Math 220 A – Lecture 8

October 21, 2020

Last time

Cauchy's Theorem (Homotopy version).

$f: U \longrightarrow \mathbb{C}$ holomorphic, γ_0, γ_1 piecewise C^1 loops in U, $\gamma_0 \overset{U}{\sim} \gamma_1$. Then

$$\int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz$$

Remark We prove a seemingly stronger result

Cauchy's Theorem$^+$ (Homotopy version).

$(+)$ $f: U \longrightarrow \mathbb{C}$ continuous, holomorphic in $U \setminus \{a\}$

$$\Longrightarrow \int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz \quad \text{if } \gamma_0 \overset{U}{\sim} \gamma_1 \text{ are piecewise } C^1 \text{ loops.}$$

We need this stronger form to prove:

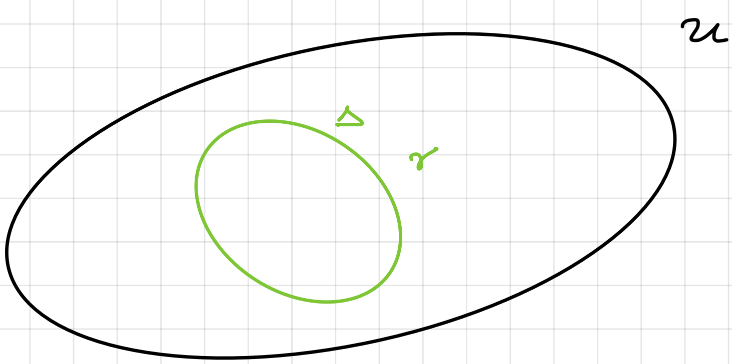
Cauchy's Integral Formula (CIF)

$f: \mathcal{U} \to \mathbb{C}$ holomorphic, $\gamma \overset{\mathcal{U}}{\sim} 0$, $a \in \mathcal{U} \setminus \{\gamma\}$

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz$$

Remark This generalizes Local Cauchy's Integral Formula.

we proved before. In that case, $\gamma = \partial \Delta$ where $\overline{\Delta} \subseteq \mathcal{U}$.



Proof of CIF

Let $$F(z) = \begin{cases} \dfrac{f(z) - f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

$\Longrightarrow$ F continuous in U, holomorphic in $U \setminus \{a\}$.

$$\Longrightarrow \int_\gamma F \, dz = 0 \quad \text{by Cauchy}^+$$

$$\Longrightarrow \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(a)}{z-a} \, dz = 0$$

$$\Longrightarrow \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} = f(a) \cdot \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = f(a) \cdot n(\gamma, a).$$

QED.

Remark

Homotopy Cauchy$^+$ $\Longrightarrow$ CIF

$\Downarrow$

Homotopy Cauchy

In fact CIF $\Longrightarrow$ Homotopy Cauchy by using CIF

for $\gamma = \gamma_0 + (-\gamma_1)$ & the function $(z-a) f(z)$

Proof of Cauchy$^+$

Recall the assumption

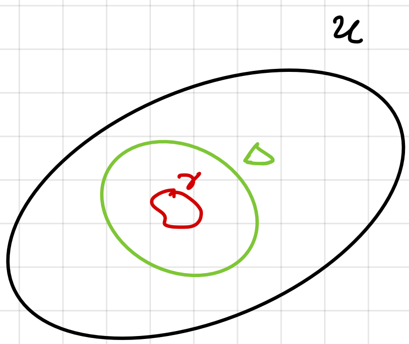
$$(+) \quad f \text{ cont } \& \ f \text{ hol. in } \mathcal{U} \setminus \{a\}.$$

For the proof we only use

$\boxed{i}$ f continuous

$\boxed{ii}$ $\forall \ \Delta \subseteq \mathcal{U}$, $\{\gamma\} \subseteq \Delta$ piecewise C^1 loop

$$\Rightarrow \int_\gamma f \, dz = 0 \qquad (*).$$

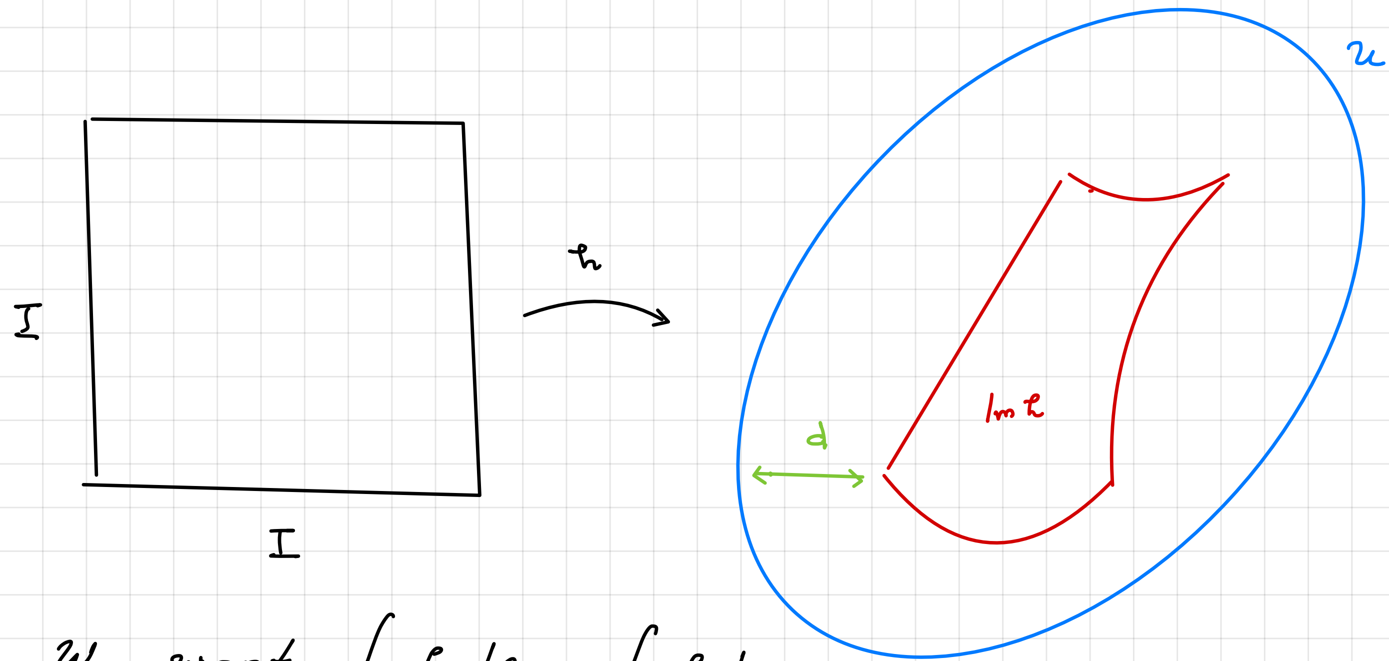


Under assumption $(+)$, item $\boxed{ii}$ follows from a previous

Corollary$^+$ $f : \Delta \longrightarrow \mathbb{C}$ continuous, holomorphic in $\Delta \setminus \{a\}$

$$\Longrightarrow \int_\gamma f \, dz = 0 \quad \forall \ \gamma \text{ piecewise } C^1 \text{ loop}$$

(see Lecture 6).



We want $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

Let $h : I \times I \longrightarrow u$ be the homotopy from γ_0 to γ_1

Im h compact, $\mathbb{C} \setminus u$ closed $\overset{\text{Rudin}}{\Longrightarrow} \exists\, d > 0$.

$$d = \operatorname{dist}(\operatorname{Im} h, \mathbb{C} \setminus u)$$

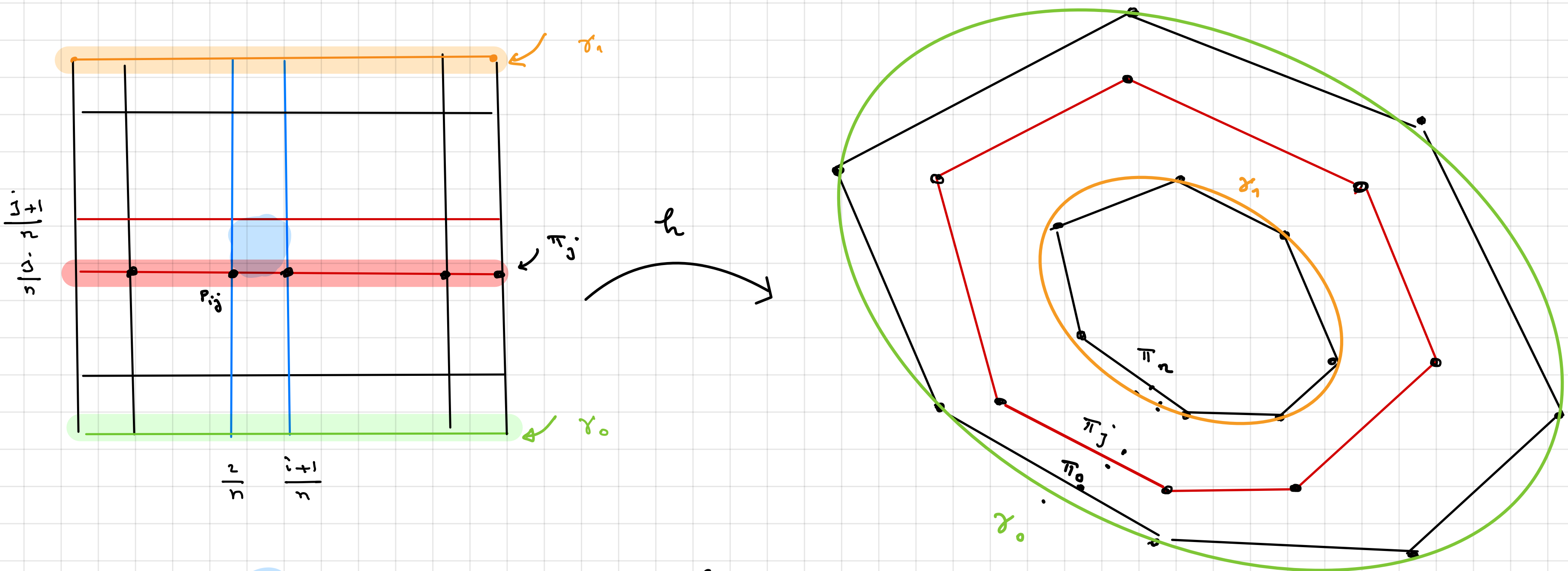
h continuous, $I \times I$ compact $\Longrightarrow$ h uniformly cont.

$\Longrightarrow \exists\, \delta > 0$ such that

$$|t - t'| < \delta,\ |s - s'| < \delta \implies |h(t,s) - h(t',s')| < d.$$

Let $n \in \mathbb{Z}_+$ with $\frac{1}{n} < \delta$. Subdivide I into equal

intervals $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ of length $< \delta$.



Let P_{ij} have coordinates $\left(\frac{i}{n}, \frac{j}{n}\right)$. Let $Q_{ij} = h(P_{ij})$.

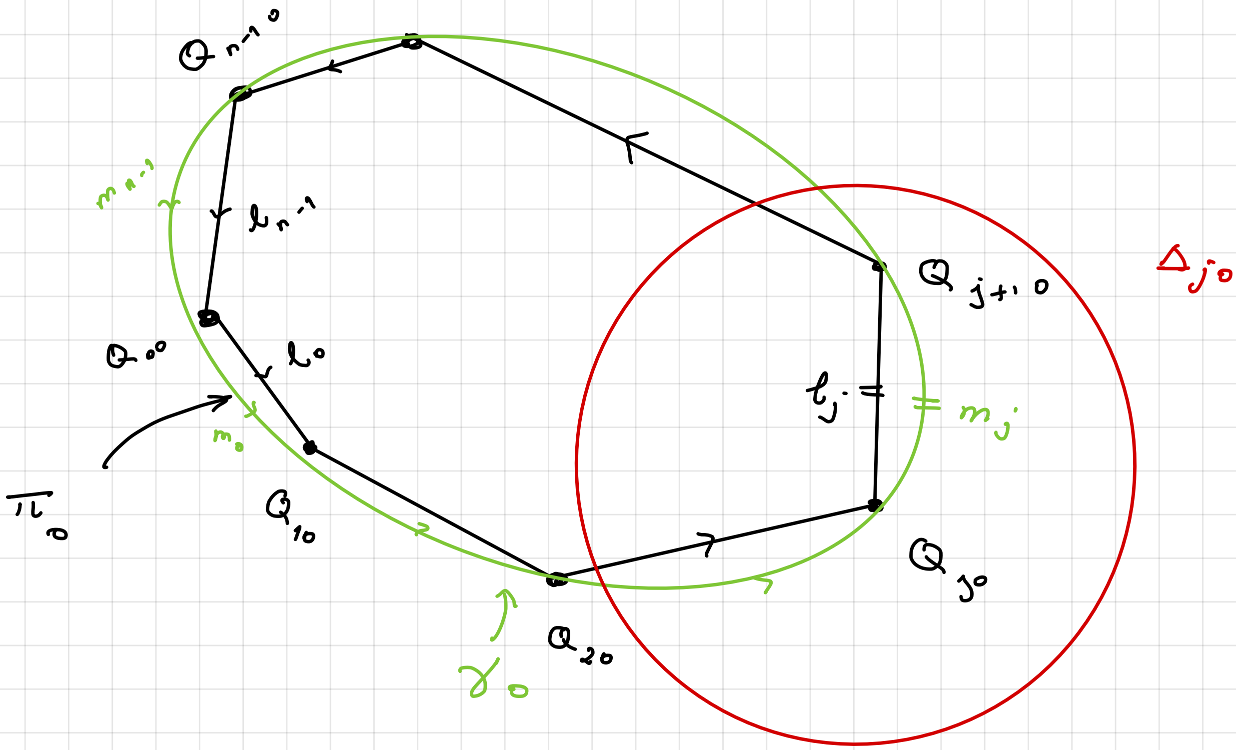
Let $R_{ij} = \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$. Let $\Delta_{ij} = \Delta(Q_{ij}, d)$

Note $\Delta_{ij} \subseteq U$. by the choice of d.

Since sides of R_{ij} have length $< \delta \implies h(R_{ij}) \subseteq \Delta_{ij}$ by uniform continuity.

Let π_j be the polygon through $Q_{0j}, Q_{1j} \dots Q_{nj} = Q_{0j}$

Claim $\boxed{a}$ $\int_{\gamma_0} f\,dz = \int_{\pi_0} f\,dz$ & $\int_{\pi_n} f\,dz = \int_{\gamma_1} f\,dz.$



Let $l_0, l_1, \dots l_{n-1}$ be the edges of the polygon π_0

$m_0, m_1, \dots m_{n-1}$ be the arcs of the curve γ_0.

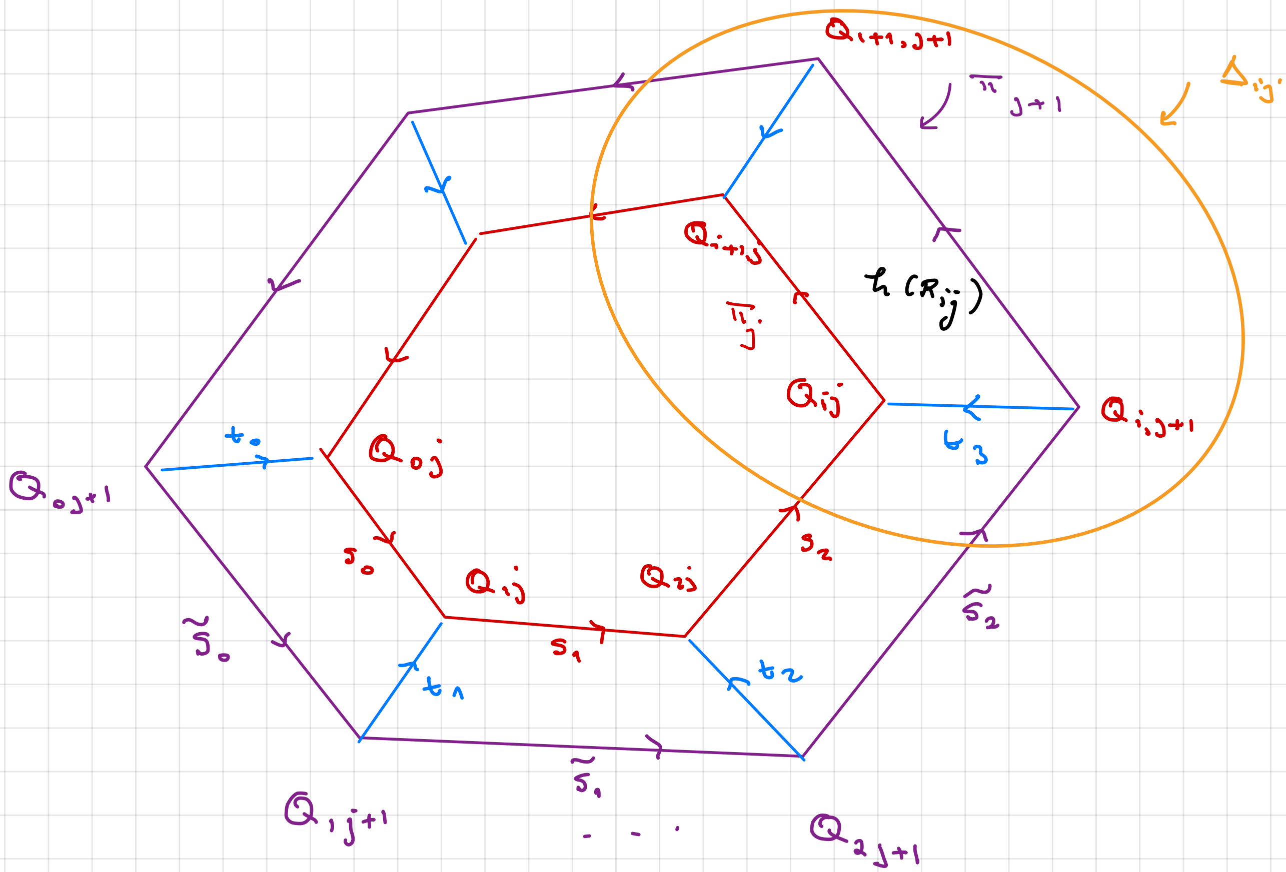
$$m_j = \gamma_0 \Big|_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}$$

By constuction both l_j, m_j are contained in $\Delta_{j0} \subseteq U$.

By $(*)$ we have $\int_{l_j + (-m_j)} f\,dz = 0 \implies \int_{l_j} f\,dz = \int_{m_j} f\,dz$

Adding for all j, we find $\int_{\pi_0} f\,dz = \int_{\gamma_0} f\,dz.$

Claim [b] $\displaystyle\int_{\pi_j} f\,dz = \int_{\pi_{j+1}} f\,dz$



Let $s_0, \dots s_{n-1}$ be the edges of π_j.

$\tilde{s}_0, \dots \tilde{s}_{n-1}$ the edges of π_{j+1}

$t_0, \dots, t_{n-1}$ the segments joining Q_{ij} to $Q_{i,j+1}$.

Since $h(R_{ij}) \subseteq \Delta_{ij} \Rightarrow \tilde{s}_i + t_{i+1} + (-s_i) + (-t_i)$ is a loop in Δ_{ij}. By $(*)$

$$\Longrightarrow \int_{\tilde{s}_i + t_{i+1} + (-s_i) + (-t_i)} f \, dz = 0$$

$$\Longrightarrow \int_{\tilde{s}_i} f \, dz - \int_{s_i} f \, dz = \int_{t_i} f \, dz - \int_{t_{i+1}} f \, dz.$$

Add these for all i: We find

$$\int_{\pi_{j+1}} f \, dz - \int_{\pi_j} f \, dz = 0 \Longrightarrow \text{Claim } \boxed{b}.$$

From Claims $\boxed{a}$ & $\boxed{b}$,

$$\int_{\gamma_0} f \, dz = \int_{\pi_0} f \, dz = \cdots = \int_{\pi_n} f \, dz = \int_{\gamma_1} f \, dz.$$

QED.