Math 220 A — Lecture 9

October 23, 2020

No lecture on Monday, Oct 26.

Last time

Cauchy's Integral Formula (CIF)

$f : U \to \mathbb{C}$ holomorphic, $\gamma \overset{U}{\sim} 0$, $a \in U \setminus \{\gamma\}$

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

Example $|a| < |b|$. We compute

$$\int_{|z|=r} \frac{e^z}{(z-a)(z-b)}\, dz$$

$\boxed{i}$ $r < |a|$, the integrand is holomorphic so answer $= 0$.

$\boxed{ii}$ $|a| < r < |b|$. Write

holomorphic in $|z| \le r$.

$$\int_{|z|=r} \frac{e^z/(z-b)}{z-a} = 2\pi i \cdot \left.\frac{e^z}{z-b}\right|_{z=a} = 2\pi i \cdot \frac{e^a}{a-b}.$$

$\boxed{iii}$ $|a| < |b| < r$

Let $\gamma_r = \{|z| = r\}$.

Let $f(z) = \dfrac{e^z}{(z-a)(z-b)}$.

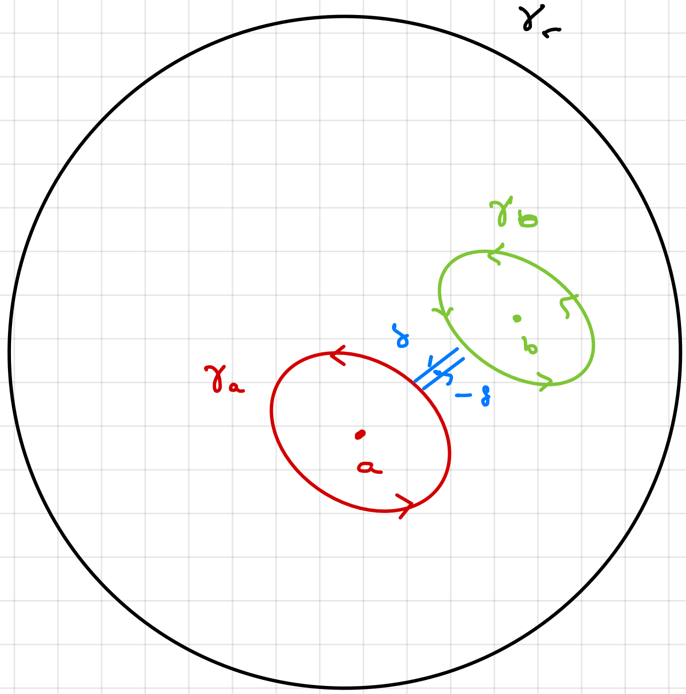
Let γ_a, γ_b be two circles

centered at a, b and δ

a segment joining them.

Let $\gamma = \gamma_a + \delta + \gamma_b + (-\delta)$.

Note $\gamma \sim \gamma_r$ in $\mathbb{C} \setminus \{a, b\}$.



By homotopy Cauchy

$$\int_{\gamma_r} f\,dz = \int_{\gamma} f\,dz = \int_{\gamma_a} f\,dz + \int_{\gamma_b} f\,dz + \cancel{\int_{\delta} f\,dz} + \cancel{\int_{-\delta} f\,dz}$$

$$= \int_{\gamma_a} \frac{e^z/z-b}{z-a}\,dz + \int_{\gamma_b} \frac{e^z/z-a}{z-b}\,dz$$

$$= 2\pi i \cdot \left.\frac{e^z}{z-b}\right|_{z=a} + 2\pi i \left.\frac{e^z}{z-a}\right|_{z=b}$$

$$= 2\pi i \cdot \frac{e^a - e^b}{a-b}.$$

Taylor Expansion

Theorem $f: U \longrightarrow \mathbb{C}$ holomorphic, $a \in U$, $\Delta(a, R) \subseteq U$.

Then in $\Delta(a, R)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad (*).$$

$\Longrightarrow$ f analytic $\Longrightarrow$ f is ∞ - many times differentiable.

ANALYTIC = HOLOMORPHIC = DIFFERENTIABLE

Proof Let $\Delta(a, R) \subseteq U$. We pick $0 < r < R$. Let $z \in \Delta(a, r)$. By CIF

$$f(z) = \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{t-z}\, dt$$

Key : $$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \frac{1}{t-a} \cdot \frac{1}{1-\frac{z-a}{t-a}}$$

$$= \frac{1}{t-a} \sum_{k=0}^{\infty} \frac{(z-a)^k}{(t-a)^k} \quad \text{converges since}$$

$$\left|\frac{z-a}{t-a}\right| = \frac{|z-a|}{r} < 1.$$

$$\Rightarrow \frac{f(t)}{t-z} = \sum_{k=0}^{\infty} f(t) \cdot \frac{(z-a)^k}{(t-a)^{k+1}} . \quad (+)$$

Claim This converges uniformly in t over $|t-a| = r$.

Indeed, let $f_k(t) = f(t) \cdot \frac{(z-a)^k}{(t-a)^{k+1}}$.

$$\Rightarrow |f_k(t)| \leq M \cdot \frac{|z-a|^k}{r^{k+1}} = M_k, \quad |f(t)| \leq M \text{ for } |t-a| = r$$

Note $\sum M_k < \infty$ since $|z-a| < r$. Thus the claim follows

by Weierstraß M-test

Since the convergence is uniform, we can integrate (Rudin)

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{t-z} \, dt \overset{(+)}{=}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}} \, dt \cdot (z-a)^k$$

$$= \sum_{k=0}^{\infty} a_k (z-a)^k .$$

Def A holomorphic $f : \mathbb{C} \longrightarrow \mathbb{C}$ is said to be entire.

Remark f entire $\overset{\text{Taylor}}{\Longrightarrow} f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in \mathbb{C}$

Remark $f : U \longrightarrow \mathbb{C}$, $\overline{\Delta}(a, r) \subseteq U$.

$$a_k = \frac{f^{(k)}(a)}{k!} = \underbrace{\frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}}\, dt}_{\text{from the proof of the Theorem.}}$$

Lecture 2/3

Thus
$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

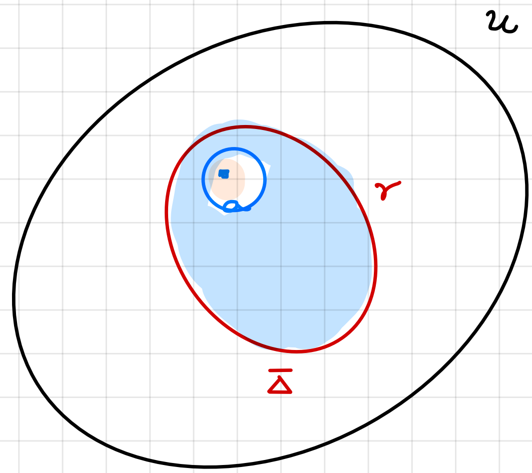
This is local CIF for derivatives.

Cauchy's Integral Formula (for derivatives)

If $\overline{\Delta} \subseteq U$, $a \in \Delta$, $f : U \longrightarrow \mathbb{C}$ holomorphic

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\partial\Delta} \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

Proof



If a is the center of Δ we showed this on the previous page.

If a is not the center then let γ_a be a small circle centered at a. Then $\gamma_a \sim \gamma$ where $\gamma = \partial\Delta$. We have

$$\underbrace{f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\gamma_a} \frac{f(t)}{(t-a)^{k+1}}\, dt}_{\gamma_a \text{ has center } a} \underbrace{=}_{\substack{\text{homotopy} \\ \text{Cauchy}}} \frac{k!}{2\pi i} \int_{\gamma} \frac{f(t)}{(t-a)^{k+1}}\, dt$$

Remark (Homotopy version)

$$f : \mathcal{U} \longrightarrow \mathbb{C}, \quad \gamma \overset{\mathcal{U}}{\sim} 0, \quad a \in \mathbb{C} \setminus \{\gamma\}$$

$$n(\gamma, a)\, f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

The case $\gamma = \partial \Delta$, $\overline{\Delta} \subseteq \mathcal{U}$ is considered above

A possible proof is via Conway $\overline{IV}$. 2.2 / HWK 3

Exercise 7. Another proof is via the residue theorem to be stated later.

Example

$$\int_{|z|=r} \frac{e^z}{(z-a)^k}\, dz \quad , \; r \neq |a|$$

If $|a| > r$ the answer is 0 because the integrand is holomorphic

If $r > |a|$, apply CIF for derivatives:

$$\frac{1}{(k-1)!} \cdot 2\pi i \cdot \partial^{(k-1)} e^z \Big/_{z=a} = \frac{e^a}{(k-1)!} \cdot 2\pi i$$

Cauchy's Estimate

Let $f: U \to \mathbb{C}$ holomorphic, $\overline{\Delta}(a, R) \subseteq U$. Let

$$M_R = \sup_{|z-a|=R} |f(z)|.$$

Then

$$|f^{(k)}(a)| \leq k! \, \frac{M_R}{R^k}.$$

Proof By CIF for derivatives

$$|f^{(k)}(a)| = \left| \frac{k!}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^{k+1}} \, dz \right|$$

$$\leq \frac{k!}{2\pi} \cdot \frac{M_R}{R^{k+1}} \cdot \text{length } |z-a|=R$$

$$= \frac{k!}{2\pi} \cdot \frac{M_R}{R^{k+1}} \cdot 2\pi R = k! \, \frac{M_R}{R^k}.$$

Liouville's theorem

If $f: \mathbb{C} \to \mathbb{C}$ entire & bounded $\Rightarrow$ f constant.

We prove this next time.