frac{z}{w}\right)^k$

$$= \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}. \quad (1)$$

The convergence is uniform in w since $\left|\frac{z}{w}\right| = \frac{|z|}{S} < 1$. We can define $M_k = \dfrac{|z|^k}{S^{k+1}}$, $f_k(z) = \dfrac{z^k}{w^{k+1}}$ and invoke Weierstraß M-test to conclude uniform convergence.

We can multiply by $f(w)$. Uniform convergence still holds. (Use $M_k = \dfrac{|z|^k}{S^{k+1}} \cdot \sup_{\gamma_S} |f|$.)

We can then integrate term by term. (Rudin). Thus

$$\frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w - z}\, dw \overset{(1)}{=} \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w^{k+1}}\, dw \cdot z^k$$

$$= \sum_{k=0}^{\infty} a_k z^k. \quad (*)$$

$\boxed{ii}$ over γ_s, we use a different expansion

$$\frac{1}{w - z} = -\frac{1}{z} \cdot \frac{1}{1 - \frac{w}{z}} = \sum_{k=0}^{\infty} -\frac{1}{z}\left(\frac{w}{z}\right)^k$$

$$= \sum_{k=0}^{\infty} -\frac{w^k}{z^{k+1}}. \quad (2)$$

Here $\left|\frac{w}{z}\right| = \frac{1}{|z|} < 1$. By the same arguments

$$-\frac{1}{2\pi i}\int_{\gamma_3}\frac{f(w)}{w-z}\,dw \overset{(2)}{=} \sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma_3} f(w)\,w^{k}\,dw \cdot z^{-k-1}$$

$$= \sum_{k=0}^{\infty} a_{-k-1}\, z^{-k-1}$$

$$= \sum_{k=-\infty}^{-1} a_k\, z^{k}. \quad (**)$$

(+), (*), (**). imply the Theorem.

Math 220 A — Lecture 13

November 9, 2020

Logistics

Wed, Nov 11 — holiday — no lecture

Office Hour — Wed 4-5PM (discuss homework, midterm)

Questions about the midterm?

Last time — Laurent expansion

Theorem Let $f: \Delta(a; r, R) \longrightarrow \mathbb{C}$ holomorphic. Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \quad \text{can be represented as}$$

Laurent series, converging absolutely & uniformly on compact subsets of $\Delta(a; r, R)$.

Today — classification of singularities Conway $\underline{V}$. 1.

— characterization of singularities.

Types of singularities

$f : \Delta^*(a, R) \longrightarrow \mathbb{C}$, holomorphic.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{Laurent series.}$$

Terminology [i] the coefficient of $(z-a)^{-1}$:

$$a_{-1} = \operatorname*{Res}_{z=a} f = \text{residue}$$

[ii]

$$\sum_{n=-\infty}^{-1} a_n (z-a)^n = \text{principal part.}$$

Three cases

[A] $a_k = 0 \quad \forall\, k < 0 \iff$ Taylor expansion

$\iff f$ extends holomorphically across a

Removable singularity

[B] $a_k = 0 \quad \forall\, k < -N, \quad a_{-N} \neq 0$

Pole of order N.

[C] $a_k \neq 0$, $k < 0$ happens infinitely often

Essential singularity

Case A a removable singularity

Theorem A TFAE $f: \Delta^*(a, R) \longrightarrow \mathbb{C}$ holomorphic

[i] f extends holomorphically across a

[ii] f extends continuously across a

[iii] f bounded near a

[iv] $\lim_{z \to a} f(z) \cdot (z-a) = 0$.

Proof [i] $\Rightarrow$ [ii] $\Rightarrow$ [iii] $\Rightarrow$ [iv] is obvious

[iv] $\Rightarrow$ [i]. WLOG $a = 0$, else work with $f(z+a)$.

We show $a_k = 0 \ \forall k < 0$. Fix $\varepsilon > 0$. Since

$$\lim_{z \to 0} z f(z) = 0 \Rightarrow |f(z)| < \frac{\varepsilon}{|z|} \text{ if } |z| < \delta.$$

We have for $0 < r < \delta < R$:

$$|a_k| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}} dz \right| \leq \frac{1}{2\pi} \cdot \frac{\varepsilon}{r} \cdot \frac{1}{r^{k+1}} \cdot 2\pi r$$

$$= \frac{\varepsilon}{r^{k+1}}.$$

If $k = -1$: $|a_{-1}| < \varepsilon$ $\forall \varepsilon > 0 \Rightarrow a_{-1} = 0$.

If $k < -1$, take $\varepsilon = 1$, $|a_k| < \frac{1}{r^{k+1}}$. Make $r \to 0$

to obtain $a_k = 0$. since $k < -1$.

Example $f: U \longrightarrow \mathbb{C}$ holomorphic

$$g(z) = \begin{cases} \frac{f(z) - f(a)}{z - a}, & z \neq a \\ f'(a), & z = a \end{cases}$$ is holomorphic

Indeed $\frac{f(z) - f(a)}{z - a}$ has a removable singularity at a

by item $\boxed{\text{IV}}$. & g is the continuous / holomorphic

extension across a.

Case B a pole of order N.

$$f(z) = (z-a)^{-N} g(z)$$ holomorphic

$$g(z) = \sum_{k=0}^{\infty} a_{k-N} (z-a)^k, \qquad g(a) = a_{-N} \neq 0.$$

Zeros versus Poles

$$\frac{1}{f(z)} = (z-a)^N \cdot \frac{1}{g(z)}, \quad \frac{1}{g} \text{ holomorphic near } a$$

f pole of order N at $a \iff \frac{1}{f}$ zero of order N at a

Lemma B $f : \Delta^*(a, R) \to \mathbb{C}$ holomorphic TFAE

$\boxed{\text{i}}$ a is a pole

$\boxed{\text{ii}}$ $\lim_{z \to a} f(z) = \infty$.

Proof $\boxed{\text{i}} \Rightarrow \boxed{\text{ii}}$ Write $f(z) = (z-a)^{-N} g(z)$.

Since $g(a) \neq 0 \Rightarrow |g(z)| \geq M > 0$ in $|z-a| < \delta$.

$$\Rightarrow |f(z)| = \frac{|g(z)|}{|z-a|^N} \geq \frac{M}{|z-a|^N}.$$ Make $z \to a$ to

conclude $\lim_{z \to a} f(z) = \infty$.

$\boxed{ii} \Rightarrow \boxed{i}$ Note $\lim_{z \to a} f(z) = \infty \Rightarrow \lim_{z \to a} \frac{1}{f(z)} = 0$

$\Rightarrow \frac{1}{f}$ bounded near a $\Rightarrow \frac{1}{f}$ can be extended across a holomorphically. Note the extension vanishes at a, say of order N $\Rightarrow f$ has a pole at a of order N.

Definition $S \subseteq U$ discrete. A function f holomorphic in $U \setminus S$, with at worst poles at S is called meromorphic.

Example

polynomials

$\boxed{i}$ $f(z) = \frac{P(z)}{Q(z)}$, $U = \mathbb{C}$ meromorphic.

$\boxed{ii}$ $f(z) = \frac{1}{\sin \frac{1}{z}}$, $U = \mathbb{C}^\times$.

Check $z = \frac{1}{n\pi}$, $n \in \mathbb{Z}^\times$ are poles. These do not accumulate in $\mathcal{U} = \mathbb{C} \setminus \{0\}$. Thus f meromorphic in $\mathcal{U} = \mathbb{C}^{\times}$.

Case C $\quad f : \Delta^*(a, R) \longrightarrow \mathbb{C}$ holomorphic

a essential singularity $\quad$ e.g. $f(z) = e^{\frac{1}{z}}$.

Example $\quad f(z) = e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k}$. $\Rightarrow$

$\Rightarrow$ $a = 0$ is essential singularity.

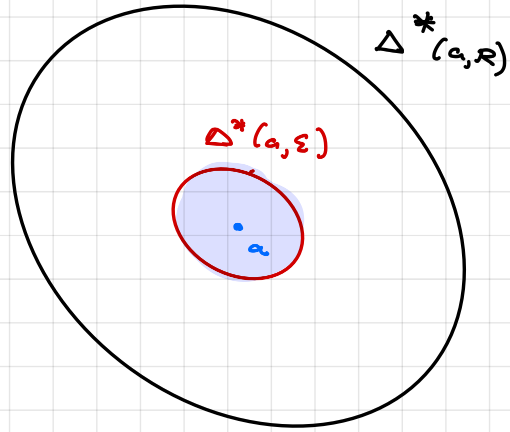
Remark $\quad f$ cannot be bounded or go to ∞.

(see cases A & B)

Question $\quad$ How does f behave near a?

Theorem (Big Picard Theorem) — 220 C.

$\forall\ \Delta^*(a,\varepsilon) \subseteq \Delta^*(a,R)$, $\quad f(\Delta^*(a,\varepsilon)) = \mathbb{C}$ or $\mathbb{C} \setminus \{\text{point}\}$.



Example $\quad f(z) = e^{\frac{1}{z}}$. $\quad a = 0$.

Claim $\quad f(\Delta^*(0,\varepsilon)) = \mathbb{C} \setminus \{0\}$. $\quad \forall\ \varepsilon > 0$

Proof $\quad y \neq 0: \quad y = e^{\frac{1}{z}}, \quad z \in \Delta^*(0,\varepsilon)$.

$$\iff \frac{1}{z} = \log y + 2n\pi i \quad \text{for any choice of log}$$

$$\iff z = \frac{1}{\log y + 2n\pi i} \in \Delta^*(0,\varepsilon) \quad \text{if } n >> 0.$$

Theorem (Casorati – Weierstraß) $\quad f: \Delta^*(a,R) \to \mathbb{C}$

$\boxed{i}$ f has essential singularity at a

$\boxed{ii}$ $\forall\ \Delta^*(a,\varepsilon) \subseteq \Delta^*(a,R)$, $\quad f(\Delta^*(a,\varepsilon))$ is dense in $\mathbb{C}$.

Proof $\boxed{I} \Rightarrow \boxed{II}$ Assume for some $\varepsilon > 0$, the set

$f(\Delta^*(a,\varepsilon))$ is not dense in $\mathbb{C}$. Then $\exists\ \Delta(\lambda, R)$

$$(*).\quad f(\Delta^*(a,\varepsilon)) \cap \Delta(\lambda, R) = \phi.$$

Define $g = \dfrac{1}{f-\lambda}$ in $\Delta^*(a,\varepsilon)$. By $(*)$ we know

$$|f-\lambda| \geq R \text{ in } \Delta^*(a,\varepsilon) \Rightarrow |g| \leq \frac{1}{R} \text{ in } \Delta^*(a,\varepsilon)$$

Thm A
$\Rightarrow$ a is removable singularity for g. But

$$f = \lambda + \frac{1}{g}. \quad (+)$$

If a is not a zero for $g \Rightarrow \dfrac{1}{g}$ holomorphic $\overset{(+)}{\Longrightarrow}$

f extends holomorphically across $a \Rightarrow$ removable singularity.

If a is a zero for $g \Rightarrow \dfrac{1}{g}$ has a pole at $a \overset{(+)}{\Longrightarrow}$

$\Rightarrow f$ has pole at a.

Both cases are impossible.

$\boxed{ii} \Rightarrow \boxed{i}$ Assume a removable singularity $\Rightarrow$

Thm A
$\Rightarrow$ f bounded near a $\Rightarrow$ $\exists\, M>0,\ \varepsilon>0$ with

$$|f(z)| < M \quad \text{in} \quad \Delta^*(a,\varepsilon)$$

$\Rightarrow f(\Delta^*(a,\varepsilon))$ cannot be dense.

Assume a pole $\overset{\text{Lemma B}}{\Rightarrow}$ $\lim\limits_{z\to a} f(z) = \infty$ $\Rightarrow$

$\Rightarrow \exists\, \varepsilon>0$, $|f(z)| \geq 1$ in $\Delta^*(a,\varepsilon)$ $\Rightarrow$

$\Rightarrow f(\Delta^*(a,\varepsilon))$ cannot be dense.

Thus a is essential singularity.

Felice Casorati

1835 – 1890

Karl Weierstraß

1815 – 1897

Émile Picard

1856 – 1941

Math 220 A — Lecture 14

November 13, 2020

1. Residues (Conway V. 2)

a singularity for f

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \quad \text{Laurent series}$$

$$a_{-1} = \operatorname{Res}(f, a) = \text{residue}$$

Problem. Compute $\operatorname{Res}(f, a)$

Method 0 Laurent expansion

Example $f(z) = \dfrac{z}{\sin^4 z}$, $\operatorname{Res}(f, 0) = ?$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)$$

$$\sin^4 z = z^4\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)^4$$

$$= z^4\left(1 - \frac{4z^2}{6} + \dots\right)$$

$$f(z) = \frac{z}{z^4\left(1 - \frac{4z^2}{6} + \dots\right)} = \frac{1}{z^3} \cdot \left(1 + \frac{4z^2}{6} + \dots\right)$$

$$= \frac{1}{z^3} + \frac{4}{6} \cdot \frac{1}{z} + \dots$$

$$\Rightarrow \text{Res}\,(f, 0) = \frac{2}{3}.$$

Method 1* $\quad f(z) = \dfrac{g(z)}{h(z)}$, g, h holomorphic

Assume a simple zero for $h \Rightarrow a$ simple pole for f.

$$\text{Res}\,(f, a) = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to a} (z-a) \frac{g(z)}{h(z) - h(a)}$$

$$= \lim_{z \to a} \frac{g(z)}{\dfrac{h(z) - h(a)}{z - a}} = \frac{g(a)}{h'(a)}$$

Conclusion: $\quad \text{Res}\,(f, a) = \dfrac{g(a)}{h'(a)}.$

Example $\quad f(z) = \dfrac{z - \sin z}{z^2 \sin z}$

- poles $\quad z = 0, \quad z = n\pi, \; n \neq 0, \; n \in \mathbb{Z}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \Rightarrow \frac{\sin z}{z} \to 1 \quad \text{as } z \to 0$$

- $z = 0$ is removable since $\quad \Rightarrow \dfrac{z - \sin z}{z^3} \to \dfrac{1}{3!}$ as $z \to 0$

$$\lim_{z \to 0} \frac{z - \sin z}{z^2 \sin z} = \lim_{z \to 0} \underbrace{\frac{z - \sin z}{z^3}}_{\to \frac{1}{6}} \cdot \underbrace{\frac{z}{\sin z}}_{\to 1.} = \frac{1}{6}.$$

Since $z = 0$ is removable $\Rightarrow \boxed{\operatorname{Res}(f, 0) = 0.}$

- $z = n\pi, \; n \neq 0$. Take $\quad g(z) = \dfrac{z - \sin z}{z^2}$

$$h(z) = \sin z$$

$$\Rightarrow g(n\pi) = \frac{1}{n\pi}, \quad h'(n\pi) = \cos z \Big|_{z = n\pi} = (-1)^n.$$

$$\Rightarrow \boxed{\operatorname{Res}(f, n\pi)} = \frac{g(n\pi)}{h'(n\pi)} = \boxed{\frac{1}{n\pi} \cdot (-1)^n.}$$

Method 2

holomorphic

$$f(z) = \frac{g(z)}{(z-a)^k} \Rightarrow \operatorname{Res}(f,a) = \frac{g^{(k-1)}(a)}{(k-1)!}$$

Write $\displaystyle g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(z-a)^n.$

coeff. of $(z-a)^{-1}$ in $f \iff$ coeff. of $(z-a)^{k-1}$ in g.

This equals $\dfrac{g^{(k-1)}(a)}{(k-1)!}$

Example

$$f(z) = \frac{z}{(z^2+1)^2} \Rightarrow \operatorname{Res}(f,i) = ?$$

$f(z) = \dfrac{g(z)}{(z-i)^2}$, $g(z) = \dfrac{z}{(z+i)^2}$. $\Rightarrow$ $g'(i) = 0$ (check)

$\operatorname{Res}(f,i) = g'(i) = 0.$

2. Residue Theorem (Conway V. 2)

Toy Example $\quad f: \Delta^*(a,R) \longrightarrow \mathbb{C}$, holomorphic.

$$\Rightarrow \int_{\gamma_s} f(z)\,dz = 2\pi i \operatorname{Res}(f,a). \quad \text{where } \gamma_s = \partial\Delta(a,s).$$

Proof Write

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k.$$

This converges uniformly on compact sets, so we can integrate

$$\Rightarrow \int_{\gamma_s} f\,dz = \sum_{k=-\infty}^{\infty} a_k \int_{\gamma_s} (z-a)^k\,dz$$

$k \neq -1$, integral $= 0$

$k = -1$, integral $= 2\pi i$

$$= 2\pi i\, a_{-1} = 2\pi i \operatorname{Res}(f,a).$$

$k \neq -1$: $(z-a)^k$ admits a primitive $\dfrac{(z-a)^{k+1}}{k+1}$. $\Rightarrow$ zero integral.

$k = -1$: $\displaystyle\int_{\gamma_s} \frac{dz}{z-a} = 2\pi i\, n(\gamma_s, a) = 2\pi i$

Residue Theorem $\mathcal{U} \subseteq \mathbb{C}$ open connected, S discrete

- $\gamma \overset{\mathcal{U}}{\sim} 0$, $\{\gamma\} \subseteq \mathcal{U} \setminus S$.

- f holomorphic in $\mathcal{U} \setminus S$, singularities at S.

Then

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$$

Remarks

$\boxed{i}$ $S = \emptyset \Rightarrow \int_\gamma f \, dz = 0 \Rightarrow$ Cauchy's Theorem

$\boxed{ii}$ $S = \{a\}$, $\gamma = \gamma_r$ = small circle near a $\Rightarrow$

recovers the toy example.

$\boxed{iii}$ $S = \{a\}$, $f(z) = \dfrac{g(z)}{(z-a)^{k+1}}$, g holomorphic, $\gamma \overset{\mathcal{U}}{\sim} 0$

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \frac{1}{2\pi i} \int_\gamma \frac{g(z)}{(z-a)^{k+1}} \, dz = \operatorname{Res}(g, a) \cdot n(\gamma, a)$$

$$= \frac{g^{(k)}(a)}{k!} \cdot n(\gamma, a) \quad \text{by Method 2.}$$

This recovers CIF for derivatives.

[IV] The sum in RHS is finite

Claim $\{ s \in S,\ n(\gamma, s) \neq 0 \}$ finite

Proof $W = \{ z \in \mathbb{C} \setminus \gamma : n(\gamma, z) \neq 0 \}$.

- W = union of components of $\mathbb{C} \setminus \gamma$ = open (Lecture 7)

- W bounded (Lecture 7)

- $W \subseteq \mathcal{U}$. Indeed if $z \in W$, $z \notin \mathcal{U}$ $\Rightarrow$ $n(\gamma, z) \neq 0$. But

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} = 0 \text{ by Cauchy}$$

using $\zeta \longrightarrow \dfrac{1}{\zeta - z}$ holomorphic in $\mathcal{U}$, $\gamma \overset{\mathcal{U}}{\sim} 0$.

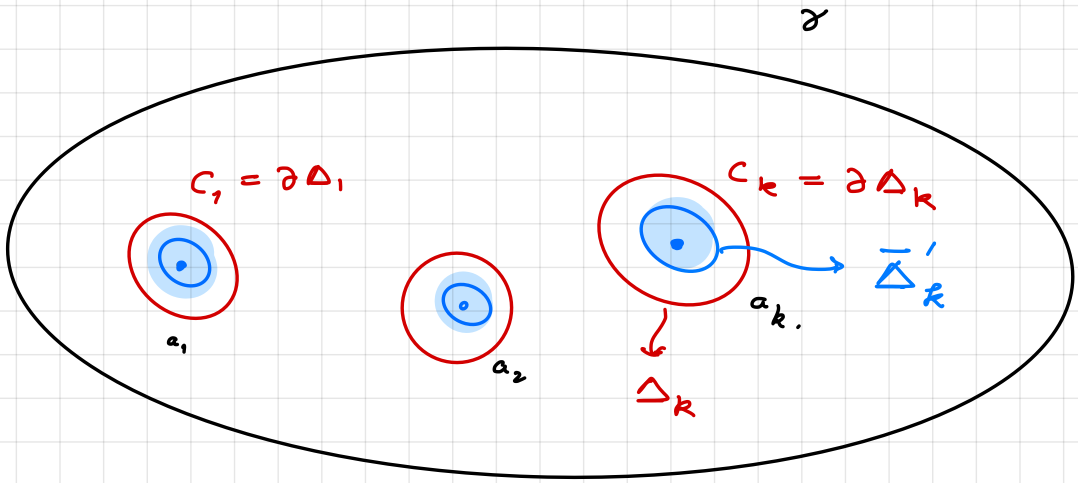
$K = W \cup \{\gamma\} \subseteq \mathcal{U}$ closed & bounded

$\Rightarrow$ K compact in $\mathcal{U}$, S discrete in $\mathcal{U}$

$\Rightarrow$ $K \cap S$ = finite.

$\boxed{\checkmark}$ Naive "proof" — γ simple closed curve

Let $S = \{a_1, \dots, a_k\}$.



Let $C_i = \partial \Delta_i$ be circles centered at a_i, $\Delta_i \subseteq \mathcal{U}$.

Let $\overline{\Delta}'_i \subseteq \Delta_i$. Let $\mathcal{U}' = \mathcal{U} \setminus \bigcup_{i=1}^{k} \overline{\Delta}'_i$ = open.

Let $\eta = \sum_{i=1}^{k} \partial C_i$. Assume we could show

$\gamma \overset{\mathcal{U}'}{\sim} \eta$ and $n(\gamma, a_i) = 1$.

Then by Cauchy applied to $f|_{\mathcal{U}'}$ we'd have

$$\int_\gamma f\,dz = \int_\eta f\,dz = \sum_{i=1}^{k} \int_{C_i} f\,dz$$

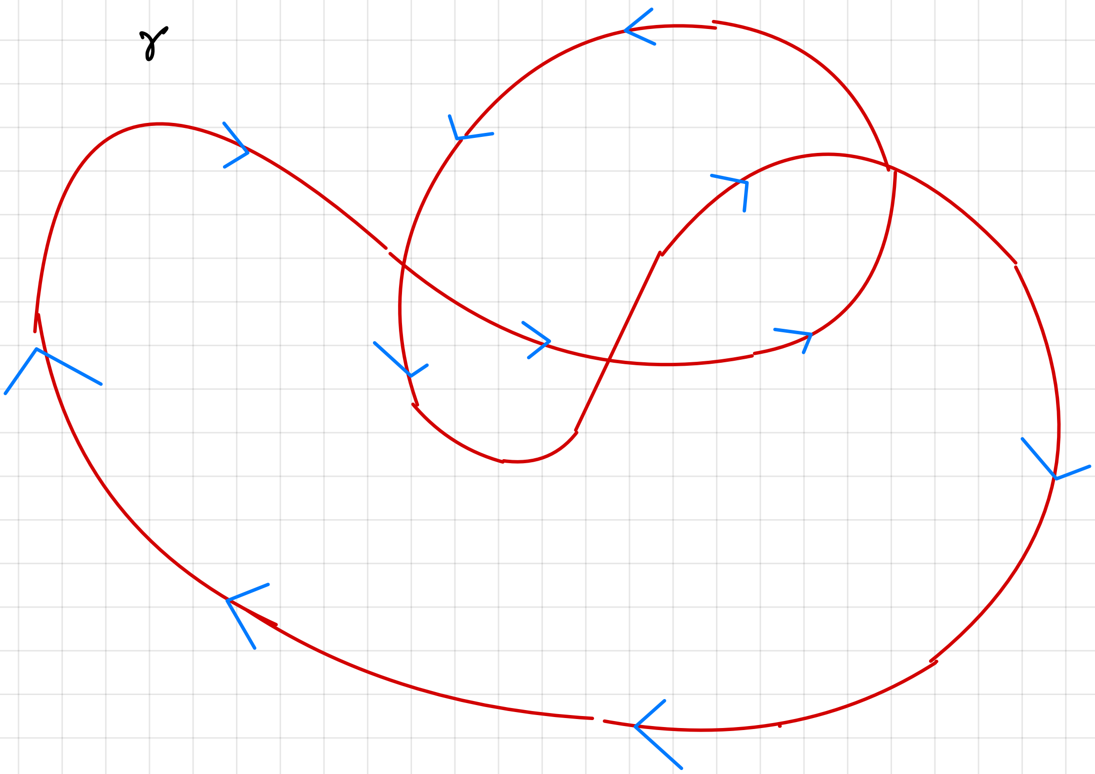
$$= 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, a_i) \quad \text{(toy example).}$$

$$= 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, a_i)\, n(\gamma, a_i).$$

Issues : $\boxed{a}$ η is not a path, but chain

$\boxed{b}$ $\gamma \overset{u'}{\sim} \eta$ and $n(\gamma, a_i) = 1$ need proofs

$\boxed{c}$ how about more complicated curves?



The proof of the residue theorem requires new ideas.

(next time)

Math 220 A - Lecture 15

November 16, 2020

Last time We wish to prove:

Residue Theorem $U \subseteq \mathbb{C}$ open connected, S discrete

- $\gamma \overset{U}{\sim} 0$, $\{\gamma\} \subseteq U \setminus S$.

- f holomorphic in $U \setminus S$, singularities at S.

Then

$$\frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$$

Example

$$\int_{|z|=3} \frac{z+1}{z^2(z-1)}\, dz$$

Take $U = \Delta(0,4)$, $S = \{0,1\}$, $f(z) = \dfrac{z+1}{z^2(z-1)}$.

- $\operatorname{Res}(f,0) = \operatorname{Res}_{z=0} \dfrac{\frac{z+1}{z-1}}{z^2} = \left(\dfrac{z+1}{z-1}\right)'\Big/_{z=0} = -2$

by Method 2 of computing residues last time

- $\operatorname{Res}(f,1) = \operatorname{Res}_{z=1} \dfrac{z+1}{z^2(z-1)} = \dfrac{(z+1)\big/_{z=1}}{(z^2(z-1))'\big/_{z=1}} = \dfrac{2}{1} = 2$

by Method 1 of computing residues last time.

Thus $\displaystyle\int_{|z|=3} f\, dz = 2\pi i\,(\operatorname{Res}(f,0) + \operatorname{Res}(f,1))$

$$= 2\pi i\,(-2+2) = 0.$$

1. Proof of the Residue Theorem

Terminology $\quad U^* \subseteq \mathbb{C}$, $\gamma^* = \sum_{i=1}^{\ell} m_i \gamma_i$ $\quad C'$-chain

[a] $$\int_{\gamma^*} f\, dz = \sum_{i=1}^{\ell} m_i \int_{\gamma_i} f\, dz$$

[b] $$n(\gamma^*, a) = \sum_{i=1}^{\ell} m_i\, n(\gamma_i, a)$$

Definition $\quad \gamma^* \overset{U^*}{\approx} 0$ if $n(\gamma^*, a) = 0 \quad \forall\, a \notin U^*$.

(we say γ^* is null homologous in U^*).

Remark [1] γ^* loop in U^*. Then

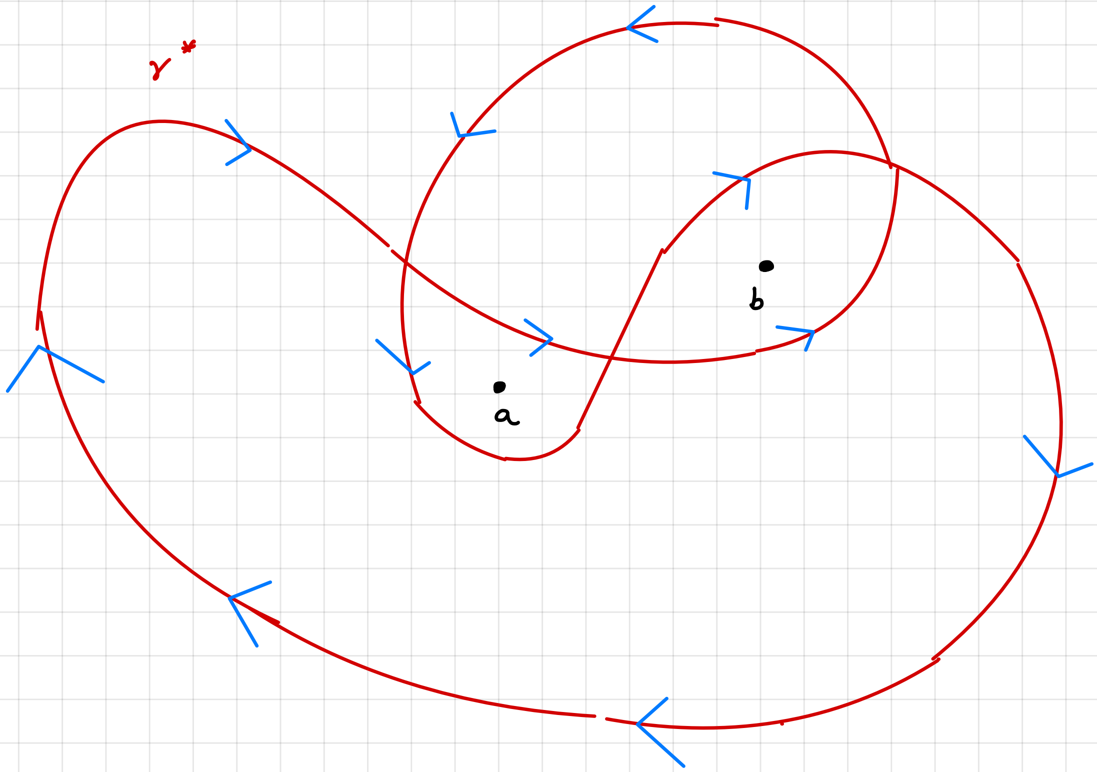
$$\gamma^* \overset{U^*}{\sim} 0 \implies \gamma^* \overset{U^*}{\approx} 0.$$

Indeed if $a \notin U^*$ then

$$n(\gamma^*, a) = \frac{1}{2\pi i} \int_{\gamma^*} \frac{dw}{w-a} = 0$$

by homotopy form of Cauchy applied to $\gamma^* \overset{U^*}{\sim} 0$ and to the holomorphic function $\dfrac{1}{w-a}$ in U^*. $(a \notin U^*)$

$\boxed{u}$ the converse is false $U^* = \mathbb{C} \setminus \{a, b\}$



Check $\gamma^* \overset{U^*}{\approx} 0$. Indeed $n(\gamma^*, a) = n(\gamma^*, b) = 0$. To see this,

find two subloops of γ^* going clockwise &

counterclockwise once around a (same for b).

However $\gamma^* \overset{U^*}{\not\sim} 0$.

Remark* In algebraic topology, one learns that 1st homology

is the abelianization of π_1. (which is defined via homotopy).

Enhanced Cauchy's Theorem

We seek to prove a "homology" version of Cauchy:

Theorem $f: U^* \longrightarrow \mathbb{C}$ holomorphic, $\gamma^* \overset{U^*}{\approx} 0 \implies \int_{\gamma^*} f\,dz = 0$.

Of course, this implies the homotopy version of the theorem proved in previous lectures.

Remark We show next

Enhanced Cauchy Theorem $\implies$ Residue Theorem.

Proof of residue theorem

We let f holomorphic in $\mathcal{U} \setminus S$, $\gamma \overset{\mathcal{U}}{\sim} 0$. We want

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$$

Last time we saw RHS is finite since

$$\{ s \in S : n(\gamma, s) \neq 0 \} \text{ is finite.}$$

Enumerate this set to be $\{a_1 \dots a_k\}$, $m_i = n(\gamma, a_i) \neq 0$.

Let Δ_i be small disjoint discs near a_i. $\Delta_i \subseteq \mathcal{U}$.

Define
- $\mathcal{U}^* = \mathcal{U} \setminus S$
- $\gamma^* = \gamma + \sum_{i=1}^{k} (-m_i) \, C_i$ where $C_i = \partial \Delta_i$

(positive orientation)

Claim: $\gamma^* \overset{\mathcal{U}^*}{\approx} 0$

Enhanced Cauchy for $(\mathcal{U}^*, \gamma^*)$ $\Rightarrow$ $\int_{\gamma^*} f \, dz = 0$

$$\Rightarrow \frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{i=1}^{k} m_i \cdot \frac{1}{2\pi i} \int_{C_i} f \, dz$$

$$= \sum_{i=1}^{k} m_i \operatorname{Res}(f, a_i) \text{ by toy example}$$

last time. QED.

Proof of the claim. Want $n(\gamma^*, a) = 0$ if $a \notin U^*$.

$\boxed{i}$ if $a \notin U$. Note $\gamma \overset{U}{\sim} 0 \Rightarrow \gamma \overset{U}{\approx} 0 \Rightarrow n(\gamma, a) = 0$.

Also $a \notin \Delta_i \Rightarrow n(c_i, a) = 0$. Then

$$n(\gamma^*, a) = \underbrace{n(\gamma, a)}_{0} + \sum (-m_i) \underbrace{n(c_i, a)}_{0} = 0.$$

$\boxed{ii}$ if $a \in S$. Note that $n(c_i, a) = \begin{cases} 0 & \text{if } a \neq a_i \\ 1 & \text{if } a = a_i \end{cases}$

If $a = a_i \Rightarrow n(\gamma^*, a) = \underbrace{n(\gamma, a)}_{m_i} + (-m_i) \underbrace{n(c_i, a)}_{1} = m_i + (-m_i) = 0.$

If $a \neq a_i \; \forall i \Rightarrow n(\gamma, a) = 0$ by definition of the a_i's

$$\Rightarrow n(\gamma^*, a) = n(\gamma, a) + \sum (-m_i) \underbrace{n(c_i, a)}_{0} = 0.$$

Remarks

[i] Proof of residue thm only requires $\gamma \overset{u}{\approx} 0$ not $\gamma \overset{u}{\sim} 0$. $\rightsquigarrow$ improvement of hypothesis.

[ii] Residue Theorem $\Rightarrow$ Enhanced CIF for derivatives.

Let $\gamma \overset{u}{\approx} 0$. Apply the residue theorem: $S = \{a\}$.

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{k+1}}\, dz = n(\gamma, a) \operatorname*{Res}_{z=a} \frac{f(z)}{(z-a)^{k+1}}$$

$$= n(\gamma, a) \cdot \frac{f^{(k)}(a)}{k!}$$

(using Method 2 from last time)

2. Proof of Enhanced Cauchy's Theorem

- change notation $U \longleftrightarrow U^*$, $\gamma \longleftrightarrow \gamma^*$
- modify statement slightly

Theorem (enhanced CIF)

$\gamma \overset{U}{\approx} 0$, $f: U \longrightarrow \mathbb{C}$ holomorphic, $a \in U \setminus \{\gamma\}$

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz = n(\gamma, a)\, f(a).$$

Remark Using the above for $f^{new}(z) = f(z)\cdot(z-a)$, $f^{new}(a) = 0$

we obtain $\gamma \overset{U}{\approx} 0 \Rightarrow \int_\gamma f\, dz = 0$. This is Enhanced Cauchy.

Remark TFAE:

Enhanced CIF $\overset{\text{above}}{\Longrightarrow}$ Enhanced Cauchy's theorem

$\overset{\text{page 6–8}}{\Longrightarrow}$ Residue Theorem

$\overset{\text{page 9}}{\Longrightarrow}$ Enhanced CIF for derivatives

$k=0$

Math 220 A — Lecture 16

November 18, 2020

1. Last time

The proof of Residue thm / Enhanced Cauchy requires:

Theorem (enhanced CIF / Conway IV. 5)

$\gamma \overset{U}{\approx} 0$, $f: U \longrightarrow \mathbb{C}$ holomorphic, $a \in U \setminus \{\gamma\}$

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a}\, dw = n(\gamma, a)\, f(a).$$

Rewriting

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a}\, dw = \frac{1}{2\pi i} \int_\gamma \frac{f(a)}{w-a}\, dw$$

$$\Longleftrightarrow \int_\gamma \underbrace{\frac{f(w) - f(a)}{w-a}}_{\varphi(a,w)}\, dw = 0$$

Auxiliary function $\varphi : U \times U \longrightarrow \mathbb{C}$

$$\varphi(z,w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w}, & z \neq w. \\ f'(z), & z = w. \end{cases}$$

Want: $\displaystyle\int_\gamma \varphi(z,w)\, dw = 0 \quad \forall z \in U \quad (*)$

Apply $(*)$ to $z = a \in U \setminus \{\gamma\}$ to conclude enhanced CIF.

Claims

[i] φ continuous in $U \times U$

[ii] $z \longrightarrow \varphi(z,w)$ holomorphic $\forall w \in U$. fixed.

Proof of [ii] This was explained in Lecture 13 as an application of Removable Singularity Theorem.

Proof of $\boxed{\text{II}}$ φ continuous in $\mathcal{U} \times \mathcal{U}$. Recall

$$\varphi(z,w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w}, & z \neq w. \\ f'(z), & z = w. \end{cases}$$

- Continuity is clear at points where $z \neq w$.
- We show continuity at (a,a). We have

$$|\varphi(z,w) - \varphi(a,a)| = \left| \frac{1}{w-z} \int_z^w f'(t)\,dt - f'(a) \right|$$

$$= \frac{1}{|w-z|} \left| \int_z^w f'(t) - f'(a)\,dt \right|$$

$$\leq \sup_{t \in [z,w]} |f'(t) - f'(a)| < \varepsilon$$

if $z, w \in \Delta(a,\delta)$.

This holds in $\Delta(a,\delta)$ for some δ, because f' is continuous (in fact holomorphic).

Proof of (*) Want $\int_\gamma \varphi(z,w)\,dw = 0$ if $\gamma \overset{u}{\approx} 0$.

Question: How do we make use of $\gamma \overset{u}{\approx} 0$?

Answer. Define

$$V = \{ z \in \mathbb{C} \setminus \gamma,\ n(\gamma, z) = 0 \}.$$

- $U \cup V = \mathbb{C}$. (this is the only place where $\gamma \overset{u}{\approx} 0$ is used).

Indeed if $z \notin U \Rightarrow n(\gamma, z) = 0$ since $\gamma \overset{u}{\approx} 0$. Also $z \in \mathbb{C} \setminus \{\gamma\}$.

- V open. Indeed, by Lecture 7, V is union of components of $\mathbb{C} \setminus \{\gamma\} =$ open $\Rightarrow$ V open.

- V unbounded. In fact, by Lecture 7, $\exists\ R >> 0$ with $\{ |z| > R \} \subseteq V$.

Define $h : \mathbb{C} \longrightarrow \mathbb{C}$

$$h(z) = \begin{cases} \int_\gamma \varphi(z,w)\, dw, & z \in \mathcal{U}. \\ \int_\gamma \frac{f(w)}{w-z}\, dw, & z \in V \end{cases}$$

Claims [a] h well-defined

[b] h bounded, $\lim_{z \to \infty} h(z) = 0$

[c] h entire

Conclusion By Liouville $\Longrightarrow$ h constant $\overset{[b]}{\Longrightarrow}$ $h \equiv 0$.

Thus if $z \in \mathcal{U} \Longrightarrow h(z) = \int_\gamma \varphi(z,w)\, dw = 0 \Longrightarrow (*)$.

QED.

Proof of [a] h well-defined. Take $z \in U \cap V$. We show

$$\int_\gamma \varphi(z,w)\, dw = \int_\gamma \frac{f(w)}{w-z}\, dw.$$

$$\Longleftrightarrow \int_\gamma \frac{f(z)}{w-z}\, dw = 0 \Longleftrightarrow f(z)\, n(\gamma, z) = 0$$ which is

true since $n(\gamma, z) = 0$ for $z \in V$.

Proof of $\boxed{b}$

Let $K > 0$ such that $\{\gamma\} \subseteq \Delta(0, K)$ by compactness.

We have $|w - z| \geq |z| - |w| \geq |z| - K$ if $w \in \{\gamma\}$.

If $R >> 0$, $|z| \geq R \Rightarrow z \in V$. then

$$|h(z)| = \left| \int_\gamma \frac{f(w)}{w - z} \, dw \right| \leq \underbrace{\operatorname{length}(\gamma) \cdot \sup_{\{\gamma\}} |f| \cdot \frac{1}{|z| - K}}_{\downarrow \atop 0 \text{ as } z \to \infty.}$$

Since h is continuous by $\boxed{c}$ $\Rightarrow$ h bounded.

why?

- $\lim_{z \to \infty} h(z) = 0 \Rightarrow \exists \alpha, \; |h(z)| \leq 1$ if $|z| \geq \alpha$
- h continuous $\Rightarrow \exists M, \; |h(z)| \leq M$ if $|z| \leq \alpha$

$\Rightarrow$

$\Rightarrow |h| \leq \max(1, M).$

Proof of $\boxed{c}$ **(h) entire**

Recall Conway Exercise IV. 2. 2. / HWK3 #7.

Key statement $\varphi : U \times \{\gamma\} \longrightarrow \mathbb{C}$

- φ continuous
- $z \longrightarrow \varphi(z, w)$ holomorphic $\forall\, w \in \{\gamma\}$.

Then $g(z) = \int_\gamma \varphi(z,w)\, dw$ holomorphic.

Proof See Solution Set 3.

Alternatively, let $\bar{R} \subseteq U$. Then

$$\int_{\partial R} g\, dz = \int_{\partial R}\int_\gamma \varphi(z,w)\, dw\, dz$$

Fubini's theorem
φ continuous

$$= \int_\gamma \underbrace{\int_{\partial R} \varphi(z,w)\, dz}_{0}\, dw$$

Goursat's lemma or Cauchy

$$= \int_\gamma 0\, dw = 0$$

$\Rightarrow g$ admits a primitive in any disc $\Delta \subseteq U$, $g = G'$

$\Rightarrow g$ holomorphic (G = holomorphic = ∞-many times differentiable)

Back to $\boxed{C}$. Apply Key Statement to

- the set $\mathcal{U}$, for $\varphi = \phi \Rightarrow h$ holomorphic in $\mathcal{U}$
- the set V, for $\varphi(z, w) = \dfrac{f(w)}{w - z} : V \times \{\gamma\} \longrightarrow \mathbb{C}$

$\Rightarrow h$ holomorphic in V.

Thus h is entire. QED.

2. Applications of the Residue theorem to real analysis

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_{s\in S} \mathrm{Res}(f,s)\cdot n(\gamma,s), \quad \gamma \overset{u.}{\approx} 0$$

Applications

[a] trigonometric functions

[b] rational functions

[c] Fourier integrals

[d] logarithmic integrals

[e] Mellin transforms

Poisson: "Je n'ai remarqué aucune intégrale qui ne fût pas déjà connue"

$\boxed{a}$ Trigonometric integrals

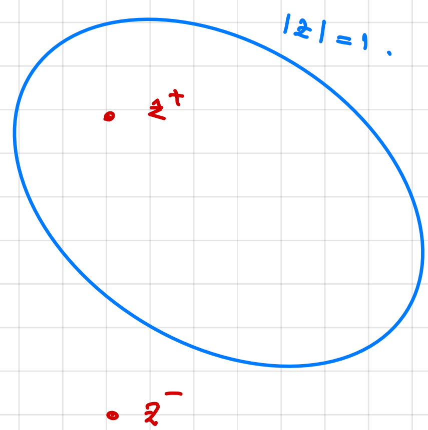
Example $a > 1$, $a \in \mathbb{R}$. $\quad I = \displaystyle\int_0^{2\pi} \frac{dt}{a + \sin t}$.

$$z = e^{it} \implies \frac{dz}{iz} = dt$$

$$\sin t = \frac{z - z^{-1}}{2i}.$$

By substitution, we find

$$I = \int_{|z|=1} \frac{2\, dz}{z^2 - 1 + 2aiz}.$$



poles $z^2 - 1 + 2aiz = 0$

$$\implies z^{\pm} = -ai \pm i\sqrt{a^2-1}$$

Note $|z^+| < 1$, $|z^-| > 1$. Thus

Method 1

$$I = 2\pi i \operatorname{Res}(f, z^+) = 2\pi i \cdot \frac{2}{(z^2 - 1 + 2aiz)'}\Bigg|_{z = z^+},$$

$$= 2\pi i \cdot \frac{2}{2z + 2ai}\Bigg|_{z=z^+} = \boxed{\frac{2\pi}{\sqrt{a^2-1}}}.$$

Math 220 A – Lecture 17

November 20, 2020

Office hour next week : Tuesday 2-3:30 PM.

Applications of the Residue theorem to real analysis

- a trigonometric functions
- b rational functions
- c Fourier integrals
- d logarithmic integrals
- e Mellin transforms

[6] Rational functions $I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx.$,

Require : (1) Q has no zeros on the real axis

(2) $\deg P - \deg Q \leq -2.$

Claim : The integral converges absolutely

Write $f(x) = \frac{P(x)}{Q(x)}$.

By (2) $\Rightarrow \lim_{|x| \to \infty} x^2 f(x) = \alpha \Rightarrow \exists\, R > 0$ with

$$|f(x)| < \frac{\alpha + 1}{x^2} \quad \text{for } |x| > R. \qquad (*)$$

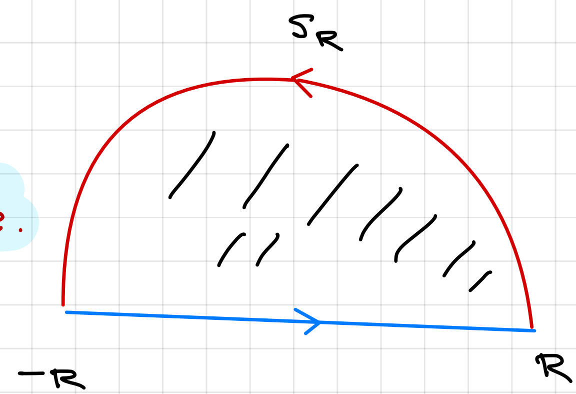
By the comparison test $\Rightarrow \int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$ QED.

Conclusion $I = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx.$

Strategy

$\boxed{\text{I}}$ $f(z) = \dfrac{P(z)}{Q(z)}$

$$\gamma_R = [-R, R] \cup S_R .$$



$\boxed{\text{II}}$ Residue theorem

$$\int_{-R}^{R} f(x)\,dx + \int_{S_R} f\,dz = \int_{\gamma_R} f\,dz = 2\pi i \sum_{\substack{a_j \in \mathcal{H}^+ \\ |a_j| < R.}} \operatorname{Res}(f, a_j).$$

$\boxed{\text{III}}$ Make $R \to \infty$. Show $\displaystyle\lim_{R\to\infty} \int_{S_R} f\,dz = 0$

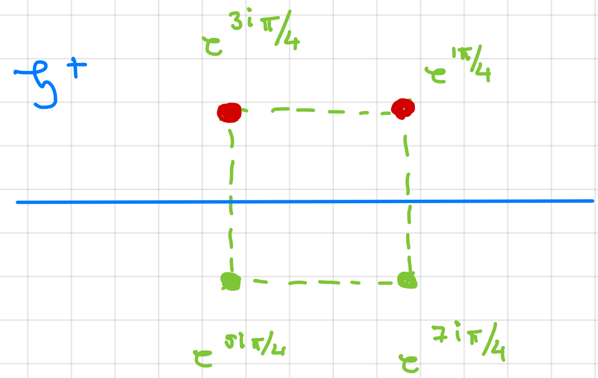
$$\left| \int_{S_R} f\,dz \right| \leq \pi R \cdot \frac{\alpha + 1}{R^2} \longrightarrow 0 \text{ as } R \to \infty. \text{ using } (*).$$

From $\boxed{\text{II}}$, we obtain

Conclusion

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a_j \in \mathcal{H}^+} \operatorname{Res}\left(\frac{P}{Q}, a_j\right).$$

Example $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$



Poles at $z^4+1=0$

$$\Rightarrow \quad z_k = e^{\frac{\pi i}{4}(2k+1)}, \quad k=0,1,2,3.$$

Only $e^{\pi i/4}$, $e^{3\pi i/4} \in \mathfrak{H}^+$.

By Method 1,

$$\operatorname*{Res}_{z=z_k} \frac{1}{z^4+1} = \frac{1}{4z^3}\Big/_{z=z_k} = \frac{1}{4z_k^3} = -\frac{z_k}{4}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \left(\operatorname*{Res}_{z=e^{\pi i/4}} \frac{1}{z^4+1} + \operatorname*{Res}_{z=e^{3\pi i/4}} \frac{1}{z^4+1} \right)$$

$$= 2\pi i \left(-\frac{1}{4} e^{\pi i/4} - \frac{1}{4} e^{3\pi i/4} \right)$$

$$= \frac{\pi}{\sqrt{2}}.$$

[C] Fourier Integrals I

$$I = \int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx \quad \text{(use upper half plane)}$$

$$I = \int_{-\infty}^{\infty} f(x)\, e^{-ix}\, dx \quad \text{(use lower half plane)}$$

Require (1) f extends meromorphically to $\mathfrak{H}^+$

(2) no poles on the real axis.

(3) $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

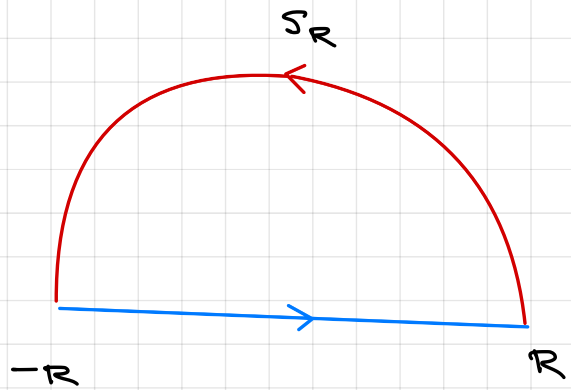
Convergence: By (3), $\int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx$ converges absolutely

Thus

$$I = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, e^{ix}\, dx.$$

Strategy Use the same contour

$$\gamma_R = [-R, R] \cup S_R.$$



By the residue theorem

$$\int_{-R}^{R} f(x)\, e^{ix} dx + \int_{S_R} f \, dz = \int_{\gamma_R} f \, dz = 2\pi i \sum_{z = a_j} \operatorname{Res}\left(f(z) e^{iz}\right)$$

Make $R \to \infty$. Assume moreover

$$(4) \quad \lim_{\substack{z \to \infty \\ z \in \overline{\mathfrak{H}^+}}} f(z) = 0 .$$

The next lemma shows $\lim_{R \to \infty} \int_{S_R} f \, dz = 0$.

Conclusion

$$\int_{-\infty}^{\infty} f(x)\, e^{ix} dx = 2\pi i \sum_{a_j \in \mathfrak{H}^+} \operatorname{Res}\left(f(z)\, e^{iz}, a_j\right).$$

Lemma If $\lim\limits_{\substack{z\to\infty \\ z\in \overline{\mathbb{H}}^+}} |f(z)| = 0$ then

$$\lim_{R\to\infty} \int_{S_R} f(z)\, e^{iz}\, dz = 0$$

Proof Write $z = Re^{it}$, $0 \le t \le \pi$.

$$M_R = \sup_{z\in S_R} |f(z)| \,, \quad M_R \longrightarrow 0 \text{ as } R \longrightarrow \infty$$

$$\left| \int_{S_R} f(z)\, e^{iz}\, dz \right| = \left| \int_0^{\pi} f(Re^{it})\, e^{iRe^{it}} \cdot iRe^{it}\, dt \right|$$

$$\le \int_0^{\pi} M_R \cdot \left| e^{iRe^{it}} \right| \cdot R\, dt$$

$$= \int_0^{\pi} M_R \left| e^{iR(\cos t + i\sin t)} \right| R\, dt$$

$$= \int_0^{\pi} R M_R \left| e^{iR\cos t}\, e^{-R\sin t} \right| dt$$

$$= \int_0^{\pi} R M_R\, e^{-R\sin t}\, dt$$

$$= 2\int_0^{\pi/2} R M_R\, e^{-R\sin t}\, dt$$

Claim

$$\le 2\int_0^{\pi/2} R M_R\, e^{-R\cdot\frac{2}{\pi}t} = \pi M_R (1 - e^{-R}) \longrightarrow 0$$

as $R \longrightarrow \infty$.

Claim $\frac{2}{\pi} \leq \frac{\sin t}{t} \quad \forall\, t \in (0, \frac{\pi}{2}]$

Proof $f(t) = \frac{\sin t}{t}$.

$$f(\pi/2) = \frac{2}{\pi}$$

We show f is decreasing. Then $f(t) \geq f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \Rightarrow$

$\Rightarrow \frac{\sin t}{t} \geq \frac{2}{\pi}$

To this end, compute $f'(t) = \dfrac{t\cos t - \sin t}{t^2} \leq 0$

$$\Longleftrightarrow \quad t\cos t \leq \sin t$$

$$\Longleftarrow \quad \tan t - t \geq 0.$$

Let

$$g(t) = \tan t - t, \quad g(0) = 0$$

We compute $g'(t) = \dfrac{1}{\cos^2 t} - 1 \geq 0 \Rightarrow g \nearrow \Rightarrow$

$\Rightarrow g(t) \geq g(0) = 0$ as needed. QED

Example $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \operatorname{Re} I. = \frac{\pi}{e}.$

Let $f(z) = \dfrac{1}{1+z^2}$, $z = i$ is the only pole in $\mathcal{H}^+$.

Method 1

$$I = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{1+z^2}\right)\Big|_{z=i} =$$

$$= 2\pi i \cdot \left.\frac{e^{iz}}{2z}\right|_{z=i}$$

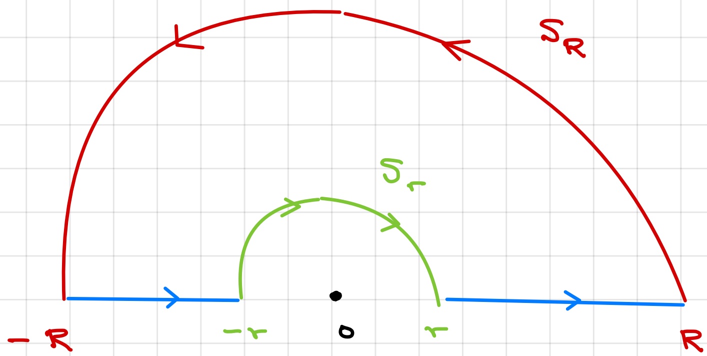
$$= \cancel{2}\pi \cancel{i} \cdot \frac{e^{-1}}{\cancel{2}\cancel{i}} = \frac{\pi}{e}.$$

Fourier Integrals – Part II. – Poles on the real axis

Example $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

- issues at 0 & ∞.

- $\displaystyle I = \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{\sin x}{x}\,dx$



Strategy

- $f(z) = \dfrac{e^{iz}}{z}$

- $\gamma_R = S_R + [-R,-r] + (-S_r) + [r,R]$

$$0 = \int_{\gamma_R} f\,dz = \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R \frac{e^{iz}}{z}\,dz + \int_{-R}^{-r} \frac{e^{iz}}{z}\,dz$$

$$= \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R \frac{e^{iz}}{z} - \frac{e^{-iz}}{z}\,dz$$

$$= \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R 2i\,\frac{\sin z}{z}\,dz$$

Make $r\longrightarrow 0$, $R\longrightarrow\infty$. By the claim:

$$0 = 0 - i\pi + 2i\int_0^\infty \frac{\sin x}{x}\,dx \Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Claims $\boxed{a}$ $$\lim_{R\to\infty} \int_{S_R} \frac{e^{iz}}{z}\, dz = 0$$

$\boxed{b}$ $$\lim_{r\to 0} \int_{S_r} \frac{e^{iz}}{z}\, dz = i\pi$$

Claim $\boxed{a}$ follows from the previous Lemma applied to $f(z) = \frac{1}{z}$.

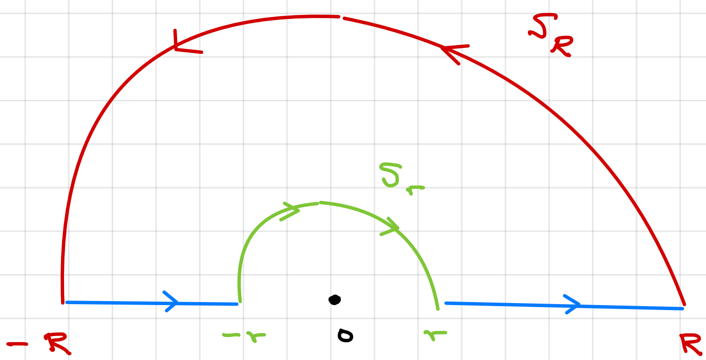
Claim $\boxed{b}$ requires a proof. We will go over the proof next time.

Math 220 A — Lecture 18

November 23, 2020

Fourier Integrals – Part II – Poles on the real axis

Example $I = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$



$f(z) = \frac{e^{iz}}{z}$.

$\gamma = S_R + [-R, -r] + (-S_r) + [r, R]$

$$\underbrace{\int_{S_R} f\,dz}_{\downarrow\; 0} - \underbrace{\int_{S_r} f\,dz}_{\downarrow\; i\pi} + \underbrace{2i\int_r^R \frac{\sin z}{z}\,dz}_{\downarrow\; 2i I} = \int_\gamma f\,dz = 0$$

as $r \to 0$, $R \to \infty$.

Claims [a] $\lim_{R\to\infty} \int_{S_R} \frac{e^{iz}}{z}\,dz = 0$

[b] $\lim_{r\to 0} \int_{S_r} \frac{e^{iz}}{z}\,dz = i\pi$

$\Rightarrow \quad I = \frac{\pi}{2}$.

Part $\boxed{a}$ is a consequence of Lemma last time:

for $g(z) = \frac{1}{z}$:

Lemma If $\lim\limits_{\substack{z \to \infty \\ z \in \overline{\mathbb{H}}^+}} |g(z)| = 0$ then

$$\lim_{R \to \infty} \int_{S_R} g(z) e^{iz} \, dz = 0$$

Part $\boxed{b}$ uses the next lemma for $g(z) = \frac{1}{z}$

Lemma Let g have simple pole at 0. Then

$$\lim_{r \to 0} \int_{S_r} g(z) e^{iz} \, dz = \pi i \operatorname{Res}(g, 0).$$

Proof Since g has a simple pole at 0, write

Taylor series

$$g(z) = \frac{\alpha}{z} + G(z)$$

$\alpha = \operatorname{Res}(g, 0)$, G holomorphic near 0

Taylor

$$e^{iz} = 1 + zF(z), \quad F \text{ holomorphic near } 0$$

$$e^{iz} g(z) = \left(\frac{\alpha}{z} + G\right)\left(1 + zF\right) = \frac{\alpha}{z} + H,$$

$H = G + zFG + \alpha F$ holomorphic near 0

$\Rightarrow$ H bounded near 0. $\Rightarrow \exists M, \delta : |H(z)| \leq M$ if $|z| \leq \delta$.

Compute $$\int_{S_r} e^{iz} g(z)\, dz = \int_{S_r} \frac{\alpha}{z} + H \, dz.$$

Note $$\alpha \int_{S_r} \frac{dz}{z} = \alpha \int_0^\pi \frac{d(re^{it})}{re^{it}} = \alpha \int_0^\pi i\, dt = \pi i \alpha$$

$$\left| \int_{S_r} H\, dz \right| \leq M \cdot \pi r \longrightarrow 0 \quad \text{as } r \to 0.$$

Thus $$\int_{S_r} e^{iz} g(z)\, dz \longrightarrow \pi i \alpha \quad \text{as } r \to 0, \text{ as claimed.}$$

Applications of the Residue theorem to real analysis

- a) trigonometric functions
- b) rational functions
- c) Fourier integrals
- d) logarithmic integrals
- e) Mellin transforms

d) Logarithmic integrals

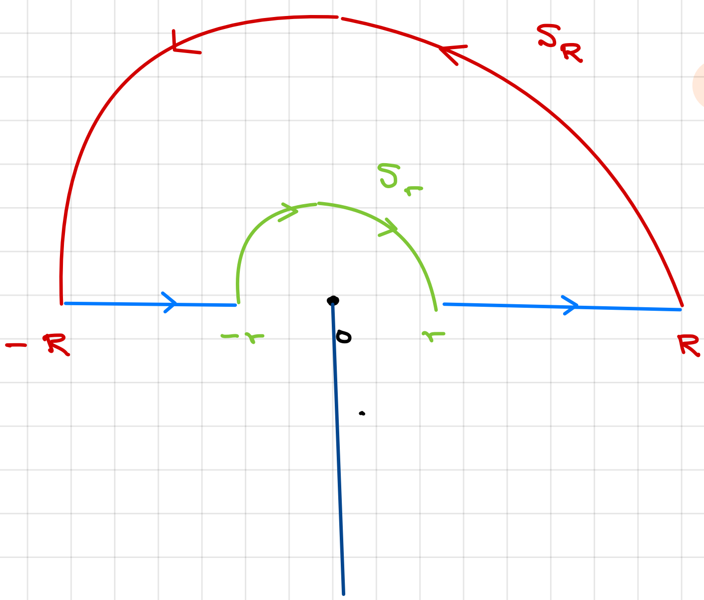
$$\int_0^\infty R(x)\,\log x\,dx$$

R = even rational function without real poles

Example $R(x) = \dfrac{1}{1+x^2} \Longrightarrow \displaystyle\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$

HWK $R(x) = \dfrac{1}{(1+x^2)^2} \Longrightarrow \displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx.$

Issues:
- logarithm undefined at 0 (use circle S_r)
- holomorphic extension for logarithm



Requires branch cut!

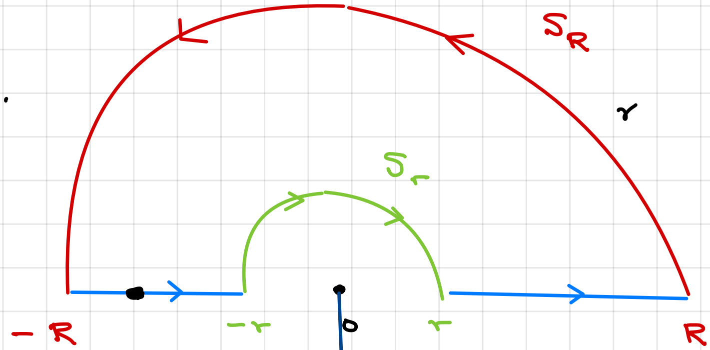
Define for $z = re^{it}$

$$\ell(z) = \log r + it$$

$$-\frac{\pi}{2} < t < \frac{3\pi}{2}$$

- $\gamma = S_R + [-R, -r] + (-S_r) + [r, R]$

- $f(z) = \dfrac{\ell(z)}{1+z^2}$ has pole at i



Residue theorem

Method 1

$$\operatorname{Res}(f, i) = \operatorname*{Res}_{z=i} \frac{\ell(z)}{1+z^2} = \left.\frac{\ell(z)}{2z}\right|_{z=i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}.$$

Residue thm:

$$\int_\gamma f\,dz = 2\pi i \operatorname{Res}(f, i) = i\,\frac{\pi^2}{2}. \qquad (*).$$

$$\int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R f(x)\,dx + \int_{-R}^{-r} f(x)\,dx.$$

We make $r \to 0$, $R \to \infty$.

Segment integrals

see the definition of ℓ

$$\int_r^R \frac{\ell(x)}{1+x^2}\,dx + \int_{-R}^{-r} \frac{\ell(x)}{1+x^2}\,dx = \int_r^R \frac{\log x}{1+x^2}\,dx + \int_{-R}^{-r} \frac{\log|x| + i\pi}{1+x^2}\,dx$$

$$= 2\int_r^R \frac{\log x}{1+x^2}\,dx + i\pi \int_{-R}^{-r} \frac{dx}{1+x^2}$$

$$\xrightarrow[R\to\infty]{r\to 0} 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi\, \arctan x \Big|_{x=-\pi}^{x=0}$$

$$= 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi \cdot \frac{\pi}{2}$$

Claim

$$\lim_{\substack{\rho\to 0\\ \rho\to\infty}} \int_{S_\rho} \frac{\ell(z)}{1+z^2}\,dz = 0.$$

Conclusion From $(*)$ we get as $r\to 0$, $R\to\infty$:

$$\frac{i\pi^2}{2} = 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi \cdot \frac{\pi}{2}$$

$$\Longrightarrow \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$$

Proof of the claim $\quad z = \rho e^{it}, \quad 0 \le t \le \pi$

$$\left| \int_{S_\rho} \frac{\ell(z)}{1+z^2}\, dz \right| = \left| \int_0^\pi \frac{\log\rho + it}{1+\rho^2 e^{2it}} \cdot \rho e^{it}\, i\, dt \right|$$

$$\le \int_0^\pi \frac{|\log\rho| + \pi}{|1+\rho^2 e^{2it}|} \cdot \rho \, dt$$

$$\le \int_0^\pi \frac{|\log\rho| + \pi}{|\rho^2 - 1|} \cdot \rho \, dt$$

$$= \pi \cdot \frac{\rho\, |\log\rho|}{|\rho^2-1|} + \pi^2 \cdot \frac{\rho}{|\rho^2-1|} \longrightarrow 0 .$$

As $\rho \to \infty$, $\dfrac{\rho \log \rho}{\rho^2 - 1}$ and $\dfrac{\rho}{\rho^2-1} \longrightarrow 0$.

As $\rho \to 0$, the same is true.

The only term that requires justification is

$$\rho \log \rho = -\frac{w}{e^{w}} \longrightarrow 0 \text{ as } w \to \infty, \quad \text{where } \rho = e^{-w}, \ \rho \to 0.$$

Applications of the Residue theorem to real analysis

- [a] trigonometric functions
- [b] rational functions
- [c] Fourier integrals
- [d] logarithmic integrals
- [e] Mellin transforms

[e] Mellin transforms

$$\int_0^\infty \frac{R(x)}{x^\alpha}\,dx \quad , \quad 0 < \alpha < 1$$

R = rational function, no poles on positive real axis

Useful in prime counting.

Example $\quad R(x) = \dfrac{1}{x+1} \implies \displaystyle\int_0^\infty \frac{dx}{x^\alpha (x+1)} = \frac{\pi}{\sin \pi\alpha}$

(next time)

Homework $\quad R(x) = \dfrac{1}{x^n+1} \implies \displaystyle\int_0^\infty \frac{dx}{x^\alpha (x^n+1)}$

Remark

(i) Fourier transform

$$f \rightsquigarrow \mathcal{F}f(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x s}\, dx$$

(ii) Laplace transform

$$f \rightsquigarrow \mathcal{L}f(s) = \int_{0}^{\infty} f(x)\, e^{-sx}\, dx$$

(iii) Mellin transform

$$f \rightsquigarrow \mathcal{M}f(s) = \int_{0}^{\infty} f(x)\, \underbrace{x^{s-1}}_{x^{-\alpha} \text{ on previous page}}\, dx$$

Remark (will not use)

The Mellin transform of $f(x) = e^{-x}$ is known as Γ - function

$$\Gamma(s) = \int_{0}^{\infty} e^{-x} x^{s-1}\, dx$$

Hjalmar Mellin (1854 – 1933)

Finnish mathematician

Math 220 A — Lecture 19

November 25, 2020

Applications of the Residue theorem to real analysis

a. trigonometric functions

b. rational functions

c. Fourier integrals

d. logarithmic integrals

e. Mellin transforms

Mellin transforms: $\displaystyle\int_0^\infty \frac{R(x)}{x^\alpha}\,dx$

R = rational function, no poles on positive real axis

Example $\quad R(x) = \dfrac{1}{x+1} \implies \boxed{\displaystyle\int_0^\infty \frac{dx}{x^\alpha(x+1)} = \frac{\pi}{\sin \pi\alpha}}$

for $0 < \alpha < 1$

Homework $\quad R(x) = \dfrac{1}{x^n+1} \implies \displaystyle\int_0^\infty \frac{dx}{x^\alpha(x^n+1)}$

Convergence uses $0 < \alpha < 1$.

- $0 < x < 1$: $\displaystyle\int_0^1 \frac{dx}{x^\alpha(x+1)} < \int_0^1 \frac{dx}{x^\alpha} = \frac{x^{-\alpha+1}}{-\alpha+1}\Bigg|_{x=0}^{x=1} < \infty$

- $1 < x < \infty$: $\displaystyle\int_1^\infty \frac{dx}{x^\alpha(x+1)} < \int_1^\infty \frac{dx}{x^{\alpha+1}} = \frac{x^{-\alpha}}{-\alpha}\Bigg|_{x=1}^{x=\infty} < \infty$

$$I = \int_0^\infty \frac{dx}{x^\alpha(x+1)} = \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{dx}{x^\alpha(x+1)}.$$

$$I = \int_0^\infty \frac{dx}{x^\alpha (x+1)}$$

Question : [a] What function ?

[b] What contour ?

Issues [i] extend x^α holomorphically

$z^\alpha = \exp(\alpha\, \ell(z)) \rightsquigarrow$ branch cut along $[0, \infty)$.

For $z = re^{it}$, $\ell(z) = \log r + it$, $0 < t < 2\pi$

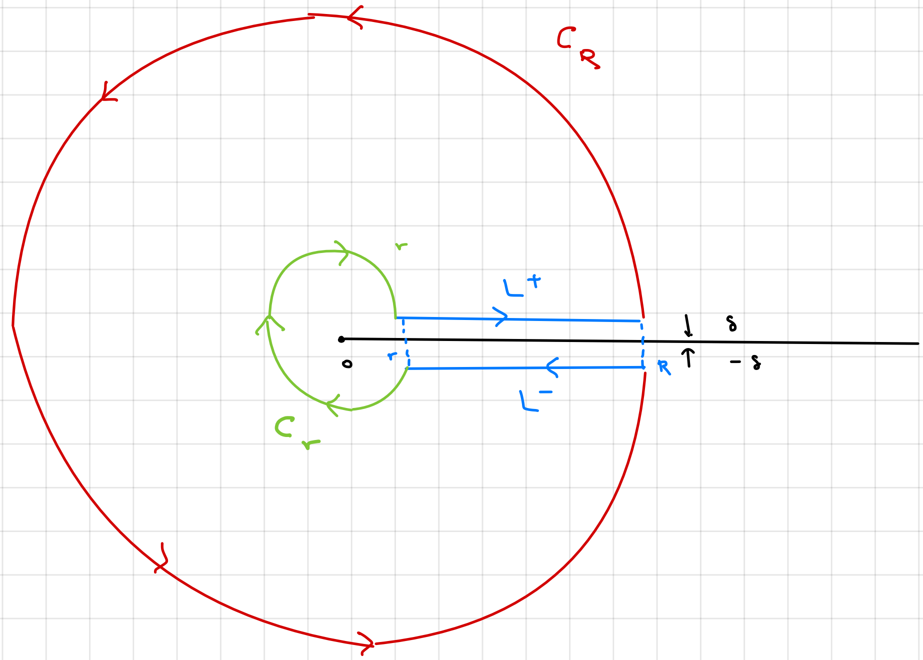
[ii] pole at 0 — use C_r to isolate the pole

Remark It is precisely the fact that we cut along $[0, \infty)$ (= domain of integration) that allows us to calculate I.

Before, we were cutting away from domain of integration

Solutions (a) $f(z) = \dfrac{1}{z^{\alpha}(z+1)}$

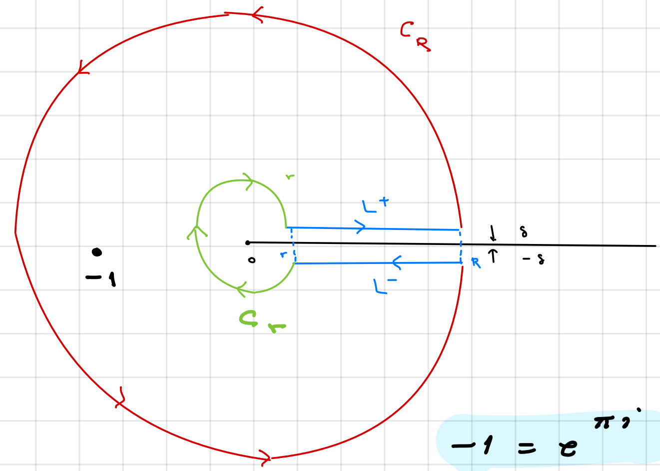
(b) γ = key-hole contour



$$\gamma = C_R + (-L^-) + (-C_r) + L^+$$

Residue theorem

$$f(z) = \frac{1}{z^{\alpha}(z+1)} \quad . \quad \text{pole at } -1.$$



$-1 = e^{\pi i}$

Method 1

$$\operatorname{Res}(f, -1) = \operatorname*{Res}_{z=-1} \frac{\frac{1}{z^{\alpha}}}{z+1} = \frac{1}{(-1)^{\alpha}} = \frac{1}{e^{\pi i \alpha}} = e^{-\pi i \alpha}$$

$$\int_{\gamma} f\, dz = 2\pi i \operatorname{Res}(f, -1) = 2\pi i \exp(-\alpha \pi i).$$

$$\int_{C_R} f\, dz - \int_{C_r} f\, dz + \int_{L^+} f\, dz - \int_{L^-} f\, dz \qquad (R).$$

Make $r \to 0$, $R \to \infty$, $\delta \to 0$.

Claim $\boxed{a}$ $\lim_{\substack{\rho \to 0 \\ \rho \to \infty}} \int_{C_\rho} \frac{dz}{z^\alpha (z+1)} = 0$

Claim $\boxed{b}$ $\lim_{\substack{\delta \to 0 \\ r \to 0 \\ R \to \infty}} \int_{L^+} \frac{dz}{z^\alpha (z+1)} = I$

$\boxed{c}$ $\lim_{\substack{\delta \to 0 \\ r \to 0 \\ R \to \infty}} \int_{L^-} \frac{dz}{z^\alpha (z+1)} = e^{-2\pi i \alpha} I$

Conclude In (R) make $\delta \to 0$, $r \to 0$, $R \to \infty$:

$$0 - 0 + I - e^{-2\pi i \alpha} I = e^{-\pi i \alpha} \cdot 2\pi i$$

$$I = \frac{2\pi i \; e^{-\pi i \alpha}}{1 - e^{-2\pi i \alpha}} = \frac{2\pi i}{e^{\pi i \alpha} - e^{-\pi i \alpha}} = \frac{\pi}{\sin \pi \alpha}.$$

Proof of $\boxed{a}$

$$\left| \int_{C_\rho} \frac{dz}{z^\alpha (z+1)} \right| \le 2\pi\rho \cdot \frac{1}{\rho^\alpha |\rho - 1|} \longrightarrow 0$$

as $\rho \to 0$ or $\rho \to \infty$. because $0 < \alpha < 1$.

Proof of $\boxed{b}$ $\quad g(z) = \frac{1}{z^{-\alpha}}$, $L^+ = \{ t + i\delta : r \le t \le R \}$.

$$\lim_{\delta \to 0} \int_{L^+} \frac{g(z)}{z+1} dz \overset{(*)}{=} \int_r^R \frac{t^{-\alpha}}{t+1} dt \longrightarrow I.$$

as $r \to 0$
$R \to \infty$.

Why $(*)$?

$$\int_{L^+} \frac{g(z)}{z+1} dz = \int_r^R \frac{g(t+i\delta)}{1+t+i\delta} dt \xrightarrow{\delta \to 0} \int_r^R \frac{t^{-\alpha}}{1+t} dt.$$

Define

$$G(t,\delta) = \begin{cases} \dfrac{g(t+i\delta)}{1+t+i\delta} - \dfrac{t^{-\alpha}}{1+t}, & \delta \ne 0. \\ 0, & \delta = 0. \end{cases}$$

$r \le t \le R$, $\quad 0 \le \delta \le 1$.

G continuous (uniformly). Given any ε, $\exists\, \tau > 0$ such that if

$$|\delta - 0| < \tau,\ |t - t| < \tau \Rightarrow |G(t,\delta) - \underbrace{G(t,0)}_{0}| < \varepsilon.$$

$$\Rightarrow |G(t,\delta)| < \varepsilon. \Rightarrow \left|\int_r^R G(t,\delta)\,dt\right| \leq (R-r)\cdot\varepsilon \text{ as } |\delta| < \tau.$$

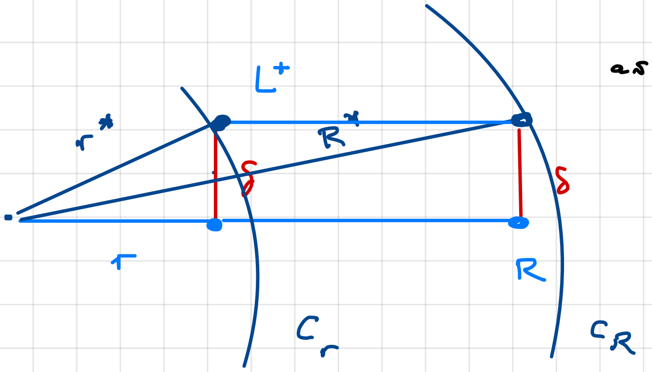
Remark If we wish to parametrize L^+ by $r \leq t \leq R$,

we'd need to use circles C_R, C_r of radii

$R^* = \sqrt{R^2+\delta^2}$, $r^* = \sqrt{r^2+\delta^2}$. The argument in $\boxed{\iota}$

still applies since $r^* \to 0$, $R^* \to \infty$.

as $r \to 0$, $\delta \to 0$, $R \to \infty$.



Proof of $\boxed{c}$

Difference $g(t - i\delta) \longrightarrow t^{-\alpha} \cdot e^{-2\pi i\alpha}$.

The rest of the proof is the same as $\boxed{b}$.

Indeed $g(t - i\delta) = (t - i\delta)^{-\alpha} = \exp(-\alpha\, \ell(t - i\delta))$

$$\xrightarrow{\delta \to 0} \exp(-\alpha \log t - 2\pi i\alpha) = t^{-\alpha} \cdot e^{-2\pi i\alpha}.$$

This explains the extra factor $e^{-2\pi i\alpha}$ in the answer to $\boxed{c}$.

II. Residues at ∞ & Shadows of Riemann Surfaces

[A]. Topology on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

- basic neighborhood of ∞

$$\mathcal{U} = \{\infty\} \cup \{ |z| > R \} \text{ for some } R.$$

- $\hat{\mathbb{C}}$ is a topological space

- $\hat{\mathbb{C}}$ compact

Remark $f: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, $z \longrightarrow \dfrac{1}{z}$, $f(0) = \infty$

$f(\infty) = 0$

(punctured) neighborhoods of 0 $\qquad |z| < r$

$z \longrightarrow \dfrac{1}{z}$.

$\left|\dfrac{1}{z}\right| > \dfrac{1}{r}$

(punctured) neighborhoods of ∞

$\boxed{B.}$ Singularities & Residues at ∞

Recall Conway $\overline{V}$. 1.13 — Pset 5 / Problem 6

If $f : \{|z| > R\} \longrightarrow \mathbb{C}$ holomorphic

$\Longrightarrow$ ∞ is isolated singularity

Types

$\boxed{i}$ removable

$\boxed{ii}$ pole $\Longrightarrow$ $g(z) = f\left(\frac{1}{z}\right)$.

$\boxed{iii}$ essential Inspect singularity at 0.

Example $f(z) = \dfrac{z^5 + 2}{z - 1}$. $\rightarrow$ poles at $1, \infty \in \hat{\mathbb{C}}$.

has a pole at $z = 1$. Inspect ∞.

$$g(z) = f\left(\frac{1}{z}\right) = \frac{\frac{1}{z^5} + 2}{\frac{1}{z} - 1} = \frac{1 + 2z^5}{1 - z} \cdot \frac{1}{z^4} \quad \text{pole at } z = 0$$

$\Longrightarrow$ f has a pole at ∞.

Residue at ∞ $\quad \mathrm{Res}(f,\infty) = ?$

Beware

$$\mathrm{Res}(f,\infty) \neq \mathrm{Res}(g,0).$$

Instead

$$\mathrm{Res}(f,\infty) := -\frac{1}{2\pi i}\int_{|z|=\rho} f\, dz \qquad \text{where } \rho > R.$$

By Homotopy Cauchy this does not depend on ρ.

Question

Why do we care about the residue at ∞?

Homework Example

$$\int_{|z|=5} \frac{z^3}{(1-z)(2-z)(3-z)(4-z)}\, dz = -2\pi i\, \mathrm{Res}(\_\_, \infty).$$

This is better than computing 4 different residues.

Next time we will answer the following:

Question How do we calculate the residue at ∞?

Math 220 A — Lecture 20

November 30, 2020

Last time — Residue at ∞

If $f : \{|z| > R\} \to \mathbb{C}$ holomorphic, ∞ isolated singularity

$$\Updownarrow$$

$g : \Delta^*(0, \frac{1}{R}) \to \mathbb{C}$, $g(z) = f(\frac{1}{z})$, 0 isolated singularity

Beware

$$\mathrm{Res}(f, \infty) \neq \mathrm{Res}(g, 0).$$

Instead define

$$\mathrm{Res}(f, \infty) := -\frac{1}{2\pi i} \int_{|z| = \rho} f \, dz \qquad \text{where } \rho > R.$$

By Homotopy Cauchy this does not depend on $\rho > R$

Example

$$\int_{|z|=11} \frac{z^9 \, dz}{(z-1) \cdots (z-10)} = -2\pi i \, \mathrm{Res}(\text{—}, \infty).$$

Question How do we compute the residue at ∞?

Answer $$\operatorname{Res}(f, \infty) = -\operatorname*{Res}_{w=0}\left(g(w) \cdot \frac{1}{w^2}\right)$$

Proof Let ρ be sufficiently large. Then

$z = \frac{1}{w}$.

$$\operatorname{Res}(f, \infty) = -\frac{1}{2\pi i}\int_{|z|=\rho} f \, dz = \qquad dz = -\frac{dw}{w^2}.$$

$$= \frac{1}{2\pi i}\int_{|w|=1/\rho} g \, \frac{-dw}{w^2} \quad \text{(change variables)}$$

(the change of orientation yields an extra sign).

$$= \operatorname*{Res}_{w=0}\left(g(w) \cdot \frac{-1}{w^2}\right)$$

using the usual residue theorem.

Residue Theorem for $\hat{\mathbb{C}}$

If f has isolated singularities only at $a_1, \ldots, a_k \in \mathbb{C}$ and possibly at ∞ then

$$\sum_{a \in \hat{\mathbb{C}}} \operatorname{Res}(f, a) = 0.$$

Proof Let ρ be large enough, $\rho > |a_j|$ for all j.

$$\operatorname{Res}(f, \infty) = -\frac{1}{2\pi i} \int_{|z|=\rho} f \, dz \qquad \text{(definition)}$$

$$= -\sum_j \operatorname{Res}(f, a_j) \qquad \text{(usual residue theorem)}$$

$$\Longrightarrow \sum_{a \in \hat{\mathbb{C}}} \operatorname{Res}(f, a) = 0.$$

Remark* This generalizes correctly to other compact Riemann surfaces.

Example $\quad f(z) = \dfrac{z^5+2}{z-1}.$

Last lecture, we saw that f has pole at $z=1$ and $z=\infty$.

$$\mathrm{Res}(f,1) \overset{\text{Method 1}}{=} \left.\frac{z^5+2}{(z-1)'}\right|_{z=1} = 3.$$

$$\mathrm{Res}(f,\infty) = \mathrm{Res}_{w=0}\left(g(w)\cdot\frac{-1}{w^2}\right)$$

$$z = 1/w \quad \Rightarrow g(w) = f\left(\frac{1}{w}\right) = \frac{1/w^5+2}{1/w - 1} = \frac{1+2w^5}{1-w}\cdot\frac{1}{w^4}.$$

Thus
$$\mathrm{Res}(f,\infty) = \mathrm{Res}_{w=0}\ \frac{1+2w^5}{1-w}\cdot\frac{1}{w^4}\cdot\frac{-1}{w^2}$$

$$= \mathrm{Coeff}_{w^5} - \frac{1+2w^5}{1-w}$$

$$= \mathrm{Coeff}_{w^5} -(1+2w^5)(1+w+w^2+w^3+w^4+w^5+\dots)$$

$$= -(1+2) = -3$$

This is consistent with the residue theorem on $\hat{\mathbb{C}}$.

Example (Lagrange)

$$f(z) = \frac{P(z)}{Q(z)} .$$ Assume that

- $\deg P = p$, $\deg Q = q$, $p \le q - 2$
- Q has simple roots $\alpha_1, \dots, \alpha_q$

f has poles at $\alpha_1, \dots \alpha_q$ and possibly at ∞.

- $$\operatorname{Res}(f, \alpha_i) \overset{\text{Method 1}}{=} \frac{P(\alpha_i)}{Q'(\alpha_i)}$$
- $\operatorname{Res}(f, \infty) = 0$ (next page).

Residue Theorem for $\hat{\mathbb{C}}$ $\Longrightarrow$ $$\sum_{i=1}^{q} \frac{P(\alpha_i)}{Q'(\alpha_i)} = 0$$

When $P(z) = z^p$, $Q(z) = \prod_{i=1}^{q} (z - \alpha_i)$, this gives

$$\sum_{i=1}^{q} \frac{\alpha_i^{\,p}}{\prod_{j \neq i} (\alpha_j - \alpha_i)} = 0 \qquad \forall\ p \le q-2.$$

$\forall\ \alpha_1, \dots \alpha_q$ distinct

Proof $\operatorname{Res}\left(\frac{P}{Q}, \infty\right) = 0$ if $p \le q-2$.

Write $P = a_0 z^p + \dots + a_p$, $a_0 \neq 0$

$Q = b_0 z^q + \dots + b_q$, $b_0 \neq 0$.

$$\operatorname{Res}\left(\frac{P}{Q}, \infty\right) = \operatorname*{Res}_{w=0}\left(\frac{a_0 \frac{1}{w^p} + a_1 \frac{1}{w^{p-1}} + \dots + a_p}{b_0 \frac{1}{w^q} + b_1 \frac{1}{w^{q-1}} + \dots + b_q} \cdot \frac{-1}{w^2}\right)$$

$$= \operatorname*{Res}_{w=0}\left(\frac{w^q}{w^p} \cdot \frac{a_0 + a_1 w + \dots + a_p w^p}{b_0 + b_1 w + \dots + b_q w^q} \cdot \frac{-1}{w^2}\right)$$

$$= -\operatorname*{Res}_{w=0}\left(w^{q-p-2} \cdot \frac{a_0 + a_1 w + \dots + a_p w^p}{b_0 + b_1 w + \dots + b_q w^q}\right)$$

$$= 0.$$

holomorphic near 0 since

$p+2 \le q$

Remark* (will not use)

Better to speak about residue of forms

$$f\,dz \quad \xleftrightarrow{\text{versus}} \quad f$$

Example $f(z) = \frac{1}{z}$. Clearly $\operatorname{Res}_z(f, 0) = 1$. But if we change coordinates

$$z = \lambda w \Rightarrow f = \frac{1}{\lambda w} \Rightarrow \operatorname{Res}_w(f, 0) = \frac{1}{\lambda}.$$

However if we work with forms, these issues are absent

$$f\,dz = \frac{dz}{z} = \frac{d(\lambda w)}{\lambda w} = \frac{dw}{w}.$$

Residues of forms are coordinate-independent!

This can be seen from $\operatorname{Res}(f, a) = \frac{1}{2\pi i} \int_{\partial\Delta(a,\varepsilon)} f\,dz$ using change of variables formula.

This independence applies to the residue at ∞ as well:

$$\operatorname*{Res}_{z=\infty}(f\,dz) = \operatorname*{Res}_{w=0}\left(g(w)\cdot d\left(\tfrac{1}{w}\right)\right) \qquad z = 1/w$$

$$= \operatorname*{Res}_{w=0}\left(g(w)\cdot \frac{-dw}{w^2}\right).$$

This justifies the choice of sign in the definition of the residue at ∞.

2. Applications of the Residue Theorem

[a] Argument Principle

[b] Rouché's Theorem

Conway V.3

Eugene Rouché

(1832 – 1910)

[a] The Argument Principle

Order $f : U \to \mathbb{C}$ meromorphic, $U \subseteq \mathbb{C}$, $a \in U$.

$$\mathrm{ord}(f, a) = \begin{cases} n, & a \text{ zero of order } n \\ -n, & a \text{ pole of order } n \\ 0, & \text{otherwise} \end{cases}$$

Remarks [i] $\mathrm{ord}(f, a) = n \iff f = (z-a)^n g$

where g holomorphic near a, $g(a) \neq 0$

This follows by inspecting the Taylor / Laurent expansion.

[ii] $\mathrm{ord}(fg, a) = \mathrm{ord}(f, a) + \mathrm{ord}(g, a)$

Indeed, let $\mathrm{ord}(f, a) = m$, $\mathrm{ord}(g, a) = n$.

Write $f = (z-a)^m F$, $g = (z-a)^n G$, $F(a), G(a) \neq 0$

$\Rightarrow fg = (z-a)^{m+n} FG$ with $FG(a) \neq 0$.

$\overset{[i]}{\Longrightarrow}$ $\mathrm{ord}(fg, a) = m + n = \mathrm{ord}(f, a) + \mathrm{ord}(g, a)$.

Math 220 A – Lecture 21

December 2, 2020

[0] Last time   Conway V.3.

$f : U \longrightarrow \mathbb{C}$ meromorphic, $U \subseteq \mathbb{C}$, $a \in U$.

Def

$$\operatorname{ord}(f, a) = \begin{cases} n, & a \text{ zero of order } n \\ -n, & a \text{ pole of order } n \\ 0, & \text{otherwise} \end{cases}$$

Remark $\operatorname{ord}(f, a) = k \iff f = (z-a)^k g$

where g holomorphic near a, $g(a) \neq 0$

This definition treats zeros & poles equally.

Question Find poles & residues of $\frac{f'}{f}$

Answer Poles of $\frac{f'}{f}$ come from zeros or poles of f.

Let a be a zero / pole with $\operatorname{ord}(f, a) = k$.

$\Longrightarrow f = (z-a)^k g$, g holomorphic, $g(a) \neq 0$.

$$\Longrightarrow \frac{f'}{f} = \frac{k(z-a)^{k-1} g + (z-a)^k g'}{(z-a)^k g} = \frac{k}{z-a} + \frac{g'}{g}$$

Since $g \neq 0$ near a $\Longrightarrow \frac{g'}{g}$ holomorphic near a

$\Longrightarrow \frac{f'}{f}$ has simple pole and

$$\operatorname{Res}\left(\frac{f'}{f}, a\right) = k = \operatorname{ord}(f, a)$$

$\boxed{1.}$ Argument Principle / Conway V. 3

Theorem Given f meromorphic in U, $\gamma \overset{U}{\sim} 0$, avoiding the zeros and poles of f, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_a n(\gamma, a)\,\mathrm{ord}\,(f, a)$$

This follows by the Residue Theorem & above discussion.

Remarks $\boxed{1}$ In practice, γ is a circle or a simple closed curve with $\mathrm{Int}\,\gamma \subseteq U$. Then

$$n(\gamma, a) = \begin{cases} 1, & a \in \mathrm{Int}\,\gamma \\ 0, & a \in \mathrm{Ext}\,\gamma \end{cases}$$

Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \#\text{ Zeroes} - \#\text{ Poles in } \mathrm{Int}\,\gamma.$$

(counted with multiplicity)

[ii] Why is it called "argument principle"?

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int_\gamma d\log f$$

$$= \frac{1}{2\pi i}\,\Delta \log f$$

$$= \frac{1}{2\pi \cancel{i}}\,\Delta\left(\underbrace{\cancel{\log|f|}}_{0} + \cancel{i}\,\mathrm{Arg} f\right)$$

$$= \frac{1}{2\pi}\,\Delta\,\mathrm{Arg} f$$

[iii] Enhanced version $g : U \to \mathbb{C}$ holomorphic

f meromorphic in U, $\gamma \overset{U}{\sim} 0$ avoiding the zeros

and poles of f,

$$\frac{1}{2\pi i}\int_\gamma g\,\frac{f'}{f}\,dz = \sum_a g(a)\cdot n(\gamma,a)\cdot \mathrm{ord}(f,a)$$

If γ is simple closed, $\operatorname{Int}\gamma \subseteq \mathcal{U}$, then

Int γ Int γ

$$\frac{1}{2\pi i}\int_\gamma g\,\frac{f'}{f}\,dz = \sum g\,(\text{zeros } f) - g\,(\text{poles of } f)$$

appear with multiplicity

Proof   We apply the Residue theorem.

We show $\operatorname{Res}\left(g\cdot\frac{f'}{f}, a\right) = g(a)\ \operatorname{ord}(f,a)$

Let $\operatorname{ord}(f,a) = k$. We know from page 2:

$$\frac{f'}{f} = \frac{k}{z-a} + F, \qquad F, G \text{ holomorphic near } a$$

$$g = g(a) + (z-a)\,G \qquad (\text{Taylor expansion})$$

$$\Longrightarrow\ g\cdot\frac{f'}{f} = \left(\frac{k}{z-a} + F\right)\left(g(a) + (z-a)\,G\right)$$

$$= \frac{k\ g(a)}{z-a} + H \quad \text{where } H \text{ holomorphic near } a$$

$$\Longrightarrow\ \operatorname{Res}\left(g\cdot\frac{f'}{f}, a\right) = k\,g(a) = \operatorname{ord}(f,a)\cdot g(a).$$

2. Applications (Conway V.3)

Let $f: U \to \mathbb{C}$ holomorphic, $\overline{\Delta} \subseteq U$. such that $f|_{\overline{\Delta}}$ injective. Let $V = f(\Delta)$ = open. Then

$$f : \Delta \longrightarrow V \quad \text{bijection.}$$

Proposition The following integral formula for the inverse function holds

$$f^{-1}(q) = \frac{1}{2\pi i} \int_{\partial\Delta} z \cdot \frac{f'(z)}{f(z)-q}\, dz \quad \forall q \in V.$$

In particular $f^{-1}: V \longrightarrow \Delta$ is holomorphic.

Proof Apply the enhanced Argument Principle to $f - q$ and $g(z) = z$. Since f injective, $\exists!\ p \in \Delta$ with $f(p) = q$. $\Rightarrow f^{-1}(q) = p$. But

$$\frac{1}{2\pi i} \int_{\partial\Delta} z \, \frac{f'(z)}{f(z)-q}\, dz = g(p) = p = f^{-1}(q).$$

no zeros on $\partial\Delta$ since $q \in f(\Delta) \Rightarrow q \notin f(\partial\Delta)$ as $f|_{\overline{\Delta}}$ injective.

Recall from Lecture 16

Key statement $\varphi : \mathcal{U} \times \{\gamma\} \longrightarrow \mathbb{C}$

- φ continuous
- $z \longrightarrow \varphi(z, w)$ holomorphic $\forall\, w \in \{\gamma\}$.

Then $g(z) = \int_\gamma \varphi(z, w)\, dw$ holomorphic.

Apply this to $\varphi : \underset{\substack{\omega\\ \eta}}{\Delta} \times \underset{\substack{\omega\\ z}}{\partial\Delta} \longrightarrow \mathbb{C}$

$$\varphi(\eta, z) = z \cdot \frac{f'(z)}{f(z) - \eta} \quad \text{continuous \&}$$

holomorphic in η $\forall\, z \in \partial\Delta$. Then

$$f^{-1}(\eta) = \frac{1}{2\pi i} \int_{\partial\Delta} \varphi(\eta, z)\, dz = \text{holomorphic in } \eta.$$

Remark

This extends a result from Lecture 11. concerning holomorphicity of the inverse (removes $f' \neq 0$).

3. Further Applications of the Argument Principle

Elliptic functions

- studied by Abel, Jacobi, Weierstraß
- connected with arclength of ellipse

  elliptic integrals

  elliptic curves

- rich theory
- we will only say a few words about them

(More in Math 220 B & C)

Carl Gustav Jacob Jacobi (1804 –1851)

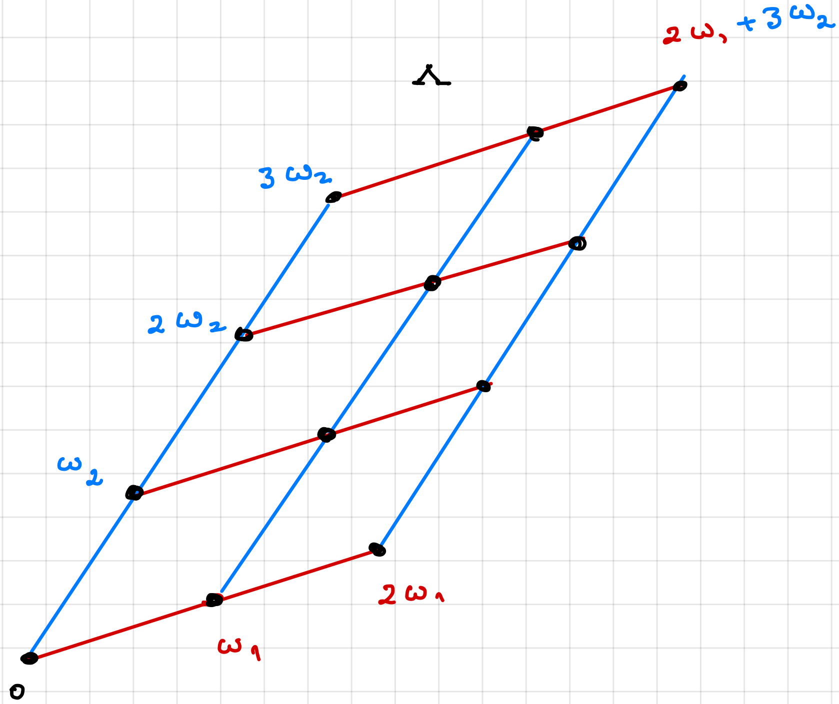
Jacobian, Jacobi symbol, Jacobi identity, symbol ∂

Karl Weierstraß (1815 – 1897)

Definition

Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. Define the lattice

$$\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2 = \{ m\,\omega_1 + n\,\omega_2 \;:\; m,n \in \mathbb{Z} \}$$



Def An elliptic function f satisfies

$\boxed{i}$ f meromorphic on $\mathbb{C}$

$\boxed{ii}$ f periodic, $f(z) = f(z+\omega_1) = f(z+\omega_2)$

Note that in fact $\forall\, \lambda \in \Lambda$, $f(z) = f(z+\lambda)$

Remarks $\boxed{i}$ The best-known elliptic function is

Weierstraß

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

We will study this function in detail later in 226 B & C.

$\boxed{ii}$ f elliptic $\Rightarrow$ f' elliptic.

Indeed $f(z) = f(z+\lambda) \Rightarrow f'(z) = f'(z+\lambda)$. $\forall \lambda \in \Lambda$

Math 220 A – Lecture 22

December 4, 2020

$\boxed{0}$ Last time In real analysis we encounter periodic functions. In complex analysis:

Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, $\dfrac{\omega_1}{\omega_2} \notin \mathbb{R}$.

Def An elliptic function f satisfies

$\boxed{i}$ f meromorphic on $\mathbb{C}$

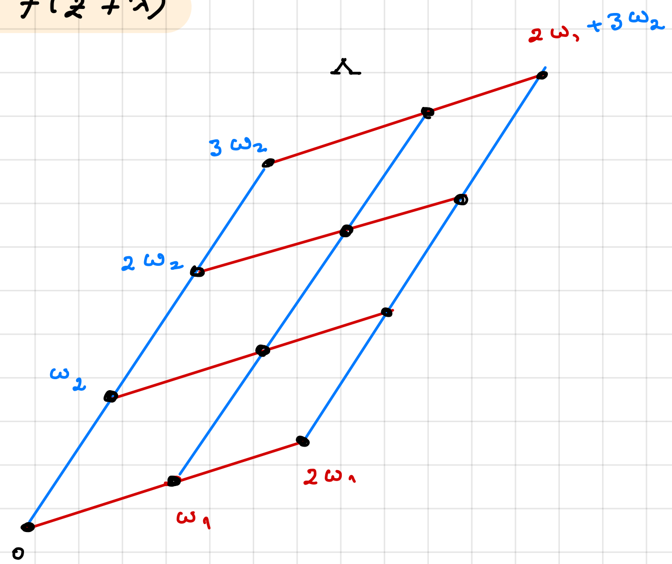
$\boxed{ii}$ f doubly periodic:

$$f(z) = f(z + \omega_1) = f(z + \omega_2) \quad \forall z$$

Remark

$$\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2 = \{ m\,\omega_1 + n\,\omega_2 \;:\; m, n \in \mathbb{Z} \}$$

$(*)$: $\forall \lambda \in \Lambda$, $f(z) = f(z + \lambda)$



[1] Basic Properties of Elliptic Functions

Note that Λ is a subgroup of $\mathbb{C}$.

Define

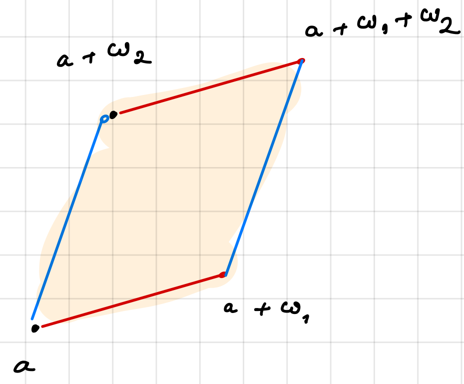
$$z \equiv w \mod \Lambda \iff z - w \in \Lambda.$$

$$z \equiv w \mod \Lambda \overset{(*)}{\Longrightarrow} f(z) = f(w).$$

Remark f is determined by values mod Λ

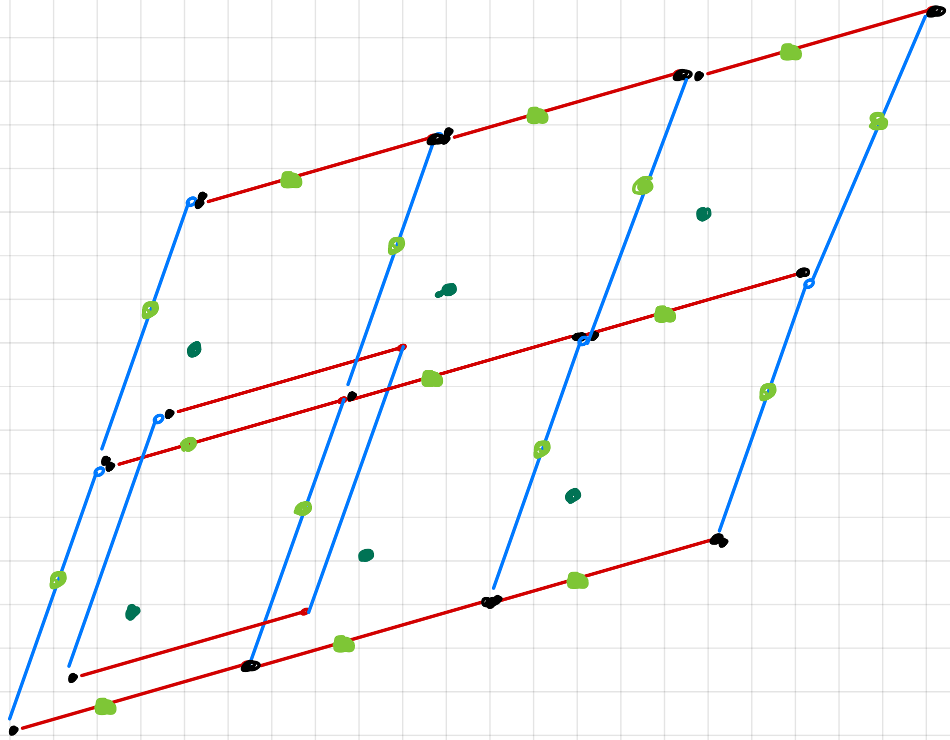
We will restrict f to a parallelogram.

$$P_a = \{ a + t_1 \omega_1 + t_2 \omega_2 : 0 \le t_1 \le 1, \ 0 \le t_2 \le 1 \}.$$



Each point in $\mathbb{C}$ is congruent to a point in P_a.

(see next picture)



Claim $\exists\ a$ such that ∂P_a contains no zeroes / poles.

Proof Start with any a. Since P_a is compact & zeroes / poles are discrete $\Rightarrow \exists$ finitely many of them in P_a. A suitable translation would ensure ∂P_a avoids them.

Write $P = P_a$ where P is chosen as above.

Remark

If f holomorphic in $\mathbb{C}$ $\Rightarrow$ $f|_P$ continuous

P compact
$\Rightarrow$ $f|_P$ bounded

periodic
$\Rightarrow$ f bounded

Liouville
$\Rightarrow$ f constant

Thus in general f will have poles.

Notation zeros in P : $\alpha_1 \dots \alpha_k$ (w/ multiplicity)

poles in P : $\beta_1 \dots \beta_\ell$ (w/ multiplicity)

Theorem [i] $k = \ell$: $\#\,\text{zeroes}(f) = \#\,\text{Poles}(f)$

[ii] $$\sum_{i=1}^{k} \alpha_i \equiv \sum_{i=1}^{k} \beta_i \mod \Lambda.$$

Remark Given $\alpha_1 \dots \alpha_k$, $\beta_1 \dots \beta_k$ with

$$\sum_i \alpha_i \equiv \sum_i \beta_i \mod \Lambda$$

there is an elliptic function with these zeroes/poles.

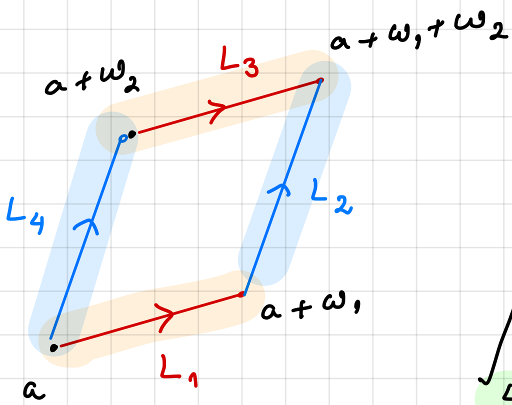
This is not obvious. $\rightsquigarrow$ Abel-Jacobi theory

Proof $\boxed{c}$ By the Argument Principle

$$\frac{1}{2\pi i}\int_{\partial P} \frac{f'}{f}\, dz = \#\text{ Zeroes}(f) - \#\text{ Poles}(f) \text{ in } P.$$

We show $\int_{\partial P} \frac{f'}{f}\, dz = 0$. Let $\partial P = L_1 + L_2 + (-L_3) + (-L_4)$

We show $\int_{L_1} \frac{f'}{f}\, dz = \int_{L_3} \frac{f'}{f}\, dz$ & $\int_{L_2} \frac{f'}{f}\, dz = \int_{L_4} \frac{f'}{f}\, dz$.



Both claims follow by periodicity.

$$\int_{L_1} \frac{f'}{f}\, dz = \int_{L_1} \frac{f'}{f}(z+\omega_2)\, dz = \int_{L_3} \frac{f'}{f}\, dz$$

$(L_3 = L_1 + \omega_2)$.

$\boxed{\text{ii}}$ We use the Enhanced Argument Principle $(g(z) = z)$.

$$\frac{1}{2\pi i}\int_{\partial P} z\,\frac{f'}{f}\,dz = \sum_{i=1}^{k}\alpha_i - \sum_{i=1}^{k}\beta_i .$$

We show
$$\frac{1}{2\pi i}\left(\int_{L_1} z\,\frac{f'}{f}\,dz - \int_{L_3} z\,\frac{f'}{f}\,dz\right) \in \Lambda \quad \text{and}$$

$$\frac{1}{2\pi i}\left(\int_{L_2} z\,\frac{f'}{f}\,dz - \int_{L_4} z\,\frac{f'}{f}\,dz\right) \in \Lambda$$

This will complete the proof.

We only consider 1st expression. $L_3 = L_1 + \omega_2$

f periodic

$$\frac{1}{2\pi i}\left(\int_{L_1} z\,\frac{f'}{f}\,dz - \int_{L_3} z\,\frac{f'}{f}\,dz\right) = \frac{1}{2\pi i}\left(\int_{L_1} \cancel{z}\,\frac{f'}{f}\,dz - \int_{L_1} (\cancel{z}+\omega_2)\,\frac{f'}{f}\,dz\right)$$

$$= -\frac{1}{2\pi i}\,\omega_2 \cdot \int_{L_1}\frac{f'}{f}\,dz$$

$w = f(z)$.

$$= -\left(\frac{1}{2\pi i}\int_{f(L_1)}\frac{dw}{w}\right)\cdot \omega_2$$

$$= -\underbrace{n\,(f(L_1), 0)}_{\text{integer}}\,\omega_2 \in \Lambda .$$

Note that $f(L_1)$ is a loop (by periodicity), not containing 0.

2. Rouché's theorem (Conway V. 3).

Idea $f, g : U \to \mathbb{C}$ holomorphic

$$f = g + \text{lower order terms}$$

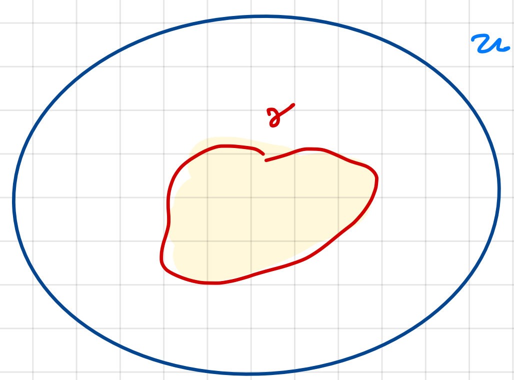
g : dominant term

$\Longrightarrow$ # zeroes (f) = # zeroes (g). (w/ multiplicity).

We can ignore the lower order terms.

Setup: $\gamma \subseteq U$ simple closed curve, $\operatorname{Int} \gamma \subseteq U$.

e.g. $\gamma = \partial \Delta$, $\overline{\Delta} \subseteq U$.



Theorem $f, g : U \to \mathbb{C}$ holomorphic, γ as above.

If $|f-g| < |g|$ on γ $\Rightarrow$

$$\# \text{zeroes}(f) = \# \text{zeroes}(g). \text{ in } \operatorname{Int}(\gamma).$$

(w/ multiplicity)

Note that $f \neq 0$ & $g \neq 0$ on γ.

Remark Conway's version is more general but less useful in practice.

Conway.

- f, g meromorphic
- $|f-g| < |f| + |g|$ on γ

$$\Rightarrow \quad \# \text{zeroes}(f) - \# \text{Poles}(f) = \# \text{zeros}(g) - \# \text{Poles}(g)$$

in $\operatorname{Int} \gamma$.

Eugene Rouché

(1832 – 1910)

Example

dominant term

[1] $$f = z^5 + 24z^3 + 2z^2 + 3z + 1$$

How many roots in $|z|<1$.

Let $g = 24z^3$ and $\gamma = \{|z|=1\}$.

We verify $|f-g| < |g|$ when $|z|=1$.

Note $|g| = 24\,|z|^3 = 24$.

triangle inequality

$$|f-g| = |z^5 + 2z^2 + 3z + 1| \le |z|^5 + 2|z|^2 + 3|z| + 1$$

$$= 1 + 2 + 3 + 1 = 6.$$

$\Longrightarrow$ $|f-g| < |g|$ on γ

$\Longrightarrow$ # Zeroes(f) = # zeroes$(g) = 3$ in $\{|z|<1\}$.

Example [ii] Fundamental Theorem of Algebra

$$f = z^n + a_1 z^{n-1} + \dots + a_n$$

$g = z^n$ = dominant term when $|z|$ large.

$$f - g = a_1 z^{n-1} + \dots + a_n.$$

When $|z| = R$,

$$|f-g| \le |a_1| R^{n-1} + \dots + |a_n| < R^n = |z|^n = |g|.$$

This happens for R large as $\lim_{R \to \infty} \dfrac{R^n}{|a_1| R^{n-1} + \dots + |a_n|} = \infty$.

By Rouché:

$$\# \text{ Zeroes } (f) = \# \text{ Zeroes } (g) = n \text{ in } \Delta(0,R). \ \forall R \gg 0$$

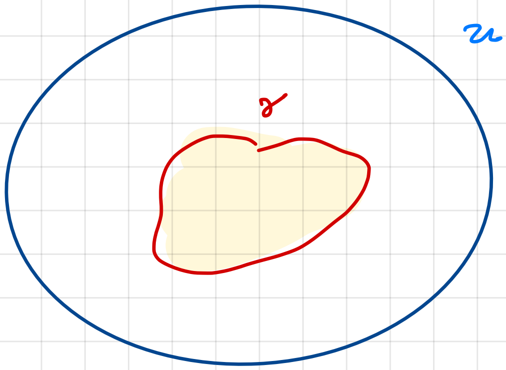
$\Rightarrow \# \text{ Zeroes } (f) = n.$ in $\mathbb{C}$

Math 220 A – Lecture 23

December 7, 2020

[1.] Last time — Rouché's theorem

γ simple closed curve, $\gamma \overset{\mathcal{U}}{\sim} 0$



e.g. $\gamma = \partial\Delta$, $\overline{\Delta} \subseteq \mathcal{U}$.

Theorem $f, g : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, γ as above.

If $|f-g| < |g|$ on γ $\Rightarrow$

$$\# \text{ zeroes}(f) = \# \text{ zeroes}(g). \quad \text{in } \operatorname{Int}(\gamma).$$

(w/ multiplicity)

What does the hypothesis mean?

$$f = \underbrace{g}_{\text{dominant term}} + \underbrace{(f-g)}_{\text{lower order terms}}$$

Proof (see Conway for a different proof)

Let $h_t = g + t(f-g)$, $0 \le t \le 1$.

Want $t \longrightarrow \#$ zeroes (h_t) in Int γ is continuous in t.

This implies $\#$ zeroes (h_t) = constant.

Since $h_0 = g$, $h_1 = f \Rightarrow \#$ zeroes $(f) = \#$ zeroes (g).

To show continuity, we use the Argument Principle

$$\# \text{ zeroes } (h_t) = \frac{1}{2\pi i} \int_\gamma \frac{h_t'(z)}{h_t(z)}\, dz .$$

Note $|h_t| = |g + t(f-g)| \ge |g| - |t|\,|f-g|$

$$\ge |g| - |f-g| > 0 \text{ on } \gamma.$$

Set $\psi(t, z) = \dfrac{h_t'(z)}{h_t(z)} ; [0,1] \times \{\gamma\} \longrightarrow \mathbb{C}$.

Note ψ is continuous.

Key Fact $\psi : [0,1] \times \{\gamma\} \longrightarrow \mathbb{C}$ continuous

$$\Rightarrow \quad \Phi(t) = \int_{\gamma} \psi(t, z)\, dz \text{ is continuous in } t.$$

Quick proof: Since $[0,1] \times \{\gamma\}$ is compact, ψ is uniformly continuous.

Fix $\varepsilon > 0$. Then $\exists\, \delta > 0$ with

$$|t - t'| < \delta \quad \Rightarrow \quad |\psi(t, z) - \psi(t', z)| < \varepsilon.$$

$$\Rightarrow \quad |\phi(t) - \phi(t')| = \left| \int_{\gamma} \psi(t, z) - \psi(t', z)\, dz \right|$$

$$< \varepsilon \cdot \text{length}(\gamma)$$

$\Rightarrow \phi$ continuous.

Applications

[I] Find the location of zeroes of holomorphic fns

e.g. $f = z^5 + 24z^3 + 2z^2 + 3z + 1$ (last time)

We can also use this for nonpolynomial functions.

Example

$$f(z) = e^z - 5z^3 + 1. \quad , \quad \gamma = \{|z| = 1\}.$$

Dominant term $g(z) = -5z^3.$

Indeed $|g| = 5$ for $|z| = 1.$

$$|f-g| = |e^z + 1| \le |e^z| + 1 = e^{\operatorname{Re} z} + 1$$

$$\le e^{|z|} + 1 = e + 1 < 5 = |g|$$

$\Rightarrow$ # zeroes (f) = # zeroes $(g) = 3$ in $\Delta(0,1)$.

III abstract applications

Example

$h : U \longrightarrow \mathbb{C}$, $\overline{\Delta}(0,1) \subseteq U$, $|h(z)| < 1$, $|z| = 1$.

$\Longrightarrow$ h has one fixed point in $\Delta(0,1)$.

Proof We show $h(z) = z \iff h(z) - z = 0$ has a unique solution in $\Delta(0,1)$.

Let $f(z) = h(z) - z$, $g(z) = -z$, $\gamma = \{|z| = 1\}$.

Then

$$|f - g| = |h| < 1 = |g| \quad \text{on } \gamma$$

$\Longrightarrow$ # zeros(f) = # zeros$(g) = 1$. $\Longrightarrow$

$\Longrightarrow$ h has a unique fixed point in $\Delta(0,1)$.

Remark Hurwitz' theorem will be another abstract application of Rouché!

[2] Sequences of holomorphic functions (Conway $\overline{VII}$).

Outline — notions of convergence

— Weierstraß' theorem

— Hurwitz's theorem $\Longleftarrow$ Rouché

[a] Types of convergence

Question What is the correct notion of convergence for holomorphic functions?

$f_n : \mathcal{U} \longrightarrow \mathbb{C}$, $f : \mathcal{U} \longrightarrow \mathbb{C}$ be any functions.

Math 140B [i] pointwise convergence

$f_n \longrightarrow f$ iff $\forall\, x \in \mathcal{U}$, $f_n(x) \longrightarrow f(x)$.

[ii] uniform convergence

$f_n \rightrightarrows f$ iff $\sup_{\mathcal{U}} |f_n - f| \longrightarrow 0$ as $n \longrightarrow \infty$.

Issues $\boxed{i}$ Pointwise convergence is not well behaved under differentiation or even integration. (Baby Rudin)

The pointwise limit of continuous functions need not be continuous (Baby Rudin / Math 140B).

$\boxed{ii}$ Uniform convergence is better. But the notion is strong. For instance, take.

$$f_n(z) = \frac{z}{n}, \quad f(z) = 0, \quad f_n \not\rightrightarrows f \text{ on } \mathbb{C}.$$

We consider slightly weaker notions.

Better
[a] uniform convergence on compact sets

[b] local uniform convergence

[a] **Notation** : $f_n \xrightarrow{c} f$ or $f_n \overset{c}{\rightrightarrows} f$

Definition $\forall\ K \subseteq \mathcal{U}$ compact, $\sup_K |f_n - f| \longrightarrow 0$ as $n \to \infty$.

[b] **Notation** $f_n \overset{l.u.}{\rightrightarrows} f$

Definition : $\forall\ x \in \mathcal{U}$ $\underbrace{\exists\ \Delta(x, r_x) \subseteq \mathcal{U}}_{\text{local}}$ with $\underbrace{f_n \rightrightarrows f}_{\text{uniform converg.}}$ in $\Delta(x, r_x)$.

Claim [a] = [b].

Thus $f_n \xrightarrow{c} f$, $f_n \overset{c}{\rightrightarrows} f$, $f_n \overset{l.u.}{\rightrightarrows} f$ mean the same thing.

Proof [a] $\Rightarrow$ [b]. If [a] holds for all K, take

$K = \overline{\Delta}(x, r_x) \subseteq \mathcal{U}$., K compact. This choice of K yields [b].

$\boxed{b} \Longrightarrow \boxed{a}$. Let $f_n \overset{l.u.}{\rightrightarrows} f$. Take K compact in $\mathcal{U}$.

For $x \in K$, $\exists\ \Delta(x, r_x)$ with $f_n \rightrightarrows f$ in $\Delta(x, r_x)$.

Since $K \subseteq \bigcup_{x \in K} \Delta(x, r_x) \Longrightarrow K \subseteq \bigcup_{i=1}^{N} \Delta(x_i, r_{x_i})$

by compactness. Since

$$\sup_K |f_n - f| \leq \max_{1 \leq i \leq N} \left(\sup_{\Delta(x_i, r_{x_i})} |f_n - f| \right) \longrightarrow 0$$

$\Longrightarrow f_n \rightrightarrows f$ in K $\Rightarrow f_n \overset{c}{\rightrightarrows} f$.

Example $f_n = \frac{z}{n}$, $f = 0$, $f_n \overset{c}{\rightrightarrows} f$ in $\mathbb{C}$.

Indeed, $\sup_K |f_n - f| = \sup_{z \in K} \left| \frac{z}{n} \right| \leq \frac{M}{n} \longrightarrow 0$.

so $f_n \overset{c}{\rightrightarrows} f$. This was the example disallowed before.

Remark (Continuity & Math 140B).

f_n continuous & $f_n \rightrightarrows f$ then f continuous

f_n continuous & $f_n \overset{l.u}{\rightrightarrows} f$ then f continuous.

(because continuity is a local concept).

Important Convention

$\mathcal{C}(\mathcal{U})$ = continuous functions in $\mathcal{U}$

$\mathcal{O}(\mathcal{U})$ = holomorphic functions in $\mathcal{U}$

We will always consider local uniform convergence. for both $\mathcal{O}(\mathcal{U})$ and $\mathcal{C}(\mathcal{U})$.

[b] Weierstraß' Theorem

Let $f_n : U \longrightarrow \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$. Then

[1] f holomorphic

[2] $f_n^{(k)} \overset{l.u.}{\rightrightarrows} f^{(k)}$

Remark [i] $\mathcal{O}(U) \hookrightarrow \mathcal{C}(U)$ "closed" under local uniform limits.

[ii] integration is not an issue

If $f_n \overset{l.u.}{\rightrightarrows} f$ then $\int_\gamma f_n \, dz \longrightarrow \int_\gamma f \, dz$. since $\{\gamma\}$ compact.

[iii] the statement fails in real analysis. (Baby Rudin or Math 140 B for examples).

The proof will be given next time.

Math 220 A – Lecture 24

December 9, 2020

[0] Last time Conway VII.2.

$f_n : \mathcal{U} \to \mathbb{C}$, $f : \mathcal{U} \to \mathbb{C}$

$$f_n \overset{l.u.}{\rightrightarrows} f \iff f_n \overset{c}{\rightrightarrows} f$$

$$\iff \forall z \in \mathcal{U},\ \exists\, \Delta(z, r_z) \subseteq \mathcal{U},\ f_n \rightrightarrows f \text{ in } \Delta(z, r_z).$$

$$\iff f_n \rightrightarrows f \text{ in } K \quad \forall K \subseteq \mathcal{U}. \text{ compact}$$

[1] Weierstraß' Theorem

Let $f_n : \mathcal{U} \to \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$. Then

[1] f holomorphic

[2] $f_n^{(k)} \overset{l.u.}{\rightrightarrows} f^{(k)}$

Proof [1] Let $\bar{R} \subseteq \mathcal{U}$ closed rectangle. , ∂R = compact.

Since $f_n \overset{l.u.}{\rightrightarrows} f \implies \displaystyle\int_{\partial R} f_n \, dz \longrightarrow \int_{\partial R} f \, dz$

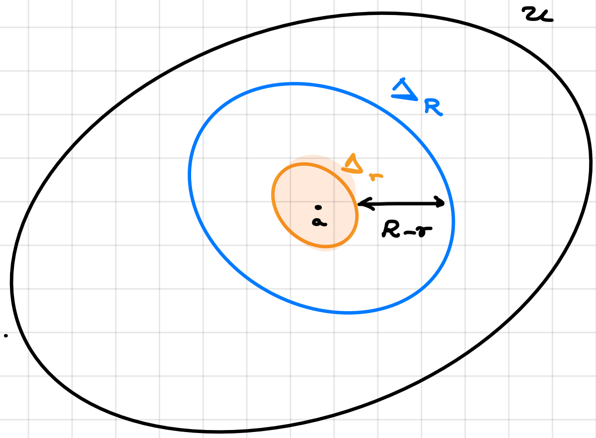
Since f_n holomorphic $\implies \displaystyle\int_{\partial R} f_n \, dz = 0.$ (Lecture 5)

$\implies \displaystyle\int_{\partial R} f \, dz = 0 \overset{\text{Lecture 6}}{\implies} f$ admits a primitive F in any disc in $\mathcal{U}$.

$\implies f = F' =$ holomorphic in any disc $\implies f$ holomorphic.

[ii] By induction, suffices to show

$$f_n' \overset{\text{l.u.}}{\rightrightarrows} f' \quad \text{in } \mathcal{U}.$$



Let $a \in \mathcal{U}$, $\underbrace{\overline{\Delta}(a,r)}_{=\,\overline{\Delta}_r} \subseteq \underbrace{\overline{\Delta}(a,R)}_{=\,\overline{\Delta}_R} \subseteq \mathcal{U}$.

$r < R$.

Suffices $f_n' \rightrightarrows f'$ in $\overline{\Delta}_r$.

$$|w - z| \geq |w| - |z| \geq R - r.$$

We use CIF for $z \in \overline{\Delta}_r$

$$\left| f_n'(z) - f'(z) \right| = \left| \frac{1}{2\pi i} \int_{\partial \Delta_R} \frac{f_n(w) - f(w)}{(w-z)^2}\, dw \right|$$

$$\leq \frac{1}{\cancel{2\pi}} \cdot \sup_{\partial \Delta_R} |f_n - f| \cdot \frac{1}{(R-r)^2} \cdot \cancel{2\pi} R$$

Thus $\displaystyle \sup_{\overline{\Delta}_r} |f_n' - f'| \leq \frac{R}{(R-r)^2} \cdot \sup_{\partial \Delta_R} |f_n - f| \longrightarrow 0$ as $n \to \infty$.

$$\Rightarrow f_n' \overset{\text{l.u.}}{\rightrightarrows} f'.$$

Series $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic. Assume

$(*)$ $\forall\ K \subseteq \mathcal{U}$ compact $\exists\ M_n(K)$, $|f_n| \leq M_n(K)$. over K. & $\sum_{n=1}^{\infty} M_n(K) < \infty$.

M-test
$\Longrightarrow$ $f = \sum_{n=1}^{\infty} f_n$ converges absolutely & uniformly on every K.

Weierstraß Thm
$\Longrightarrow$ f holomorphic & $f' = \sum_{n=1}^{\infty} f_n'$

Remark We have seen a particular case of this for power series. (Lecture 2).

Example (ζ- function)

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ gives a holomorphic function in $Re\, s > 1$.

Take $f_n(s) = \frac{1}{n^s}$. holomorphic in s.

Take $K \subseteq \{ Re\, s > 1 \}$. Since $Re : K \longrightarrow \mathbb{R}$ is continuous,

it achieves a minimum on K $\Rightarrow$ $Re\, z \geq \alpha \quad \forall z \in K, \; \alpha > 1$.

$$|f_n| = \left| \frac{1}{n^s} \right| = \left| \frac{1}{n^x} \cdot \cancel{\frac{1}{n^{iy}}} \right| = \frac{1}{n^x} \cdot 1. \leq \underbrace{\frac{1}{n^\alpha}}_{M_n}. \quad \text{where } s = x + iy$$

$\sum_{n=1}^{\infty} M_n < \infty$. by real analysis $\Longrightarrow$ $\sum_{n=1}^{\infty} f_n$ holomorphic in s

$\Rightarrow$ ζ holomorphic in s, $Re\, s > 1$.

Remarks $\boxed{i}$ We have seen $\zeta(2n) = \frac{(-1)^{n+1} (2\pi)^{2n} B_{2n}}{2\,(2n)!}$ (HWK6)

$\boxed{ii}$ This can be extended holomorphically to $s \neq 1$

(requires work).

[2.] Hurwitz' Theorem

$f_n : \mathcal{U} \to \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$, $\bar{V} \subseteq \mathcal{U}$ compact

If $f|_{\partial V}$ has no zeroes,

$$\#\underset{\bar{V}}{\text{zeroes}}(f_n) = \#\underset{\bar{V}}{\text{zeroes}}(f) \quad \forall n \geq N.$$

Proof

[i] Most useful case (Conway)

$V = \bar{\Delta}(a, R)$

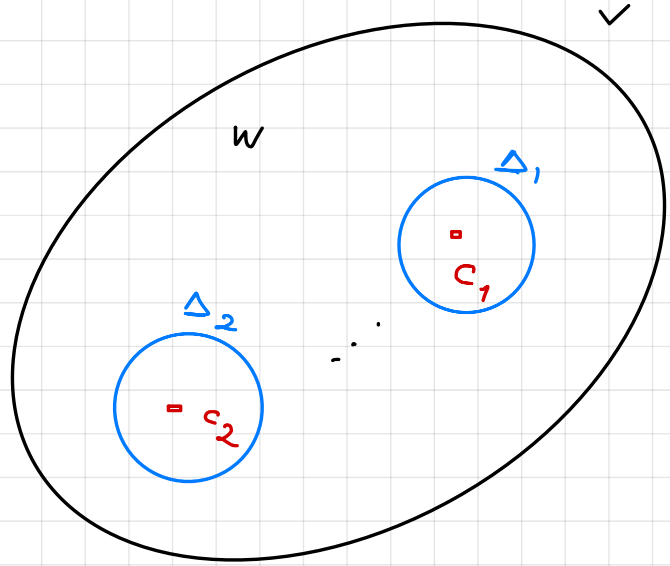
Since $f|_{\partial V}$ has no zeroes $\Rightarrow \varepsilon = \min_{\partial V} |f| > 0$.

Since $f_n \rightrightarrows f$ over $\partial V \Rightarrow \exists N$ s.t. over ∂V $\forall n \geq N$.

$$|f_n - f|_{\partial V} < \varepsilon \leq |f|_{\partial V} \Rightarrow |f_n - f| < |f| \text{ over } \partial V.$$

$\Rightarrow$ Rouché: $\#\text{zeroes}(f) = \#\text{zeroes}(f_n)$ in $\bar{V}$.

(ii) General case



$\overline{V}$ compact $\Rightarrow$ f has finitely many zeroes $c_1 \dots c_k$ in $\overline{V}$.

Surround c_j by small disjoint discs Δ_j, $W = V \setminus \bigcup_j \overline{\Delta}_j$

$\Rightarrow$ f has no zeroes in $\overline{W}$. $\Rightarrow$ $\exists N$ s.t. $\forall n \geq N$, f_n has no zeroes in $\overline{W}$. (If $\varepsilon = \min_{\overline{W}} |f| > 0$ $\Rightarrow$ $\exists N$, s.t. $\forall n \geq N$ $|f_n - f| < \varepsilon$ in $\overline{W}$ $\Rightarrow$ $f_n \neq 0$ in $\overline{W}$ for $n \geq N$.)

$$\Rightarrow \#\, \text{Zeroes}_{\overline{V}}(f) = \sum_{j=1}^{k} \#\, \text{Zeroes}_{\overline{\Delta}_j}(f) =$$

for n large by 1st case applied to f_n on $\overline{\Delta}_j$.

$$= \sum_{j=1}^{k} \#\, \text{Zeroes}(f_n) =$$

using f_n has no zeroes in $\overline{W}$

$$= \#\, \text{Zeroes}(f_n) \text{ for } n \geq N.$$

Corollary A $f_n \overset{l.u.}{\rightrightarrows} f$, f_n holomorphic in $\mathcal{U}$,

If f_n is zero free $\forall n$ $\implies$ f zero - free or $f \equiv 0$.

This fails in real analysis, $f_n = x^2 + \frac{1}{n} \rightrightarrows f = x^2$.

Proof Indeed if $f \not\equiv 0$, let a be chosen so that

$f(a) = 0$. Let $V = \overline{\Delta}(a, r)$, $f|_{\partial V}$ has no zeroes.

(Argue by contradiction, otherwise zeroes of f would accumulate).

Hurwitz

$\implies$ $\underbrace{\# \text{zeroes}_{\overline{V}} (f_n)}_{0.} = \# \text{Zeroes}_{\overline{V}} (f) \geq 1$. $\forall$ $n \geq N$.

assumption. $\swarrow$ a is a zero - contradiction.

$\implies$ f is zero - free

Example $\mathcal{U} = \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$

- $f_n(z) = z$, $f(z) = z$, $f_n \overset{l.u.}{\rightrightarrows} f$, f zero free.
- $f_n(z) = \frac{z}{n}$, $f(z) = 0$, $f_n \overset{l.u.}{\rightrightarrows} f$, $f \equiv 0$.

Both possibilities occur.

Corollary B $f_n \overset{l.u.}{\rightrightarrows} f$, f_n holomorphic in $\mathcal{U}$,

If f_n are injective $\forall n$ $\Rightarrow$ f injective or f constant.

Proof Assume f not injective, $f(a) = f(b)$, $a \neq b$.

$\tilde{f}_n = f_n - f_n(a)$.

$\tilde{f} = f - f(a)$.

Since $\left.\begin{array}{l} f_n(a) \longrightarrow f(a) \\ f_n \overset{l.u}{\rightrightarrows} f \end{array}\right\} \Rightarrow \tilde{f}_n \overset{l.u.}{\rightrightarrows} \tilde{f}$.

f_n injective $\Rightarrow$ $\tilde{f}_n$ zero free on $\tilde{\mathcal{U}} = \mathcal{U} \setminus \{a\}$.

Corollary A

$\Rightarrow$ $\tilde{f}$ is zero free on $\tilde{\mathcal{U}}$ or $\tilde{f} \equiv 0$. on $\tilde{\mathcal{U}}$

Note that $\tilde{f}(b) = f(b) - f(a) = 0$ $\Rightarrow$ $\tilde{f}$ is not zero-free

in $\tilde{\mathcal{U}}$. Thus $\tilde{f} \equiv 0$ in $\tilde{\mathcal{U}}$ $\Rightarrow$ f constant.

Adolf Hurwitz (1859 - 1919)

Riemann - Hurwitz formula, Hurwitz automorphism thm