Solutions: Homework 1

Problem 1. *If U, V are open sets, $g : V \to U$ is holomorphic, $u : U \to \mathbb{R}$ harmonic, then $u \circ g : V \to \mathbb{R}$ is harmonic in V.*

Proof. The statement is local. Let $a \in V$ and let Δ be an open disc in U around $g(a)$. Let

$$W = g^{-1}(\Delta) \subset V$$

which is an open neighborbood of a. We show that $u \circ g$ is harmonic in W. Indeed, since Δ is simply connected, we can write

$$u|_{\Delta} = \text{Re } f$$

for a holomorphic function f in Δ. Then

$$u \circ g = \text{Re}(f \circ g)$$

in $W = g^{-1}(\Delta)$, and $f \circ g$ is holomorphic in W. Thus $u \circ g$ is harmonic in W being the real part of a holomorphic function. □

Problem 2. *Let $u : G \to \mathbb{R}$ be a nonconstant harmonic function in a region $G \subset \mathbb{C}$. Show that u is an open map.*

Proof. Since open sets are unions of open balls, it suffices to check that images of open balls are open. Let $\Delta(a, r) \subset G$. Since $\Delta(a, r)$ is simply connected, there exists $v : \Delta(a; r) \to \mathbb{R}$ such that $f = u + iv$ defined on $\Delta(a, r)$ is holomorphic. Since f is holomorphic, $f(\Delta(a, r))$ is open in $\mathbb{C}$. Note that Re: $\mathbb{C} \to \mathbb{R}$ given by

$$\text{Re } (x + iy) = x$$

is an open map since the images of balls $\Delta(\alpha, r)$ are open intervals $(\text{Re } \alpha - r, \text{Re } \alpha + r)$. Note that

$$u(\Delta(a, r)) = \text{Re } (f(\Delta(a, r))),$$

and hence is open. So u is an open map. □

Problem 3. *Let $u : G \to \mathbb{R}$ be harmonic, and $\overline{\Delta}(a, r) \subset G$. Let*

$$M = \sup_{|z-a|=r} |u(z)|.$$

(i) Show that

$$u(a) = \frac{1}{\pi r^2} \int\int_{\Delta(a,r)} u(x, y) dx dy.$$

(ii) Show that the derivatives u_x and u_y are also harmonic. Therefore

$$u_x(a) = \frac{1}{\pi r^2} \int\int_{\Delta(a,r)} u_x(x, y) dx dy.$$

(iii) Use Green's theorem in part (ii) and deduce that

$$|u_x(a)| \leq \frac{2}{r} M.$$

Derive the similar statement for u_y.
(iv) Using induction, show that for $i + j = n$ then the higher derivatives satisfy the estimates

$$|\partial_x^i \partial_y^j u(a)| \leq C_n r^{-n} M$$

for some constant C_n.
(v) Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and $|u(z)| \leq A(1 + |z|^m)$ then u is a polynomial.
(vi) Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and bounded in absolute value then u is constant.

Proof. (i) Changing into polar coordinates, we have

$$\frac{1}{\pi r^2} \int\int_{\Delta(a,r)} u(x + iy) dx dy = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u(a + \rho e^{i\theta}) \rho d\theta d\rho$$

Since u is harmonic, for any $0 < \rho < r$, we have

$$\int_0^{2\pi} u(a + \rho e^{i\theta}) d\theta = 2\pi u(a)$$

Hence we have

$$\frac{1}{\pi r^2} \int\int_{\Delta(a,r)} u(x + iy) dx dy = \frac{2u(a)}{r^2} \int_0^r \rho d\rho = u(a)$$

(ii) Since u is harmonic, u is infinitely differentiable. So,

$$(u_x)_{yy} = (u_{yy})_x = (-u_{xx})_x = -(u_x)_{xx}.$$

Hence

$$(u_x)_{xx} + (u_x)_{yy} = 0$$

and u_x is therefore harmonic. Similarly, u_y is also harmonic. So, by part (i)

$$u_x(a) = \frac{1}{\pi r^2} \int\int_{\Delta(a,r)} u_x(x, y) dx dy$$

(iii) By Green's theorem, we have

$$u_x(a) = \frac{1}{\pi r^2} \int\int_{\Delta(a,r)} u_x(x, y) dx dy = \frac{1}{\pi r^2} \int_{\partial\Delta(a,r)} u(x, y) dy$$

Then, changing x to $\mathrm{Re}(a) + r\cos\theta$ and y to $\mathrm{Im}(a) + r\sin\theta$, we have

$$u_x(a) = \frac{1}{\pi r^2} \int_0^{2\pi} u(a + re^{i\theta}) r\cos\theta d\theta = \frac{1}{\pi r} \int_0^{2\pi} u(a + re^{i\theta}) \cos\theta d\theta$$

Hence, we have

$$|u_x(a)| \leq \frac{M}{\pi r} \int_0^{2\pi} |\cos\theta| d\theta = \frac{4M}{\pi r} \leq \frac{2}{r} M.$$

Similarly,

$$|u_y(a)| \leq \frac{2}{r} M.$$

(iv) We proceed by induction on n. It is true for $n = 1$ with $C_1 = 2$. Assume that the inequalities are true for $n - 1$. Since $u_{xx} = -u_{yy}$ on G, if j is even, we have

$$|\partial_x^i \partial_y^j u(a)| = |\partial_x^n u(a)|$$

and if j is odd, we have

$$|\partial_x^i \partial_y^j u(a)| = |\partial_x^{n-1} u_y(a)|.$$

In either case, we have, by our induction hypothesis applied to $r/2$ instead of r :

$$|\partial_x^i \partial_y^j u(a)| = |\partial_x^{n-1} v(a)| \leq \frac{C_{n-1}}{(r/2)^{n-1}} \sup_{|z-a|=r/2} |v(z)|$$

where $v = u_x$ if j is even and $= u_y$ if j is odd. In either case, by (iii) above, we have for any $z_0 \in \partial\Delta(a, r/2)$,

$$|v(z_0)| \leq \frac{2}{r/2} \sup_{|z-z_0|=r/2} |u(z)|$$

Since for any $z_0 \in \partial\Delta(a, r/2)$, we have $\partial\Delta(z_0, r/2) \subset \overline{\Delta}(a, r)$, by the Maximum principle, we obtain that

$$|v(z_0)| \leq \frac{4M}{r}$$

and hence

$$\sup_{|z-a|=r/2} |v(z)| \leq \frac{4M}{r}.$$

Putting this back into the first inequality above, we get

$$|\partial_x^i \partial_y^j u(a)| \leq \frac{2^{n+1} C_{n-1}}{r^n} M$$

Putting $C_n = 2^{n+1} C_{n-1}$, we are done.
(v) Let $a \in \mathbb{R}^2$. For any $r > 0$, we have

$$\sup_{|z-a|=r} |u(z)| \leq A(1 + (|a| + r)^m)$$

So, by part (iv) above, we have for $i + j > m$,

$$|\partial_x^i \partial_y^j u(0)| \leq \frac{AC_n}{r^{i+j}} (1 + (|a| + r)^m)$$

Letting $r \to \infty$, we have

$$\partial_x^i \partial_y^j u(a) = 0$$

for all $a \in \mathbb{R}^2$, whenever $i + j > m$. In particular,

$$\partial_x^{m+1} u \equiv 0.$$

This implies that

$$u = x^m f_m(y) + x^{m-1} f_{m-1}(y) + ... + f_0(y)$$

for some f_i's from $\mathbb{R} \to \mathbb{R}$. Similarly, $\partial_y^{m+1} u \equiv 0$ and hence the f_i's are polynomials in y of degree $\leq m$. This shows that u is a polynomial.
(vi) This is clear from part (v), with $m = 0$.

$\square$

Problem 4. *Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and bounded from below or from above then u is constant.*

Proof. Assume that u is bounded from below. If u is bounded from above, we work instead with $-u$. Thus $u \geq m$ for some m. Replacing u by $u - m$ we may assume that $u(z) \geq 0$ for all $z \in \mathbb{R}^2$. Since $\mathbb{C}$ is simply connected, there exists f entire such that Re $f = u$. Let $g = e^{-f}$. Then g is also entire, and for all $z \in \mathbb{C}, |g(z)| = e^{-u(z)} \leq 1$. By Liouville's theorem, g is constant, and hence so is f. This implies that u is constant. □

Problem 5. *Let $U \subset \mathbb{C}$ be open connected.*

(i) Show that if $h : U \to \mathbb{C}$ is holomorphic and nowhere zero in U, then

$$u(z) = \log |h(z)|$$

is harmonic in U.

(ii) Assume that every harmonic function in U admits a harmonic conjugate. Show that U is simply connected.

Proof. (i) Let $V = \mathbb{C} \setminus \{0\}$. The function $h : U \to V$ is holomorphic and

$$v : V \to \mathbb{R}, \quad v(z) = \log |z|$$

is harmonic so by Problem 1, $u = v \circ h$ is harmonic in U.

The statement that v is harmonic was proven in Math 220A, Homework 2, Problem 3. A direct proof is as follows. For $(x, y) \neq (0, 0)$, we have $v(x + iy) = \frac{1}{2} \log(x^2 + y^2)$. Then

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Then

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

and hence v is harmonic.

(ii) Let h be an arbitrary nowhere zero holomorphic function in U. We show that $h = e^g$ for some holomorphic function g in U. Thus U is simply connected, by Math 220B, Lecture 26.

Let $u(z) = \log|h(z)|$. By (i), u is harmonic in U. By hypothesis u admits a harmonic conjugate w. Thus $g = u + iw$ is holomorphic in U. Note that

$$|e^g| = e^{\mathrm{Re}\ g} = e^u = |h(z)|$$

so

$$|he^{-g}| = 1.$$

The image of the holomorphic function he^{-g} is a subset of the unit circle. This contradicts the open mapping theorem, unless he^{-g} is a constant c with $|c| = 1$. Letting $c = e^{i\alpha}$, we find

$$h = ce^g = e^{g+i\alpha},$$

as claimed above.

$\square$