HW7 - SOLUTIONS

Q1. (i) Let

$$f(z) = \frac{\log z}{(1+z^2)^2}$$

where $\log(z) := \log|z| + i\arg(z)$ and $-\frac{\pi}{2} < \arg(z) < \frac{3}{2}\pi$. Consider

$$\gamma = C_R \cup S_2 \cup C_r^\star \cup S_1,$$

where C_r and C_R are the half circles of radii r and R, and S_1, S_2 are the segments $[r, R]$ and $[-R, -r]$. The star decorating C_r indicates the reversed orientation. Then f has a pole at i with multiplicity 2 inside the region enclosed by the contour. It follows

$$\int_\gamma f(z)\, dz = 2\pi i \operatorname{Res}(f, i).$$

Note

$$\operatorname{Res}(f, i) = \frac{d}{dz}\frac{\log z}{(z+i)^2}|_{z=i} = \frac{1}{4}\left(i + \frac{\pi}{2}\right).$$

Also,

$$\begin{aligned}\left|\int_{C_R} \frac{\log z}{(1+z^2)^2} dz\right| &\leq \int_0^\pi \frac{\log R + i\theta}{(R^2-1)^2} R d\theta \\ &= O\left(\frac{\log R}{R^3}\right) + O\left(\frac{1}{R^3}\right) \to 0 \text{ as } R \to \infty,\end{aligned}$$

and

$$\begin{aligned}\left|\int_{C_r} \frac{\log z}{(1+z^2)^2} dz\right| &\leq \int_0^\pi \frac{\log r + i\theta}{1} r d\theta \\ &= \pi\left(r\log r + \frac{r}{2}\right) \to 0 \text{ as } r \to 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_r^R \frac{\log x}{(1+x^2)^2} dx + \int_{-R}^{-r} \frac{\log x}{(1+x^2)^2} dx &= \int_r^R \frac{\log x}{(1+x^2)^2} dx + \int_r^R \frac{\log x + i\pi}{(1+x^2)^2} dx \\ &= 2\int_r^R \frac{\log x}{(1+x^2)^2} dx + i\pi \int_r^R \frac{1}{(1+x^2)^2} dx.\end{aligned}$$

Note that when $r \to 0$ and $R \to \infty$, we have

$$i\pi \int_r^R \frac{1}{(1+x^2)^2} dx \to i\pi \int_0^\infty \frac{1}{(1+x^2)^2} dx = i\pi \int_0^{\pi/2} \cos^2\theta d\theta = i\frac{\pi^2}{4}.$$

Combing all these together, we have

$$\int_0^\infty \frac{\log x}{(1+x^2)^2} = -\frac{\pi}{4}.$$

(ii) Let

$$f(z)=\frac{z^{\alpha}}{1+z^{n}}=\frac{\exp(\alpha\cdot\log(z))}{1+z^{n}},$$

where log denotes the branch of the logarithm with argument in $(0,2\pi)$, so that we cut along the positive real axis. Let γ be the keyhole contour made up of four curves $S_R,$ S_r, L_1 and L_2. The line segments L_1, L_2 are used at height δ and $-\delta$ respectively, and S_r, S_R are parts of the circles with radii

$$r^{*}=\sqrt{r^{2}+\delta^{2}},\quad R^{*}=\sqrt{R^{2}+\delta^{2}}.$$

Write $\xi=e^{\frac{\pi}{n}i}$. Then f has all simple poles ξ^{2k+1} for $k=0,\ldots,n-1$ inside the region enclosed by the contours. Thus,

$$\int_{\gamma}f(z)\,dz=2\pi i\sum_{k=0}^{n-1}\operatorname{Res}\left(f,\xi^{2k+1}\right).$$

We have

$$\operatorname{Res}\left(f,\xi^{2k+1}\right)=\frac{g(\xi^{2k+1})}{h'(\xi^{2k+1})}$$

where $g=z^{\alpha}$ and $h=z^{n}+1$. This yields

$$\operatorname{Res}\left(f,\xi^{2k+1}\right)=\frac{1}{n}\cdot\frac{(\xi^{2k+1})^{\alpha}}{(\xi^{2k+1})^{n-1}}=-\frac{1}{n}(\xi^{2k+1})^{\alpha}\cdot\xi^{2k+1}.$$

We used that $\xi^{n}=-1$ here. We have

$$(\xi^{2k+1})^{\alpha}=\exp(\alpha\cdot\log(\xi^{2k+1}))=\exp\left(\alpha\cdot\frac{\pi i(2k+1)}{n}\right).$$

This is valid since $\frac{2k+1}{n}\pi\in(0,2\pi)$ as required by the branch we chose. Therefore,

$$\operatorname{Res}\left(f,\xi^{2k+1}\right)=-\frac{1}{n}\exp\left((\alpha+1)\cdot\frac{\pi i(2k+1)}{n}\right).$$

Thus

$$\begin{aligned}\sum_{k=0}^{n-1}\operatorname{Res}\left(f,\xi^{2k+1}\right)&=-\frac{1}{n}\sum_{k=0}^{n-1}\exp\left((\alpha+1)\cdot\frac{\pi i(2k+1)}{n}\right)\\&=-\frac{1}{n}\cdot\exp\left((\alpha+1)\cdot\frac{\pi i}{n}\right)\cdot\frac{\exp((\alpha+1)\frac{2\pi i n}{n})-1}{\exp((\alpha+1)\frac{2\pi i}{n})-1}\\&=-\frac{1}{n}\cdot(\exp(2\pi i\alpha)-1)\cdot\frac{\exp\left((\alpha+1)\cdot\frac{\pi i}{n}\right)}{\exp((\alpha+1)\frac{2\pi i}{n})-1}\\&=-\frac{1}{n}\frac{e^{2\pi\alpha i}-1}{2i\sin\left(\frac{\pi}{n}(\alpha+1)\right)}\end{aligned}$$

We have for ρ standing for either r or R, and $\rho^{\star}$ standing for the radii r^{*} and R^{*} of S_{ρ} that

$$\left|\int_{S_{\rho}}\frac{z^{\alpha}}{1+z^{n}}dz\right|\leq\frac{\rho^{*\alpha}}{|\rho^{*n}-1|}\cdot2\pi\rho^{\star}$$

As $r \to 0, \delta \to 0, R \to \infty$ we have $r^\star \to 0$ or $R^\star \to \infty$. Since

$$\lim_{x\to\infty \text{ or } x\to 0} \frac{x^{\alpha+1}}{x^n - 1} = 0$$

when $0 < \alpha + 1 < n$, we obtain

$$\int_{S_\rho} \frac{z^\alpha}{1+z^n} dz \to 0$$

for both $\rho = r$ and $\rho = R$.

Finally, recall that log was chosen so that $0 \le Arg\,(z) < 2\pi$. Then,

$$\begin{aligned}
\int_{L_1} f(z) = \int_r^R \frac{(x+i\delta)^\alpha}{1+(x+i\delta)^n} dx &= \int_r^R \frac{e^{\alpha \operatorname{Log}(x+i\delta)}}{1+(x+i\delta)^n} dx \\
&= \int_r^R \frac{e^{\alpha \operatorname{Log}|x+i\delta| + \alpha i \operatorname{Arg}(x+i\delta)}}{1+(x+i\delta)^n} dx \\
&= \int_r^R \frac{|x+i\delta|^\alpha \, e^{\alpha i \operatorname{Arg}(x+i\delta)}}{1+(x+i\delta)^n} dx \\
&\to \int_0^\infty \frac{x^\alpha}{1+(x+i\delta)^n} dx \text{ as } \delta,\, r \to 0 \text{ and } R \to \infty.
\end{aligned}$$

This limit is justified by the same argument as done in class. Similarly,

$$\begin{aligned}
\int_{L_2} f(z) = \int_R^r \frac{(x-i\delta)^\alpha}{1+(x-i\delta)^n} dx &= -\int_r^R \frac{e^{\alpha \operatorname{Log}(x-i\delta)}}{1+(x-i\delta)^n} dx \\
&= \int_r^R \frac{e^{\alpha \operatorname{Log}|x-i\delta| + \alpha i \operatorname{Arg}(x-i\delta)}}{1+(x-i\delta)^n} dx \\
&= \int_r^R \frac{|x-i\delta|^\alpha \, e^{\alpha i \operatorname{Arg}(x-i\delta)}}{1+(x-i\delta)^n} dx \\
&\to -e^{2\pi\alpha i} \int_0^\infty \frac{x^\alpha}{1+(x-i\delta)^n} dx \text{ as } \delta,\, r \to 0 \text{ and } R \to \infty.
\end{aligned}$$

By combing all these, we have

$$\begin{aligned}
\left(1 - e^{2\pi\alpha i}\right) \int_0^\infty \frac{x^\alpha}{1+x^n} dx &= 2\pi i \left(-\frac{i}{n} \frac{\xi^{2n\alpha} - 1}{2i \sin\left(\frac{\pi}{n}(\alpha+1)\right)} \right) \\
\int_0^\infty \frac{x^\alpha}{1+x^n} dx &= \frac{\pi}{n \sin\left(\frac{\pi}{n}(\alpha+1)\right)}.
\end{aligned}$$

Q2. (i) Let $\{a_1, \ldots, a_N\}$ be isolated singularities in $\mathbb{C}$. By Residue theorem,

$$\frac{1}{2\pi i} \int_{|z|=R} f dz = \sum_{|a_i|<R} \operatorname{Res}(f(z)dz, a_i).$$

The required result follows using the residue theorem for $\hat{\mathbb{C}}$, which implies

$$\sum_{|a_i|<R} \operatorname{Res}(f(z)dz, a_i) = -\sum_{|a_j|>R} \operatorname{Res}(f(z)dz, a_j) - \operatorname{Res}(f(z)dz, \infty).$$

(ii) The function $f(z) = (z-a)^k$ can have an isolated singularity only at a and ∞. By definition, we have

$$\mathrm{Res}(f(z)dz, a) = \begin{cases} 1 & \text{if } k = -1 \\ 0 & \text{if } k \neq -1 \end{cases}.$$

Using the residue theorem for $\hat{\mathbb{C}}$, we get

$$\mathrm{Res}(f(z)dz, \infty) = -\mathrm{Res}(f(z)dz, a) = \begin{cases} -1 & \text{if } k = -1 \\ 0 & \text{if } k \neq -1 \end{cases}.$$

Remark: This can also be seen directly by computing the residue at $z = 0$ of

$$-f\left(\frac{1}{z}\right)\frac{dz}{z^2} = -(1-az)^k z^{-k-2}\, dz.$$

When $k = -1$, this expression becomes

$$-\frac{dz}{z} \cdot \frac{1}{1-az} = -\frac{dz}{z}(1 + az + a^2z^2 + \ldots) = -\frac{dz}{z} + \ldots$$

and the residue is clearly -1.

When $k \leq -2$, the function $(1-az)^k z^{-k-2}$ is holomorphic at $z = 0$ so the residue vanishes.

When $k \geq 0$, the residue is found by extracting the z^{-1} coefficient in $(1-az)^k z^{-k-2}$ or the coefficient of z^{k+1} in $(1-az)^k$. The latter also vanishes by the binomial theorem.

(iii) The function $f(z) = \frac{z^3}{(z-1)(z-2)(z-3)(z-4)}$ has isolated singularities at $\{1, 2, 3, 4, \infty\}$. Using (i) we know that the given integral equals

$$\int_{|z|=5} f dz = -2\pi i\ \mathrm{Res}(f(z)dz, \infty).$$

Let $g : \Delta^*(0, 1/R) \to \mathbb{C}$ be defined as $g(z) = f(1/z)$, where $R > 4$. In our case

$$g(z) = \frac{z}{(1-z)(1-2z)(1-3z)(1-4z)}.$$

Thus

$$\frac{g(z)}{z^2}\, dz = \frac{dz}{z(1-z)(1-2z)(1-3z)(1-4z)} = \frac{dz}{z}\,(1 + \text{higher order terms}) = \frac{dz}{z} + \ldots.$$

The residue at $z = 0$ of the latter expression equals 1. Thus

$$\mathrm{Res}(f(z)dz, \infty) = -\mathrm{Res}\left(\frac{g(z)}{z^2}dz, 0\right) = -1$$

and therefore

$$\int_{|z|=5} f dz = 2\pi i \mathrm{Res}\left(\frac{g(z)}{z^2}dz, 0\right) = 2\pi i.$$

Q3. Suppose h is a meromorphic function on $\mathbb{C} \cup \{\infty\}$. We first show that h can only have finitely many zeroes and poles. In fact, it suffices to argue for the poles since by working with $\frac{1}{h}$ instead we can derive the same statement for the zeros. Assume that h has infinitely many poles $a_j \in \mathbb{C} \cup \{\infty\}$.

- if a_j is a bounded sequence, then a_j will have a convergent subsequence but this contradicts the fact that the poles of a meromorphic are discrete (by definition);
- if a_j is unbounded, then a_j will have a subsequence converging to ∞, again contradicting that the poles of a meromorphic function are discrete in $\mathbb{C} \cup \{\infty\}$.

We now show that h is a rational function. Let $(q_1 \ldots, q_n)$ be the poles of h on $\mathbb{C}$ (enumerated with multiplicities). Then let

$$\phi(z) = h(z) \prod_j (z - q_j).$$

This function has no poles on $\mathbb{C}$, hence it is holomorphic on $\mathbb{C}$ and has possibly a pole at ∞. By Problem Set 5, Problem 5, such a function ϕ is necessarily a polynomial, completing the proof.

Q4. Consider the region $|z| < 2$ and let $f = z^4 + 5z + 3$ and $g = z^4$. Over the boundary circle $|z| = 2$ we have

$$|f - g| = |5z + 3| \leq 5|z| + 3 = 13 < |g| = |z|^4 = 16.$$

By Rouché f has as many zeros as g in $|z| < 2$, that is, f has exactly four zeros. When $|z| < 1$, take $f = z^4 + 5z + 3$ and $h = 5z$. In this case, for $|z| = 1$, we have

$$|f - h| = |z^4 + 3| \leq |z|^4 + 3 = 4 < |h| = 5.$$

Therefore, f has as many zeroes as h in $|z| \leq 1$, namely one zero. Thus f has 3 zeros in the region $1 < |z| < 2$.

Q5. Let

$$f(z) = z + e^{-z} - \lambda, \;\; g(z) = z - \lambda.$$

Consider γ the boundary of the half disc of radius R contained in the right half plane Re $z > 0$. We assume that the radius $R > \lambda + 1$. Then, if z is on the half circle, it follows

$$\begin{aligned} |f - g| &= \left|e^{-z}\right| \\ &= e^{-\operatorname{Re}(z)} \\ &\leq 1 < R - \lambda \leq |z - \lambda| = |g|. \end{aligned}$$

Furthermore, if z on the diameter of the half circle lying on y-axis from $-Ri$ to Ri, then it follows

$$\begin{aligned} |f-g| &= \left|e^{-z}\right| \\ &= e^{-\operatorname{Re}(z)} \\ &= 1 < \lambda \leq \sqrt{\lambda^2 + |\operatorname{Im}(z)|^2} = |g|\,. \end{aligned}$$

Hence, by Rouché's Theorem, f has only one solution inside the the half circle contour with a radius R. By taking $R \to \infty$, we conclude that f has only one solution on the half plane $\{z : \operatorname{Re}(z) > 0\}$.

Q6. Let $f(z) = z^4 + 3z^2 + z + 1$ and $g(z) = 3z^2 + 1$. For z on the unit circle, it follows

$$|f-g| = |z^4 + z| \leq |z|^4 + |z| \leq 2$$

and

$$|g| = |3z^2 + 1| \geq 3|z|^2 - 1 = 2.$$

Thus

$$|f-g| \leq |g|$$

on the unit circle.

We claim that equality cannot in fact occur. Assume otherwise. Note that if

$$|a+b| = |a| + |b|$$

then $a = bt$ for t real and nonnegative or $b = 0$. (Just let $t = a/b$, rewrite the above as $|t+1| = |t| + 1$, which implies $t \in \mathbb{R}_{\geq 0}$). In our case, we must have equality throughout. In particular, we must have $|g| = 2$ so

$$|3z^2 + 1| + |-1| = |g| + 1 = 3 = |3z^2|.$$

By our remark, z^2 is negative real. Since $|z^2| = 1$ we must have $z^2 = -1$. Thus $z = \pm i$. However in this case, it can be seen that $|f-g| = |z^4+z| = |1 \pm i| = \sqrt{2} \neq 2$.

Thus

$$|f-g| < |g|$$

on the unit circle. By Rouché's Theorem, we conclude that number of roots of f is the same as number of roots of g inside the unit disc which is 2.

Q7. We claim that $f = z^n + a_1 z^{n-1} + \ldots + a_n$ has $n-1$ roots in the disc $|z| < 1$. Indeed, take $g = a_1 z^{n-1}$ and compute for $|z| = 1$:

$$|f-g| = |z^n + a_2 z^{n-2} + \ldots + a_n| \leq |z|^n + |a_2||z|^{n-2} + \ldots + |a_n| = 1 + |a_2| + \ldots + |a_n|$$

$$< |a_1| = |g|.$$

Thus by Rouché, f has $n-1$ roots $z_1, \ldots, z_{n-1}$ with $|z_i| < 1$, and one root $|z_n| > 1$. Assume that f is reducible so that

$$f = f_1 f_2.$$

Without loss of generality, we may assume z_n is a root of f_2. The roots of f_1 must be among $z_1, \ldots, z_{n-1}$. As f is monic, f_1 is also monic. Writing $\alpha \in \mathbb{Z}$ for the free term of f_1 we must have α is the product of the roots of f_1, hence $|\alpha| < 1$ by the above discussion regarding the roots of f_1. This means $\alpha = 0$ so $f_1(0) = 0 \implies f(0) = a_n = 0$, which is a contradiction.