

Lecture 4 1/11/21.

Internal direct sums.

→ if $N_1, \dots, N_m \subseteq M$ are submodules

then $N_1 + \dots + N_m =$

$$\{n_1 + \dots + n_m \mid n_i \in N_i\}.$$

which is an R -submodule,

the sum of the submodules.

Thm. M an R -module.

N_1, \dots, N_m R -submodules of M . if

(i) $N_1 + \dots + N_m = M$ and

(ii) $N_i \cap (N_1 + \dots + N_{i-1} + N_{i+1} + \dots + N_m) = 0$
for all i .

Then $M \cong N_1 \oplus \dots \oplus N_m$.

and we say M is an internal direct sum of the N_i .

Pf. By the group version,

$$\phi: N_1 \oplus \dots \oplus N_m \longrightarrow M$$

$$(n_1, \dots, n_m) \longmapsto n_1 + \dots + n_m$$

is an \cong of Abelian groups

Now notice ϕ is a homomorphism of R -modules.

Def. A surjective module homomorphism $f: N \rightarrow M$ is split if there is a homomorphism $g: M \rightarrow N$ s.t. $f \circ g = 1_M$.

Lemma if $f: N \rightarrow M$ is split then $N \cong M \oplus (\ker f)$

Pf. let $M' = g(M)$. $f \circ g = 1_M$
 $\Rightarrow g$ is injective. Then $M' \cong M$
We'll show N is internal direct sum of M' and $(\ker f) = K$.

* $M' \cap K = 0$: if $x \in M' \cap K$,
 $x = g(y)$ $y \in M$, $f(x) = 0$.

$$f(x) = f(g(y)) = y = 0$$

$$\text{So } x = g(y) = g(0) = 0.$$

$$\bullet M' + K = N :$$

If $y \in N$, consider $y - g(f(y))$.

$$\begin{aligned} \text{Then } f(y - g(f(y))) &= \\ &= f(y) - \overline{f(g(f(y)))} \\ &= f(y) - f(y) = 0. \end{aligned}$$

$$\text{So } y - g(f(y)) \in K = \ker(f).$$

$$\begin{aligned} \text{So } y &= g(f(y)) + (y - g(f(y))) \\ &\in M' + K. \end{aligned}$$

$$\begin{aligned} \text{So } N &\cong M' \oplus K \\ &\cong M \oplus K. \end{aligned}$$

Cor. If $f: M \rightarrow F$

is a R -module surjection,

and if F is free then

f is split, so $M \cong F \oplus (\ker f)$.

Pf. We want $g: F \rightarrow M$

s.t. $f \circ g = 1_F$. Let $\{e_\alpha\}$
be a basis for F . For each
 α fix $m_\alpha \in M$ s.t. $f(m_\alpha) = e_\alpha$.

Now there is a unique $g: F \rightarrow M$

s.t. $g(e_\alpha) = m_\alpha$ and so

$f \circ g(e_\alpha) = e_\alpha$ for all α ,

So $f \circ g = 1_F$. Now apply
the lemma to get $M \cong F \oplus \ker f$.

Now: Let R be commutative.
Goal: understand f.g. modules over
a PID R .

Def. If M is an R -module, $m \in M$
 $\text{ann}_R(m) = \{r \in R \mid rm = 0\}$
the annihilator of m .

The annihilator of M is

$\text{ann}_R(M) = \{r \in R \mid rm = 0 \ \forall m \in M\}$

$$= \bigcap_{m \in M} \text{ann}_R(m).$$

Note $\text{ann}_R(m) \supset \text{ann}_R(M)$ are ideals of R .

Def. Let R be an integral domain, M an R -module.

$m \in M$ is torsion if $\text{ann}_R(m) \neq 0$.

i.e. $rm = 0$ for some $r \neq 0$ in R .

Otherwise m is non-torsion.

$$\text{Tors}(M) = \{ m \in M \mid m \text{ is torsion} \}.$$

M is torsion if $M = \text{Tors}(M)$

and M is torsionfree if $\text{Tors}(M) = 0$.

Lemma. R an integral domain, M a module.

① $\text{Tors}(M)$ is a submodule of M .

② $M/\text{Tors}(M)$ is torsionfree.

Pf. if $m_1, m_2 \in \text{Tors}(M)$,

$$r_1 m_1 = 0 \quad r_2 m_2 = 0 \quad \text{of } r_1, r_2 \in R$$

$$\text{Then } r_1 r_2 (m_1 - m_2) = 0$$

and $r_1 r_2 \neq 0$ so $m_1 - m_2 \in \text{Tors}(M)$.

Also $r_1 (sm_1) = 0 \quad \forall s \in R$, so $sm_1 \in \text{Tors}(M)$.

②. if $r \neq 0$ and $r(m + \text{Tors}(M)) = 0$
then $rm + \text{Tors}(M) = 0$
So $rm \in \text{Tors}(M)$.

Then $s(rm) = 0$ $s \neq 0$.

Now $sr \neq 0$ so $m \in \text{Tors}(M)$

So $m + \text{Tors}(M) = 0$.

So $M/\text{Tors}(M)$ is torsion free.

Ex. $\mathbb{Q} = \mathbb{Q}$.

A module M is torsion iff every element of M has finite order. M is torsion free iff all $m \neq 0$ in M has infinite order. e.g. \mathbb{Q} is torsion free.

e.g. $\mathbb{Q}/(m)$ is torsion with annihilator $= (m)$.

Modules over PID.

Thm. Let R be a PID.

Let M be a f.g. R -module.

Then

$$\textcircled{1} \quad M \cong \overbrace{R \oplus \dots \oplus R}^r \oplus R/(p_1^{e_1}) \oplus \dots \oplus R/(p_m^{e_m})$$

for some primes p_1, \dots, p_m .

$$e_i \geq 1.$$

r is the rank of M

$p_1^{e_1}, \dots, p_m^{e_m}$ are the
elementary divisors.

$\textcircled{2}$ Also, r and the
elementary divisors are
unique, up to rearranging
the elementary divisors or
replacing p_i by an associate.

Rules.

① M being f.g. is essential.

\mathcal{Q} is a \mathcal{U} -module which is not a direct sum of cyclic \mathcal{U} -modules (so not free).

\mathcal{Q} can't be an internal direct sum $N \oplus P$

since if $0 \neq N \subseteq \mathcal{Q}$

$0 \neq P \subseteq \mathcal{Q}$ then

$$N \cap P \neq 0.$$

② R being a PID is essential, since

$$Rx + Ry \subseteq K[x, y]$$

is not a direct sum of
cyclics.