MATH 200B MIDTERM SOLUTIONS

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Problem 1. Let R be an integral domain. Recall that a left $R-$module M is called divisible if for all $x \in M$, and $0 \neq r \in R$, there exists $y \in M$ such that $ry = x$.

(a). Let M be any left $R-$module and let N be a torsion left $R-$module. Prove that $M \otimes_R N$ is again a torsion left $R-$module.

(b). Let M be a divisible left $R-$module and again let N be a torsion left $R-$module. Prove that $M \otimes_R N = 0$.

Proof. (a). Let $\alpha = m_1 \otimes n_1 + m_2 \otimes n_2 + \cdots m_t \otimes n_t$ be an element in $M \otimes N$. Suppose all the summands are torsion elements, then there exist nonzero elements $r_1, \cdots r_t$in R such that $r_i(m_i \otimes n_i) = 0$. Note that R is an integral domain so $r = r_1 r_2 \cdots r_t$ is nonzero. We easily see that $r\alpha = 0$, so α is also a torsion element. Thus, it suffices to show that pure tensors are torsion elements. Let $m \otimes n$ be an element in $M \otimes N$. Since N is torsion there is some $0 \neq r$ such that $rn = 0$. Now $r \cdot (m \otimes n) = m \otimes (r \cdot n) = 0$.

(b). Again, it suffices to show that pure tensors are 0 (by applying a similar argument as in the beginning of part a, noting that $\alpha = 0$ if all the summands are 0). With the notations above, we may find $m_0 \in M$ such that $r \cdot m_0 = m$. Then $m \otimes n = (r \cdot m_0) \otimes n = r \cdot (m_0 \otimes n) = m_0 \otimes (r \cdot n) = 0$. $\square$

Problem 2. Let R be a PID. Suppose that there exists a nonzero finitely generated divisible $R-$module M. Prove that R is a field.

Proof. By the classification theorem we may write M as $R^t \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots R/(a_m)$, with $a_1 | a_2 | \cdots a_m$. First we show that in this case M is torsion-free. Consider the element $\mathbf{1} = (1, 1, \cdots, \hat{1}, \hat{1}, \cdots, \hat{1})$ and $a_m \in R$. By assumption there is an element $x = (x_1 \cdots, x_t, \widehat{x_{t+1}}, \cdots \widehat{x_{t+m}})$ such that $a_m \cdot x = \mathbf{1}$. But this is not possible since the last component of the left hand side is $\hat{0}$, whereas the last component of the right hand side is $\hat{1}$. Thus M should be torsion-free and hence free, with $t > 0$. Now again take $\mathbf{1} = (1, 1, \cdots, 1) \in M$. For any $0 \neq r \in R$ there exists $x = (x_1 \cdots, x_t) \in M = R^t$ such that $rx = \mathbf{1}$. Looking at the first component, we draw $r \cdot x_1 = 1$. So r is invertible. $\square$

Problem 3. A matrix $A \in M_2(F)$ has a square root if there is $B \in M_2(F)$ such that $B^2 = A$. Let F be an algebraically closed field of characteristic 2. Which matrices $A \in M_2(F)$ have a square root?

Proof. We may assume that A_0 is the Jordan canonical form of A, with $SAS^{-1} = A_0$, for some invertible matrix S. Note that $A = B^2 \iff SAS^{-1} = SB^2S^{-1} \iff A_0 = B_0^2$ $(B_0 = SBS^{-1})$. Thus, A has a square root if and only if A_0 has a square root. Now consider the Jordan blocks of A_0.

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(1). A_0 has 2 Jordan blocks. That is, A is diagonalizable. We assume A_0 is of the following form:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \zeta \end{pmatrix}$$

Since F is algebraically closed we may find $\sqrt{\lambda}$ (by this we mean THE root of the equation $x^2 - \lambda = 0$ in F) and $\sqrt{\zeta}$ in F. Then one sees easily that

$$\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\zeta} \end{pmatrix}$$

is a square root of A_0. Thus all diagonalizable matrices have square roots.

(2). A_0 has only one Jordan block. So A_0 is of the form $\lambda I + N$, where N is the following matrix:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Suppose B is a square root of A_0. Then, B is not diagonalizable (otherwise we draw a contradiction quickly). The Jordan canonical form of B should be $\zeta I + N$ for some ζ, in other words $T(\zeta I + N)T^{-1} = B$ for some T. Note that $(\zeta I + N)^2$ is $\zeta^2 I$, because $N^2 = 0$ and $\text{char}(F) = 2$. Thus $A_0 = B^2 = (T(\zeta I + N)T^{-1})^2 = \zeta^2 I$. This means A_0 is a diagonal matrix, which is absurd.

We conclude: A matirx A has a square root if and only if it is diagonalizable. $\square$