Math 220A - Lecture 1

October 2, 2020

Cauchy

Riemann

Weierstraß

Let $\mathcal{U} \subseteq \mathbb{C}$ open & connected set.

Def $f : \mathcal{U} \longrightarrow \mathbb{C}$ is complex differentiable (CD) provided $\forall z \in \mathcal{U}$.

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} := f'(z).$$

Examples

(i) f, g complex differentiable $\Rightarrow f+g, fg$ complex diff. (CD)

(ii) $1, z, z^2, \dots z^n \dots$ are CD.

CD = complex differentiable

RD = real differentiable

Remark

We have seen the same definition for $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}$.

The two definitions have different consequences. For instance

i. If f is CD $\Rightarrow f'$ is CD. $\Rightarrow f''$ is CD $\Rightarrow \dots$

If f RD. this statement fails.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0 & , x = 0 \end{cases} \Rightarrow f' = \begin{cases} 2x \sin \frac{1}{x^2} + \cos \frac{1}{x^2} \cdot \frac{-2}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

f' is not even continuous at 0.

ii. f CD. $\Rightarrow f = \sum \frac{f^{(n)}(a)}{n!} (z-a)^n$ Taylor in some $\Delta(a, r)$.

f RD. $\Rightarrow f \neq$ Taylor exansion.

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function is C^∞ and $f^{(k)}(0) = 0$.

The Taylor series of f is zero but $f(x) \neq 0$ for x near 0

Thus f is not equal to its Taylor expansion.

$\boxed{iii}$ f CD on $\mathcal{U} = \mathbb{C}$ + f bounded $\Rightarrow$ f constant

This fails for RD. $f(x) = \sin x, \cos x$

$\boxed{iv}$ f, g CD & $f = g$ on $\underbrace{V \subseteq \mathcal{U}}_{\text{open}}$ $\Rightarrow$ $f \equiv g$. in $\mathcal{U}$.

This fails for RD.

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A more appropriate comparison is with functions of two real variables.

Identify $\mathbb{C} \cong \mathbb{R}^2$, $z = x + iy \longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Def $f : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is said to be real diff. (RD). provided

$\forall z \in \mathcal{U}$ $\exists$ $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is $\mathbb{R}$-linear & $A = Df(z)$.

$$\lim_{h \to 0} \frac{\| f(z+h) - f(z) - Ah \|}{\| h \|} = 0.$$ (Rudin, 9.11).

Remark If f is CD. $\Rightarrow$ f is RD. Indeed, we can take

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad A = \text{multiplication by } f'(z).$$

Remark If f is RD. Write $f = u + iv$. $\Rightarrow \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist

Claim $A = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ = Jacobian matrix

Indeed if f is RD $\Rightarrow \lim\limits_{h \to 0} \frac{\| f(x+h, y) - f(x,y) - hA\begin{bmatrix} 1 \\ 0 \end{bmatrix} \|}{h} =$

$\Rightarrow A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = f_x = \begin{bmatrix} u_x \\ v_x \end{bmatrix}$. Similarly $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_y \\ v_y \end{bmatrix}$

Remark Conversely, if u_x, u_y, v_x, v_y all exist $\not\Rightarrow$ f is RD.

Take $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

If u_x, u_y, v_x, v_y exist & are continuous $\Rightarrow$ f is RD. (Rudin 9.21).

Lemma $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\mathbb{R}$-linear map. TFAE.

$\boxed{a}$ A is $\mathbb{C}$-linear

$\boxed{b}$ $Az = \alpha . z$ for some $\alpha \in \mathbb{C}$.

$\boxed{c}$ $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for some $a, b \in \mathbb{R}$.

$\alpha = a + bi$

Proof $\boxed{a} \Rightarrow \boxed{b}$ $\alpha = A(1)$. $A(z) = A(z \cdot 1) = z\, A(1) = z\alpha$.

$\boxed{b} \Rightarrow \boxed{a}$ clear.

$\boxed{b} \Rightarrow \boxed{c}$ $\alpha = a + bi$. Then $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\, 1 = \alpha = \begin{bmatrix} a \\ b \end{bmatrix}$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\, i = \alpha i = ai - b = \begin{bmatrix} -b \\ a \end{bmatrix}$$

$\boxed{c} \Rightarrow \boxed{b}$. If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \Rightarrow$ set $\alpha = a + bi$. Apply part $\boxed{b}$ to conclude $Az = \alpha z$.

Remark $\det A = a^2 + b^2 \Rightarrow$ either $\det A > 0$, or else $A = 0$. $\Rightarrow$ either $A = 0$ or A is orientation preserving.

Remark The lemma shows that TFAE

[i] f is CD.

[ii] f is RD. & $Df(z)$ is $\mathbb{C}$-linear.

Remark (Cauchy - Rieman equation).

If f is CD $\Rightarrow Df(z) = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$ is $\mathbb{C}$-linear

$$\Rightarrow \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases} \quad (\text{Cauchy - Riemann}).$$

Remark Conversely, if u, v are C^1 & satisfy Cauchy - Riemann equation

$\Rightarrow f = u + iv$ is CD.

Indeed, f is RD + $Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ is $\mathbb{C}$-linear

since it has the shape in Lemma $\boxed{c}$.

Harmonic functions

Assume u, v satisfy Cauchy - Riemann (CR)

Assume u, v are C^{∞}.

$$u_x = v_y \implies u_{xx} = v_{yx}$$

$$u_y = -v_x \qquad u_{yy} = -v_{xy}$$

$$\implies u_{xx} + u_{yy} = 0.$$

$\implies u, v$ are harmonic

$$\implies v_{xx} + v_{yy} = 0.$$

Conclusion If f is CD $\implies \operatorname{Re} f = u$, $\operatorname{Im} f = v$ are harmonic

(u, v) are said to be harmonic conjugates

Notation

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y}\right).$$

$$z = x + iy \implies x = \frac{1}{2}(z + \bar{z})$$

$$\bar{z} = x - iy \implies y = \frac{1}{2i}(z - \bar{z})$$

Think of $z, \bar{z}$ as being independent variables. Apply chain rule:

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x} \cdot \frac{1}{2} + \frac{\partial}{\partial y} \cdot \frac{1}{2i} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y}\right).$$

The case $\frac{\partial}{\partial \bar{z}}$ is similar.

"f only depends on z not on $\bar{z}$."

Lemma f is CD $\implies \dfrac{\partial f}{\partial \bar{z}} = 0.$

Proof $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + i f_y) \overset{?}{=} 0 \iff f_x \overset{?}{=} -i f_y .$

$$\iff u_x + i v_x = -i(u_y + i v_y)$$

$$\iff \begin{array}{l} u_x = v_y \\ v_x = -u_y \end{array}$$

These are the Cauchy - Riemann equations.