Math 220 A – Lecture 22

December 4, 2020

$\boxed{0}$ **Last time** In real analysis we encounter periodic functions. In complex analysis:

Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$.

Def An elliptic function f satisfies

$\boxed{i}$ f meromorphic on $\mathbb{C}$

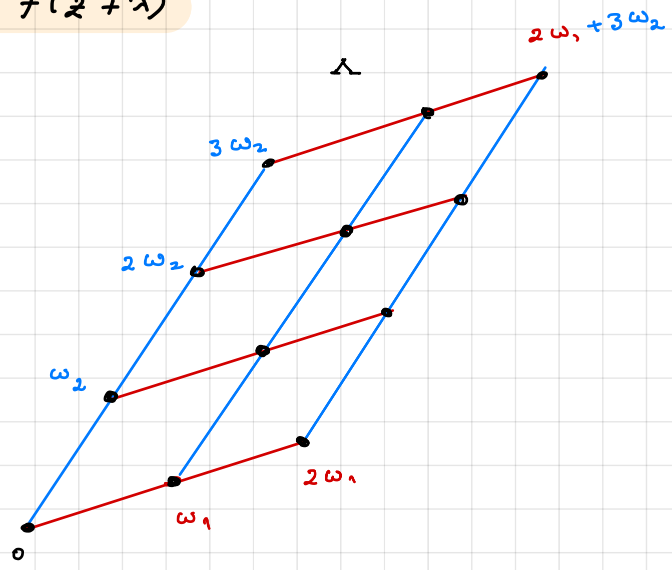
$\boxed{ii}$ f doubly periodic:

$$f(z) = f(z + \omega_1) = f(z + \omega_2) \quad \forall z$$

Remark

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{ m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \}$$

$(*)$: $\forall \lambda \in \Lambda, \ f(z) = f(z + \lambda)$



[1] Basic Properties of Elliptic Functions

Note that Λ is a subgroup of $\mathbb{C}$.

Define

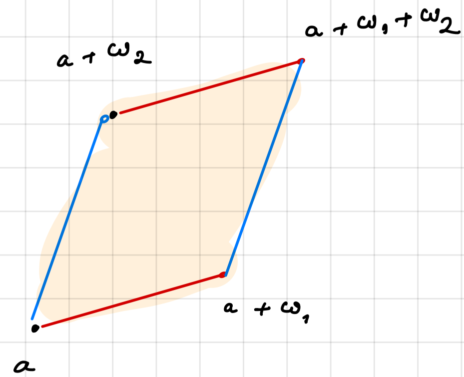
$$z \equiv w \mod \Lambda \iff z - w \in \Lambda.$$

$$z \equiv w \mod \Lambda \overset{(*)}{\Longrightarrow} f(z) = f(w).$$

Remark f is determined by values mod Λ

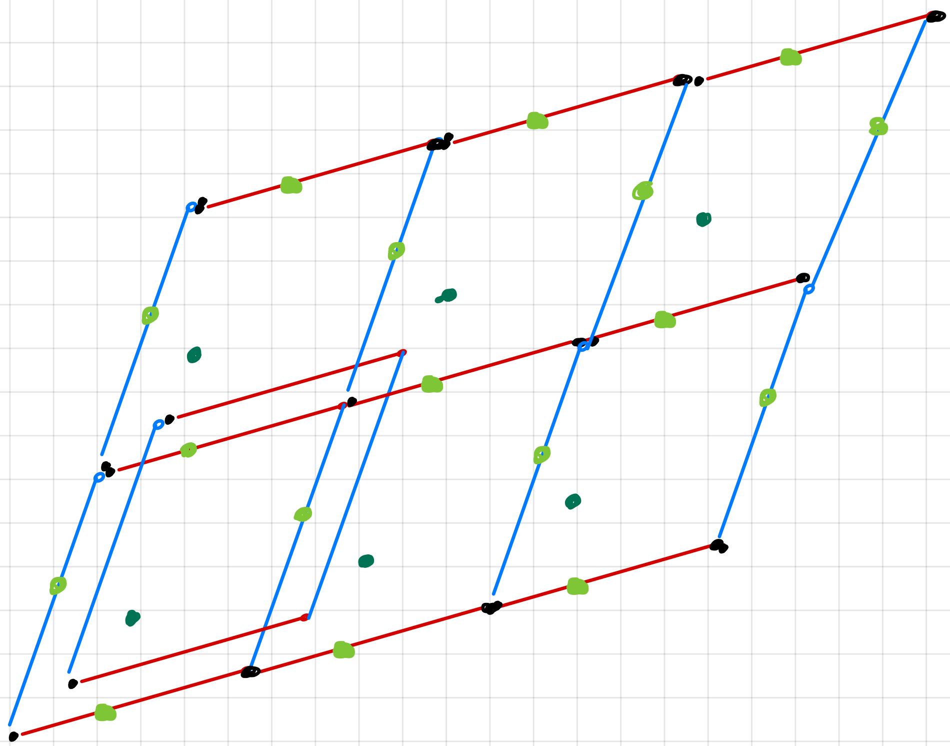
We will restrict f to a parallelogram.

$$P_a = \{ a + t_1 \omega_1 + t_2 \omega_2 : 0 \le t_1 \le 1,\ 0 \le t_2 \le 1 \}.$$



Each point in $\mathbb{C}$ is congruent to a point in P_a.

(see next picture)



Claim $\exists$ a such that ∂P_a contains no zeroes / poles.

Proof Start with any a. Since P_a is compact & zeroes / poles are discrete $\Rightarrow \exists$ finitely many of them in P_a. A suitable translation would ensure ∂P_a avoids them.

Write $P = P_a$ where P is chosen as above.

Remark

If f holomorphic in $\mathbb{C}$ $\Rightarrow$ $f|_P$ continuous

$\overset{P \text{ compact}}{\Rightarrow}$ $f|_P$ bounded

$\overset{\text{periodic}}{\Rightarrow}$ f bounded

$\overset{\text{Liouville}}{\Rightarrow}$ f constant

Thus in general f will have poles.

Notation zeros in P : $\alpha_1 \dots \alpha_k$ (w/ multiplicity)

poles in P : $\beta_1 \dots \beta_\ell$ (w/ multiplicity)

Theorem [i] $k = \ell$: # zeroes (f) = # Poles (f)

[ii] $\displaystyle\sum_{i=1}^{k} \alpha_i \equiv \sum_{i=1}^{k} \beta_i \mod \Lambda.$

Remark Given $\alpha_1 \dots \alpha_k$, $\beta_1 \dots \beta_k$ with

$$\sum_i \alpha_i \equiv \sum_i \beta_i \mod \Lambda$$

there is an elliptic function with these zeroes/poles.

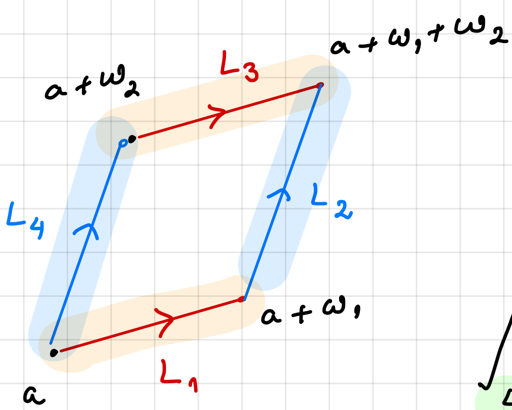
This is not obvious. $\rightsquigarrow$ Abel–Jacobi theory

Proof $\boxed{C}$ By the Argument Principle

$$\frac{1}{2\pi i} \int_{\partial P} \frac{f'}{f}\, dz = \#\ \text{Zeroes}(f) - \#\ \text{Poles}(f) \text{ in } P.$$

We show $\displaystyle\int_{\partial P} \frac{f'}{f}\, dz = 0$. Let $\partial P = L_1 + L_2 + (-L_3) + (-L_4)$

We show $\displaystyle\int_{L_1} \frac{f'}{f}\, dz = \int_{L_3} \frac{f'}{f}\, dz$ & $\displaystyle\int_{L_2} \frac{f'}{f}\, dz = \int_{L_4} \frac{f'}{f}\, dz$.



Both claims follow by periodicity.

$$\int_{L_1} \frac{f'}{f}\, dz = \int_{L_1} \frac{f'}{f}(z+\omega_2)\, dz = \int_{L_3} \frac{f'}{f}\, dz$$

$(L_3 = L_1 + \omega_2)$.

$\boxed{c)}$ We use the Enhanced Argument Principle $(g(z)=z)$.

$$\frac{1}{2\pi i}\int_{\partial P} z\,\frac{f'}{f}\,dz = \sum_{i=1}^{k}\alpha_i - \sum_{i=1}^{k}\beta_i .$$

We show
$$\frac{1}{2\pi i}\left(\int_{L_1} z\,\frac{f'}{f}\,dz - \int_{L_3} z\,\frac{f'}{f}\,dz\right) \in \Lambda \quad \text{and}$$

$$\frac{1}{2\pi i}\left(\int_{L_2} z\,\frac{f'}{f}\,dz - \int_{L_4} z\,\frac{f'}{f}\,dz\right) \in \Lambda$$

This will complete the proof.

We only consider 1st expression. $L_3 = L_1 + \omega_2$

f periodic

$$\frac{1}{2\pi i}\left(\int_{L_1} z\,\frac{f'}{f}\,dz - \int_{L_3} z\,\frac{f'}{f}\,dz\right) = \frac{1}{2\pi i}\left(\int_{L_1} \cancel{z}\,\frac{f'}{f}\,dz - \int_{L_1} (\cancel{z}+\omega_2)\,\frac{f'}{f}\,dz\right)$$

$$= -\frac{1}{2\pi i}\,\omega_2 \cdot \int_{L_1}\frac{f'}{f}\,dz$$

$w = f(z).$

$$= -\left(\frac{1}{2\pi i}\int_{f(L_1)}\frac{dw}{w}\right)\cdot \omega_2$$

$$= -\underbrace{n\,(f(L_1), 0)}_{\text{integer}}\,\omega_2 \in \Lambda .$$

Note that $f(L_1)$ is a loop (by periodicity). not containing 0.

2. Rouché's theorem (Conway V. 3).

Idea $f, g : U \longrightarrow \mathbb{C}$ holomorphic

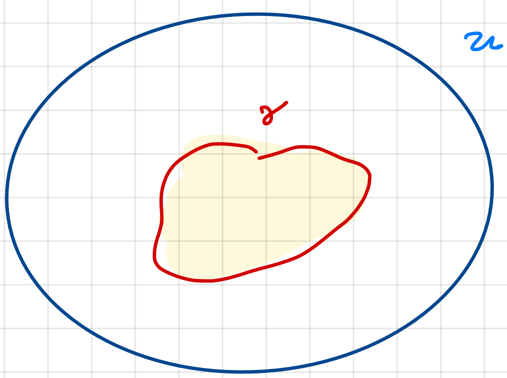
$$f = \underbrace{g}_{\text{dominant term}} + \text{lower order terms}$$

$\Longrightarrow$ # zeroes (f) = # zeroes (g). (w/ multiplicity).

We can ignore the lower order terms.

Setup: $\gamma \subseteq \mathcal{U}$ simple closed curve, $\operatorname{Int} \gamma \subseteq \mathcal{U}$.

e.g. $\gamma = \partial \Delta$, $\overline{\Delta} \subseteq \mathcal{U}$.



Theorem $f, g : U \longrightarrow \mathbb{C}$ holomorphic, γ as above.

If $|f-g| < |g|$ on γ $\Rightarrow$

$$\# \text{zeroes}(f) = \# \text{zeroes}(g). \text{ in } \operatorname{Int}(\gamma).$$

(w/ multiplicity)

Note that $f \neq 0$ & $g \neq 0$ on γ.

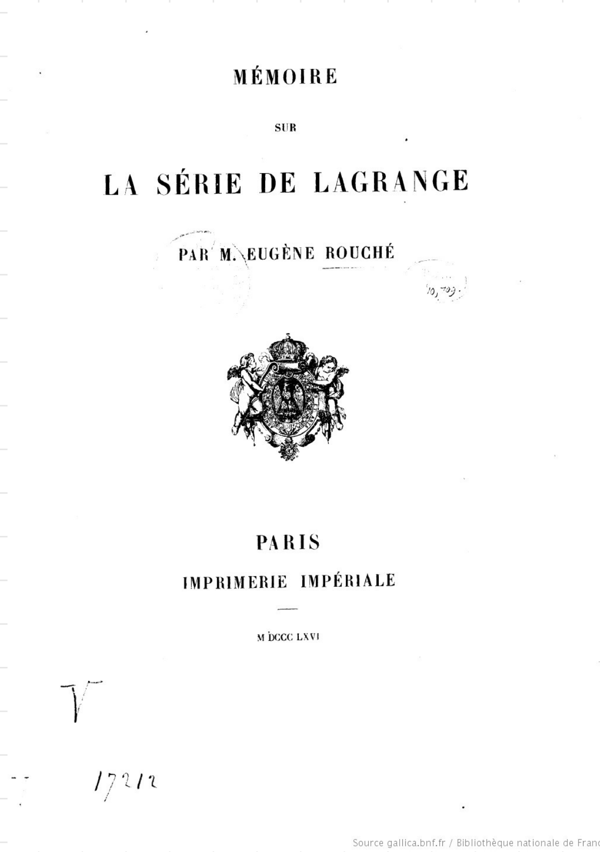
Remark Conway's version is more general but less useful in practice.

Conway.

- f, g meromorphic
- $|f-g| < |f| + |g|$ on γ

$$\Rightarrow \# \text{zeroes}(f) - \# \text{Poles}(f) = \# \text{zeros}(g) - \# \text{Poles}(g)$$

in $\operatorname{Int} \gamma$.



MÉMOIRE

SUR

LA SÉRIE DE LAGRANGE

PAR M. EUGÈNE ROUCHÉ

PARIS

IMPRIMERIE IMPÉRIALE

M DCCC LXVI

Eugene Rouché

(1832 – 1910)

Example

dominant term

$\boxed{1}$ $f = z^5 + 24z^3 + 2z^2 + 3z + 1$

How many roots in $|z|<1$.

Let $g = 24z^3$. and $\gamma = \{|z|=1\}$.

We verify $|f-g| < |g|$. when $|z|=1$.

Note $|g| = 24\,|z|^3 = 24$.

triangle inequality

$$|f-g| = |z^5 + 2z^2 + 3z + 1| \le |z|^5 + 2|z|^2 + 3|z| + 1$$

$$= 1 + 2 + 3 + 1 = 6.$$

$\Rightarrow$ $|f-g| < |g|$ on γ

$\Rightarrow$ # Zeroes(f) = # zeroes $(g) = 3$ in $\{|z|<1\}$.

Example [ii] Fundamental Theorem of Algebra

$$f = z^n + a_1 z^{n-1} + \dots + a_n$$

$g = z^n$ = dominant term when $|z|$ large.

$$f - g = a_1 z^{n-1} + \dots + a_n.$$

When $|z| = R$,

$$|f-g| \le |a_1| R^{n-1} + \dots + |a_n| < R^n = |z|^n = |g|.$$

This happens for R large as $\lim\limits_{R \to \infty} \dfrac{R^n}{|a_1| R^{n-1} + \dots + |a_n|} = \infty$.

By Rouché:

$\#$ Zeroes $(f) = \#$ Zeroes $(g) = n$ in $\Delta(0,R)$. $\forall R \gg 0$

$\Rightarrow \#$ Zeroes $(f) = n$. in $\mathbb{C}$