Math 220 A - Lecture 2

Oct 7, 2020

Last time

[i] $f: U \longrightarrow \mathbb{C}$ is holomorphic provided $\forall z \in U$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

[ii] $f = u + iv$ holomorphic $\Longrightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(Cauchy Riemann)

$$\Longleftrightarrow \frac{\partial f}{\partial \bar{z}} = 0.$$

[iii] u, v are harmonic conjugates

$$Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Today:

[i] geometric consequences

[ii] analytic functions & power series

[iii] logarithm

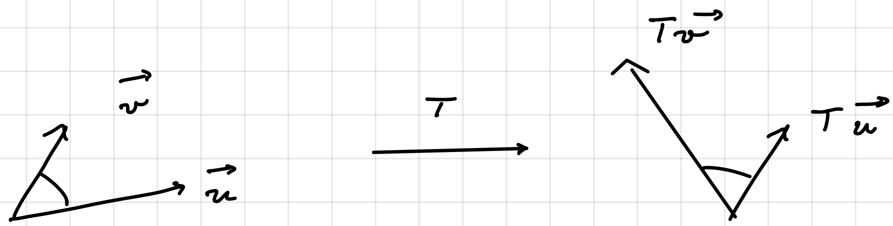
1. Geometric consequences / Conformal maps

Def $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\mathbb{R}$-linear, invertible

$\boxed{\text{i}}$ T is orientation preserving if $\det T > 0$.

$\boxed{\text{ii}}$ T is angle preserving if for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$

$$\measuredangle(\vec{u}, \vec{v}) = \measuredangle(T\vec{u}, T\vec{v}).$$



Remark $T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is both orientation & angle preserving (unless $a = b = 0$).

- $\det T = a^2 + b^2 > 0$ if $(a, b) \neq (0, 0)$

- $^tT\,T = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = |\alpha|^2 \cdot \mathbb{1}$, $\alpha = a + bi$

$$\Rightarrow \quad Tu \cdot Tv = {}^tT\,Tu \cdot v = |\alpha|^2\, u\cdot v$$

If $u = v \Rightarrow Tv \cdot Tv = |\alpha|^2\, v\cdot v \Rightarrow \|Tv\| = |\alpha|\cdot\|v\|$

$$\cos\measuredangle(u,v) = \frac{u\cdot v}{\|u\|\cdot\|v\|}$$

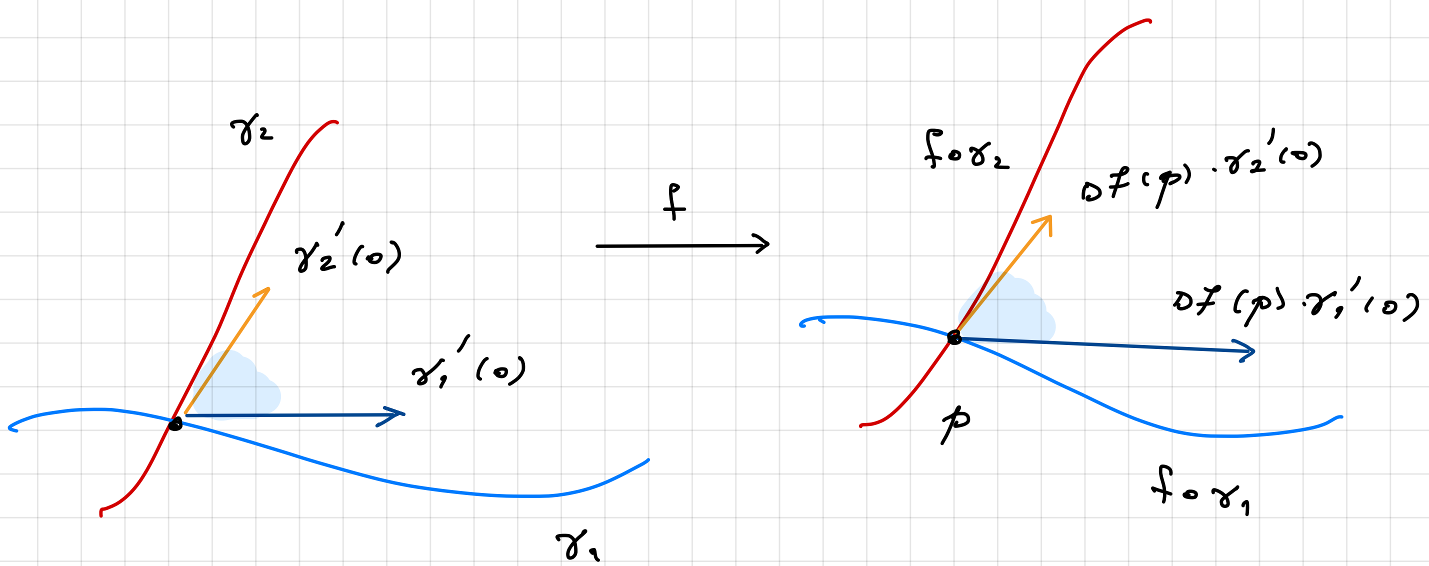
$$\cos\measuredangle(Tu,Tv) = \frac{Tu\cdot Tv}{\|Tu\|\cdot\|Tv\|} = \frac{|\alpha|^2\, u\cdot v}{|\alpha|\cdot\|u\|\cdot|\alpha|\cdot\|v\|}$$

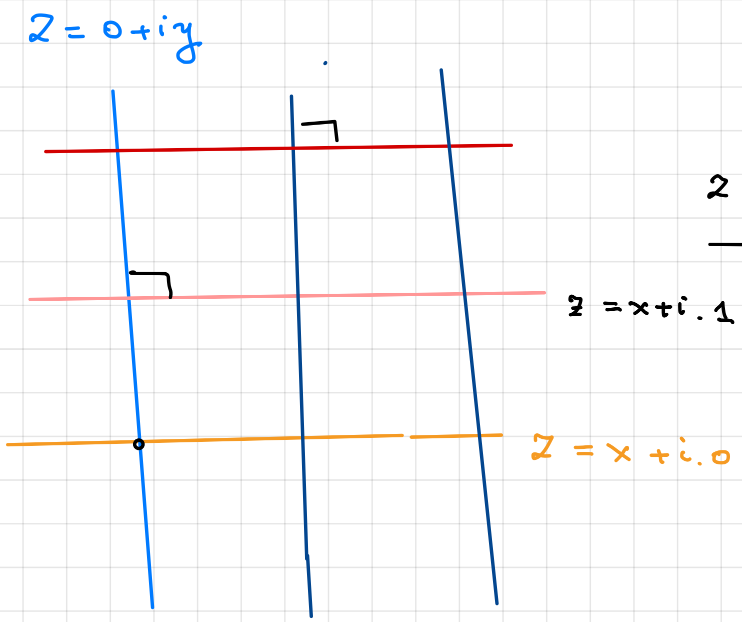
$$\Rightarrow \measuredangle(u,v) = \measuredangle(Tu,Tv)$$

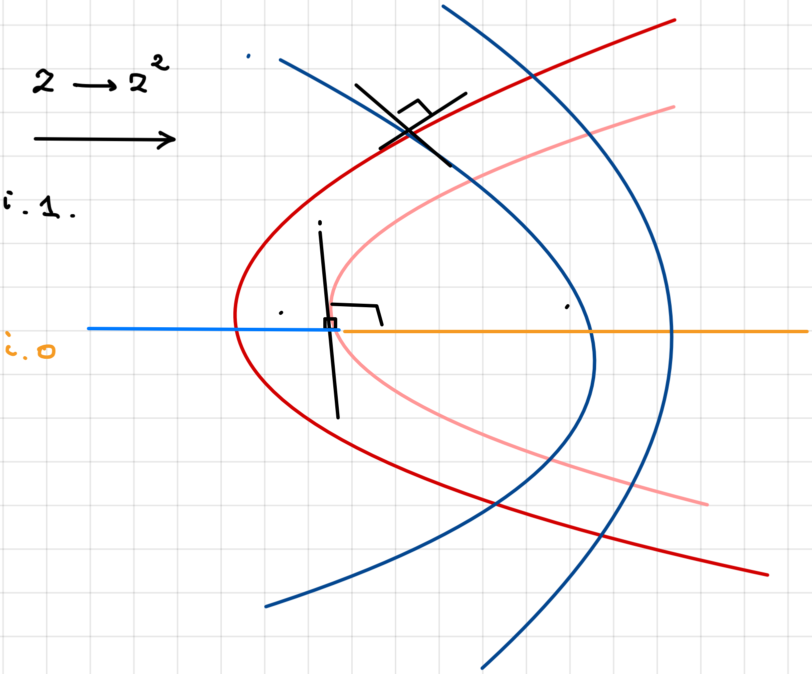
Remark f holomorphic $\Rightarrow$ either $f'(z) = 0$ or else

$Df(z)$ is orientation & angle preserving.

$\Rightarrow$ "f preserves angles" (f "conformal")







$f(z) = z^2$

$z = x + i1 \Rightarrow z^2 = \underbrace{x^2 - 1}_{u} + \underbrace{2x}_{v}\, i \Rightarrow$

$\Rightarrow u = -1 + \frac{v^2}{4}$ parabola

$z = 0 + i \cdot y \Rightarrow$ half line

$z = 1 + i \cdot y \Rightarrow z^2 = \underbrace{1 - y^2}_{u} + \underbrace{2y}_{v}\, i$

$\Rightarrow u = 1 - \frac{v^2}{4}$ parabola

<u>Check</u> : Angles are preserved.

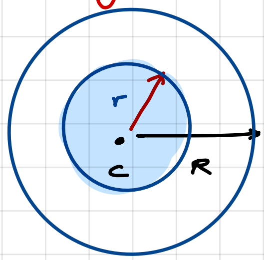
$\boxed{2}$ Power series & Analytic functions $\qquad c \in \mathbb{C},\ a_n \in \mathbb{C}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n \qquad (*)$$

Def / Theorem (Abel) $\exists\, R \quad 0 \le R \le \infty$ such that

$\boxed{i}$ if $|z-c| < R \implies (*)$ converges.

If $0 \le r < R \implies (*)$ converges absolutely & uniformly in $\Delta(c, r)$.



$\boxed{ii}$ if $|z-c| > R \implies (*)$ diverges.

Furthermore $\quad R^{-1} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}. \qquad R$ = radius of convergence.

Definition $f : U \longrightarrow \mathbb{C}$ is analytic if $\forall\, z_0 \in U \ \exists\, R > 0$

such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{in} \quad \Delta(z_0, R) \subseteq U.$$

Proof WLOG $c = 0$, else work $z^{new} = z - c$.

$\sum_{n=0}^{\infty} a_n z^n$. We set $R^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Let $|z| < r$

$\boxed{i}$. Let $r < \rho < R$. $\Rightarrow \limsup \sqrt[n]{|a_n|} = \frac{1}{R} < \frac{1}{\rho} \Rightarrow$

$$\Rightarrow \sqrt[n]{|a_n|} < \frac{1}{\rho} \quad \text{if } n \geq N.$$

$$\Rightarrow |a_n| < \frac{1}{\rho^n} \quad \text{if } n \geq N$$

$$\Rightarrow \underbrace{|a_n z^n|}_{f_n(z).} < \underbrace{\left(\frac{r}{\rho}\right)^n}_{M_n} \quad \text{if } n \geq N.$$

By Weierstraß M-test

$|f_n| \leq M_n$, $\sum_{n=0}^{\infty} M_n < \infty \Rightarrow \sum_{n=0}^{\infty} f_n$ converges absolutely & uniformly.

$\Rightarrow \sum_n a_n z^n$ converges absolutely & uniformly in $\Delta(0, r)$.

$\boxed{ii}$ If $|z| > \rho > R \Rightarrow \limsup \sqrt[n]{|a_n|} = \frac{1}{R} > \frac{1}{\rho}$

$$\Rightarrow \sqrt[n]{|a_n|} > \frac{1}{\rho} \text{ for infinitely many } n\text{'s}.$$

$$\Rightarrow |a_n| > \frac{1}{\rho^n} \text{ for infinitely many } n\text{'s}.$$

$$\Rightarrow |a_n z^n| > \Big(\underbrace{\frac{|z|}{\rho}}_{>1}\Big)^n \text{ for infinitely many } n\text{'s}.$$

$$\Rightarrow a_n z^n \not\to 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n \text{ diverges}$$

Differentiation

Recall that if $f_n \longrightarrow f$ it doesn't follow $f_n' \longrightarrow f'$ in general. However, for power series we have

Theorem (Rudin 8.1).

If $\sum_{n\geq 0} a_n (z-c)^n$ has radius of convergence R, then $\sum_{n\geq 1} n a_n (z-c)^{n-1}$ has radius of convergence R as well.

Furthermore, if

$$f(z) = \sum_{n\geq 0} a_n (z-c)^n \quad \text{in} \quad \Delta(c,R)$$

$$\Longrightarrow f'(z) = \sum_{n\geq 1} n a_n (z-c)^{n-1} \quad \text{in} \quad \Delta(c,R).$$

Corollary $f^{(k)}(z) = \sum_{n\geq k} a_n n(n-1)\dots(n-k+1)(z-c)^{n-k}$

$$\overset{z=c}{\Longrightarrow} f^{(k)}(c) = a_k k! \Rightarrow a_k = \frac{f^{(k)}(c)}{k!}$$

$$\Rightarrow f(z) = \sum_{n\geq 0} \frac{f^{(n)}(c)}{n!}(z-c)^n \quad \text{if } f \text{ is analytic in } \Delta(c,R).$$

Remark If f is analytic $\overset{\text{Thm}}{\Longrightarrow}$ f is holomorphic.

Examples : exp, cos, sin

(i) $e^z := 1 + z + \frac{z^2}{2} + \dots + \frac{z^n}{n!} + \dots$, $R = \infty$. $f(z)$.

$$\limsup_{n\to\infty} \sqrt[n]{n!} \geq \limsup_{n\to\infty} \sqrt[n]{\left(\frac{n}{2}\right)^{n/2}} = \limsup_{n\to\infty} \sqrt{\frac{n}{2}} = \infty.$$

$$f'(z) = 0 + 1 + z + \dots + \frac{z^{n-1}}{(n-1)!} + \dots = f(z)$$

$\Longrightarrow$ $(e^z)' = e^z$.

$\Longrightarrow$ $e^{z+c} = e^z \cdot e^c$. (Both sides satisfy $y' = y$, $y(0) = e^c$ so they are equal)

(ii) $\cos z := \dfrac{e^{iz} + e^{-iz}}{2} = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \dots$

$\sin z := \dfrac{e^{iz} - e^{-iz}}{2i} = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \dots$

$(\sin z)' = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \dots = \cos z$.

$\sin^2 z + \cos^2 z = 1$.

Beware ! $\sin z$, $\cos z$ are not bounded functions as $z \in \mathbb{C}$

$$\cos in\pi = \frac{e^{-n\pi} + e^{n\pi}}{2} \longrightarrow \infty \quad \text{as } n \longrightarrow \pm\infty.$$

(iii) z^n can be defined for all $n \in \mathbb{Z}$, if $z \neq 0$.

3 Logarithm

Remark

$e^{2\pi i} = 1 \implies$ exponential is not invertible.

$$\log 1 = 0, \pm 2\pi i, \pm 4\pi i, \dots, \pm 2n\pi i$$

Question Define $\log z$. ?

Remark Issues also arise with $\sqrt[n]{z}$ and z^α.

These are related to the logarithm.

$$\sqrt[n]{z} \longleftrightarrow z^\alpha \quad \text{for } \alpha = \frac{1}{n}$$

$$z^\alpha := \exp(\alpha \log z)$$

Def A logarithm function $\ell : U \longrightarrow \mathbb{C}$ is a continuous function such that

$$e^{\ell(z)} = z \quad \forall\, z \in U.$$

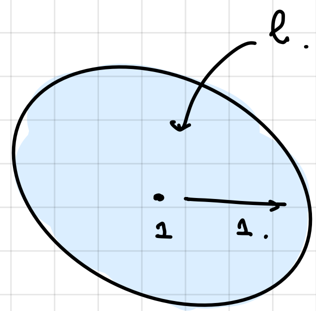
Naturally, for this to make sense, we need $U \subseteq \mathbb{C} \setminus 0$.

Any two logarithms ℓ', ℓ on U differ by $2\pi i n$, $n \in \mathbb{Z}$.

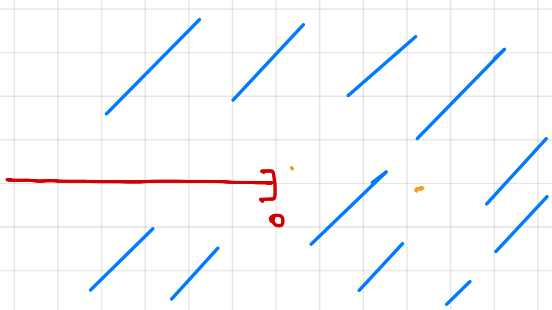
Example A $U = \Delta(1,1)$,

$$\ell(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

HWK : ℓ is a logarithm in U.



Example B $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$



$z \in U$

$z = r e^{i\theta}$, $\theta \in (-\pi, \pi)$.

$r \neq 0$

$$\operatorname{Log} z = \log r + i\theta$$

$$\Rightarrow e^{\operatorname{Log} z} = e^{\log r + i\theta} = r e^{i\theta} = z \Rightarrow \operatorname{Log} \text{ is a logarithm in } \mathbb{C} \setminus \mathbb{R}_{\leq 0}.$$

Remark The two examples above give the same answer in $\Delta(1,1)$.

Indeed the two logarithms $\ell(z)$ and $\mathcal{L}og\, z$ differ by $2\pi i n$ $\Rightarrow$ $\mathcal{L}og\, z - \ell(z) = 2\pi i n$. Set $z=1$

$$\Rightarrow \underbrace{\mathcal{L}og\, 1}_{0} - \underbrace{\ell(1)}_{0} = 2\pi i n \Rightarrow n = 0$$

$$\Rightarrow \mathcal{L}og\, z = \ell(z) \text{ in } \Delta(1,1).$$