Solutions: Homework 5

Problem 1. *If h is meromorphic on $\mathbb{C}$, and omits three values then h is constant.*

Proof. Suppose that h omits three distinct values, say $a, b, c \in \mathbb{C}$. The function g defined by

$$g(z) = \begin{cases} \frac{1}{h(z)-a} & \text{if } z \text{ is not a pole of } h \\ 0 & \text{if } z \text{ is a pole of } h \end{cases}$$

is entire. Indeed, the only need to argue around the the poles of h, but we have seen in Math 220A take taking inverses turns poles into zeroes. The function h does not take the values $\frac{1}{b-a}$ and $\frac{1}{c-a}$. By the Little Picard Theorem, we have that g is a constant function, and hence h is also constant. $\square$

Problem 2. *Let $n \geq 3$. If f, g are entire such that $f^n + g^n = 1$, show that f, g are constant.*

Proof. since $f^n + g^n = 1$, either $f \not\equiv 0$ or $g \not\equiv 0$. Suppose that $g \not\equiv 0$. Then f/g is meromorphic, and we have

$$g^n \left(\left(\frac{f}{g} \right)^n + 1 \right) = 1$$

which shows that $(f/g)^n$ can never take the value -1. This is equivalent to saying that f/g can never take any of the values $e^{\frac{\pi i(2k+1)}{n}}$ for $0 \leq k \leq n-1$, and hence f/g is a meromorphic function that omits at least n values. Since $n \geq 3$, by Problem 1, we get that f/g is constant. Thus

$$f = cg$$

for some $c \in \mathbb{C}$ which implies that

$$g^n(c^n + 1) = 1.$$

Thus $g(z) = (1 + c^n)^{-n}$ up to an ambiguity coming from roots of unity, and hence g is constant by continuity. Then $f = cg$ is also constant. $\square$

Problem 3. *Let f, g be two nonconstant entire functions, P, Q two nonconstant polynomials such that*

$$e^f + P = e^g + Q.$$

Show that $P = Q$.

Proof. We have

$$P - Q = e^g(1 - e^{f-g})$$

Suppose $P \neq Q$. Then, the polynomial $P - Q$ has only finitely many zeros and so $1 - e^{f-g}$ has only finitely many zeros and omits also the value 1. By Great Picard (see the Lemma in Lecture 12), this is impossible unless $1 - e^{f-g}$ is constant. This implies that $P - Q = ce^g$ for some $c \in \mathbb{C}$. If $c \neq 0$, since ce^g has no zeros, we have that the polynomial $P - Q$ is constant, and hence ce^g is constant, which gives g constant, contradicting our hypothesis. Hence $c = 0$ and thus $P = Q$. □

Problem 4. *If h is a nonconstant polynomial and f is a nonconstant entire function, show that he^f does not omit any values.*

Proof. Assume he^f omits the value α. Note that $\alpha = 0$ is impossible since h has at least a root. Thus $\alpha \neq 0$, so by replacing h by h/α, we may assume he^f omits the value 1. Then $1 - he^f$ omits the value 0, so it can be written in the form e^g. Thus

$$1 - he^f = e^g \implies e^{-f} - h = e^{g-f} - 0.$$

By Problem 3, this shows $h = 0$, a contradiction. The only exception will occur if $g - f$ is constant c. But then

$$e^{-f} - h = e^{g-f} = e^c \implies e^{-f} = h + e^c.$$

This is however impossible as well. The polynomial $h + e^c$ has at least one root, while e^{-f} never vanishes, a contradiction.

□

Problem 5. *Let f be entire such that $f \circ f$ has no fixed points. Show that $f(z) = z + a$ for some a.*

Proof. Let

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}$$

Note that f also has no fixed points. Indeed, if $f(z_0) = z_0$ for some $z_0 \in \mathbb{C}$, then $f(f(z_0)) = f(z_0) = z_0$, which is not possible as $f \circ f$ has no fixed points. Hence $f(z) - z$ has no zeros and so g is entire.

Now, note that since $f \circ f$ has no fixed points, the numerator above can never be zero, and hence g has no zeros. Suppose there exists $z_0 \in \mathbb{C}$ such that $g(z_0) = 1$. Then,

$$f(f(z_0)) - z_0 = f(z_0) - z_0 \implies f(f(z_0)) = f(z_0)$$

which is not possible since then $f(z_0)$ would be a fixed point of f. Hence g also omits the value 1. By Little Picard's Theorem, we have $g = c$ for some $c \in \mathbb{C} \setminus \{0, 1\}$.

This shows that

$$f(f(z)) = z + c(f(z) - z) = z(1 - c) + cf(z)$$

Differentiating this expression, we get

$$f'(f(z))f'(z) = 1 - c + cf'(z)$$

This hows that if $f'(z) = 0$ for some $z \in \mathbb{C}$, then $c = 1$, a contradiction. So f' has no zeros, and so $f' \circ f$ has no zeros. Now note that

$$f'(f(z)) = c + \frac{1 - c}{f'(z)}$$

Since $c \neq 1$, we have $1 - c \neq 0$ and hence $\frac{1-c}{f'(z)}$ is never zero, which shows that $f' \circ f$ never attains the value c. So, $f' \circ f$ omits the values 0 and c, and hence by Little Picard Theorem, we have $f' \circ f$ is constant. Since $f \circ f$ has no fixed points, f cannot be constant. Since the image of f is dense (it omits at most one value), and $f' \circ f$ is constant, it follows that f' has to be constant.

This shows that f is linear, so that $f(z) = bz + a$ for some $a, b \in \mathbb{C} \setminus \{0\}$. Now,

$$(f \circ f)(z) - z = b(bz + a) + a - z = (b^2 - 1)z + a(b + 1)$$

This cannot have a zero in $\mathbb{C}$ iff $b^2 - 1 = 0$ and $a(b + 1) \neq 0$, which just implies that $b = 1$. Hence $f(z) = z + a$ for some $a \in \mathbb{C} \setminus \{0\}$. $\square$