Math 281a – Problem Set # 4
Module 5

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Due @ November 13th, 2021

Problem 1

Let $X_1, X_2, \cdots, X_n$ be i.i.d. from Uniform distribution $[-1, 1]$. Consider the l_2 norm squared defined as

$$\|X\|_2^2 = \sum_{i=1}^n X_i^2.$$

Establish a concentration of measure (tail inequality on both sides) for the above defined $\|X\|_2^2$.

Problem 2

Let X be a random variable with mean zero such that

$$P(|X| > t) \leq 2\exp\{-2t/\lambda\}$$

for some $\lambda > 0$. Show that the following is then true for any positive integer $k \geq 1$

$$E[|X|^k] \leq \lambda^k k!.$$

Hints: recall that

$$E|X| = \int_0^\infty P(|X| > t)dt$$

as well as

$$\int_0^\infty e^{-u} u^{k-1} du = \Gamma(k) = k!$$

with Γ denoting the Gamma function : click here for Gamma on Wiki.

Problem 3

Let X be sub-Gaussian with parameter σ^2. Show that Z is sub-Exponential with parameter $16\sigma^2$, where

$$Z = X^2 - E[X^2].$$

Hint: Begin by utilizing the following expansion

$$E[e^{tW}] = 1 + \sum_{k=2}^{\infty} \frac{t^k E[W^k]}{k!}$$

for a suitably chosen W. Moreover, recall the following corollary of Jensen's inequality

$$E[X+Y]^k \leq E[X^k] + E[Y^k]$$

for any X, Y and integer $k \geq 1$.

Problem 4

Let $X_1, \cdots, X_n$ be n independent sub-Gaussian random variables, each with sub-Gaussian parameter σ^2. Show that

(a)

$$E\left[\max_{1\leq i\leq n} X_i\right] \leq \sigma\sqrt{2\log(n)}$$

(b)

$$P\left(\max_{1\leq i\leq n} X_i > t\right) \leq n \exp\left\{-\frac{t^2}{2\sigma^2}\right\}$$

Hint: Note that

$$E[W] = \frac{1}{t} E[\log e^{tW}]$$

for any W. Moreover, Jensen's inequality and a union bound will prove useful in this problem.