HW3 - SOLUTIONS

Q1. The problem is asking for the Mittag-Leffler solution associated to the poles $z = n$ and principal part $\frac{\sqrt{n}}{z-n}$. We determine the convergence enhancing factors by looking at the Laurent expansion

$$\frac{\sqrt{n}}{z-n} = -\frac{\sqrt{n}}{n} \cdot \frac{1}{1-\frac{z}{n}} = -\frac{1}{\sqrt{n}} \cdot \left(1 + \frac{z}{n} + \frac{z^2}{n^2} + \dots\right).$$

We have

$$\left|\frac{\sqrt{n}}{z-n} + \frac{1}{\sqrt{n}}\right| = |z| \cdot \frac{1}{\sqrt{n}|z-n|}.$$

We let $r_n = n^{\frac{1}{4}}$, and estimate for $|z| < r_n$ that

$$\left|\frac{\sqrt{n}}{z-n} + \frac{1}{\sqrt{n}}\right| = |z| \cdot \frac{1}{\sqrt{n}|z-n|} \le r_n \cdot \frac{1}{\sqrt{n}(n-r_n)} := c_n.$$

Note that

$$\sum_{n=1}^{\infty} c_n < \infty$$

by the limit comparison test. Indeed,

$$\lim_{n\to\infty} \frac{c_n}{n^{\frac{5}{4}}} = 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}} < \infty$. By the proof of Mittag-Leffler, the series

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{\sqrt{n}}{z-n} + \frac{1}{\sqrt{n}}\right)$$

defines a meromorphic function with poles at $z = n$ and residues $\sqrt{n}$.

Q2. Let $\{a_n\}$ be a sequence of distinct complex numbers with $a_n \to \infty$. Let $g(z)$ be a solution to the Weiestrass problem with zeros at a_n of order 1. Let

$$g(z) = B_{n,1}(z-a_n) + B_{n,2}(z-a_n)^2 + \cdots$$

be the Taylor expansion at $z = a_n$ with $B_{n,1} \neq 0$. Let h be a solution to the Mittag-Leffler's problem with poles at a_n with Laurent principal part $\frac{C_{n,1}}{z-a_n}$ where

$$C_{n,1}B_{n,1} = A_n.$$

Define $f(z) = g(z)h(z)$ on $\mathbb{C} - \{a_n\}$. Note that near a_n we have

$$f = \left(B_{n,1}(z-a_n) + B_{n,2}(z-a_n)^2 + \cdots\right)\left(\frac{C_{n,1}}{z-a_n} + \dots\right)$$
$$= B_{n,1}C_{n,1} + \text{ powers of } (z-a_n) = A_n + \text{ powers of } (z-a_n).$$

Therefore a_n is a removable singularity, and we can extend f to an entire function such that $f(a_n) = A_n$.

Q3. The Weiestrass zeta function. Let ω_1, ω_2 be two non-zero complex number with $\omega_1/\omega_2 \notin \mathbb{R}$. Let Λ be the lattice generated by ω_1 and ω_2. Let

$$\sigma(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}\right) = z \prod_{\lambda \in \Lambda \setminus \{0\}} E_2\left(\frac{z}{\lambda}\right)$$

(i) Recall that for infinite products $f = \prod_{i=1}^{\infty} f_i$, the logarithmic derivative f'/f is holomorphic away from the zeros of f and it can be obtained using the formula

$$\frac{f'}{f} = \sum_{i=1}^{\infty} \frac{f_i'}{f_i}.$$

Note that

$$\frac{E_2'(z/\lambda)}{E_2(z/\lambda)} = \frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2},$$

and $(z)'/(z) = 1/z$. Applying the above formula to $\zeta = \sigma'/\sigma$, we obtain the expression

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right).$$

The function ζ is a solution to the Mittag-Lefler problem for Λ with the singular part at $\lambda \in \Lambda$ given by $\frac{1}{z-\lambda}$.

(ii) Pick a sequence of positive integers $\{p_n\}$ satisfying the convergence condition

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \qquad \forall r > 0.$$

Corresponding to such a choice of p_n, we have the solution to Weiestrass Factorization problem given by

$$f(z) = \prod_n E_{p_n}\left(\frac{z}{a_n}\right),$$

which has zeros, $\{a_n\}$, of order 1. Recall that $E_p(z) = (1-z)\exp(z + z^2/2 + \cdots z^p/p)$ and

$$\frac{E_p'(z/a)}{E_p(z/a)} = \frac{1}{z-a} + \frac{1}{a} + \frac{z}{a^2} + \cdots + \frac{z^{p-1}}{a^p}.$$

Thus the logarithmic derivative of f can be written as

$$\frac{f'(z)}{f(z)} = \sum_n \left(\frac{1}{z-a_n} + \frac{1}{a_n} + \frac{z}{a_n^2} + \cdots + \frac{z^{p-1}}{a_n^p}\right).$$

Looking at the singular part of the above summand, we see that the logarithmic derivative f'/f satisfies Mittag-Leffler problem for the simple poles at $\{a_n\}$ with residue 1.

Q4.

(i) Note that local uniform and absolute convergence of the series follows from these properties proven for the infinite product σ in the previous problem set, and the fact that these properties are preserved by taking derivatives and logarithmic derivatives (away from the poles). Thus, by taking the derivative of each term of the series defining ζ, for $z \notin \Lambda$, we get the required expression

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

(ii) Recall that Λ is the lattice generated by ω_1 and ω_2. For any fixed $\mu = a\omega_1 + b\omega_2$, where a and b are integers, the series

$$\sum_{\lambda \in \Lambda \setminus \{0,\mu\}} \left(\frac{1}{(\lambda-\mu)^2} - \frac{1}{\lambda^2} \right)$$

is well defined and absolutely convergent. This uses the fact that each term in the series behaves as λ^{-3} for $\lambda >> 0$, as shown in the discussion in Lecture 10, Example iv or via the direct calculation

$$\frac{1}{(\lambda-\mu)^2} - \frac{1}{\lambda^2} = \frac{2\lambda\mu - \mu^2}{\lambda^2(\lambda-\mu)^2} \approx \frac{1}{\lambda^3}.$$

Moreover, the above series equals zero since

$$\left(\frac{1}{(\lambda-\mu)^2} - \frac{1}{\lambda^2} \right) + \left(\frac{1}{((-\lambda+\mu)-\mu)^2} - \frac{1}{(-\lambda+\mu)^2} \right) = 0.$$

Note that we have a series expansion for $\wp$ which absolutely convergent. We will subtract the above absolutely convergent series to the series expansion of $\wp(z+\mu)$ to obtain the required answer :

$$\begin{aligned}
\wp(z+\mu) &= \frac{1}{(z+\mu)^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda+\mu)^2} - \frac{1}{\lambda^2} \right) \\
&= \frac{1}{(z+\mu)^2} + \left(\frac{1}{z^2} - \frac{1}{\mu^2} \right) + \sum_{\lambda \in \Lambda \setminus \{0,\mu\}} \left(\frac{1}{(z-\lambda+\mu)^2} - \frac{1}{(\lambda-\mu)^2} \right) \\
&= \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{\mu\}} \left(\frac{1}{(z-\lambda+\mu)^2} - \frac{1}{(\lambda-\mu)^2} \right) \\
&= \wp(z).
\end{aligned}$$

The last equality follows by defining new indexing variable $\lambda' = \lambda - \mu$.

(iii) Since $\wp$ is periodic meromorphic function on $\mathbb{C} \setminus \Lambda$, then so is $\wp'$. Fix any $\mu \in \Lambda$, and apply chain rule to obtain

$$\wp'(z) = \frac{d}{dz}\wp(z) = \frac{d}{dz}\wp(z+\mu) = \wp'(z+\mu)\frac{d}{dz}(z+\mu) = \wp'(z+\mu).$$

(iv) The function $\wp$ is an even function since

$$\begin{aligned}\wp(-z) &= \frac{1}{(-z)^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(-z-\lambda)^2} - \frac{1}{\lambda^2}\right)\\ &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right)\\ &= \frac{1}{z^2} + \sum_{\lambda'\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda')^2} - \frac{1}{\lambda'^2}\right) = \wp(z).\end{aligned}$$

The last equality is obtained by noting that the above series is absolutely convergent and by reindexing $\lambda' = -\lambda$.

(v) Note that the function $\wp(z) - \frac{1}{z^2}$ has a removable singularity at $z = 0$ since the series

$$h(z) = \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

is absolutely and unformly convergent in a small compact ball containing 0. This follows by noting that the terms in the above series grows as λ^{-3} as $\lambda \to \infty$, as shown in Lecture 10, Example iv. Thus, we may substitute $z = 0$ to obtain $h(0) = 0$. Therefore, using that the fact the $\wp$ (hence h) is even, we get

$$\wp(z) = \frac{1}{z^2} + az^2 + bz^4 + \cdots$$

and $\wp'(z) = \frac{-2}{z^3} + 2az + 4bz^3 + \cdots$ for some constants a and b.

(vi) The Laurent principal tail at $z = 0$ for the functions $\wp^3$ and $\wp'^2$ are

$$\begin{aligned}\wp^3 &= \frac{1}{z^6} + \frac{3a}{z^2} + 3b + O(z^2)\\ \wp'^2 &= \frac{4}{z^6} - \frac{8a}{z^2} - 16b + O(z^2).\end{aligned}$$

Comparing the coefficients carefully, we get

$$\wp'^2 = 4\wp^3 - 20a\wp - 28b + O(z^2).$$

Let $A = -20a$, $B = -28b$, and

$$f(z) := \wp'^2 - (4\wp^3 + A\wp + B).$$

Thus f clearly has no Laurent principal tail and $f(0) = 0$.

(vii) Since $\wp$ and $\wp'$ are double periodic, then so is f. Moreover since f vanishes at $z = 0$, f has removable singularities at all the lattice points $\lambda \in \Lambda$ (and $f(\lambda) = 0$). But this implies that the function f is entire and bounded (because it is double periodic and the values), hence $f \equiv 0$ by Liouville's theorem. This argument was also Remark on page 5, Lecture 22 in Math 220A.