Math 220C, Problem Set 1. Due Friday, April 2.

1. If U, V are open sets in $\mathbb{C}$, $g : V \to U$ is holomorphic, $u : U \to \mathbb{R}$ harmonic, then $u \circ g : V \to \mathbb{R}$ is harmonic in V.

Hint: This requires no calculation. Write u locally as the real part of a holomorphic function.

2. (*Open Mapping Theorem.*) Let $u : U \to \mathbb{R}$ be a nonconstant harmonic function in a region $U \subset \mathbb{C}$. Show that u is an open map.

Hint: Write u locally as the real part of a holomorphic function.

3. (*Mean Value Property and Cauchy's estimates.*) Let $u : U \to \mathbb{R}$ be harmonic, and $\overline{\Delta}(a, r) \subset U$. Let

$$M = \sup_{|z-a|=r} |u(z)|.$$

(i) Show that

$$u(a) = \frac{1}{\pi r^2} \int\int_{\Delta(a,r)} u(x, y)\, dx\, dy.$$

(ii) Show that the derivatives u_x and u_y are also harmonic. Therefore

$$u_x(a) = \frac{1}{\pi r^2} \int\int_{\Delta(a,r)} u_x(x, y)\, dx\, dy.$$

(iii) Use Green's theorem in part (ii) to deduce that

$$|u_x(a)| \leq \frac{2}{r} M.$$

Derive the similar statement for u_y.

(iv) Using induction, show that for $i + j = n$ then the higher derivatives satisfy the estimates

$$|\partial_x^i \partial_y^j u(a)| \leq C_n r^{-n} M$$

for some constant C_n.

(v) (*Generalized Liouville's theorem.*) Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and $|u(z)| \leq A(1 + |z|^m)$ then u is a polynomial.

(vi) If $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and bounded (in absolute value), then u is constant.

4. Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and bounded either from below or from above, then u is constant.

Hint: Reduce to the case when $u \geq 0$. Write u as the real part of an entire function f and work with e^{-f}.

5. Let $U \subset \mathbb{C}$ be open connected.

(i) Show that if $h : U \to \mathbb{C}$ is holomorphic and nowhere zero in U, then

$$u(z) = \log |h(z)|$$

is harmonic in U.

There are several ways to solve this, some requiring almost no calculation.

(ii) Assume that every harmonic function in U admits a harmonic conjugate. Show that U is simply connected.

Hint: If h is a nowhere zero holomorphic function in U, consider the harmonic function $\log |h(z)|$ and use the hypothesis to show that h admits a logarithm. Conclude using a result from Math 220B.