Math 220 B – Lecture 8

January 22, 2021

Weierstraß Problem for arbitrary regions

Question Given $\mathcal{U} \subseteq \mathbb{C}$, $\{a_n\} \subseteq \mathcal{U}$ without limit point in $\mathcal{U}$, find f holomorphic in $\mathcal{U}$ with zeroes only at $\{a_n\}$.

The sequence $\{a_n\}$ may contain repetitions according to multiplicities of the zeroes.

Main Theorem The Weierstraß Problem can be solved in $\mathcal{U}$.

Remark $\boxed{i}$ It is not true any two solutions f_1, f_2

satisfy $f_1 = e^h f_2$

Counterexample $U = \mathbb{C}^\times$, $f_2 = 1$, $f_1 = z$

h would have to be a logarithm, which is undefined in $\mathbb{C}^\times$.

$\boxed{ii}$ Any meromorphic function in U is quotient of two holomorphic functions.

The same proof for $U = \mathbb{C}$ works for all U.

How to prove Weierstrass for $\mathcal{U}$?

We could again try

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Convergence used $a_n \longrightarrow \infty$.

Indeed, if we wish to have

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \quad \text{we'd need} \quad \frac{r}{|a_n|} \longrightarrow 0 \Rightarrow a_n \longrightarrow \infty.$$

Since $a_n \in \mathcal{U}$, this may not be the case. e.g. if $\mathcal{U}$ is bounded. How to deal with bounded regions for instance?

New ideas

(1) Use biholomorphisms to change the region $\mathcal{U}$ e.g. via $z \longrightarrow 1/z$. If $\mathcal{U}$ were bounded, the new region would be unbounded.

(2) Think of $\mathcal{U} \subseteq \mathbb{C}$ as $\mathcal{U} \subseteq \hat{\mathbb{C}}$ & prescribe values at ∞ as well.

New idea

Even for unbounded regions, we can try new functions:

$$f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{a_n - b_n}{z - b_n}\right)$$ for good choices of b_n.

This also has zeroes at $z = a_n$ since $E_n(1) = 0$.

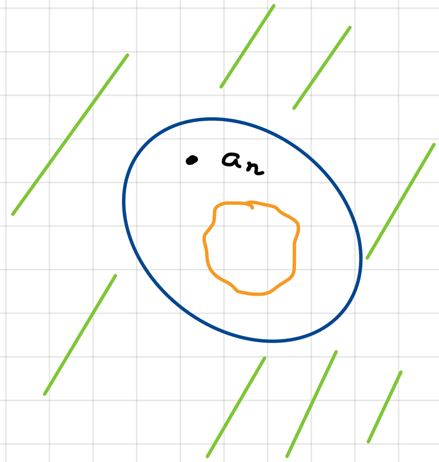
Weierstraß Problem in $\mathcal{U} \subseteq \mathbb{C}$

Step (1) Assume $\exists R > 0$

neighborhood of ∞.

[i] $\{|z| \geq R\} \subseteq \mathcal{U}$

[ii] $|a_n| \leq R \quad \forall n.$



Construct f holomorphic in $\mathcal{U}$ such that

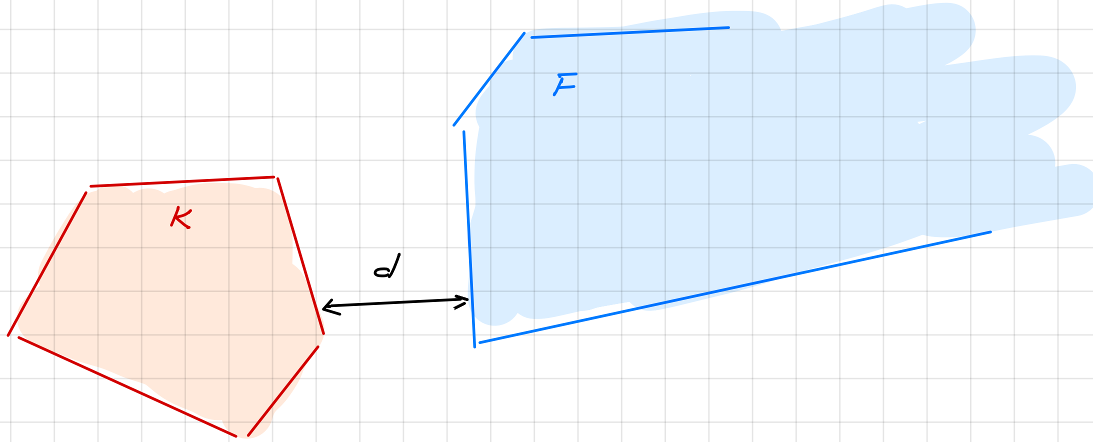
[i] f has zeroes at a_n

[ii] $\lim_{z \to \infty} f(z) = 1.$

Step (2) General case. Uses easy trick.

Use $z \longrightarrow 1/z$. to reduce to Step 1.

Topological Fact used in the Proof (Rudin)



$K \cap F = \Phi$, $K \neq \Phi$ compact, $F \neq \Phi$ closed.

$$d = dist(K, F) = \inf_{\substack{k \in K \\ f \in F}} |k - f| > 0$$

Proof Assume $d = 0$. Then $\exists\, k_n \in K$, $f_n \in F$ with

$$|k_n - f_n| \longrightarrow 0$$

Passing to a subsequence, assume $k_n \longrightarrow k \in K$.

It follows that $f_n \longrightarrow k$ as well.

Since F closed, $k \in F$. Thus $k \in K \cap F = \Phi$. contradiction.

Step 1 $\exists\, R>0,\ \{|z|\geq R\}\subseteq \mathcal{U}$ & $|a_n|\leq R$.

[i] f zeroes at a_n only

[ii] $\lim_{z\to\infty} f(z) = 1$

Note $K = \mathbb{C}\setminus\mathcal{U} \subseteq \{|z|\leq R\} \Rightarrow K$ bounded & closed

$\Rightarrow K$ compact.

Since $|a_n - z|$ is continuous, $\exists\, b_n\in K$ with

$$|a_n - b_n| = \min_{z\in K} |a_n - z|.$$

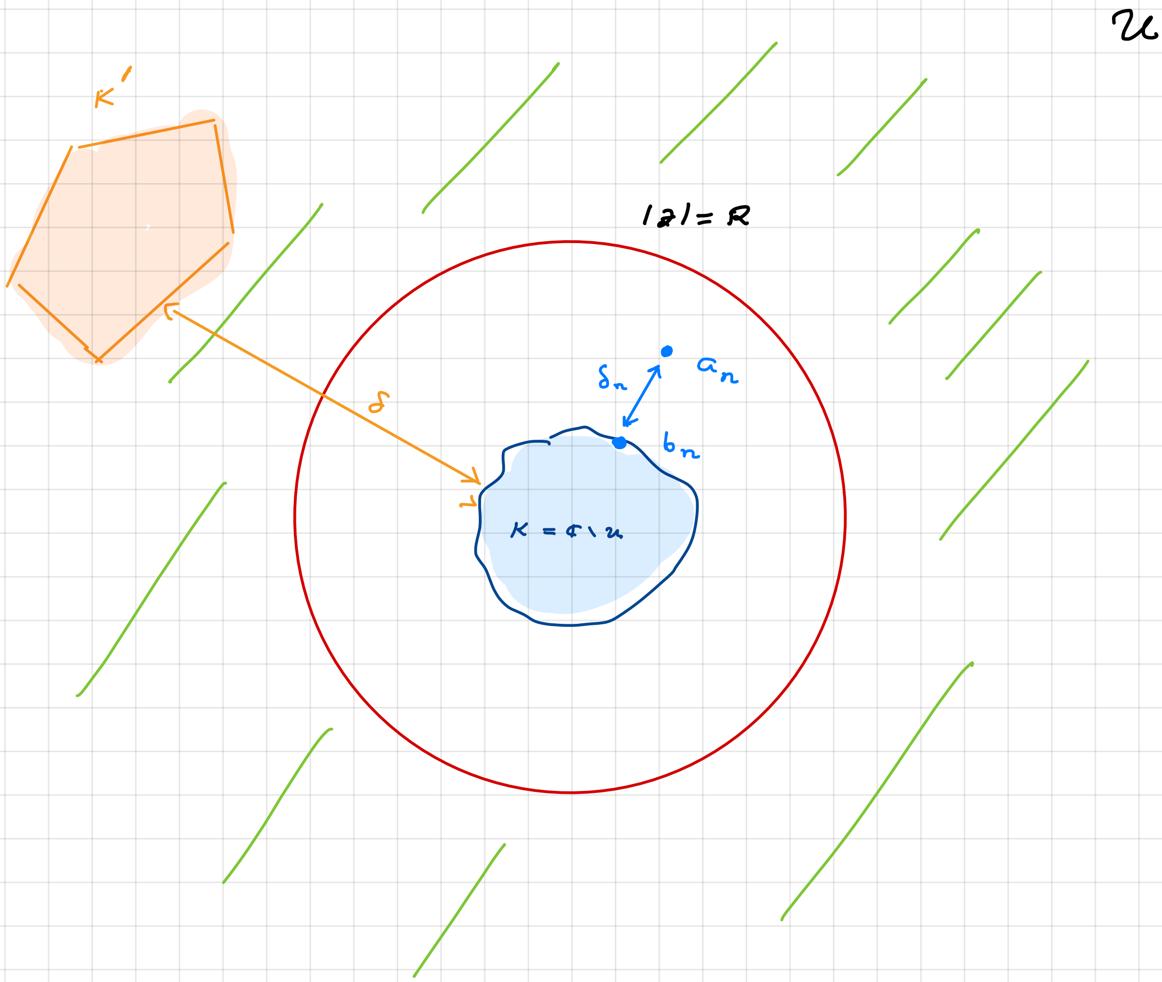
Write $\delta_n = |a_n - b_n| > 0$ since $a_n \in \mathcal{U}$, $b_n \notin \mathcal{U}$.

Claim $\delta_n \longrightarrow 0$.

Proof Assume otherwise. Then $\exists\,\varepsilon\ \forall N\ \exists\, n\geq N$ with

$$|\delta_n| \geq \varepsilon.$$

Passing to a subsequence we may assume $|\delta_n|\geq\varepsilon\ \forall n$.



Now $\{a_n\} \subseteq \bar{\Delta}(0,R) =$ compact. Passing to a subsequence we may assume $a_n \to a$. Since $\{a_n\}$ has no limit point in $\mathcal{U}$ $\Rightarrow a \in K$. Then by the definition of b_n:

$$|a_n - a| \geq |a_n - b_n| = \delta_n > \varepsilon.$$

This contradicts $a_n \to a$. Thus $\delta_n \to 0$.

Claim $f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{a_n - b_n}{z - b_n}\right)$ converges absolutely & locally uniformly in $\mathcal{U}$. & vanishes only at a_n.

Proof It suffices to show

$$\sum_{n=1}^{\infty} \left| E_n\left(\frac{a_n - b_n}{z - b_n}\right) - 1 \right|$$

converges absolutely & locally uniformly in $\mathcal{U}$. To this end, let $K' \subseteq \mathcal{U}^-$ compact.

Let $\delta = d(K, K') > 0$ since $K \cap K' = \Phi$.

For $z \in K' \Rightarrow |z - b_n| \geq \delta. \Rightarrow$

$$\left| \frac{a_n - b_n}{z - b_n} \right| \leq \frac{\delta_n}{\delta} \leq \frac{1}{2} \text{ if } n \geq N \text{ since } \delta_n \longrightarrow 0.$$

Recall

$$|1 - E_p(w)| \leq |w|^{p+1}. \quad \forall \ |w| \leq 1.$$

Thus

$$\left| 1 - E_n\left(\frac{a_n - b_n}{z - b_n}\right) \right| \leq \left| \frac{a_n - b_n}{z - b_n} \right|^{n+1} \leq \frac{1}{2^{n+1}} \quad \forall z \in K', n \geq N$$

We conclude by Weierstraß M-test since $\sum_n \frac{1}{2^{n+1}} < \infty$.

Proof of [ii] $\lim_{z \to \infty} f(z) = 1$.

Equivalently $\lim_{z \to 0} f\left(\frac{1}{z}\right) = 1$.

We compute

$$g(z) = f\left(\frac{1}{z}\right) = \prod_{n=1}^{\infty} E_n\left(\frac{a_n - b_n}{1/z - b_n}\right) = \prod_{n=1}^{\infty} E_n\left(\frac{z(a_n - b_n)}{1 - z b_n}\right). \quad (*)$$

We show the product $(*)$ converges absolutely & locally uniformly in $\Delta\left(0, \frac{1}{R}\right)$. The limit will be holomorphic at $z = 0$ hence continuous. Then

$$\lim_{z \to 0} g(z) = g(0) = 1. \Rightarrow \lim_{z \to 0} f\left(\frac{1}{z}\right) = 1.$$

To show convergence, let $\overline{\Delta}(0,\rho) \subseteq \Delta\left(0, \frac{1}{R}\right)$. $\Rightarrow \rho R < 1$.

We have for $z \in \overline{\Delta}(0,\rho)$:

$|z| \leq \rho,\ |b_n| \leq R.$

$$\left| \frac{z(a_n - b_n)}{1 - z b_n} \right| \leq \frac{\rho \delta_n}{|1 - z b_n|} \leq \frac{\rho \delta_n}{1 - |z||b_n|} \leq \frac{\rho \delta_n}{1 - \rho R} \leq \frac{1}{2}.$$

for $n \geq N$ since $\delta_n \to 0$.

Then

$$\left| 1 - E_n\left(\frac{z(a_n - b_n)}{1 - z b_n} \right) \right| \leq \left| \frac{z(a_n - b_n)}{1 - z b_n} \right|^{n+1} \leq \frac{1}{2^{n+1}} \Rightarrow$$

$\Rightarrow$ Weierstraß M-test

$$\sum_{n=1}^{\infty} \left| 1 - E_n\left(\frac{z(a_n - b_n)}{1 - z b_n} \right) \right|$$ converges absolutely & locally uniformly in $\Delta\left(0, \frac{1}{R}\right)$.

Case (2) General case

WLOG $0 \in \mathcal{U}$ & $a_n \neq 0$

Indeed we may take $a \in \mathcal{U}$, $a \neq a_n$ $\forall n$. Let

$$\mathcal{U}^{new} = \{ u - a,\ u \in \mathcal{U}\},\quad a_n^{new} = a_n - a,$$

$\Rightarrow$ $0 \in \mathcal{U}^{new}$, $a_n^{new} \neq 0$. If f^{new} solves Weierstraß for $(\mathcal{U}^{new}, \{a_n^{new}\})$ let $f(z) = f^{new}(z-a)$. solves Weierstraß for $(\mathcal{U}, \{a_n\})$.

Trick to reduce to Case 1

Define $\tilde{\mathcal{U}} = \{ \frac{1}{z} : z \in \mathcal{U} \setminus \{0\}\}$. This is open by the open mapping theorem for $\mathcal{U} \setminus \{0\}$, $z \longrightarrow 1/z$.

Let $\tilde{a}_n = \frac{1}{a_n} \in \tilde{\mathcal{U}}$

Claim $(\tilde{\mathcal{U}}, \{\tilde{a}_n\})$ satisfies Step 1.

Let $\tilde{f}$ be the solution to Weierstraß for $(\tilde{\mathcal{U}}, \{\tilde{a}_n\})$

Let $f(z) = \tilde{f}\left(\frac{1}{z}\right)$ = holomorphic in $\mathcal{U} \setminus \{0\}$.

Since $\lim_{z \to \infty} \tilde{f}(z) = 1 \implies \lim_{z \to 0} f(z) = 1$. Thus 0 is removable

singularity and f extends to $\mathcal{U}$. Its zeroes are only at a_n.

Proof of the claim

Since $0 \in \mathcal{U} \Rightarrow \exists\, \varepsilon$ with $\Delta(0, \varepsilon) \subseteq \mathcal{U}$.

$\implies \{ |z| \geq \frac{1}{\varepsilon} \} \subseteq \tilde{\mathcal{U}}$.

Since $0 \in \mathcal{U}$ & $\{a_n\}^{\neq 0}$ do not have 0 as limit point

$\implies \exists\, \varepsilon'$ with $|a_n| \geq \varepsilon' \Rightarrow |\tilde{a}_n| \leq \frac{1}{\varepsilon'}$.

Let $R = \max(1/\varepsilon, 1/\varepsilon')$. $\Rightarrow |a_n| \leq R$ & $\{ |z| \geq R \} \subseteq \mathcal{U}$.

Exercise

Follow the above proof for $u = \mathbb{C}$. What function f does the proof produce?