Math 220, Problem Set 3.

0. (*Slight change in grading policy.*) We discussed in Lecture 7 the possibility of dropping the lowest problem set. Do you agree/disagree?

1. (*Mittag-Leffler. Monday, January 25.*) Using the procedure described in lecture, write down a meromorphic function on $\mathbb{C}$ only with simple poles at $z = n$ and residues equal to $\sqrt{n}$ for each $n \in \mathbb{Z}_{\geq 0}$. Please justify your answer.

2. (*Generalized Weierstraß problem. Monday, January 25.*) Let $\{a_n\}$ be distinct complex numbers with $a_n \to \infty$. Fix complex numbers $\{A_n\}$. Show that there exists an entire function f such that

$$f(a_n) = A_n.$$

This is a special case of Conway VIII.3.5, page 209, which you may wish consult for hints.

The next two questions are a modified version of Conway VIII.3, Problems 2 and 3.

3. (*The Weierstraß zeta function. Monday, January 25, but more realistically Wednesday, January 27.*) Let ω_1, ω_2 be two non-zero complex numbers such that $\omega_2/\omega_1 \notin \mathbb{R}$:

$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

Recall the Weierstraß σ-function introduced in the previous homework as a solution to a Weierstraß problem for Λ:

$$\sigma(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{1}{2} \cdot \frac{z^2}{\lambda^2}\right) = z \prod_{\lambda \in \Lambda \setminus \{0\}} E_2\left(\frac{z}{\lambda}\right).$$

(i) Weierstraß also defined the function ζ (not to be confused with Riemann's zeta) by taking logarithmic derivatives

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}.$$

Show that

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right).$$

Thus, the Weierstraß ζ function solves a Mittag-Leffler problem for Λ. Which one?

(We have seen this in class, of course. Here, we are just making the connection between σ and ζ.)

(ii) Let a_n be a sequence of nonzero distinct complex numbers with $a_n \to \infty$. Can you generalize (i) to a statement relating the solution to the Weierstraß Factorization for the set $\{a_n\}$ to the solution to a certain Mittag-Leffler problem for $\{a_n\}$? Your statement should involve the logarithmic derivative.

(Again this doesn't require much: you only need to formulate the statement and show why it holds via a short computation.)

4. (*The Weierstraß $\wp$-function. This uses only the methods of Math 220A.*) Probably the best known Weierstraß function is the function *"pe"* obtained from the derivative of the function ζ:

$$\wp(z) = -\zeta'(z).$$

(i) By direct calculation, show that

$$\wp = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

(There is no need to verify local uniform and absolute convergence of the series, this follows for free from the theorems on taking derivatives we have developed and the fact that the product giving σ converges absolutely locally uniformly.)

(ii) Show that $\wp$ is an elliptic function in the sense defined in Math 220A, Lecture 21. That is, show that $\wp$ is meromorphic and doubly periodic

$$\wp(z) = \wp(z + \omega_1) = \wp(z + \omega_2).$$

(iii) Using (ii), show that $\wp'$ is doubly periodic as well.

(iv) Using the definition, show that $\wp$ is even.

(v) Show that the Laurent expansions around 0 of $\wp$ and $\wp'$ take the form

$$\wp = \frac{1}{z^2} + az^2 + bz^4 + \dots$$
$$\wp' = -\frac{2}{z^3} + 2az + 4bz^3 + \dots$$

for certain $a, b \in \mathbb{C}$.

(vi) Using undetermined coefficients show that there exist constants A, B such that

$$\wp'^2 - (4\wp^3 + A\wp + B)$$

has no Laurent principal tail at $z = 0$ and vanishes at $z = 0$.

(vii) Conclude from (vi) and double periodicity that

$$\wp'^2 - (4\wp^3 + A\wp + B) = 0.$$

Remark: The function $\wp$ is truly very interesting. There are connections between $\wp$ (naturally arising in *complex analysis*) and *algebraic geometry* (cubic/elliptic curves), *number theory* (modular forms), and *mathematical physics* that one may not have guessed at first:

(a) $\wp$ and $\wp'$ parametrize the cubic (elliptic) curves

$$y^2 = 4x^3 + Ax + B$$

for suitable A, B via $y = \wp'(z), x = \wp(z)$. The expressions A, B depend on ω_1, ω_2 and are called Eisenstein series. They are examples of modular forms.

(b) using (a), one can show that $\wp$ provides a solution for the equation of waves in shallow waters (Korteweg-de Vries equation):

$$\wp''' = 12\wp\wp'.$$