Math 220C - Lecture 8

April 14, 2021

Plan — short discussion of Dirichlet Problem

— begin Chp XI — Jensen's formula

Last time G bounded, $f : \partial G \longrightarrow \mathbb{R}$ continuous

- **Perron family**

$$\mathcal{P}(G, f) = \left\{ \varphi : G \longrightarrow \mathbb{R} \text{ subharmonic}, \ \limsup_{z \to a} \varphi(z) \le f(a) \ \forall \ a \in \partial G. \right\}$$

- **Perron function** $u : G \longrightarrow \mathbb{R}$

$$u(z) = \sup\left\{ \varphi(z), \ \varphi \in \mathcal{P}(G, f) \right\}$$

- **Theorem**

The Perron function u is harmonic

Question Does the Perron function solve Dirichlet Problem?

What is the issue?

We know u is harmonic in G.

We need to show $\lim_{z \to a} u(z) = f(a)$ $\forall\, a \in \partial G$.

Answer (HWK 3, #2) **NO!**

If $G = \Delta(0,1) \setminus \{0\}$, we show that the Dirichlet Problem does not always admit a solution.

Better answer In special cases, it does!

Terminology (differs from Conway $\overline{X}$. 4)

Let G be bounded. Let $a \in \partial G$.

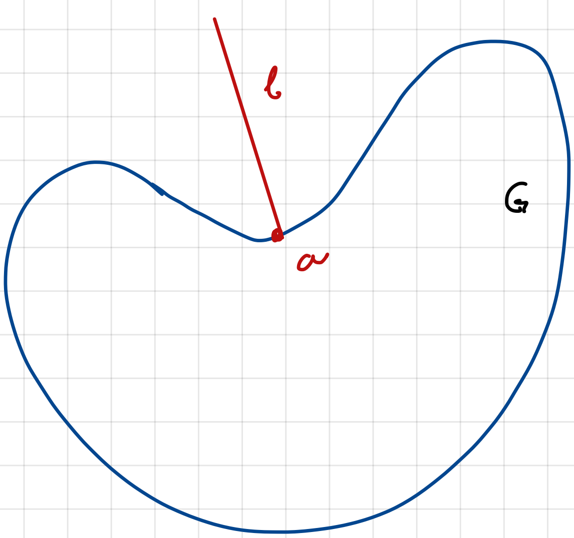
$\omega : \overline{G} \longrightarrow \mathbb{R}$ continuous in $\overline{G}$, harmonic in G,

$\omega(a) = 0$, $\omega > 0$ in $\partial G \setminus \{a\}$

ω is said to be a barrier at a.

The terminology is due to Lebesgue.

Example (HWK 3, # 5) Many reasonable domains satisfy this definition. For instance, if $\exists$ ℓ segment



$\ell \cap \overline{G} = \{a\}$ then there is a barrier at a.

Theorem The Dirichlet Problem can be always be solved in G.

$\iff \forall\, a \in \partial G$, $\exists$ barrier at a.

The Perron function solves the Dirichlet Problem.

Remark „$\Longrightarrow$" HWK 3, #4

„$\Longleftarrow$" A proof is given in the Appendix to the lecture.

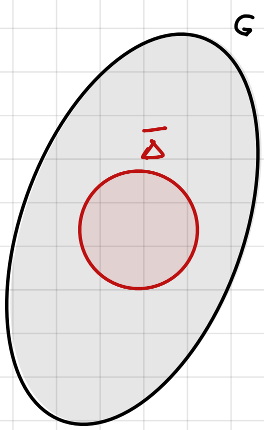
& video on Canvas.

§2. Jensen's Formula

$f : G \longrightarrow \mathbb{C}$ holomorphic, f nowhere zero in G, $\overline{\Delta}(0,r) \subseteq G$.

Recall from HWK 1

> **5.** Let $U \subset \mathbb{C}$ be open connected.
>
> (i) Show that if $h : U \to \mathbb{C}$ is holomorphic and nowhere zero in U, then
> $$u(z) = \log|h(z)|$$
> is harmonic in U.



Mean Value Property for $\log|f|$ gives

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|\,dt.$$

Question What if f has zeros?

The zeros of f will give corrections to the formula.

Theorem $f: G \longrightarrow \mathbb{C}$ holomorphic, $\overline{\Delta}(0,r) \subseteq G$, $f(0) \neq 0$.

Let $a_1, \dots, a_k$ be the zeroes of f in $\Delta(0,r)$. Then

$$\log |f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt.$$

Proof Shrinking G, we may assume $G = \Delta(0,R)$

We may assume $r = 1$. Indeed, otherwise let

$$f^{new}(z) = f(rz) \quad \text{defined in} \quad G^{new} = \Delta\left(0, \frac{R}{r}\right) \supseteq \overline{\Delta}(0,1).$$

When f is holomorphic in $\Delta(0,R) \supseteq \overline{\Delta}(0,1)$, we show

$$\log |f(0)| - \sum_{k=1}^{n} \log |a_k| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, dt. \qquad (*)$$

Proof of (*) Let

- $a_1, \dots, a_k$ be zeroes of f in $\Delta = \Delta(0,1)$
- $b_1, \dots, b_m$ be zeroes of f on $\partial\Delta$.

Recall $\varphi_a : \bar{\Delta} \longrightarrow \bar{\Delta}$, $\partial\Delta \longrightarrow \partial\Delta$, $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$.

Let
$$F(z) = f(z) \Big/ \prod_{j=1}^{k} \varphi_{a_j}(z) \cdot \prod_{j=1}^{m} \frac{b_j}{b_j - z}$$

Note that F has no zeroes in $\bar{\Delta}$ & in fact in a neighborhood of $\bar{\Delta}$. Note

$$F(0) = f(0) \Big/ \prod_{j=1}^{m} (-a_j)$$

By the previous observation applied to F

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{it})| \, dt. \quad (1).$$

By substitution, we find

$$\log |F(0)| = \log |f(0)| - \sum_{j=1}^{k} \log |a_k| \qquad (2)$$

$$\int_0^{2\pi} \log |F(e^{it})| \, dt = \int_0^{2\pi} \log |f(e^{it})| \, dt$$

0 (see below)

$$- \sum_{j=1}^{k} \int_0^{2\pi} \log |\varphi_{a_j}(e^{it})| \, dt$$

0 (Claim)

$$+ \sum_{j=1}^{m} \int_0^{2\pi} \log \left| \frac{b_j}{b_j - e^{it}} \right| dt$$

$$= \int_0^{2\pi} \log |f(e^{it})| \, dt. \qquad (3)$$

Here we used $\varphi_{a_j} : \partial\Delta \longrightarrow \partial\Delta$ so that

$$|\varphi_{a_j}(e^{it})| = 1 \Rightarrow \log |\varphi_{a_j}(e^{it})| = 0.$$

Jensen's formula follows from (1), (2), (3).

Claim

$$\int_0^{2\pi} \log \left| \frac{b}{b - e^{it}} \right| dt = 0 \quad \forall \ |b| = 1.$$

Proof of the claim

Let $b = e^{i\alpha}$. Then

$$\int_0^{2\pi} \log\left|\frac{b}{b-e^{it}}\right| dt = \int_0^{2\pi} \log\left|\frac{e^{i\alpha}}{e^{i\alpha}-e^{it}}\right| dt$$

$$= \int_0^{2\pi} \log\left|\frac{1}{1-e^{i(t-\alpha)}}\right| dt \qquad t \to t+\alpha$$

$$= \int_0^{2\pi} \log \frac{1}{|1-e^{it}|} dt$$

$$= -\int_0^{2\pi} \log|1-e^{it}|\, dt \overset{?}{=} 0.$$

We note that

$$|1-e^{it}|^2 = (1-\cos t)^2 + \sin^2 t = 2 - 2\cos t = 4\sin^2\frac{t}{2}.$$

We need to show

$$\int_0^{2\pi} \log\left|2\sin\frac{t}{2}\right| dt = 0 \quad \overset{t=2u}{\Longleftrightarrow}$$

$$\Longleftrightarrow \int_0^{\pi} \log|2\sin u|\, du = 0$$

$$\Longleftrightarrow \int_0^{\pi} \log 2\, du + \int_0^{\pi} \log\sin u\, du = 0$$

$$\Longleftrightarrow \boxed{\int_0^{\pi} \log\sin u\, du = -\pi\log 2.}$$

Calculation

$$\int_0^{\pi} \log \sin u \, du = -\pi \log 2.$$

Convergence

$$\int_0^{\pi} \log \sin u \, du \le \int_0^{\pi} \log u \, du = u \log u - u \Big|_{u=0}^{u=\pi} < \infty.$$

This uses $\lim_{u \to 0} u \log u = 0$.

Evaluation

$$I = \int_0^{\pi} \log \sin u \, du \overset{u = 2v}{=}$$

$$= 2 \int_0^{\pi/2} \log \sin 2v \, dv = \quad \sin 2v = 2 \sin v \cos v.$$

$$= 2 \int_0^{\pi/2} \log 2 \, dv + 2 \int_0^{\pi/2} \log \sin v \, dv + 2 \int_0^{\pi/2} \log \cos v \, dv$$

$$= \pi \log 2 + 2 \int_0^{\pi/2} \log \sin v \, dv + 2 \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} + v\right) dv$$

$$= \pi \log 2 + 2 \int_0^{\pi} \log \sin v \, dv$$

$$= \pi \log 2 + 2I \implies I = -\pi \log 2.$$

SUR UN NOUVEL ET IMPORTANT THÉORÈME DE LA THÉORIE DES FONCTIONS

PAR

J. L. W. V. JENSEN.

Monsieur le Professeur,

Lors de votre dernier séjour à Copenhague j'ai eu honneur de vous entretenir au sujet d'une intégrale définie appelée, si je ne me trompe, à jouer un rôle dans la théorie des fonctions analytiques. Comme il me parut que cette question vous interéssa vivement, je profiterai de cette occasion — l'envoi des deux petits mémoires[1] destinés à votre Journal — pour vous communiquer le développement détaillé de mon théorème.

Soit $z = re^{\theta i}$ une variable complexe, et α un nombre complexe différent de zéro, on a pour $r < |\alpha|$,

$$l\left(1 - \frac{z}{\alpha}\right) = -\sum_{\nu=1}^{\infty} \frac{1}{\nu}\left(\frac{z}{\alpha}\right)^{\nu}$$

où l désigne la valeur principale du logarithme. En prenant les parties réelles des deux membres et en observant que l'on a $\Re(a) = \frac{1}{2}(a + \dot{a})$,[2] on trouve

$$(1) \qquad l\left|1 - \frac{z}{\alpha}\right| = -\sum_{\nu=1}^{\infty} \frac{r^{\nu}}{2\nu}\left(\frac{e^{\nu\theta i}}{\alpha^{\nu}} + \frac{e^{-\nu\theta i}}{\dot{\alpha}^{\nu}}\right), \qquad r = |z| < |\alpha|.$$

[1] (1) *Sur les fonctions entières.*

(2) *Note sur une condition nécessaire et suffisante pour que tous les zéros d'une fonction entière soient réels.*

[2] Ici et dans la suite je désigne toujours par $\Re(a)$ la partie réelle et par $\dot{a}$ la valeur conjuguée de a.

Acta mathematica. 22. Imprimé le 6 mars 1899.

Johan Jensen (1859-1925) was a Danish mathematician. He worked as a telephone engineer, a job that he took to support himself while he pursued mathematics.

Jensen found the formula while unsuccessfully trying to prove the Riemann hypothesis.

He is also known for Jensen's inequality (about convex functions).