Math 220C - Lecture 10

April 19, 2021

§0. Last time

$f: \mathbb{C} \longrightarrow \mathbb{C}$ entire function

Main Question Establish relationship between

$\{$ Growth of $f \}$ $\longleftrightarrow$ $\{$ Distributions of zeros $\}$

Subquestion: How do we interpret the two sides mathematically?

§1. Left hand side

Order Recall $M(R) = \sup_{|z|=R} |f(z)|$. & we defined

$$\lambda(f) = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}$$

Intuitively, " $f(z) \sim e^{|z|^{\lambda}}$ ".

Question How to prove a function f has order λ?

We need to show two statements:

$\boxed{I}$ $\forall\, \varepsilon > 0\ \exists\, r$ such that $|f(z)| < e^{|z|^{\lambda+\varepsilon}}$ $\forall\, |z| > r$

This shows $M(R) < e^{R^{\lambda+\varepsilon}}$ $\forall\, R > r$ &

$$\lambda(f) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \leq \lambda + \varepsilon \quad \forall\, \varepsilon. \overset{\varepsilon\to 0}{\Longrightarrow} \lambda(f) \leq \lambda$$

$\boxed{II}$ $\forall\, \varepsilon > 0$ $\exists\, z_n \longrightarrow \infty$ with $|f(z_n)| > e^{|z_n|^{\lambda-\varepsilon}}$

This shows

$$\lambda(f) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \geq \limsup_{n\to\infty} \frac{\log\log|f(z_n)|}{\log|z_n|} \geq \lambda - \varepsilon$$

$$\overset{\varepsilon\to 0}{\Longrightarrow} \lambda(f) \geq \lambda.$$

Properties

$\boxed{I}$ $\lambda(z^m) = 0$, $M(R) = R^m \Rightarrow \lambda = 0$.

$\boxed{II}$ $\lambda(e^P) = \deg P$ (exercise)

$\boxed{III}$ $\lambda(fg) \leq \max(\lambda(f), \lambda(g))$ (HWK 4)

§2. Right hand side & Distribution (growth) of zeroes

Assume f has zeroes at

$$|a_1| \le |a_2| \le \dots \le |a_n| \le \dots \quad . \quad a_n \longrightarrow \infty, \quad a_n \neq 0$$

Several quantities attached to growth of zeroes:

$\boxed{\text{i}}$ rank $= p$

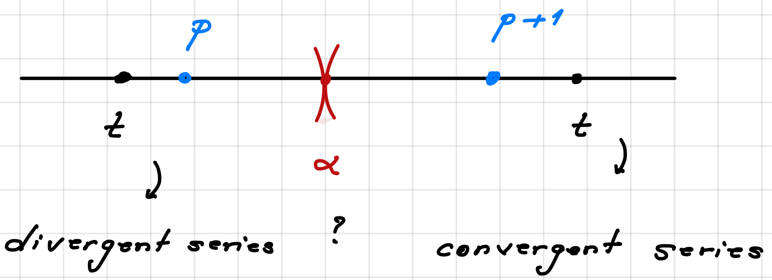
The smallest integer p such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$.

If such a p doesn't exist, $p = \infty$.

$\boxed{\text{ii}}$ critical exponent (HWK 4, #5)

$\displaystyle \alpha = \inf\left\{ t > 0 : \sum \frac{1}{|a_n|^t} < \infty \right\}$ may not be an integer

By the homework



Thus by definition $p \le \alpha \le p+1$.

If $\alpha \notin \mathbb{Z}$ then α determines p uniquely.

[iii] $N(R)$ = # zeroes in $\Delta(0,R)$ with multiplicity

Fact* (we will not use/prove)

$$\alpha = \limsup_{R\to\infty} \frac{\log N(R)}{\log R}$$

Example* Let $a_n = n^3$, $n>0$. Then

$$N(R) = \#\{n : n^3 < R\} \sim R^{1/3} \Rightarrow \frac{\log N(R)}{\log R} \longrightarrow \frac{1}{3}$$

Note

$$\sum \frac{1}{n^{3t}} < \infty \iff 3t > 1 \iff t > \frac{1}{3} \quad \text{so } \alpha = \frac{1}{3}.$$

harmonic series

Upshot We have defined the following quantities measuring growth / distribution of zeroes

$N(R)$, α, p.

Note $N(R)$ determines α, α determines p if $\alpha \notin \mathbb{Z}$.

Best for us: p (or h to be defined next).

Small variation — Genus of an entire function

Let f has zeroes at $a_1, a_2 \ldots, a_n, \ldots, a_k \neq 0$.

where $\{a_n\}$ has rank p. $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$

Recall Weierstraß Factorization

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right).$$

Recall

$$E_p(z) = \begin{cases} 1-z, & p=0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right), & p>0 \end{cases}$$

Define

$$h = \text{genus}(f) = \begin{cases} \max(p, q) & \text{if } g \text{ polynomial of degree } q \\ \infty & \text{if } g \text{ not polynomial or } p=\infty. \end{cases}$$

If the exponential e^g doesn't appear then $h = p$.

In general $p \leq h$.

Example (Math 220B)

$$\sin z = z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$ factorization of sine.

Rewrite this as

$$\sin z = z \prod_{n=1}^{\infty}\left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} \left(1 + \frac{z}{n\pi}\right) e^{-z/n\pi}$$

$$= z \prod_{n=1}^{\infty} E_1\left(\frac{z}{n\pi}\right) E_1\left(-\frac{z}{n\pi}\right)$$

$\Rightarrow$ g doesn't appear. Thus genus $h = p$.

The zeroes are at $n\pi$, $n \in \mathbb{Z}$. We want

$$\sum_{n \neq 0} \frac{1}{|n\pi|^{p+1}} < \infty \iff p+1 > 1 \iff p > 0.$$ Thus the

harmonic series

smallest p equals 1.

The genus of $z \longrightarrow \sin z$ equals 1.

§3. Revisiting the Main Question (now made precise)

Establish relationship between

$$\{\text{Growth of } f\} \longleftrightarrow \{\text{Growth of zeroes}\}$$

measured by λ measured by h = genus.

Answer **Theorem** (Hadamard)

$$h \leq \lambda \leq h+1$$

Remarks $\boxed{i}$ If $\lambda \notin \mathbb{Z}$ then λ determines h uniquely.

$\boxed{ii}$ If e^g doesn't appear then $h = p$ so in this case.

$$p \leq \lambda \leq p+1$$

$\boxed{iii}$ We have $p \leq h \leq \lambda$ so the order bounds the p in the Weierstraß Factorization. The statement that we can take $p \leq \lambda$ is called Hadamard Factorization.

Conclusion

$\exists$ connections between

- $M(R)$ and λ by definition $\lambda = \limsup\limits_{R\to\infty} \dfrac{\log\log M(R)}{\log R}$
- $N(R)$, α, p as we saw above
- λ and $h = \max(p, q)$ via Hadamard $h \leq \lambda \leq h+1$

Next

- proof that $\lambda \leq h+1$
- proof that $h \leq \lambda$
- Applications

Jacques Hadamard

1865 - 1963 (age 97)

Proved the prime number theorem

Institutions

University of Bordeaux
Sorbonne
College de France
École Polytechnique
École Centrale Paris

Doctoral advisor

Émile Picard

Doctoral students

Maurice Fréchet
André Weil