Math 220C - Lecture 18

May 7, 2021

Announcements

(1) Review – Friday, May 14

(2) Office Hours:

Wed, May 12, 4 – 5

Fri, May 14, 1 – 2

Mon, May 17, 3 – 4

(3) Qualifying Exam – Tuesday, May 18, 5 – 8

(4) No lecture/office hours on Wednesday, May 19

(5) Plenty of Practice Exams linked on website

(6) Gradescope

10 lectures left – Minicourse on Riemann Surfaces

First Goal – Introduction & basic properties

– So far, we have done complex analysis for domains $G \subseteq \mathbb{C}$ & studied holomorphic functions

– Many results carry over if we replace $G \subseteq \mathbb{C}$ by Riemann surfaces.

– The subject merges ideas from Complex Analysis with Geometry & Topology

– Connections w/ many fields

topology
differential geometry
algebraic geometry
arithmetic geometry
number theory
dynamics
....

Historically, Riemann Surfaces arose from attempts to understand analytic continuation of multi-valued functions

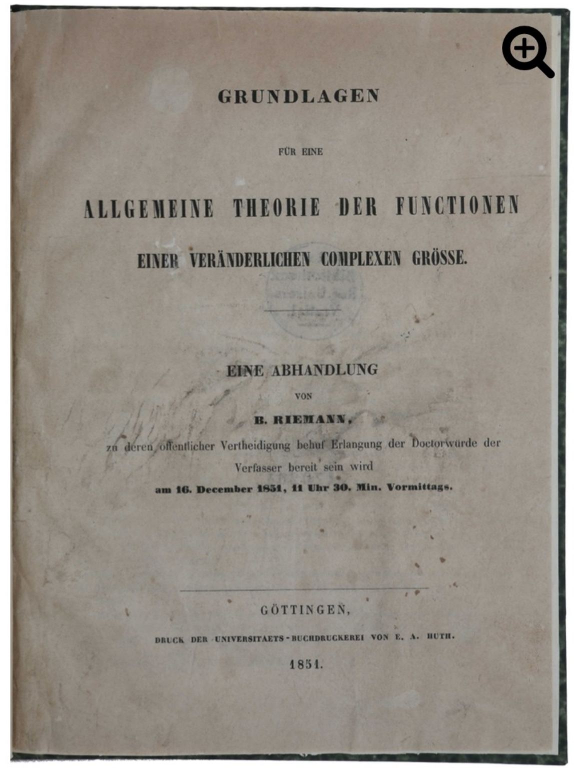
e.g. log ; algebraic functions

See Conway $\underline{\overline{IX}}$.

Riemann Surfaces – first defined by Riemann in his dissertation 1851

– the same dissertation considered the Riemann Mapping Theorem (Math 220B).

Bernhard Riemann (1826–1866)



GRUNDLAGEN

FÜR EINE

ALLGEMEINE THEORIE DER FUNCTIONEN

EINER VERÄNDERLICHEN COMPLEXEN GRÖSSE.

EINE ABHANDLUNG

VON

B. RIEMANN,

zu deren öffentlicher Vertheidigung behuf Erlangung der Doctorwürde der Verfasser bereit sein wird

am 16. December 1851, 11 Uhr 30. Min. Vormittags.

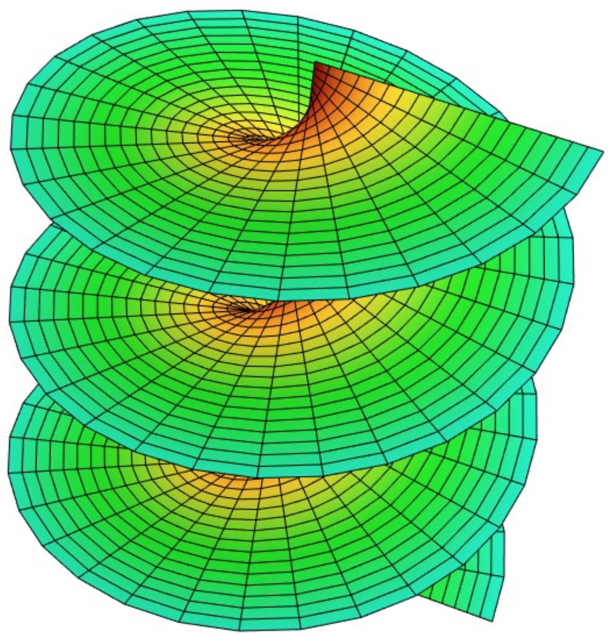
GÖTTINGEN,

DRUCK DER UNIVERSITAETS-BUCHDRUCKEREI VON E. A. HUTH.

1851.

Riemann surfaces were discovered by Riemann and introduced in his doctoral dissertation (1851). This transformed the field of complex analysis, merging it with topology and algebraic geometry.

"We restrict the variables x, y to a finite domain by considering as the locus of the point O no longer the plane A but a surface T spread over the plane"



"We admit the possibility ... of covering the same part of the plane several times. However in such a case, we assume that those parts of the surface lying on top of one another are not connected by a line. Thus a fold or a splitting of parts of the surface cannot occur."

– Klein : "Riemann's methods were regarded almost with distrust by other mathematicians".

– Ahlfors : " Riemann's writings are full of almost cryptic messages to the future."

§ 1. Sheaves

Sheaves in agriculture – a collection of stalks bundled together

Sheaves in mathematics

– we seek to formalize the concept of "function-like objects" e.g. holomorphic functions on Riemann surfaces

– the most elegant way of doing so is via sheaf theory

Definition Let X be a topological space. A *presheaf* of sets, abelian groups, rings ... is an assignment

$$U \rightsquigarrow \mathcal{F}(U)$$

of sets, abelian groups, rings... for all $U \subseteq X$ open.

& restriction maps

$$\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

which should be homomorphisms of ... . We require

(i) $\rho_{UU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is the identity

(ii) $\forall\ W \subseteq V \subseteq U$ we have

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV} : \begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"'] \arrow[rr, bend right, "\rho_{UW}"'] & \mathcal{F}(V) \arrow[r, "\rho_{VW}"'] & \mathcal{F}(W) \end{tikzcd}$$

Terminology

(i) elements $s \in \mathcal{F}(U)$ are called sections.

(ii) restriction maps $\rho_{UV}(s) = s|_V$.

Definition A presheaf $\mathcal{F} \to X$ is a **sheaf** provided

$\forall$ $\mathcal{U} = \bigcup_i \mathcal{U}_i$ open cover, $s_i \in \mathcal{F}(\mathcal{U}_i)$ with

$$s_i\big|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_j\big|_{\mathcal{U}_i \cap \mathcal{U}_j}$$

$\Rightarrow$ $\exists!$ $s \in \mathcal{F}(\mathcal{U})$ such that $s\big|_{\mathcal{U}_i} = s_i$.

Examples

[i] X topological space, $\mathcal{F} = \mathcal{C}$ is the sheaf:

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U}) = \{ f : \mathcal{U} \longrightarrow \mathbb{C} \text{ continuous} \}.$$

with the usual restriction maps $\mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(V)$, $f \longrightarrow f\big|_V$.

[ii] $X \subseteq \mathbb{R}^n$ open, $\mathcal{F} = \mathcal{C}^k$, $0 \leq k \leq \infty$, $k = \omega$

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U}) = \{ f : \mathcal{U} \longrightarrow \mathbb{C} \text{ of class } \mathcal{C}^k \}$$

is a sheaf.

III $G \subseteq \mathbb{C}$ open, $\mathcal{F} = \mathcal{O}_G$, $U \subseteq G$ open

$$\mathcal{O}_G(U) = \{ f : U \longrightarrow \mathbb{C} \text{ holomorphic} \} \text{ is a sheaf.}$$

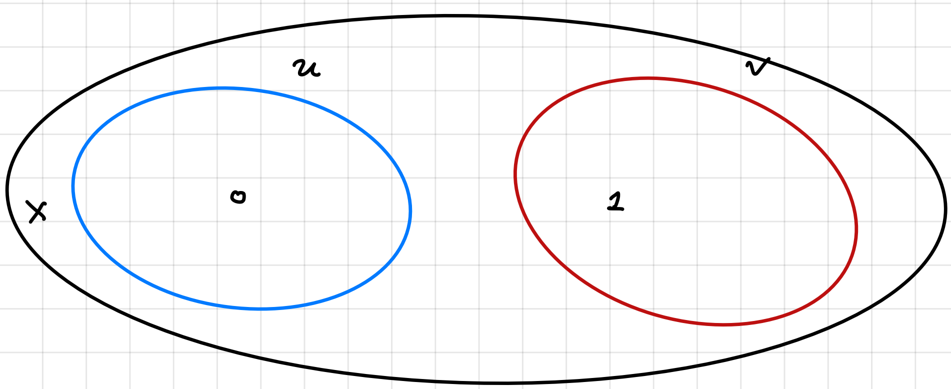
IV $p \in X$ topological space. The skyscraper sheaf

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

V The constant presheaf over X = top. space

$$\underline{\mathbb{C}}(U) := \{ f : U \longrightarrow \mathbb{C} \text{ constant} \} \text{ is not a sheaf.}$$

Why? Assume $U, V \subseteq X$



$U \cap V = \emptyset$

$f_1 \equiv 0$ on U

$f_2 \equiv 1$ on V

$\Rightarrow f_1|_{U \cap V} = f_2|_{U \cap V}$.

Let $W = U \cup V$. Gluing fails.

However

$\underline{\mathbb{C}}^{sh}: \mathcal{U} \longrightarrow \{f: \mathcal{U} \longrightarrow \mathbb{C} \text{ locally constant}\}$ is a sheaf.

VII Restriction of sheaves to open sets

$\mathcal{F} \longrightarrow X$ sheaf, $\mathcal{U} \subseteq X$ open

Define $\mathcal{F}/_{\mathcal{U}}$ a sheaf over $\mathcal{U}$ via

$\mathcal{F}/_{\mathcal{U}}(V) = \mathcal{F}(V)$ for $V \subseteq \mathcal{U}$ open. Note that

$V \subseteq X$ is also open since $\mathcal{U} \subseteq X$ is open, so the above makes sense.

Sheaves were discovered by Leray in the 40s as POW.

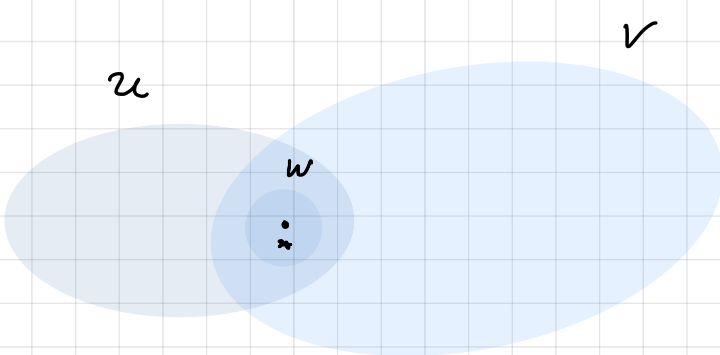
His papers were sent to Hopf in Zürich for publication.

Stalks & Germs

$\mathcal{F} \longrightarrow X$ presheaf, $x \in X$

Consider pairs (U, s), consisting of $x \in U \subseteq X$ open and $s \in \mathcal{F}(U)$ a section.

$(U, s) \sim (V, t)$ provided $\exists\, x \in W \subseteq U \cap V$ open with $s/_W = t/_W$.



This is an equivalence relation.

The stalk of $\mathcal{F}_x$ is the set of equivalence classes.

An equivalence class is called a germ.

We have $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$