Math 220, Practice problems for the midterm.

Please review the homework questions in addition to these practice problems.

1. Let $f : U \to \mathbb{C}$ be a holomorphic function defined in a connected open set U. Assume that for each $z \in U$, there are positive integers m and n (that may depend on z) such that

$$f(z)^m = \overline{f(z)}^n.$$

Show that f is a constant.

2. (*You could solve this problem on Monday.*) Find the Laurent expansions around 0 for the function

$$f(z) = \frac{1}{z^2 + 5z + 4}$$

valid in three different regions of the complex plane.

3. Using Cauchy's integral formula, calculate the following integrals:

(i)

$$\int_{|z-1|=1} \frac{\sin(\pi z)}{(z^2-1)^2}\, dz$$

(ii)

$$\int_{|z-1|=a} \frac{e^z}{z^2 - 2z}\, dz.$$

5. Let Δ be the open unit disc. Let $f : \overline{\Delta} \to \mathbb{C}$ be a nonconstant continuous function on the closed unit disc, holomorphic on the open disc Δ. Assume that $f(\partial\Delta) \subset \partial\Delta$.

(i) Show that $f(\Delta) \subset \Delta$.
(ii) Show that f must have a zero inside Δ.

6. (*We should cover the material for this on Wednesday.*)

(i) Find the residue at $z = -1 - i$ for the function

$$f(z) = \frac{z\,\mathrm{Log}(z)}{(z+1+i)^2}.$$

Here, the principal branch of the logarithm is used.

(ii) For what value of a, does the function

$$\frac{1}{e^z - 1} + \frac{a}{\sin z}$$

has a removable singularity at the origin?

7. Assume that $f : \overline{\Delta} \to \mathbb{C}$ is a continuous function defined on a closed disc $|z| \leq r$ and holomorphic inside the disc $|z| < r$. Prove that Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for $|z| < r$.

Hint: This requires an argument; the usual Cauchy integral formula stated in class does not apply directly. Instead use circles of radii $r - \frac{1}{n}$ *and let* $n \to \infty$.

8. (*We should cover the material for this on Wednesday. Not required for the midterm.)* Prove the Casorati-Weierstrass theorem: if $f : \Delta \setminus \{0\} \to \mathbb{C}$ is a holomorphic function on the punctured unit disc with an essential singularity at the origin, then $f(\Delta \setminus \{0\})$ is dense in $\mathbb{C}$.

9. Let U be open and connected, and let f, g be holomorphic functions such that $f(z)g(z) = 0$. Show that either f or g is identically zero on U.

10. (*We should cover the material for this on Wednesday. Not required for the midterm.*) Show that there is no meromorphic function f on the unit disc $\Delta(0, 1)$ such that f' has a pole of order exactly one at $z = 0$.

11. Consider the holomorphic function

$$f(z) = e^z + ie^{-z}$$

over the closed rectangle R with corners

$$\pm 1 \pm i\frac{\pi}{2}.$$

Find the maximum of f and confirm that it lies over the boundary of R. Where does the minimum occur?

12. Show that a function $f : \mathbb{C} \to \mathbb{C}$ which is entire and doubly periodic must be constant. A function f is doubly periodic provided

$$f(z) = f(z + \omega_1) = f(z + \omega_2)$$

for complex numbers ω_1, ω_2 such that $\omega_1/\omega_2 \notin \mathbb{R}$.

13. Let f be an entire function such that $|f(z)| \leq e^{\mathrm{Re} z}$. Show that either $f = 0$ or else f has no zeros in $\mathbb{C}$.

14. Suppose that f is entire and $\frac{f(z)}{1+|z|^{1/2}}$ is bounded. Prove that f is constant.