#### Math 220B - Winter 2021 - Final Exam

Name:				
Student ID:				

#### **Instructions:**

There are 7 questions which are worth 60 points.

You may not use any books, notes, internet or collaborate with anybody on this exam. If you use a homework problem you will need to reprove it.

Please upload your answers in Gradescope. There is a 15 minute buffer period to upload the answers in Gradescope. You may not work on the exam during the buffer period.

Question	Score	Maximum
1		5
2		7
3		8
4		10
5		10
6		10
7		10
Total		60

# Problem 1. [5 points.]

Show that

Show that 
$$f(z)=\prod_{n=1}^{\infty}\left(1+n^2z^n\right)$$
 defines a holomorphic function in the unit disc  $\Delta(0,1)$ .

# Problem 2. [7 points.]

Let  $f:\mathbb{C}\to\mathbb{C}$  be an entire function which takes real values on the real line  $f(\mathbb{R})\subset\mathbb{R}$ . Show that

$$\overline{f(z)} = f(\bar{z}).$$

## Problem 3. [8 points.]

Find a biholomorphism between the strip  $S = \{z = x + iy : -\pi < y < \pi\}$  and the first quadrant  $Q = \{z = x + iy : x > 0, y > 0\}.$ 

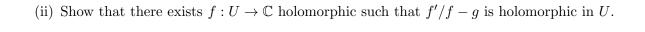
### **Problem 4.** [10 points; 3, 3, 4.]

Let  $U \subset \mathbb{C}$  be a simply connected open set. Let  $\{a_n\}_{n\geq 1}$  be a sequence in U without limit points in U. Let  $\{m_n\}_{n\geq 1}$  be a sequence of positive integers.

Let g be a meromorphic function in U with simple poles only at  $a_n$ , and with residues equal to  $m_n$  at  $a_n$ .

Show that there exists a holomorphic function  $h: U \to \mathbb{C}$  such that h'/h = g on  $U \setminus \{a_1, a_2, \ldots\}$ .

(i) For  $f: U \to \mathbb{C}$  holomorphic, with a zero of order m at  $a \in U$ , show that f'/f has a simple pole at a with residue equal to m.



(iii) Show that there exists  $h:U\to\mathbb{C}$  holomorphic such that h'/h=g.

### Problem 5. [10 points.]

Let U be a simply connected proper subset of  $\mathbb{C}$ . Let  $a \in U$ . Let  $f: U \to U$  be holomorphic, such that

$$f(a) = a, \quad |f'(a)| = 1.$$

Show that f is a biholomorphism of U.

## Problem 6. [10 points.]

Let  $\mathcal{F}$  be a normal family of holomorphic functions in the unit disc  $\Delta$ . Show that the family

$$\mathcal{G} = \{ f : \Delta \to \mathbb{C} \text{ holomorphic }, \ f(0) = 0, \ f' \in \mathcal{F} \}$$

is also normal.

#### Problem 7. [10 points.]

Let f,g be two entire functions. Let A,B be two disjoint nonempty compact sets. Assume  $\mathbb{C}\setminus (A\cup B)$  is connected. Show that there exists a polynomial p such that

$$|p(z) - f(z)| < \frac{1}{1000} \text{ for } z \in A$$

and

$$|p(z) - g(z)| > 1000 \text{ for } z \in B.$$