Problem 1. [*6 points.*]

Let $P_1, \ldots, P_m$ be points on the unit circle. Show that there is a point Q on the unit circle such that

$$P_1Q \cdot P_2Q \cdot \ldots \cdot P_mQ \geq 1.$$

Hint: the above expression is the modulus of a holomorphic function.

Problem 2. [*10 points; 3, 7.*]

(i) Let $f : U \to \mathbb{C}$ be continuous, and let $R = [a, b] \times [c, d] \subset U$ be any rectangle. For n sufficiently large, consider the smaller rectangles

$$R_n = \left[a + \frac{1}{n}, b - \frac{1}{n}\right] \times \left[c + \frac{1}{n}, d - \frac{1}{n}\right].$$

Show that

$$\int_{\partial R_n} f(z)\, dz \to \int_{\partial R} f(z)\, dz.$$

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be continuous and holomorphic on $\mathbb{C} \setminus [0, 1]$. Show that f is entire.

Problem 3. [*10 points.*]

Let $p_1, \ldots, p_n$ be polynomials, and let $f : \mathbb{C} \to \mathbb{C}$ be entire such that

$$f(z)^n + f(z)^{n-1} p_1(z) + \ldots + p_n(z) = 0.$$

Show that f is a polynomial.

Problem 4. [*7 points.*]

Let $f : \Delta \setminus \{a\} \to \mathbb{C}$ be a holomorphic function on a punctured disc centered at a, having $z = a$ as an essential singularity. Show that f is not injective.