Math 220A - Lecture 5

October 14, 2020

Last time – existence of primitives

$U \subseteq \mathbb{C}$ open & connected, $f : U \longrightarrow \mathbb{C}$ continuous

Proposition A TFAE

$\boxed{i}$ f admits a primitive

$\boxed{ii}$ $\int_\gamma f\, dz = 0$ $\forall \gamma$ piecewise C^1 loop.

Corollary $\nexists$ no logarithm in $U = \mathbb{C}^\times$.

Proposition B If $U = \Delta$ = disc. TFAE

$\boxed{i}$ f admits primitive

$\boxed{ii}$ $\int_{\partial R} f\, dz = 0$ for all rectangles $\bar{R} \subseteq U$.

Compare:

Prop. A	Prop B.
$U \subseteq \mathbb{C}$	$U = \Delta$
γ piecewise C^1	$\gamma = \partial R$

Proposition C If $f : U \to \mathbb{C}$ holomorphic $\Rightarrow \int_{\partial R} f\, dz = 0$

for all rectangles $R \subseteq U$. (Goursat's lemma).

Remark f' is NOT assumed to be continuous.

If f' is continuous an easier proof is possible.

Corollary $f : \Delta \to \mathbb{C}$ holomorphic

$\overset{B+C}{\Longrightarrow}$ f admits a primitive.

$\overset{A}{\Longrightarrow}$ $\int_\gamma f\, dz = 0 \quad \forall \gamma$ piecewise C^1 loops.

This is a form of Cauchy's theorem.

DÉMONSTRATION DU THÉORÈME DE CAUCHY.

Extrait d'une lettre adressée à M. Hermite

PAR

E. GOURSAT

À TOULOUSE.

Voici une démonstration du théorème de CAUCHY, qui me parait un peu plus simple que les démonstrations habituelles. Elle repose uniquement sur la définition de la dérivée et sur cette remarque que les intégrales définies $\int dz$, $\int z dz$, prises le long d'un contour fermé quelconque, sont nulles.

Soit $f(z)$ une fonction de la variable complexe z, uniforme et continue, ainsi que sa première dérivée $f'(z)$, à l'intérieur d'une aire A limitée par un contour fermé C, simple ou multiple, et sur ce contour lui-même; j'admettrai de plus que ce contour a une *longueur* finie. Imaginons que l'on partage l'aire A en parties plus petites par des parallèles équidistantes à deux directions rectangulaires; l'intégrale $\int f(z)dz$, prise le long du contour total dans le sens direct, est égale à la somme des intégrales $\int f(z)dz$, prises dans le même sens le long du contour de chacune des parties en lesquelles on a subdivisé l'aire A. Il suffit de remarquer qu'en ajoutant ces dernières les parties de l'intégrale qui proviennent des lignes auxiliaires se détruisent, comme étant prises deux fois dans des sens

Edouard Goursat

1858 – 1936

J'ai reconnu depuis longtemps que la demonstration du theoreme de Cauchy, que j'ai donnee en 1883, ne supposait pas la continuite de la derivee."

(I have recognized for a long time that the demonstration of Cauchy's theorem which I gave in 1883 didn't really presuppose the continuity of the derivative.)

Proposition A TFAE

$\boxed{i}$ f admits a primitive

$\boxed{ii}$ $\int_\gamma f\,dz = 0$ $\forall\, \gamma$ piecewise C^1 loop.

Proof $\boxed{i} \Rightarrow \boxed{ii}$ follows by path independence

$\boxed{ii} \Rightarrow \boxed{i}$. Fix $p \in U$. Let

$$F(q) = \int_\gamma f\,dz$$ where γ is a piecewise C^1 path in U joining p to q.

This is well-defined $\Longleftrightarrow \int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$.

$\Longleftrightarrow \int_\eta f\,dz = 0$ where $\eta = \gamma_1 + (-\gamma_2)$.

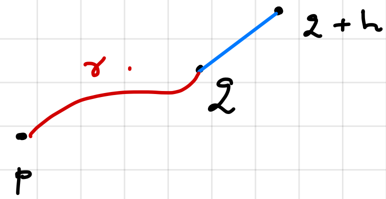
which is true by assumption $\boxed{i}$.

Claim $F' = f$.

Proof Fix $q \in U$. Let $\varepsilon > 0$. Let $\delta > 0$ with

$$(*) \quad |f(z) - f(q)| < \varepsilon \quad \text{if} \quad z \in \Delta(q, \delta).$$

We compute



$$\left|\frac{F(q+h)-F(q)}{h}-f(q)\right| = \left|\frac{1}{h}\int_q^{q+h} f(z)\,dz - f(q)\right|$$

$$= \frac{1}{|h|}\left|\int_q^{q+h}(f(z)-f(q))\,dz\right|$$

$< \varepsilon$ by $(*)$ if $|h|<\delta$.

$$\leq \frac{1}{|h|}\cdot \text{length}([q,q+h])\cdot \varepsilon.$$

$$= \frac{1}{|h|}\cdot |h|\cdot \varepsilon = \varepsilon \implies F'=f$$

Question Why can we always find a piecewise C^1 path?

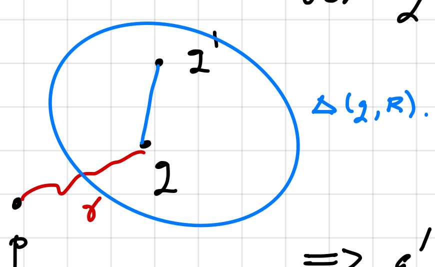
Let

$$\mathcal{X} = \{q\in U : \exists \text{ piecewise } C^1 \text{ path from } p \text{ to } q\}.$$

$\mathcal{X}\neq\emptyset$ since $p\in\mathcal{X}$.

$\mathcal{X}$ open. Let $q\in\mathcal{X} \implies \exists\, R>0$ with $\Delta(q,R)\subseteq U$.

For $q'\in\Delta(q,R)$, join p to q (since $q\in\mathcal{X}$)



join q to q' (via line segment).

$\implies q'\in\mathcal{X} \implies \Delta(q,R)\subseteq\mathcal{X} \implies \mathcal{X}$ open.

$\mathcal{H}$ closed. Let $q \in \partial \mathcal{H}$. We show $q \in \mathcal{H}$.

Let $q' \in \mathcal{H}$, $q' \in \Delta(q, \varepsilon) \subseteq \mathcal{U}$.

Join p to q' by a piecewise C' path. &

q' to q by line segment thus joining p to q

$\Rightarrow q \in \mathcal{H}$.

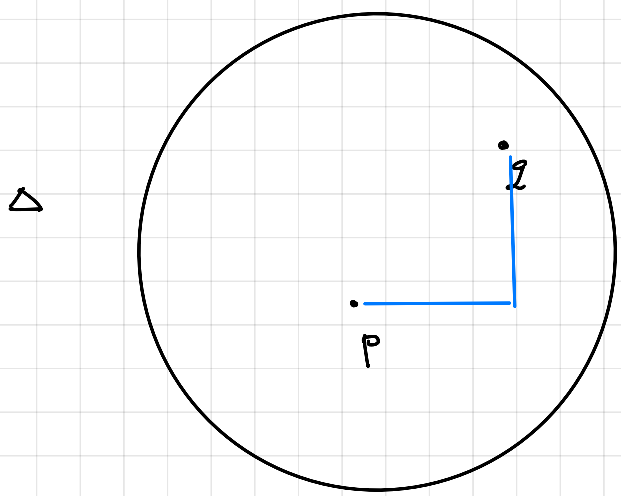
Since $\mathcal{U}$ connected $\Rightarrow \mathcal{H} = \mathcal{U}$.

Proposition B If $U = \Delta$ = disc. TFAE

[i] f admits primitive

[ii] $\int_{\partial R} f\,dz = 0$ for all rectangles $\bar{R} \subseteq U$.

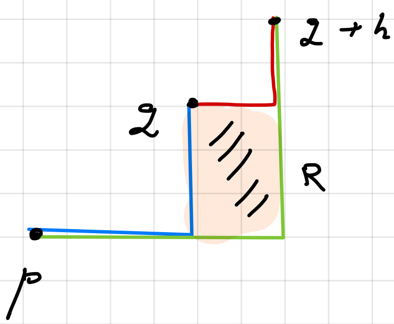
Proof We only need [ii] $\Rightarrow$ [i]. Let $p \in U$.



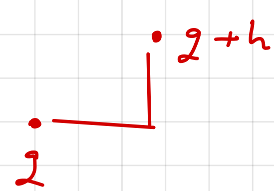
Define $F(q) = \int_\gamma f\,dz$ where γ is a path from p to q, consisting of 2 segments parallel to the axes. Such a path exists since $U = \Delta$ = disc.

The proof $F' = f$ is similar. We have

$$F(q+h) - F(q) = \int_p^{q+h} f\,dz - \int_p^{q} f\,dz = \int_q^{q+h} f\,dz$$

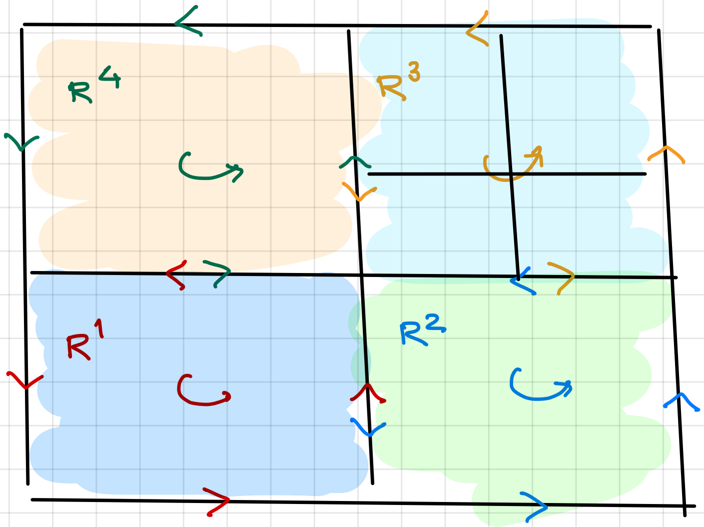


because $\int_{\partial R} f\,dz = 0$.



For the red path from q to $q+h$, the same argument applies, the length of the path $\leq 2|h|$.

Proposition C If $f: U \to \mathbb{C}$ holomorphic $\Rightarrow \int_{\partial R} f\, dz = 0$ for all rectangles $\bar{R} \subseteq U$. (Goursat's lemma).



Proof Let $A = \left| \int_{\partial R} f\, dz \right|$.

Let $\varepsilon > 0$ arbitrary. Wish $A = 0$. We will show

$$A < K\varepsilon \quad \forall \varepsilon > 0.$$

for some $K > 0$.

Subdivide rectangle R into 4 equal rectangles R^1, R^2, R^3, R^4.

$$\Rightarrow A = \left| \int_{\partial R} f\, dz \right| = \left| \sum_{j=1}^{4} \int_{\partial R^j} f\, dz \right| \leq \sum_{j=1}^{4} \left| \int_{\partial R^j} f\, dz \right|.$$

$\Rightarrow \exists$ rectangle (out of R^1, R^2, R^3, R^4), call it $R^{(1)}$, with

$$\frac{A}{4} \leq \left| \int_{\partial R^{(1)}} f\, dz \right|$$

Continue inductively. We obtain a sequence of rectangles

$$R \supseteq R^{(1)} \supseteq R^{(2)} \supseteq \dots \quad , \ \operatorname{diam} R^{(n)} \longrightarrow 0.$$

such that $$\frac{A}{4^n} \le \left| \int_{\partial R^{(n)}} f \, dz \right| .$$

By compactness, $\bigcap_{n=0}^{\infty} R^{(n)} = \{c\}$. Since f is holomorphic

$$\left| \underbrace{\frac{f(z) - f(c)}{z - c} - f'(c)}_{\chi(z)} \right| < \varepsilon \ \text{ if } z \in \Delta(c, \delta), \text{ for some } \delta > 0.$$

$$\Rightarrow \ |\chi(z)| < \varepsilon \quad \& \quad f(z) = f(c) + (z-c) f'(c) + (z-c)\chi(z).$$

$$\Rightarrow \frac{A}{4^n} \le \left| \int_{\partial R^{(n)}} f \, dz \right| = \left| \int_{\partial R^{(n)}} \underbrace{f(c) + (z-c) f'(c)}_{\substack{0 \ \text{admits} \\ \text{primitive}}} + (z-c)\chi(z) \, dz \right|$$

$$= \left| \int_{\partial R^{(n)}} (z-c)\, \chi(z) \, dz \right|$$

$R^{(n)} \subseteq \Delta(c, \delta)$ if $n \gg 0$.

$$\le \operatorname{diam}(R^{(n)}) . \varepsilon \ \operatorname{length}(\partial R^{(n)}).$$

$$= \varepsilon . \frac{\operatorname{diam}(R)}{2^n} . \frac{\operatorname{length}(\partial R)}{2^n} = \frac{\varepsilon}{4^n} K.$$

$$\Rightarrow \ A < K\varepsilon \ \ \forall \varepsilon > 0 \ \Rightarrow \ A = 0.$$