

## HOMEWORK 1

DUE APRIL 7, 2021 AT 11:59PM

1. Let  $\mathcal{C}$  be a category, and let  $U_1$  and  $U_2$  be objects in  $\mathcal{C}$ . Suppose  $U_1$  and  $U_2$  are both universally attracting/final objects. Show that there is a unique isomorphism  $i : U_1 \longrightarrow U_2$ . (For future reference, the same is true if they're both universally repelling/initial objects, with the same proof.)
2. What are the initial and final objects in the following categories, if they exist?
  - (a) The category of sets.
  - (b) The category of rings with unit.
  - (c) The category of commutative rings with unit.
  - (d) The category of topological spaces.
  - (e) The category of open sets in a topological space  $X$ .
3. Let  $R$  be a commutative ring (with  $1 \neq 0$ ) and  $\{M_i\}_{i \in I}$  be a collection of  $R$ -modules, that is objects in the category  $R\text{-}\mathbf{mod}$  of  $R$ -modules.
  - (a) Show that arbitrary direct sums and arbitrary direct products exist in the category of abelian groups.
  - (b) Consider  $\bigoplus_{i \in I} M_i$  and  $\prod_{i \in I} M_i$  in the category of abelian groups. Prove that they become  $R$ -modules with  $r \cdot (m_i)_{i \in I} = (r \cdot m_i)_{i \in I}$ .
  - (c) Show that  $\bigoplus_{i \in I} M_i$  is the direct sum in  $R\text{-}\mathbf{mod}$  of  $\{M_i\}_{i \in I}$  and  $\prod_{i \in I} M_i$  is the direct product in  $R\text{-}\mathbf{mod}$  of  $\{M_i\}_{i \in I}$ .
  - (d) Show that, for every  $R$ -module  $N$ ,

$$\mathrm{Hom}_R \left( \bigoplus_{i \in I} M_i, N \right) \simeq \prod_{i \in I} \mathrm{Hom}_R(M_i, N)$$

and

$$\mathrm{Hom}_R \left( N, \prod_{i \in I} M_i \right) \simeq \prod_{i \in I} \mathrm{Hom}_R(N, M_i).$$

- (e) Show that, for every  $R$ -module  $N$ ,

$$N \otimes_R \left( \bigoplus_{i \in I} M_i \right) \simeq \bigoplus_{i \in I} (N \otimes_R M_i).$$

- (f) Does the tensor product also commute with direct products? Prove or give a counterexample.
- (g) Is the tensor product of two free  $R$ -modules also free as an  $R$ -module? Prove or give a counterexample.

4. Let  $\mathcal{R}$  be the category of rings with  $1 \neq 0$  (but not necessarily commutative). If  $R_1$  and  $R_2$  are rings, then let  $R_1 \times R_2$  be their set-theoretic product, which can also be given the natural structure of a ring.

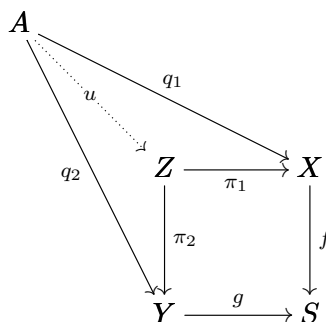
(a) Show that  $R_1 \times R_2$  is the product of  $R_1$  and  $R_2$  in  $\mathcal{R}$ .

(b) Show that  $R_1 \times R_2 \simeq R_1 \oplus R_2$  is not the coproduct of  $R_1$  and  $R_2$  in  $\mathcal{R}$ .

Note: We'll see later that coproducts do exist in the category  $\mathcal{R}_{\text{comm}}$  of commutative rings; they're called tensor products.

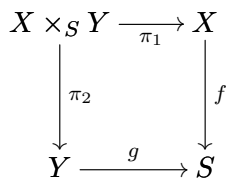
5. Let  $\mathcal{C}$  be a category. Let  $X, Y, S$  be objects in  $\mathcal{C}$  and  $f : X \rightarrow S, g : Y \rightarrow S$  be morphisms in  $\mathcal{C}$ . A *fiber(ed) product* of  $f$  and  $g$  in  $\mathcal{C}$  (or by abuse of terminology, fiber product of  $X$  and  $Y$  over  $S$ ) is an object  $Z$  in  $\mathcal{C}$  together with morphisms  $\pi_1 : Z \rightarrow X$  and  $\pi_2 : Z \rightarrow Y$  such that

- (i)  $g \circ \pi_2 = f \circ \pi_1$ ;  
 (ii) for any object  $A$  in  $\mathcal{C}$  and any morphisms  $q_1 : A \rightarrow X, q_2 : A \rightarrow Y$  such that  $g \circ q_2 = f \circ q_1$  there exists a unique morphism  $u : A \rightarrow Z$  such that the diagram



is commutative.

Show that, if it exists, the fiber product  $Z$  of  $f$  and  $g$  is unique up to isomorphism. The fiber product is denoted  $X \times_S Y$  and the diagram



is called a *fibered/pullback/Cartesian diagram*.

6. (a) Show in the category of sets

$$X \times_S Y = \{(x, y) \in X \times Y; f(x) = g(y)\}.$$

That is, show that the right hand side equipped with the obvious maps to  $X$  and  $Y$  satisfies the universal property of the fibered products.

- (b) If  $X$  is a topological space show that fibered products always exist in the category of open sets of  $X$  by describing what a fibered product is. (Should be a one-word description.)