Math 200b Winter 2021 Homework 3

Due 1/29/2020 by midnight on Gradescope

1. Let F be a field. Prove that if $n \leq 3$, two matrices $A, B \in M_n(F)$ are similar if and only if $\mathrm{charpoly}(A) = \mathrm{charpoly}(B)$ and $\mathrm{minpoly}(A) = \mathrm{minpoly}(B)$. Give an example to show this result does not hold for matrices in $M_4(F)$ in general.

2. Let F be an algebraically closed field. Consider the matrix

$$M = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in M_3(F).$$

Find the minimal and characteristic polynomials of M, and the rational and Jordan canonical forms of M. (The answers may depend on the characteristic of F.)

3. Consider the three matrices

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Are any of these matrices similar to each other over $\mathbb{C}$? Justify your answer.

4. let F be a field. Find representatives of each of the similarity classes of matrices $A \in \mathrm{GL}_2(F)$ such that A has multiplicative order exactly 4 in this group, and thus calculate exactly how many such similarity classes there are, when

(a) $F = \mathbb{Q}$.

(b) $F = \mathbb{C}$.

(c) F is a field of characteristic 2.

5. Let F be a field. Let $A \in M_n(F)$ and let $f = \text{charpoly}(A)$ and $g = \text{minpoly}(A)$. Let A^t be the transpose of A.

(a) Prove that $\text{charpoly}(A^t) = f$ and $\text{minpoly}(A^t) = g$.

(b) Prove that $f = g$ if and only if A is similar to the companion matrix C_f.

(c) Show that $(C_h)^t$ is similar to C_h for any companion matrix C_h.

(d) Show that A is similar to A^t.

6. Let $J \in M_n(\mathbb{C})$ be a single Jordan block corresponding to the eigenvalue $\lambda \in \mathbb{C}$.

(a) If $\lambda \neq 0$, prove that the Jordan canonical form of J^2 is also a single Jordan block, with eigenvalue λ^2.

(b) If $\lambda = 0$, prove that the Jordan form of J^2 consists of two Jordan blocks, of size $n/2$ and $n/2$ if n is even and of size $(n+1)/2$ and $(n-1)/2$ if n is odd.

(c) Determine necessary and sufficient conditions for a matrix $M \in M_n(\mathbb{C})$ to have a square root, that is for there to exist $N \in M_n(\mathbb{C})$ such that $N^2 = M$.