Math 220C, Problem Set 3. Due Friday, April 16.

1. (*Subharmonic functions.*) Show that if $u_n : G \to \mathbb{R}$ are subharmonic/superhamonic converging locally uniformly to $u : G \to \mathbb{R}$, then u is also subharmonic/superharmonic.

2. (*Poisson modification/Bumping.*) Let $\phi : G \to \mathbb{R}$ be subharmonic, and let $\overline{\Delta} \subset G$ be a closed disc in G. Consider the Poisson modification $\widetilde{\phi} : G \to \mathbb{R}$ of ϕ along Δ. Show that $\widetilde{\phi}$ is also subharmonic.

Hint: To check the mean value inequality at points in $\partial\Delta$, you may wish to use $\phi \le \widetilde{\phi}$.

3. (*Dirichlet Problem is not always solvable.*) Show that the Dirichlet Problem cannot be solved in the punctured disc $G = \Delta(0, 1) \setminus \{0\}$.

That is, exhibit a continuous function $f : \partial G \to \mathbb{R}$ which cannot be obtained as the boundary value of a harmonic function in G, continuous in $\overline{G}$.

Hint: You may wish to remember Homework 2, Problem 2.

Question: When is the Dirichlet Problem solvable?

4. (*Barriers and the Dirichlet Problem.*) Let G be open connected, $\zeta_0 \in \partial G$. We say G admits a *barrier* at ζ_0 provided there exists $\omega : \overline{G} \to \mathbb{R}$ continuous in $\overline{G}$, harmonic in G, such that $\omega(\zeta_0) = 0$ and $\omega > 0$ on $\partial G \setminus \{\zeta_0\}$.

Show that if the Dirichlet Problem is solvable in G, then G admits a barrier at each $\zeta_0 \in \partial G$.

Hint: Let $f(z) = |z - \zeta_0|$ be the boundary value, and let ω be the solution to the Dirichlet Problem for f.

Remark: The converse is also true. One can show that

Theorem. If G is bounded and admits a barrier at each $\zeta_0 \in \partial G$, then the Dirichlet Problem is solvable. In this case, the Perron function is the solution to the Dirichlet Problem.

If you are interested, you may wish to read the Addendum to Lecture 8 and/or watch the video posted in Canvas for a proof.

5. (*Barriers.*) Let G be open connected, $\zeta_0 \in \partial G$, and let ℓ be a half-line starting at ζ_0 that intersects $\overline{G}$ only at ζ_0. Let $\zeta_1 \neq \zeta_0$ be a point on the half line ℓ.

The terminology *barrier* is due to Lebesgue. First draw a picture for yourself to visualize what is happening.

Show that ζ_0 is a *barrier* for G.

Hint: For suitable α, let

$$\omega(z) = \operatorname{Im} e^{i\alpha} \sqrt{\frac{z - \zeta_0}{z - \zeta_1}}.$$

6. (*Qualifying Exam 2020. Review of Math 220B.*) Let $\mathcal{H}$ be the family of harmonic functions $h : \Delta \to \mathbb{R}$ with $h(0) = 1$ and $h(z) > 0$ for $z \in \Delta$. Show that every sequence in $\mathcal{H}$ admits a subsequence that converges locally uniformly to a function in $\mathcal{H}$.

Hint: Write h as the real part of a holomorphic function. The Cayley transform $\frac{1-z}{1+z}$ maps $\{z : \operatorname{Re} z > 0\}$ to Δ.

Remark: Thus, the family $\mathcal{H}$ is "sequentially compact".