<u>Math</u> <u>220 A</u> – <u>Lecture</u> <u>7</u>

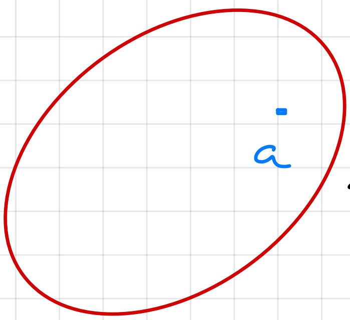
October 19, 2020

Last time : Winding number (index)

- γ piecewise C^1 loop, $a \notin \{\gamma\}$.

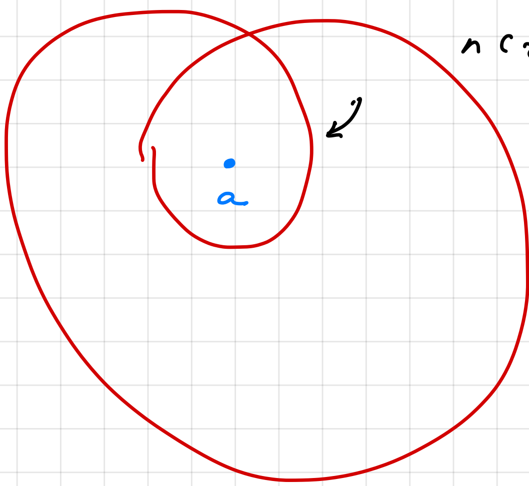
$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$$

γ



$n(\gamma, a) = 1$.

γ



$n(\gamma, a) = 2$

Properties

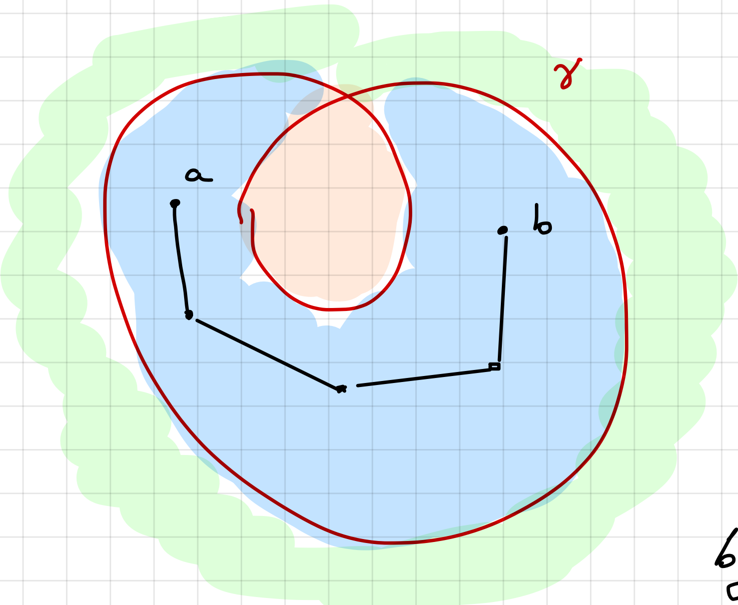
$\boxed{1}$ $n(-\gamma, a) = -n(\gamma, a)$ (change of orientation)

Proof:

$$\int_{-\gamma} \frac{dz}{z-a} = -\int_\gamma \frac{dz}{z-a}$$

$\boxed{ii}$ $n(\gamma, -) : \mathbb{C} \setminus \{\gamma\} \longrightarrow \mathbb{Z}$ is locally constant

$n(\gamma, a) = 0$ for a in the unbounded component of $\mathbb{C} \setminus \{\gamma\}$.



Proof.

Let R be a component of $\mathbb{C} \setminus \{\gamma\}$. If $a, b \in \bar{R}$

$\Rightarrow a, b$ can be joined by a polygonal path in R.

This is the same argument used in the past to show we can join by piecewise C^1 path. Suffices to show if $\overline{ab} \subseteq R \Rightarrow n(\gamma, a) = n(\gamma, b)$

$$\Longleftrightarrow \int_\gamma dz \left(\frac{1}{z-a} - \frac{1}{z-b} \right) = 0 .$$

This is true since $\mathrm{Log}\, \dfrac{z-a}{z-b}$ is a primitive of the integrand. We showed last time $\operatorname{Log} \frac{z-a}{z-b}$ is well defined in $\mathbb{C} \setminus \overline{ab}$.

If U is the unbounded component, let $R >> 0$ such that $\{\gamma\} \subseteq \Delta(0, R)$. Let m be the value of $n(\gamma, \_)$ on U. Pick $|a| \geq 2R$.

$a \in U$. $\Rightarrow$ $|z-a| \geq |a| - |z| \geq 2R - R = R$ if $z \in \{\gamma\}$ $\Rightarrow$

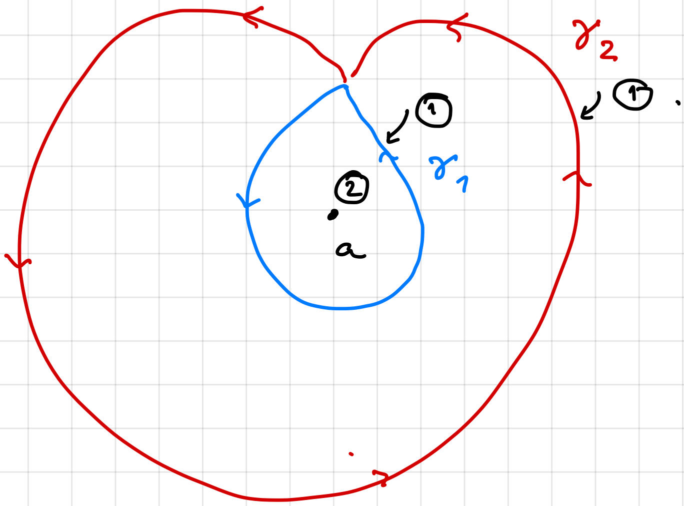
$$|m| = |n(\gamma, a)| = \frac{1}{2\pi} \left| \int_\gamma \frac{dz}{z-a} \right| \leq$$

$$\leq \frac{1}{2\pi} \cdot \frac{1}{R} \cdot \text{length}(\gamma).$$

Make $R \longrightarrow \infty$ $\Rightarrow$ $n(\gamma, a) = m = 0$.

$\boxed{iii}$ $\gamma = \gamma_1 + \gamma_2$

$\Rightarrow n(\gamma, a) = n(\gamma_1, a) + n(\gamma_2, a)$



Proof:

$$\int_{\gamma} \frac{dz}{z-a} = \int_{\gamma_1} \frac{dz}{z-a} + \int_{\gamma_2} \frac{dz}{z-a}.$$

Rudiments of algebraic topology

$$\pi_1(X) = \text{(based) loops in } X \Big/ \underset{\text{homotopy}}{\sim}$$

$$\pi_1(\mathbb{C} \setminus \{a\}) \cong \mathbb{Z} \quad \text{isomorphism}$$

$$\gamma \longrightarrow n(\gamma, a).$$

Two questions arise

[a] Can we define integrals over γ continuous?

[b] $\gamma_1 \sim \gamma_2 \overset{?}{\Longrightarrow} n(\gamma_1, a) = n(\gamma_2, a).$

Answer to [a] YES. If f holomorphic, γ continuous we define $\int_\gamma f\,dz$. For instance by analytic continuation

We will not pursue this here.

Answer to [b] YES. Cauchy's Theorem (Homotopy)

Conway $\overline{IV}$. 6.

We reparametrize so that the domain is $I = [0,1]$.

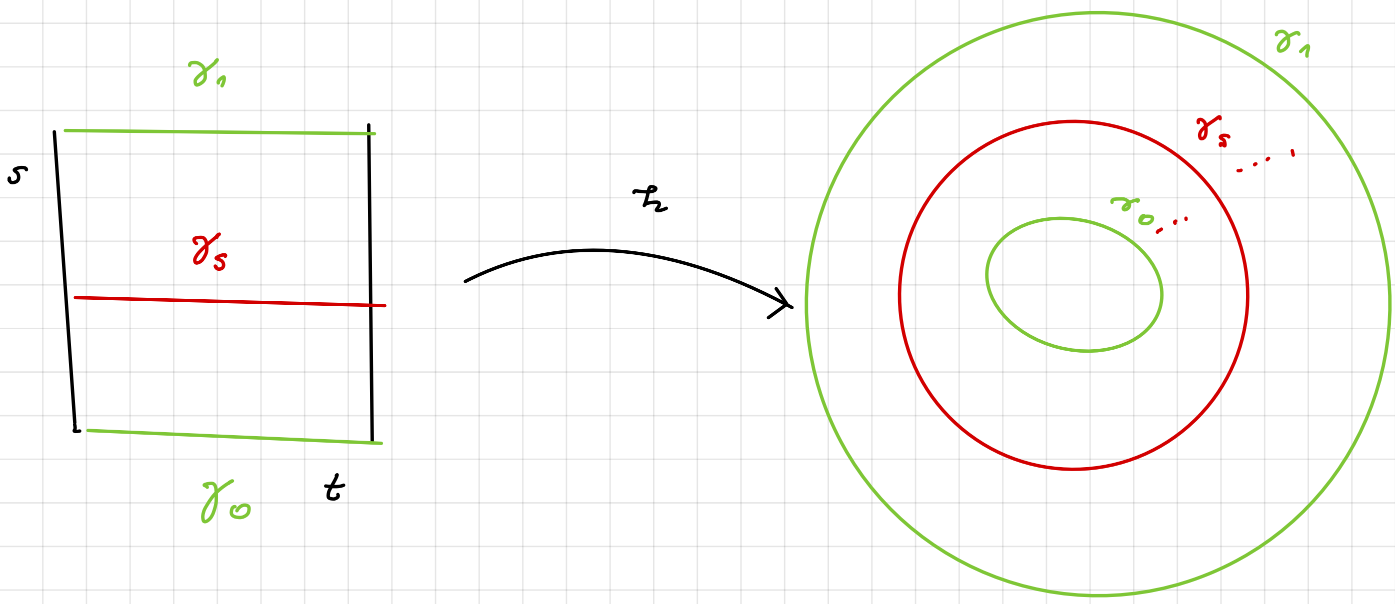
Homotopy $\gamma_0, \gamma_1 : I \longrightarrow U$ continuous loops

$\gamma_0 \overset{U}{\sim} \gamma_1$ if $\exists\ h : I \times I \longrightarrow U$ continuous

$$h(t,0) = \gamma_0(t), \quad h(t,1) = \gamma_1(t).$$

$$h(0,s) = h(1,s).$$

$\Longrightarrow \quad \gamma_s(t) = h(t,s).$ continuous loop.

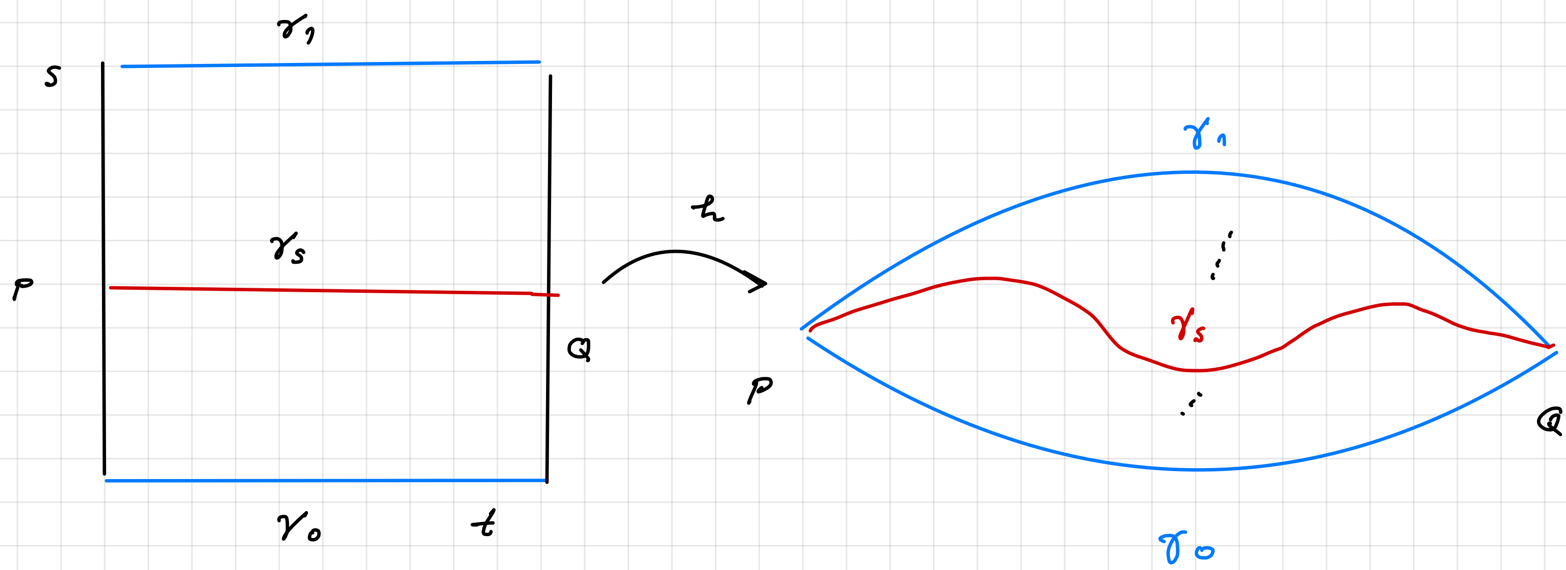


Def $\gamma_0, \gamma_1 : I \longrightarrow \mathcal{U}$ continuous paths from P to Q

$\gamma_0 \overset{\mathcal{U}}{\underset{FEP}{\sim}} \gamma_1$ if $\exists\, h : I \times I \longrightarrow \mathcal{U}$ continuous

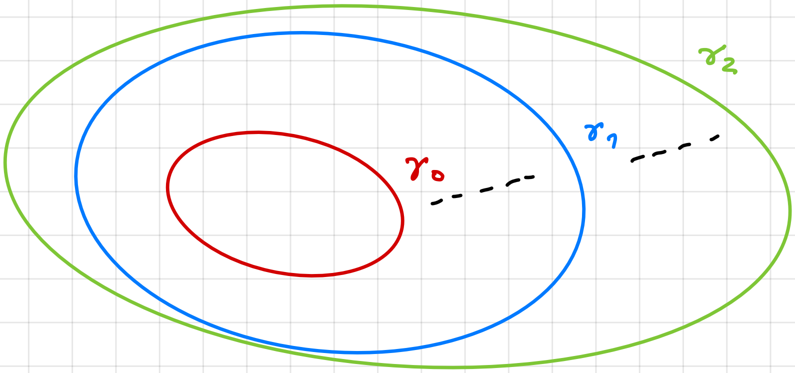
$$h(t,0) = \gamma_0(t) \quad , \quad h(t,1) = \gamma_1(t)$$

$$h(0,s) = P \quad , \quad h(1,s) = Q .$$



Remark $\boxed{a}$ $\sim$ is an equivalence relation.

$$\gamma_0 \overset{U}{\sim} \gamma_1, \ \gamma_1 \overset{U}{\sim} \gamma_2 \Rightarrow$$



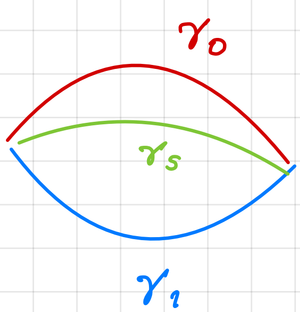
$$\Rightarrow \gamma_0 \overset{U}{\sim} \gamma_2$$

constant loop

$\boxed{b}$ Check $\gamma + (-\gamma) \overset{U}{\sim} 0$. $\forall \gamma$ path in U

$\boxed{c}$ If $\gamma_0 \overset{FEP}{\sim} \gamma_1$, let $\gamma = \gamma_0 + (-\gamma_1)$ loop

$\Rightarrow \gamma \overset{U}{\sim} 0$. as loops. Indeed let



$$\Gamma_s = \gamma_s + (-\gamma_1).$$

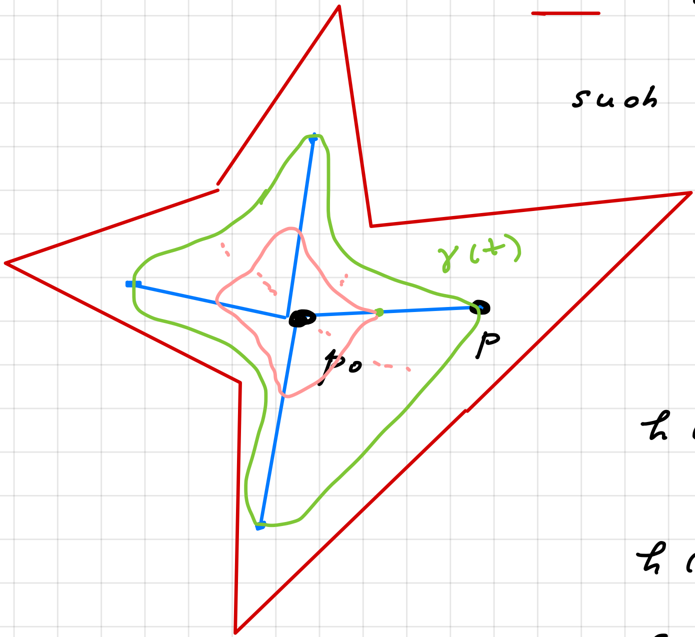
$\Gamma_0 = \gamma$. By $\boxed{b}$, $\Gamma_1 \sim 0$.

By $\boxed{a}$ $\Rightarrow \gamma \overset{U}{\sim} 0$.

Def U is *simply connected* if $\forall \gamma$ loop in U,

$\gamma \overset{U}{\sim} 0 \Leftrightarrow \pi_1(U) = 0$.

Example $\mathcal{U}$ is star convex $\Rightarrow$ $\mathcal{U}$ simply connected



Def $\mathcal{U}$ star convex if $\exists\, p_0 \in \mathcal{U}$ such that $\forall\, p \in \mathcal{U} \Rightarrow \overline{p_0 p} \subseteq \mathcal{U}$.

Let γ be a loop in $\mathcal{U}$.

$$h(t,s) = s\, p_0 + (1-s)\,\gamma(t) \subseteq \mathcal{U}$$

$$h(t,0) = \gamma(t)$$

$$h(t,1) = p_0 \qquad \Rightarrow \gamma \sim 0.$$

Cauchy's Theorem (Homotopy version)

$f: U \to \mathbb{C}$ holomorphic, $\gamma_0 \overset{U}{\sim} \gamma_1$ piecewise C' loops in U $\Rightarrow$ $\displaystyle\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

Remarks [i] $\gamma \overset{U}{\sim} 0 \Rightarrow \displaystyle\int_\gamma f\,dz = \int_0 f\,dz = 0.$

If U simply connected $\Rightarrow \displaystyle\int_\gamma f\,dz = 0 \quad \forall \gamma$ C' loop in U.

[ii] γ_1, γ_2 piecewise C' paths, $\gamma_1 \overset{FEP}{\sim} \gamma_2$

$\Rightarrow \displaystyle\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz.$ Indeed let $\gamma = \gamma_1 + (-\gamma_2)$.

By [i] $\Rightarrow \displaystyle\int_\gamma f\,dz = 0 \Rightarrow \int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz.$

[iii] $\gamma_0 \overset{U}{\sim} \gamma_1$, $U \subseteq \mathbb{C} \setminus \{a\}$ piecewise C' loops in $U \subseteq \mathbb{C}\setminus\{a\}$ $\Rightarrow \displaystyle\int_{\gamma_0} \frac{dz}{z-a} = \int_{\gamma_1} \frac{dz}{z-a}$

$\Rightarrow n(\gamma_0, a) = n(\gamma_1, a).$

This proves a previous assertion.

Remark The homotopy in Cauchy's theorem is not assumed to be C^1.

Existence of primitives in simply connected sets

If U simply connected, $f : U \longrightarrow \mathbb{C}$ holomorphic

$$\Rightarrow \int_\gamma f\,dz = 0 \quad \text{by Remark } \boxed{1}$$

$\Rightarrow$ Prop A, f has a primitive

Corollary Any holomorphic function in a simply connected set admits a primitive.

Take $f(z) = \frac{1}{z}$. A primitive is a branch of logarithm.

Corollary Let $U \subseteq \mathbb{C} \setminus \{0\}$ simply connected. We can define a branch of logarithm in U.