HW6 - SOLUTIONS

Q1. By Method 1 proved in class, we compute

$$\mathrm{Res}_{z=a}\frac{1}{f(z)} = \frac{1}{f'(a)}.$$

The residue theorem gives

$$\int_{|z-a|=r}\frac{dz}{f(z)} = 2\pi i\cdot \mathrm{Res}_{z=a}\frac{1}{f} = \frac{2\pi i}{f'(a)}.$$

The radius r is chosen small enough tso that there are no other zeros of f in $\Delta(a,r)$.

Q2. By the residue theorem

$$\int_{|z|=4}\frac{e^z}{(z-1)^2(z-3)^2}\,dz = 2\pi i\left(\mathrm{Res}_{z=1}\frac{e^z}{(z-1)^2(z-3)^2} + \mathrm{Res}_{z=3}\frac{e^z}{(z-1)^2(z-3)^2}\right).$$

For the first residue, we let $g(z) = \frac{e^z}{(z-3)^2} \implies g'(z) = \frac{(z-5)e^z}{(z-3)^3}$. Then

$$\mathrm{Res}_{z=1}\frac{e^z}{(z-1)^2(z-3)^2} = \mathrm{Res}_{z=1}\frac{g(z)}{(z-1)^2} = g'(1) = \frac{e}{2}.$$

The second residue is similar. Setting

$$h(z) = \frac{e^z}{(z-1)^2} \implies h'(z) = \frac{e^z(z-3)}{(z-1)^3}$$

we find

$$\mathrm{Res}_{z=3}\frac{e^z}{(z-1)^2(z-3)^2} = \mathrm{Res}_{z=3}\frac{h(z)}{(z-3)^2} = h'(3) = 0.$$

The integral is $2\pi i\cdot\frac{e}{2} = \pi i e$.

Q3. We use the residue theorem

$$\frac{1}{2\pi i}\int_\gamma f(z)g(z)dz = \sum_i \mathrm{Res}(fg, a_i)n(\gamma, a_i).$$

To complete the proof, we show

$$\mathrm{Res}(fg, a) = g(a)\mathrm{Res}(f, a).$$

Indeed, in a neighborhood of a, we have for $a_{-1} = \mathrm{Res}\ (f, a)$ that

$$f(z) = \frac{a_{-1}}{z-a} + F(z)$$

$$g(z) = g(a) + (z-a)\cdot G(z),$$

where F, G are holomorphic. Then

$$fg = \left(\frac{a_{-1}}{z-a} + F(z)\right)\cdot(g(a) + (z-a)G(z)) = \frac{a_{-1}g(a)}{z-a} + H(z),$$

where $H(z) = g(a)F(z) + a_{-1}G(z) + (z-a)F(z)G(z)$ is holomorphic near a. Thus

$$\mathrm{Res}(fg, a) = a_{-1}g(a) = \mathrm{Res}(f, a) \cdot g(a).$$

Q4. We have

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z + z^{-1}}{2}$$

where $z = e^{it}$. Thus $dz/z = i\,dt$ and

$$\int_0^{2\pi} \cos^{2n}(t)\,dt = \int_{|z|=1} \left(\frac{z+z^{-1}}{2}\right)^{2n} \frac{dz}{iz} = \frac{1}{2^{2n}i} \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}}\,dz$$

$$= \frac{\pi}{2^{2n-1}} \cdot \mathrm{Res}\left(\frac{(z^2+1)^{2n}}{z^{2n+1}}, 0\right) = \frac{\pi}{2^{2n-1}} \cdot \mathrm{Coeff}_{z^{2n}}(z^2+1)^{2n} = \frac{\pi}{2^{2n-1}} \cdot \binom{2n}{n},$$

where in the last line we used the binomial theorem.

Q5. The function $f(z)$ has poles at $z = 0$ with order $2n$ and $z = 2p\pi i$ for $p \in \mathbb{Z} \setminus \{0\}$ of order 1. Note that

$$\frac{1}{z^{2n}\,(e^z - 1)} = \frac{1}{z^{2n+1}} \left(\sum_{j=0}^{\infty} B_j \frac{z^j}{j!}\right) = \sum_{j=0}^{\infty} \frac{B_j}{j!} z^{j-2n-1}.$$

It follows that

$$\mathrm{Res}\,(f, 0) = \frac{B_{2n}}{(2n)!}.$$

Also, writing $g(z) = \frac{1}{z^{2n}}$ and $h(z) = e^z - 1$, we have for $p \neq 0$:

$$\mathrm{Res}\,(f, 2p\pi i) = \mathrm{Res}\left(\frac{g}{h}, 2p\pi i\right) = \frac{g(2\pi ip)}{h'(2\pi ip)} = \frac{1}{(2\pi pi)^{2n}} \cdot \frac{1}{e^{2\pi ip}} = \frac{(-1)^n}{(2\pi)^{2n}p^{2n}}.$$

Then, it follows

$$\int_{\gamma_m} f\,(z)\,dz = 2\pi i \left(\sum_{p=-m, p\neq 0}^{m} \frac{(-1)^n}{(2\pi)^{2n}p^{2n}} + \frac{B_{2n}}{(2n)!}\right).$$

We show

$$\int_{\gamma_m} f\,dz \to 0 \quad \text{as } m \to \infty.$$

Indeed, we claim that

$$|f(z)| \leq \frac{2}{((2m+1)\pi)^{2n}} := K_m$$

over γ_m. To prove the estimate, we show

$$|z^{2n}(e^z - 1)| \geq \frac{((2m+1)\pi)^{2n}}{2}.$$

Consider for instance the sides given by $z = t \pm (2m+1)i\pi$. We have

$$e^z - 1 = -e^t - 1$$

so that

$$|z^{2n}(e^z - 1)| = |z|^{2n}(e^t + 1) \geq ((2m+1)\pi)^{2n}.$$

The side $z = (2m+1)\pi + it$, we have

$|z^{2n}(e^z - 1)| \geq |z|^{2n} \cdot (|e^z| - 1) \geq ((2m+1)\pi)^{2n} \cdot (e^{(2m+1)\pi} - 1) \geq ((2m+1)\pi)^{2n}.$

Finally, for the side $z = -(2m+1)\pi + it$, we have

$|z^{2n}(e^z - 1)| \geq |z|^{2n} \cdot (1 - |e^z|) \geq ((2m+1)\pi)^{2n} \cdot (1 - e^{-(2m+1)\pi}) \geq ((2m+1)\pi)^{2n} \cdot \frac{1}{2}.$

The above estimate implies

$$\left| \int_{\gamma_m} f\, dz \right| \leq K_m \cdot \text{length}(\gamma_m) = K_m \cdot 8(2m+1)\pi = \frac{16}{((2m+1)\pi)^{2n-1}} \to 0$$

as $m \to \infty$.

In conclusion, we have

$$2 \sum_{p=1}^{\infty} \frac{(-1)^n}{p^{2n}(2\pi)^{2n}} + \frac{B_{2n}}{(2n)!} = \lim_{m\to\infty} \int_{\gamma_m} f(z)\, dz = 0.$$

Therefore,

$$\sum_{p=1}^{\infty} \frac{1}{p^{2n}} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2\,(2n)!}.$$

Q6. The function

$$f(z) = \frac{\pi \cot \pi z}{z^2 - a^2} = \frac{\pi \cos \pi z}{\sin(\pi z)(z^2 - a^2)}$$

has poles at $z = \pm a$ and at $z = n$ for $n \in \mathbb{Z}$. The residues are found by the rules given in class. First,

$$\text{Res}(f, a) = \frac{\pi \cot \pi z}{(z^2 - a^2)'}\Big|_{z=a} = \frac{\pi \cot \pi a}{2a}.$$

Similarly,

$$\text{Res}(f, -a) = \frac{\pi \cot \pi a}{2a}.$$

The residue at $z = n$, write $g = \frac{\pi \cos \pi z}{z^2 - a^2}$ and $h = \sin(\pi z)$ so that

$$\text{Res}(f, n) = \text{Res}\left(\frac{g}{h}, n\right) = \frac{g(n)}{h'(n)} = \frac{1}{n^2 - a^2}.$$

We have

$$\frac{1}{2\pi i} \int_{\gamma_n} f(z)\, dz = \sum_{m=-n}^{n} \text{Res}(f, m) + \text{Res}(f, a) + \text{Res}(f, -a)$$

$$= \frac{1}{a^2} + 2 \sum_{m=1}^{n} \frac{1}{m^2 - a^2} + \frac{\pi \cot \pi a}{a}.$$

To complete the proof we show that

$$\lim_{n\to\infty} \int_{\gamma_n} f(z)\, dz = 0.$$

To this end, we **claim** that

$$|\cot \pi z| \leq 3$$

over γ_n. This implies

$$|f(z)| \le \frac{3\pi}{|z^2 - a^2|} \le \frac{3\pi}{n^2 - |a|^2}$$

using $|z| \ge n$ and the triangle inequality for n sufficiently large. Thus by the basic estimate

$$\left| \int_{\gamma_n} f\, dz \right| \le \frac{3\pi}{n^2 - |a|^2} \cdot \mathrm{length}(\gamma_n) = \frac{3\pi}{n^2 - |a|^2} \cdot 2(4n+1) \to 0,$$

as $n \to \infty$.

To show the claim, it suffices by the fact that cotangent is odd, to consider only two sides of the rectangle, for instance the sides:

$$y = n, |x| \le n + \frac{1}{2} \text{ and } x = n + \frac{1}{2}, |y| \le n.$$

We compute

$$|\cot \pi z| = \frac{|e^{\pi i z} + e^{-\pi i z}|}{|e^{\pi i z} - e^{-\pi i z}|} = \left| \frac{e^{2\pi i z} + 1}{e^{-2\pi i z} - 1} \right| = \left| 1 + \frac{2}{e^{-2\pi i z} - 1} \right| \le 1 + \frac{2}{|e^{-2\pi i z} - 1|}.$$

We will show $|e^{-2\pi i z} - 1| > 1$ over the two sides, completing the argument. Indeed, over the side $y = n$, we have

$$|e^{-2\pi i z} - 1| = |e^{-2\pi i x} e^{2\pi n} - 1| \ge e^{2\pi n} - 1 > 1.$$

Over the side $x = n + \frac{1}{2}$, we have

$$|e^{-2\pi i z} - 1| = |e^{-2\pi i (n+1/2) + 2\pi y} - 1| = |-e^{2\pi y} - 1| = e^{2\pi y} + 1 > 1.$$

Q7. The function

$$f(z) = \frac{\pi \cot \pi z}{(z+a)^2} = \frac{\pi \cos \pi z}{\sin(\pi z)(z+a)^2}$$

has poles at $z = -a$ and at $z = n$ for $n \in \mathbb{Z}$. The residues are found by the rules given in class. First,

$$\mathrm{Res}(f, -a) = \frac{d}{dz} \pi \cot \pi z|_{z=-a} = -\frac{\pi^2}{\sin^2 \pi a}.$$

The residue at $z = n$, write $g = \pi \cos \pi z$ and $h = \sin(\pi z)(z+a)^2$ so that

$$\mathrm{Res}(f, n) = \mathrm{Res}\left(\frac{g}{h}, n\right) = \frac{g(n)}{h'(n)} = \frac{1}{(n+a)^2}.$$

We have

$$\frac{1}{2\pi i} \int_{\gamma_n} \frac{\pi \cot \pi z}{(z+a)^2}\, dz = \sum_{n \in \mathbb{Z}} \mathrm{Res}(f, n) + \mathrm{Res}(f, -a) = \frac{1}{(n+a)^2} - \frac{\pi^2}{\sin^2 \pi a}.$$

To complete the proof we show that

$$\lim_{n \to \infty} \int_{\gamma_n} f(z)\, dz = 0.$$

To this end, we recall by the previous problem that

$$|\cot \pi z| \le 3$$

over γ_n. This implies

$$|f(z)| \leq \frac{3\pi}{|z+a|^2} \leq \frac{3\pi}{(n-|a|)^2}$$

using $|z| \geq n$ and the triangle inequality $|z-a| \geq |z| - |a| \geq n - |a| > 0$ for n sufficiently large. Thus by the basic estimate

$$\left|\int_{\gamma_n} f\,dz\right| \leq \frac{3\pi}{(n-|a|)^2} \cdot \operatorname{length}(\gamma_n) = \frac{3\pi}{(n-|a|)^2} \cdot 2(4n+1) \to 0,$$

as $n \to \infty$.

Q8.

(a) Let

$$f(z) = \frac{1-e^{2iz}}{z^2}.$$

The function f is holomorphic inside the region bounded by

$$\gamma = C_R \cup S_2 \cup C_r^\star \cup S_1,$$

where C_r and C_R are the half circles of radii r and R, and S_1, S_2 are the segments $[r, R]$ and $[-R, -r]$. The star decorating C_r indicates the reversed orientation. Hence,

$$\int_\gamma f(z)\,dz = 0.$$

Moreoever,

$$\begin{aligned}
\int_{S_1} f + \int_{S_2} f &= \int_r^R \frac{1-e^{2ix}}{x^2}dx + \int_{-R}^{-r} \frac{1-e^{2ix}}{x^2}dx \\
&= \int_r^R \frac{2-\left(e^{2ix}+e^{-2ix}\right)}{x^2}dx \\
&= \int_r^R \frac{2-2\cos 2x}{x^2}dx \\
&= 4\int_r^R \frac{\sin^2 x}{x^2}dx.
\end{aligned}$$

Also,

$$\begin{aligned}
\left|\int_{C_R} \frac{1-e^{2iz}}{z^2}dz\right| &\leq \int_0^\pi \frac{1+|e^{2iRe^{i\theta}}|}{R^2}Rd\theta \\
&= \int_0^\pi \frac{1+e^{-2R\sin\theta}}{R}d\theta \\
&\leq \int_0^\pi \frac{1+1}{R}d\theta \\
&\leq \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.
\end{aligned}$$

Finally,

$$\begin{aligned}\int_{C_r} \frac{1-e^{2iz}}{z^2}dz &= \int_{C_r} \frac{e^{2iz}-1-2iz}{z^2}dz + \int_{C_r} \frac{2i}{z}dz \\ &= \int_{C_r} \frac{e^{2iz}-1-2iz}{z^2}dz - 2\pi.\end{aligned}$$

By examining the Taylor expansion, $\frac{e^{2iz}-1-2iz}{z^2}$ has a removable singularity at $z = 0$, and hence $\frac{e^{2iz}-1-2iz}{z^2}$ can be extended to an entire function. Particularly, $\frac{e^{2iz}-1-2iz}{z^2}$ is bounded by M on for $|z| < r$. Therefore, we can show that

$$\left|\int_{C_r} \frac{e^{2iz}-1-2iz}{z^2}dz\right| \leq Mr\pi \to 0 \text{ as } r \to 0.$$

By taking $r \to 0$ and $R \to \infty$, it follows

$$\int_0^\infty \frac{\sin^2 x}{x^2} = \frac{\pi}{2}.$$