HOMEWORK 3

DUE APRIL 21, 2021 AT 11:59PM

Unless otherwise specified from now on all rings are commutative and have unit.

1. Let p be a prime number. For $n \geq m$ let $f_{nm} : \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z}$ be the canonical projection, i.e. $f_{nm}(a \bmod p^n) = a \bmod p^m$.

(a) Show that $\{\mathbb{Z}/p^n\mathbb{Z}\}$ with homomorphisms f_{nm} forms an inverse system of commutative rings. Let $\mathbb{Z}_p$ denote $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$

(b) Find the canonical image of $\mathbb{Z}$ in $\mathbb{Z}_p$ and show that $\mathbb{Z}_p$ is an integral domain.

(c) Show that $\mathbb{Z}_p$ is a local ring and an principal ideal domain.

The ring $\mathbb{Z}_p$ is called the ring of p-adic integers.

2. Let p be a prime and let R be the set of formal power series in p :

$$R = \left\{ \sum_{n=0}^{\infty} a_n p^n \, ; a_n = 0, 1, \ldots, p-1 \right\}.$$

(a) Show that R is a commutative ring under the addition and multiplication of power series (do show that multiplication makes sense!).

(b) Show that $\mathbb{Z}_p$ is naturally isomorphic to R.

3. Let $\mathbb{N}$ be the set of positive integers ordered by divisibility. Observe that

$$\{\mathbb{Z}/n\mathbb{Z}\}_{n \in \mathbb{N}}$$

forms an inverse system of commutative rings with the canonical homomorphisms $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$. Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. Show that

$$\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p.$$

4. Let R be a local ring with maximal ideal $\mathfrak{m}$.

(a) If $r \in \mathfrak{m}$, show that $1 + r \in R^\times$.

(b) Let T be a finitely generated R-module such that $\mathfrak{m}T = T$. Show that $T = \{0\}$.

(c) Let M and N be finitely generated R-modules. Let $f : M \longrightarrow N$ be an R-module homomorphism, and let $\bar{f} : M/\mathfrak{m}M \longrightarrow N/\mathfrak{m}N$ be the induced homomorphism. Show that f is surjective if and only if $\bar{f}$ is surjective.
Hint: for one direction, apply part (b) to $N/\operatorname{Im} f$.

5. Let G be a finite group. For any G-module M, let

$$C(M) = \{ \text{ functions } f : G \longrightarrow M\},$$

$$Z(M) = \{f \in C(M);\ f(\sigma\tau) = \sigma f(\tau) + f(\sigma)\ \forall \sigma, \tau \in G\},$$

$$B(M) = \{f \in C(M);\, \exists m \in M \text{ so that } f(\sigma) = \sigma m - m\, \forall \sigma \in G\}\,.$$

(a) Check that $B(M)$ and $Z(M)$ are abelian groups and that $B(M) \subseteq Z(M)$.

(b) Let $H(M) = Z(M)/B(M)$. If G acts trivially on M, show that as abelian groups $H(M) \cong \mathrm{Hom}(G, M)$ where $\mathrm{Hom}(G, M)$ is the set of all group homomorphisms $G \longrightarrow M$.

(c) For any short exact sequence of G-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

show that there is an exact sequence

$$0 \longrightarrow M^G \longrightarrow N^G \longrightarrow P^G \longrightarrow H(M) \longrightarrow H(N) \longrightarrow H(P).$$

6. Let p be a prime number. For any nonzero integer a we denote by $\mathrm{ord}_p(a)$ the largest exponent n such that $p^n \mid a$.

(a) For $a \in \mathbb{Z}$, define the *p-adic absolute value* of a to be $|a|_p = \dfrac{1}{p^{\mathrm{ord}_p(a)}}$. Show that

$$|a + b|_p \le \max\{|a|_p, |b|_p\}$$

for all $a, b \in \mathbb{Z}$. Under what conditions is this an equality?

(b) Show that

$$(a, b) \mapsto |a - b|_p, \quad a, b, \in \mathbb{Z},$$

defines a metric on $\mathbb{Z}$ which makes $\mathbb{Z}$ into a topological group (that is, with respect to this topology, addition $+ : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is continuous).

(c) Extend the absolute value $|\cdot|_p$ from $\mathbb{Z}$ to one on the p-adic integers $\mathbb{Z}_p$ in a natural way so that, in the topology on $\mathbb{Z}_p$ every Cauchy sequence in $\mathbb{Z}$ converges in $\mathbb{Z}_p$.

(d) Let $\{a_i\}_{i=1}^{\infty}$ be an infinite sequence in $\mathbb{Z}_p$. Under which conditions will the sum $\sum a_i$ converge?