Math 220 A – Lecture 15

November 16, 2020

Last time We wish to prove:

Residue Theorem $U \subseteq \mathbb{C}$ open connected, S discrete

- $\gamma \overset{U}{\sim} 0$, $\{\gamma\} \subseteq U \setminus S$.
- f holomorphic in $U \setminus S$, singularities at S.

Then

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$$

Example

$$\int_{|z|=3} \frac{z+1}{z^2(z-1)}\,dz$$

Take $U = \Delta(0,4)$, $S = \{0,1\}$, $f(z) = \dfrac{z+1}{z^2(z-1)}$.

- $\operatorname{Res}(f,0) = \operatorname{Res}_{z=0} \dfrac{\frac{z+1}{z-1}}{z^2} = \left(\dfrac{z+1}{z-1}\right)'\Big|_{z=0} = -2$

by Method 2 of computing residues last time

- $\operatorname{Res}(f,1) = \operatorname{Res}_{z=1} \dfrac{z+1}{z^2(z-1)} = \dfrac{(z+1)\big|_{z=1}}{\left(z^2(z-1)\right)'\big|_{z=1}} = \dfrac{2}{1} = 2$

by Method 1 of computing residues last time.

Thus $\displaystyle\int_{|z|=3} f\,dz = 2\pi i\left(\operatorname{Res}(f,0) + \operatorname{Res}(f,1)\right)$

$$= 2\pi i\,(-2+2) = 0.$$

1. Proof of the Residue Theorem

Terminology $U^* \subseteq \mathbb{C}$, $\gamma^* = \sum_{i=1}^{\ell} m_i \gamma_i$ C^1-chain

[a] $$\int_{\gamma^*} f\, dz = \sum_{i=1}^{\ell} m_i \int_{\gamma_i} f\, dz$$

[b] $$n(\gamma^*, a) = \sum_{i=1}^{\ell} m_i\, n(\gamma_i, a)$$

Definition $\gamma^* \overset{U^*}{\approx} 0$ if $n(\gamma^*, a) = 0 \quad \forall a \notin U^*$.

(we say γ^* is null homologous in U^*).

Remark [1] γ^* loop in U^*. Then

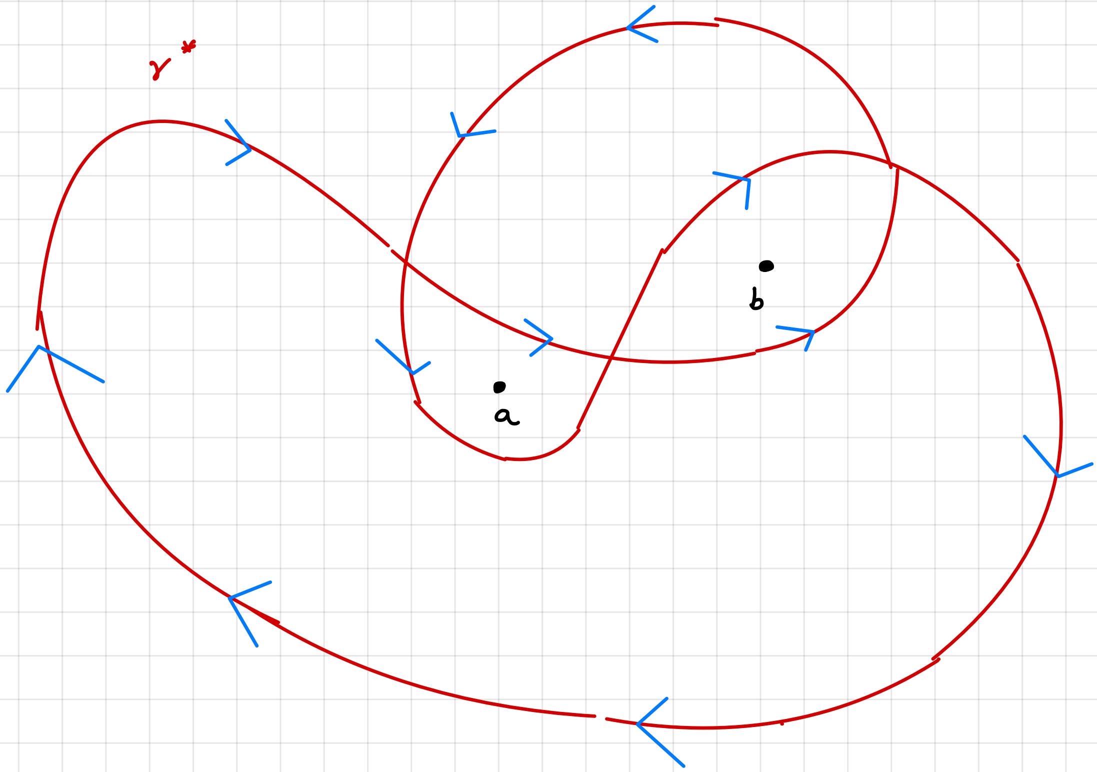
$$\gamma^* \overset{U^*}{\sim} 0 \implies \gamma^* \overset{U^*}{\approx} 0.$$

Indeed if $a \notin U^*$ then

$$n(\gamma^*, a) = \frac{1}{2\pi i} \int_{\gamma^*} \frac{dw}{w-a} = 0$$

by homotopy form of Cauchy applied to $\gamma^* \overset{U^*}{\sim} 0$ and to the holomorphic function $\dfrac{1}{w-a}$ in U^*. $(a \notin U^*)$

$\boxed{\text{ii}}$ the converse is false $U^* = \mathbb{C} \setminus \{a, b\}$



Check $\gamma^* \overset{U^*}{\approx} 0$. Indeed $n(\gamma^*, a) = n(\gamma^*, b) = 0$. To see this,

find two subloops of γ^* going clockwise &

counter clockwise once around a (same for b).

However $\gamma^* \overset{U^*}{\not\sim} 0$.

Remark* In algebraic topology, one learns that 1st homology

is the abelianization of π_1. (which is defined via homotopy).

Enhanced Cauchy's Theorem

We seek to prove a "homology" version of Cauchy:

Theorem $f : U^* \longrightarrow \mathbb{C}$ holomorphic, $\gamma^* \overset{U^*}{\approx} 0 \implies \int_{\gamma^*} f \, dz = 0.$

Of course, this implies the homotopy version of the theorem proved in previous lectures.

Remark We show next

Enhanced Cauchy Theorem $\implies$ Residue Theorem.

Proof of residue theorem We let f holomorphic in $\mathcal{U} \setminus S$, $\gamma \overset{\mathcal{U}}{\sim} 0$. We want

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$$

Last time we saw RHS is finite since

$$\{ s \in S : n(\gamma, s) \neq 0 \} \text{ is finite.}$$

Enumerate this set to be $\{a_1, \dots a_k\}$, $m_i = n(\gamma, a_i) \neq 0$.

Let Δ_i be small disjoint discs near a_i. $\Delta_i \subseteq \mathcal{U}$.

Define
- $\mathcal{U}^* = \mathcal{U} \setminus S$
- $\gamma^* = \gamma + \sum_{i=1}^{k} (-m_i) \, C_i$ where $C_i = \partial \Delta_i$

(positive orientation)

Claim $\gamma^* \overset{\mathcal{U}^*}{\approx} 0$

Enhanced Cauchy for $(\mathcal{U}^*, \gamma^*)$ $\Rightarrow \int_{\gamma^*} f \, dz = 0$

$$\Rightarrow \frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{i=1}^{k} m_i \cdot \frac{1}{2\pi i} \int_{C_i} f \, dz$$

$$= \sum_{i=1}^{k} m_i \operatorname{Res}(f, a_i) \text{ by toy example}$$

last time. QED.

Proof of the claim Want $n(\gamma^*, a) = 0$ if $a \notin U^*$.

$\boxed{i}$ if $a \notin U$. Note $\gamma \overset{U}{\sim} 0 \Rightarrow \gamma \overset{U}{\approx} 0 \Rightarrow n(\gamma, a) = 0$.

Also $a \notin \Delta_i \Rightarrow n(c_i, a) = 0$ Then

$$n(\gamma^*, a) = \underbrace{n(\gamma, a)}_{0} + \sum (-m_i) \underbrace{n(c_i, a)}_{0} = 0.$$

$\boxed{ii}$ if $a \in S$. Note that $n(c_i, a) = \begin{cases} 0 & \text{if } a \neq a_i \\ 1 & \text{if } a = a_i \end{cases}$

If $a = a_i \Rightarrow n(\gamma^*, a) = \underbrace{n(\gamma, a)}_{m_i} + (-m_i) \underbrace{n(c_i, a)}_{1} = m_i + (-m_i) = 0.$

If $a \neq a_i \ \forall i \Rightarrow n(\gamma, a) = 0$ by definition of the a_i's

$$\Rightarrow n(\gamma^*, a) = n(\gamma, a) + \sum (-m_i) \underbrace{n(c_i, a)}_{0} = 0.$$

Remarks

I. Proof of residue thm only requires $\gamma \overset{u}{\approx} 0$ not $\gamma \overset{u}{\sim} 0$. $\rightsquigarrow$ improvement of hypothesis.

II. Residue Theorem $\Rightarrow$ Enhanced CIF for derivatives.

Let $\gamma \overset{u}{\approx} 0$. Apply the residue theorem: $S = \{a\}$.

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{k+1}}\, dz = n(\gamma, a) \operatorname*{Res}_{z=a} \frac{f(z)}{(z-a)^{k+1}}$$

$$= n(\gamma, a) \cdot \frac{f^{(k)}(a)}{k!}$$

(using Method 2 from last time)

2. Proof of Enhanced Cauchy's Theorem

- change notation $U \longleftrightarrow U^*$, $\gamma \longleftrightarrow \gamma^*$

- modify statement slightly

Theorem (enhanced CIF)

$\gamma \overset{U}{\approx} 0$, $f : U \longrightarrow \mathbb{C}$ holomorphic, $a \in U$.

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = n(\gamma, a)\, f(a).$$

Remark Using the above for $f^{new}(z) = f(z)\cdot(z-a)$, $f^{new}(a) = 0$

we obtain $\gamma \overset{U}{\approx} 0 \Rightarrow \int_\gamma f\,dz = 0$. This is Enhanced Cauchy.

Remark TFAE:

Enhanced CIF $\Rightarrow$ Enhanced Cauchy's theorem

above

$\Rightarrow$ Residue Theorem

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$\Rightarrow$ Enhanced CIF for derivatives

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$k=0$

Next time we give Dixon's proof of Enhanced CIF.

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A BRIEF PROOF OF CAUCHY'S INTEGRAL THEOREM

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ABSTRACT. A short proof of Cauchy's theorem for circuits homologous to 0 is presented. The proof uses elementary local properties of analytic functions but no additional geometric or topological arguments.

The object of this note is to present a very short and transparent proof of Cauchy's theorem for circuits homologous to 0. The proof is based on simple 'local' properties of analytic functions that can be derived from Cauchy's theorem for analytic functions on a disc, and it may be compared with the treatment in Ahlfors [**1**, pp. 137–145]. It is apparent from this proof that this version of Cauchy's theorem is not only much more natural than the homotopic version which appears in several recent textbooks; it is also much easier to prove (contra Dieudonné [**2**, p. 192]). It is reasonable to argue that the concept of homotopy in connection with Cauchy's theorem is as extraneous as the notion of Jordan curve.

We recall that if γ is a circuit (="continuous, piecewise smooth, closed curve"), and $w\in\mathbf{C}$ does not lie on γ, then the *index* of w with respect to γ is $\operatorname{Ind}(\gamma, w)=(2\pi i)^{-1}\int_\gamma (z-w)^{-1}dz$. It is easily proved that $E=\{w\in\mathbf{C}\mid \operatorname{Ind}(\gamma, w)=0\}$ contains a neighbourhood of ∞ and is open (see [**1**, p. 116]). In the following proof we give full references to the 'local' properties used in order to emphasize the elementary nature of the proof.

CAUCHY'S THEOREM. *Let D be an open subset of $\mathbf{C}$ and let γ be a circuit in D. Suppose that γ is homologous to 0 in D, i.e. each $w\notin D$ lies in the set E defined above. Then, for each f analytic on D*:

(i) $\int_\gamma f(z)dz=0$;

(ii) $\operatorname{Ind}(\gamma, w)f(w)=(2\pi i)^{-1}\int_\gamma (z-w)^{-1}f(z)dz$ *for all* $w\in D$ *not lying on* γ.

PROOF. Consider $g: D\times D\to\mathbf{C}$ defined by $g(w, z)=(f(z)-f(w))/(z-w)$ for $z\neq w$ and $g(w, w)=f'(w)$. Then g is continuous, and for each fixed z, $w\mapsto g(w, z)$ is analytic [**1**, p. 124]. Define $h:\mathbf{C}\to\mathbf{C}$ by $h(w)$

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