Math 220A - Fall 2016 - Midterm

Problem 1.

Let $P_1, \ldots, P_n$ be points on the unit circle. Show that there is a point Q on the unit circle such that
$$P_1Q \cdot P_2Q \cdot \ldots \cdot P_nQ \geq 1.$$

`Solution:` *Let $\Delta(0,1)$ be the unit disc, and let $z_1, \ldots, z_n$ be the coordinates of the points $P_1, \ldots, P_n$ so that $|z_i| = 1$. Define*
$$f : \Delta(0,1) \to \mathbb{C}$$
by
$$f(z) = (z - z_1) \cdot \ldots \cdot (z - z_n).$$
Note that f is entire, and that
$$|f(0)| = |z_1 \cdots z_n| = 1.$$
By the maximum modulus principle, $|f|$ achieves the maximum over $\overline{\Delta}$ at a point w on the boundary of Δ. Since $|f(0)| = 1$, it follows that there exits $|w| = 1$ such that $|f(w)| \geq |f(0)| = 1$. This w corresponds to a point Q on the unit disc, and
$$P_1Q \cdot \ldots \cdot P_nQ = |w - z_1| \cdot \ldots \cdot |w - z_n| = |f(w)| \geq 1.$$

Problem 2.

(i) Let f be continuous in an open set U and let $\bar{R} = [a,b] \times [c,d] \subset U$ be a rectangle. Let $R_n = [a+\frac{1}{n}, b-\frac{1}{n}] \times [c+\frac{1}{n}, d-\frac{1}{n}]$ be a rectangle contained in R for n large enough. Note that $R_n \to R$ in the obvious sense. Show that

$$\int_{\partial R_n} f(z)\,dz \to \int_{\partial R} f(z)\,dz.$$

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be continuous and holomorphic on $\mathbb{C} \setminus [0,1]$. Show that f is entire.

`Solution:`

(i) *We show that over each side of the rectangles we obtain convergence. It suffices to treat one side at a time, say for instance the lower horizontal side* $L = [a,b] \times \{c\}$ *and the corresponding sides* $L_n = [a+\frac{1}{n}, b-\frac{1}{n}] \times \{c+\frac{1}{n}\}$. *Thus we show*

$$\int_{L_n} f(z)\,dz \to \int_L f(z)\,dz.$$

We parametrize the sides by $t \in [0,1]$ *via*

$$t \to ((1-t)a + tb, c), \quad t \to \left(\left(a+\frac{1}{n}\right)(1-t) + t\left(b-\frac{1}{n}\right), c+\frac{1}{n}\right).$$

We have

$$\int_L f(z)\,dz = \int_0^1 g(t)(b-a)\,dt$$

$$\int_{L_n} f(z)\,dz = \int_0^1 g_n(t)\cdot\left(b-a-\frac{2}{n}\right)\,dt,$$

where

$$g(t) = f((1-t)a+tb, c), \quad g_n(t) = f\left(\left(a+\frac{1}{n}\right)(1-t) + t\left(b-\frac{1}{n}\right), c+\frac{1}{n}\right).$$

We claim that

$$g_n \to g$$

uniformly. Indeed, since f *is uniformly continuous over* $[a,b]\times[c,d]$ *for each* $\epsilon > 0$, *we can find* $\delta > 0$ *such that*

$$|f(x,y) - f(x',y')| < \epsilon$$

if $|x-x'| < \delta, |y-y'| < \delta$. *Then set*

$$x = \left(a+\frac{1}{n}\right)(1-t) + t\left(b-\frac{1}{n}\right), x' = a(1-t)+bt, y = c+\frac{1}{n}, y' = c,$$

and pick n *so that* $\frac{1}{n} < \delta$ *to conclude. In particular, since we work over compact sets,*

$$g_n \to g, b-a-\frac{1}{n} \to b-a$$

both converge uniformly, and since we work over compact sets we can multiply to still obtain uniform convergence. This result is typically covered in Math 140B. Thus

$$g_n(t)\left(b-a-\frac{2}{n}\right) \to g(t)(b-a)$$

uniformly, and integrating we obtain the claim.

(ii) *We show f admits a primitive in $\mathbb{C}$ so that $f = F'$ for an entire function F. The derivative of an entire function is entire as shown in class, hence f is entire as well.*

To this end, since f is continuous, we only need to check that

$$\int_{\partial R} f(z)\,dz = 0$$

for all rectangles $R \subset \mathbb{C}$.

- *If the rectangle R does not intersect the segment $[0,1]$, then the statement follows by Goursat's lemma.*
- *If the rectangle R intersects the segment $[0,1]$, then we can split R into finitely many subrectangles R_j, oriented compatibly, such that R_j either doesn't intersect $[0,1]$, or else it has one corner at 0 or at 1 but is otherwise disjoint from $[0,1]$, or else it has one side contained within the segment $[0,1]$. We have*

$$\int_{\partial R} f(z)\,dz = \sum_j \int_{\partial R_j} f(z)\,dz.$$

We claim that for all j we have $\int_{\partial R_j} f(z)\,dz = 0$.

  - *This is clear if the rectangle R_j doesn't intersect $[0,1]$ as already explained above.*
  - *If R_j intersects $[0,1]$ in one corner or if it has a side contained along the segment $[0,1]$, then pick a sequence of rectangles $R_j^n \to R_j$ in $\mathbb{C} \setminus [0,1]$. Then*

$$\int_{\partial R_j^n} f(z)\,dz = 0$$

by Goursat, and thus by part (i), we have

$$\int_{\partial R} f(z)\,dz = 0,$$

as well.

Problem 3.

Let $p_1, \ldots, p_n$ be polynomials, and let $f : \mathbb{C} \to \mathbb{C}$ be entire such that

$$f(z)^n + f(z)^{n-1}p_1(z) + \ldots + p_n(z) = 0.$$

Show that f is a polynomial.

`Solution:` *Let d be the maximum of the degrees of $p_1, \ldots, p_n$. Since $\lim_{z\to\infty} \frac{p_i(z)}{z^{d+1}} = 0$, there must exists $R > 0$ such that*

$$|p_i(z)| \le |z|^{d+1}$$

if $|z| \ge R$, and for all i. We have

$$f(z)^n = -p_1(z)f(z)^{n-1} - \ldots - p_n(z).$$

Taking absolute values we conclude that for $|z| \ge R$ we have

$$|f(z)|^n = |p_1(z)f(z)^{n-1} + \ldots + p_n(z)| \le |f(z)|^{n-1}|p_1(z)| + \ldots + |p_n(z)| \le |z|^{d+1}(|f(z)|^{n-1} + \ldots + 1).$$

We claim

$$|f(z)| \le \max(1, n|z|^{d+1}).$$

Assume otherwise, so that in particular $|f(z)| > 1$ and $|f(z)| > n|z|^{d+1}$. Then

$$|z|^{d+1}(|f(z)|^{n-1} + \ldots + 1) < |z|^{d+1}(|f(z)|^{n-1} + \ldots + |f(z)|^{n-1}) = n|f(z)|^{n-1}|z|^{d+1} < |f(z)|^n$$

contradicting the above inequality. Therefore,

$$|f(z)| \le n|z|^{d+1},$$

for $|z| \ge \max(R, (1/n)^{1/(d+1)}) := \rho$. This implies f is a polynomial by the extended Liouiville theorem of Homework 2, Problem 2.

Problem 4.

Let $f : \Delta \setminus \{a\} \to \mathbb{C}$ be a holomorphic function on a punctured disc centered at a, having $z = a$ as an essential singularity. Show that f is not injective.

`Solution:` *Clearly, f is not constant. Let $\Delta = \Delta(a, r)$. By the Casoratti-Weierstrass theorem, the set*

$$U = f\left(\Delta\left(a, \frac{r}{2}\right) \setminus \{a\}\right)$$

is dense in $\mathbb{C}$. By the open mapping theorem,

$$V = f\left(\Delta\left(a, \frac{r}{2}, r\right)\right)$$

is open. Since U is dense, it follows that $U \cap V \neq \emptyset$, proving that f is not injective.