Math 220B - Winter 2021 - Midterm

Name: ____________________________________

Student ID: ________________________________

Instructions:

There are 5 questions which are worth 50 points.

You may not use any books, notes or internet. If you use a homework problem you will need to reprove it.

Please upload your answers in Gradescope before Tuesday, February 16, at noon.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
Total		50

Problem 1. [*10 points; 2, 4, 4.*]

(i) Give an example of an entire function with simple zeroes only at $z = \sqrt{n}$ for each $n \in \mathbb{Z}_{\geq 0}$, and no other zeroes.

(ii) Give an example of a meromorphic function function in $\mathbb{C}$ with poles only at $z = -\sqrt{n}$ and principal parts $\frac{1}{z+\sqrt{n}}$, for $n \in \mathbb{Z}_{\geq 0}$.

(iii) Consider $\{a_n\}, \{b_n\}$ two sequences of complex numbers without common terms, such that

$$\sum_{n=1}^{\infty} |a_n - b_n| < \infty$$

and $b_n \to \infty$ as $n \to \infty$. Show that the product

$$f(z) = \prod_{n=1}^{\infty} \frac{z - a_n}{z - b_n}$$

defines a holomorphic function in the open set $\mathbb{C} \setminus \{b_1, b_2, \ldots\}$, and determine its zeros.

Problem 2. [*10 points.*]

Let $f : \Delta(0,1) \setminus \{0\} \to \mathbb{C}$ be a holomorphic function on the punctured unit disc. Let

$$f_n : \Delta(0,1) \setminus \{0\} \to \mathbb{C}, \quad f_n(z) = f\left(\frac{z}{n}\right).$$

Show that the family $\mathcal{F} = \{f_n : n \geq 1\}$ is normal iff f has a removable singularity at the origin.

Problem 3. [*10 points; 7, 3.*]

Let $f : \Delta(0,1) \to \mathbb{C}$ be such that $\operatorname{Re} f(z) > 0$ for all $z \in \Delta(0,1)$, and assume that $f(0) = 1$.

(i) Show that for all $z \in \Delta(0,1)$ we have

$$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}.$$

(ii) Find the maximum and minimum value of $\left|f\left(\frac{1}{2}\right)\right|$.

Problem 4. [*10 points; 4, 6.*]

Recall the function

$$G(z) = \prod_{n=1}^{\infty} E_1\left(-\frac{z}{n}\right).$$

(i) Show that there exists an entire function h such that

$$\left(z + \frac{1}{2}\right) G(z) G\left(z + \frac{1}{2}\right) = e^{h(z)} G(2z).$$

(ii) Show furthermore that $h(z) = az + b$.

Remark: The constants a, b can be found explicitly, by setting $z = 0$ and $z = 1/2$ and computing the relevant values of G from the values of the Γ-function. You can try it for yourself if you are interested. It's good practice with the Γ-function.

Problem 5. [*10 points.*]

Let $a, b \neq 0$ and $a, b \in \Delta(0,1)$. Consider the twice punctured discs

$$D_1 = \Delta(0,1) \setminus \{0, a\}\,, \quad D_2 = \Delta(0,1) \setminus \{0, b\}\,.$$

Find a necessary and sufficient condition for D_1, D_2 to be biholomorphic, and determine all biholomorphic maps

$$f : D_1 \to D_2.$$