

Math 280 A

Fall '21

#1

Administrative

(See Canvas for details)

- Lectures recorded by EVT and podcast
system.

↑
sign up!

↑
Media Gallery
in
Canvas

- Piazza for discussion
- Gradescope for Homework

- Reference list at
math.ucsd.edu/~pfitz
click on 280A link
- Course Grade based on HW
assignments (7 or 8 in total).

Math 280 content

A : Goal is SLLN (Chap. 1-7)

- probability space
- random variable
- expectation (= integration)
- independence
- modes of convergence

B:

- CLT $\bar{X}_n = \mu + \frac{\sigma}{\sqrt{n}} Z + \dots$
- Martingale (key tool in modern probability)

(chapters 8-10)

C:

Markov Chains

Brownian Motion

Ergodic Theory or Poisson process

...

Preview Borel's Strong Law of Large Numbers

X_1, X_2, \dots i.i.d. ^{symmetric} Bernoulli r.v.'s

$$P(X_k = 1) = P(X_k = -1) = 1/2$$

$$\therefore E(X_k) = 0$$

$$\text{Var}(X_k) = 1$$



Define $\bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$ ("running average")

SLLN : $\bar{X}_n \rightarrow 0$ a.s. , $n \rightarrow \infty$

That is

$$P\left(\lim_{n \rightarrow \infty} \bar{X}_n = 0\right) = 1$$



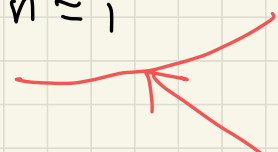
Notice

$$E(\bar{X}_n) = 0$$

$$\text{Var}(\bar{X}_n) = 1/n$$

$$E[\bar{X}_n^4] = \frac{3}{n^2} - \frac{2}{n^3} \leq \frac{3}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} E(\bar{X}_n^4) \leq \sum_{n=1}^{\infty} \frac{3}{n^2} = \frac{\pi^2}{2} < \infty$$

$$\therefore E\left(\sum_{n=1}^{\infty} \bar{X}_n^4\right) < \infty$$


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If this sum diverges
we assign it value $+\infty$

$$\therefore P\left(\sum_1^{\infty} \bar{X}_n^4 < \infty\right) = 1$$

$$\therefore P\left(\lim_{n \rightarrow \infty} \bar{X}_n^4 = 0\right) = 1$$

$$\therefore P\left(\lim_{n \rightarrow \infty} \bar{X}_n = 0\right) = 1$$

(Needs only $E(X_k) = 0, E(X_k^4) < \infty$)

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σ -field (aka σ -algebra)

Ω non-empty set "sample space"

A σ -field is a collection $\mathcal{B} \subset \mathcal{P}(\Omega)$ such that

$$(1) \quad \emptyset \in \mathcal{B}$$

$$(2) \quad B \in \mathcal{B} \Rightarrow B^c \in \mathcal{B}$$

$$(3) \quad B_1, B_2, \dots \in \mathcal{B} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$$

(2) is "closure under formation of complements"

(3) is "closure under formation of countable unions"

Because of (1), if B_1, B_2, \dots, B_n are $\in \mathcal{B}$
then so is $\bigcup_{k=1}^n B_k$

— use $B_1, B_2, B_3, \dots, B_n, \emptyset, \emptyset, \dots$

Ex. 1

$$\Omega = (0, 1]$$

$$\mathcal{A} = \left\{ \text{finite unions of intervals shaped like } (a, b] \right\}$$

\mathcal{A} is a field $((a, a] = \emptyset)$ but

\mathcal{A} is not a σ -field:

$$\bigcup_{n=1}^{\infty} (0, 1 - 1/n] = (0, 1) \notin \mathcal{A}$$

Ex. 2 Generators

Ω arbitrary

$$\mathcal{C} \subset \mathcal{P}(\Omega)$$

$$\begin{aligned}\sigma(\mathcal{C}) &= \bigcap \{ \mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-field, } \mathcal{B} \supset \mathcal{C} \} \\ &= \sigma\text{-field generated by } \mathcal{C} \\ &= \text{least } \sigma\text{-field containing } \mathcal{C}\end{aligned}$$

If $A \subset \Omega$ then

$$A \in \sigma(\mathcal{C}) \quad \text{iff} \quad A \in \mathcal{B} \text{ for each } \sigma\text{-field } \mathcal{B} \supset \mathcal{C}$$

Another way to say this :

$\sigma(\mathcal{C})$ is a σ -field containing \mathcal{C} , and if

\mathcal{B} is another σ -field containing \mathcal{C} then

$$\sigma(\mathcal{C}) \subset \mathcal{B}.$$

Important Special Case

\mathcal{A} as in ex. 1

$\sigma(\mathcal{A})$ is called the Borel σ -field on $(0,1]$

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Similarly

$$\mathcal{B}(\mathbb{R}) = \sigma\left(\{(-\infty, b] : b \in \mathbb{R}\}\right).$$