Solutions: Homework 2

Problem 0. *Prove the following formula for the Laplacian in polar coordinates*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Proof. Using $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Note that we have

$$r\frac{\partial}{\partial r} = r\cos\theta\frac{\partial}{\partial x} + r\sin\theta\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Differentiating the first equation above with respect to r, we get

$$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)$$

$$= \cos\theta\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} + \sin\theta\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}$$

$$= \cos^2\theta\frac{\partial^2}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y} + \sin^2\theta\frac{\partial^2}{\partial y^2}$$

So we have

$$r^2\frac{\partial^2}{\partial r^2} = r^2\cos^2\theta\frac{\partial^2}{\partial x^2} + 2r^2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y} + r^2\sin^2\theta\frac{\partial^2}{\partial y^2} = x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2}$$

Differentiating $\partial/\partial\theta$ with respect to θ, we obtain

$$\frac{\partial^2}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$$

$$= -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - y\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} + x\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}$$

$$= -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial y \partial x} + x^2\frac{\partial^2}{\partial y^2}$$

Combining all of the above, we get

$$r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} = (x^2+y^2)\Delta = r^2\Delta$$

This completes the proof. $\square$

Problem 1. *Assume that $\phi : G \to \mathbb{R}$ is a function of class $\mathcal{C}^2$ such that*

$$\Delta\phi \geq 0.$$

Let $a \in G$ and $\overline{\Delta}(a, R) \subset G$. Define for $0 \leq r < R$ the function

$$h(r) = \frac{1}{2\pi}\int_0^{2\pi} \phi(a + re^{it})dt$$

(i) Show that h is non-decreasing.
(ii) Using (i), show that ϕ is subharmonic

$$\phi(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(a + re^{it})dt.$$

(iii) If ϕ is harmonic in $G \setminus \{a\}$, show that

$$\left(r\frac{\partial}{\partial r}\right)^2 h = 0$$

and conclude that

$$h(r) = \alpha \log r + \beta.$$

(iv) Which of the following functions are subharmonic? superharmonic? harmonic? neither?
(a) $f(x, y) = x^2 + y^2$
(b) $f(x, y) = x^2 - y^2$
(a) $f(x, y) = x^2 + y$.

Proof. (i) Let

$$u : [0, R] \times [0, 2\pi] \to \mathbb{R}, \quad u(r, t) = \phi(a + re^{it}).$$

The function u is $\mathcal{C}^2$. Hence $\frac{\partial u}{\partial r}$ exists and is a continuous function on $[0, R] \times [0, 2\pi]$, and by the Leibniz's rule (differentiation under the integral sign), we have

$$r\frac{\partial h}{\partial r} = \int_0^{2\pi} r\frac{\partial}{\partial r}u(r,t)dt.$$

Applying Leibniz's rule again above, which can be done since u is $\mathcal{C}^2$, we have

$$\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = \int_0^{2\pi} \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)u(r,t)dt$$

So we have

$$(rh'(r))' = \int_0^{2\pi} \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}dt = \int_0^{2\pi} r\Delta u - \frac{1}{r}\frac{\partial^2 u}{\partial t^2}dt$$

Since $\Delta u \geq 0$, we have

$$(rh'(r))' \geq -\frac{1}{r}\int_0^{2\pi} \frac{\partial^2 u}{\partial t^2}dt = -\frac{1}{r}\left(\left(\frac{\partial u}{\partial t}\right)(a+re^{2\pi i}) - \left(\frac{\partial u}{\partial t}\right)(a+re^{0})\right) = 0.$$

This implies that $rh'(r)$ is nondecreasing, and hence $rh'(r) \geq 0h'(0) = 0$. So, $h'(r) \geq 0$ for all $0 < r < R$. This implies that h is non-decreasing.

(ii) Now, since h is non-decreasing we have

$$h(0) = \phi(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(a + re^{it})dt = h(r)$$

Thus ϕ is subharmonic.

(iii) From part (i), if ϕ is harmonic, we have

$$(rh'(r))' = \int_0^{2\pi} r\Delta u - \frac{1}{r}\frac{\partial^2 u}{\partial t^2}dt = -\int_0^{2\pi} \frac{1}{r}\frac{\partial^2 u}{\partial t^2}dt = 0$$

Thus, we have $rh'(r) = \alpha$ for some constant α. We obtain

$$h'(r) = \frac{\alpha}{r}$$

and hence $h(r) = \alpha \log r + \beta$ for some constant β.

(iv) By direct computation, we have

(a) $\Delta f = f_{xx} + f_{yy} = 2 + 2 = 4 > 0$, and so f is subharmonic by part (ii) above.
(b) $\Delta f = f_{xx} + f_{yy} = 2 - 2 = 0$, and so f is harmonic.
(c) $\Delta f = f_{xx} + f_{yy} = 2 + 0 = 2 > 0$, and so f is subharmonic by part (ii) above. $\square$

Problem 3. *Show that if $u : \Delta(0,1)\backslash\{0\} \to \mathbb{R}$ is harmonic and $\lim_{z\to 0} u(z)$ exists, then u can be extended to a harmonic function on Δ.*

Proof. Suppose $u : \Delta(0,1) \setminus \{0\} \to \mathbb{R}$ is harmonic. Then

$$\int_0^{2\pi} u(re^{it})dt = a \log r + b$$

for some constants a, b and for $0 < r < 1$. Extend u to $\Delta(0,1)$ by defining

$$u(0) = \lim_{z\to 0} u(z).$$

Then u is continuous on $\Delta(0,1)$. For $\epsilon > 0$ arbitrary, there exists $r_\epsilon > 0$ such that $|u(z) - u(0)| \le \epsilon$ for all $|z| < r_\epsilon$. Then,

$$|a \log r + b - 2\pi u(0)| = \left| \int_0^{2\pi} (u(re^{it}) - u(0))dt \right| \le 2\pi\epsilon$$

for all $0 < r < r_\epsilon$. Letting $r \to 0$, we obtain that this can happen only if $a = 0$ and $b = 2\pi u(0)$. Substituting, we obtain

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})dt$$

for all $r > 0$. Hence, u is harmonic on Δ. $\square$

Problem 3. *Let Δ denote the open unit disc, and let $\Delta' = \left\{z \in \mathbb{C} : \left|z + \frac{2}{5}\right| < \frac{2}{5}\right\}$ denote the open disc of center $-\frac{2}{5}$ and radius $\frac{2}{5}$. Let $\Omega = \Delta \setminus \overline{\Delta'}$.*

Find, with justification, an explicit continuous functions $h : \overline{\Omega} \to \mathbb{R}$, harmonic in Ω, and with boundary values $h = 0$ on $\partial\Delta$ and $h = 1$ on $\partial\Delta'$.

Proof. Consider the automorphism $\phi : \Delta \to \Delta$ given by

$$\phi(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z}.$$

We claim

(i) ϕ sends $\partial\Delta'$ to $\partial\Delta''$.

(ii) ϕ sends the disc Δ' into the disc $\Delta'' = \Delta\left(0, \frac{1}{2}\right)$ centered at 0.

(iii) $\phi : \Omega \to \Omega'$ is a biholomorphism, where

$$\Omega' = \Delta \setminus \overline{\Delta}''.$$

To see (i), pick three points on $\partial\Delta'$ e.g. $0, -\frac{4}{5}, -\frac{2}{5} + \frac{2}{5}i$ and note that their images lie in $\partial\Delta''$:

$$\phi(0) = \frac{1}{2}, \quad \phi\left(-\frac{4}{5}\right) = -\frac{1}{2}, \quad \phi\left(-\frac{2}{5} + \frac{2}{5}i\right) = \frac{1}{2}\frac{1+4i}{i+4}.$$

Since Möbius transforms send generalized circles to generalized circles, it follows that ϕ sends $\partial\Delta'$ to $\partial\Delta''$.

To see (ii), we use the maximum modulus principle. Note that $|\phi(z)| = \frac{1}{2}$ for $z \in \partial\Delta'$ by (i). Thus by maximum modulus, it follows $|\phi(z)| < \frac{1}{2}$ for $z \in \Delta'$ giving (ii).

For (iii), note that Ω is connected, and its image $\phi(\Omega)$ is connected and contained in $\Delta \setminus \partial\Delta''$. The latter set has two connected components Ω'' and Δ''. Thus the biholomorphism ϕ maps Δ' and Ω to either Δ'' or Ω'. Picking points in each of the two domains, we see that

$$\phi : \Omega \to \Omega', \quad \phi : \Delta' \to \Delta''.$$

Thus ϕ establishes a biholomorphism between Ω and Ω'.

Let $g = h \circ \phi^{-1} : \Omega' \to \mathbb{R}$ which is continuous over $\overline{\Omega}'$, harmonic in Ω', and by construction

$$g = 0 \text{ on } \partial\Delta, \quad g = 1 \text{ on } \partial\Delta''.$$

Such a harmonic function is unique, as shown in class. By inspection, we see that

$$g(z) = \frac{1}{\log 1/2} \log|z|$$

satisfies the boundary requirements, and we have seen (for instance in the previous homework) that g is harmonic. Thus

$$h = g \circ \phi \implies h(z) = \frac{1}{\log\frac{1}{2}} \log\left|\frac{z + \frac{1}{2}}{1 + \frac{1}{2}z}\right| = -\frac{1}{\log 2} \log\left|\frac{2z+1}{z+2}\right|.$$

$\square$

Problem 4. *(i) Give an example of a harmonic function in the half plane $u : \{z : Re\ z > 0\} \to \mathbb{R}$ such that*

$$\lim_{z \to iy} u(z) = 1 \textit{ for } y > 0, \quad \lim_{z \to iy} u(z) = -1 \textit{ for } y < 0.$$

(ii) Using part (i) and a Cayley transform, give an example of a harmonic function on the unit disc $u : \Delta \to \mathbb{R}$ such that

$$\lim_{r \to 1} u(re^{it}) = 1 \textit{ if } 0 < t < \pi, \ \lim_{r \to 1} u(re^{it}) = -1 \textit{ if } \pi < t < 2\pi.$$

Proof. (i) Let $G = \{z : \text{Re } z > 0\}$. Let

$$\partial G^+ = \{z : \text{Re } z = 0, \text{Im } z > 0\}, \quad \partial G^- = \{z : \text{Re } z = 0, \text{Im } z < 0\}.$$

We first note that the function $h : G \to \mathbb{R}$ given by

$$h(z) = \frac{2}{\pi} \text{Im}(\text{Log } z)$$

where Log denotes the principal branch of the logarithm, is obviously harmonic. Using

$$\text{Log } z = \log r + i\theta$$

for $\theta \in (-\pi, \pi)$, we see that we also have

$$h(z) \to 1 \text{ as } z \to z_0 \text{ for } z_0 \in \partial G^+$$

and

$$h(z) \to -1 \text{ as } z \to z_0 \text{ for } z_0 \in \partial G^-.$$

(ii) The Möbius transformation ϕ given by

$$\phi(z) = \frac{1+z}{1-z}$$

is holomorphic and sends $\Delta \to G$, as we have seen before. Furthermore,

$$\partial\Delta \cap \{z : \text{Im } z > 0\} \to \partial G^+, \quad \partial\Delta \cap \{z : \text{Im } z < 0\} \to \partial G^-.$$

This was verified in Lecture 20, Math 220B for the usual Cayley transform from Δ to the upper half plane, but in our case, ϕ is just a rotation of the usual Cayley transform by $\frac{\pi}{2}$..

In particular, the function $u = h \circ \phi$ is harmonic on Δ. We have

$$u(z) = \frac{2}{\pi}\text{Im Log}\frac{1+z}{1-z}.$$

Using part (i) and the observation about the boundary behavior of ϕ, it follows that

$$\lim_{r\to 1} u(re^{it}) = 1 \quad \text{if} \quad 0 < t < \pi, \qquad \lim_{r\to 1} u(re^{it}) = -1 \quad \text{if} \quad \pi < t < 2\pi.$$

□

Problem 5. *Let $G \subset \mathbb{C}$ be a symmetric region with respect to the real axis, and let*

$$G^+ = G \cap \{Im\ z > 0\}$$

be the part in the upper half plane. Moreover, assume that u is harmonic on G^+ and that

$$\lim_{z\to z_0} u(z) = 0$$

for any point $z_0 \in G \cap \mathbb{R}$. Show that u extends to a harmonic function on G, and the extension satisfies

$$u(\overline{z}) = -u(z).$$

Proof. Extend u to G by defining $u(z) = 0$ for $z \in G \cap \mathbb{R}$ and $u(z) = -u(\overline{z})$ for $z \in G^-$. Clearly u is continuous on G.

We claim u is harmonic. Clearly u is harmonic in G^+. Furthermore, u is harmonic on G^- because for any $(x_0, y_0) \in G^-$ we have

$$u_x(x_0, y_0) = -u_x(x_0, -y_0), \quad u_y(x_0, -y_0) = u_y(x_0, y_0),$$

$$u_{xx}(x_0, y_0) = -u_{xx}(x_0, -y_0), \quad u_{yy}(x_0, y_0) = -u_{yy}(x_0, -y_0)$$

and thus

$$u_{xx} + u_{yy} = 0$$

on G^- from the Laplace equation in G^+. Finally, let $a \in G \cap \mathbb{R}$. Let $R > 0$ be such that $\overline{\Delta}(a; R) \subset G$. Let $0 < r < R$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{\pi} u(a + re^{i\theta})d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} u(a + re^{i\theta})d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi} u(a + re^{i\theta})d\theta - \frac{1}{2\pi}\int_{\pi}^{2\pi} u(a + re^{-i\theta})d\theta = 0 = u(a).$$

By the Mean Value Property, u is harmonic around any point on $G \cap \mathbb{R}$, hence harmonic in G.

$\square$