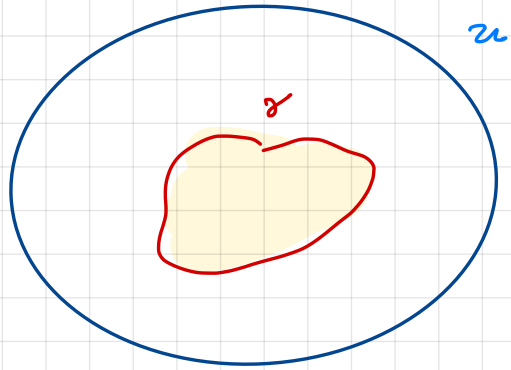
Math 220 A - Lecture 23

December 7, 2020

[1] Last time – Rouché's theorem

γ simple closed curve, $\gamma \overset{\mathcal{U}}{\sim} 0$

e.g. $\gamma = \partial \Delta$, $\overline{\Delta} \subseteq \mathcal{U}$.



Theorem $f, g : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, γ as above.

If $|f-g| < |g|$ on γ $\Rightarrow$

$$\#\ \text{zeroes}(f) = \#\ \text{zeroes}(g). \quad \text{in } \mathrm{Int}(\gamma).$$

(w/ multiplicity)

What does the hypothesis mean?

$$f = \underbrace{g}_{\text{dominant term}} + \underbrace{(f-g)}_{\text{lower order terms}}$$

Proof (see Conway for a different proof)

Let $h_t = g + t(f-g)$, $0 \le t \le 1$.

Want $t \longrightarrow \#$ zeroes (h_t) is continuous in t.
in Int γ

This implies $\#$ zeroes (h_t) = constant.

Since $h_0 = g$, $h_1 = f$ $\Rightarrow$ $\#$ zeroes $(f) = \#$ zeroes (g).

To show continuity, we use the Argument Principle

$$\# \text{ zeroes } (h_t) = \frac{1}{2\pi i} \int_\gamma \frac{h_t'(z)}{h_t(z)} \, dz .$$

Note $|h_t| = |g + t(f-g)| \ge |g| - |t|\,|f-g|$

$$\ge |g| - |f-g| > 0 \text{ on } \gamma.$$

Set $\varphi(t, z) = \dfrac{h_t'(z)}{h_t(z)} : [0,1] \times \{\gamma\} \longrightarrow \mathbb{C}$.

Note φ is continuous.

Key Fact $\psi : [0,1] \times \{\gamma\} \longrightarrow \mathbb{C}$ continuous

$$\Longrightarrow \quad \Phi(t) = \int_{\gamma} \psi(t, z)\, dz \text{ is continuous in } t.$$

Quick proof: Since $[0,1] \times \{\gamma\}$ is compact, ψ is uniformly continuous.

Fix $\varepsilon > 0$. then $\exists\, \delta > 0$ with

$$|t - t'| < \delta \Longrightarrow |\psi(t, z) - \psi(t', z)| < \varepsilon.$$

$$\Longrightarrow |\phi(t) - \phi(t')| = \left| \int_{\gamma} \psi(t, z) - \psi(t', z)\, dz \right|$$

$$< \varepsilon \cdot \text{length}(\gamma)$$

$\Longrightarrow \phi$ continuous.

Applications

[1] Find the location of zeroes of holomorphic fns

e.g. $f = z^5 + 24z^3 + 2z^2 + 3z + 1$ (last time)

We can also use this for nonpolymomial functions.

Example

$$f(z) = e^z - 5z^3 + 1. \quad , \quad \gamma = \{ |z| = 1 \}.$$

Dominant term $g(z) = -5z^3.$

Indeed $|g| = 5$ for $|z| = 1.$

$$|f-g| = |e^z + 1| \le |e^z| + 1 = e^{\operatorname{Re} z} + 1$$

$$\le e^{|z|} + 1 = e + 1 < 5 = |g|$$

$\Rightarrow$ # zeroes (f) = # zeroes $(g) = 3$ in $\Delta(0,1)$.

III abstract applications

Example

$h : U \longrightarrow \mathbb{C}$, $\overline{\Delta}(0,1) \subseteq U$, $|h(z)| < 1$, $|z| = 1$.

$\Longrightarrow$ h has one fixed point in $\Delta(0,1)$.

Proof We show $h(z) = z \iff h(z) - z = 0$ has a unique solution in $\Delta(0,1)$.

Let $f(z) = h(z) - z$, $g(z) = -z$, $\gamma = \{|z| = 1\}$.

Then

$$|f - g| = |h| < 1 = |g| \text{ on } \gamma$$

$\Longrightarrow$ # zeros(f) = # zeros$(g) = 1$. $\Longrightarrow$

$\Longrightarrow$ h has a unique fixed point in $\Delta(0,1)$.

Remark Hurwitz' theorem will be another abstract application of Rouché!

[2] Sequences of holomorphic functions (Conway $\overline{VII}$).

Outline — notions of convergence

— Weierstraß' theorem

— Hurwitz's theorem $\Longleftarrow$ Rouché

[a] Types of convergence

Question What is the correct notion of convergence for holomorphic functions?

$f_n : \mathcal{U} \longrightarrow \mathbb{C}$, $f : \mathcal{U} \longrightarrow \mathbb{C}$ be any functions.

Math 140 B [i] pointwise convergence

$f_n \longrightarrow f$ iff $\forall x \in \mathcal{U}$, $f_n(x) \longrightarrow f(x)$.

[ii] uniform convergence

$f_n \rightrightarrows f$ iff $\sup_{\mathcal{U}} |f_n - f| \longrightarrow 0$ as $n \longrightarrow \infty$.

Issues $\boxed{i}$ Pointwise convergence is not well behaved under differentiation or even integration. (Baby Rudin)

The pointwise limit of continuous functions need not be continuous (Baby Rudin / Math 140B).

$\boxed{ii}$ Uniform convergence is better. But the notion is strong. For instance, take

$$f_n(z) = \frac{z}{n}, \quad f(z) = 0, \quad f_n \not\rightrightarrows f \text{ on } \mathbb{C}.$$

We consider slightly weaker notions.

Better [a] uniform convergence on compact sets

[b] local uniform convergence

[a] **Notation**: $f_n \xrightarrow{c} f$ or $f_n \overset{c}{\rightrightarrows} f$

Definition $\forall K \subseteq \mathcal{U}$ compact, $\sup_K |f_n - f| \longrightarrow 0$ as $n \to \infty$.

[b] **Notation** $f_n \overset{l.u.}{\rightrightarrows} f$

Definition: $\forall x \in \mathcal{U}$ $\underbrace{\exists\ \Delta(x, r_x) \subseteq \mathcal{U}}_{\text{local}}$ with $\underbrace{f_n \rightrightarrows f}_{\text{uniform converg.}}$ in $\Delta(x, r_x)$.

Claim [a] = [b].

Thus $f_n \xrightarrow{c} f$, $f_n \overset{c}{\rightrightarrows} f$, $f_n \overset{l.u.}{\rightrightarrows} f$ mean the same thing.

Proof [a] $\Rightarrow$ [b]. If [a] holds for all K, take

$K = \overline{\Delta}(x, r_x) \subseteq \mathcal{U}$., K compact. This choice of K yields [b].

$\boxed{b} \Longrightarrow \boxed{a}$. Let $f_n \overset{l.u.}{\rightrightarrows} f$. Take K compact in $\mathcal{U}$.

For $x \in K$, $\exists\ \Delta(x, r_x)$ with $f_n \rightrightarrows f$ in $\Delta(x, r_x)$.

Since $K \subseteq \bigcup_{x \in K} \Delta(x, r_x) \Longrightarrow K \subseteq \bigcup_{i=1}^{N} \Delta(x_i, r_{x_i})$

by compactness. Since

$$\sup_K |f_n - f| \le \max_{1 \le i \le N} \left(\sup_{\Delta(x_i, r_{x_i})} |f_n - f| \right) \longrightarrow 0$$

$\Longrightarrow f_n \rightrightarrows f$ in K $\Longrightarrow f_n \overset{c}{\rightrightarrows} f$.

Example $f_n = \frac{z}{n}$, $f = 0$, $f_n \overset{c}{\rightrightarrows} f$ in $\mathbb{C}$.

Indeed, $\sup_K |f_n - f| = \sup_{z \in K} \left|\frac{z}{n}\right| \le \frac{M}{n} \longrightarrow 0$.

so $f_n \overset{c}{\rightrightarrows} f$. This was the example disallowed before.

Remark (Continuity & Math 140B).

f_n continuous & $f_n \rightrightarrows f$ then f continuous

f_n continuous & $f_n \overset{l.u}{\rightrightarrows} f$ then f continuous.

(because continuity is a local concept).

Important Convention

$\mathcal{C}(\mathcal{U})$ = continuous functions in $\mathcal{U}$

$\mathcal{O}(\mathcal{U})$ = holomorphic functions in $\mathcal{U}$

We will always consider local uniform convergence for both $\mathcal{O}(\mathcal{U})$ and $\mathcal{C}(\mathcal{U})$.

[6] Weierstraß' Theorem

Let $f_n : U \longrightarrow \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$. Then

[1] f holomorphic

[2] $f_n^{(k)} \overset{l.u.}{\rightrightarrows} f^{(k)}$

Remark [i] $\mathcal{O}(U) \hookrightarrow \mathcal{C}(U)$ "closed." under local uniform limits.

[ii] integration is not an issue

If $f_n \overset{l.u.}{\rightrightarrows} f$ then $\int_\gamma f_n \, dz \longrightarrow \int_\gamma f \, dz$. since $\{\gamma\}$ compact.

[iii] the statement fails in real analysis. (Baby Rudin or Math 140B for examples).

The proof will be given next time.