Math 220, Problem Set 2.

1. (*Applications of the* Γ*-function. Wednesday, January 13.*) Let

$$R(z) = \frac{P(z)}{Q(z)}$$

be a rational function without poles at the positive integers.

Remark: To put things into perspective, recall that Problem 5 on the Final Exam for Math 220A gave a formula for the infinite sum $\sum_{k=-\infty}^{\infty} R(k)$ under certain assumptions. In this problem, we study the infinite product $\prod_{k=1}^{\infty} R(k)$.

Assume that P, Q are polynomials of the same degree, with leading term equal to 1. Write

$$P(z) = \prod_{i=1}^{d} (z - a_i), \quad Q(z) = \prod_{i=1}^{d} (z - b_i)$$

and furthermore assume that

$$\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} b_i.$$

(One can see that without these assumptions the product does not converge.)

(i) Using the definition of the function G, show that

$$\prod_{k=1}^{\infty} R(k) = \frac{G(-a_1) \cdots G(-a_d)}{G(-b_1) \cdots G(-b_d)}.$$

Express the product $\prod_{k=1}^{\infty} R(k)$ in terms of the Γ function.

(ii) Using (i), compute the product

$$\prod_{n=1}^{\infty} \frac{n^2 + n - 4/9}{n^2 + n - 5/16}.$$

You can simplify the answer so that only sine's are involved.

2. (*Factorization of trigonometric functions. Monday, January 11.*) This is a version of Problem 3, Conway VII.6 with some hints.

Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Show that

$$\frac{\sin \pi(z + \alpha)}{\sin \pi\alpha} = e^{\pi z \cot \pi\alpha} \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n - \alpha}\right) e^{\frac{z}{n-\alpha}} = e^{\pi z \cot \pi\alpha} \prod_{n=-\infty}^{\infty} E_1\left(\frac{z}{n - \alpha}\right).$$

Use this to find a factorization of the function

$$\cos\left(\frac{\pi}{4} z\right) - \sin\left(\frac{\pi}{4} z\right).$$

Hint: You will need the usual strategy given in class: examine the zeros and take logarithmic derivatives. Also recall the identities of Problem Set 6 from Math 220A.

For the next two questions, you will only need to know that the Weierstraß problem admits a solution. (Feel free to use the explicit form of the solution if it helps you, though this is not strictly speaking needed.)

3. (*"Greatest common divisor." Friday, January 15.*) This is Conway VII.5, Problem 3. Assume that f and g are entire functions. Show that there exist entire functions h, F and G such that

$$f(z) = h(z)F(z), \; g(z) = h(z)G(z)$$

with F, G having no common zeroes.

4. (*Roots. Friday, January 15.*) Let f be an entire function and $n \geq 1$. Show that there exists an entire function g such that $g^n = f$ if and only if the orders of all zeroes of f are divisible by n.

The final question introduces a new entire function that was not part of the traditional arsenal of examples in Math 220A.

5. (*The Weierstraß σ-function. Friday, January 15.*) This is a modified version of Conway VII.5, Problem 13.

Let ω_1, ω_2 be two non-zero complex numbers such that $\omega_2/\omega_1 \notin \mathbb{R}$. Let

$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

Solve the Weierstraß problem of finding entire functions with simple zeroes at the lattice points in Λ.

To this end, define the Weierstraß σ-function as the infinite product

$$\sigma(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{1}{2} \cdot \frac{z^2}{\lambda^2}\right) = z \prod_{\lambda \in \Lambda \setminus \{0\}} E_2\left(\frac{z}{\lambda}\right).$$

(i) Show that $\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3}$ converges.

Hint: Show that there exists a number $c > 0$ such that

$$|n\omega_1 + m\omega_2| \geq c(|n| + |m|),$$

for all real numbers n, m. Show that the number of integer solutions of $|n|+|m| = k$ is equal to $4k$. Conclude that

$$\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3} \leq 4c^{-3} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

(ii) Show that σ is an entire function with simple zeroes only at the points of Λ.

Remark: The σ function is important in the study of elliptic functions and the study of Riemann surfaces of genus 2.