Math 220C, Problem Set 2. Due Friday, April 9.

0. (*Laplacian in polar coordinates.*) Prove, but do not hand in, the following formula for the Laplacian in polar coordinates

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

This is a repeated application of the chain rule.

1. (*Subharmonic functions.*) Assume that $\phi : G \to \mathbb{R}$ is a function of class $\mathcal{C}^2$ such that

$$\Delta\phi \geq 0.$$

Let $a \in G$ and $\overline{\Delta}(a, R) \subset G$. Define for $0 \leq r < R$ the function

$$h(r) = \frac{1}{2\pi}\int_0^{2\pi} \phi(a + re^{it})\, dt$$

(i) Show that h is non-decreasing.

Hint: Let $u(r, t) = \phi(a + re^{it})$. Show that

$$\left(r\frac{\partial}{\partial r}\right)^2 h = \frac{1}{2\pi}\int_0^{2\pi}\left(r\frac{\partial}{\partial r}\right)^2 \phi\, dt.$$

Using the expression for Δ in polar coordinates, and the fact that the integral of $\frac{\partial^2 u}{\partial \theta^2}$ vanishes, conclude that $rh'(r)$ is non-decreasing. Conclude that $h'(r) \geq 0$.

(ii) Using (i), show that ϕ is subharmonic, that is ϕ satisfies the mean value inequality

$$\phi(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(a + re^{it})\, dt.$$

(iii) If ϕ is in fact harmonic in $G \setminus \{a\}$, show that

$$\left(r\frac{\partial}{\partial r}\right)^2 h = 0$$

and conclude that

$$h(r) = \alpha \log r + \beta.$$

(iv) Which of the following functions are subharmonic? harmonic? neither?

(a) $f(x, y) = x^2 + y^2$

(b) $f(x, y) = x^2 - y^2$

(c) $f(x, y) = x^2 + y$

2. (*Removable Singularities.*) Show that if $u : \Delta(0, 1) \setminus \{0\} \to \mathbb{R}$ is harmonic and $\lim_{z\to 0} u(z)$ exists and is finite, then u can be extended to a harmonic function on Δ.

3. (*Dirichlet Problem. Qualifying Exam, Fall 2020.*) Let Δ denote the open unit disc, and let $\Delta' = \left\{z \in \mathbb{C} : \left|z + \frac{2}{5}\right| < \frac{2}{5}\right\}$ denote the open disc of center $-\frac{2}{5}$ and radius $\frac{2}{5}$. Let $\Omega = \Delta \setminus \overline{\Delta'}$.

Find, with justification, an explicit continuous functions $h : \overline{\Omega} \to \mathbb{R}$, harmonic in Ω, and with boundary values $h = 0$ on $\partial\Delta$ and $h = 1$ on $\partial\Delta'$.

Hint: Recenter.

4. (*Harmonic functions on the disc.*)

(i) Give an example of a harmonic function in the half plane $u : \{z : \text{Re } z > 0\} \to \mathbb{R}$ such that

$$\lim_{z \to iy} u(z) = 1 \text{ for } y > 0, \quad \lim_{z \to iy} u(z) = -1 \text{ for } y < 0.$$

Hint: Your function should be the imaginary part of a familiar function.

(ii) Using part (i), give an example of a harmonic function on the unit disc $u : \Delta \to \mathbb{R}$ such that

$$\lim_{r \to 1} u(re^{it}) = 1 \text{ if } 0 < t < \pi, \ \lim_{r \to 1} u(re^{it}) = -1 \text{ if } \pi < t < 2\pi.$$

5. (*Schwarz Reflection.*) Let $G \subset \mathbb{C}$ be a symmetric region with respect to the real axis, and let

$$G^+ = G \cap \{\text{Im } z > 0\}$$

be the part in the upper half plane. Moreover, assume that u is harmonic on G^+ and that

$$\lim_{z \to z_0} u(z) = 0$$

for any point $z_0 \in G \cap \mathbb{R}$. Show that u extends to a harmonic function on G, and the extension satisfies

$$u(\bar{z}) = -u(z).$$

Hint: To verify the extension is harmonic, use the mean value property.