HOMEWORK 4

DUE APRIL 28, 2021 AT 11:59PM

1. Let p be a prime and let $\mathbb{F}_p$ be a field with p elements.

(a) Show that $\mathbb{F}_p$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. (Use the shortest proof possible...)

(b) Let $K/\mathbb{F}_p$ be a finite extension. Then K is a finite dimensional vector space over $\mathbb{F}_p$ and hence has $q = p^n$ elements for some n. Show that $K/\mathbb{F}_p$ is Galois with cyclic Galois group generated by

$$\phi : K \longrightarrow K, x \mapsto x^p.$$

As you probably already know, ϕ is called the p-th power Frobenius.

(c) Show that for each $n \geq 1$, there is exactly one field K, with $\mathbb{F}_p \subseteq K \subseteq \bar{\mathbb{F}}_p$ of degree n over $\mathbb{F}_p$.

(d) Show that $\mathrm{Gal}\left(\bar{\mathbb{F}}_p/\mathbb{F}_p\right) \cong \hat{\mathbb{Z}}$. (It is also true that $\mathrm{Gal}\left(\bar{\mathbb{F}}_q/\mathbb{F}_q\right) \cong \hat{\mathbb{Z}}$ with the same proof.) And yes, $\hat{\mathbb{Z}}$ is the object defined in the last homework.

From Atiyah-MacDonald:

Chapter 1: 15, 17 - 21