Math 220 A – Lecture 20

November 30, 2020

Last time — Residue at ∞

If $f : \{|z| > R\} \to \mathbb{C}$ holomorphic, ∞ isolated singularity

$$\Updownarrow$$

$g : \Delta^*(0, \frac{1}{R}) \to \mathbb{C}$, $g(z) = f(\frac{1}{z})$, 0 isolated singularity

Beware

$$\mathrm{Res}(f, \infty) \neq \mathrm{Res}(g, 0).$$

Instead define

$$\mathrm{Res}(f, \infty) := -\frac{1}{2\pi i} \int_{|z| = \rho} f \, dz \qquad \text{where } \rho > R.$$

By Homotopy Cauchy this does not depend on $\rho > R$

Example

$$\int_{|z|=11} \frac{z^9 \, dz}{(z-1) \cdots (z-10)} = -2\pi i \, \mathrm{Res}(\_\_, \infty).$$

Question How do we compute the residue at ∞?

Answer $\operatorname{Res}(f, \infty) = - \operatorname{Res}_{w=0} \left(g(w) \cdot \frac{1}{w^2} \right)$

Proof Let ρ be sufficiently large. Then

$$\operatorname{Res}(f, \infty) = - \frac{1}{2\pi i} \int_{|z|=\rho} f \, dz =$$

$z = \frac{1}{w}$. $\quad dz = - \frac{dw}{w^2}$.

$$= \frac{1}{2\pi i} \int_{|w| = 1/\rho} g \, \frac{-dw}{w^2} \quad \text{(change variables)}$$

(the change of orientation yields an extra sign).

$$= \operatorname{Res}_{w=0} \left(g(w) \cdot \frac{-1}{w^2} \right)$$

using the usual residue theorem.

Residue Theorem for $\hat{\mathbb{C}}$

If f has isolated singularities only at $a_1, \ldots, a_k \in \mathbb{C}$

and possibly at ∞ then

$$\sum_{a \in \hat{\mathbb{C}}} \operatorname{Res}(f, a) = 0.$$

Proof Let ρ be large enough, $\rho > |a_j|$ for all j.

$$\operatorname{Res}(f, \infty) = -\frac{1}{2\pi i} \int_{|z|=\rho} f \, dz \qquad \text{(definition)}$$

$$= -\sum_j \operatorname{Res}(f, a_j) \qquad \text{(usual residue theorem)}$$

$$\implies \sum_{a \in \hat{\mathbb{C}}} \operatorname{Res}(f, a) = 0.$$

Remark* This generalizes correctly to other compact Riemann surfaces.

Example $f(z) = \dfrac{z^5+2}{z-1}$.

Last lecture, we saw that f has pole at $z = 1$ and $z = \infty$.

$$\mathrm{Res}(f, 1) \overset{\text{Method 1}}{=} \left.\frac{z^5+2}{(z-1)'}\right|_{z=1} = 3.$$

$$\mathrm{Res}(f, \infty) = \underset{w=0}{\mathrm{Res}}\left(g(w) \cdot \frac{-1}{w^2}\right)$$

$$z = 1/w \quad \Rightarrow g(w) = f\left(\frac{1}{w}\right) = \frac{1/w^5 + 2}{1/w - 1} = \frac{1+2w^5}{1-w} \cdot \frac{1}{w^4}.$$

Thus

$$\mathrm{Res}(f, \infty) = \underset{w=0}{\mathrm{Res}} \frac{1+2w^5}{1-w} \cdot \frac{1}{w^4} \cdot \frac{-1}{w^2}$$

$$= \mathrm{Coeff}_{w^5} - \frac{1+2w^5}{1-w}$$

$$= \mathrm{Coeff}_{w^5} - (1+2w^5)(1+w+w^2+w^3+w^4+w^5+\dots)$$

$$= -(1+2) = -3$$

This is consistent with the residue theorem on $\hat{\mathbb{C}}$.

Example (Lagrange)

$$f(z) = \frac{P(z)}{Q(z)}.$$ Assume that

- $\deg P = p$, $\deg Q = q$, $p \le q-2$
- Q has simple roots $\alpha_1, \dots, \alpha_q$

f has poles at $\alpha_1, \dots \alpha_q$ and possibly at ∞.

- $$\operatorname{Res}(f, \alpha_i) \overset{\text{Method 1}}{=} \frac{P(\alpha_i)}{Q'(\alpha_i)}$$

- $\operatorname{Res}(f, \infty) = 0$ (next page).

Residue Theorem for $\hat{\mathbb{C}}$ $\Longrightarrow$ $$\sum_{i=1}^{q} \frac{P(\alpha_i)}{Q'(\alpha_i)} = 0$$

When $P(z) = z^p$, $Q(z) = \prod_{i=1}^{q} (z - \alpha_i)$, this gives

$$\sum_{i=1}^{q} \frac{\alpha_i^{\,p}}{\prod_{j \neq i} (\alpha_j - \alpha_i)} = 0 \qquad \forall\ p \le q-2.$$

$\forall\ \alpha_1, \dots \alpha_q$ distinct

Proof $\operatorname{Res}\left(\frac{P}{Q}, \infty\right) = 0$ if $p \leq q-2$.

Write $P = a_0 z^p + \dots + a_p$, $a_0 \neq 0$

$Q = b_0 z^q + \dots + b_q$, $b_0 \neq 0$.

$$\operatorname{Res}\left(\frac{P}{Q}, \infty\right) = \operatorname{Res}_{w=0}\left(\frac{a_0 \frac{1}{w^p} + a_1 \frac{1}{w^{p-1}} + \dots + a_p}{b_0 \frac{1}{w^q} + b_1 \frac{1}{w^{q-1}} + \dots + b_q} \cdot \frac{-1}{w^2}\right)$$

$$= \operatorname{Res}_{w=0}\left(\frac{w^q}{w^p} \cdot \frac{a_0 + a_1 w + \dots + a_p w^p}{b_0 + b_1 w + \dots + b_q w^q} \cdot \frac{-1}{w^2}\right)$$

$$= -\operatorname{Res}_{w=0}\left(\underbrace{w^{q-p-2} \cdot \frac{a_0 + a_1 w + \dots + a_p w^p}{b_0 + b_1 w + \dots + b_q w^q}}_{\text{holomorphic near } 0 \text{ since } p+2 \leq q}\right)$$

$= 0$.

Remark* (will not use)

Better to speak about residue of forms

$$f\,dz \xleftrightarrow{\text{versus}} f$$

Example $f(z) = \frac{1}{z}$. Clearly $\operatorname{Res}_z(f, 0) = 1$. But if we change coordinates

$$z = \lambda w \Rightarrow f = \frac{1}{\lambda w} \Rightarrow \operatorname{Res}_w(f, 0) = \frac{1}{\lambda}.$$

However if we work with forms, these issues are absent

$$f\,dz = \frac{dz}{z} = \frac{d(\lambda w)}{\lambda w} = \frac{dw}{w}.$$

Residues of forms are coordinate-independent!

This can be seen from $\operatorname{Res}(f, a) = \frac{1}{2\pi i} \int_{\partial\Delta(a,\varepsilon)} f\,dz$

using change of variables formula.

This independence applies to the residue at ∞ as well:

$$\operatorname*{Res}_{z=\infty} (f\,dz) = \operatorname*{Res}_{w=0} \left(g(w) \cdot d\left(\frac{1}{w}\right)\right) \qquad z = 1/w$$

$$= \operatorname*{Res}_{w=0} \left(g(w) \cdot \frac{-dw}{w^2} \right).$$

This justifies the choice of sign in the definition of the residue at ∞.

2. Applications of the Residue Theorem

a Argument Principle

b Rouché's Theorem

Conway V.3

Eugene Rouché

(1832 – 1910)

[a] The Argument Principle

Order $f : U \longrightarrow \mathbb{C}$ meromorphic, $U \subseteq \mathbb{C}$, $a \in U$.

$$\mathrm{ord}(f,a) = \begin{cases} n, & a \text{ zero of order } n \\ -n, & a \text{ pole of order } n \\ 0, & \text{otherwise} \end{cases}$$

Remarks [i] $\mathrm{ord}(f,a) = n \iff f = (z-a)^n g$

where g holomorphic near a, $g(a) \neq 0$

This follows by inspecting the Taylor / Laurent expansion.

[ii] $\mathrm{ord}(fg, a) = \mathrm{ord}(f,a) + \mathrm{ord}(g,a)$

Indeed, let $\mathrm{ord}(f,a) = m$, $\mathrm{ord}(g,a) = n$.

Write $f = (z-a)^m F$, $g = (z-a)^n G$, $F(a), G(a) \neq 0$

$\Rightarrow fg = (z-a)^{m+n} FG$ with $FG(a) \neq 0$.

$\overset{[i]}{\Longrightarrow}$ $\mathrm{ord}(fg, a) = m+n = \mathrm{ord}(f,a) + \mathrm{ord}(g,a)$.