Math 220B - Lecture 1

January 4, 2021

0 Logistics

(1) Zoom lectures — MWF 3 – 3:50 PM.

(2) Office Hour — W 4 – 5:30 PM

(3) PSets — due "Fridays," weekly

(4) Grades — 30% HWK
30% midterm
40% Final

(5) Midterm — take home, Feb 12

(6) Final — March 17, 3–6 PM

(7) Canvas / Gradescope / Website

math.ucsd.edu/~dopra/220w21.html

(8) Attendance

III Topics to be covered

Part I : Sequences / Series / Products

(1) infinite products of holomorphic functions

Weierstraß Problem

(2) sequences & series of meromorphic functions

Mittag-Leffler Problem

(3) sequences of hol functions, Montel families

Part II : Geometric aspects / Conformal maps

(4) Schwarz Lemma, automorphisms of $\Delta, \mathfrak{H}^+, \Delta^*, \ldots$

(5) Riemann mapping theorem

Part III : Further topics (if time)

(6) Runge's theorem

(7) Schwarz Reflection

(8) harmonic functions

(9) Hadamard factorization

(10) Little & Big Picard.

Some of these will only be covered in Math 220C.

2. Three Motivating Questions for Part I

Math 220A, Lecture 10 : $f \not\equiv 0$ entire has countably many zeroes that do not accumulate.

Weierstraß Problem

Given a sequence of distinct $\{a_n\}$, $a_n \to \infty$ and positive integers $\{m_n\}$, is there an entire function with zeroes only at $\{a_n\}$ with order exactly $\{m_n\}$. ?

Weierstraß$^+$ Problem

Given $\{a_n\}, \{m_n\}$ as above, $\{A_{nj}\}_{0 \le j < m_n}$ is there an entire function f with

$$f^{(j)}(a_n) = A_{nj} \quad \forall\ 0 \le j < m_n$$

Mittag - Leffler Problem

Take $\{a_n\}$ as above.

We can always find a meromorphic function f in $\mathbb{C}$ with poles only at a_n. e.g. take g solving Weierstraß at $\{a_n\}$ and set $f = 1/g$.

Mittag - Leffler asks if we can furthermore prescribe the Laurent principal parts.

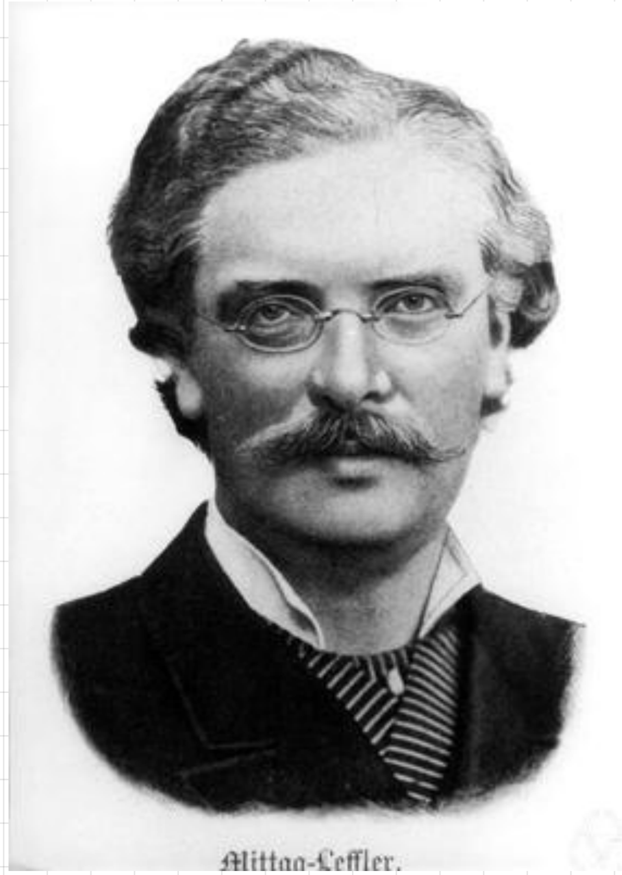
Given $\{a_n\}$ distinct, $a_n \longrightarrow \infty$, and polynomials $p_n\left(\frac{1}{z-a_n}\right)$ without constant terms, is there a meromorphic function in $\mathbb{C}$ with poles only at a_n and Laurent expansion

$$f = p_n\left(\frac{1}{z-a_n}\right) + \dots \quad \text{near } a_n.$$

Weierstraß – Poincaré Problem

Is any meromorphic function a quotient of two holomorphic functions?

Remark The three questions above can be asked & answered for all $\mathcal{U} \subseteq \mathbb{C}$ open & connected.



Karl Weierstraß

1815 – 1897

Gösta Mittag-Leffler

1846 – 1927

We will also illustrate general theory e.g.

$\boxed{a}$ factorization of sine, Γ - function

$\updownarrow$

Weierstraß problem.

$\boxed{b}$ elliptic functions - Weierstraß $\wp$ function

$\updownarrow$

Mittag - Leffler

Tools — sequences, series, products of

holomorphic & meromorphic functions.

Last quarter [I] sequences

of holomorphic functions.

[II] series

This quarter

[I] Weierstraß requires infinite products of holomorphic functions.

Intuitively, this makes sense. We could try to solve Weierstraß by setting $f(z) = \prod_{n=1}^{\infty} (z - a_n)$ but convergence is an issue

[II] Mittag - Leffler requires infinite sums of meromorphic functions.

$\boxed{3}$ Quick Review of the last lectures in Math 220A

Sequences $\{f_n\}$ holomorphic in $\mathcal{U} \subseteq \mathbb{C}$

Recall that the notion of convergence we considered was

local uniform convergence $\Longleftrightarrow$ convergence on compact subsets

$$f_n \overset{l.u.}{\rightrightarrows} f \qquad \Longleftrightarrow \qquad f_n \overset{c}{\rightrightarrows} f$$

Weierstraß Convergence Theorem

Let $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$. Then

$\boxed{1}$ f holomorphic

$\boxed{2}$ $f_n^{(k)} \overset{l.u.}{\rightrightarrows} f^{(k)}$

Series $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic . Assume

(*) $\forall\ K \subseteq \mathcal{U}$ compact $\exists\ M_n(K)$, $|f_n| \le M_n(K)$.

over K. & $\sum_{n=1}^{\infty} M_n(K) < \infty$.

M-test
$\Longrightarrow f = \sum_{n=1}^{\infty} f_n$ converges absolutely & uniformly on every K

Weierstraß
Thm
$\Longrightarrow f$ holomorphic & $f' = \sum_{n=1}^{\infty} f_n'$

This quarter

$\boxed{I}$ infinite products $\prod_{n=1}^{\infty} f_n(z)$ Weierstraß

$\boxed{II}$ series of meromorphic functions $\sum_{n=1}^{\infty} f_n(z)$

Mittag-Leffler

[4] Infinite Products

Main Question Given $f_k : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, how do we define $f(z) = \prod_{k=1}^{\infty} f_k(z)$? Furthermore,

[i] Is f holomorphic?

[ii] Is it true that $\text{Zero}(f) = \bigcup_k \text{Zero}(f_k)$?

Step back: Given $p_k \in \mathbb{C}$, how to define

$$P = \prod_{k=1}^{\infty} p_k \; ?$$

Wrong answer Form the partial products

$$P_n = \prod_{k=1}^{n} p_k \quad \text{and define} \quad P = \lim_{n \to \infty} P_n$$

Issues

[i] If $p_\ell = 0 \implies P = 0$ no matter what the other p_n's are. Thus one term would determine convergence of the product which is unfair.

[ii] We could have $P = 0$ even though $p_k \neq 0 \ \forall k$

e.g. $\prod_{k=1}^{\infty} \frac{1}{k} = 0$. Thus we have no control over the zeroes of a product of functions.

Question What kind of products will we consider?

Definition $\prod_{k=1}^{\infty} p_k = P$ converges iff $\exists M$ such that $\lim_{n\to\infty} \prod_{k=M}^{n} p_k$ exists and equals $\hat{P} \neq 0$. We then set

$$P = p_1 \cdots p_{M-1} \hat{P}$$

Remarks $\boxed{\text{i}}$ the value of $\hat{P}$ is independent of M (check)

$\boxed{\text{ii}}$ in the infinite products above only finitely many terms can be 0. ($\hat{P} \neq 0 \Rightarrow p_k \neq 0$ for $k \geq M$).

$\boxed{\text{iii}}$ With this definition we have control over the zeros.

Indeed

$$P = 0 \iff p_1 \cdots p_{M-1} \hat{P} = 0 \quad (\hat{P} \neq 0)$$

$$\iff p_1 = 0 \text{ or } \ldots \text{ or } p_{M-1} = 0$$

$$\iff \exists\, k \text{ with } p_k = 0.$$

Thus this behaves in the same fashion as finite products.

Math 220B — Lecture 2

January 6, 2021

Last time – Infinite products   Conway VII. 5

Given $p_k \in \mathbb{C}$, define $P = \prod_{k=1}^{\infty} p_k$ convergent product

if $\exists N$ with

$$\lim_{n\to\infty} \prod_{k=N}^{n} p_k = \hat{P} \neq 0 \quad \text{and set}$$

$$P = p_1 \cdots p_{N-1} \hat{P} = \text{value independent of } N.$$

Remarks $\boxed{i}$ $\exists$ finitely many zero terms

$\boxed{ii}$ $P = 0 \iff \exists\, k$ with $p_k = 0$

$\boxed{iii}$ Note

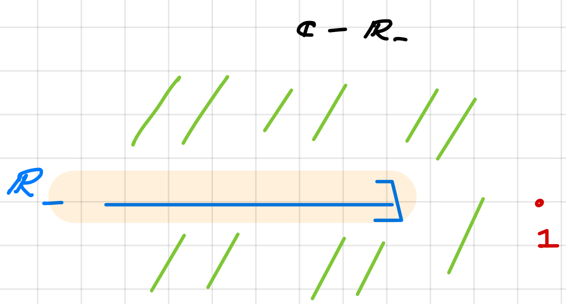
$$p_n = \frac{\prod_{k=N}^{n} p_k}{\prod_{k=N}^{n-1} p_k} \longrightarrow \frac{\hat{P}}{\hat{P}} = 1. \quad \text{as } n \to \infty.$$

Henceforth, we will assume $p_k = 1 + a_k$, $a_k \to 0$.

$$\prod_{k=1}^{\infty} (1 + a_k)$$

We seek to connect infinite products to infinite series.

Recall Principal branch of logarithm $z \neq 0$, $z \in \mathbb{C} \setminus \mathbb{R}_-$



$$\operatorname{Log}(z) = \log r + i\theta$$

$$\theta \in (-\pi, \pi)$$

$\operatorname{Log}(1+z)$. makes sense if z small since $1+z \notin \mathbb{R}_-$.

Lemma $\prod_{k=1}^{\infty} (1+a_k)$ converges $\iff$ $\exists\, N > 0$ such that $\sum_{k=N}^{\infty} \operatorname{Log}(1+a_k)$ converges

Proof Write

$$S_n = \sum_{k=N}^{n} \operatorname{Log}(1+a_k)$$

$$\Longrightarrow \qquad e^{S_n} = P_n.$$

$$P_n = \prod_{k=N}^{n} (1+a_k)$$

Proof "$\Leftarrow$": If $s_n \to s$, $p_n = e^{s_n} \to e^s = \hat{P} \neq 0$.

"$\Rightarrow$" Assume $p_n \to \hat{p}$. We wish to show $s_n \to s$.

Pick α such that $\hat{p} \notin \mathbb{R}_{\geq 0} e^{i\alpha}$. We use the branch Log_α

$$\mathrm{Log}_\alpha z = \log r + i\theta, \quad \theta \in (\alpha, \alpha + 2\pi)$$

$$e^{s_n} = p_n \implies s_n = \mathrm{Log}_\alpha p_n + 2\pi i \ell_n, \quad \ell_n \in \mathbb{Z}.$$

$$\begin{array}{c} e^{s_n} = p_n \\ \| \\ e^{\mathrm{Log}_\alpha p_n} \end{array}$$

We claim $\ell_n = \ell_{n-1}$ if $n \gg 0$ $\implies \exists\, \ell$, $\ell_n = \ell$.

$$\implies s_n = \mathrm{Log}_\alpha p_n + 2\pi i \ell_n \longrightarrow \mathrm{Log}_\alpha \hat{p} + 2\pi i \ell := s$$

To prove the claim, consider

$$\underbrace{s_n - s_{n-1}}_{\downarrow \atop 0} = \underbrace{\mathrm{Log}_\alpha p_n - \mathrm{Log}_\alpha p_{n-1}}_{\downarrow \atop 0} + 2\pi i (\ell_n - \ell_{n-1}) \quad \text{as } n \to \infty$$

Note $s_n - s_{n-1} = \mathrm{Log}(1 + a_n) \longrightarrow \mathrm{Log}\, 1 = 0$

$$\mathrm{Log}_\alpha p_n - \mathrm{Log}_\alpha p_{n-1} \longrightarrow \mathrm{Log}_\alpha \hat{p} - \mathrm{Log}_\alpha \hat{p} = 0$$

This shows
$$\left.\begin{array}{l} \ell_n - \ell_{n-1} \longrightarrow 0 \text{ as } n \to \infty \\ \ell_n - \ell_{n-1} \in \mathbb{Z} \end{array}\right\} \implies \ell_n = \ell_{n-1} \text{ if } n \gg 0.$$

Absolute convergence

Question How do we define absolutely convergent products $\prod_{k=1}^{\infty} p_k$

Wrong Answer: $\prod_{k=1}^{\infty} |p_k|$ converges

But then for $p_k = (-1)^k$, $\prod_{k=1}^{\infty} (-1)^k$ converges absolutely which is absurd.

Def $\prod_{k=1}^{\infty} (1+a_k)$ converges absolutely iff $\exists N$ such that

$$\sum_{k=N}^{\infty} \operatorname{Log}(1+a_k) \text{ converges absolutely.}$$

Lemma TFAE

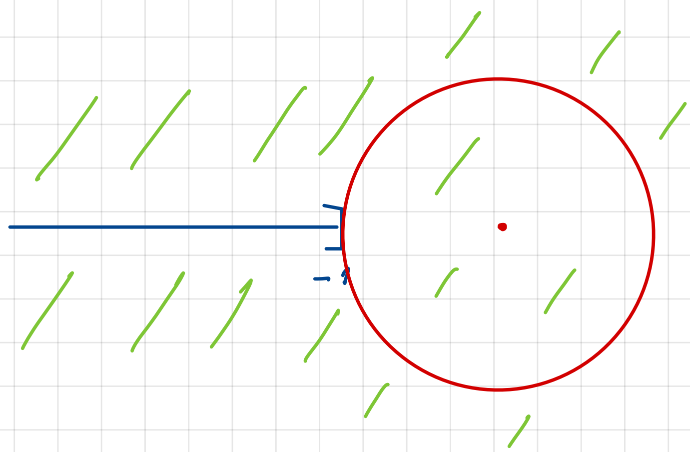
[i] $\prod_{k=1}^{\infty}(1+a_k)$ converges absolutely

[ii] $\sum_{k=1}^{\infty} a_k$ converges absolutely

[iii] $\prod_{k=1}^{\infty}(1+|a_k|)$ converges

Proof Consider Taylor expansion in $\Delta(0,1) \subseteq \mathbb{C} \setminus (-\infty,-1]$

$$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$



$$\Downarrow$$

$$\frac{\mathrm{Log}(1+z)}{z} = 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots$$

$\Rightarrow \lim_{z\to 0} \frac{\mathrm{Log}(1+z)}{z} = 1 \Rightarrow \exists\, \rho > 0$ such that if $|z| < \rho,\ z \neq 0.$

$$\frac{1}{2} \le \left| \frac{\mathrm{Log}(1+z)}{z} \right| \le \frac{3}{2}$$

Important inequality $\exists\, \rho$ s.t. if $|z| < \rho$

$$\frac{1}{2}|z| \le |\mathrm{Log}(1+z)| \le \frac{3}{2}|z|.$$

$\boxed{i} \iff \boxed{ii}$ By defn, $\prod_{k=1}^{\infty} (1+a_k)$ converges absolutely

$\iff \sum_{k=N}^{\infty} \operatorname{Log}(1+a_k)$ converges absolutely

$\iff \sum_{k=N}^{\infty} a_k$ converges absolutely (comparison test + important inequality)

Finally,

$\boxed{ii} \iff \sum_{k=N}^{\infty} |a_k|$ converges absolutely

$\iff \prod_{k=1}^{\infty} (1+|a_k|)$ converges absolutely by $\boxed{i} \iff \boxed{ii}$ for $\tilde{a}_k = |a_k|$

$\iff \prod_{k=1}^{\infty} (1+|a_k|)$ converges $\iff \boxed{iii}$

indeed, absolute convergence of the product is superfluous

$$\sum_{k=N}^{\infty} |\operatorname{Log}(1+|a_k|)| = \sum_{k=N}^{\infty} \operatorname{Log}(1+|a_k|)$$

Remark (Rearrangements).

Math 140A we learned that if $\sum_{k=1}^{\infty} b_k$ is absolutely convergent then $\forall\ \sigma: \mathbb{N} \longrightarrow \mathbb{N}$ bijection then $\sum_{k=1}^{\infty} b_{\sigma(k)}$ converges to the same sum.

The same happens for absolutely convergent products $\prod_{k=1}^{\infty} p_k$ can be rearranged, $b_k = \operatorname{Log}(1+a_k)$, $p_k = 1+a_k$.

2. Infinite Products of Holomorphic Functions

$f_k : U \longrightarrow \mathbb{C}$ holomorphic, $U \subseteq \mathbb{C}$

Assumption $\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly

Terminology $\sum_{k=1}^{\infty} f_k$ converges absolutely locally uniformly.

Define

$$(*) \quad F(z) = \prod_{k=1}^{\infty} (1 + f_k(z)).$$

Remark $(*)$ converges absolutely $\forall z \in U \Longrightarrow$ can rearrange the product.

Proposition Under the above **Assumption**

$\boxed{i}$ the partial products of $(*)$ converge locally uniformly.

$\boxed{ii}$ F is holomorphic

$\boxed{iii}$ $F(z_0) = 0 \iff \exists\, k$ with $1 + f_k(z_0) = 0$

Proof will be given next time.

Examples $\boxed{i}$ $z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$ defines an entire function with zeroes only at the integers. & nowhere else.

Indeed, apply the Proposition to $f_k(z) = \frac{z^2}{k^2}$.

$\boxed{ii}$ $\prod_{k=1}^{\infty} (1 + q^k z)$ is an entire function if $|q| < 1$ with zeroes only at $z = -q^{-k}$.

Apply the Proposition to $f_k(z) = q^k z$.

Math 220 B – Lecture 3

January 8, 2021

Last time Conway VII. 5.

$f_k : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $\mathcal{U} \subseteq \mathbb{C}$

Assumption $\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly

Then

(*) $$F(z) = \prod_{k=1}^{\infty} (1 + f_k(z)).$$

converges absolutely for all $z \in \mathcal{U}$.

Remark [i] By Cauchy's criterion (Math 1408).

$\forall K \subseteq \mathcal{U}$ compact $\forall \varepsilon > 0$ $\exists N_{K,\varepsilon}$ if $m > n > N$

$$\Longrightarrow \sum_{k=n}^{m} |f_k(z)| < \varepsilon \quad \text{for } z \in K$$

[ii] In practice, instead of Assumption above we might

check.

$$\sum_{k=n}^{m} \sup_{z \in K} |f_k(z)| < \varepsilon. \Longleftrightarrow \sum_{k=1}^{\infty} \sup_{K} |f_k| < \infty.$$

"normal convergence"!

This is simply the Weierstraß M-test, with $M_k(K) = \sup_K |f_k(z)|$

so $\boxed{ii} \Rightarrow \boxed{i}$.

Proposition Assume $\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly.

$\boxed{i}$ the partial products of $(*)$ converge locally uniformly to F

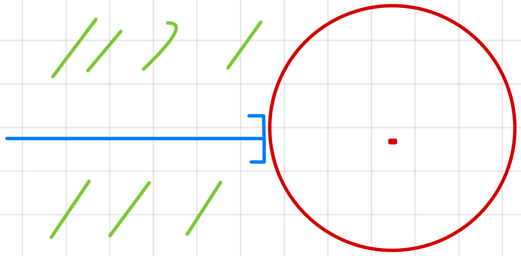
$\boxed{ii}$ F is holomorphic

$\boxed{iii}$ $F(z_0) = 0 \iff \exists\, k$ with $1 + f_k(z_0) = 0$

Furthermore,

$$\operatorname{ord}(F, z_0) = \sum_{k=1}^{\infty} \operatorname{ord}(1 + f_k, z_0).$$

Proof Recall from last time



$\log(1+z)$ is continuous in $\Delta(0,1)$

Important inequality: $\exists\, \rho < 1$ such that

$$|\log(1+z)| \le \frac{3}{2}|z| \qquad \text{if } |z| \le \rho$$

Proof of $\boxed{C}$ Let $K \subseteq \mathcal{U}$ compact. By Remark $\boxed{C}$

$\exists N$ such that if $k \geq N \Rightarrow |f_k(z)| < \rho$ for $z \in K$

$\Longrightarrow$ by important inequality

$$|\log(1+f_k(z))| \leq \frac{3}{2}|f_k(z)| \text{ for } z \in K, \; k \geq N.$$

Since $\sum_{k=N}^{\infty} |f_k|$ converges uniformly by assumption,

comparison test $\Rightarrow \sum_{k=N} \log(1+f_k(z))$ converges (absolutely). uniformly on K.

Write $G_n = \sum_{k=N}^{n} \log(1+f_k(z)) \underset{K}{\rightrightarrows} G$.

Note that G_n is continuous since $\log(1+w)$ is continuous

for $|w| < \rho$. Thus G is also continuous.

Since $G_n \underset{K}{\rightrightarrows} G$, by the claim $\boxed{a}$ below

$$e^{G_n} \underset{K}{\rightrightarrows} e^{G} \Rightarrow \prod_{k=N}^{n} (1+f_k(z)) \underset{K}{\rightrightarrows} e^{G}.$$

$$\Longrightarrow \prod_{k=1}^{n} (1+f_k(z)) \underset{K}{\rightrightarrows} e^{G}(1+f_1(z)) \cdots (1+f_{N-1}(z)). \quad (+)$$

(Uniform convergence after multiplication uses claim $\boxed{b}$ below)

Thus

$$F = e^{G}(1+f_1)\dots(1+f_{N-1}) \text{ in } K.$$ & the convergence is

uniform, in K, completing the proof.

$\boxed{ii}$ F holomorphic by $\boxed{i}$ (local uniform convergence)

& Weierstraß convergence theorem.

$\boxed{iii}$ Recall from last time that

$$F(z_0) = 0 \iff \exists\, k \text{ with } 1 + f_k(z_0) = 0.$$

To prove the assertion about orders, consider $(+)$

in $K = \bar{\Delta}$, Δ neighb. of z_0

$$F(z_0) = \underbrace{e^{G(z_0)}}_{\neq 0}(1+f_1(z_0)) \dots (1+f_{N-1}(z_0))$$

$$\Rightarrow \operatorname{ord}(F, z_0) = \sum_{k=1}^{N-1} \operatorname{ord}(1+f_k, z_0)$$

$$= \sum_{k=1}^{\infty} \operatorname{ord}(1+f_k, z_0).$$

using that $1+f_k \neq 0$ for $k \geq N$ (because $|f_k| < p < 1$ in K.)

Remark Analyzing the proof, we see the argument only requires

$$\sum_k \left| \operatorname{Log}(1+f_k) \right| \text{ converges locally uniformly.}$$

The following standard claims were used in the proof:

Claim $\boxed{a}$ Let u_n be continuous, $u_n \underset{K}{\rightrightarrows} u$. Then $e^{u_n} \underset{K}{\rightrightarrows} e^u$.

$\boxed{b}$ If $u_n \underset{K}{\rightrightarrows} u$, $v_n \underset{K}{\rightrightarrows} v$ (u_n, v_n continuous). Then

$$u_n v_n \underset{K}{\rightrightarrows} uv.$$

Proof $\boxed{a}$ Suffices to show $\sup_K |e^{u_n} - e^u| \longrightarrow 0$.

Compute

$$\sup_K |e^{u_n} - e^{u}| = \sup_K |e^{u}| \cdot |e^{u_n - u} - 1|$$

$$\leq \sup_K |e^{u}| \cdot \sup_K |e^{u_n - u} - 1|$$

$$= M \cdot \sup_K |e^{u_n - u} - 1| < \varepsilon M \quad \text{for } n \geq N.$$

Why? By continuity, $\exists\, \delta > 0$: $|e^{w} - 1| < \varepsilon$ if $|w| < \delta$.

Since $u_n \underset{K}{\rightrightarrows} u \Rightarrow \exists\, N$ with $|u_n - u| < \delta$ on K

$$\Rightarrow |e^{u_n - u} - 1| < \varepsilon$$

Proof of [b] We show $\sup_K |u_n v_n - uv| \longrightarrow 0$.

Indeed by triangle inequality

$$\sup_K |u_n v_n - uv| \leq \sup_K |(u_n - u)(v_n - v)| + \sup_K |u(v_n - v)|$$

$$+ \sup_K |v(u_n - u)|$$

$$\leq \sup_K |u_n - u| \cdot \sup_K |v_n - v| + \sup_K |u| \cdot \sup_K |v_n - v| + \sup_K |v| \cdot \sup_K |u_n - u|$$

$\longrightarrow 0$ since $\sup_K |u_n - u| \longrightarrow 0$ and $\sup_K |v_n - v| \longrightarrow 0$.

Logarithmic derivative

Taking derivatives of products is messy. It is easier to take logarithmic derivatives

h holomorphic $\Rightarrow$ $\dfrac{h'}{h}$ = logarithmic derivative

= holomorphic away from Zero(h)

Addition formula

$$h = fg \implies \frac{h'}{h} = \frac{f'}{f} + \frac{g'}{g}.$$

$$h' = f'g + fg' \Rightarrow \frac{h'}{h} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

Inductively $\quad h = f_1 \cdots f_s \Rightarrow \dfrac{h'}{h} = \dfrac{f_1'}{f_1} + \cdots + \dfrac{f_s'}{f_s}$

We prove the same for infinite products.

[i] $g_k : \mathcal{U} \to \mathbb{C}$ holomorphic, $f_k = 1 + g_k$.

[ii] $\sum_{k=1}^{\infty} |g_k|$ converges locally uniformly in $\mathcal{U}$

Proposition $h = \prod_{k=1}^{\infty} f_k$. Away from $\mathrm{Zero}(h)$:

$$\frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

The RHS converges locally uniformly on $\mathcal{U} \setminus \mathrm{Zero}(h)$.

Proof Recall from (+) in the previous Proof that

for $K = \bar{\Delta} \subseteq \mathcal{U}$, Δ neighborhood of an arbitrary point

$\exists\ N$ with

$$F_n = \prod_{k=N}^{n} f_k \xrightarrow[\Delta]{l.u.} F = e^G \quad \text{on } \Delta$$

$$G = \sum_{k=N}^{\infty} \operatorname{Log}(1 + g_k)$$

Note $h = f_1 \cdots f_{N-1} \prod_{k=N}^{\infty} f_k = f_1 \cdots f_{N-1}\, e^G.$

finite case

$$\Rightarrow \frac{h'}{h} = \frac{f_1'}{f_1} + \cdots + \frac{f_{N-1}'}{f_{N-1}} + \frac{(e^G)'}{e^G}.$$

$$\Rightarrow \frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

We need to show $\displaystyle\sum_{k=N}^{n} \frac{f_k'}{f_k} \overset{\text{l.u.}}{\underset{\Delta}{\rightrightarrows}} \frac{(e^G)'}{e^G}.$

To see this, by the finite case again

$$\sum_{k=N}^{n} \frac{f_k'}{f_k} = \frac{F_n'}{F_n}. \quad \text{so we show} \quad \frac{F_n'}{F_n} \overset{\text{l.u.}}{\underset{\Delta}{\rightrightarrows}} \frac{(e^G)'}{e^G}.$$

Note $F_n \overset{\text{l.u.}}{\underset{\Delta}{\rightrightarrows}} e^G \neq 0$ so by Weierstraß $F_n' \overset{\text{l.u.}}{\underset{\Delta}{\rightrightarrows}} (e^G)'.$

We finish using Claim $\boxed{b}$ above (products) &

Claim $\boxed{c}$ below.

Claim $\boxed{C}$ If $u_n \underset{K}{\rightrightarrows} u$, u_n continuous, u nowhere zero

$$\Rightarrow \frac{1}{u_n} \underset{K}{\rightrightarrows} \frac{1}{u}.$$

Proof We show $\sup_K \left| \frac{1}{u_n} - \frac{1}{u} \right| \longrightarrow 0.$ for $n \to \infty$.

Compute $$\sup_K \left| \frac{1}{u_n} - \frac{1}{u} \right| = \sup_K \frac{|u_n - u|}{|u| \cdot |u_n|}$$

$$\leq \sup_K |u_n - u| \cdot \frac{1}{\inf_K |u|} \cdot \frac{1}{\inf_K |u_n|} \longrightarrow 0$$

Note $$\inf_K |u_n| \geq \inf_K |u| - \sup_K |u - u_n| \longrightarrow \inf_K |u|. > 0$$

where we used $\sup_K |u_n - u| \longrightarrow 0.$

Math 220 B — Lecture 4

January 11, 2021

0. Last time

$f_k : \mathcal{U} \to \mathbb{C}$ holomorphic

$\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly

(1) $$h(z) = \prod_{k=1}^{\infty} (1 + f_k(z))$$ holomorphic

(2) $$\frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{1+f_k}$$

The series on RHS converges absolutely locally uniformly on $\mathcal{U} \setminus \text{Zero}(h)$.

Remark

If $\sum_{k=1}^{\infty} \left| \operatorname{Log}(1+f_k) \right|$ converges locally uniformly

the same conclusions hold.

Today $\boxed{i}$ factorization of sine Conway VII.6.

Euler, 1734

"De Summis Serierum Reciprocarum"

$\boxed{ii}$ Γ - function Conway VII. 7

Euler, Bernoulli, Gauss, Legendre, Weierstraß

These two topics are naturally connected

1. Factorization of sine (Euler, 1734)

Theorem

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

Idea. Both sides have the same zeroes (with multiplicity)

Question When do two entire functions have exactly the same zeroes?

Lemma If $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ entire, with the same zeroes and multiplicities. Then $f = g e^h$ for some $h : \mathbb{C} \longrightarrow \mathbb{C}$ entire.

Proof Let $H = \frac{f}{g}$. $\Rightarrow$ H entire without zeroes by hypothesis. We show $H = e^{h}$.

The function $\frac{H'}{H}$ is entire so it admits primitive h.

$\Rightarrow \frac{H'}{H} = h'$. Then

$$(H e^{-h})' = H' e^{-h} - H e^{-h} h' = e^{-h}(H' - H h') = 0$$

$$\Rightarrow H e^{-h} = c \neq 0 \Rightarrow H = c e^{h} = e^{\log c + h}.$$

Remark The same holds for $f, g : \mathcal{U} \longrightarrow \mathbb{C}$, $\mathcal{U}$ simply connected.

Proof of the sine factorization

(1) convergence:

Note that $\sum_{k=1}^{\infty} \left|\frac{z^2}{k^2}\right|$ converges locally uniformly $\Rightarrow \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$ converges.

(2) location of zeroes:

Both sides $\sin \pi z$ & $\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$ have simple zeroes at the integers & nowhere else.

(3) completing the proof

By the Lemma, $\exists$ h entire

$$\sin \pi z = e^{h} \, \pi z \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$$

We show $h \equiv 0$. Compute logarithmic derivative

$$\frac{\pi \cos \pi z}{\sin \pi z} = \frac{(e^h)'}{e^h} + \frac{\pi}{\pi z} + \sum_{k=1}^{\infty} \frac{-\frac{2z}{k^2}}{1-\frac{z^2}{k^2}}$$

$$\pi \cot \pi z = h' + \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}$$

Recall Math 220, Hwk 6:

6. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2}\, dz$$

via the residue theorem. Making $n \to \infty$, show that

$$\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.$$

Thus $h' \equiv 0 \implies h$ constant. We show $h \equiv 0$.

From

$$\frac{\sin \pi z}{\pi z} = e^{h} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \quad \text{make } z \to 0$$

$$\downarrow$$

$$1 = e^{h(0)} \cdot 1 \implies h(0) = 0 \implies h \equiv 0.$$

This completes the proof.

Remark $\quad \sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$

$\boxed{\text{i}}$ $\quad z = \frac{1}{2}$

$$1 = \frac{\pi}{2} \prod_{k=1}^{\infty} \left(1 - \frac{1}{4k^2}\right) = \frac{\pi}{2} \cdot \prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{(2k)(2k)}$$

$$\Rightarrow \frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k)(2k)}{(2k-1)(2k+1)}$$

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$ Wallis, 1655

$\boxed{\text{ii}}$ $\quad z = i$

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2}\right) = \frac{\sin \pi i}{\pi i} = \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

$\boxed{\text{iii}}$ $\quad \cos \pi z = \frac{\sin 2\pi z}{2 \sin \pi z} = \frac{2\pi z \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{k^2}\right)}{2\pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)}$

Splitting into k even/odd:

$$\cos \pi z = \prod_{l=1}^{\infty} \left(1 - \frac{4z^2}{(2l-1)^2}\right).$$

2. Γ - function — probability, statistics, combinatorics, ...

"The product $1.2.\ldots\cdot x$ is the function that must be introduced in analysis" (Gauss to Bessel, 1811)

$$\Pi x = "1.2.3\ldots\cdot x" = \Gamma(x+1)$$

"The theory of analytic factorials does not seem to have the importance some mathematicians used to attribute to it"

Weierstraß 1854

Definition $G(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$.

Remark The convergence (absolutely & locally uniformly) of the product is HWK 1, #4. There, you show

$$\sum_{n=1}^{\infty}\left|\log\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right]\right|$$ converges locally uniformly.

Properties of the function G

$\boxed{i}$ $$G(z)\,G(-z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}$$

$$= \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right)e^{-z/n}e^{z/n}$$

$$= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi z} \quad \text{by Euler.}$$

$\boxed{ii}$ $G(z-1) = z\,G(z)\,e^{\gamma}$ where γ is Euler constant.

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right).$$

We will prove this next time.

Definition

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\cdot\frac{1}{G(z)}$$

Remark G has zeroes at $-1,-2,\dots -n,\dots$

$\Rightarrow$ Γ meromorphic in $\mathbb{C}$ with zeroes at $-1,-2,\dots,-n,\dots$

Math 220B – Lecture 5

January 13, 2021

The Γ-function (Conway VII. 7)

Definition $\quad G(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}.$

Properties of the function G

$\boxed{\text{I}}$ $\quad G(z)\, G(-z) = \dfrac{\sin \pi z}{\pi z}$ (Euler)

$\boxed{\text{II}}$ $\quad G(z-1) = z\, G(z)\, e^{\gamma}$ for some constant γ.

We inspect zeroes of both sides.

Zeroes G: $-1, -2, \dots, -n, \dots$

$$\left.\begin{array}{l} zG(z):\ 0, -1, -2, \dots, -n, \dots \\ \\ G(z-1):\ 0, -1, -2, \dots, -n \end{array}\right\} \text{ have the same zeroes}$$

$\Longrightarrow \quad G(z-1) = z\, G(z)\, e^{\gamma(z)}$ for some function $\gamma(z)$.

We need $\gamma(z) = \text{constant}$. We verify $\gamma' = 0$.

Take logarithmic derivatives

$$\frac{G'(z-1)}{G(z-1)} = \frac{1}{z} + \frac{G'(z)}{G(z)} + \gamma' \quad (*)$$

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Since logarithmic derivative turns products into sums

$$\frac{G'(z)}{G(z)} = \sum_{n=1}^{\infty} \left(\frac{\frac{1}{n}}{1 + \frac{z}{n}} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right)$$

$$\Longrightarrow \frac{G'(z-1)}{G(z-1)} = \sum_{n=1}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n} \right) = \left(\frac{1}{z} - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n+1} \right)$$

$n \geq 2 \qquad n \longrightarrow n+1$

$$= \left(\frac{1}{z} - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)}$$

$$= \left(\frac{1}{z} - \cancel{1} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \cancel{1}$$

$$= \frac{1}{z} + \frac{G'(z)}{G(z)} \quad \overset{(*)}{\Rightarrow} \gamma'(z) = 0 \Rightarrow \gamma(z) = \gamma. = \text{constant}$$

$\boxed{iii}$ The above constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log(n+1)\right) \quad \text{(Euler constant)}$$

Indeed, $G(0) = 1$ by definition of the function G.

By $\boxed{ii} \Longrightarrow G(z-1) = z\,G(z)\,e^{\gamma} \overset{z=1}{\Longrightarrow} G(1) = e^{-\gamma}$.

Using the definition

$$G(1) = \prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)e^{-1/k} =$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-1/k} =$$

$$= \lim_{n\to\infty} \frac{n+1}{\cancel{n}}\cdot\frac{\cancel{n}}{\cancel{n-1}}\cdot\cancel{\ }\cdots\cancel{\ }\frac{\cancel{2}}{1}\cdot e^{-1-1/2-\dots-1/n}$$

$$= \lim_{n\to\infty} e^{-\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log(n+1)\right)} = e^{-\gamma}.$$

$$\Longrightarrow \gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log(n+1)\right)$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right)$$

Definition $\Gamma(z) = \dfrac{e^{-\gamma z}}{z} \cdot \dfrac{1}{G(z)} = \Gamma$ - function

Properties of Γ

[i] $\Gamma(1) = \dfrac{e^{-\gamma}}{G(1)} = 1$ using $G(1) = e^{-\gamma}$ from above.

[ii] $\Gamma(z+1) = z\,\Gamma(z)$ $\rightsquigarrow$ "Γ behaves like a factorial"

In particular, by induction

$$\Gamma(n) = (n-1)! \qquad \forall\, n > 0,\ n \in \mathbb{Z}.$$

This follows by direct computation

$$\Gamma(z+1) = \frac{e^{-\gamma z-\gamma}}{(z+1)} \cdot \frac{1}{G(z+1)} \overset{???}{=} \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)} \cdot z = z\,\Gamma(z)$$

$\iff$ $G(z) = (z+1)\,G(z+1)\,e^{\gamma}$ which is true.

(see above)

$\boxed{iii}$ $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$. In particular $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

To prove this, we use $\boxed{ii}$ + Euler:

$$\Gamma(z)\,\Gamma(1-z) \overset{\boxed{ii}}{=} \Gamma(z)\,(-z)\,\Gamma(-z) =$$

definition $\longrightarrow$
$$= \frac{e^{-\gamma z}}{z\,G(z)} \cdot \cancel{(-z)} \cdot \frac{e^{\gamma z}}{\cancel{(-z)}\,G(-z)}$$

$$= \frac{1}{z\,G(z)\,G(-z)} = \frac{\pi}{\sin \pi z} \quad \text{(see above)}$$

$\boxed{iv}$ $\Gamma(z) = \lim\limits_{n\to\infty} \dfrac{n!}{z\,(z+1)\dots(z+n)}\, n^{z}$ (Gauss' definition)

We use the definition

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)} =$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z}}{z} \cdot \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} e^{z/k}$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z}}{z} \cdot \prod_{k=1}^{n} \frac{k}{z+k} \cdot e^{z\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n!}{z\,(z+1)\dots(z+n)} \cdot e^{z\underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n-\gamma\right)}_{\to 0}} \cdot n^{z}$$

$$= \lim_{n\to\infty} \frac{n!}{z\,(z+1)\dots(z+n)} \cdot n^{z}.$$

Exercise (Conway VII. 7. 3)

Legendre duplication formula.

Use Gauss' definition to check

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$$

Residues

Note that $\Gamma(z) = \dfrac{e^{-\gamma z}}{z} \cdot \dfrac{1}{G(z)}$ is meromorphic with poles at $0, -1, -2, \dots$ since G has zeroes at $-1, -2, \dots$

What are the residues?

$$\mathrm{Res}(\Gamma, -n) = \lim_{z \to -n} (z+n)\,\Gamma(z) =$$

$\boxed{ii}$

$$= \lim_{z \to -n} \cancel{(z+n)} \frac{\Gamma(z+n+1)}{z\,(z+1)\cdots\cancel{(z+n)}}$$

$$= \lim_{z \to -n} \frac{\Gamma(z+n+1)}{z\,(z+1)\cdots(z+n-1)}$$

$$= \frac{\Gamma(1)}{(-n)\cdots(-1)} = \frac{1}{(-1)^n n!} = \frac{(-1)^n}{n!}$$

Remark It can be shown

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \quad \text{for } \operatorname{Re} z > 0.$$

Mellin transform of e^{-t}.

Step 1 Convergence of RHS. Fix z, $\operatorname{Re} z > 0$.

$$\left| \int_0^1 e^{-t}\, t^{z-1}\, dt \right| \le \int_0^1 |e^{-t}\, t^{z-1}|\, dt$$

since $|e^{-t}| \le 1$

$$\le \int_0^1 t^{\operatorname{Re} z - 1}\, dt$$

$$= \frac{t^{\operatorname{Re} z}}{\operatorname{Re} z} \Big|_{t=0}^{t=1} = \frac{1}{\operatorname{Re} z} \quad \text{using } \operatorname{Re} z > 0.$$

Pick A with $|t^{z-1}| \le e^{t/2}$ when $|t| > A$

$$\left| \int_A^\infty e^{-t}\, t^{z-1}\, dt \right| \le \int_A^\infty |e^{-t}\, t^{z-1}|\, dt \le \int_A^\infty e^{-t}\, e^{t/2}\, dt$$

$$= \int_A^\infty e^{-t/2}\, dt$$

$$= 2e^{-A/2} < \infty.$$

$\int_1^A e^{-t}\, t^{z-1}\, dt < \infty$ by continuity of $e^{-t}\, t^{z-1}$ in t.

Step 2 Using integration by parts, one easily shows

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = \frac{n^z \quad n!}{z\,(z+1)\cdots(z+n)}$$

Exercise – check the details.

Step 3 Make $n\to\infty$. From real analysis

$$\left(1-\frac{t}{n}\right)^n \longrightarrow e^{-t} \quad \text{as } n\to\infty.$$ This will also be explained below. We will argue that

(1) $$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt \xrightarrow{n\to\infty} \int_0^\infty e^{-t}\, t^{z-1}\,dt$$

$\Vert$ Step $\boxed{ii}$

$$\frac{n^z \quad n!}{z\,(z+1)\cdots(z+n)} \xrightarrow{n\to\infty} \Gamma(z) \quad \text{by Gauss' formula}$$

This shows $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\,dt.$

Rigorous justification of convergence in (1)

Claim $\quad 0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \le \frac{e^{-t}\, t^2}{n}$ if $0 \le t \le n$.

Assuming the claim, we prove Step 3. Compute

$$\int_0^\infty e^{-t}\, t^{z-1}\, dt - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt =$$

$$= \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{z-1}\, dt + \int_n^\infty e^{-t}\, t^{z-1}\, dt \longrightarrow 0$$

We claim both terms converge to 0 as $n \to \infty$.

term II. $\quad \int_n^\infty e^{-t} t^{z-1}\, dt \longrightarrow 0$ as $n \longrightarrow \infty$ because

$\int_0^\infty e^{-t}\, t^{z-1}\, dt$ converges by Step 1.

term I

$$\left| \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n \right) t^{z-1}\, dt \right| \overset{\text{claim}}{\le}$$

$$\le \int_0^n \frac{1}{n}\, t^2 e^{-t} \cdot t^{z-1}\, dt < \int_0^\infty \frac{1}{n} \cdot t^2 e^{-t}\, t^{z+1}\, dt$$

$$= \frac{1}{n} \underbrace{\int_0^\infty e^{-t}\, t^{z+1}\, dt}_{\text{converges by step 1.}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Proof of claim (only in the notes)

(a) first inequality.

Use $1 - y \le e^{-y}$ for $y \ge 0$.

Take

$$y = \frac{t}{n} \implies 1 - \frac{t}{n} \le e^{-t/n} \implies \left(1 - \frac{t}{n}\right)^n \le e^{-t}.$$

To see $1 - y \le e^{-y}$, let $f(y) = e^{-y} - (1-y)$,

$f(0) = 0$, $f' = -e^{-y} + 1 > 0 \implies f \nearrow \implies f(y) \ge f(0) = 0$

(b) second inequality.

The inequality to prove is

$$e^{-t} - \left(1-\frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n} \iff$$

$$\iff 1 - e^t\left(1-\frac{t}{n}\right)^n \leq \frac{t^2}{n}. \quad (*)$$

Use $e^y \geq 1+y$ for $y \geq 0$ proven just as above. Take $y = \frac{t}{n}$

Since $e^t = \left(e^{t/n}\right)^n \geq \left(1+\frac{t}{n}\right)^n$, to show $(*)$ we show

$$1 - \left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n \leq \frac{t^2}{n} \iff$$

$$\iff \left(1-\frac{t^2}{n}\right) \leq \left(1-\frac{t^2}{n^2}\right)^n. \quad \text{This is true.}$$

Indeed, use $(1-y)^n \geq 1-ny$ for $y = \frac{t^2}{n^2}$.

The last inequality can be proved by induction on n.

Math 220B — Lecture 6

January 15, 2021

1. The Weierstraß Problem

Conway VII.5.

Given (i) $\{a_n\}$ distinct, $a_n \longrightarrow \infty$.

(ii) $\{m_n\}$ positive integers

Find entire functions f with zeroes only at a_n of order m_n.

Remark This also makes sense for arbitrary regions $\mathcal{U} \subseteq \mathbb{C}$

Main Theorem

The Weierstraß problem is always solvable in $\mathbb{C}$.

Henceforth, $\{a_n\}$ will be an infinite sequence. The finite case is easy.

Corollary Every meromorphic function in $\mathbb{C}$ is quotient of two entire functions.

Proof Let h be meromorphic. Let P be the collection of poles of h listed with multiplicity. Let g be the solution to the Weierstraß problem for P.

(The set P has no limit point in $\mathbb{C}$. By Remark [11] the hypothesis of Weierstraß is satisfied.) Then f is entire. & $h = \frac{f}{g}$.

Remarks [i] Any two solutions f_1 & f_2

$$f_1 = e^{h} f_2 \,, \quad h \text{ entire.}$$

[ii] If $\{a_n\}$ has no limit point in $\mathbb{C}$ then $a_n \longrightarrow \infty$.

Indeed, if not, $\exists\, r > 0$ such that $\forall\, N \;\exists\, n \geq N$, $|a_n| \leq r$.

This means $\exists$ subsequence of $\{a_n\}$ bounded by r. Since $\bar{\Delta}(0,r)$ compact, this will have a convergent subsequence, with limit $a \in \mathbb{C}$.

[iii] Repetitions & zero terms.

We will agree from now on that $\{a_n\}$ may contain repetitions. That is, by relabelling we can repeat each zero as many times as their multiplicity.

We assume $a_n \neq 0 \;\forall n$. If 0 is a zero for f, we will add it via multiplication by z^m at the end.

2. Solution to the Weierstraß Problem

Naive Attempt. We could try $\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)$

Issue: Convergence!

Idea Try $f(z) = \prod_{n=1}^{\infty} f_n(z)$ where

f_n has zero at a_n. e.g. $f_n(z) = \left(1-\frac{z}{a_n}\right)e^{h_n}$

Hope $f(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{h_n}$ converges.

For example,

$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)$ does not converge

$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} = G(z)$ does converge.

Weierstraß elementary / primary factors

Define

$$E_p(z) = \begin{cases} 1-z & \text{if } p=0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) & \text{if } p>0. \end{cases}$$

$\Rightarrow$ E_p is entire.

Remark Zero of $E_p(z)$ is at $z=1$.

$\Rightarrow$ $E_p\left(\frac{z}{a}\right)$ has a simple zero at $z=a$.

We look for an answer of the form

zeroes of f are at a_n.

$$(*) \quad f(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) \quad \text{for suitable } p_n \geq 0$$

Issue: Can we pick p_n such that $(*)$ converges absolutely & locally uniformly.

Recall: $\sum_{n=1}^{\infty} |f_n|$ converges locally uniformly $\Rightarrow$ $\prod_{n=1}^{\infty}(1+f_n)$ converges absolutely locally uniformly.

We wish to use this for $f_n = E_{p_n}\left(\frac{z}{a_n}\right) - 1$.

Growth of the elementary factors

Lemma $|1 - E_p(z)| \le |z|^{p+1}$ if $|z| \le 1$.

Proof The proof will be given next time.

Lemma Given $a_n \longrightarrow \infty$, $a_n \neq 0$, $\exists$ p_n natural numbers (not unique) such that

$$\forall r > 0 \implies \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty.$$

Proof

For instance, take $p_n = n-1$. Let $r > 0$.

Since $a_n \longrightarrow \infty$, $\exists$ N such that $|a_n| \ge \frac{r}{2}$ if $n \ge N$.

$$\implies \frac{r}{|a_n|} \le \frac{1}{2} \implies \left(\frac{r}{|a_n|} \right)^n \le \frac{1}{2^n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \implies \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^n < \infty$. by comparison test.

Weierstraß Factorization

Thm Let $a_n \to \infty$, $a_n \neq 0$. Pick p_n as in the previous Lemma:

$$\forall r > 0 \implies \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty .$$

Then $\displaystyle\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ converges absolutely & locally uniformly

to an entire function with zeroes at a_n and no other zeroes.

Proof Let $f_n = E_{p_n}\left(\frac{z}{a_n}\right) - 1$. Pick K compact, $K \subseteq \Delta(0,r)$.

for some r. We will argue that $\displaystyle\prod_{n=1}^{\infty}(1+f_n)$ converges locally uniformly.

It suffices to show $\displaystyle\sum_{n=1}^{\infty} |f_n|$ converges uniformly on $\Delta(0,r)$.

Note for $\Delta(0,r)$:

$$|f_n(z)| = \left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right| \overset{1^{st}\text{ Lemma}}{\leq} \left|\frac{z}{a_n}\right|^{p_n+1} \leq \left(\frac{r}{|a_n|}\right)^{p_n+1}.$$

This requires $\left|\frac{z}{a_n}\right| \leq \frac{r}{|a_n|} \leq 1$ which is true for $n \geq N$ since $a_n \to \infty$.

Since $\displaystyle\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \implies$ Weierstraß M-test $\displaystyle\sum_{n=1}^{\infty}|f_n|$ converges

uniformly in $\Delta(0,r)$, as needed

$\longrightarrow$ $\prod_{n=1} E_{p_n}\left(\frac{z}{a_n}\right)$ converges absolutely & locally uniformly

The statement about zeroes follows from Lecture 3 & the fact that $E_{p_n}\left(\frac{z}{a_n}\right)$ vanishes only at $z = a_n$.

Corollary Any (not identically 0) entire function can be written as

$$f(z) = z^m e^{h} \prod_n E_{p_n}\left(\frac{z}{a_n}\right), \quad h = \text{entire}.$$

for a non-unique choice of p_n & h.

Remark For the same function f, several p_n's may work.

Changing p_n into $\tilde{p}_n$ can be absorbed in the exponential.

Proof WLOG we may assume $f(0) \neq 0$. Else if $\operatorname{ord}(f,0) = m$ we add the factor z^m.

Let $\{a_n\}$ be the zeroes of f listed with multiplicity. Both f and $\prod_n E_{p_n}\left(\frac{z}{a_n}\right)$ solve the Weierstraß problem. Apply Remark $\boxed{L}$ to conclude.

Remark Weierstraß' theorem allows us to define functions which were not even thinked at before.

Poincaré: "Weierstraß' most important contribution to the theory of complex variables is the discovery of primary factors."

Example

$\boxed{i}$ $$Q(z) = \prod_{k=1}^{\infty} (1+q^k z) = \prod_{k=1}^{\infty} E_0(-q^k z)$$

Note $p_k = 0$ & $\sum_{k=1}^{\infty} (r q^k)^{p_k+1} < \infty$. so the hypothesis of Weierstraß factorization holds.

$\boxed{ii}$ $$G(z) = \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}} = \prod_{k=1}^{\infty} E_1\left(-\frac{z}{k}\right).$$

Note $p_k = 1$ & $\sum \left(\frac{r}{k}\right)^{p_k+1} < \infty$.

$\boxed{iii}$ $$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1-\frac{z^2}{k^2}\right)$$

$$= \pi z \prod_{k=1}^{\infty} \left(1-\frac{z}{k}\right) e^{z/k} \left(1+\frac{z}{k}\right) e^{-z/k}$$

$$= \pi z \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} E_1\left(\frac{z}{k}\right)$$

$\boxed{iv}$ Do we get any new examples we didn't know?

Yes. See hwk for the Weierstraß σ- function.

Math 220B — Lecture 7

January 20, 2021

Last time

(i) We defined the elementary/primary factors

$$E_p(z) = \begin{cases} 1 - z & , p = 0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) & , p > 0 \end{cases}$$

(ii) We saw that given $a_n \longrightarrow \infty$, $a_n \neq 0$,

$$f(z) = z^m e^h \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

are entire with zeroes at a_n.

(iii) The p_n's are chosen so that

$$\forall\, r > 0 \ , \quad \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$$

Remark [i] Convergence requires the estimate

$$|1 - E_p(z)| \leq |z|^{p+1} \quad \text{if } |z| \leq 1. \quad \text{(next)}$$

[ii] Analogy: The factorization.

$$f(z) = z^m e^h \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

is reminiscent of the factorization of integers into primes.

$$\text{primes} \longleftrightarrow E_p$$

$$\text{units} \longleftrightarrow e^h$$

Difference. Not canonical / uniqueness of p_n's.

We can however ask questions with arithmetic flavor.

Wedderburn: Can we write $1 = Af + Bg$ when f, g have no common zeroes?

Remarks We have freedom in the choice of p_n.

Question Is there a canonical choice?

Assume $\exists\, h \in \mathbb{Z}_{\geq 0}$ with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} < \infty$.

If such h exists, pick the smallest one. This is called genus of the canonical product $\displaystyle\prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right)$

Example

$\boxed{\text{I}}$ $$Q(z) = \prod_{k=1}^{\infty}(1+q^k z) = \prod_{k=1}^{\infty} E_0(-q^k z)$$

genus 0

$\boxed{\text{II}}$ $$G(z) = \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k} = \prod_{k=1}^{\infty} E_1\left(-\frac{z}{k}\right)$$

genus 1

$\boxed{\text{III}}$ $$\sigma = z \prod_{\lambda \in \Lambda \setminus \{0\}} E_2\left(\frac{z}{\lambda}\right)$$ genus 2. (HWK)

Remark

The genus controls the growth of zeroes via the expression $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}}$.

Remarkably, genus controls the growth of entire functions (Hadamard factorization theorem). This will be covered in Math 220C.

Proof of the estimate

$$|1 - E_p(z)| \leq |z|^{p+1} \quad \text{for} \quad |z| \leq 1.$$

where $E_p(z) = (1-z)\, e^{u}$, $\quad u = z + \frac{z^2}{2} + \dots + \frac{z^p}{p}$

Write $E_p(z) = \sum_{k=0}^{\infty} a_k z^k$.

By definition $E_p(0) = 1 \implies a_0 = 1$.

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k.$$

Claim

[i] $a_1 = a_2 = \dots = a_k = 0$

[ii] a_k real & $a_k \leq 0 \quad \forall\, k \geq p+1$.

[iii] $\sum_{k=p+1}^{\infty} a_k = -1$.

Assuming the Claim, we compute

$$|E_p(z) - 1| = \left| \sum_{k=1}^{\infty} a_k z^k \right| \overset{[i]}{=} \left| \sum_{k=p+1}^{\infty} a_k z^k \right|$$

$$= |z|^{p+1} \left| \sum_{k=p+1}^{\infty} a_k z^{k-p-1} \right|$$

$$\le |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k| \, |z|^{k-p-1} \qquad |z| \le 1$$

$$\le |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k| \, ,$$

$$\overset{\boxed{II}}{=} -|z|^{p+1} \sum_{k=p+1}^{\infty} a_k \overset{\boxed{III}}{=} |z|^{p+1}$$

Proof of the claim

$\boxed{I}$ $\quad E_p(z) = (1-z)\, e^{u}, \qquad u = z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}.$

Note $\quad u' = 1 + z + \dots + z^{p-1} \implies (1-z)\, u' = 1 - z^p.$

Compute

$$E_p'(z) = \left((1-z)\, e^{u}\right)' =$$

$$= -e^{u} + (1-z)\, u' e^{u}$$

$$= -e^{u} + (1-z^p)\, e^{u}$$

$$= -z^p e^{u} \qquad (1)$$

Since

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \implies E_p'(z) = \sum_{k=1}^{\infty} k\, a_k z^{k-1}. \qquad (2)$$

The terms in (1) have powers of z^p.

Comparing with (2) we see $a_k = 0 \quad \forall \ 1 \leq k \leq p$.

$\boxed{ii}$ Also for $k \geq p+1$,

$$a_k = -\frac{1}{k} \cdot \text{Coefficient of } z^{k-p-1} \text{ in } e^{u}.$$

Since

$$e^{u} = e^{z} \cdot e^{z^2/2} \cdot \ldots \cdot e^{z^p/p} \quad \& \text{ using the}$$

expansion of the exponential, we see that

real number

$$\text{Coefficient of } z^{k-p-1} \text{ in } e^{u} \geq 0 \implies a_k \leq 0.$$

$\boxed{iii}$ Set $z = 1$:

$$0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} a_k \implies \sum_{k=p+1}^{\infty} a_k = -1.$$

Further remarks – Looking forward (not needed)

[i] A divisor is a formal sum

$$D = \sum_{p \in \mathbb{C}} n_p [p] \quad \text{where} \quad n_p \in \mathbb{Z}$$

We require that this sum be locally finite.

A divisor is non-negative (effective) if $n_p \geq 0 \;\; \forall p$.

Example $D = 3[a] + 5[b]$ is a divisor.

[ii] Any entire function gives rise to a divisor

Indeed,

$$\operatorname{div}(f) = \sum_{p \text{ zero for } f} \operatorname{ord}(f, p)\,[p]$$

Example

$$f = (z-a)^3 (z-b)^5 \Rightarrow \operatorname{div}(f) = 3[a] + 5[b]$$

[iii] Weierstraß Problem can be rephrased

Every effective divisor is the divisor on an entire function

$$D \geq 0 , \quad D = \operatorname{div}(f).$$

[iv] For a meromorphic function f

$$\operatorname{div}(f) = \sum_{\substack{p \text{ zero or} \\ \text{pole}}} \operatorname{ord}(f, p)\,[p]$$

"principal divisor"

Question Is every divisor the divisor of a meromorphic function?

Yes For a general divisor D we can separate

$$D = D_+ - D_- , \quad D_+, D_- \text{ nonnegative.}$$

Write $D_+ = \operatorname{div} f_+$, $D_- = \operatorname{div} f_-$ & set $f = f_+ / f_-$

Then $\quad \operatorname{div}(f) = \operatorname{div}(f_+) - \operatorname{div}(f_-)$ (check)

$$= D_+ - D_- = D.$$

$\boxed{\text{IV}}$ These questions naturally lead to sheaf cohomology.

(Math 220C).

<u>Next time</u> the Weierstraß problem in $\mathcal{U} \subseteq \mathbb{C}$.

This is a bit more involved.

Math 220 B — Lecture 8

January 22, 2021

Weierstraß Problem for arbitrary regions

Question Given $\mathcal{U} \subseteq \mathbb{C}$, $\{a_n\} \subseteq \mathcal{U}$ without limit point in $\mathcal{U}$, find f holomorphic in $\mathcal{U}$ with zeroes only at $\{a_n\}$.

The sequence $\{a_n\}$ may contain repetitions according to multiplicities of the zeroes.

Main Theorem The Weierstraß Problem can be solved in $\mathcal{U}$.

Remark $\boxed{i}$ It is not true any two solutions f_1, f_2 satisfy

$$f_1 = e^h f_2$$

Counterexample $U = \mathbb{C}^\times$, $f_2 = 1$, $f_1 = z$

h would have to be a logarithm, which is undefined in $\mathbb{C}^\times$.

$\boxed{ii}$ Any meromorphic function in U is quotient of two holomorphic functions.

The same proof for $U = \mathbb{C}$ works for all U.

How to prove Weierstraß for $\mathcal{U}$?

We could again try

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Convergence used $a_n \longrightarrow \infty$.

Indeed, if we wish to have

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \quad \text{we'd need } \frac{r}{|a_n|} \longrightarrow 0 \Rightarrow a_n \longrightarrow \infty.$$

Since $a_n \in \mathcal{U}$, this may not be the case. e.g. if $\mathcal{U}$ is bounded. How to deal with bounded regions for instance?

New ideas

(1) Use biholomorphisms to change the region $\mathcal{U}$ e.g. via $z \longrightarrow 1/z$. If $\mathcal{U}$ were bounded, the new region would be unbounded.

(2) Think of $\mathcal{U} \subseteq \mathbb{C}$ as $\mathcal{U} \subseteq \hat{\mathbb{C}}$ & prescribe values at ∞ as well.

New idea

Even for unbounded regions, we can try new functions:

$$f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{a_n - b_n}{z - b_n}\right)$$

for good choices of b_n.

This also has zeroes at $z = a_n$ since $E_n(1) = 0$.

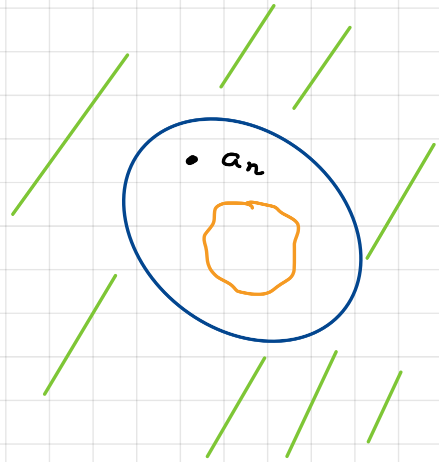
Weierstraß Problem in $\mathcal{U} \subseteq \mathbb{C}$

Step (1) Assume $\exists\, R > 0$

neighborhood of ∞.

$\boxed{i}$ $\{ |z| \geq R \} \subseteq \mathcal{U}$

$\boxed{ii}$ $|a_n| \leq R \quad \forall\, n.$



Construct f holomorphic in $\mathcal{U}$ such that

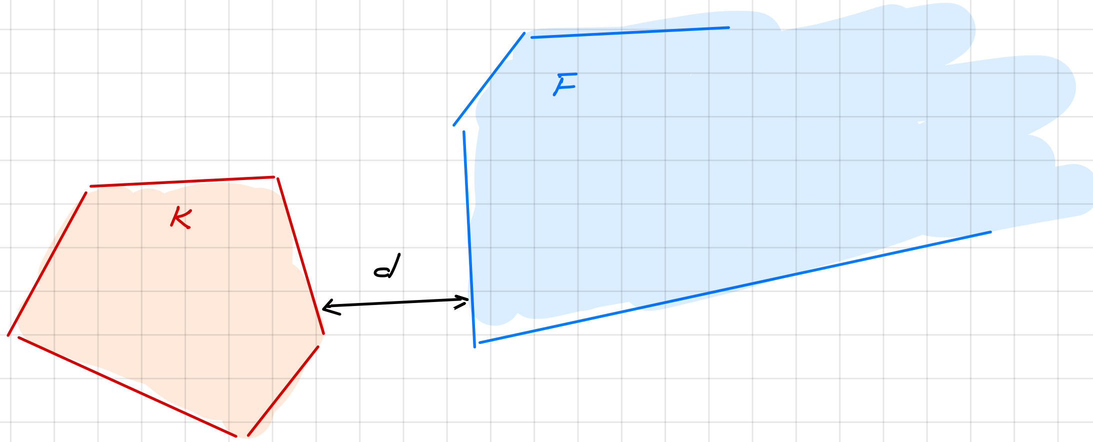
$\boxed{i}$ f has zeroes at a_n

$\boxed{ii}$ $\lim\limits_{z \to \infty} f(z) = 1.$

Step (2) General case. Uses easy trick.

Use $z \longrightarrow 1/z$ to reduce to Step 1.

Topological Fact used in the Proof (Rudin)



$K \cap F = \Phi$, $K \neq \Phi$ compact, $F \neq \Phi$ closed.

$$d = \mathrm{dist}(K, F) = \inf_{\substack{k \in K \\ f \in F}} |k - f| > 0$$

Proof Assume $d = 0$. Then $\exists\, k_n \in K$, $f_n \in F$ with

$$|k_n - f_n| \longrightarrow 0$$

Passing to a subsequence, assume $k_n \longrightarrow k. \in K.$

It follows that $f_n \longrightarrow k$ as well.

Since F closed, $k \in F$. Thus $k \in K \cap F = \Phi$. contradiction.

Step 1 $\exists\, R>0$, $\{|z| \ge R\} \subseteq \mathcal{U}$ & $|a_n| \le R$.

[i] f zeroes at a_n only

[ii] $\lim_{z\to\infty} f(z) = 1$

Note $K = \mathbb{C} \setminus \mathcal{U} \subseteq \{|z| \le R\} \Rightarrow K$ bounded & closed

$\Rightarrow K$ compact.

Since $|a_n - z|$ is continuous, $\exists\, b_n \in K$ with

$$|a_n - b_n| = \min_{z\in K} |a_n - z|.$$

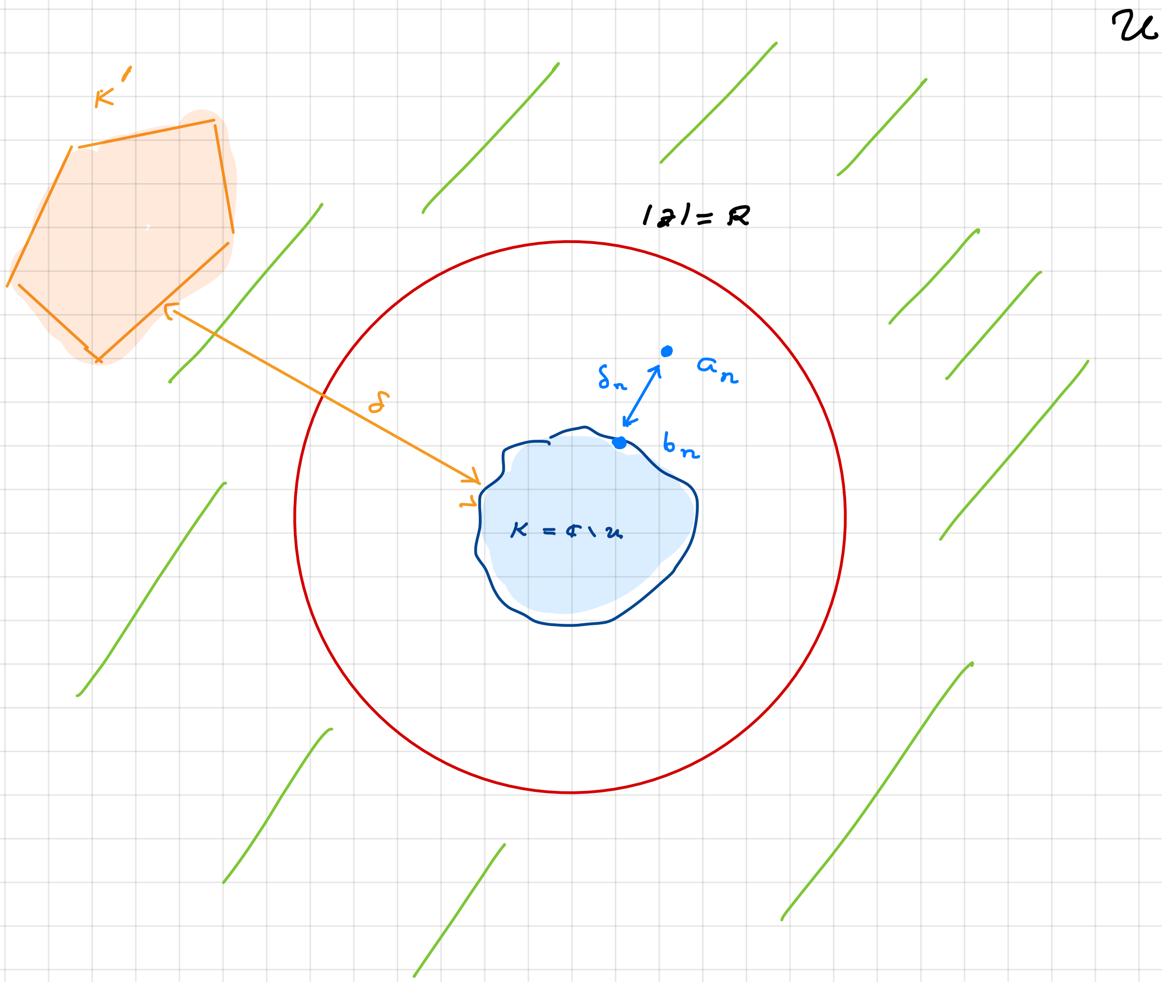
Write $\delta_n = |a_n - b_n| > 0$ since $a_n \in \mathcal{U}$, $b_n \notin \mathcal{U}$.

Claim $\delta_n \longrightarrow 0$.

Proof Assume otherwise. Then $\exists\, \varepsilon$ $\forall N$ $\exists\, n \ge N$ with

$$|\delta_n| \ge \varepsilon.$$

Passing to a subsequence we may assume $|\delta_n| \ge \varepsilon$ $\forall n$.



Note $\{a_n\} \subseteq \bar{\Delta}(0, R)$ = compact. Passing to a subsequence we may assume $a_n \longrightarrow a$. Since $\{a_n\}$ has no limit point in $\mathcal{U}$ $\Rightarrow$ $a \in K$. Then by the definition of b_n:

$$|a_n - a| \geq |a_n - b_n| = \delta_n > \varepsilon.$$

This contradicts $a_n \longrightarrow a$. Thus $\delta_n \longrightarrow 0$.

Claim $f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{a_n - b_n}{z - b_n}\right)$ converges absolutely & locally uniformly in $\mathcal{U}$. & vanishes only at a_n.

Proof It suffices to show

$$\sum_{n=1}^{\infty} \left| E_n\left(\frac{a_n - b_n}{z - b_n}\right) - 1 \right|$$ converges absolutely & locally uniformly in $\mathcal{U}$. To this end, let $K' \subseteq \mathcal{U}^-$ compact.

Let $\delta = d(K, K') > 0$ since $K \cap K' = \Phi$.

For $z \in K' \Rightarrow |z - b_n| \geq \delta. \Rightarrow$

$$\left| \frac{a_n - b_n}{z - b_n} \right| \leq \frac{\delta_n}{\delta} \leq \frac{1}{2} \text{ if } n \geq N \text{ since } \delta_n \longrightarrow 0.$$

Recall

$$|1 - E_p(w)| \leq |w|^{p+1}. \quad \forall \ |w| \leq 1.$$

Thus

$$\left| 1 - E_n\left(\frac{a_n - b_n}{z - b_n}\right)\right| \leq \left| \frac{a_n - b_n}{z - b_n} \right|^{n+1} \leq \frac{1}{2^{n+1}} \quad \forall z \in K', n \geq N$$

We conclude by Weierstraß M-test since $\sum_n \frac{1}{2^{n+1}} < \infty$.

Proof of [ii] $\lim\limits_{z \to \infty} f(z) = 1.$

Equivalently $\lim\limits_{z \to 0} f\left(\frac{1}{z}\right) = 1.$

We compute

$$g(z) = f\left(\frac{1}{z}\right) = \prod_{n=1}^{\infty} E_n\left(\frac{a_n - b_n}{1/z - b_n}\right) = \prod_{n=1}^{\infty} E_n\left(\frac{z(a_n - b_n)}{1 - z b_n}\right). \quad (*)$$

We show the product $(*)$ converges absolutely & locally uniformly in $\Delta\left(0, \frac{1}{R}\right)$. The limit will be holomorphic at $z=0$ hence continuous. Then

$$\lim_{z \to 0} g(z) = g(0) = 1. \Rightarrow \lim_{z \to 0} f\left(\frac{1}{z}\right) = 1.$$

To show convergence, let $\overline{\Delta}(0,\rho) \subseteq \Delta\left(0, \frac{1}{R}\right)$. $\Rightarrow \rho R < 1$.

We have for $z \in \overline{\Delta}(0,\rho)$:

$|z| \leq \rho,\ |b_n| \leq R.$

$$\left|\frac{z(a_n - b_n)}{1 - z b_n}\right| \leq \frac{\rho\, \delta_n}{|1 - z b_n|} \leq \frac{\rho\, \delta_n}{1 - |z||b_n|} \leq \frac{\rho\, \delta_n}{1 - \rho R} \leq \frac{1}{2}.$$

for $n \geq N$ since $\delta_n \to 0$.

Then

$$\left|1 - E_n\left(\frac{z(a_n - b_n)}{1 - z b_n}\right)\right| \leq \left|\frac{z(a_n - b_n)}{1 - z b_n}\right|^{n+1} \leq \frac{1}{2^{n+1}} \Rightarrow$$

$\Rightarrow$ Weierstraß M-test

$$\sum_{n=1}^{\infty} \left|1 - E_n\left(\frac{z(a_n - b_n)}{1 - z b_n}\right)\right|$$ converges absolutely & locally

uniformly in $\Delta\left(0, \frac{1}{R}\right)$.

Case (2) General case

WLOG $0 \in \mathcal{U}$ & $a_n \neq 0$

Indeed we may take $a \in \mathcal{U}$, $a \neq a_n$ $\forall n$. Let

$$\mathcal{U}^{new} = \{ u - a,\ u \in \mathcal{U} \},\quad a_n^{new} = a_n - a,$$

$\Rightarrow$ $0 \in \mathcal{U}^{new}$, $a_n^{new} \neq 0$. If f^{new} solves Weierstraß for $(\mathcal{U}^{new}, \{a_n^{new}\})$ let $f(z) = f^{new}(z - a)$. solves Weierstraß for $(\mathcal{U}, \{a_n\})$.

Trick to reduce to Case 1

Define $\tilde{\mathcal{U}} = \left\{ \frac{1}{z} : z \in \mathcal{U} \setminus \{0\} \right\}$. This is open by the open mapping theorem for $\mathcal{U} \setminus \{0\}$, $z \longrightarrow 1/z$.

Let $\tilde{a}_n = \frac{1}{a_n} \in \tilde{\mathcal{U}}$

Claim $(\tilde{\mathcal{U}}, \{\tilde{a}_n\})$ satisfies Step 1.

Let $\tilde{f}$ be the solution to Weierstraß for $(\tilde{\mathcal{U}}, \{\tilde{a}_n\})$

Let $f(z) = \tilde{f}\left(\frac{1}{z}\right)$ = holomorphic in $\mathcal{U} \setminus \{0\}$.

Since $\lim_{z \to \infty} \tilde{f}(z) = 1 \implies \lim_{z \to 0} f(z) = 1$. Thus 0 is removable singularity and f extends to $\mathcal{U}$. Its zeroes are only at a_n.

Proof of the claim

Since $0 \in \mathcal{U} \Rightarrow \exists\, \varepsilon$ with $\Delta(0, \varepsilon) \subseteq \mathcal{U}$.

$\implies \{ |z| \geq \frac{1}{\varepsilon} \} \subseteq \tilde{\mathcal{U}}$.

Since $0 \in \mathcal{U}$ & $\{a_n\}^{\neq 0}$ do not have 0 as limit point

$\implies \exists\, \varepsilon'$ with $|a_n| \geq \varepsilon' \Rightarrow |\tilde{a}_n| \leq \frac{1}{\varepsilon'}$.

Let $R = \max(1/\varepsilon, 1/\varepsilon')$. $\Rightarrow |a_n| \leq R$ & $\{ |z| \geq R \} \subseteq \mathcal{U}$.

Exercise

Follow the above proof for $n = \mathbb{C}$. What function f does the proof produce?

Math 220B — Lecture 9

January 25, 2021

The Mittag-Leffler Problem Conway VIII.3 simplified.

Weierstraß Problem

Given (i) $\{a_n\}$ distinct, $a_n \longrightarrow \infty$.

(ii) $\{m_n\}$ positive integers

Find entire functions f with zeroes only at a_n of order m_n.

Answer We can always solve the Weierstraß Problem. & we even have a factorization of the solution.

Remark The function $1/f$ is meromorphic & its poles are only at a_n & their order equals m_n.

The Mittag-Leffler Problem asks a sharper question.

The Mittag-Leffler (ML) Problem for $\mathbb{C}$

Given (i) $\{a_n\}$ distinct, $a_n \longrightarrow \infty$.

(ii) Laurent principal parts (singular parts)

$$g_n(z) = \frac{A_{nm_n}}{(z-a_n)^{m_n}} + \frac{A_{nm_n-1}}{(z-a_n)^{m_n-1}} + \cdots + \frac{A_{n1}}{z-a_n}$$

Main Theorem We can always find meromorphic function f with poles only at a_n & Laurent principal parts g_n near a_n.

Remark If f_1, f_2 are two solutions $\Rightarrow$ $f_1 - f_2$ = entire since the singular parts at a_n cancel out

$$f_1 = f_2 + h$$

Remark This makes sense for $\mathcal{U} \subseteq \mathbb{C}$.

- student of Hermite & Weierstraß
- Nobel Prize committee
- founder of Acta Math.

Gösta Mittag-Leffler

1846 – 1927

SUR LA REPRÉSENTATION ANALYTIQUE

DES

FONCTIONS MONOGÈNES UNIFORMES

D'UNE VARIABLE INDÉPENDANTE

PAR

G. MITTAG-LEFFLER
À STOCKHOLM.

Les recherches dont je vais exposer ici l'ensemble, ont été publiées auparavant, quant à leurs traits les plus essentiels, dans le Bulletin (Öfversigt) des travaux de l'Académie royale des sciences de Suède, ainsi que dans les Comptes-rendus hebdomadaires de l'Académie des sciences à Paris. Leur but est de faire parvenir, dans un certain sens, la théorie des fonctions analytiques uniformes d'une variable, à ce degré d'achèvement auquel la théorie des fonctions rationnelles est arrivée depuis longtemps.

Soit x une grandeur variable complexe à variabilité illimitée, et x' un point donné fini (1) dans le domaine de la variable x. Soit enfin R une quantité positive donnée. Je dis que l'ensemble des points x remplissant la condition $|x - x'| < R$, constitue le *voisinage* ou *l'entourage* ou les *environs du point* x' (2) *correspondant à* R. Chacun de ces points est dit *appartenir au voisinage* ou *à l'entourage* ou *aux environs* R, ou être

(1) C'est-à-dire représentant une valeur donnée finie.

(2) Cf.: *Zur Functionenlehre, von* K. WEIERSTRASS. Monatsbericht der Königl. Akademie der Wissenschaften zu Berlin, August 1880, pag. 4.

Acta Math 4 (1884)

Remarks Connections:

$$
\begin{tikzcd}
 & \text{Generalized Weierstraß} \arrow[d, Rightarrow, "\text{Clear}"] \\
\text{Mittag Leffler} \arrow[ur, Rightarrow, "\text{HWK3}"] \arrow[r, Rightarrow, "f \rightsquigarrow 1/f"'] & \text{Weierstraß Existence Problem} \\
 & \text{Weierstraß Factorization} \arrow[u, Rightarrow, "\text{Previous Lectures}"']
\end{tikzcd}
$$

HWK3, Problem 2

2. (*Generalized Weierstraß problem. Monday, January 25.*) Let $\{a_n\}$ be distinct complex numbers with $a_n \to \infty$. Fix complex numbers $\{A_n\}$. Show that there exists an entire function f such that

$$f(a_n) = A_n.$$

Further Connections

In HWK3, Problem 3 we will see that we can derive Mittag-Leffler for simple poles from Weierstraß factorization.

Discussion of the proof

Given $\{a_n\}$, $a_n \to \infty$, g_n = Laurent principal parts

we try $f = \sum_{n=1}^{\infty} g_n$ as solution to Mittag-Leffler

Issue: As usual, this may not converge

New idea Pick h_n entire functions & argue

$$f = \sum_{n=1}^{\infty} (g_n - h_n) \text{ converges}$$

Since h_n are entire, we are not changing the Laurent principal parts.

Compare this to Weierstraß

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \quad \text{vs.} \quad \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{h_n}$$

may not converge could converge.

Terminology

$$\sum_{n=1}^{\infty} (g_n - h_n) = \text{Mittag-Leffler series}$$

$$h_n = \text{convergence enhancing corrections}$$

The h_n's are not unique!

Remark WLOG $a_n \neq 0$. $\forall n$.

The contributions of the poles at 0 are added at the end:

$$\frac{A_m}{z^m} + \cdots + \frac{A_1}{z} + \text{Solution with } a_n \neq 0.$$

Proof The proof is part of the theorem. Conway VIII. 3.

Fix $\boxed{I}$ $r_n \to \infty$, $r_n < |a_n|$

$\boxed{II}$ c_n , $\sum_{n=1}^{\infty} c_n < \infty$

e.g. $c_n = \frac{1}{2^n}$, $c_n = \frac{1}{n^2}$, ...

Consider $$g_n(z) = \frac{A_{nm_n}}{(z-a_n)^{m_n}} + \frac{A_{nm_n-1}}{(z-a_n)^{m_n-1}} + \cdots + \frac{A_{n1}}{z-a_n}$$

Since $a_n \neq 0$, g_n is holomorphic at $z=0$ in $\Delta(0, |a_n|)$

We can Taylor expand g_n in $\Delta(0, |a_n|)$ around 0.

Since $\bar{\Delta}(0, r_n) \subseteq \Delta(0, |a_n|)$, the Taylor series of g_n

converges uniformly in $\bar{\Delta}(0, r_n)$. We can pick a

Taylor polynomial h_n such that

$$|g_n - h_n| < c_n \quad \text{in } \bar{\Delta}(0, r_n).$$

Let $f = \sum_{k=1}^{\infty} (g_k - h_k)$

We show

Claim f meromorphic with poles only at a_k & principal parts g_k near a_k. $\Rightarrow$ f solves Mittag-Leffler.

Proof Let $r > 0$.

Since $r_k \longrightarrow \infty$, $\Rightarrow r_k > r$ if $k \geq N$. Then

$|g_k - h_k| < c_k$ in $\bar{\Delta}(0,r) \subseteq \Delta(0,r_k)$ if $k \geq N$..

By Weierstraß M-test $\sum_{k=N}^{\infty} (g_k - h_k)$ converges uniformly in $\bar{\Delta}(0,r)$. Note that since $|a_k| > r_k > r$

$\Rightarrow$ $g_k - h_k$ holomorphic in $\Delta(0,r)$. Thus the sum

polynomial

the pole a_k is not in $\Delta(0,r)$

$$\sum_{k=N}^{\infty} (g_k - h_k)$$

is holomorphic in $\Delta(0,r)$.

The sum $\sum_{k=1}^{N-1} (g_k - h_k)$ is meromorphic as a finite sum of meromorphic functions in $\Delta(0,r)$. The poles are only at those a_j's with $|a_j| < r$ and the Laurent principal parts are g_j. This is because h_k are polynomials. so they do not contribute to the Laurent principal parts.

$$\text{Thus } f = \underbrace{\sum_{k=1}^{N-1} (g_k - h_k)}_{\text{meromorphic}} + \underbrace{\sum_{k=N}^{\infty} (g_k - h_k)}_{\text{holomorphic}}$$

is meromorphic with poles at $|a_j| < r$ for all $\Delta(0,r)$.

Varying r we get the claim & finish the proof.

Summary of the proof

Step 1 Pick $r_n \longrightarrow \infty$, $|a_n| > r_n$

c_n, $\sum c_n < \infty$

Step 2 Taylor expand g_n near 0

Pick Taylor polynomial h_n with

$|g_n - h_n| < c_n$ in $\Delta(0, r_n)$

Step 3 $$f = \sum_{n=1}^{\infty} (g_n - h_n)$$

Examples (will be repeated next time)

$\boxed{\text{II}}$ Poles at $-n \in \mathbb{Z}$, principal parts $\dfrac{1}{z+n}$.

For $n \neq 0$, we expand $\dfrac{1}{z+n}$ at $z=0$.

$$g_n = \frac{1}{z+n} = \frac{1}{n} \cdot \frac{1}{1+\frac{z}{n}} = \frac{1}{n}\left(1 - \frac{z}{n} + \frac{z^2}{n^2} - \cdots\right)$$

$$= \frac{1}{n} - \frac{z}{n^2} + \frac{z^2}{n^3} - \cdots$$

Let $h_n = \dfrac{1}{n} \Rightarrow g_n - h_n = \dfrac{1}{z+n} - \dfrac{1}{n} = \dfrac{z}{n(z+n)}$.

Let $r_n = \sqrt{|n|}$ If $|z| \leq r_n$ then

$$\Rightarrow |g_n - h_n| = \frac{|z|}{n\,|z+n|} \leq \frac{\sqrt{n}}{n\,(n-\sqrt{n})} = c_n \quad \text{if } n>0$$

Note $\displaystyle\lim_{n\to\infty} \frac{c_n}{n^{-3/2}} = 1$ & $\displaystyle\sum_{n=}^{\infty} n^{-3/2} < \infty$. Thus $\displaystyle\sum_{n=1}^{\infty} c_n < \infty$.

A similar argument works for $n < \infty$.

$$f = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \frac{1}{z}$$ is the solution

we need to add this at the end for $n=0$.

to the Mittag - Leffler Problem.

Remark

Note that the n & $-n$ terms can be collected

$$f = \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) + \frac{1}{z}$$

$$= \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} + \frac{1}{z} = \pi \cot \pi z$$ by

HWK 6 in Math 220A.

Math 220 B — Lecture 10

January 27, 2021

Last time – Mittag-Leffler Problem

Given

- $a_n \to \infty$ distinct and
- Laurent principal parts g_n

find f meromorphic with poles at a_n & principal parts g_n at a_n

Construction

Step 1 Expand g_n into Taylor series at 0

Step 2 Pick h_n a Taylor polynomial & check

$$|g_n - h_n| < c_n \quad \text{in } \Delta(0, r_n) \quad \text{with } \sum_n c_n < \infty.$$

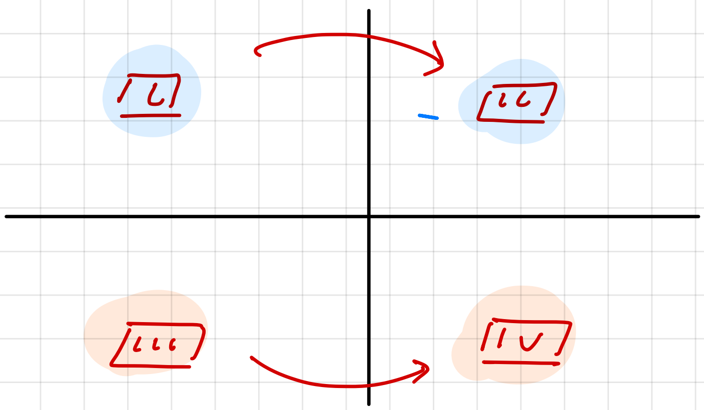
and $r_n < |a_n|$, $r_n \to \infty$

Step 3 Solution

$$f = \sum_{n=1}^{\infty} (g_n - h_n) + \text{add Laurent principal part at } 0.$$

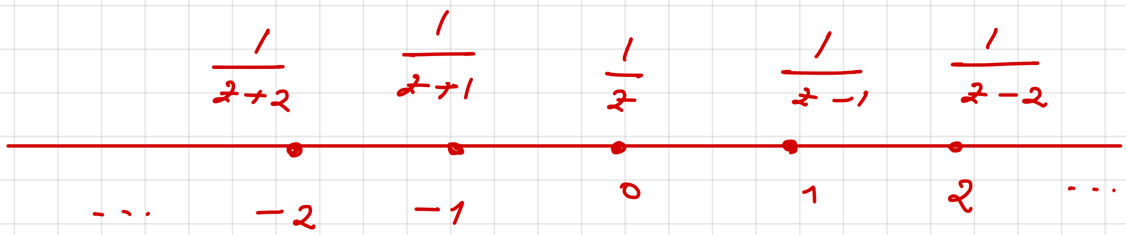
Today – 4 historically important examples

– we group them in pairs of two



poles　　principal parts

Example $\boxed{1}$ $\left(-n, \dfrac{1}{z+n}\right)$, $n \in \mathbb{Z}$.



Step 1 Taylor expand:

$$g_n = \frac{1}{z+n} = \frac{1}{n}\cdot\frac{1}{1+\frac{z}{n}} = \frac{1}{n}\left(1-\frac{z}{n}+\frac{z^2}{n^2}-\cdots\right)$$

$$= \frac{1}{n}-\frac{z}{n^2}+\frac{z^2}{n^3}-\cdots$$

$$h_n = \frac{1}{n}, \quad n \neq 0$$

Step 2 Let $r_n = \frac{1}{2}|n|^{1/2}$. If $|z| \le r_n$:

$$|g_n - h_n| = \left|\frac{1}{z+n}-\frac{1}{n}\right| = \frac{|z|}{|n|\,|n+z|} \le \frac{r_n}{|n|(|n|-r_n)} = c_n$$

Since $\lim\limits_{n\to\infty} \dfrac{c_n}{|n|^{-3/2}} < \infty$ and $\sum\limits_{n} \dfrac{1}{|n|^{3/2}} < \infty$. $\Longrightarrow \sum\limits_{n=1}^{\infty} c_n < \infty$.

Step 3   Mittag - Loffler solution

$$f = \sum_{n \neq 0} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \frac{1}{z}.$$

Collecting the terms for n & $-n$ we find

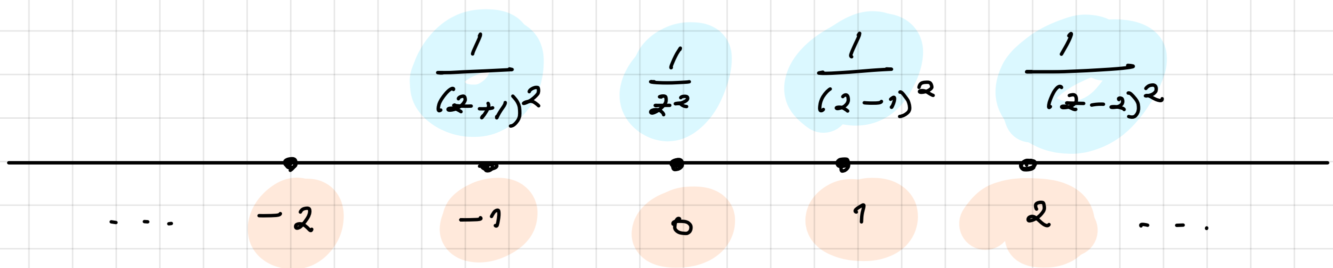
$$f = \sum_{n>0} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) + \frac{1}{z}$$

Math 220A, HWK6.

$$= \sum_{n>0} \frac{2z}{z^2 - n^2} + \frac{1}{z} = \pi \cot \pi z$$

11 Poles at $-n \in \mathbb{Z}$, principal parts $\frac{1}{(z+n)^2}$.

$$\left(-n, \frac{1}{(z+n)^2}\right)$$



Step 1

$$g_n = \frac{1}{(z+n)^2}$$

$$h_n = 0$$

Step 2 $\quad r_n = \frac{1}{2}|n|^{1/2}$. If $|z| \le r_n$

$$|g_n - h_n| = \left|\frac{1}{(z+n)^2}\right| \le \frac{1}{(|n|-r_n)^2} =: c_n .$$

$$\lim_{n\to\infty} \frac{c_n}{|n|^{-2}} = 1 \quad \& \quad \sum_{n\neq 0} \frac{1}{n^2} < \infty \Rightarrow \sum_{n\neq 0} c_n < \infty$$

Step 3 Mittag - Leffler function

$$f = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}.$$

We have seen $f = \dfrac{\pi^2}{\sin^2 \pi z}$ in Math 220A, HWK6, #7.

6. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2} \, dz$$

via the residue theorem. Making $n \to \infty$, show that

$$\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.$$

7. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners

$$\pm \left(n + \frac{1}{2} \right) \pm ni.$$

Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{(z+a)^2} \, dz$$

via the residue theorem, and use this to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

Hwk 6, Math 220A

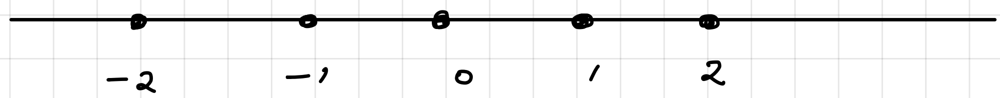
Remark Compare $\boxed{i}$ & $\boxed{ii}$

$$\left(-n, \frac{1}{z+n}\right) \longleftrightarrow \left(-n, \frac{1}{(z+n)^2}\right)$$

$$\pi \cot \pi z \longleftrightarrow \frac{\pi^2}{\sin^2 \pi z}$$

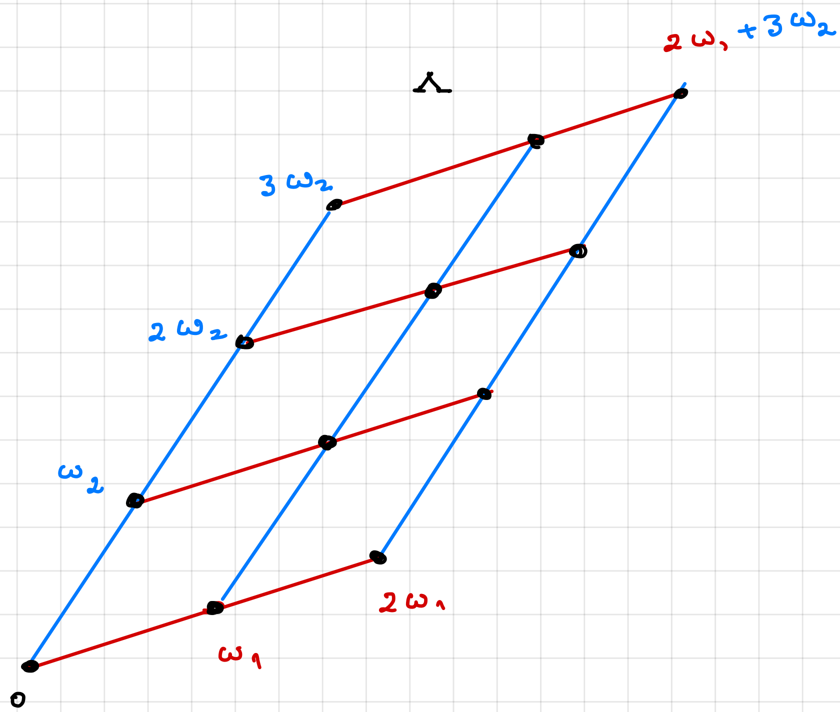
These are related by differentiation (up to a sign).

For the next examples, we replace

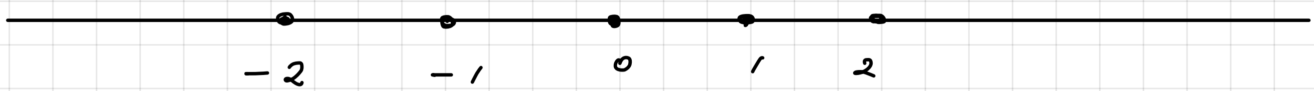


by the lattice

$$\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2 = \{ m\,\omega_1 + n\,\omega_2 \ : m, n \in \mathbb{Z} \}, \quad \frac{\omega_1}{\omega_2} \notin \mathbb{R}$$



Main Difference



$$\sum_{n \neq 0} \frac{1}{|n|^{\alpha}} \quad \text{converges if } \alpha = 2$$

$$\text{if } \alpha > 1.$$

For the lattice,

$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{|\lambda|^{\alpha}} \quad \text{converges if } \alpha = 3 \quad (\text{HWK } 2)$$

$$\text{if } \alpha > 2.$$

$\boxed{111}$

Poles at $\lambda \in \Lambda$, principal parts $\dfrac{1}{z-\lambda}$.

$$\left(\lambda, \frac{1}{z-\lambda}\right)_{\lambda \in \Lambda}.$$

Step 1 $\quad \lambda \neq 0$

Taylor expand

$$g_\lambda = \frac{1}{z-\lambda} = \frac{1}{\lambda} \cdot \frac{-1}{1-\frac{z}{\lambda}}$$

$$= \frac{-1}{\lambda}\left(1 + \frac{z}{\lambda} + \frac{z^2}{\lambda^2} + \dots\right)$$

$$= -\frac{1}{\lambda} - \frac{z}{\lambda^2} - \frac{z^2}{\lambda^3} - \dots$$

$$h_\lambda = -\frac{1}{\lambda} - \frac{z}{\lambda^2}$$

Step 2 $\quad$ Let $\quad r_\lambda = \min\left(\frac{1}{2}|\lambda|, \ |\lambda|^{1/4}\right)$.

If $\ |z| \leq r_\lambda \ $ then

$$\left| g_\lambda - h_\lambda \right| = \left| \sum_{k=2}^{\infty} \frac{z^k}{\lambda^{k+1}} \right|$$

$$= \frac{|z|^2}{|\lambda|^3} \sum_{k=0}^{\infty} \left|\frac{z}{\lambda}\right|^k \leq \frac{r_\lambda^2}{|\lambda|^3} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} =$$

$$= 2 \cdot \frac{r_\lambda^2}{|\lambda|^3} \leq 2 \cdot \frac{1}{|\lambda|^{5/2}} = c_\lambda \, .$$

Since $\sum_{\lambda \neq 0} \frac{1}{|\lambda|^{5/2}} < \infty$. we get $\sum_{\lambda \neq 0} c_\lambda < \infty$.

Step 3 Mittag - Leffler solution

$$\zeta = \frac{1}{z} + \sum_{\lambda \neq 0} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right)$$

Weierstraß ζ - function (HWK 3, #3)

$\boxed{\text{IV}}$ Poles at $\lambda \in \Lambda$, principal parts $\dfrac{1}{(z-\lambda)^2}$.

$$\left(\lambda, \frac{1}{(z-\lambda)^2}\right)_{\lambda \in \Lambda}$$

Step 1 $\quad \lambda \neq 0$

$$g_\lambda = \frac{1}{(z-\lambda)^2} = \frac{1}{\lambda^2} \cdot \frac{1}{\left(1 - \frac{z}{\lambda}\right)^2} =$$

$$= \frac{1}{\lambda^2}\left(1 + \frac{2z}{\lambda} + \frac{3z^2}{\lambda^2} + \cdots\right)$$

$$= \frac{1}{\lambda^2} + \frac{2z}{\lambda^3} + \frac{3z^2}{\lambda^3} + \cdots$$

$$\frac{1}{(1-w)^2} = 1 + 2w + 3w^2 + \cdots$$

$$h_\lambda = \frac{1}{\lambda^2}.$$

Step 2 $\quad r_\lambda = \min\left(\frac{|\lambda|}{2}, |\lambda|^{1/4}\right)$

$$|h_\lambda - g_\lambda| = \left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| = \left|\frac{z^2 - 2z\lambda}{\lambda^2 (z-\lambda)^2}\right|$$

$$\leq \frac{r_\lambda^2 + 2r_\lambda|\lambda|}{|\lambda|^2 (|\lambda| - r_\lambda)^2} \overset{r_\lambda \leq \frac{|\lambda|}{2}}{\leq} 4 \cdot \frac{r_\lambda^2 + 2r_\lambda|\lambda|}{|\lambda|^4} = c_\lambda.$$

Note

$$\lim_{\lambda \to \infty} \frac{c_\lambda}{|\lambda|^{-5/2}} < \infty \implies \sum c_\lambda \sim \sum \frac{1}{|\lambda|^{5/2}} < \infty .$$

Step 3 The Mittag-Leffler solution

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) = \text{Weierstraß}$$

$\wp$ function.

Homework 3, #4.

Compare $\boxed{III}$ & $\boxed{IV}$

$$\left(\lambda, \frac{1}{z-\lambda}\right) \xleftrightarrow{-\text{derivative}} \left(\lambda, \frac{1}{(z-\lambda)^2}\right)$$

$$\zeta \longleftrightarrow \wp = -\zeta'$$

$\boxed{I}$ $\boxed{II}$

$$\left(-n, \frac{1}{z+n}\right) \xleftrightarrow{-\text{derivative}} \left(-n, \frac{1}{(z+n)^2}\right)$$

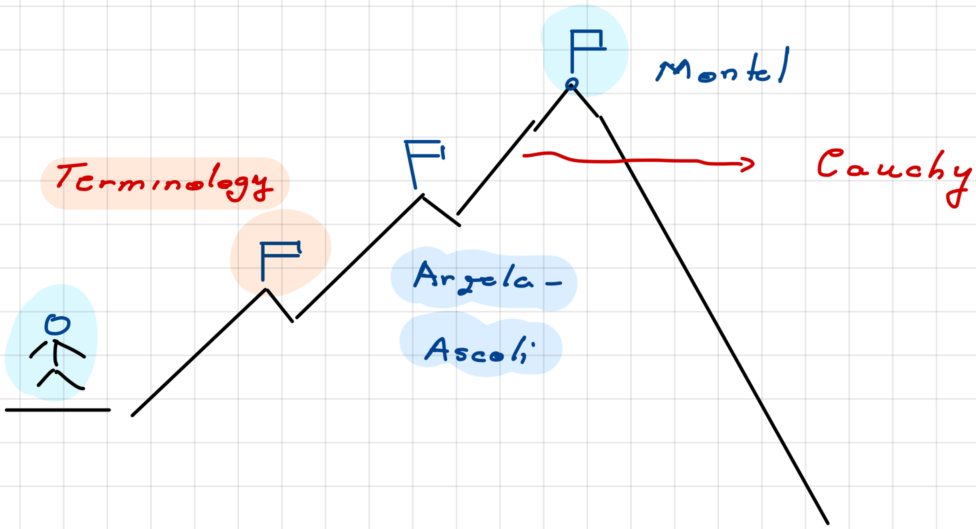
$$\pi \cot \pi z \longleftrightarrow \frac{\pi^2}{\sin^2 \pi z} = -(\pi \cot \pi z)'.$$

Remark The Mittag Leffler Problem makes sense for all $u \subseteq \mathbb{C}$. We will not give the details here (but see Conway VIII. 3)

Math 220B — Lecture 11

January 29, 2021

Next few lectures — Normal Families   Conway VII.1 & 2.



Why climb the mountain — Motivation

Sequences of complex numbers

$\{a_n\}$ bounded $\Longrightarrow$ $\exists$ convergent subsequence

Indeed, if $|a_n| \leq M \Rightarrow a_n \in \overline{\Delta}(0,M)$. The closed disc

$\overline{\Delta}(0,M)$ is compact.

We wish to make similar statements for sequences of functions (continuous or holomorphic).

Dream Statement

Given a "bounded" sequence of functions, there exists a "convergent" subsequence.

Question A What could "—" mean?

Question B Is this connected to compactness?

Answer is "yes" but it has no consequences for the current lecture.

Remark Dream statement makes sense in

[i] real analysis (continuous functions) ⇝ Arzela-Ascoli

[ii] complex analysis (holomorphic functions). ⇝ Montel.

We will investigate both.

Question A $f_n : U \longrightarrow \mathbb{C}$ "convergent" could mean

[i] pointwise ↙ weak

[ii] uniform ↙ strong

[iii] local uniform ↙ OK for us

$\Updownarrow$

[iv] uniform convergence on compact sets ↙ OK for us

"bounded" could mean

[i] pointwise bounded $\swarrow$ weak

$\forall x \in \mathcal{U} \quad \exists M(x) \quad$ with $\quad |f_n(x)| < M(x) \quad \forall n$

[ii] uniformly bounded $\swarrow$ strong

$\exists M \quad \forall x \in \mathcal{U} \quad |f_n(x)| < M \quad \forall n$

[iii] locally uniformly bounded. $\swarrow$ OK for us

$\forall x \ \exists \Delta_x \subseteq \mathcal{U}$ neighborhood of x, such that the restrictions $f_n/_{\Delta_x}$ are uniformly bounded.

OK for us $\swarrow$

[iv] uniformly bounded on compact sets

$\forall K \quad \exists M(K), \quad |f_n(x)| \leq M(K) \quad \forall x \in K \quad \forall n$

Remark We have $\boxed{III} \Leftrightarrow \boxed{IV}$ that is,

locally uniformly bounded $\Leftrightarrow$

uniformly bounded on each compact

Why? $\Leftarrow$ If $x \in \mathcal{U}$, let $K = \overline{\Delta}_x$ be a compact neighborhood of x.

$\Rightarrow$ For all $x \in \mathcal{U}$, $\exists\ \Delta_x$ where $f_n / _{\Delta_x}$ are bounded by M_x.

Then $K \subseteq \bigcup_{x \in K} \Delta_x$. $\Rightarrow$ $K \subseteq \bigcup_i \Delta_{x_i}$ and let

$$M = \max(M_{x_1}, \dots, M_{x_n}) > 0.$$

This is a bound for all f_n's over K.

Example

[i] $f_n(x) = \sin nx$ uniformly bounded by 1 in $\mathbb{R}$.

[ii] $f_n(z) = z^n$ in $\Delta(0,1)$ uniformly bounded by 1

[iii] $f_n(z) = n z^n$ locally uniformly bound in $\Delta(0,1)$

but not uniformly bounded.

Proof

For $0 \le r < 1$: $|f_n(z)| \le n r^n$. in $\Delta(0,r)$. Since

$$\lim_{n\to\infty} n r^n = 0 \Rightarrow \{n r^n\} \text{ is bounded by } M \Rightarrow$$

$$\Rightarrow |f_n(z)| \le M \text{ in } \Delta(0,r).$$

Each $K \subseteq \Delta(0,1)$ compact, $K \subseteq \overline{\Delta}(0,r)$ for $r<1 \Rightarrow$

$\Rightarrow \{f_n\}$ uniformly bounded (locally/on compacts).

Since $f_n\left(\frac{1}{\sqrt[n]{2}}\right) = \frac{n}{2} \longrightarrow \infty$.

$\Rightarrow \{f_n\}$ not uniformly bounded.

Dream Statement Revisited

$f_n : \mathcal{U} \longrightarrow \mathbb{C}$ locally uniformly bounded

$\Longrightarrow f_n$ admits a locally convergent subsequence

---

Question Could this be true?

Example No.

Let $\mathcal{U} = \mathbb{R}$. The sequence

$$f_n(x) = \sin nx$$

is uniformly bounded, but we can't get a convergent subsequence not even pointwise.

Question C1 Could this be true in complex analysis i.e. holomorphic functions? **YES**

Question C2 What is the correct statement in real analysis i.e. continuous functions?

Answer to C1

Main Theorem (Montel)

$f_n : U \longrightarrow \mathbb{C}$ holomorphic & locally uniformly bounded

$\Longrightarrow$ f_n admits a locally uniformly convergent subsequence.

More generally – Families

$\mathcal{F}$ family of continuous or holomorphic functions.

Required for applications (Riemann-mapping & Picard's theorems)

---

[i] Any sequence determines $\mathcal{F} = \{f_1, f_2 \dots f_n \dots\}$ = family.

[ii] $\mathcal{F} = \{ f : \Delta(0,1) \longrightarrow \mathbb{C}$ holomorphic

$$f(z) = \sum_{k=1}^{\infty} a_k z^k, \quad |a_k| \le k \}$$

[iii] $\mathcal{F} = \{ f : \Delta(0,1) \longrightarrow \mathbb{C}$ holomorphic, $f(0) = 1$, $\operatorname{Re} f > 0 \}$

Def $\mathcal{F}$ is normal if all sequences in $\mathcal{F}$ admit a locally uniformly convergent subsequence.

Remark The limit does not have to be in $\mathcal{F}$.

Example

[1] $\mathcal{F}$ normal family of holomorphic functions

$\Longrightarrow$ $\mathcal{F}'$ is normal where $\mathcal{F}' = \{ f' : f \in \mathcal{F} \}$

Proof Definition + Weierstraß Convergence

Let $\{f_n'\} \subseteq \mathcal{F}'$ be a sequence with $f_n \in \mathcal{F}$.

Pick a subsequence $f_{n_k} \overset{l.u.}{\rightrightarrows} f$ By Weierstraß, $f_{n_k}' \overset{l.u.}{\rightrightarrows} f'$ showing $\mathcal{F}'$ is normal.

Remark

We can define $\mathcal{F}$ uniformly bounded, locally uniformly bounded etc just as before.

Examples

$\boxed{1}$ $\mathcal{F} = \left\{ f : \Delta(0,1) \longrightarrow \mathbb{C} \text{ holomorphic}, \ f = \sum_{k=1}^{\infty} a_k z^k, \ |a_k| \leq k \right\}$

locally uniformly bounded.

Indeed, since all compacts $K \subseteq \overline{\Delta}(0,r)$ suffices to work over $\overline{\Delta}(0,r)$. Then

$$|f(z)| \leq \sum_{k=1}^{\infty} |a_k| \, |z|^k \leq \sum_{k=1}^{\infty} k \, r^k = \frac{r}{(1-r)^2} \quad \forall \, |z| \leq r$$
$$\forall \, f \in \mathcal{F}$$

$\Longrightarrow$ $\mathcal{F}$ locally uniformly bounded.

$\boxed{ii}$ $\mathcal{F}$ family of holomorphic functions in $\mathcal{U}$

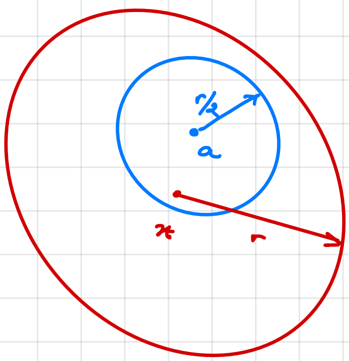
$\mathcal{F}$ locally uniformly bounded. $\Rightarrow$

$\mathcal{F}'$ locally uniformly bounded.

Proof Cauchy's estimates.

Take $x \in \mathcal{U}$. $\Rightarrow$ $\exists\, \Delta(x,r) \subseteq \mathcal{U}$ such that $\forall f \in \mathcal{F}$:

$$|f| \leq M \quad \text{over} \quad \Delta(x,r).$$



We bound $|f'|$ over $\Delta(x, r/2)$.

Let $a \in \Delta(x, r/2)$. By Cauchy's estimate

$$|f'(a)| \leq \frac{\sup |f| \text{ over } \overline{\Delta}(a, r/2)}{r/2} \leq \frac{M}{r/2}.$$

where we used $\overline{\Delta}(a, r/2) \subseteq \Delta(x,r)$.

$\boxed{iii}$ We have seen that $\mathcal{F} = \{z^n\}_n$ is uniformly bounded but $\mathcal{F}' = \{n z^{n-1}\}_n$ is not uniformly bounded

Montel Rephrased (Dream Statement)

$\mathcal{F}$ family of holomorphic functions in $\mathcal{U} \subseteq \mathbb{C}$

$\mathcal{F}$ locally uniformly bounded $\iff$ $\mathcal{F}$ normal.

Remark Both sides are well behaved under taking derivatives as we noted.

"Une suite infinie de fonctions analytiques et bornées à l'intérior d'un domaine simplement connexe, admet au moins une fonction limite à l'intérior de ce domaine"

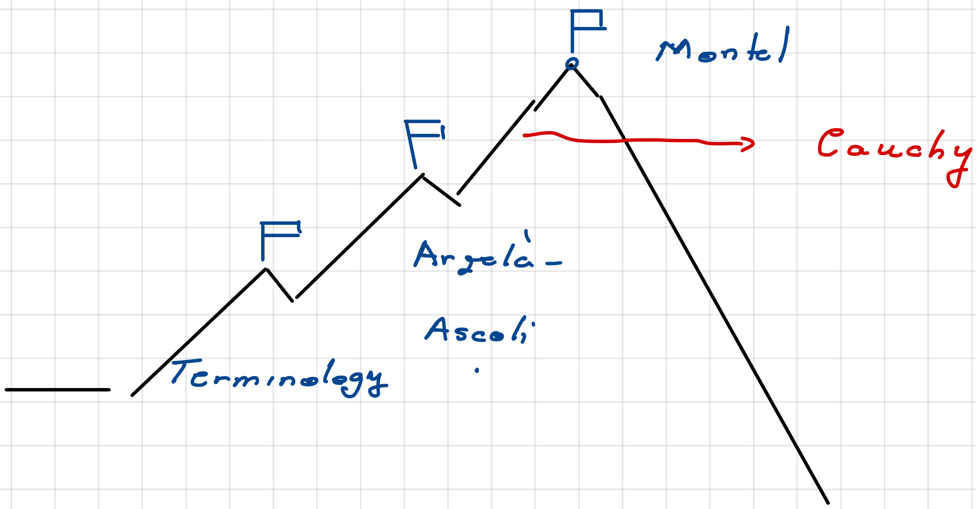
P. Montel 1907

Paul Montel (1876 - 1975) studied normal families of functions. He proved the above theorem in his thesis in 1907. In 1927 he published a monograph on normal families

Students: Cartan, Dieudonné

Math 220B — Lecture 12

February 1, 2021



I. Last time

Point #1 All notions we use are local e.g.

local boundedness, local uniform convergence, local equicontinuity (today)

Point #2 Work with families

$\mathcal{F}$ family of continuous or holomorphic functions in $\mathcal{U}$

Definitions

locally uniformly

$\mathcal{F}$ normal $\iff$ every sequence in $\mathcal{F}$ has convergent subsequence

$\mathcal{F}$ locally bounded (LB) $\iff$ $\forall x \; \exists \Delta_x \subseteq \mathcal{U}$, $\mathcal{F}|_{\Delta_x}$ uniformly bounded

i.e. $\exists M > 0 \; \forall f \in \mathcal{F}$, $|f| \leq M$ in Δ_x.

Montel's Theorem $\mathcal{F}$ family of holomorphic functions in $\mathcal{U}$.

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ locally bounded.

This fails in real analysis,

$\mathcal{F} = \{\sin nx\}_n$ locally bounded in $\mathbb{R}$ & not normal.

(we can't even arrange pointwise convergence)

Question c.2. What is the correct statement in real analysis i.e. continuous functions?

Remark

This requires the notion of equicontinuity.

There will be several versions.

II. Notions of Equicontinuity

strongest

1. $\mathcal{F}$ equicontinuous on $\mathcal{U}$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall \ |x-y| < \delta \quad \forall f \in \mathcal{F} : \quad |f(x) - f(y)| < \varepsilon .$$

Main Point: If $\mathcal{F} = \{f\}$ this says f uniformly continuous.

In general, this says

all $f \in \mathcal{F}$ are uniformly continuous, "uniformly".

that is, the same δ in the definition of uniform continuity works for all $f \in \mathcal{F}$, uniformly.

[I] Fix $M > 0$. The family

$$\mathcal{F} = \{ f : (0,1) \longrightarrow \mathbb{R},\ |f(x) - f(y)| \le M\, |x-y| \}$$ equicontinuous.

Suffices to take $\delta = \dfrac{\varepsilon}{M}$ and note

$$|x-y| < \delta \implies |f(x) - f(y)| \le M\, |x-y| < \varepsilon \quad \forall f \in \mathcal{F}.$$

[II] $\mathcal{F} = \{ f = \sum_{k=0}^{2021} a_k x^k ,\ |a_k| \le 1 \}$ equicontinuous on $[-1,1]$.

$$\left| \frac{f(x) - f(y)}{x-y} \right| = \left| \sum_{k=0}^{2021} a_k \left(x^{k-1} + \dots + y^{k-1} \right) \right|$$

$$\le \sum_{k=0}^{202} |a_k| \left(|x|^{k-1} + \dots + |y|^{k-1} \right)$$

$$\le \sum_{k=0}^{2021} 1 \cdot \underbrace{(1 + \dots + 1)}_{k} = \sum_{k=0}^{2021} k = M$$ & use part [I]

[III] $\mathcal{F} = \{ f_n \}$; $f_n(x) = nx$ not equicontinuous in $[0,1]$.

[IV] See also the Proposition at the end of lecture.

Variations

I equicontinuous

II equicontinuous at each point (Conway).

$\forall\, x \in \mathcal{U} \quad \forall\, \varepsilon > 0 \quad \exists\, \Delta(x,\delta) \text{ s.t. } \forall\, y \in \Delta(x,\delta) \Rightarrow |f(y) - f(x)| < \varepsilon.$
$\forall\, f \in \mathcal{F}$

When $\mathcal{F} = \{f\}$ this says f is continuous at each point.

III locally equicontinuous

$\forall\, x \quad \exists\, \Delta_x \subseteq \mathcal{U}, \quad \mathcal{F}/_{\Delta_x}$ is equicontinuous

---

IV equicontinuous on all compacts (Rudin, Ahlfors, us)

$\forall\, K \subseteq \mathcal{U}$ compact, $\mathcal{F}/_K$ equicontinuous

$\boxed{ii}$ – $\boxed{iii}$ – $\boxed{iv}$ are equivalent.

$\boxed{iv} \Rightarrow \boxed{iii}$ Just use $K = \overline{\Delta}_*$ where Δ_* is

a bounded neighborhood of $*$ in $\mathcal{U}$.

$\boxed{iii} \Rightarrow \boxed{ii}$ clear from definitions

$\boxed{ii} \Rightarrow \boxed{iv}$ requires a compactness argument

(HWK 4, #6 or Conway VII.1).

III. Question C Characterization of normality?

Theorem (Arzelà - Ascoli)

$\mathcal{F}$ family of continuous functions

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ is locally equicontinuous & locally bounded.

Theorem (Montel) $\mathcal{F}$ family of holomorphic functions.

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ locally bounded.

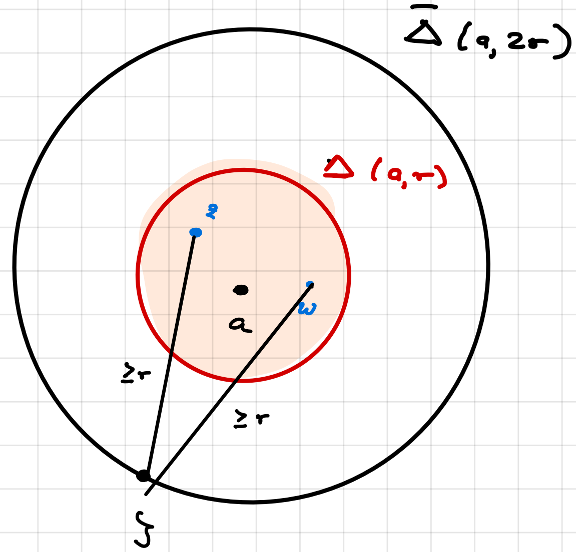
Question D Why is local equicontinuity needed in real analysis?

Question E Why is local equicontinuity NOT needed in complex analysis?

Answer to E

Proposition $\mathcal{F}$ family of holomorphic functions.

$\mathcal{F}$ is locally bounded $\Longrightarrow$ $\mathcal{F}$ is locally equicontinuous.



Proof Fix $a \in \mathcal{U}$.

$\Longrightarrow \exists\ \bar{\Delta}(a, 2r)$ such that

$\mathcal{F}/_{\bar{\Delta}(a,2r)}$ is bounded by M.

Claim $\mathcal{F}/_{\Delta(a,r)}$ is equicontinuous.

Fix $\varepsilon > 0$. Let $z, w \in \Delta(a,r)$. Take $f \in \mathcal{F}$.

$$|f(z) - f(w)| = \left| \frac{1}{2\pi i} \int_{|\zeta - a| = 2r} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{|\zeta - a| = 2r} \frac{f(\zeta)}{\zeta - w} d\zeta \right| \quad \swarrow \text{ Cauchy's formula}$$

$$= \frac{1}{2\pi} \left| \int_{|\zeta - a| = 2r} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta \right|$$

$$= \frac{1}{2\pi} \left| \int_{|\zeta - a| = 2r} f(\zeta) \cdot \frac{z - w}{\underbrace{(\zeta - z)}_{\geq r} \underbrace{(\zeta - w)}_{\geq r}} d\zeta \right|$$

$$\leq \frac{1}{2\pi} \cdot M \cdot |z-w| \cdot \frac{1}{r^2} \cdot 2\pi \cdot (2r)$$

$$= \frac{2M}{r} \cdot |z-w| = K\,|z-w| \quad \text{for} \quad K = \frac{2M}{r}.$$

The Claim follows by Example $\boxed{c}$ above, or directly,

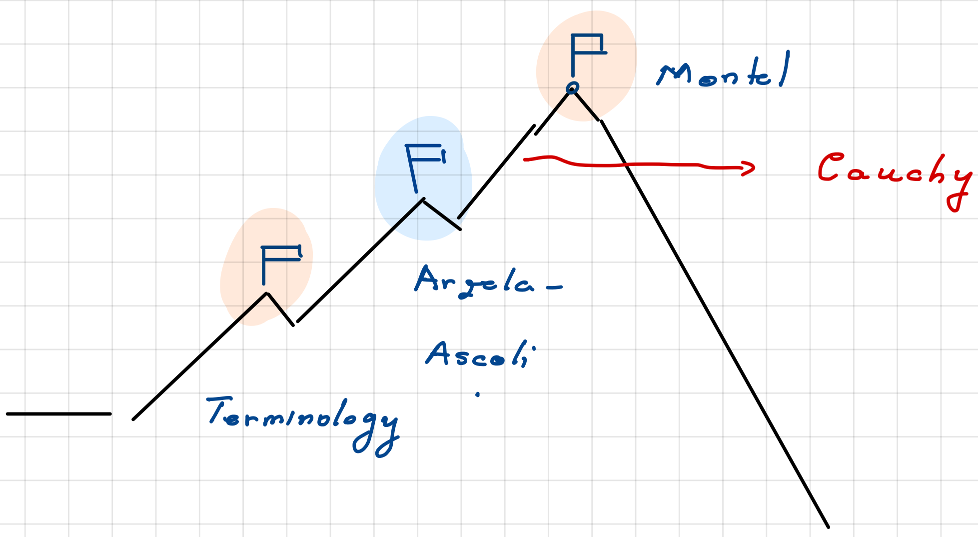
let $\delta = \frac{\varepsilon}{K}$. If $|z-w| < \delta \Rightarrow |f(z) - f(w)| \leq K\,|z-w| < \varepsilon$.

QED

Conclusion

Proposition + Arzelà - Ascoli $\Longrightarrow$ Montel

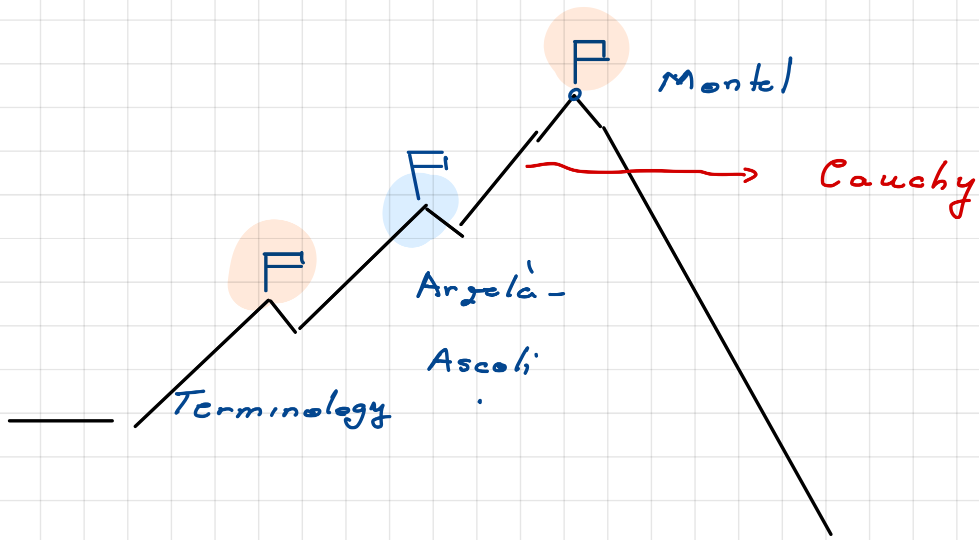
We only prove Arzelà - Ascoli (next time)



Math 220 B — Lecture 13

February 3, 2021

Last time



Arzelá-Ascoli $\mathcal{F}$ family of continuous functions in $\mathcal{U}$

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ locally equicontinuous and

locally bounded.

Today · — we give the proof.

All functions today are continuous.

Notation & Preliminaries

$f : U \longrightarrow \mathbb{C}$ continuous, $K \subseteq U$ compact

$$\|f\|_K = \sup_{z \in K} |f(z)|$$

Note

[i] $\|f+g\|_K \leq \|f\|_K + \|g\|_K$

[ii] $f_n \overset{K}{\rightrightarrows} f \iff \|f_n - f\|_K \longrightarrow 0$ as $n \longrightarrow \infty$.

Def f_n is uniformly Cauchy in K if

$\forall \varepsilon > 0 \quad \exists N \quad \forall n, m \geq N, \quad \|f_n - f_m\|_K < \varepsilon.$

Lemma f_n converges uniformly in K

$\iff$ f_n uniformly Cauchy in K.

Proof We will only use "$\Leftarrow$" so we only give its proof.

Fix $\varepsilon > 0 \Rightarrow \exists N$ with $|f_n(z) - f_m(z)| < \varepsilon$ $\forall n, m \geq N$.

$\forall z \in K$. $(*)$

Thus $\{f_n(z)\}$ is Cauchy for fixed z. Then $\{f_n(z)\}$ converges pointwise to $f(z)$. Make $m \to \infty$ in $(*)$ to conclude that

$\forall \varepsilon \quad \exists N$ with $|f_n(z) - f(z)| \leq \varepsilon$ $\forall n \geq N$, $z \in K$.

Thus $f_n \rightrightarrows f$ in K.

Proof of Arzelà – Ascoli

„$\Longrightarrow$" Let $\mathcal{F}$ be normal.

(1) $\mathcal{F}$ locally bounded

Let $K \subseteq \mathcal{U}$ compact. We show $\mathcal{F}|_K$ bounded. i.e.

$$\exists\, M > 0 \quad \forall f \in \mathcal{F} \implies \|f\|_K < M.$$

Assume not for a contradiction. Then

$$\forall\, M > 0 \quad \exists\, f_M \in \mathcal{F} \text{ with } \|f_M\|_K \geq M$$

Letting $M = n.$, we obtain a sequence f_n with $\|f_n\|_K \geq n$.

Since $\mathcal{F}$ normal, we can find a subsequence $f_{n_k} \overset{K}{\rightrightarrows} f$

Thus $\|f_{n_k} - f\|_K < 1$ if k sufficiently large.

Note f_{n_k} continuous $\Rightarrow$ f continuous. so $\|f\|_K < M$. Then

$$M. > \|f\|_K \geq \|f_{n_k}\|_K - \|f_{n_k} - f\|_K \geq n_k - 1 \longrightarrow \infty \text{ as } k \to \infty$$

This gives a contradiction.

(2) $\mathcal{F}$ locally equicontinuous

Let $K \subseteq U$ compact. We show $\mathcal{F}/_K$ equicontinuous.

that is $\forall \varepsilon\ \exists \delta : \forall\ x, y \in K,\ |x-y| < \delta\ \forall f \in \mathcal{F}$ then

$$|f(x) - f(y)| < \varepsilon.$$

Assume not, then

$$\exists \varepsilon \quad \forall \delta \quad \exists\ x_\delta, y_\delta \in K \text{ with } |x_\delta - y_\delta| < \delta\ \exists f_\delta \in \mathcal{F} \text{ but}$$

$$|f_\delta(x_\delta) - f_\delta(y_\delta)| \geq \varepsilon.$$

Take $\delta = \frac{1}{n}$. Then

$$\exists\ x_n, y_n \in K, \quad |x_n - y_n| < \frac{1}{n} \quad \exists f_n \in \mathcal{F} \quad \text{with}$$

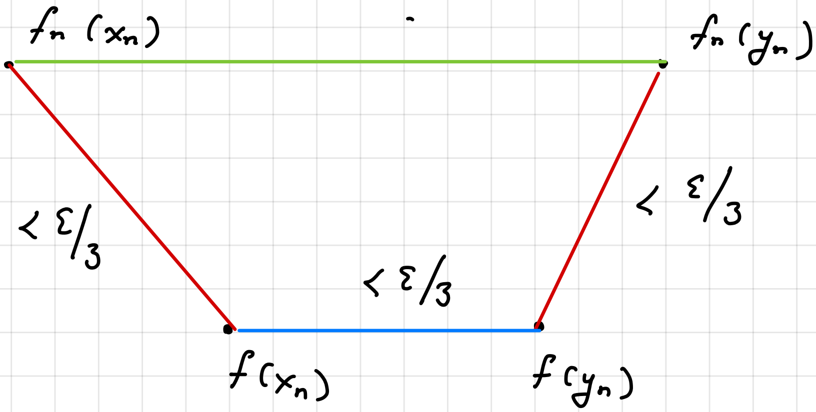
$$|f_n(x_n) - f_n(y_n)| \geq \varepsilon.$$

After passing to a subsequence & relabelling, we arrange

[i] $f_n \overset{K}{\rightrightarrows} f$ because $\mathcal{F}$ normal

[ii] $|x_n - y_n| < \frac{1}{n}$

[iii] $|f_n(x_n) - f_n(y_n)| \geq \varepsilon.$



Using f_n continuous, $f_n \rightrightarrows f$ we get f continuous.

Since K compact $\Rightarrow f|_K$ uniformly continuous.

Then $\exists\ \tau > 0$ with

$$|x-y| < \tau\ ,\quad x, y \in K \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon/3 . \qquad (1)$$

Let N be so that $\forall\ n \geq N$, we have $\frac{1}{n} < \tau$ and

$$\| f_n - f \|_K < \varepsilon/3 \qquad . \ (2)$$

Then $|x_n - y_n| < \frac{1}{n} < \tau \Rightarrow |f(x_n) - f(y_n)| < \varepsilon/3$ by (1).

$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{3}$ & $|f_n(y_n) - f(y_n)| < \varepsilon/3$ by (2).

By triangle inequality (see picture)

$$|f_n(x_n) - f_n(y_n)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

contradicting $\boxed{\text{UC}}$

The Converse

Assume $\mathcal{F}$ is locally equicontinuous & locally bounded.

$$\overset{?}{\Longrightarrow} \quad \mathcal{F} \text{ normal}$$

Let $f_n \in \mathcal{F}$. We wish to find a subsequence converging locally uniformly?

How do we find such a subsequence?

Plan $\boxed{\text{I}}$ arrange pointwise convergence of f_n

$\boxed{\text{II}}$ show local uniform convergence

Better Plan $\boxed{\text{I}}$ arrange pointwise convergence of f_n only at a countable dense set

$\boxed{\text{II}}$ show local uniform convergence

Let $\{a_k\}$ be the set of points in $\mathcal{U}$ with rational coordinates enumerated in any order. Dense!

Claim $\boxed{\text{i}}$ After passing to a subsequence of f_n & relabelling, we may assume

$(*)$ $\forall k$, the sequence $f_n(a_k)$ converges as $n \longrightarrow \infty$.

Claim $\boxed{\text{ii}}$ If $\{f_n\}$ equicontinuous & $(*)$ $\Rightarrow$ f_n converges locally uniformly.

We win!

Proof of Claim $\boxed{2}$ Cantor diagonalization

We only use pointwise boundedness of $\{f_n\}$.

Consider $f_1(a_1)\ \ f_2(a_1)\ \ \dots\ \ f_n(a_1)\ \dots$ bounded

Find a subsequence

$(S_1)\qquad f_{11},\ f_{12},\ \dots\ f_{1n},\ \dots$ ← converges at a_1

Look at the values of (S_1) at a_2 & repeat. We find

$\underline{\underline{(S_2)}}\qquad f_{21},\ f_{22},\ \dots\ f_{2n}\ \dots$ ← converges at a_2 and a_1

Look at the values of (S_2) at a_3 & repeat.

We obtain an array:

$$\begin{array}{ccccccccl}
 & f_1 & f_2 & \cdots & & f_n & \cdots & & \text{converges} \\
(S_1) & \boxed{f_{11}} & f_{12} & f_{13} & \cdots & f_{1n} & \cdots & & \text{at } a_1 \\
(S_2) & f_{21} & \boxed{f_{22}} & f_{23} & \cdots & f_{2n} & \cdots & & \text{at } a_1, a_2 \\
(S_3) & f_{31} & f_{32} & \boxed{f_{33}} & \cdots & f_{3n} & \cdots & & \text{at } a_1, a_2, a_3 \\
 & \cdots & & & \ddots & & & &
\end{array}$$

Each row is a subsequence of the previous one.

Consider the diagonal subsequence

$$f_{11},\ f_{22},\ f_{33},\ \cdots\ f_{nn}\ \cdots$$

It is a subsequence of the original sequence. & converges at each a_k. Indeed

$$\underbrace{f_{11},\ f_{22},\ \cdots\ f_{k-1,k-1},}_{\text{initial terms}}\ \Big/\ \underbrace{f_{kk},\ f_{k+1,k+1},\ \cdots}_{\text{part of } (S_k) \text{ so we have convergence at } a_k.}$$

Proof of Claim [II]

Know [a] $\{a_k\}$ dense in $\mathcal{U}$ and

$\forall k$, the sequence $\{f_n(a_k)\}_n$ converges

[b] f_n locally equicontinuous

Wish $\forall\, \alpha \in \mathcal{U}$, $\exists$ Δ = bounded open ball in $\mathcal{U}$, $\alpha \in \mathcal{U}$

$f_n/_{\bar{\Delta}}$ converges uniformly.

---

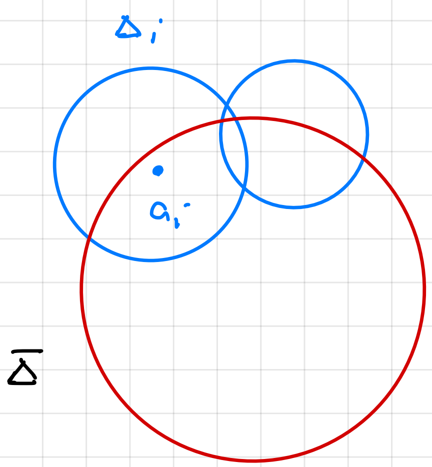
(1) $\forall \alpha$ $\exists$ $\alpha \in \bar{\Delta}$, $\mathcal{F}/_{\bar{\Delta}}$ equicontinuous.

Thus $\forall\, \varepsilon$ $\exists\, \delta$: $\forall\, |x-y| < \delta$, $x, y \in \bar{\Delta}$, $\forall\, f \in \mathcal{F}$

$$|f(x) - f(y)| < \varepsilon/3$$

---

(2) $\bar{\Delta}$ can be covered by $\Delta_i = \Delta(a_i, \delta)$ for $a_i \in \bar{\Delta}$.



This because $\{a_i\}_i \cap \bar{\Delta}$ is dense in $\bar{\Delta}$.

By compactness, we may assume

$$\bar{\Delta} \subseteq \bigcup_{i=1}^{\ell} \Delta(a_i, \delta).$$

(3) Since $\{f_n(a_i)\}_{i=\overline{1,\ell}}$ is convergent, it is Cauchy. Hence

$$\forall \varepsilon \quad \exists N \quad \forall n,m \geq N \quad \forall 1 \leq i \leq \ell$$

$$|f_n(a_i) - f_m(a_i)| < \varepsilon/3$$

---

(4) Let $z \in \overline{\Delta}$ By (2), $\exists i$ with $|z - a_i| < \delta$. Let $n,m \geq N$.

as in (3). Then

$$|f_n(z) - f_m(z)| \leq \underbrace{|f_n(z) - f_n(a_i)|}_{\text{use (1)}} + \underbrace{|f_n(a_i) - f_m(a_i)|}_{\text{use (3)}} + \underbrace{|f_m(a_i) - f_m(z)|}_{\text{use (1)}}$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

---

(5) <u>Conclusion</u> $\|f_n - f_m\|_{\overline{\Delta}} < \varepsilon \quad \forall n,m \geq N$.

$\Longrightarrow \{f_n\}$ uniformly Cauchy in $\overline{\Delta}$

$\overset{\text{Lemma}}{\Longrightarrow} \{f_n\}$ converges uniformly in $\overline{\Delta}$.

This completes the proof.

Remark The converse only used pointwise boundedness

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ pointwise bounded + locally equicont

$\iff$ $\mathcal{F}$ locally bounded + locally equicont.

The second version bears connections with Montel & it is more uniform.

Math 220B — Lecture 14

February 5, 2021

0. Logistics

(1) Poll regarding Math 220C

[i] MW 3 – 4:20

[ii] live / recorded / half live – half recorded?

(2) Midterm – Friday 12 – take home

will cover everything up to and including Monday

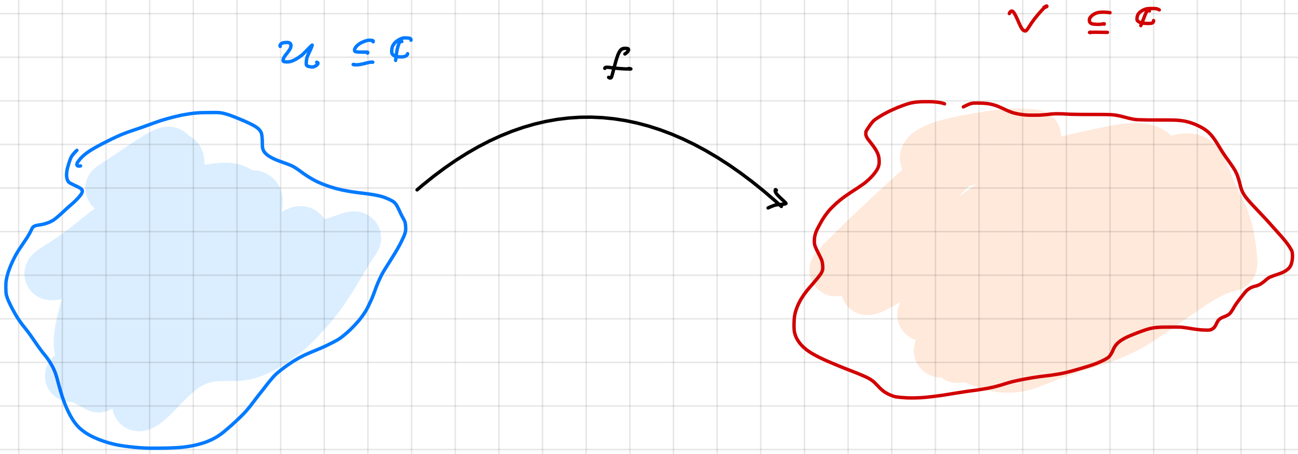
Conflicts ?

Topics we covered:

- Infinite Products, Γ function, sine
- Weierstraß factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz lemma & applications

220B Part II — Mapping Theory

The goal is to frame the discussion & formulate guiding questions.



Given $\mathcal{U}, V \subseteq \mathbb{C}$ we wish to study holomorphic

$$f : \mathcal{U} \longrightarrow V.$$

This may be too general. We can ask

[i] f injective

[ii] f finite to one

[iii] f bijective

[iv] f proper ... etc.

We will focus on bijective holomorphic maps.

Remark

[a] Final Exam, Math 220A, we showed

Let $U \subset \mathbb{C}$ be an open set containing 0. Let $f : U \to \mathbb{C}$ be an injective holomorphic function. Show that $f'(0) \neq 0$.

The same argument works for any U & any point of U:

$f: U \longrightarrow V$ injective holomorphic $\Rightarrow$ f' has no zeroes.

[b] In Math 220A, Lecture 11, we showed

Example $f: U \longrightarrow V$ bijective, holomorphic & $f'(a) \neq 0$ $\forall\, a \in U$. Then f^{-1} holomorphic.

Conclusion $f: U \longrightarrow V$ holomorphic & bijective

$\Rightarrow$ f^{-1} holomorphic

Biholomorphism = holomorphic + bijective

We focus on biholomorphisms

Question A Given $U, V \subseteq \mathbb{C}$ are U, V biholomorphic?

Remark This has implications in topology & differential geometry. In particular U, V are

homeomorphic, diffeomorphic

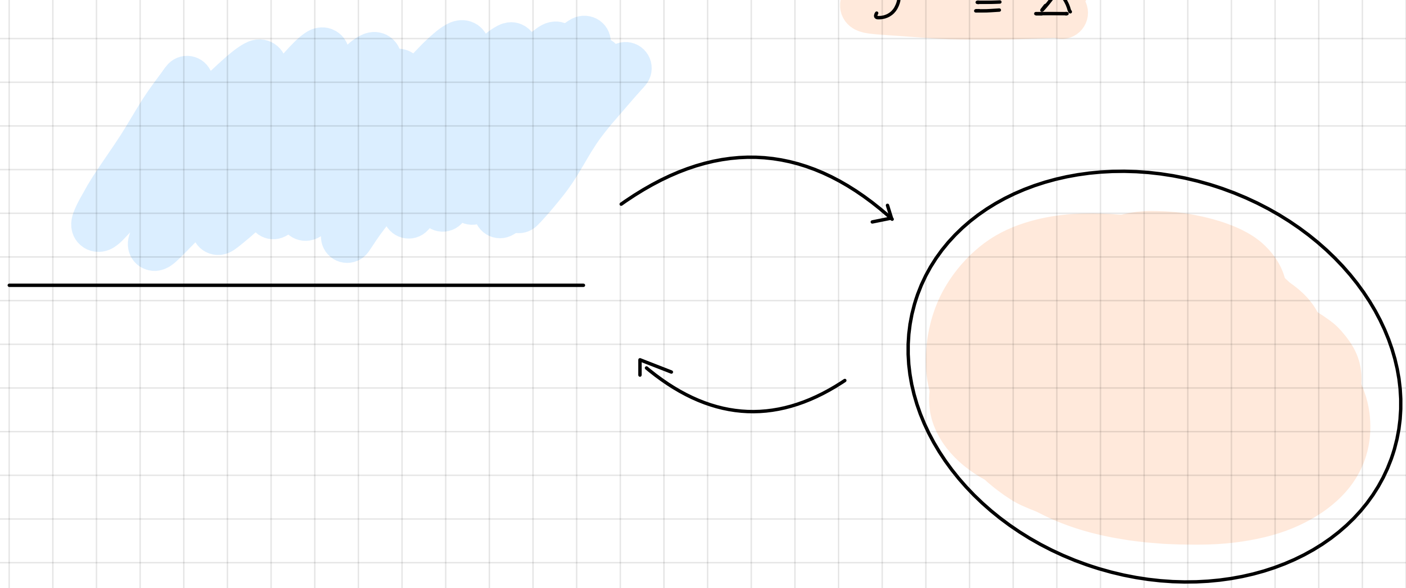
Examples

$\boxed{I}$ $U = \mathbb{C}$, $V = \Delta(0,1)$, $U \not\cong V$. This follows by Liouville's theorem.

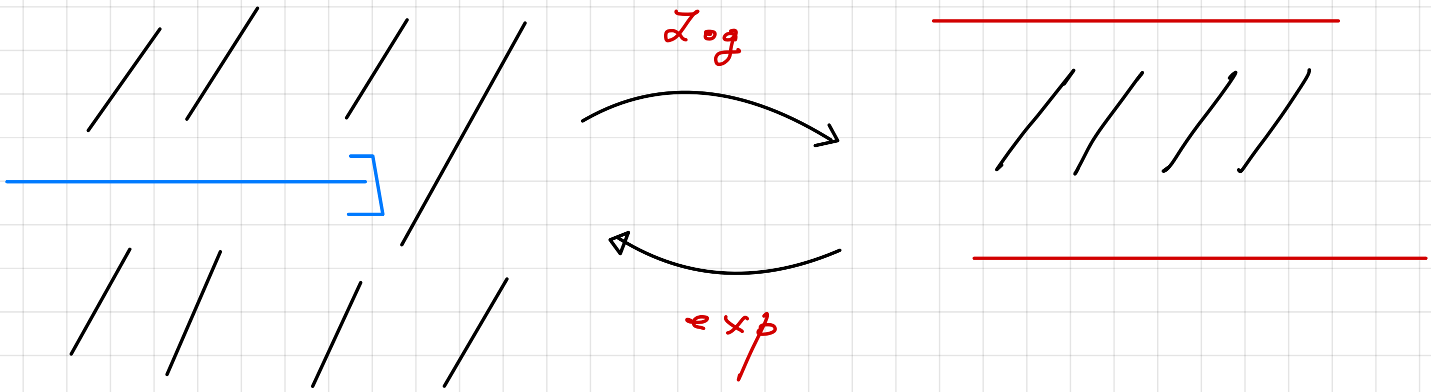
$\boxed{II}$ $U = \mathfrak{H}^+$, $V = \Delta$, $c : \mathfrak{H}^+ \longrightarrow \Delta$ Math 220A

Cayley transform: $c(z) = \dfrac{z-i}{z+i}$, $c^{-1}(w) = i \cdot \dfrac{1-w}{1+w}$.

$\mathfrak{H}^+ \cong \Delta$



III $\quad \mathcal{U} = \mathbb{C} \setminus \mathbb{R}_{\leq 0}, \quad V = \text{strip } -\pi < \operatorname{Im} z < \pi$



This is Homework 2, Math 220A.

Very Important Theorem (Riemann Mapping Theorem)

Given $\mathcal{U}, V \neq \mathbb{C}$, $\mathcal{U}, V$ simply connected $\Rightarrow \mathcal{U}, V$ are biholomorphic.

In particular, if $V = \Delta(0,1)$, then any $\mathcal{U} \neq \mathbb{C}$ simply connected then $\mathcal{U}$ is biholomorphic to $\Delta(0,1)$.

— Riemann's dissertation (1851) sketched a proof

— Refereed by Gauss

"The whole is a solid work of high quality, not merely fulfilling the requirements usually set for doctoral thesis, but far surpassing them."

— it took the effort of many great minds

Weierstraß, Carathéodory, Hilbert, Schwarz, Koebe, Fejér, Riesz & others to finalize the proof.

Question B

Given $\mathcal{U}, \mathcal{V} \subseteq \mathbb{C}$ biholomorphic can we construct

$\boxed{i}$ one biholomorphism $\mathcal{U} \longrightarrow \mathcal{V}$ explicitly?

$\boxed{ii}$ all biholomorphism $\mathcal{U} \longrightarrow \mathcal{V}$ explicitly?

Special cases of $\boxed{i}$

We saw some specific examples above e.g. the Cayley transform for $\mathcal{G}^+$ and $\Delta(0,1)$.

When $\mathcal{U} = V$, Question B $\boxed{ii}$ becomes.

Question C

What are all biholomorphisms $f : \mathcal{U} \longrightarrow \mathcal{U}$?

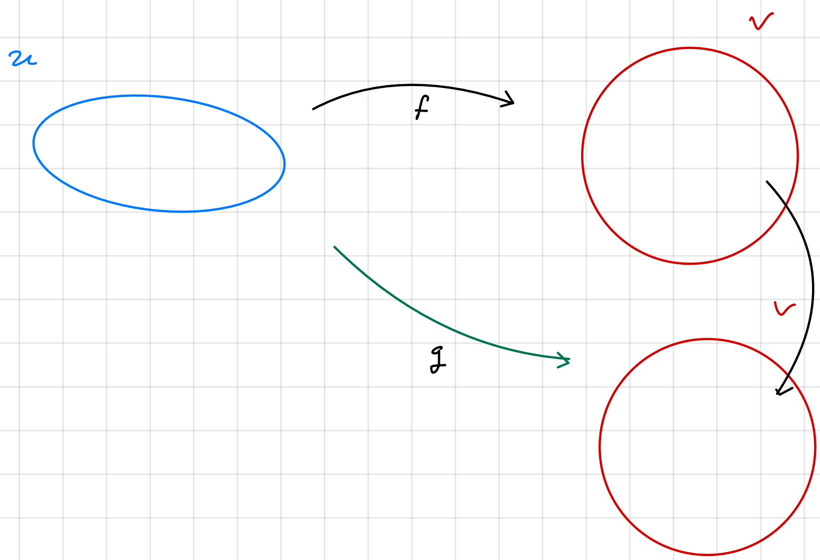
Remarks

$\boxed{i}$ $\mathrm{Aut}(\mathcal{U}) = \{ f : \mathcal{U} \longrightarrow \mathcal{U} : f$ holomorphic & bijective $\}$

is a group. Indeed $f \in \mathrm{Aut}(\mathcal{U}) \Rightarrow f^{-1} \in \mathrm{Aut}(\mathcal{U})$ using that f^{-1} is automatically holomorphic by the above remarks.

$\boxed{ii}$ Examples: We can consider this question for $\mathcal{U} = \Delta, \mathfrak{H}^+, \mathbb{C}, \Delta^\times, \mathbb{C}^\times$ etc...

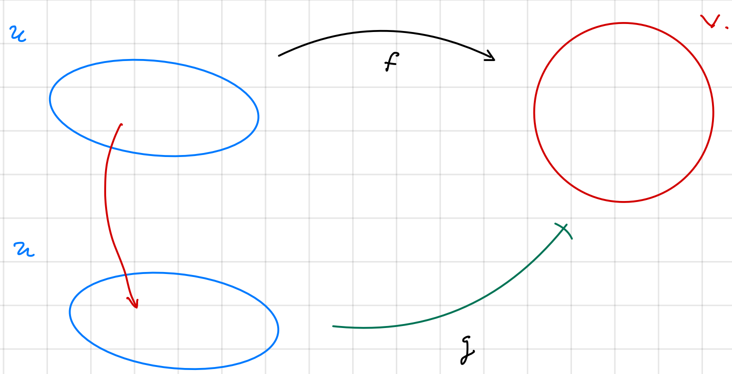
$\boxed{141}$ If $f, g : U \longrightarrow V$, f = known biholomorphism then any other biholomorphism $g : U \longrightarrow V$ differs from f by automorphisms:



$$g = \Phi \circ f$$

$$\Phi \in Aut(V)$$

Indeed, $\Phi = g \circ f^{-1}$



In the same fashion

$g = f \circ \varphi$ where $\varphi = g \circ f^{-1}$

and $\varphi \in Aut(U)$.

Thus knowledge of Question C helps with aspects of Question B.

Question D

Is the action of $Aut(\mathcal{U})$ on $\mathcal{U}$ transitive i.e.

$\forall\, a, b \in \mathcal{U} \quad \exists\, f \in Aut(\mathcal{U})$ with $f(a) = b$?

Example $\mathcal{U} = \mathbb{C} \cup \{\infty\}$. FLT are automorphisms of $\mathcal{U}$

& action is transitive. (Math 220A)

Question E

Given $a \in \mathcal{U}$, describe $f : \mathcal{U} \longrightarrow \mathcal{U}$

biholomorphism, with $f(a) = a$.

Many other questions can be asked.

We begin the discussion with the case

$$\mathcal{U} = \Delta(0,1) = \Delta.$$

The crucial statement is <u>Schwarz Lemma</u>

<u>Theorem</u> Given $f : \Delta \longrightarrow \Delta$, $\Delta = \Delta(0,1)$ holomorphic, $f(0) = 0$.

then $\boxed{\text{i}}$ $|f'(0)| \leq 1$ and

$\boxed{\text{ii}}$ $|f(z)| \leq |z|$.

If $|f'(0)| = 1$ or if $|f(z_0)| = |z_0|$ for some $z_0 \in \Delta \setminus \{0\}$ then

f is a rotation, $f(z) = e^{i\alpha} z$. $\forall z \in \Delta$.

<u>Proof</u> – next time.

Math 220B — Lecture 15

February 8, 2021

Midterm Exam

(1) 4 - 5 Questions

- Infinite Products, Γ function, sine
- Weierstraß factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz lemma & applications

(2) Available on Friday at noon, due Tuesday at noon.

You can think about the Questions for as long as you wish in this interval.

I. Schwarz Lemma - Conway VI. 2.

$\Delta = \Delta(0,1)$

Theorem Given $f: \Delta \longrightarrow \Delta$, $f(0)=0$ then

[i] $|f'(0)| \leq 1$, and

[ii] $|f(z)| \leq |z|$

If either $|f'(0)|=1$ or $\exists\, z_0 \neq 0$ with $|f(z_0)| = |z_0|$ then f is a rotation i.e. $f(z) = e^{i\theta} z$

Proof Let
$$g(z) = \begin{cases} \dfrac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}.$$
By the removable singularity theorem (Lecture 13, Math 220A), g is holomorphic.

This uses $f(0)=0$.

Let $0 < r < 1$. Then for $|w|=r$,

$$|g(w)| = \frac{|f(w)|}{|w|} \leq \frac{1}{r} \quad \text{since } \operatorname{Im} f \subseteq \Delta.$$

By maximum modulus principle,

$$\sup_{|w|\leq r} |g(w)| = \sup_{|w|=r} |g(w)| \leq \frac{1}{r}.$$

In particular, for all $|z| < r < 1$, we have

$$|g(z)| \leq \frac{1}{r}$$

Make $r \to 1$ keeping z fixed. Then $|g(z)| \leq 1$. In particular

$|g(0)| = |f'(0)| \leq 1$ & $|f(z)| \leq |z|$.

If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for $z_0 \neq 0$ then either

$|g(0)| = 1$ or $|g(z_0)| = 1$. Since $|g(z)| \leq 1$ $\forall z$ then g must be

constant by MMP again. Thus $g(z) = e^{i\theta}$. $\Rightarrow$ $f(z) = e^{i\theta} z$.

Corollary. $f : \Delta \longrightarrow \Delta$ biholomorphism, $f(0) = 0$ then f is a rotation.

Proof Note $f(0) = 0 \Rightarrow f^{-1}(0) = 0$. We apply Schwarz to both f, f^{-1}: We obtain

$|f(z)| \leq z$ and $|f^{-1}(w)| \leq w$. Let $w = f(z)$ to get $|z| \leq |f(z)|$. Therefore $|f(z)| = |z|$ $\forall z \Rightarrow f$ rotation.

II. Automorphisms of the unit disc $\Delta = \Delta(0,1)$.

Question What can we do if we are given

$$f : \Delta \longrightarrow \Delta \quad \text{with} \quad f(0) = a \neq 0 \,, \quad |a| < 1 .$$

Key Idea

$$\exists \; \varphi_a : \Delta \longrightarrow \Delta \quad \text{with} \quad \varphi_a(a) = 0 .$$

We can then recenter f by considering $\tilde{f} = \varphi_a \circ f$.

$$\begin{tikzcd}
0 \arrow[r, bend left, "f"] \arrow[rr, bend right, "\varphi_a \circ f"'] & a \arrow[r, bend left, "\varphi_a"] & 0
\end{tikzcd}$$

Specifically

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a} z} .$$

Important Properties

[i] $\varphi_a : \Delta \longrightarrow \Delta$, $\varphi_a : \partial\Delta \longrightarrow \partial\Delta$

[ii] $\varphi_a(0) = -a$, $\varphi_a(a) = 0$

[iii] φ_a, φ_{-a} are inverses

[iv] $\varphi_a'(0) = \underbrace{1 - |a|^2}_{\text{shrinks } <1.}$, $\varphi_a'(a) = \underbrace{\dfrac{1}{1-|a|^2}}_{\text{expands } >1}$.

Proof [ii] – [iv] follow by direct calculation.

[i] Note that $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$ has pole at $\dfrac{1}{\bar{a}}$ but this is not in $\bar{\Delta}$ since $|a|<1$. Thus φ_a is holomorphic in Δ, continuous in $\bar{\Delta}$. If we show

(*) $|\varphi_a(z)| = 1$ if $|z| = 1$, by the maximum modulus, it follows $|\varphi_a(z)| < 1$ $\forall$ $|z|<1$ so $\varphi_a : \Delta \longrightarrow \Delta$.

To see (*) we show $|z-a| = |1-\bar{a}z|$ if $|z| = 1$.

Note $|1 - \bar{a} z| = |1 - a \bar{z}|$ conjugation

$$= \left|1 - \frac{a}{z}\right| \quad \text{since } z\bar{z} = |z|^2 = 1$$

$$= \frac{|z-a|}{|z|} = |z-a| \quad \text{as needed.}$$

Theorem If $f : \Delta \longrightarrow \Delta$ is biholomorphic then

$$f(z) = e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z} \quad \text{for } |a| < 1.$$

Proof

$$\begin{tikzcd} 0 \arrow[r, bend left, "\varphi_{-a}"] \arrow[rr, bend right, "f \circ \varphi_{-a}"'] & a \arrow[r, bend left, "f"] & 0 \end{tikzcd}$$

Let a be such that $f(a) = 0$. Let

$$\tilde{f} = f \circ \varphi_{-a} \Rightarrow \tilde{f}(0) = 0.$$

Note $\tilde{f}$ is a biholomorphism. Then $\tilde{f}$ is a rotation

$$\Rightarrow \tilde{f}(w) = e^{i\theta} w \Rightarrow f \circ \varphi_{-a}(w) = e^{i\theta} w \Rightarrow f(z) = e^{i\theta} \varphi_a(z).$$

setting $w = \varphi_a(z)$.

Remark We have seen φ_a's in HWK 1.

Blaschke's products

$\varphi : \Delta \longrightarrow \Delta$, $\partial\Delta \longrightarrow \partial\Delta$ then

$$f(z) = c\, z^m \prod_{k=1}^{N} \varphi_{a_k} \quad , \quad |c| = 1.$$

$\rightsquigarrow$ zeroes of f.

Exercise Assume $f : \Delta \longrightarrow \Delta$, $\partial\Delta \longrightarrow \partial\Delta$ whose only zeroes are at $1/2$ & $1/4$ with multiplicities 2 & 3. Find $|f(0)|$.

Solution $f(z) = c\, \varphi_{\frac{1}{2}}^{2}\, \varphi_{\frac{1}{4}}^{3}$. Then

$$f(0) = c \cdot \left(-\frac{1}{2}\right)^2 \cdot \left(-\frac{1}{4}\right)^3, \ |c| = 1 \implies |f(0)| = \frac{1}{2^8}.$$

III. Understanding the action of Aut (Δ) on Δ

Important Remark The action of Aut (Δ) on Δ is transitive $\forall\, a, b \in \Delta \ \exists\, f \in \operatorname{Aut} \Delta,\ f(a) = b.$

$$\begin{tikzcd}
a \arrow[r, bend left, "\varphi_a"] \arrow[rr, bend right, "f"'] & 0 \arrow[r, bend left, "\varphi_{-b}"] & b
\end{tikzcd}$$

Note $f = \varphi_{-b} \circ \varphi_a$ is an automorphism and

$$f(a) = \varphi_{-b}\, \varphi_a(a) = \varphi_{-b}(0) = b.$$

Application – Fixed points

Show if $f : \Delta \longrightarrow \Delta$ holomorphic, $f \neq \mathbb{1} \Rightarrow f$ has at most 1 fixed point.

Proof Assume $f(a) = a$ & $f(b) = b$. & $a \neq b$.

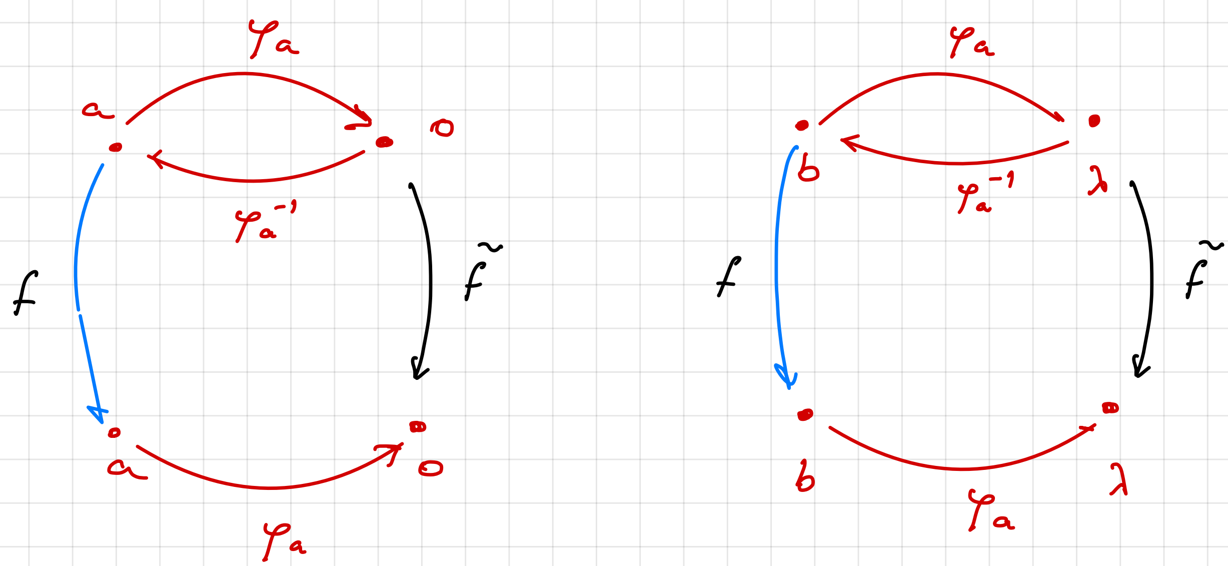
If $a = 0$ then $f(0) = 0$ & $f(b) = b$ $\Rightarrow$ f rotation via

Schwarz $f(z) = e^{i\theta} z$. Using $f(b) = b$ $\Rightarrow$ $e^{i\theta} = 1$ $\Rightarrow$

$\Rightarrow$ $f = \mathbb{1}$ which is disallowed.

For $a \neq 0$, we reduce to this case. Let

$$\tilde{f} = \varphi_a f \varphi_a^{-1}. \quad \& \quad \lambda = \varphi_a(b) \neq 0 = \varphi_a(a).$$



Then $\tilde{f}(0) = 0$ and $\tilde{f}(\lambda) = \lambda$ $\Rightarrow$ $\tilde{f} = \mathbb{1}$ $\Rightarrow$

$\Rightarrow$ $\varphi_a f \varphi_a^{-1} = \mathbb{1}$ $\Rightarrow$ $f = \mathbb{1}$. again a contradiction.

Thus f has at most one fixed point.

Recap

- if $f(0) = 0$ then
  - we proved Schwarz Lemma
  - we determined $f \in \operatorname{Aut} \Delta$, $f(0) = 0$
- if $f(0) \neq 0$
  - we determined $f \in \operatorname{Aut} \Delta$

Question Is there a version of Schwarz if $f(0) \neq 0$?

Yes — Schwarz-Pick Lemma.

- we illustrate it for derivatives

Proposition $f: \Delta \longrightarrow \Delta$ holomorphic, $\forall a \in \Delta$

$$\frac{|f'(a)|}{1-|f(a)|^2} \leq \frac{1}{1-|a|^2}.$$

Remark If $a=0$ this gives $|f'(0)| \leq 1-|f(0)|^2$.

If $f(0)=0$ this gives $|f'(0)| \leq 1$. Thus the Proposition generalizes Schwarz Lemma.

The proof will be given next time.

Remark This is naturally formulated in hyperbolic geometry.

Math 220B — Lecture 16

February 10, 2021

0. Midterm Exam

(1) 5 Questions

- Infinite Products, Γ function, sine
- Weierstraß factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz lemma & applications

(2) Available on Friday at noon, due Tuesday at noon.

You can think about the Questions for as long as you wish in this interval.

(3) Closed book / closed notes / no internet / no collaboration

(4) e-mail if questions arise

(5) you may use theorems proved in lecture but no homework problems can be used without proof.

(6) Office hour 4 – 5:30 today.

1. Last time

- if $f(0) = 0$ then
  - we proved Schwarz Lemma
  - we determined $f \in \operatorname{Aut} \Delta$, $f(0) = 0$
- if $f(0) \neq 0$
  - we determined $f \in \operatorname{Aut} \Delta$

Idea Use φ_a to recenter f so that 0 maps to 0.

Question Is there a version of Schwarz if $f(0) \neq 0$?

Yes — Schwarz-Pick Lemma.

— we illustrate it for derivatives

Schwarz - Pick $f: \Delta \longrightarrow \Delta$ holomorphic, $\forall\, a \in \Delta = \Delta(0,1)$.

$$\frac{|f'(a)|}{1-|f(a)|^2} \leq \frac{1}{1-|a|^2}.$$

Remark $f(0)=0$, $a=0$ recovers Schwarz $|f'(0)| \leq 1$.

Example Conway VI. 2. 3 $f: \Delta \longrightarrow \Delta$ holomorphic.

If $f\left(\frac{1}{2}\right) = \frac{1}{4}$, find the maximum value of $\left|f'\left(\frac{1}{2}\right)\right|$.

Proof We know this when $a = 0$ & $\alpha = f(a) = 0$.

We use $\mathrm{Aut}(\Delta)$ to reduce to this case

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, shift right=1ex, "\varphi_a"'] & \alpha \arrow[d, "\varphi_\alpha"] \\ 0 \arrow[r, "\tilde{f}"'] \arrow[u, shift right=1ex, "\varphi_{-a}"'] & 0 \end{tikzcd}$$

Let $f(a) = \alpha$. Let

$$\tilde{f} = \varphi_\alpha \circ f \circ \varphi_{-a} \Rightarrow \tilde{f}(0) = 0$$

as the diagram shows.

By Schwarz, $|\tilde{f}'(0)| \leq 1$. We compute using the chain rule

$$\tilde{f}'(0) = \varphi_\alpha'(f(\varphi_{-a}(0))) \cdot f'(\varphi_{-a}(0)) \cdot \varphi_{-a}'(0)$$

$$= \varphi_\alpha'(\alpha) \cdot f'(a) \cdot \varphi_{-a}'(0)$$

$$= \frac{1}{1-|\alpha|^2} \cdot f'(a) \cdot (1-|a|^2)$$ & $|\tilde{f}'(0)| \leq 1$ gives

$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}$$ as needed.

Remark

Schwarz $f(0)=0$	Schwarz - Pick
$\lvert f'(0)\rvert \leq 1$ $\Longleftarrow$	$\lvert f'(a)\rvert \leq \dfrac{1-\lvert f(a)\rvert^2}{1-\lvert a\rvert^2}$
$\lvert f(z)\rvert \leq \lvert z\rvert$	?

Define $\quad d(z,w) = \left\lvert \dfrac{z-w}{1-\bar{z}w} \right\rvert$ = pseudo hyperbolic distance

Schwarz - Pick $\quad$ Holomorphic maps decrease pseudo hyperbolic distance.

This will be made precise in HWK 5.

2. Further applications of Schwarz

We can use Schwarz to study other domains e.g.

[i] $\mathcal{U} = \Delta^{\times} = \Delta(0,1) \setminus \{0\}$

[ii] $\mathcal{U} = \mathfrak{h}^{+}$ = upper half plane

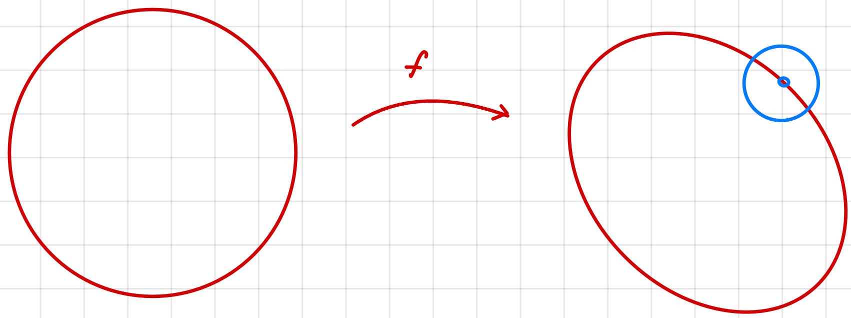
[i] Example All automorphisms of $\Delta^{\times}$ are rotations.

Proof Let $f : \Delta^{\times} \longrightarrow \Delta^{\times}$. Since $\operatorname{Im} f$ is bounded $\Rightarrow$

$\Rightarrow$ f can be extended across 0 by the removable singularity theorem. The extension $\tilde{f} : \Delta \longrightarrow \overline{\Delta}$ is holomorphic.

Its image $\operatorname{Im} \tilde{f} \subseteq \Delta$ by the open mapping theorem

(draw picture)



We claim $\tilde{f}(0) = 0$. Then $f : \Delta^\times \longrightarrow \Delta^\times$ shows $\tilde{f}$ bijective from $\Delta \longrightarrow \Delta$ hence a biholomorphism preserving 0. Then $\tilde{f}$ is a rotation.

To show $\tilde{f}(0) = 0$ assume otherwise $\tilde{f}(0) = \alpha \neq 0$.

Since $\alpha \in \Delta^\times$ we can find $a \in \Delta^\times$. $f(a) = \alpha$.

By the open mapping theorem, we can find small discs $\Delta_0, \Delta_a, \Delta_\alpha$ near $0, a, \alpha$ with $\Delta_0 \cap \Delta_a = \phi$ and.

$$\Delta_\alpha \subseteq \tilde{f}(\Delta_0), \quad \Delta_\alpha \subseteq f(\Delta_a). \quad (\text{why?}).$$

Let $b \in \Delta_\alpha \setminus \{\alpha\} \Rightarrow b \in \tilde{f}(\Delta_0) \Rightarrow b = f(u),\ u \neq 0, u \in \Delta_0$

$$\Rightarrow b \in f(\Delta_a) \Rightarrow b = f(v), \quad v \in \Delta_a$$

$\Rightarrow f(u) = f(v) = b$

$u \neq v$ since $\Delta_0 \cap \Delta_a = \phi$

$\Rightarrow f$ not injective (contradiction).

$\boxed{\text{II}}$ Upper half plane

Key idea Use $\mathfrak{H}^+ \xrightarrow{c} \Delta$, $c(z) = \dfrac{z-i}{z+i}$

$c^{-1}(z) = i \cdot \dfrac{1+z}{1-z}$.

Questions we can answer :

$\boxed{\text{I}}$ $\mathrm{Aut}(\mathfrak{H}^+)$ next time

Schwarz Lemma for $f : \mathfrak{H}^+ \longrightarrow \mathfrak{H}^+$

Schwarz-Pick for $f : \mathfrak{H}^+ \longrightarrow \mathfrak{H}^+$.

$\boxed{\text{II}}$ Biholomorphisms $\Delta \longrightarrow \mathfrak{H}^+$

Schwarz Lemma for $f : \Delta \longrightarrow \mathfrak{H}^+$

Schwarz-Pick for $f : \mathfrak{H}^+ \longrightarrow \Delta$

for derivatives or for distance ...

It is impossible to record them all.

Example $f: \Delta \to \mathfrak{H}^+$, $f(0) = i$. Show

$$|f'(0)| \leq 2.$$

Let $\tilde{f} = c \circ f$. Then $\tilde{f}(0) = 0$ since $c(i) = \left.\frac{z-i}{z+i}\right|_{z=i} = 0$

$\Rightarrow |\tilde{f}'(0)| \leq 1$ by Schwarz. We compute

$$|\tilde{f}'(0)| = |c'(f(0)) \cdot f'(0)| = |c'(i) \cdot f'(0)| \leq 1.$$

Since $c'(i) = \frac{1}{2i}$ $\Rightarrow |f'(0)| \leq 2.$

---

3. Further discussion of Aut. — Loose ends

[i] Aut $\mathbb{C}$

[ii] Aut $\hat{\mathbb{C}}$

next time

[iii] Aut Δ

[iv] Aut $\mathfrak{H}^+$

[v] Aut $\Delta^\times$

Math 220 B — Lecture 17

February 17, 2021

1. Further discussion of Aut. — Loose ends

$\boxed{i}$ $\mathrm{Aut}\ \mathbb{C} = \{az+b : a \neq 0,\ b \in \mathbb{C}\} \cong \mathrm{Aff}.$

$\boxed{ii}$ $\mathrm{Aut}\ \widehat{\mathbb{C}} = PGL_2$

$\boxed{iii}$ $\mathrm{Aut}\ \Delta \cong SU(1,1)/_{\pm 1} = PSU(1,1)$

$\boxed{iv}$ $\mathrm{Aut}\ \mathfrak{H}^+ = SL(2,\mathbb{R})/_{\pm 1}. = PSL(2,\mathbb{R})$

$\boxed{v}$ $\mathrm{Aut}\ \Delta^\times \cong$ Rotations

Case $\boxed{i}$ $\mathcal{U} = \mathbb{C}$

7. Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire and injective. Show that $f(z) = az + b$. You can solve this problem using the notions introduced in Problem 6 above.

Math 220 A, Homework 5.

Case [iii] $\quad \mathcal{U} = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Mobius transforms - Math 220A, Lecture 3.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies f_M(z) = \frac{az+b}{cz+d}, \quad f_M : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

$$f_M = f_N \iff M = \lambda N.$$

$$f_M f_N = f_{MN}.$$

f_M bijective $\iff$ M invertible since $f_M \circ f_{M^{-1}} = \mathbb{I}$

Define $PGL_2 = GL_2 / \{\lambda \cdot \mathbb{I},\ \lambda \neq 0\}$ = invertible 2×2 matrices up to scaling.

Recall from Math 220 A, Lecture 3, the action of Mobius transforms is transitively on $\hat{\mathbb{C}}$.

Theorem $\operatorname{Aut}\hat{\mathbb{C}} = PGL_2$.

Proof If $f \in \operatorname{Aut}\hat{\mathbb{C}}$, $f(\infty) = \infty$ then $f : \mathbb{C} \longrightarrow \mathbb{C}$ is bijective. Thus $f(z) = az + b = f_M$ for the matrix

$$M = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

If $f(\infty) \neq \infty$ then $f(\infty) = \lambda \in \mathbb{C}$. Let

$$g(z) = \frac{1}{f(z) - \lambda} \Rightarrow g(\infty) = \infty \Rightarrow g(z) = az + b$$

$$\Rightarrow f(z) = \lambda + \frac{1}{az+b} = \text{fractional linear transformation,}$$

as needed.

Case $\boxed{iii}$ $\quad Aut(\Delta)$.

Question: What is $Aut(\Delta)$ as an abstract group?

$f \in Aut\,\Delta$

$$f(z) = e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z} = \frac{e^{i\theta/2}}{e^{-i\theta/2}} \cdot \frac{z-a}{1-\bar{a}z} = h_M.$$

$$M = \begin{bmatrix} e^{i\theta/2} & -a\,e^{i\theta/2} \\ -\bar{a}\,e^{-i\theta/2} & e^{-i\theta/2} \end{bmatrix} = \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} \text{ invertible.}$$

Note $\det M = 1 - |a|^2 > 0$. Let $\lambda = (1-|a|^2)^{-1/2}$.

Rescale $A \longrightarrow \lambda A$, $\lambda \in \mathbb{R}$.

$$\Longrightarrow \quad A\bar{A} - B\bar{B} = |A|^2 - |B|^2 = 1.$$

$B \longrightarrow \lambda B$, $\lambda \in \mathbb{R}$.

Conclusion

$$Aut\,\Delta = \left\{ \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} : |A|^2 - |B|^2 = 1 \right\} \Big/ \pm 1$$

$$= SU(1,1)/\pm 1 = PSU(1,1).$$

Case IV Aut $\mathfrak{H}^+$

Key idea Use Cayley transform:

$$\mathfrak{H}^+ \underset{C^{-1}}{\overset{C}{\rightleftarrows}} \Delta \qquad C(z) = \frac{z-i}{z+i}$$

$$C^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

$$\begin{tikzcd} \mathfrak{H}^+ \arrow[r, "C", "\sim"'] \arrow[rrr, bend right, "g", "\sim"'] & \Delta \arrow[r, "f", "\sim"'] & \Delta \arrow[r, "C^{-1}", "\sim"'] & \mathfrak{H}^+ \end{tikzcd}$$

$g = C^{-1} \circ f \circ C$ is an automorphism

Any $g \in \mathrm{Aut}\, \mathfrak{H}^+$ is of this form for $f = C g C^{-1}$.

Compute $C^{-1}\underbrace{\begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix}}_{\mathrm{Aut}\,\Delta} C = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$\alpha = \mathrm{Re}\, A + \mathrm{Re}\, B$ $\qquad$ $\delta = \mathrm{Re}\, A - \mathrm{Re}\, B$

$\beta = \mathrm{Im}\, A - \mathrm{Im}\, B$ $\qquad$ $\gamma = -\mathrm{Im}\, A - \mathrm{Im}\, B$

$\Longrightarrow$ $\alpha, \beta, \gamma, \delta \in \mathbb{R}.$

$|A|^2 - |B|^2 = 1$ $\iff$ $(\mathrm{Re}\, A)^2 + (\mathrm{Im}\, A)^2 - (\mathrm{Re}\, B)^2 - (\mathrm{Im}\, B)^2 = 1.$

$\iff$ $\alpha\delta - \beta\gamma = 1.$

$$\Rightarrow \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2, \mathbb{R}).$$

Conclusion $\quad \mathrm{Aut}(\mathfrak{h}^+) \cong SL(2, \mathbb{R}) / \{\pm 1\} = PSL(2, \mathbb{R}).$

II Riemann Mapping theorem

GRUNDLAGEN

FÜR EINE

ALLGEMEINE THEORIE DER FUNCTIONEN

EINER

VERÄNDERLICHEN COMPLEXEN GRÖSSE.

VON

(Georg Friedrich) Bernhard
B. RIEMANN.

ZWEITER, UNVERÄNDERTER ABDRUCK.

GÖTTINGEN,
VERLAG VON ADALBERT RENTE.
1867.

Riemann's thesis was published in 1851.

"Two given simply connected planar surfaces can always be related to each other in such a way that every point of one corresponds to one point of another, which varies continuously with it, and their corresponding smaller parts are similar."

(Translation by R. Remmert).

Theorem $\mathcal{U} \neq \mathbb{C}$ simply connected $\Rightarrow$ $\mathcal{U}$ biholomorphic to the unit disc. $\Delta = \Delta(0,1)$.

Remarks $\boxed{I}$ $\mathcal{U} = \mathbb{C}$ is not biholomorphic to Δ.

By Liouville, there cannot exist a holomorphic nonconstant map $\mathbb{C} \longrightarrow \Delta$.

$\boxed{II}$ Implications in topology

$\mathcal{U}$ simply connected, $\mathcal{U} \subseteq \mathbb{C}$. $\Rightarrow$ $\mathcal{U}$ is topologically Δ i.e.

$\exists$ bicontinuous map $\mathcal{U} \longrightarrow \Delta$ (homeomorphism).

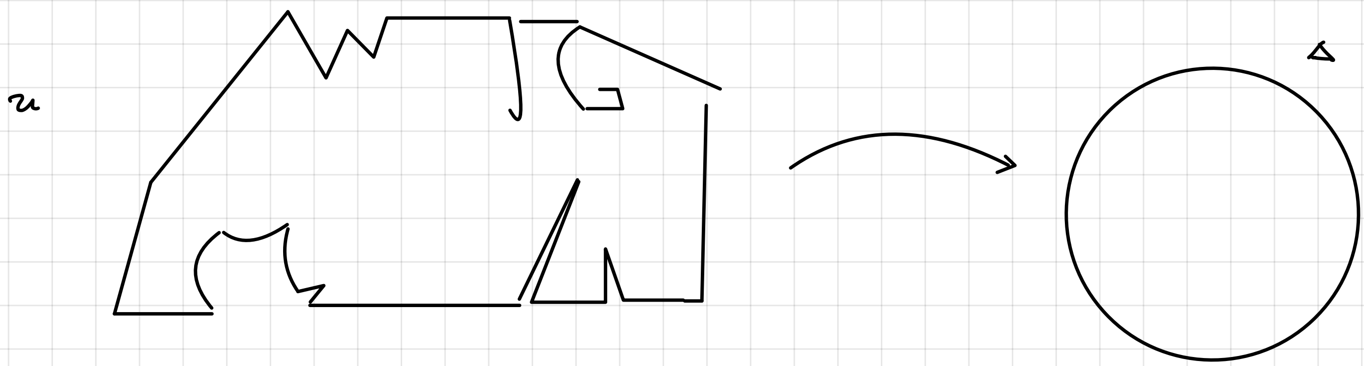
This holds even for $\mathcal{U} = \mathbb{C}$ using the map:

not holomorphic.

$$\mathbb{C} \longrightarrow \Delta, \quad z \longrightarrow \frac{z}{\sqrt{1+|z|^2}}$$

Why is the proof difficult?

Imagine the domain



It is hard to construct explicit maps (even in the topological category).

Examples

[i] $c : \mathfrak{H}^+ \longrightarrow \Delta$, $\quad c(z) = \dfrac{z-i}{z+i}$.

[ii] biholomorphism between Δ and the slit plane

$\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ (both simply connected).

$\mathbb{C}^-$

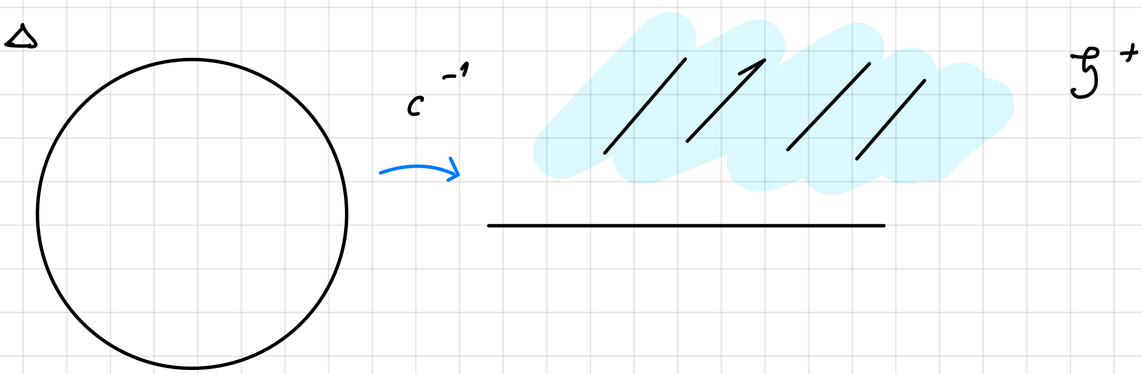
We use simple geometric moves:

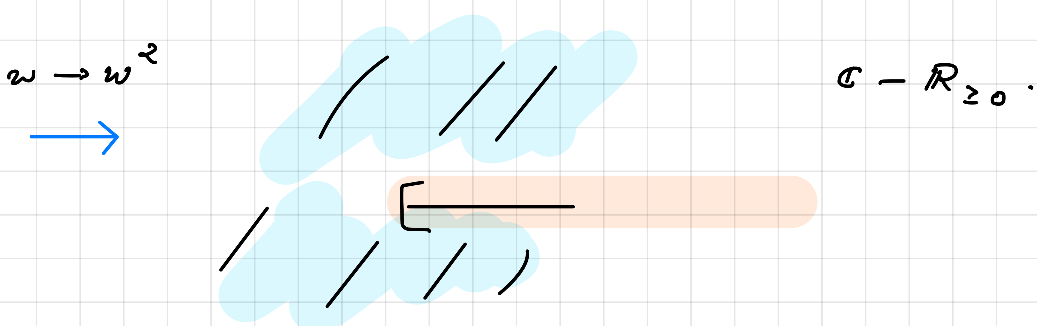
$$\Delta \longrightarrow \mathfrak{H}^+ \quad \text{via} \quad c^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

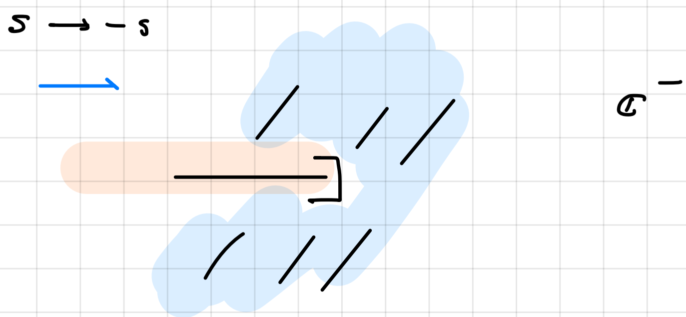
$$\mathfrak{H}^+ \longrightarrow \mathbb{C} \setminus \mathbb{R}_{\geq 0} \quad \text{via} \quad w \longrightarrow w^2.$$

$$\mathbb{C} \setminus \mathbb{R}_{\geq 0} \longrightarrow \mathbb{C} \setminus \mathbb{R}_{\leq 0} = \mathbb{C}^- \quad \text{via} \quad s \longrightarrow -s.$$

Composition: $-\left(i \cdot \frac{1+z}{1-z}\right)^2 = \left(\frac{1+z}{1-z}\right)^2 : \Delta \longrightarrow \mathbb{C}^-.$







Math 220 B — Lecture 18

February 19, 2021

§ 0. Riemann Mapping theorem

Theorem $\mathcal{U} \neq \mathbb{C}$ simply connected $\Rightarrow$ $\mathcal{U}$ biholomorphic to the unit disc. $\Delta = \Delta(0,1)$.

Ingredients in the proof

$\boxed{\text{i}}$ Montel & normal families

$\boxed{\text{ii}}$ Hurwitz's Theorem

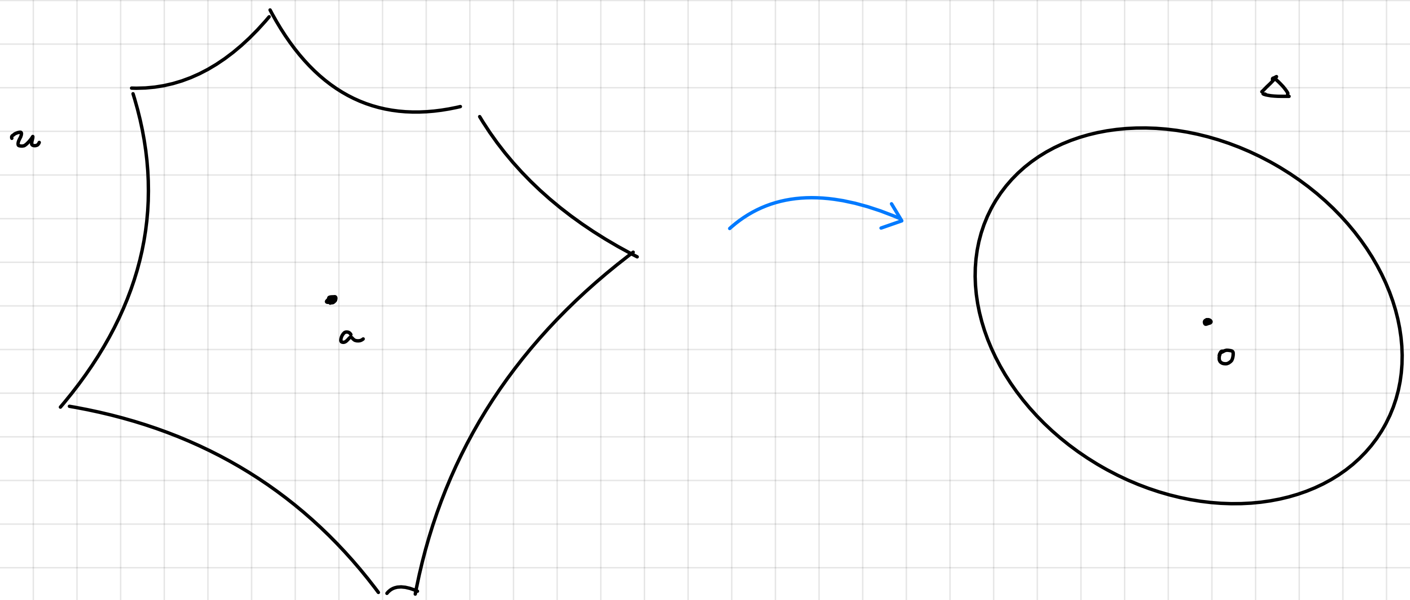
$\boxed{\text{iii}}$ Aut Δ & Schwarz Lemma

$\boxed{\text{iv}}$ Square root trick of Carathéodory – Koebe.

& standard tools: Open Mapping & Weierstraß.

§ 1. Strategy

Fix $a \in \mathcal{U}$



Want $f : \mathcal{U} \longrightarrow \Delta$ & $f(a) = 0$ & f bijective.

Goal #1 First, $f : \mathcal{U} \longrightarrow \Delta$, $f(a) = 0$, &

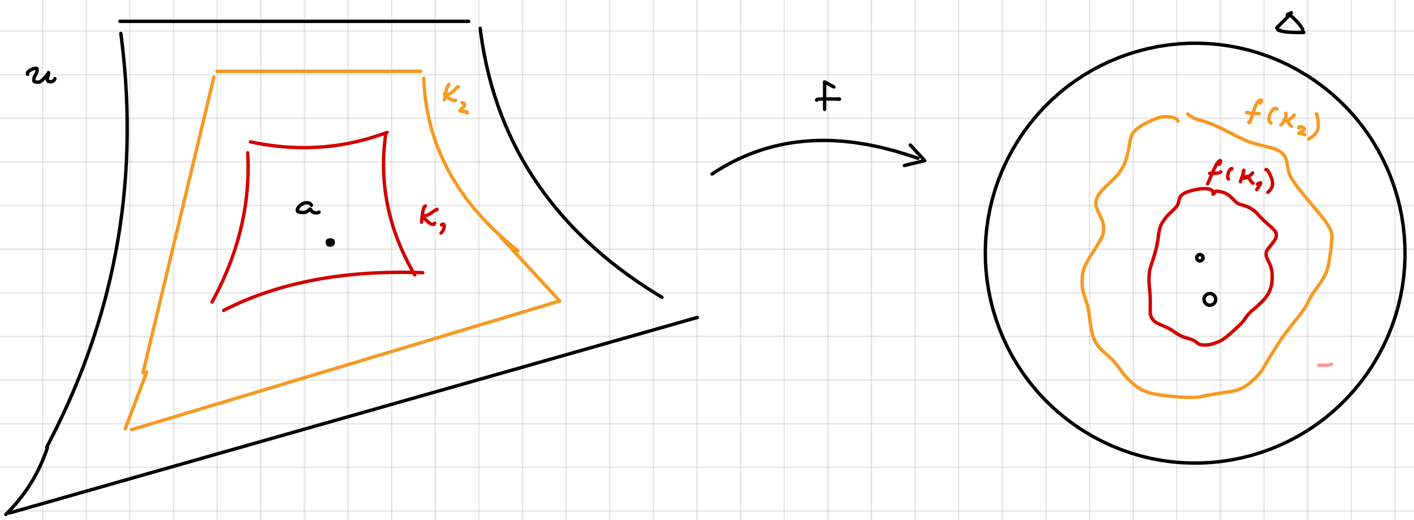
f injective

Main Actor in the Proof Consider the family

$$\mathcal{F} = \{ f : \mathcal{U} \longrightarrow \Delta,\ f(a) = 0,\ f \text{ injective} \}.$$

Want $\mathcal{F} \neq \emptyset$.

Question How to achieve f bijective?



Imagine $\mathcal{U} = \bigcup_n K_n$, $a \in K_n \subseteq \mathrm{Int}\, K_{n+1}$.

We hope $\bigcup_n f(K_n)$ cover Δ. We expect that this has a chance if $|f'(a)|$ is as large as possible.

Let $M = \sup\{|f'(a)| : f \in \mathcal{F}\}$.

Goal #2 Show $\exists f \in \mathcal{F}$ with $|f'(a)| = M$.

Goal #3 Show that for this choice, $f : \mathcal{U} \longrightarrow \Delta$ is bijective

Why might this actually work?

Example $U = \Delta$, $a = 0$.

$$\mathcal{F} = \{ f : \Delta \longrightarrow \Delta,\ f(0) = 0,\ f \text{ injective} \}.$$

By Schwarz Lemma, $|f'(0)| \le 1$. If the maximum value $|f'(0)| = 1$ then f is a rotation so f is bijective.

Remark

We can also consider points $a \in \Delta$, $a \neq 0$. Let

$$\mathcal{F} = \{ f : \Delta \longrightarrow \Delta,\ f(a) = 0,\ f \text{ injective} \}.$$

Schwarz - Pick

$$|f'(a)| \le \frac{1}{1 - |a|^2}$$ with equality iff

$$f = Rot \circ \varphi_a \implies f \text{ bijective.}$$

Question How do we use U simply connected?

Answer Math 220A, Homework 4

2. Assume $f : U \to \mathbb{C}$ is a holomorphic function on a simply connected open set U such that $f(z) \neq 0$ for all $z \in U$. Let $n \geq 2$ be an integer. Show that there is a holomorphic function $g : U \to \mathbb{C}$ such that

$$g(z)^n = f(z).$$

Hint: This has something to do with problem 1(ii).

We only need $n = 2$.

U simply connected $\Rightarrow$ any $f : U \to \mathbb{C}$ holomorphic, nowhere zero, admits a holomorphic square root $g : U \to \mathbb{C}$

$$f = g^2 \quad (*)$$

"Root domain"

- $U \subseteq \mathbb{C}$ is a root domain if $(*)$ is satisfied.

Remark simply connected $\Rightarrow$ root domain

Remark This turns out to be equivalent to $\mathcal{U}$ simply connected

We will prove: the seemingly stronger form:

Riemann Mapping Theorem

$\mathcal{U} \neq \mathbb{C}$ root domain $\Rightarrow$ $\mathcal{U}$ is biholomorphic to Δ.

§ 2. Study of the family $\mathcal{F}$

Fix $a \in \mathcal{U}$.

$$\mathcal{F} = \{ f : \mathcal{U} \longrightarrow \Delta : f \text{ holomorphic, injective, } f(a) = 0 \}.$$

Step 1 If $\mathcal{U}$ is a root domain, $\mathcal{U} \neq \mathbb{C} \Rightarrow \mathcal{F} \neq \emptyset$

Proof Let $b \notin \mathcal{U}$. which is possible since $\mathcal{U} \neq \mathbb{C}$.

Consider $h(z) = z - b$, $h : \mathcal{U} \longrightarrow \mathbb{C}$. Note $h(z) \neq 0$ for $z \in \mathcal{U}$. since $b \notin \mathcal{U}$. Thus h admits a square root

$g : \mathcal{U} \longrightarrow \mathbb{C}$, $g(z)^2 = z - b$.

Claim 1 g injective.

Indeed, if $g(z_1) = g(z_2) \Rightarrow g(z_1)^2 = g(z_2)^2 \Rightarrow$

$\Rightarrow z_1 - b = z_2 - b \Rightarrow z_1 = z_2$.

Claim 2 $g(\mathcal{U}) \cap (-g)(\mathcal{U}) = \emptyset$.

Indeed, if $\exists\, z_1, z_2 \in \mathcal{U}$ with $g(z_1) = -g(z_2)$

$\Rightarrow g(z_1)^2 = g(z_2)^2 \Rightarrow z_1 - b = z_2 - b \Rightarrow z_1 = z_2$.

But then $g(z_1) = -g(z_2) \Rightarrow g(z_1) = -g(z_1) \Rightarrow g(z_1) = 0$

$\Rightarrow g(z_1)^2 = 0 = z_1 - b \Rightarrow z_1 = b$. But $z_1 \in \mathcal{U}$, $b \notin \mathcal{U}$.

Claim 3 $\exists\, c, r$ with $|g(z) - c| > r \ \forall z \in \mathcal{U}$.

Indeed, by the open mapping theorem, $(-g)(\mathcal{U})$ is open so it contains a disc $\overline{\Delta}(c,r)$. By Claim 2,

$g(\mathcal{U}) \subseteq \mathbb{C} \setminus \overline{\Delta}(c,r) \iff |g(z) - c| > r \ \forall z \in \mathcal{U}$.

Construction Let $f(z) = \dfrac{r}{g(z) - c}$. $\Rightarrow f$ injective since g is by Claim 1 & $f : \mathcal{U} \longrightarrow \Delta(0,1)$. by Claim 3.

To achive $f(a) = 0$, define $\tilde{f}(z) = \dfrac{f(z) - f(a)}{2}$.

$\Rightarrow \tilde{f}$ injective since f is. & $\tilde{f}(a) = 0$.

Note that since f takes values in Δ, the same is true for $\tilde{f}$

$$|\tilde{f}(z)| \leq \frac{1}{2}\left(|f(z)| + |f(a)|\right) < \frac{1}{2}(1+1) = 1$$

Thus $\tilde{f} \in \mathcal{F} \Rightarrow \mathcal{F} \neq \Phi$.

Step 2 Let $M = \sup\{|f'(a)|, f \in \mathcal{F}\}$

Show: The supremum is achieved by some $f \in \mathcal{F}$.

Proof: Indeed, take $f_n \in \mathcal{F}$ with $|f_n'(a)| \longrightarrow M$. as $n \longrightarrow \infty$

The family $\mathcal{F}$ is bounded by 1 since the functions in $\mathcal{F}$ take values in Δ. $\xRightarrow{\text{Montel}}$ $\mathcal{F}$ normal. $\Rightarrow$

$\Rightarrow$ passing to a subsequence, we may assume

$f_n \rightrightarrows f$ locally uniformly.

Claim 4 f holomorphic, $f(a) = 0$, $|f'(a)| = M$.

Indeed, by Weierstrass convergence, f is holomorphic.

and $f_n' \rightrightarrows f'$ locally uniformly. In particular,

$f_n'(a) \longrightarrow f'(a)$ so $|f'(a)| = M$.

Since $f_n(a) = 0$ & $f_n \to f$ at a, we have

$f(a) = 0$.

Claim 5. $f: U \longrightarrow \Delta$ & f injective.

Indeed, f_n injective & $f_n \overset{l.u.}{\rightrightarrows} f$ shows f is either

injective or f constant by Hurwitz's theorem

(Math 220A, Lecture 24).

If f = constant $\Rightarrow f'(a) = 0 \Rightarrow M = 0 \Rightarrow$

$\Rightarrow g'(a) = 0 \quad \forall g \in \mathcal{F}$ since M is the supremum.

But if $g \in \mathcal{F}$, g injective and $g'(a) \neq 0$ by

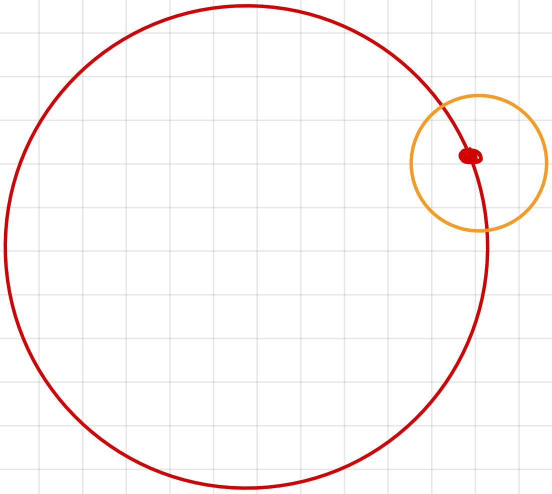
Math 220A, Final Exam, Problem 7.

Thus f injective.

Note that since $f_n \overset{l.u}{\rightrightarrows} f$ and $f_n : U \longrightarrow \Delta$

shows $f: U \longrightarrow \overline{\Delta}$. By the open mapping theorem,

$f: U \longrightarrow \Delta$ ($f \neq$ not constant).



By Claims 4 & 5, $f \in \mathcal{F}$ and $|f'(a)| = M \Rightarrow$ Step 2 ✓.

Step 3 For a function $f \in \mathcal{F}$ which achieves the supremum

f is bijective.

Proof: next time.

Math 220B - Lecture 15

February 22, 2021

Homework now due Friday, due to Office Hrs. on Wed.

Last time

Conway VII. 4.

$\mathcal{U} \neq \mathbb{C}$ root domain $\Leftarrow$ $\mathcal{U}$ simply connected

Let $a \in \mathcal{U}$. Wish $\mathcal{U} \cong \Delta$ biholomorphically.

$$\mathcal{F} = \{ f : \mathcal{U} \longrightarrow \Delta, \ f(a) = 0, \ f \text{ injective} \}$$

Step 1 $\mathcal{U} \neq \mathbb{C}$ root domain $\Longrightarrow$ $\mathcal{F} \neq \Phi$.

Step 2 Let $M = \sup \{ |f'(a)|, \ f \in \mathcal{F} \}$.

Then M is achieved by some function $f \in \mathcal{F}$.

Today

Step 3 For the extremal function f in Step 2

we show f surjective. Then f biholomorphism

If $f : U \longrightarrow \Delta$ not surjective then we show $\exists\, \tilde{f} \in \mathcal{F}$

with $|\tilde{f}'(a)| > |f'(a)|$ contradicting maximality of $|f'(a)|$.

Strategy

We will in fact show that if $f : U \longrightarrow \Delta$ not surjective

then $\exists\, \tilde{f} : U \longrightarrow \Delta$, $F : \Delta \longrightarrow \Delta$, $f = F \circ \tilde{f}$

$\tilde{f} \in \mathcal{F}$, $F(0) = 0$, $F \notin \operatorname{Aut} \Delta$.

Assume this can be done. The proof is then completed.

Indeed, by Schwarz Lemma $\Rightarrow |F'(0)| < 1$. (The inequality is strict since F is not a rotation as $F \notin \operatorname{Aut} \Delta$).

Then we indeed contradict maximality since

$$|f'(a)| = |F'(0)| \cdot |\tilde{f}'(a)| < |\tilde{f}'(a)|.$$

How do we execute the above Strategy?

Assume $f : U \longrightarrow \Delta$ is not surjective.

Let $\alpha \in \Delta \setminus f(U)$.

Construction of the function $\tilde{f}$ "square root trick".

We carry out the following moves:

(1) recenter.

The function $\varphi_\alpha \circ f : U \longrightarrow \Delta$ omits the value $\varphi_\alpha(\alpha) = 0$ since f omits α. & $\varphi_\alpha \in \mathrm{Aut}\,\Delta$.

---

(2) square root. Since U is a root domain & $\varphi_\alpha \circ f$ is nowhere zero, we can find $g : U \longrightarrow \Delta$ holomorphic with $g^2(z) = \varphi_\alpha \circ f$.

Claim g injective.

Indeed $g(z) = g(w) \Rightarrow g(z)^2 = g(w)^2 \Rightarrow \varphi_\alpha \circ f(z) = \varphi_\alpha \circ f(w)$

$\Rightarrow f(z) = f(w) \Rightarrow z = w$ since $f \in \mathcal{F}$ injective.

---

(3) recenter. Let $\beta = g(a)$. We define

$$\tilde{f} = \varphi_\beta \circ g \Rightarrow \tilde{f}(a) = \varphi_\beta\, g(a) = \varphi_\beta(\beta) = 0.$$

& $\tilde{f} : U \longrightarrow \Delta$ injective. Then $\tilde{f} \in \mathcal{F}$.

Outcome

$$g^2 = \varphi_\alpha \circ f, \quad \tilde{f} = \varphi_\beta \circ g, \quad \tilde{f} \in \mathcal{F}.$$

Comparison

$$g^2 = \varphi_\alpha \circ f \Rightarrow f = \varphi_{-\alpha} \circ g^2.$$

Let $s : \Delta \longrightarrow \Delta$, $s(w) = w^2 \Rightarrow f = \varphi_{-\alpha} \circ s \circ g.$

$$\Downarrow$$

$$\tilde{f} = \varphi_\beta \circ g \Rightarrow g = \varphi_{-\beta} \circ \tilde{f} \Rightarrow f = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \circ \tilde{f}$$

Let $F : \Delta \longrightarrow \Delta$, $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}$. $\Rightarrow f = F \circ \tilde{f}$

Claim

$F \notin \operatorname{Aut} \Delta$, $F(0) = 0$.

Indeed, if $F \in \operatorname{Aut} \Delta$, $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \in \operatorname{Aut} \Delta$

$\Rightarrow s \in \operatorname{Aut} \Delta$. But s is not even injective as $s(z) = s(-z)$.

To see $F(0) = 0$ we compute

$$F(0) = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}(0) = \varphi_{-\alpha} \circ s(\beta) = \varphi_{-\alpha}(\beta^2) = \varphi_{-\alpha}(-\alpha) = 0$$

where we used

$$\beta^2 = g(a)^2 = \varphi_\alpha \circ f(a) = \varphi_\alpha(0) = -\alpha.$$

This is exactly what we needed to complete the proof of Step 3 & the proof of Riemann Mapping.

Remarks

1. Uniqueness of the biholomorphism. Take two biholom.

$$f, g : U \longrightarrow \Delta, \quad f(a) = g(a) = 0 \quad \text{then}$$

consider $\Delta \xrightarrow[f^{-1}]{} U \xrightarrow[g]{} \Delta$, $gf^{-1}(0) = 0$, $gf^{-1} \in \mathrm{Aut}\,\Delta$.

Then

$$gf^{-1} = \mathrm{Rot} \implies g = \mathrm{Rot} \circ f.$$

Thus the biholomorphisms we constructed are unique up to rotations.

[ii] The extremal function f we constructed maximizes the derivatives at a of ALL functions $g: U \to \Delta$, $g(a) = 0$ not only the INJECTIVE ones.

Indeed if $f: U \to \Delta$ is the function we constructed, then $\forall g: U \to \Delta$, $g(a) = 0$,

$$\Delta \xrightarrow[f^{-1}]{} U \xrightarrow{g} \Delta, \quad F = g \circ f^{-1}: \Delta \to \Delta.$$

$$F(0) = 0.$$

Then $g = F \circ f \Rightarrow |g'(a)| = |F'(0)|\,|f'(a)| \leq |f'(a)|$

where we used $|F'(0)| \leq 1$ by Schwarz.

[iii] U, V simply connected, $U, V \neq \mathbb{C} \Rightarrow U, V$ are biholomorphic. ($U \cong \Delta \cong V$ transitive)

Loose ends

TFAE

$\boxed{i}$ $\mathcal{U}$ simply connected

$\boxed{ii}$ $\mathcal{U}$ is a "logarithm domain".

$\boxed{iii}$ $\mathcal{U}$ is a root domain

A "logarithm domain" is a domain where $\forall\, f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, f nowhere zero, we can define

$$\log f : \mathcal{U} \longrightarrow \mathbb{C} \text{ holomorphic.}$$

Proof

$\boxed{i} \Rightarrow \boxed{ii}$. Math 220A, PSet 4

$\boxed{ii} \Rightarrow \boxed{iii}$. Define $\sqrt{f} = \operatorname{Exp}\left(\frac{1}{2}\log f\right)$ for all $f : \mathcal{U} \to \mathbb{C}$ nowhere zero.

$\boxed{iii} \Rightarrow \boxed{i}$. If $\mathcal{U} = \mathbb{C} \Rightarrow \mathcal{U}$ simply connected

Let $U \neq \mathbb{C}$ $\Rightarrow$ let $f: U \longrightarrow \Delta$, $g: \Delta \longrightarrow U$ inverse biholomorphisms. If γ is a loop in U, then

$f \circ \gamma$ loop in Δ = simply connected $\Rightarrow$ $f \circ \gamma \overset{\Delta}{\sim} 0$

$\Rightarrow g \circ f \circ \gamma \overset{U}{\sim} g(0) \Rightarrow \gamma \overset{U}{\sim} g(0) \Rightarrow$ γ null homotopic.

Question How do we construct biholomorphism

$f: U \longrightarrow \Delta$ explicitly?

Answer : depends on U.

Some examples worth knowing

$\boxed{a}$ Lecture 17 :

$$\mathbb{C}^{-} \longrightarrow \Delta, \quad z \longrightarrow \left(\frac{1+z}{1-z}\right)^{2}.$$

We will give more examples next time.

Next : More on boundary behaviour &

Schwarz Reflection (Conway IX. 1)

After : Runge's Theorem. (Conway VIII. 1)

Math 220B – Lecture 20

February 24, 2021

Extra hints added to 1 ii. & 1 iv.

Office Hour : 4 – 5:30 today

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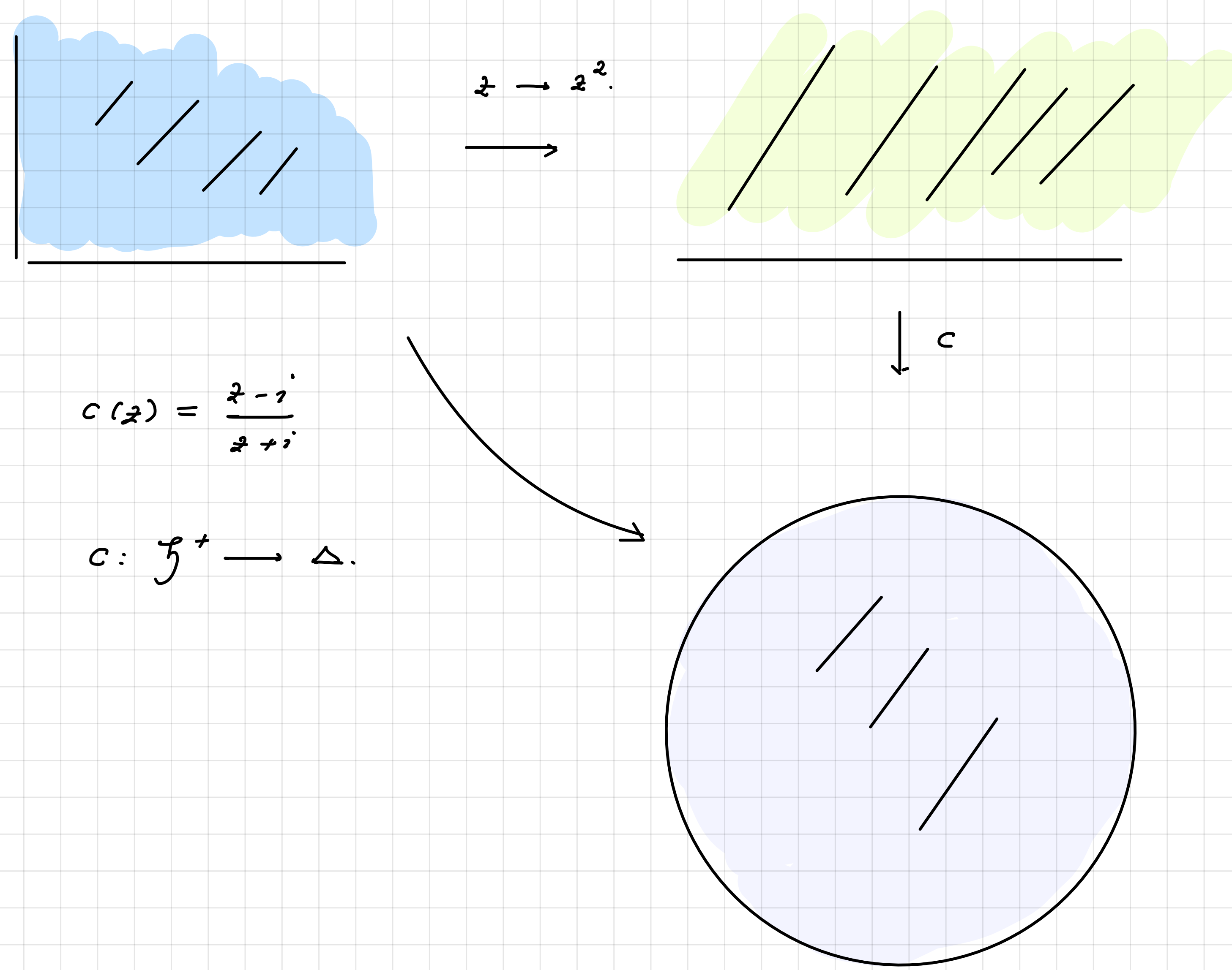
1. More examples of biholomorphisms

Example 1 Squaring in $\mathcal{Q}^+$ "Half $\mathfrak{H}^+ \longrightarrow \mathfrak{H}^+$"



$$c(z) = \frac{z-i}{z+i}$$

$$c : \mathfrak{H}^+ \longrightarrow \Delta.$$

Example [16]

Δ^+ = upper half disc (open)

Question Find $\begin{array}{c}\Delta^+\\ \rotatebox{90}{$\in$}\\ 0\end{array} \xrightarrow{\sim} \begin{array}{c}\Delta\\ \rotatebox{90}{$\in$}\\ 0\end{array}$. $\quad z \longrightarrow z^2$

Answer Not done by squaring since $0 \in \Delta$, $0 \notin \Delta^+$.

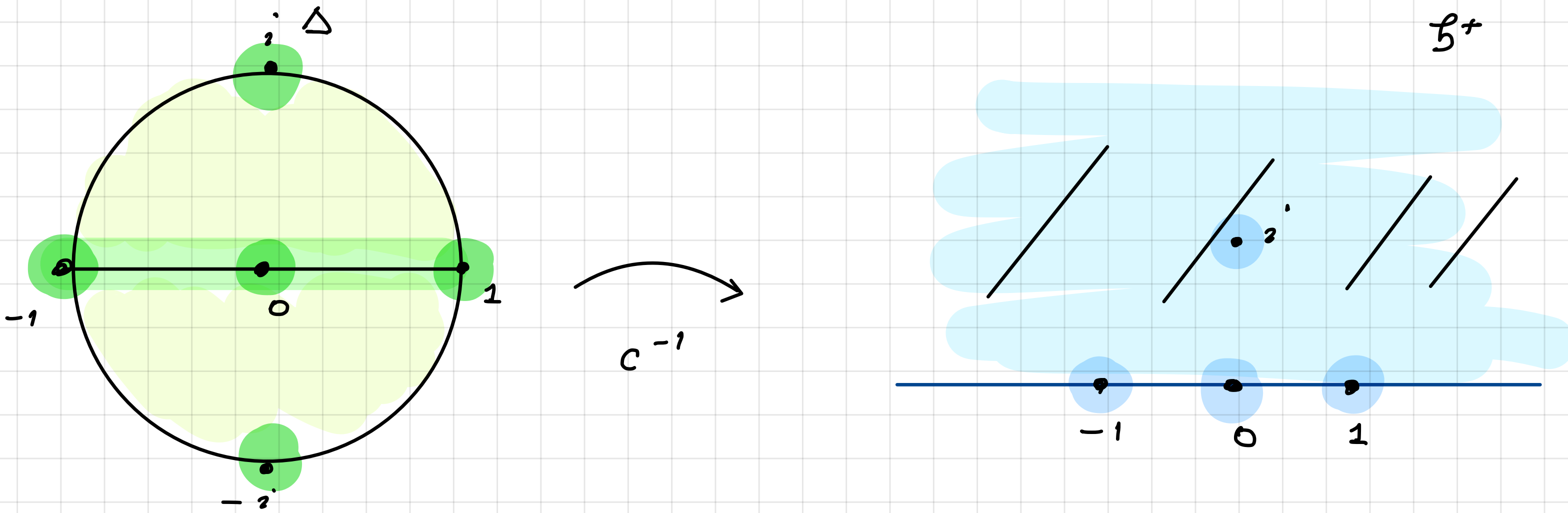
Instead We use the Cayley transform & work in $\mathfrak{H}^+$.

Idea

$$\begin{tikzcd}
\Delta \arrow[r, "c^{-1}"] & \mathfrak{H}^+ \\
\underbrace{\Delta^+}_{\text{half disc}} \arrow[u, bend left, "\text{the map we want}"] \arrow[r, "c^{-1}"] & \text{“half } \mathfrak{H}^+\text{”} \arrow[u, bend right, "\text{square}"']
\end{tikzcd}$$

Concretely Consider $c : \mathfrak{h}^+ \to \Delta$. the Cayley transform

$$c^{-1} : \Delta \to \mathfrak{h}^+ , \quad c^{-1}(z) = i \cdot \frac{1+z}{1-z}$$



Check Under c^{-1}, we map

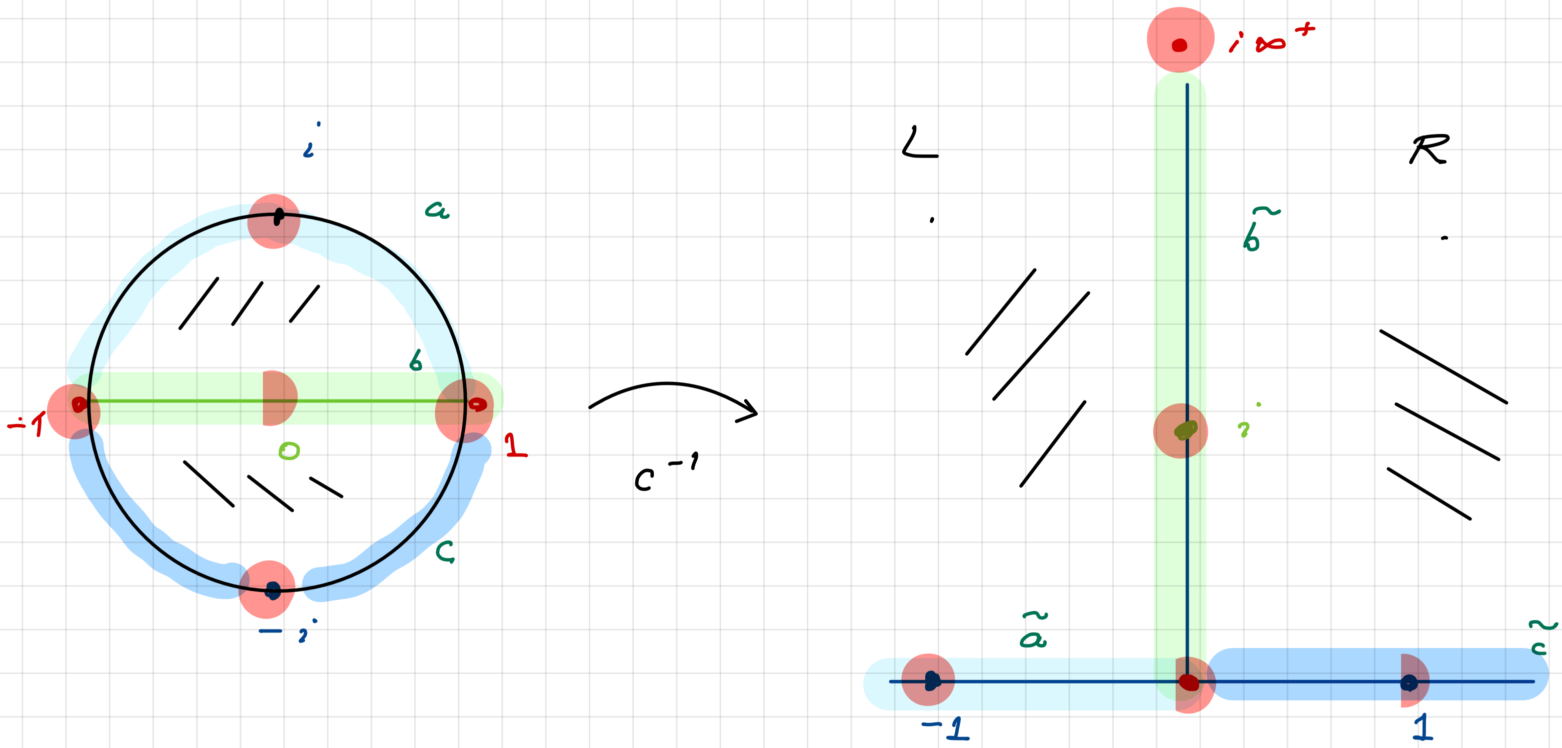
$-1 \longrightarrow 0$

$1 \longrightarrow \infty$

$0 \longrightarrow i$

$i \longrightarrow -1$

$-i \longrightarrow +1$.



Conclusions

[I] diameter b $\longrightarrow$ imaginary axis $\tilde{b}$

arc a $\longrightarrow$ negative real axis $\tilde{a}$

arc c $\longrightarrow$ positive real axis $\tilde{c}$

[II] $\Delta^+ \xrightarrow{\sim} L$ = 2nd quadrant (left)

[III] $\Delta^- \xrightarrow{\sim} R$ = 1st quadrant (right)

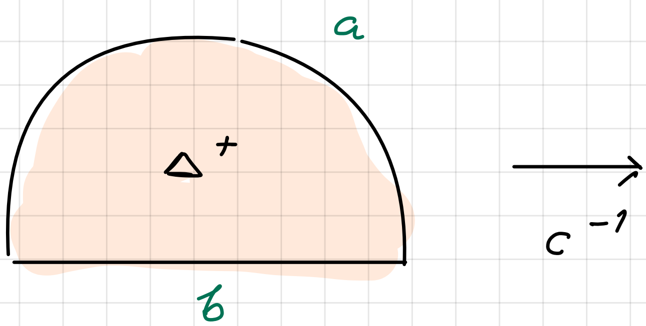
Construction of biholomorphism $\Delta^+ \longrightarrow \Delta$

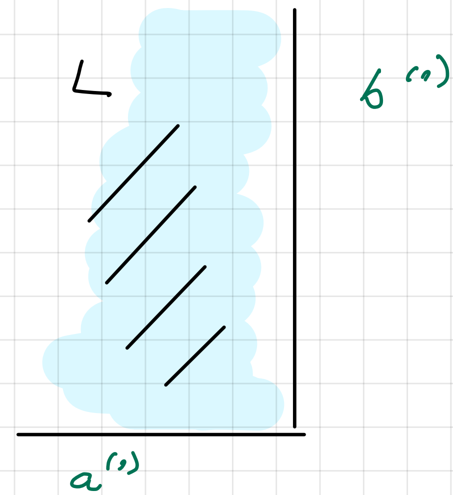
as a composition of three moves:

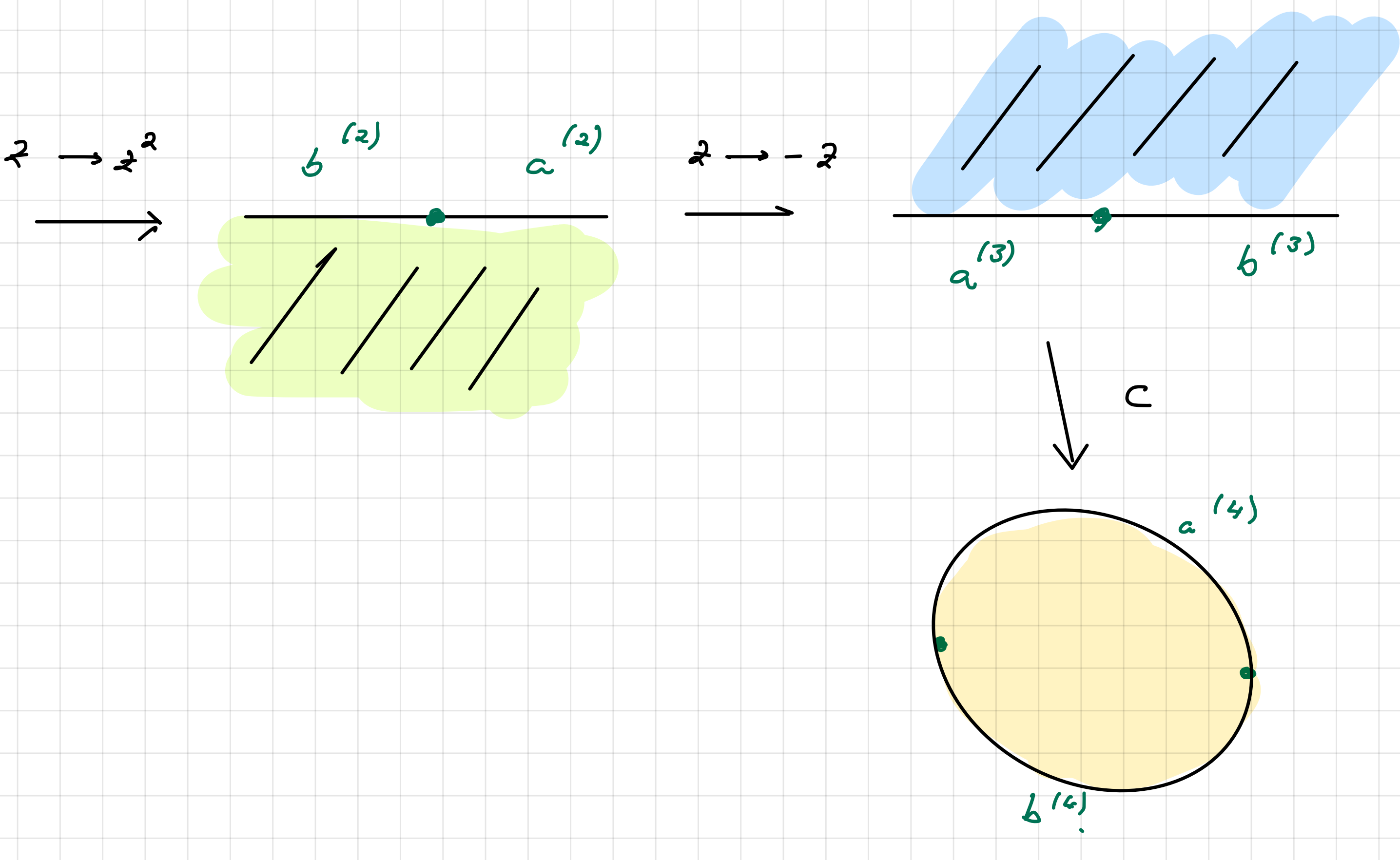
(1) $\Delta^+ \xrightarrow{C^{-1}} L$, $z \longrightarrow i \cdot \dfrac{1+z}{1-z}$.

(2) $L \longrightarrow \mathfrak{H}^+$ $\quad z \longrightarrow -z^2$.

(3) $\mathfrak{H}^+ \xrightarrow{C} \Delta$, $C(z) = \dfrac{z-i}{z+i}$







Conclusion

The biholomorphism $\Delta^+ \longrightarrow \Delta$ extends to $\partial\Delta^+ \longrightarrow \partial\Delta$ continuously & bijectively.

(the upper arc a is sent to the upper arc & the diameter b is sent to the lower arc).

2. Extension to the boundary

Question Given $f: \mathcal{U} \to \Delta$ biholomorphism, does it extend $\bar{f}: \bar{\mathcal{U}} \to \bar{\Delta}$ bicontinuously?

Answer [i] yes if $\mathcal{U}$ bounded & $\partial \mathcal{U}$ = simple closed curve.

Caratheodory's theorem

[ii] We will not give the proof in this course.

3. Beyond the boundary

Question Can we extend beyond the boundary?

The easiest instance is provided by

Schwarz Reflection Principle Conway IX. 1.

There are several versions but two stand out:

[i] reflection across line segments (book)

[ii] reflection across circular arcs (HWK6).

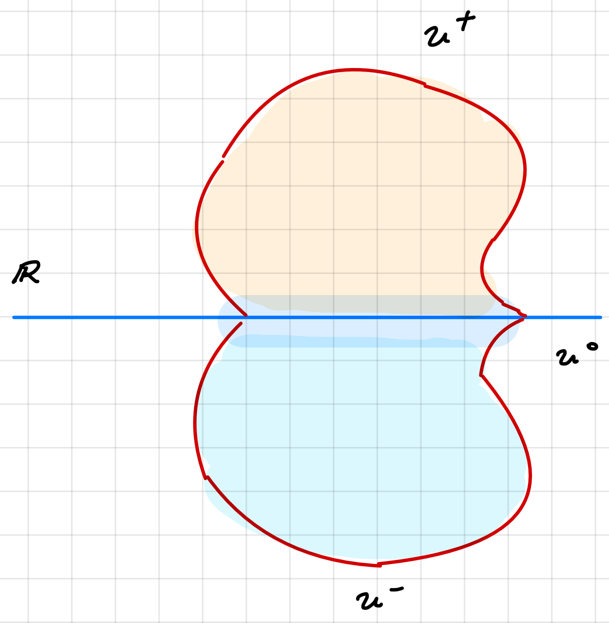
Applications

[i] biholomorphic maps between rectangles, annuli

[ii] analytic continuation ...

4. Reflection across segments

open $\mathcal{U} \subseteq \mathbb{C}$ symmetric $z \longrightarrow \bar{z}$. $\forall z \in \mathcal{U} \Rightarrow \bar{z} \in \mathcal{U}$.



$\mathcal{U}^+ = \mathcal{U} \cap \mathfrak{H}^+$

$\mathcal{U}^- = \mathcal{U} \cap \mathfrak{H}^-$

$\mathcal{U}^\circ = \mathcal{U} \cap \mathbb{R}. = (a, b)$

Given $f : \mathcal{U}^+ \longrightarrow \mathbb{C}$

[i] holomorphic in $\mathcal{U}^+$

[ii] extends continuously to $\mathcal{U}^\circ$.

[iii] such that the values $f(\mathcal{U}^\circ) \subseteq \mathbb{R}$.

Define

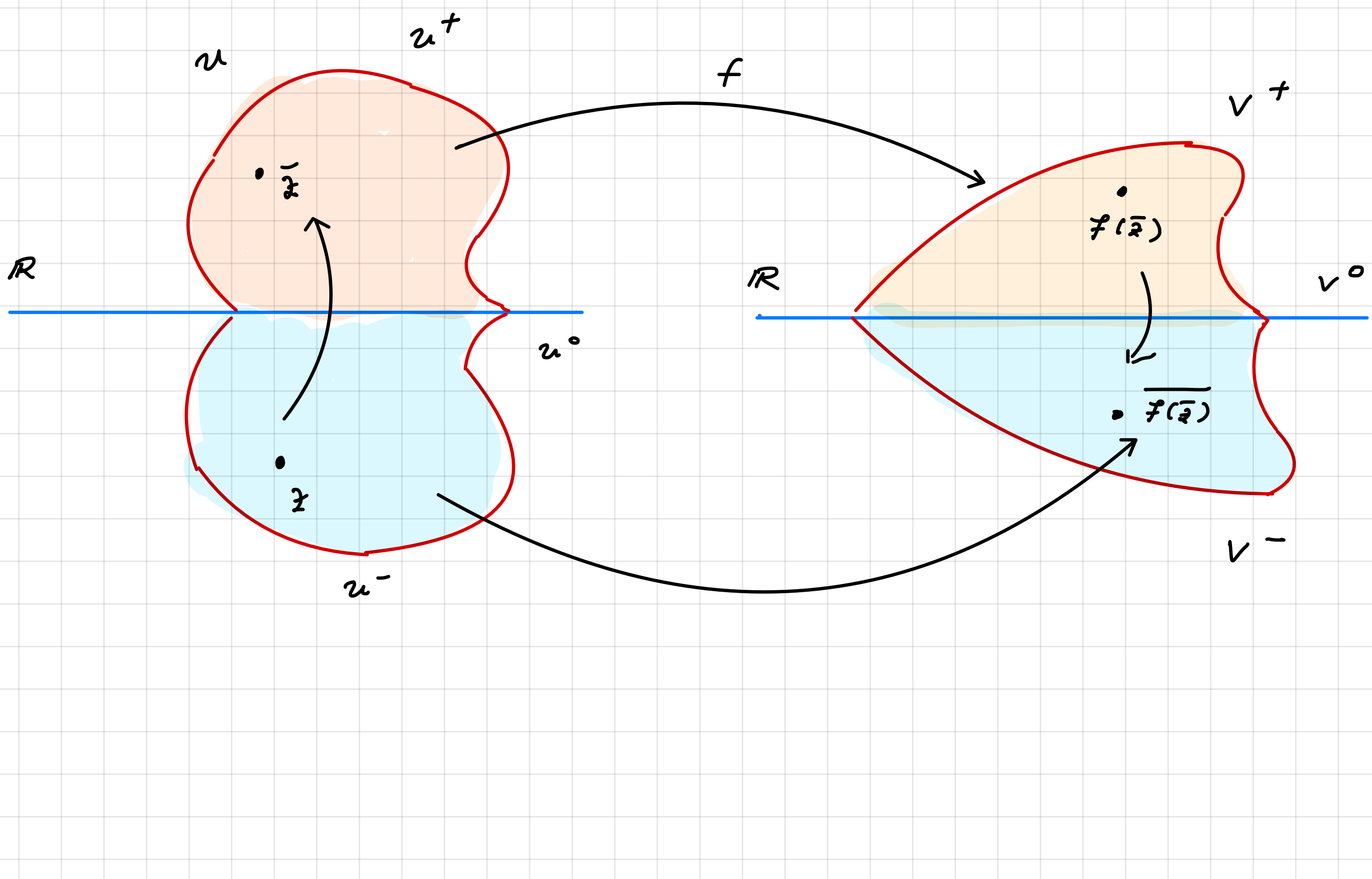
$$F(z) = \begin{cases} f(z) & \text{if } z \in \mathcal{U}^+ \\ f(z) & \text{if } z \in \mathcal{U}^\circ \\ \overline{f(\bar{z})} & \text{if } z \in \mathcal{U}^- \end{cases}$$

Theorem The function $F: U \longrightarrow \mathbb{C}$

is a holomorphic extension of f beyond the boundary.

Remarks

[1] Visualization

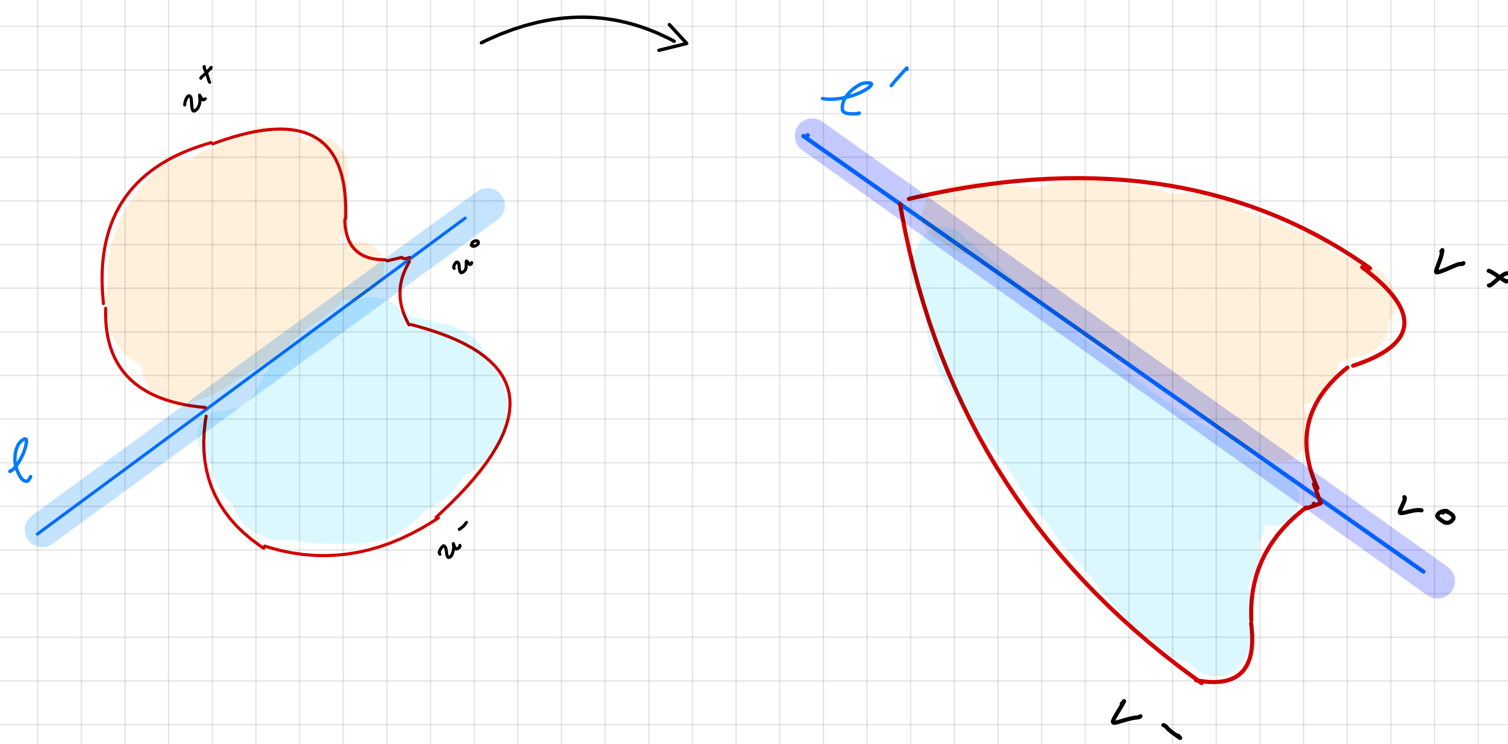


[11] The condition

$$f(U_0) \subseteq \mathbb{R}$$

ensures we reflect across real axis on both sides.

More generally, we can reflect across arbitrary lines



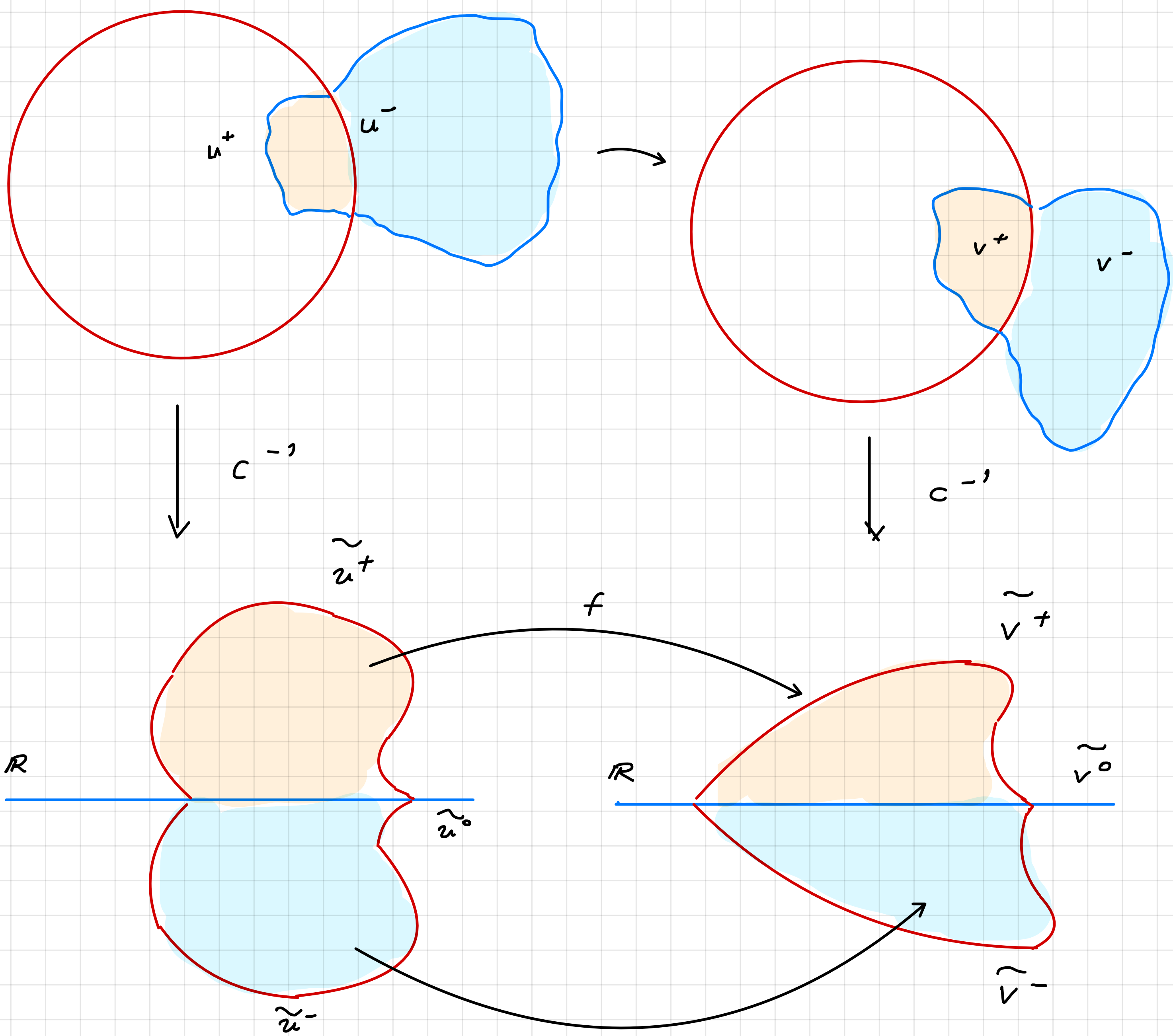
This can be deduced via rotations

$\boxed{iii}$ Using the Cayley transform

$$C : \Delta \longrightarrow \mathfrak{H}^+$$

we can also reflect across arcs in the unit disc.

(HWK 6).

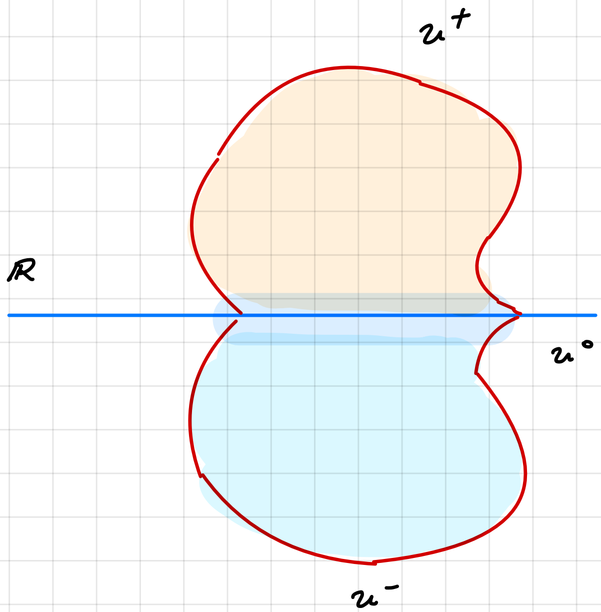


Math 220B - Lecture 21

February 26, 2021

Last time

open $\mathcal{U} \subseteq \mathbb{C}$ symmetric $z \longrightarrow \bar{z}$. $\forall z \in \mathcal{U} \Rightarrow \bar{z} \in \mathcal{U}$.



$\mathcal{U}^+ = \mathcal{U} \cap \mathfrak{H}^+$

$\mathcal{U}^- = \mathcal{U} \cap \mathfrak{H}^-$

$\mathcal{U}^\circ = \mathcal{U} \cap \mathbb{R} = (a, b)$

Given $f : \mathcal{U}^+ \longrightarrow \mathbb{C}$

[i] holomorphic in $\mathcal{U}^+$

[ii] extends continuously to $\mathcal{U}^\circ$.

[iii] such that the values $f(\mathcal{U}^\circ) \subseteq \mathbb{R}$.

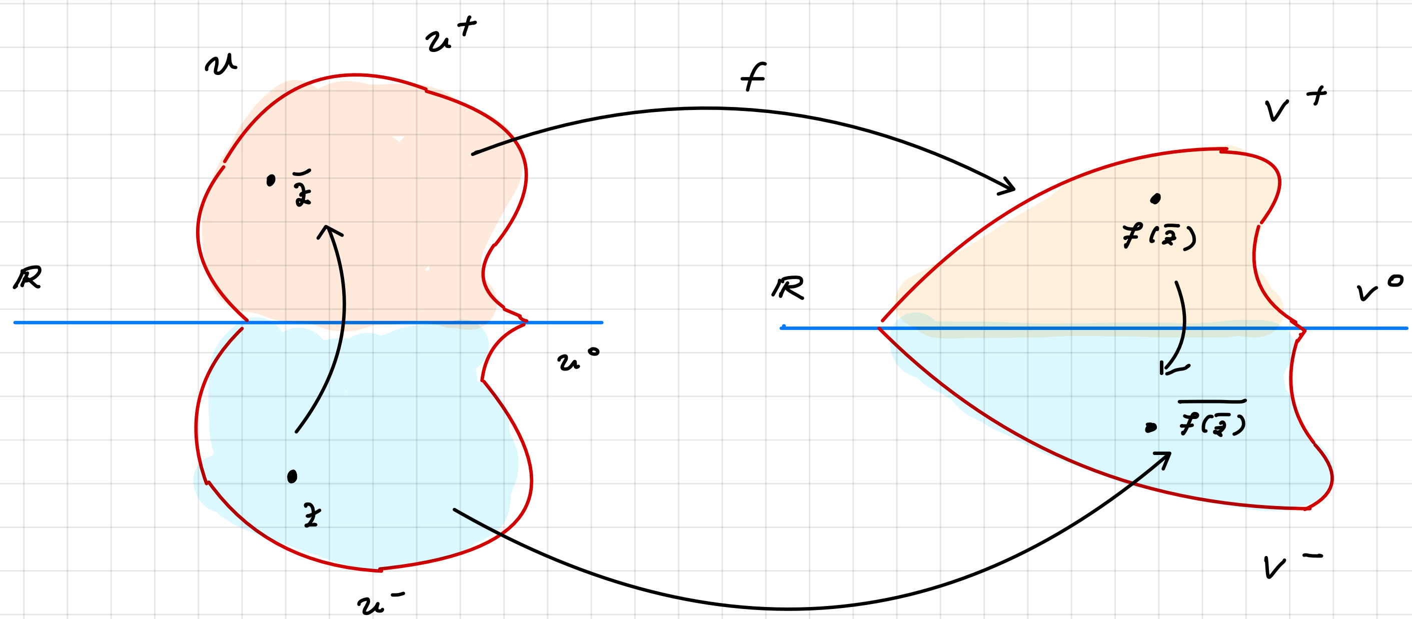
Define

$$F(z) = \begin{cases} f(z) & \text{if } z \in \mathcal{U}^+ \\ f(z) & \text{if } z \in \mathcal{U}^\circ \\ \overline{f(\bar{z})} & \text{if } z \in \mathcal{U}^- \end{cases}$$

Theorem The function $F: \mathcal{U} \longrightarrow \mathbb{C}$

is a holomorphic extension of f beyond the boundary.

Visualization



Proof of Schwarz

$\boxed{I}$ F continuous

$\boxed{II}$ F holomorphic in U^+

$\boxed{III}$ F holomorphic in U^-

$\boxed{IV}$ F holomorphic at points of U^0.

Proof of $\boxed{I}$

Let $z_0 \in U_0 \Rightarrow z_0 = \overline{z_0}$

We show $\displaystyle \lim_{\substack{z \to z_0 \\ z \in U^+}} F(z) = \lim_{\substack{z \to z_0 \\ z \in U^-}} F(z)$.

$$\Longleftrightarrow \quad \lim_{\substack{z \to z_0 \\ z \in U^+}} f(z) = \lim_{\substack{z \to z_0 \\ z \in U^-}} \overline{f(\overline{z})}$$

$$\Longleftrightarrow \quad f(z_0) = \overline{f(\overline{z_0})}$$

which holds since $z_0 = \overline{z_0}$ & $f(z_0) = \overline{f(z_0)}$

Proof of $\boxed{iii}$ We show F holomorphic in $\mathcal{U}^-$.

Let $c^- \in \mathcal{U}^-$. Let $c^+ = \overline{c^-} \in \mathcal{U}^+$. Since f is holomorphic at c^+ $\Rightarrow$ $\exists\ \Delta(c^+, r) \subseteq \mathcal{U}^+$. Taylor expand in $\Delta(c^+, r)$:

$$f(z) = \sum_{k=0}^{\infty} a_k \, (z - c^+)^k, \quad \text{radius of convergence} \geq r.$$

Let $z \in \Delta(c^-, r) = \overline{\Delta(c^+, r)}$. Then

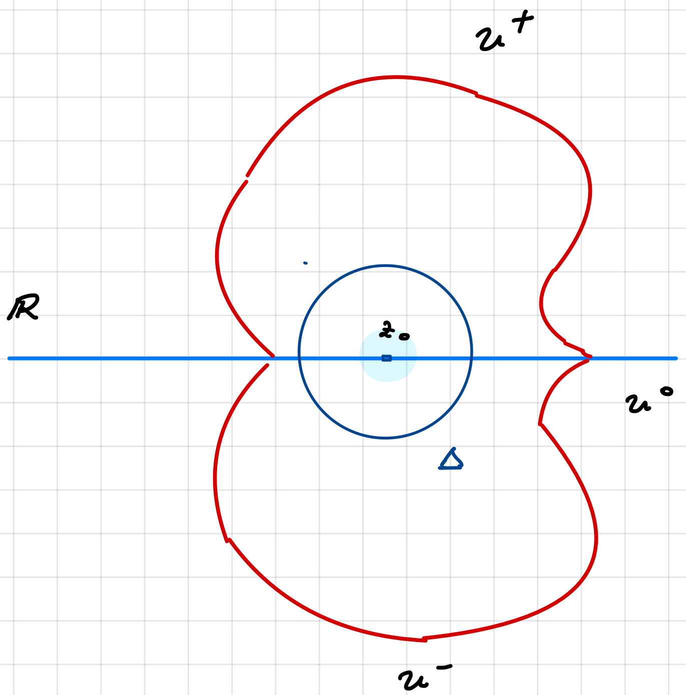
$$F(z) = \overline{f(\bar{z})} = \overline{\sum_{k=0}^{\infty} a_k \, (\bar{z} - c^+)^k}$$

$$= \sum_{k=0}^{\infty} \overline{a_k} \, (z - \overline{c^+})^k$$

$$= \sum_{k=0}^{\infty} \overline{a_k} \, (z - c^-)^k, \quad \text{radius of convergence} \geq r.$$

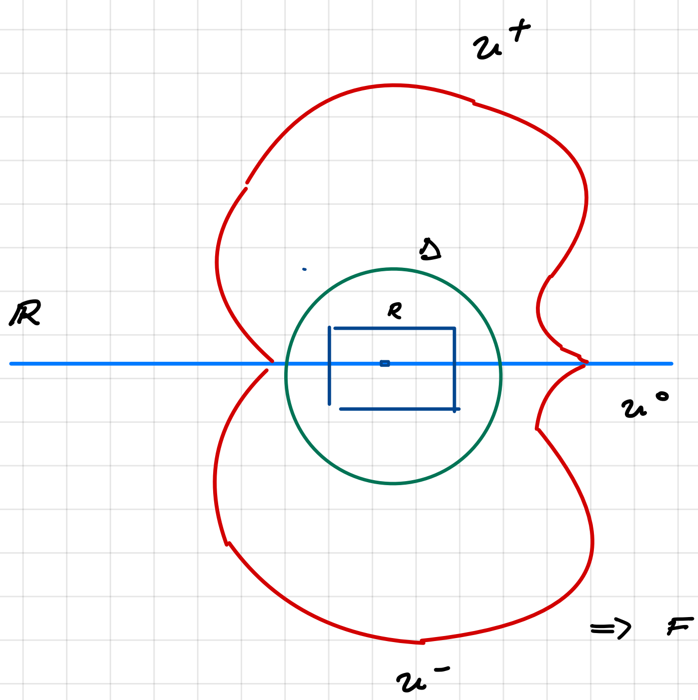
$\Rightarrow$ F holomorphic in $\mathcal{U}^-$.

Proof of $\boxed{IV}$



We show F is holomorphic in discs $z_0 \in \Delta \subseteq \mathcal{U}$ for arbitrary $z_0 \in \mathcal{U}_0$.

This will complete the proof.



Goal $\forall\ \bar{R} \subseteq \Delta$

$$\int_{\partial R} F\, dz = 0$$

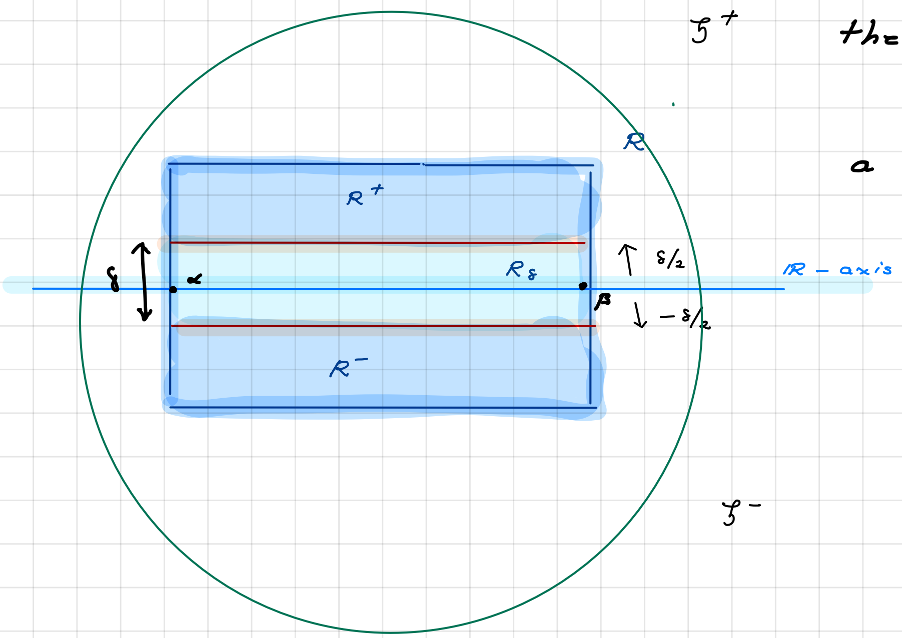
By Math 220A, Lecture 5

$\Rightarrow F = G'$ for some holomorphic G in Δ

$\Rightarrow F$ holomorphic in Δ.

If $\bar{R} \subseteq \mathcal{U}^+$ or $\bar{R} \subseteq \mathcal{U}^-$ this is clear (Goursat / Cauchy).

Assume $\bar{R}$ intersects the real axis. We assume that the intersection is not a side of R. Otherwise the argument is simpler.



We show $\exists\ K > 0$ such that for all $\varepsilon > 0$,

$$\left| \int_{\partial R} F\, dz \right| \leq K.\,\varepsilon. \implies \int_{\partial R} F\, dz = 0.$$

$\boxed{\text{i}}$ F continuous in $\bar{\Delta} \implies |F(z)| \leq M$ for all $z \in \bar{\Delta}$.

$\boxed{\text{ii}}$ F uniformly continuous in $\bar{\Delta}$ = compact.

$\implies \forall\, \varepsilon\ \exists\, \delta,\ |x - y| \leq \delta \implies |F(x) - F(y)| < \varepsilon.$

We may assume $\delta < \varepsilon$.

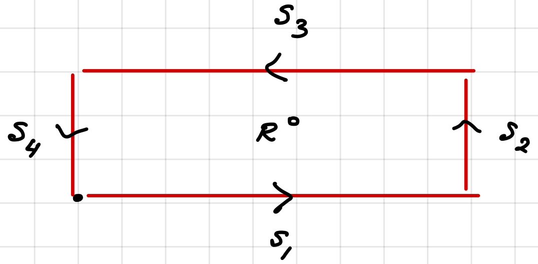
$\boxed{i}$ Construct R^+, R^-, R^0 where $R^+ \subseteq \mathcal{U}^+$, $R^- \subseteq \mathcal{U}^-$

$$R^0 = [\alpha, \beta] \times \left[-\frac{\delta}{2}, \frac{\delta}{2}\right].$$

$\boxed{ii}$ $\displaystyle\int_{\partial R^+} F\, dz = 0$, $\displaystyle\int_{\partial R^-} F\, dz = 0$ by Goursat.

$$\Longrightarrow \int_{\partial R} F\, dz = \int_{\partial R^0} F\, dz.$$

Estimates:



Sides of R^0: S_1, S_2, S_3, S_4.

(1) $$\left| \int_{S_2} F\,dz + \int_{S_4} F\,dz \right| \leq \left| \int_{S_2} F\,dz \right| + \left| \int_{S_4} F\,dz \right|$$

$$\leq M \cdot \underbrace{\text{length } S_2}_{\delta} + M \cdot \underbrace{\text{length } S_4}_{\delta}$$

$$= 2M\delta < 2M\varepsilon.$$

parametrize

(2) $$\left| \int_{S_1} F\,dz + \int_{S_3} F\,dz \right| \leq \int_{\alpha}^{\beta} \underbrace{\left| F\left(t - \frac{i\delta}{2}\right) - F\left(t + \frac{i\delta}{2}\right) \right|}_{<\varepsilon} dt$$

(uniform continuity).

$$\leq \varepsilon \cdot (\beta - \alpha) \leq \varepsilon \cdot \operatorname{diam}(\Delta)$$

(1)+(2)

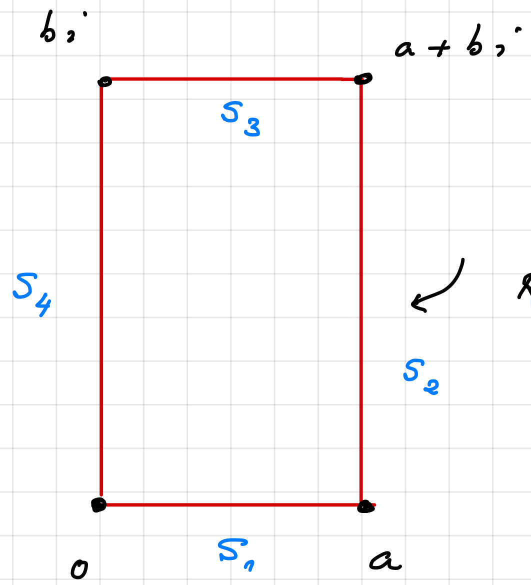
$$\Rightarrow \left| \int_{\partial R^\circ} F\,dz \right| \leq \left| \int_{S_2} F\,dz + \int_{S_4} F\,dz \right| + \left| \int_{S_1} F\,dz + \int_{S_3} F\,dz \right|$$

$$\leq 2M\varepsilon + \varepsilon \cdot \operatorname{diam}(\Delta) = K\varepsilon.$$

This completes the proof.

2. Application

Conformal maps of rectangles



$R_{a,b}$, $a, b \in \mathbb{R}$.

Example

$\exists$ biholomorphism $f : R_{a,b} \longrightarrow R_{a',b'}$ such that

[i] f extends continuously & bijectively to the boundary.

[ii] sending corners to corners & edges to edges.

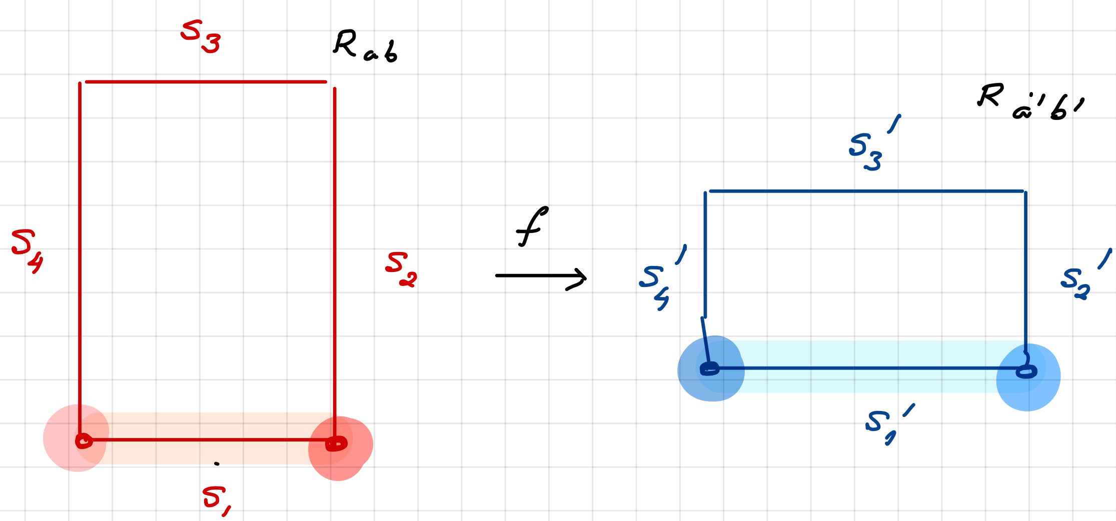
IF AND ONLY IF

$$\frac{a'}{a} = \pm \frac{b'}{b} \quad \text{or} \quad a a' = \pm b b'!$$

Remark Condition $\boxed{\text{ii}}$ is automatic by Caratheodory, while condition $\boxed{\text{iii}}$ is really need.

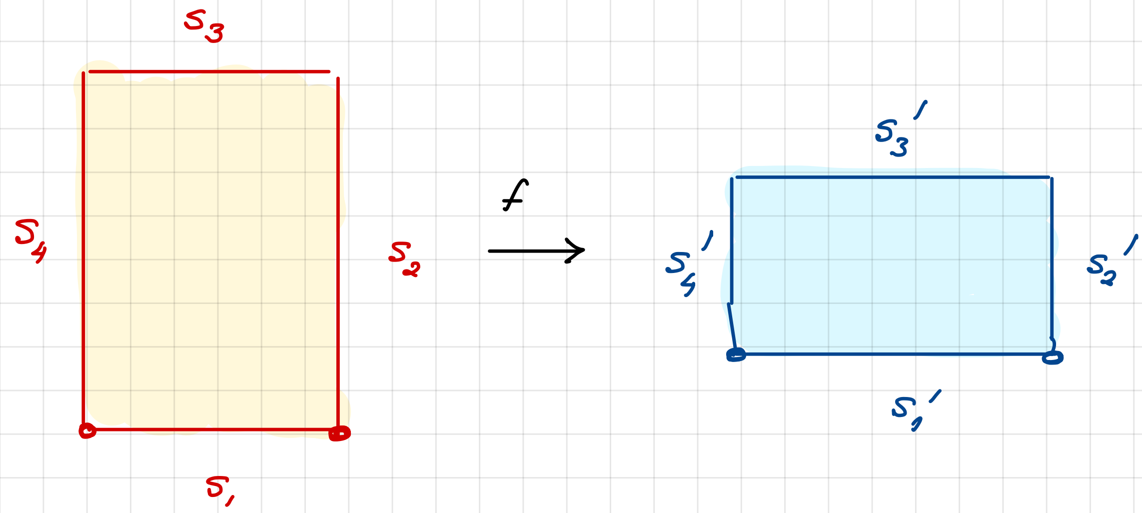
We first assume

$$f: S_1 \longrightarrow S_1', \quad 0 \longrightarrow 0, \quad a \longrightarrow a'.$$

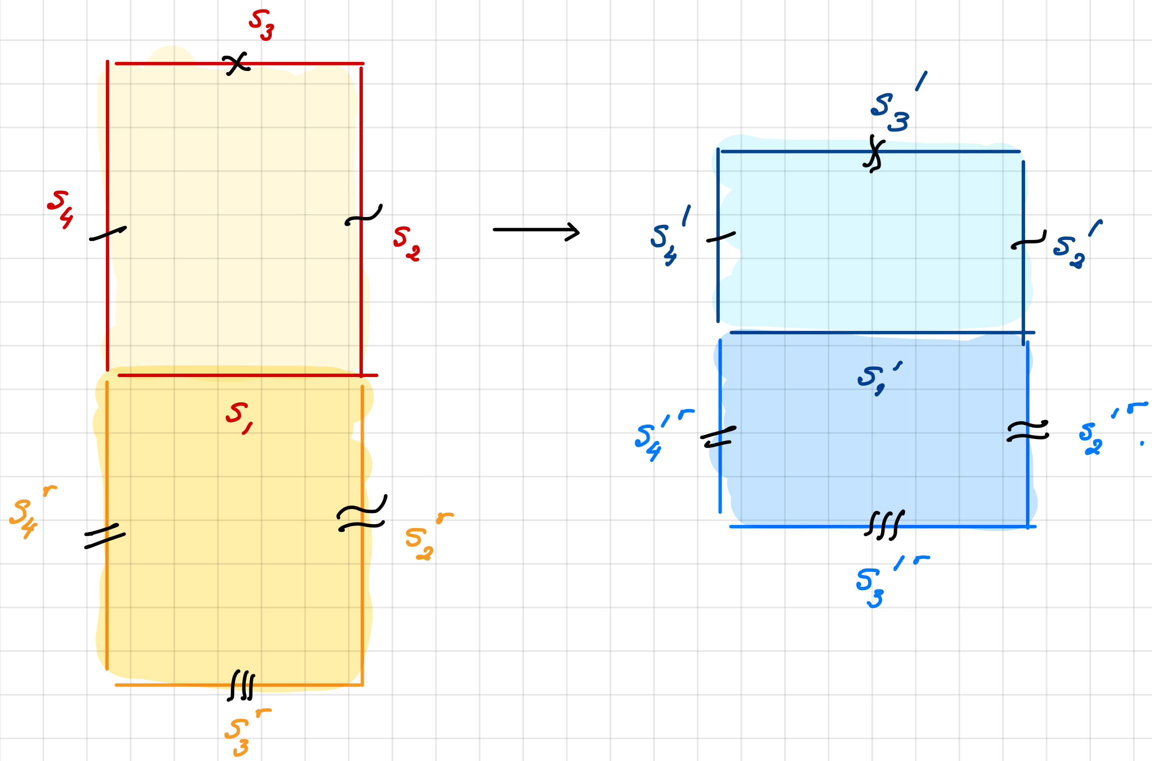


$f(0) = 0, \quad f(a) = a'$

- S_4 is sent to a side containing $f(0) = 0$, hence S_4'
- S_2 is sent to a side containing $f(a) = a'$, hence S_2'
- S_3 is sent to the remaining side S_3



We use Schwarz Reflection along S_1 & S_1'.



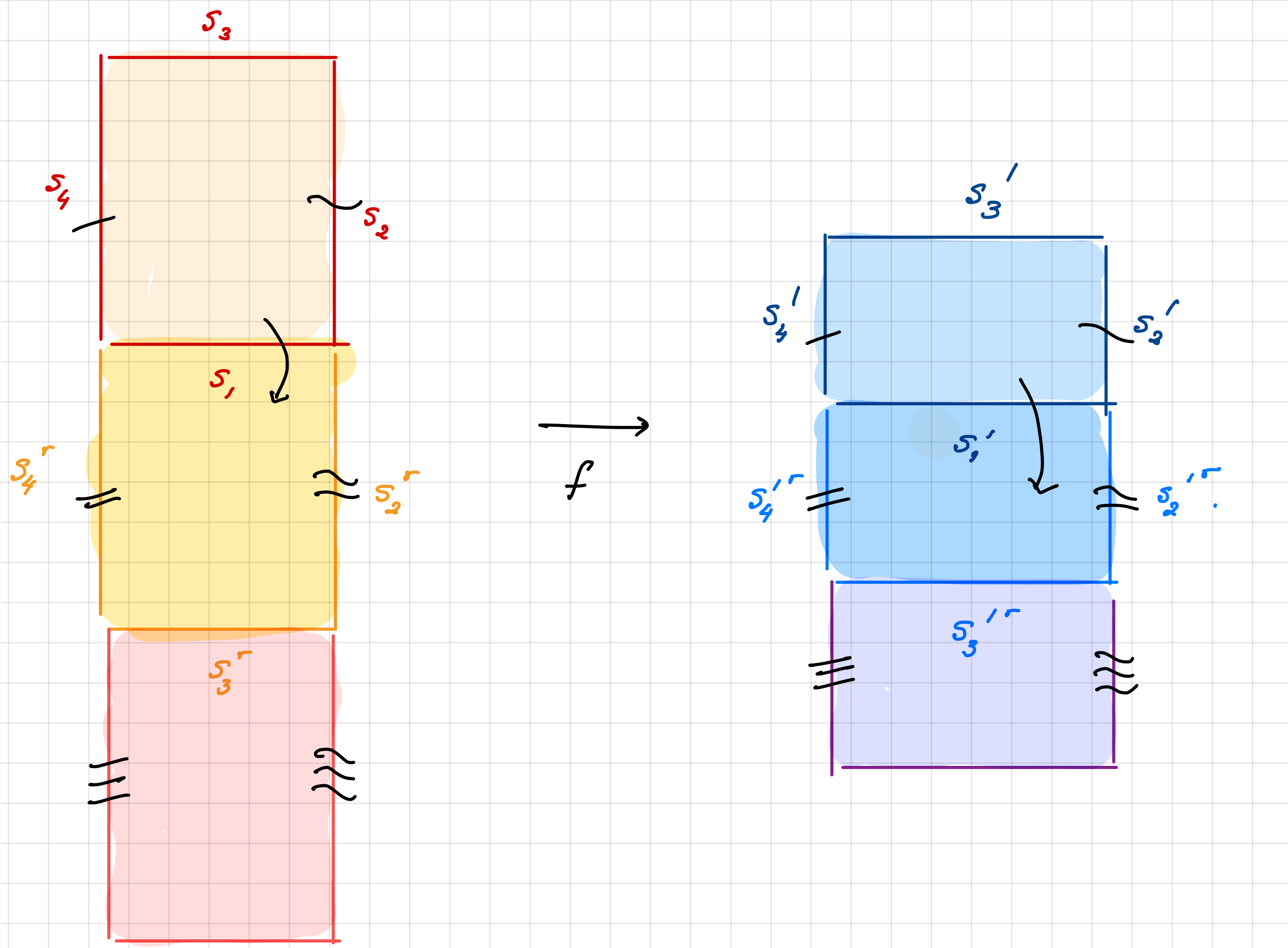
Note

$$S_4^r \longrightarrow S_4'^r, \quad S_2^r \longrightarrow S_2'^r, \quad S_3^r \longrightarrow S_3'^r.$$

from the explicit formula for the extension

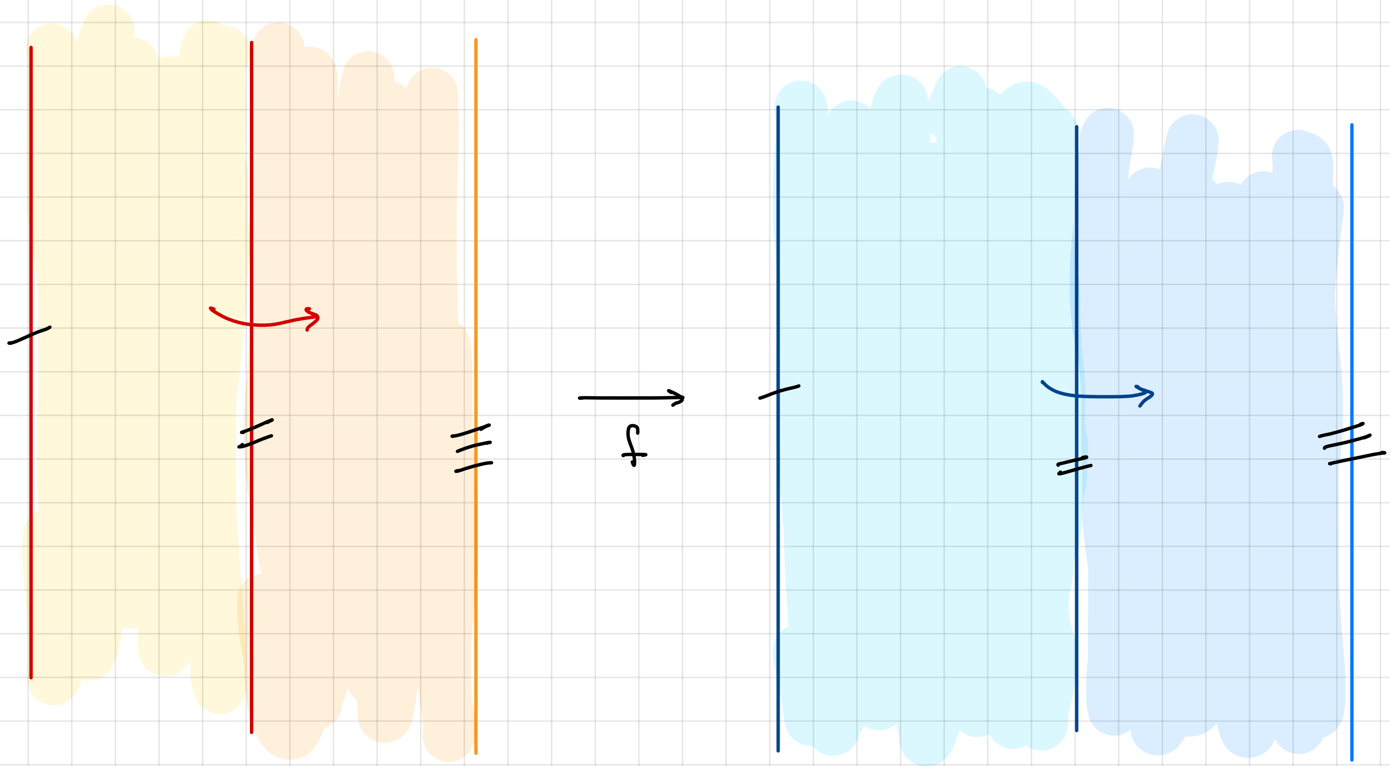
The extension is still bijective. (as the picture shows).

Reflect the new rectangle one more time, across S_3^r & $S_3'^r$.



and continue until we get two strips mapping to each other so that their boundaries are mapped respectively.

Now reflect the strips across their sides.



In the end, we obtain $f: \mathbb{C} \longrightarrow \mathbb{C}$ bijective & holomorphic.

We saw in Math 220A, PSet 5 that $f(z) = \alpha z + \beta$.

Since $f(0) = 0 \Rightarrow \beta = 0 \Rightarrow f(z) = \alpha z$.

$$\left.\begin{array}{l} f(a) = a' \Rightarrow \alpha a = a' \\ f(b) = b' \Rightarrow \alpha b = b' \end{array}\right\} \Rightarrow \frac{a'}{a} = \frac{b'}{b}.$$

The remaining cases are part of Homework 6.

Math 220B — Lecture 22

March 1, 2021

Part I : Weierstraß & Mittag-Leffler

Series & Products

Part II Riemann & Schwarz

Mapping Theory

Part III Runge ← Conway VIII. 1.

Approximation theory

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§1. Context for Runge

In real analysis (Math 140B), we learn

Weierstraß Approximation theorem

$f : [a,b] \longrightarrow \mathbb{R}$ continuous, $\exists$ P_n polynomials

$$P_n \rightrightarrows f.$$

This was proven by Weierstraß at age 70 in 1885.

There are many applications of this theorem.

e.g. in Fourier analysis, functional analysis etc.

Remark This can be generalized in $\mathbb{R}^n$:

If $K \subseteq \mathbb{R}^n$ compact, $f : K \longrightarrow \mathbb{R}$ continuous, then

$$\exists \; P_n \text{ polynomials, } P_n \rightrightarrows f \text{ in } K.$$

Runge (age 29, Ph. D. 1850, student of Weierstraß):

Question What about f holomorphic? Can it be approximated by polynomials in z?

Answer was given in 1855 as well.

Remark This doesn't follow from Weierstraß.

Weierstraß produces polynomials in x, y for $z = x + iy$.

e.g. polynomials in z and $\bar{z}$.

ZUR THEORIE DER EINDEUTIGEN ANALYTISCHEN FUNCTIONEN

VON

C. RUNGE[1]

in BERLIN.

Seit dem Bekanntwerden der Modulfunctionen, weiss man, dass der Gültigkeitsbereich einer analytischen Function nicht nothwendig von discreten Punkten begrenzt zu sein braucht, sondern dass auch continuirliche Linien als Begrenzungsstücke auftreten und einen Theil der complexen Ebene von dem Gültigkeitsbereich ausschliessen können.

Hier entsteht nun die Frage, ob der Gültigkeitsbereich analytischer Functionen seiner Form nach irgend welchen Beschränkungen unterliegt oder nicht. Diese Frage bildet, so weit sie sich auf eindeutige analytische Functionen bezieht, den Gegenstand der nachfolgenden Untersuchung. Es wird sich ergeben, dass der Gültigkeitsbereich einer eindeutigen analytischen Function d. h. die Gesammtheit aller Stellen an denen sie sich regulär oder ausserwesentlich singulär verhält keiner andern Beschränkung unterliegt als derjenigen, zusammenhängend zu sein. In dem ersten Theile

[1] Die Aufgabe, welche in dem ersten Paragraphen dieser Arbeit in eleganter Weise gelöst wird, ist nicht in meiner Abhandlung *Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante* (Acta mathematica 4, S. 1—79) behandelt worden. Diejenige Aufgabe dagegen, mit welcher sich der Verfasser in dem zweiten Paragraphen beschäftigt, ist in meiner Abhandlung aus mehreren verschiedenen Gesichtspunkten betrachtet und gelöst worden. Da jedoch der Verfasser seine Untersuchungen vor der Veröffentlichung meiner oben citirten Abhandlung machte und auch ganz andere mit dem CAUCHY'schen Integralsatze in Zusammenhang stehende Methoden braucht, so habe ich die ganze Arbeit für geeignet gehalten hier aufgenommen zu werden.

Der Herausgeber.

Acta mathematica. 6. Imprimé 29 Septembre 1884.

Acta Math 6 (1885)

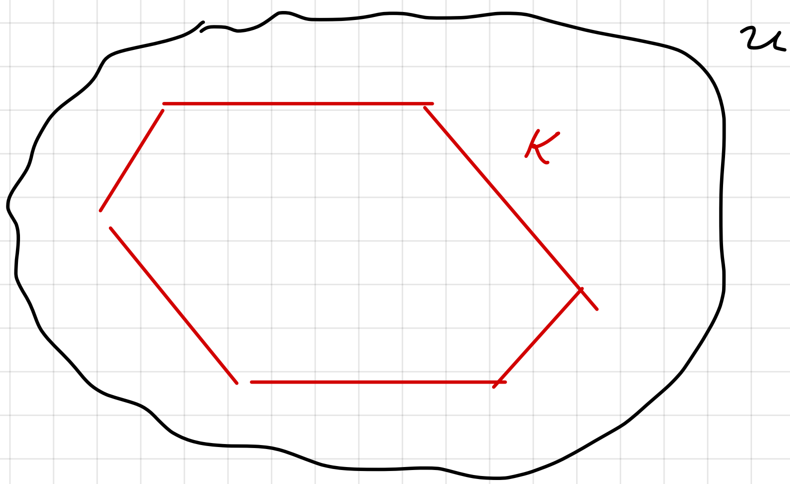
Carl Runge (1856 - 1927)

- Runge - Kutta
- Runge's Approximation
- mathematics, astrophysics, spectroscopy.

§2. Parsing the Question more carefully

Beware A holomorphic function is defined over OPEN sets. (see Math 220A).

Definition $K \subseteq \mathbb{C}$ compact. A holomorphic function in K is a function $f: K \longrightarrow \mathbb{C}$ that extends holomorphically to a neighborhood $U \supseteq K$.



Two versions of the question

Runge C (compact sets) $\quad K \subseteq \mathbb{C}$ compact

Given f holomorphic in K, are there polynomials p_n such that $p_n \rightrightarrows f$ in K ?

Runge O (open sets) $\quad \mathcal{U} \subseteq \mathbb{C}$ open

Given f holomorphic in $\mathcal{U}$, are there polynomials p_n such that $p_n \overset{l.u.}{\rightrightarrows} f$ in $\mathcal{U}$?

Emphasis

Runge C : approximation on a single compact K

Runge O : approximation on all compacts K

in the domain of a holomorphic function

Runge C is more basic.

complex analysis

Runge C $\Longrightarrow$ Runge O

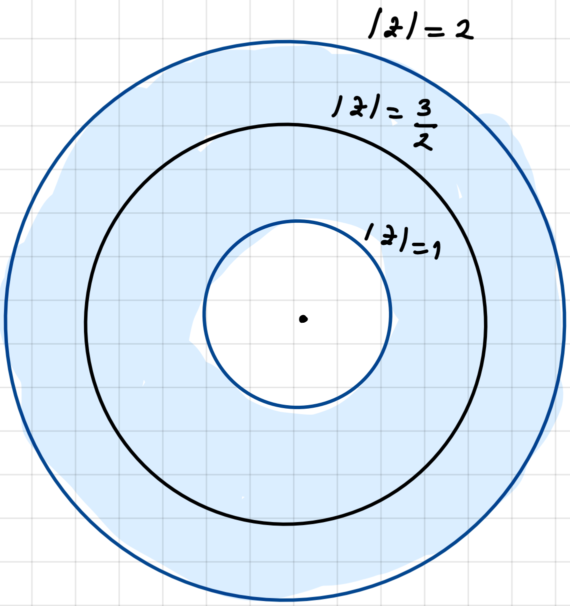
point-set topology.

The two versions are very similar.

Example Runge C.

$K = \{ 1 \le |z| \le 2 \}$, $f(z) = \frac{1}{z}$. holomorphic in K.

Can we find $P_n \rightrightarrows f$ in K ?



No ! Note f is holomorphic in $u \supseteq K$, $u = \{ \frac{1}{2} < |z| < \frac{5}{2} \}$

so "holomorphic in K".

If $P_n \rightrightarrows f$ in K then

$$\int_{|z|=\frac{3}{2}} P_n \, dz \longrightarrow \int_{|z|=\frac{3}{2}} f \, dz.$$

Note $\int_{|z|=\frac{3}{2}} P_n \, dz = 0$ & $\int_{|z|=\frac{3}{2}} f \, dz = 2\pi i$ by the residue theorem. This is a contradiction.

The failure is due to the "hole" in K.

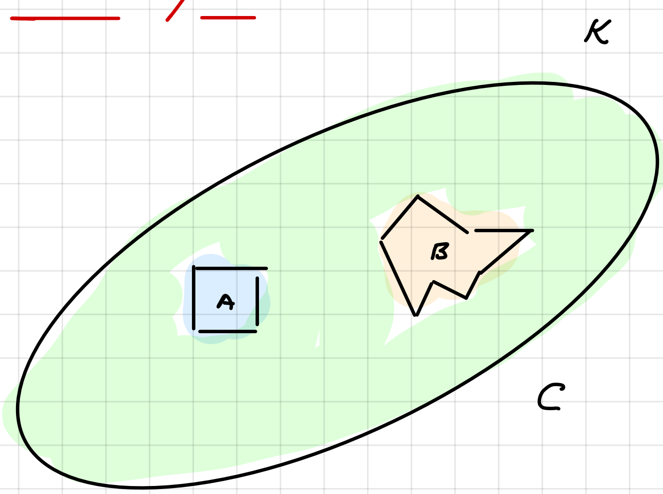
What is a "hole"?

Definition $K \subseteq \mathbb{C}$. compact

A hole is a bounded connected component of $\mathbb{C} \setminus K$.

Example

[I]



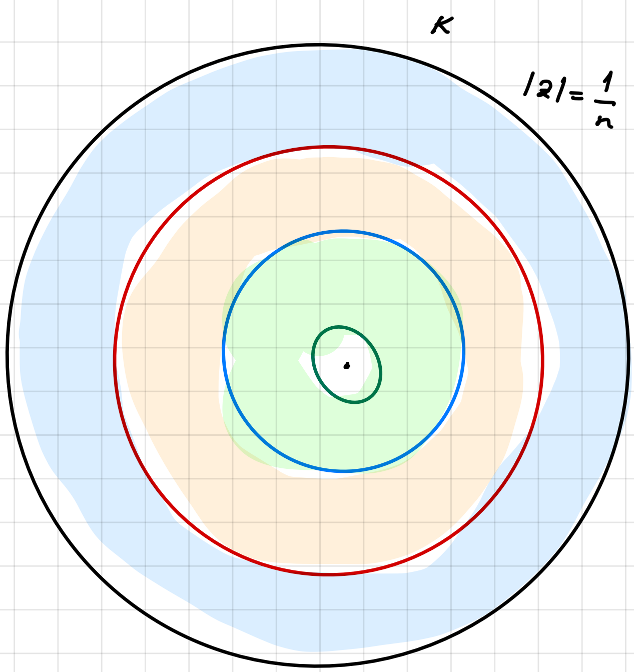
$\mathbb{C} - K = A \cup B \cup C$

C unbounded

A, B bounded

A, B are holes for K.

[II]



$$K = \bigcup_{n \geq 1} \left\{ |z| = \frac{1}{n} \right\} \cup \{0\}$$

K closed & bounded $\Rightarrow$

$\Rightarrow K$ compact.

∞-many holes

$$H_n = \left\{ \frac{1}{n+1} < |z| < \frac{1}{n} \right\}.$$

§ 3. Runge's Theorem - Compact Sets

We give three versions. The simplest version is:

Runge's Little Theorem (Case $\mathbb{C}$)

If K has no holes ($\iff \mathbb{C} \setminus K$ connected)

then $\forall f$ holomorphic in K, $\exists$ polynomials P_n

with

$$P_n \rightrightarrows f \text{ in } K.$$

Question How about arbitrary K ?

Answer Polynomial approximation fails (Example)

Are we even asking the right question?

Better Rational Approximation.

Question Given f holomorphic in K,

$\exists$ R_n rational functions, $R_n \rightrightarrows f$ in K &

poles of R_n are outside K ?

Question Can we prescribe the location of the poles of R_n ?

Runge C (Almost final) $K \subseteq \mathbb{C}$ compact.

Thm Let S be a set of points,

at least one from each hole of K.

then $\forall f$ holomorphic in K.

[i] $\exists\ R_n \rightrightarrows f$ in K

[ii] R_n are rational functions whose poles are $\overset{\text{in } \mathbb{C}}{\overbrace{\text{in}}}$ S.

Remark The poles of R_n are contained in S, but it may happen that not all points of S are poles.

Remark If K has no holes then $S = \emptyset$. Thus

R_n has no poles $\Rightarrow$ R_n have no denominators $\Rightarrow$

$\Rightarrow$ R_n are polynomials. We recover Little Runge.

Runge C Final Form

Conway VIII. 1. 7.

We replace $\mathbb{C}$ by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Thm Let $K \subseteq \mathbb{C}$ compact. Let $S \subseteq \hat{\mathbb{C}}$ be a set of points, at least one chosen from each component of $\hat{\mathbb{C}} \setminus K$.

Let f be holomorphic in K. Then

[i] $\exists\ R_n \rightrightarrows f$ in K

in $\hat{\mathbb{C}}$

[ii] R_n are rational with possible poles in S.

Remark An interesting case allowed by the Final Version is to pick $\infty \in S$ from the unbounded component.

Thus, when S consists in

- ∞ from the unbounded component of $\hat{\mathbb{C}} \setminus K$
- a point from each bounded component of $\mathbb{C} \setminus K$ (holes)

we recover Almost Final Version.

The two versions are even equivalent in this case since the condition that a rational function R have at worst a pole at ∞ is vacuous. Indeed,

$$R(z) = \frac{\prod_{i=1}^{n} (z - a_i)}{\prod_{i=1}^{m} (z - b_i)} \Rightarrow R\left(\frac{1}{z}\right) = z^{m-n} \frac{\prod (1 - a_i z)}{\prod (1 - b_i z)}$$

has at worst a pole at 0.

Summary

Runge C (Final) $\Longrightarrow$ Runge C (Almost Final)

Conway VIII.1.7

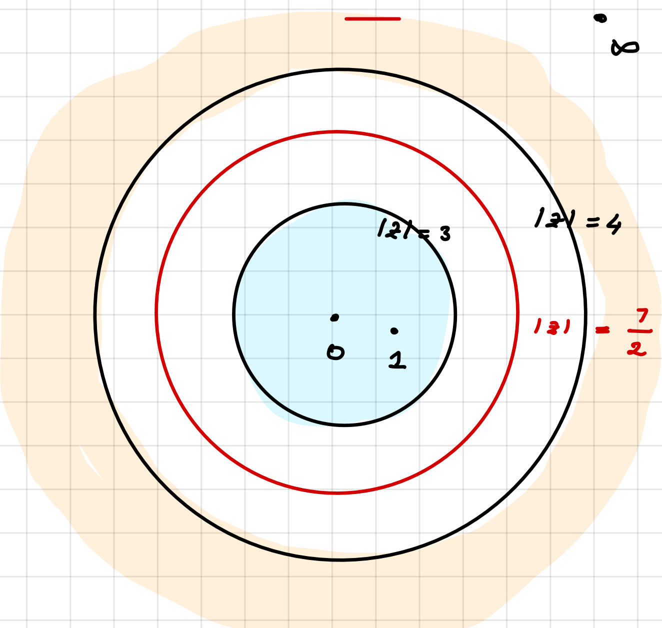
- rational approximation
- version for $\hat{\mathbb{C}}$

- rational approximation
- poles in each hole

Little Runge C

- polynomial approximation
- K has no holes

Example / Review



$$f(z) = \frac{z^3}{(z-2)(z-7)}$$

$$K = \{ 3 \leq |z| \leq 4 \}.$$

f is holomorphic in K because it extends holomorphically to

$$U = \left\{ \frac{5}{2} < |z| < \frac{9}{2} \right\} \supseteq K.$$

Can we approximate f uniformly on K by:

(1) rational functions with poles in $\mathbb{C}$ at 1 ?

YES Amost Final Version. Poles in $\hat{\mathbb{C}}$ are $1, \infty$.

(2) rational functions with poles at $0, \infty$

YES Final Version

(3) rational functions with poles at ∞ ?

NO. Such rational functions would have to be polynomials (if they had denominators, there would be poles). But if $P_n \rightrightarrows f$ then

$$\int_{|z|=\frac{7}{2}} P_n \, dz \longrightarrow \int_{|z|=\frac{7}{2}} f \, dz = 2\pi i \; \mathrm{Res}\,(f, 2)$$

$$\int_{|z|=\frac{7}{2}} P_n \, dz = 0$$

$$= 2\pi i \cdot \left. \frac{z^3}{z-7} \right|_{z=2} \neq 0$$

using the Residue theorem. Contradiction!

Math 220B — Lecture 23

March 3, 2021

Math 220C Survey

- first half: MWF 3–3:50, live
- second half: TBD.

Remaining Topics in Math 220B

- Proof of Runge C (today, Friday)
- Runge O (Monday)
- Summary & Loose Ends (Wednesday)
- Review (Friday)

Runge C (Final) $\Longrightarrow$ Runge C (Almost Final)

Conway VIII.1.7

- rational approximation
- version for $\hat{\mathbb{C}}$

- rational approximation
- poles in each hole

Little Runge C

- polynomial approximation
- K has no holes

Last time

Thm Let $K \subseteq \mathbb{C}$ compact. Let $S \subseteq \hat{\mathbb{C}}$ be a set of points, at least one chosen from each component of $\hat{\mathbb{C}} \setminus K$.

Let f be holomorphic in K. Then

$\boxed{i}$ $\exists\ R_n \rightrightarrows f$ in K

$\boxed{ii}$ R_n are rational with possible poles in S.

in $\hat{\mathbb{C}}$

Strategy

Step 1 Cauchy Integral Formula for compact sets.

Step 2 Approximation without prescribed poles

Step 3 Push the poles to prescribed location.

Step 1

Recall (Math 220A).

f holomorphic in $\mathcal{U}$, $\overline{R} \subseteq \mathcal{U}$, then

$$\frac{1}{2\pi i} \int_{\partial R} \frac{f(z)}{z-a}\, dz = \begin{cases} f(a). & \forall a \in R \\ 0 & \forall a \notin \overline{R} \end{cases}$$

We wish to do the same for any compact $K \subseteq \mathcal{U}$.

Lemma

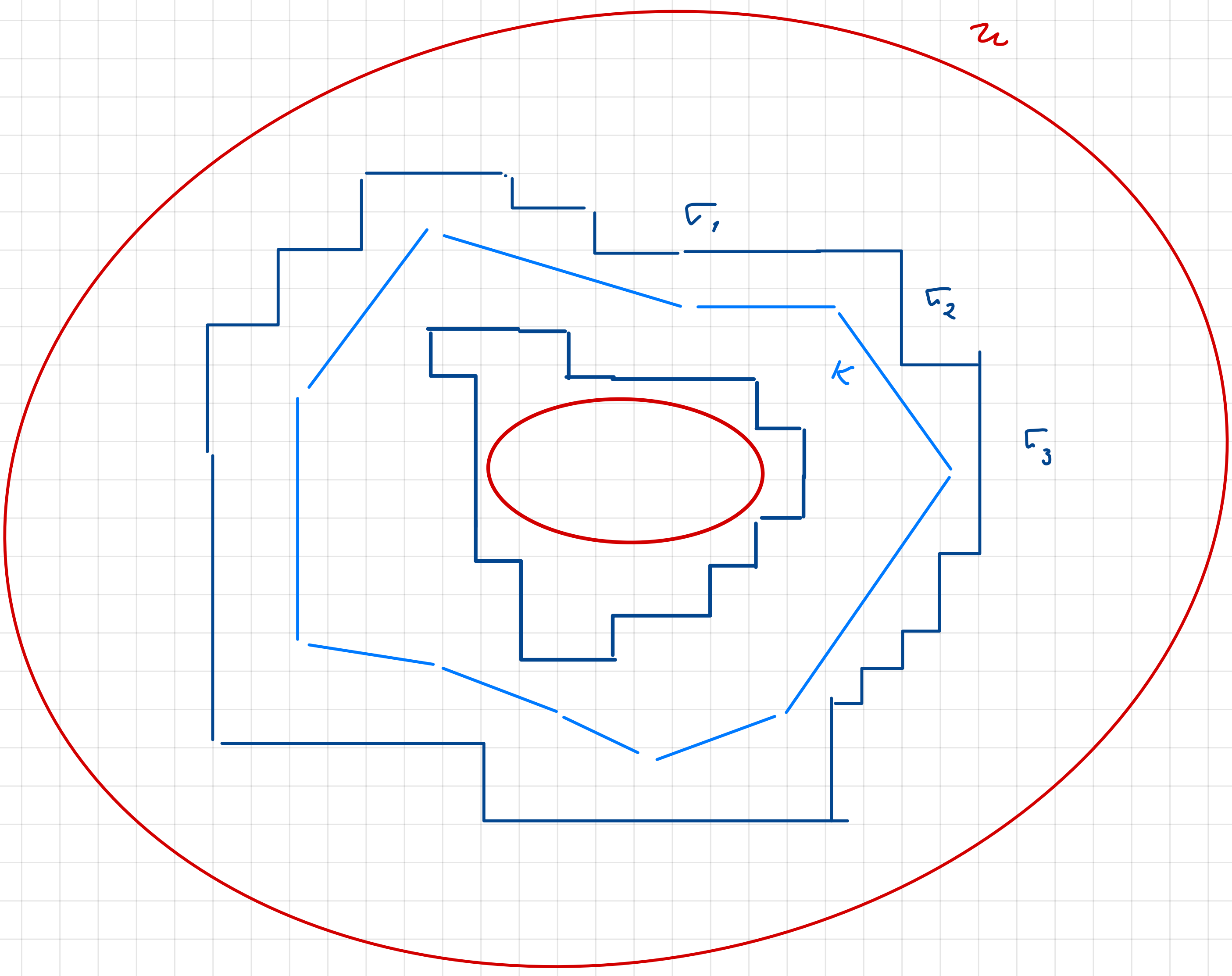
Conway VIII. 1.1.

Let $K \subseteq \mathcal{U}$ compact. There exist segments Γ_j such that

$$\Gamma = \Gamma_1 + \dots + \Gamma_n \subseteq \mathcal{U} \setminus K$$

and such that for all functions f holomorphic in $\mathcal{U}$

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz. \qquad \forall\, a \in K.$$



We will construct Γ as a union of closed polygons.

Remark If K has a simple structure this is not so bad. We'd need

$$n(\Gamma, a) = 1 \quad \forall a \in K.$$

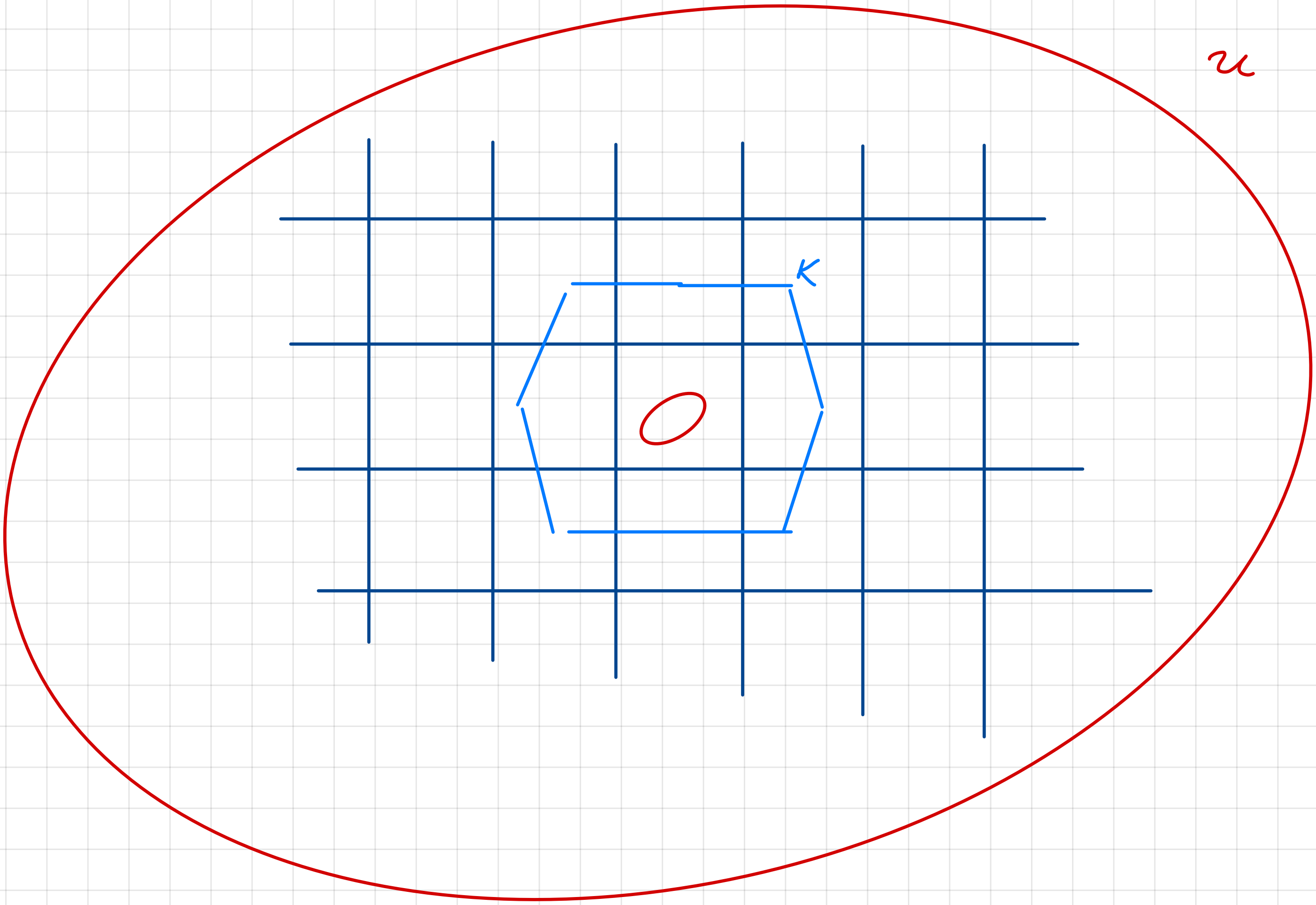
and argue using Cauchy's formula from Math 220A.

The issue is if K has complicated (fractal) structure.

Idea : Lay a grid !.

Proof

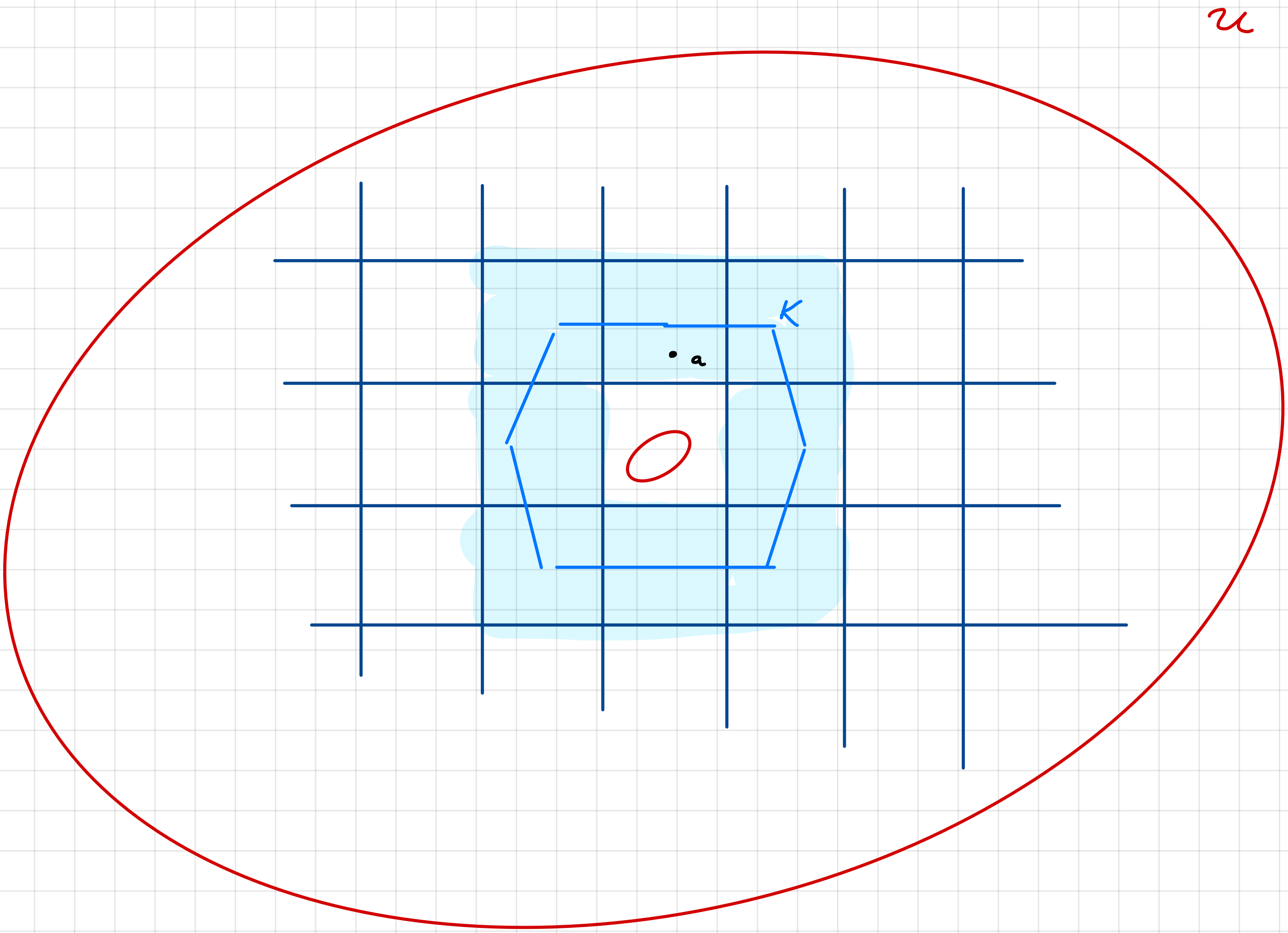
(1) Construction



WLOG $\mathcal{U} \neq \mathbb{C} \Rightarrow \mathbb{C} \setminus \mathcal{U} \neq \emptyset$ is closed. Note

$K \cap (\mathbb{C} \setminus \mathcal{U}) = \emptyset$. Let $d = d(K, \mathbb{C} \setminus \mathcal{U}) > 0$.

Lay a grid of squares of side $< \frac{d}{\sqrt{2}}$.



Consider the closed squares

$Q_1, Q_2 \dots Q_m$ that intersect K.

There are only finitely many squares since K is compact.

Claim 1 $K \subseteq \bigcup_{j=1}^{m} Q_j \subseteq \mathcal{U}$.

Proof If $k \in K$ then k is contained in a square of the grid. This square intersects K at k so it must be one of the Q_j & $k \in Q_j$. This gives the first inclusion.

For the second inclusion, let $g \in Q_j$ where $Q_j \cap K \neq \emptyset$. Let $k \in Q_j \cap K$. If $g \notin \mathcal{U} \Rightarrow$

$\Rightarrow g \in \mathbb{C} \setminus \mathcal{U}$ and $k \in K$ so

$$d(g, k) \geq d(\mathbb{C} \setminus \mathcal{U}, K) = d.$$

But $g, k \in Q_j \Rightarrow d(g, k) < \operatorname{diam}(Q_j) = d$ contradiction!

Thus $g \in \mathcal{U}$, as needed.

Construction of σ

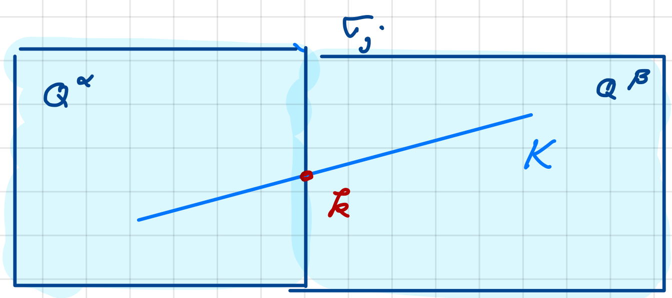
- $\sigma_1, \dots, \sigma_n$ sides of $Q_1, \dots, Q_m$ which are not shared by two squares Q_α, Q_β, $1 \leq \alpha \neq \beta \leq m$.

Claim 2 $\bigcup_{j=1}^{n} \tau_j \subseteq \mathcal{U} \setminus K.$

Proof

Note $\tau_j \subseteq \mathcal{U}$ by Claim 1. Assume $\tau_j \cap K \neq \Phi$.

Let $k \in \tau_j \cap K$. Then τ_j is a side of two squares.



These squares must intersect K necessarily since τ_j does.

These squares must be some of the Q_α, Q_β's, contradicting the definition of τ_j.

Claim 3 $\forall\, a \in \mathcal{U} \setminus \bigcup_{j=1}^{m} \partial Q_j$ then

$$\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(z)}{z-a}\, dz = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z-a}\, dz .$$

This follows because the common sides of the Q_j's cancel out, leaving only the integral over γ_j's.

Assume $a \in \operatorname{Int} Q_\ell$. By Cauchy for rectangles

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(z)}{z-a}\, dz = f(a) \quad \text{if } j = \ell$$

and 0 otherwise.

$\Rightarrow \forall\, a \in \operatorname{Int} Q_\ell$,

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\gamma_j} \frac{f(z)}{z-a}\, dz \qquad (*)$$

This is almost the Lemma. We have one more step.

Claim 4 $\forall\, a \in K$

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz \qquad (**).$$

Proof The only issue is the case when $a \notin \operatorname{Int} Q_e$.

$\Rightarrow$ a must be on a side of some Q_j b/c. $K \subseteq \bigcup_{j=1}^{m} Q_j$

by Claim 1. By Claim 2, $a \notin \Gamma_j$.

Find $a_n \longrightarrow a$ with a_n in the interior of the squares Q_s. Both sides of $(**)$ agree at a_n by $(*)$

$$f(a_n) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a_n}\, dz$$

Both sides are continuous in a. This is clear for LHS & RHS is explained below. Make $n \to \infty$ to conclude

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz,$$

proving the Lemma completely.

Continuity of RHS is a consequence of:

Key Fact (Math 220A, Homework 3, Problem 7).

$$\Phi : \overset{z}{\gamma} \times \overset{a}{\mathcal{U} \setminus \gamma} \longrightarrow \mathbb{C} \quad \text{continuous}$$

then $a \longrightarrow \int_{\gamma} \Phi(z, a)\, dz$ is continuous.

Apply to $\Phi : \gamma_j \times \mathcal{U} \setminus \gamma_j \longrightarrow \mathbb{C}$

$$\Phi(z, a) = \frac{f(z)}{z - a}, \quad z \in \gamma_j, \quad a \in \mathcal{U} \setminus \gamma_j$$

to conclude.

Step 1 is now established. Steps 2 & 3 next time.

Math 220B — Lecture 24

March 5, 2021

Where are we?

K compact, $K \subseteq \mathcal{U}$, $f : \mathcal{U} \to \mathbb{C}$ holomorphic

Wish $\forall\, \varepsilon \quad \exists\, R$ rational function with prescribed poles

in a suitable set S.

$$|f - R| < \varepsilon \quad \text{in } K$$

Conway VIII. 1.1.

Step 1 We found segments $\gamma_1, \dots, \gamma_n \subseteq \mathcal{U} \setminus K$

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(w)}{w - z}\, dw \qquad \forall\, z \in K.$$

Step 2 Find rational functions R with

Conway VIII. 1.5

$|f - R| < \varepsilon$ in K, poles of R are on the segments γ_j.

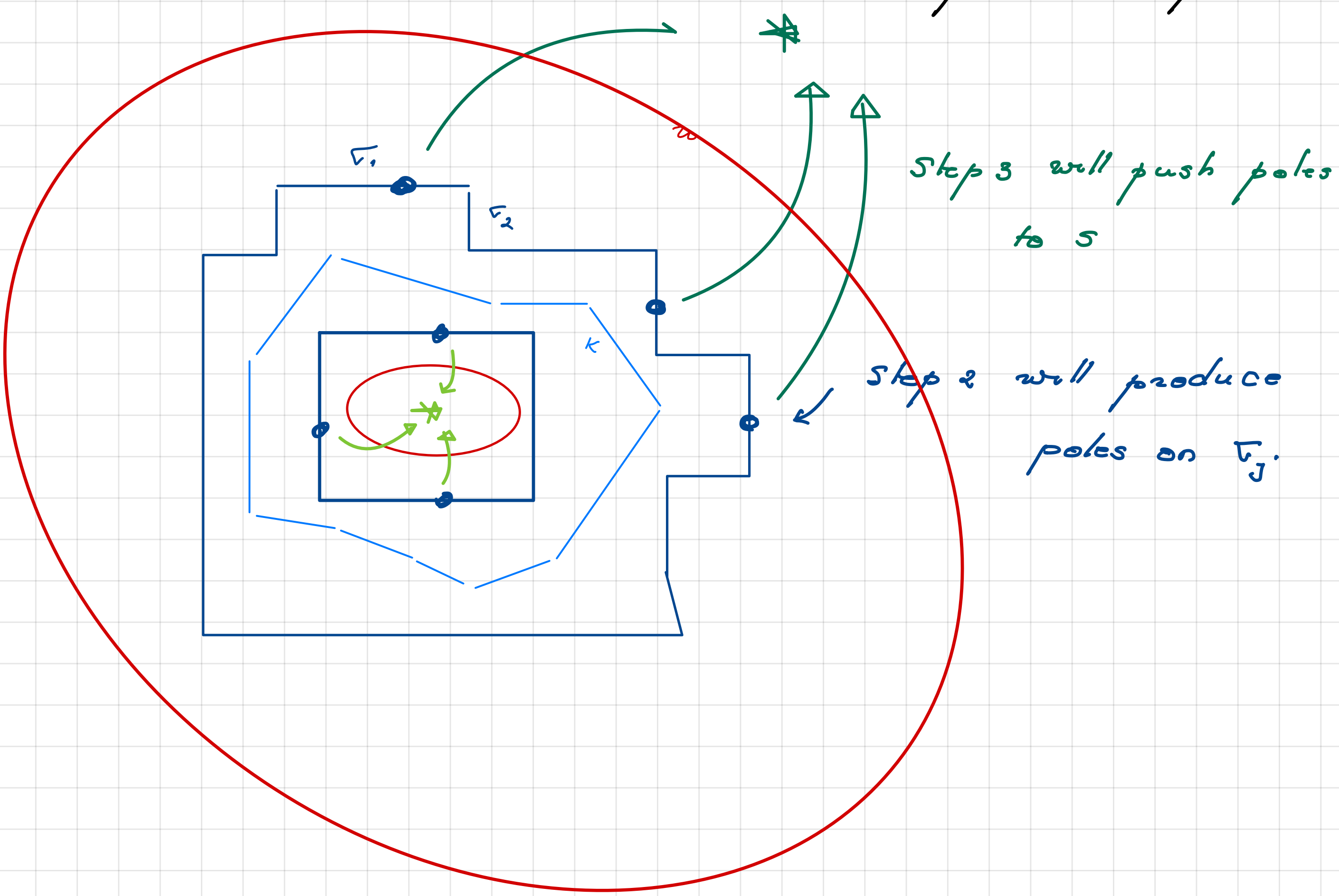
Step 3 Push the poles to prescribed locations.

Conway 1.6 – 1.13.

Visualization of the strategy

$S = \{ *, * \}$

prescribed poles



For step 2 we argue one segment Γ_j at a time showing

$$F_j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f(w)}{w-z}\, dw$$ can be approximated by

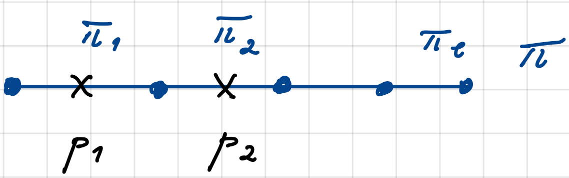
rational functions with poles in Γ_j.

Proof of Step 2

- K compact, π segment (compact), $\pi \cap K = \emptyset$
- f continuous in K

Main Claim (Conway VIII. 1.5)

$$F(z) = \int_\pi \frac{f(w)}{w-z}\, dw$$ can be approximated uniformly on K by rational functions with poles in π.



Proof Let $\varphi(w, z) = \dfrac{f(w)}{w - z} : \pi \times K \longrightarrow \mathbb{C}$, $w \in \pi$, $z \in K$.

Since $\pi \cap K = \emptyset \Rightarrow \varphi$ is continuous hence uniformly cont.

$\Rightarrow \forall \varepsilon \ \exists \delta$ such that

$$|w - w'| < \delta \Rightarrow |\varphi(w, z) - \varphi(w', z)| < \varepsilon.$$

- Subdivide π into subsegments $\pi_1, \ldots, \pi_\ell$ of length $< \delta$.
- Pick $p_k \in \pi_k$
- Let $c_k = f(p_k) \int_{\pi_k} dw$
- $R = \sum_{k=1}^{\ell} \dfrac{c_k}{p_k - z}$. rational function with pole at $p_k \in \pi$.

Claim

$$\left| F(z) - R(z) \right| = \left| \int_{\pi} \frac{f(w)}{w-z}\, dw - \sum_{k=1}^{\ell} \frac{f(p_k)}{p_k - z} \int_{\pi_k} dw \right|$$

$$\doteq \left| \sum_{k=1}^{\ell} \int_{\pi_k} \left(\frac{f(w)}{w-z} - \frac{f(p_k)}{p_k - z} \right) dw \right|$$

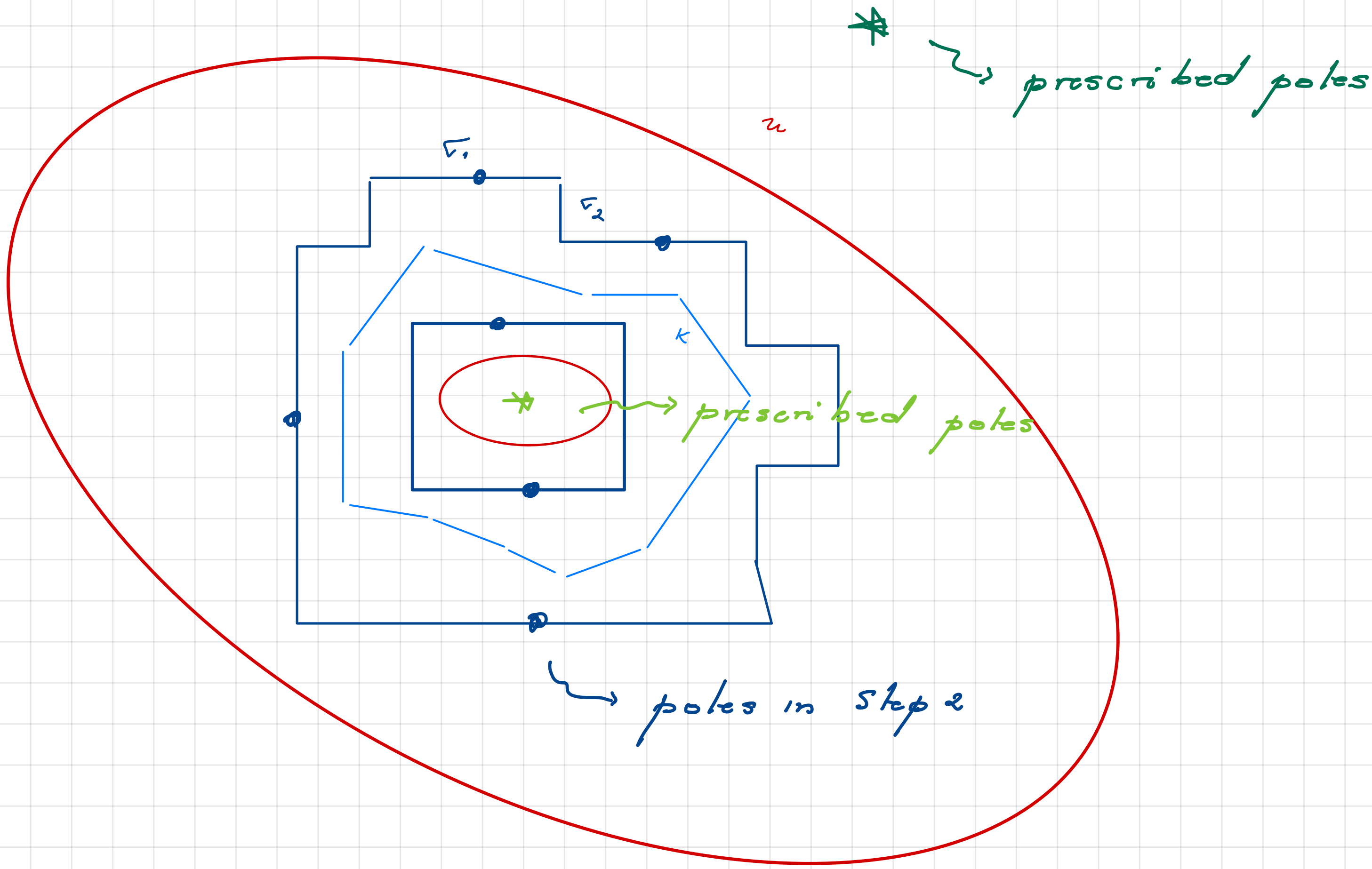
$$\leq \sum_{k=1}^{\ell} \left| \int_{\pi_k} \varphi(w,z) - \varphi(p_k, z)\, dw \right|$$

$$\leq \sum_{k=1}^{\ell} \varepsilon \cdot \operatorname{length}(\pi_k) = \varepsilon \cdot \operatorname{length}(\pi).$$

Here we used $|\varphi(w,z) - \varphi(p_k, z)| < \varepsilon$ since

$|w - p_k| < \delta$ which is true as $p_k, w \in \pi_k$, $\operatorname{length}(\pi_k) < \delta$.

The proof of Step 2 is completed.



Where are we?

- $K \subseteq \mathcal{U}$, f holomorphic
- $\exists R$ with poles in Γ_j, $|f - R| < \varepsilon$ on K.

Final Step Fix S a set of poles, one from each component of $\hat{\mathbb{C}} \setminus K$.

Push the poles from Γ_j to the points of S.

Step 3 Pole pushing to prescribed location.

Let $\hat{\mathbb{C}} \setminus K = \bigcup_i H_i$ = connected components

Let H be a fixed component.

pole produced in Step 2

prescribed location

Lemma $\forall\ a, b \in H$. Then

$\frac{1}{z-a}$ can be approximated uniformly in K by

poles at b

polynomials in $\frac{1}{z-b}$

If H is unbounded & $b = \infty$ then

$\frac{1}{z-a}$ can be approximated uniformly in K by polynomials

Polynomials in z = Rational Functions with poles possibly only at ∞.

Proof of the Lemma

- Keep b fixed & vary a. Consider the set

- $W = \{ c \in H : \frac{1}{z-c}$ can be approximated uniformly in K polynomials in $\frac{1}{z-b} \}$

We wish to prove $W = H$.

- $W \neq \Phi$. because $b \in W$.

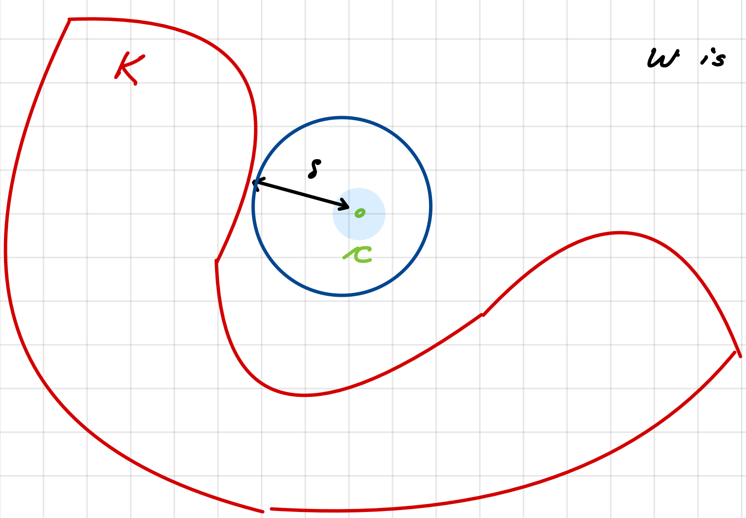
Key Claim

(*) $\forall\ c \in W$, let $\delta = d(c, K)$. Then $\Delta(c, \delta) \subseteq W$.

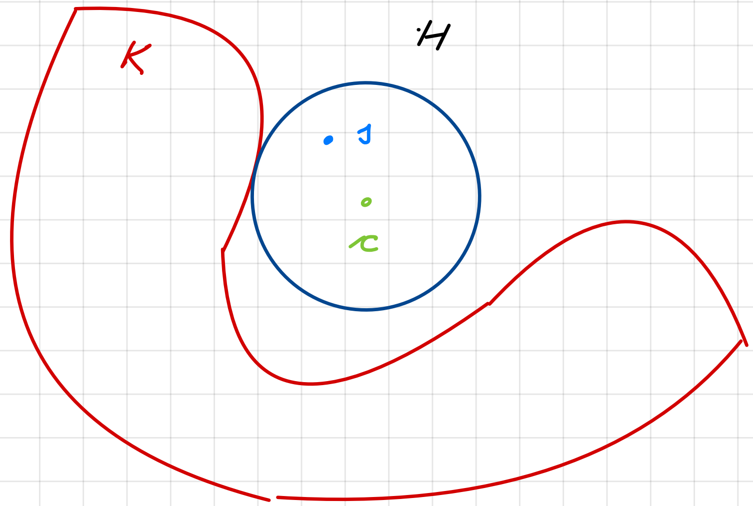
$\Downarrow$

Exercise This implies

W is closed & open hence $W = H$.



Proof of Key Claim Let $s \in \Delta(c, \delta)$. We wish to show that $s \in W$. $\Rightarrow \Delta \subseteq W$ as needed.



Idea $\dfrac{1}{z-s} \rightsquigarrow$ poly in $\dfrac{1}{z-c} \rightsquigarrow$ poly in $\dfrac{1}{z-b}$. $\Rightarrow s \in W$.

Consider the Laurent expansion of $\dfrac{1}{z-s}$ at c in $\Delta(c; \delta, \infty)$

$$\frac{1}{z-s} = \frac{1}{z-c} \cdot \frac{1}{1 - \frac{s-c}{z-c}} = \frac{1}{z-c} \sum_{k \geq 0} \left(\frac{s-c}{z-c} \right)^k = \sum_{k \geq 0} \frac{(s-c)^k}{(z-c)^{k+1}}$$

Convergence: $|z-c| > \delta > |s-c|$.

Note $z \in K$, $\delta = d(c, K) \Rightarrow K \subseteq \Delta(c; \delta, \infty)$. The Laurent expansion in $\Delta(c; \delta, \infty)$ converges locally uniformly (Math 220 A, Lecture 12).

Pick T a Laurent polynomial in $\frac{1}{z-c}$ from the Laurent expansion above so that

$$\left|\frac{1}{z-s} - T\right| < \frac{\varepsilon}{2} \quad \text{over } K.$$

Since $c \in W \Rightarrow \frac{1}{z-c}$ can be approximated by polynomials in $\frac{1}{z-b}$. The same is then true about $T =$ polynomial in $\frac{1}{z-c}$. Then

$\exists\ P$ polynomial in $\frac{1}{z-b}$ so that

$$|T - P| < \frac{\varepsilon}{2} \quad \text{in } K$$

Then $\left|\frac{1}{z-s} - P\right| \leq \left|\frac{1}{z-s} - T\right| + |T - P| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ in K. This

polynomial in $\frac{1}{z-b}$.

shows $s \in W$.

If H is unbounded. Let $K \subseteq \Delta(0,r)$

— first move the poles to $|c| > r$.

— Taylor expand $\frac{1}{z-c}$ near $z=0$ in

$$\Delta(0,|c|) \supseteq \Delta(0,r) \supseteq K$$

The Taylor series converges locally uniformly. Hence we can approximate $\frac{1}{z-c}$ by polynomials uniformly on K.

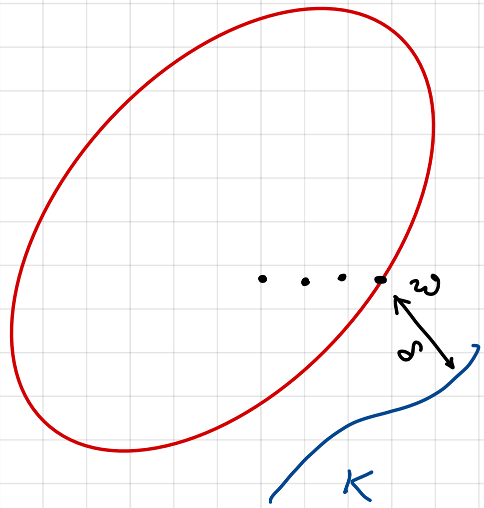
Proof of the Exercise

- W open. Indeed $\forall\, c \in W$ $\exists\, \Delta(c,\delta) \subseteq W$

by $(*)$ showing W open

- We show W closed in H.

Assume $w_n \longrightarrow w$, $w_n \in W$, $w \in H$.



Let $d(w, K) = \delta$

Fix n with $d(w, w_n) < \frac{\delta}{2}$.

$$\Rightarrow d(w_n, K) \geq d(w, K) - d(w, w_n) > \frac{\delta}{2}$$

$\Rightarrow \Delta\left(w_n, \frac{\delta}{2}\right) \subseteq W$ since $w_n \in W$ and $(*)$

$\Rightarrow w \in W$. since $w \in \Delta\left(w_n, \frac{\delta}{2}\right)$. This proves the Exercise.

Remark: This completes the proof of Runge.

Summary: start with f $\overset{\text{Step 1}}{\rightsquigarrow}$ Cauchy for compact sets

$\overset{\text{Step 2}}{\rightsquigarrow}$ rational approximation with poles in Γ_j.

$\overset{\text{Step 3}}{\rightsquigarrow}$ further approximation with prescribed poles

Math 220B — Lecture 25

March 8, 2021

§0. Last time — We established Runge C

Thm [i] $K \subseteq \mathbb{C}$ compact, $S \subseteq \hat{\mathbb{C}} \setminus K$ contains a point from each component of $\hat{\mathbb{C}} \setminus K$.

[ii] f holomorphic in K

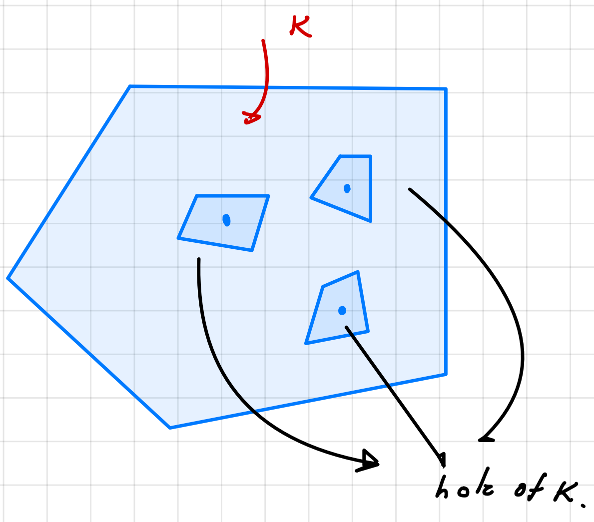
$\Rightarrow \forall \varepsilon \;\; \exists R$ rational,

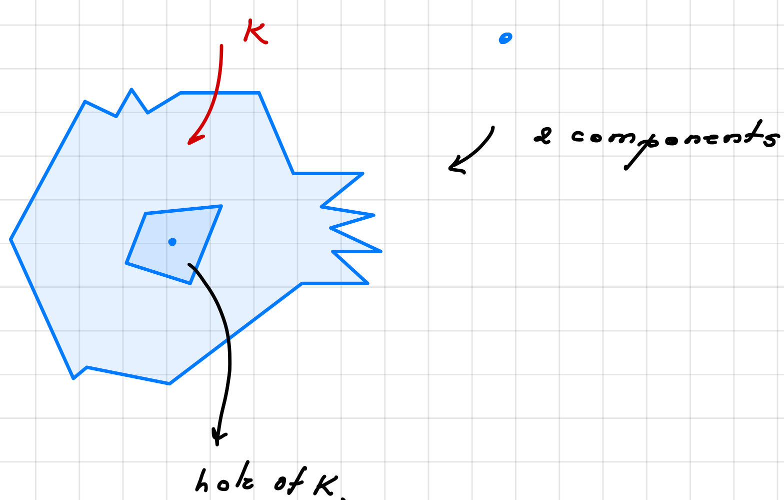
$|f - R| < \varepsilon$ in K and $\text{poles}(R) \subseteq S$.

Remark For $\varepsilon = \frac{1}{n} \Rightarrow \exists R_n$ with $|f - R_n| < \frac{1}{n}$ in K

$\Rightarrow R_n \rightrightarrows f$ in K. & $\text{poles}(R_n) \subseteq S$.

The set K can be disconnected and quite strange.





Applications

- density in spaces of functions
- new proof of Mittag-Leffler Conway VIII.3.
- polynomial convexity Conway VIII.1.
- generalizations: Mergelyan, ...

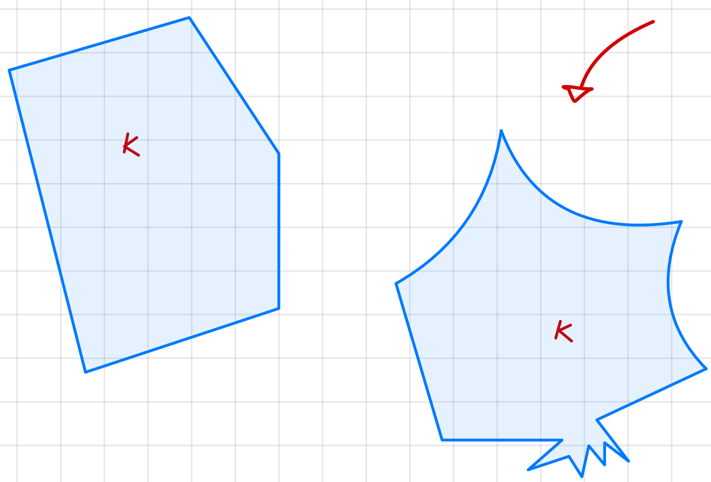
Important Special Case — Little Runge C

K has no holes $\Longrightarrow$ $\hat{\mathbb{C}} \setminus K$ has only one unbounded component & we can take $S = \{\infty\}$

$\Downarrow$

All f holomorphic in K can be approximated uniformly in K by polynomials.

no holes in K.



The set K can be disconnected

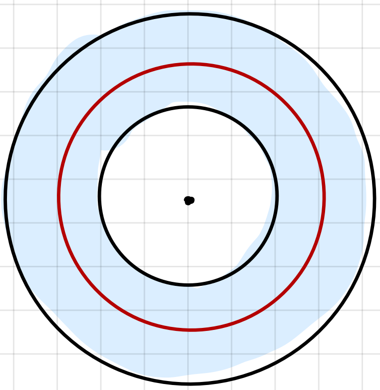
§ 1. How about the converse ?

If K has no holes $\overset{\text{Runge}}{\Longrightarrow}$ polynomial approximation holds.

If K has holes $\Longrightarrow$ polynomial approximation fails in general

How to see this ? Two methods

$\boxed{I}$ (Lecture 22) $K = \{1 \leq |z| \leq 2\}$, $f(z) = \frac{1}{z}$.



If $P_n \rightrightarrows f$ in K, P_n polynomials

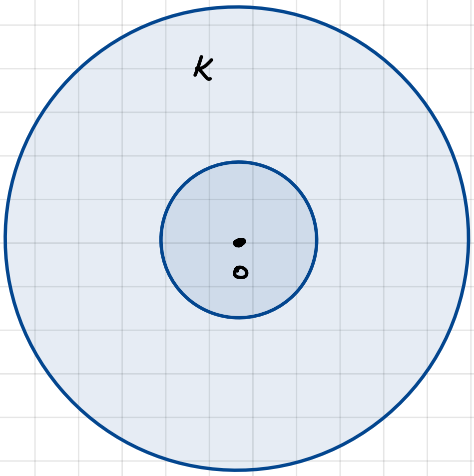
then $$\underbrace{\int_{|z|=\frac{3}{2}} P_n \, dz}_{0} \longrightarrow \int_{|z|=\frac{3}{2}} f \, dz = \underbrace{\int_{|z|=\frac{3}{2}} \frac{dz}{z}}_{2\pi i}$$

Both integrals follow by the residue theorem, for instance.

This contradiction shows f cannot be approximated uniformly in K by polynomials P_n.

$\boxed{ii}$ (New method).

$$K = \{1 \le |z| \le 2\}, \quad f(z) = \frac{1}{z}$$



Assume $P_n \rightrightarrows f$ in K, P_n polynomials.

$$\exists N : |P_N - f| < \frac{1}{4} \text{ on } K$$

$$\iff \left|P_N - \frac{1}{z}\right| < \frac{1}{4} \quad \text{on } K.$$

$$\iff |z P_N - 1| < \frac{|z|}{4} \quad \text{on } K.$$

$\Rightarrow |z P_N - 1| < \frac{|z|}{4}$ when $|z| = 1$. $\Rightarrow$ $|z P_N - 1| < \frac{1}{4}$. when $|z| = 1$.

Let $g(z) = 1 - z P_N \Rightarrow g$ entire. Note $|g(0)| = 1$ and

$$|g(z)| < \frac{1}{4} \quad \text{for } |z| = 1.$$

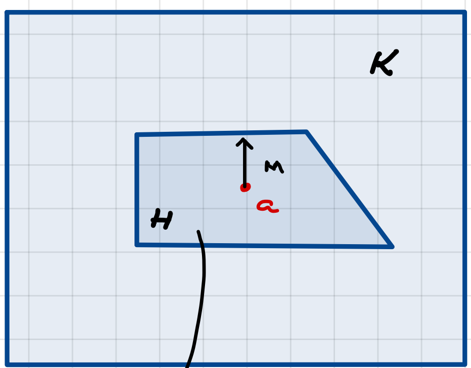
This contradicts Maximum modulus for g in $\bar{\Delta}(0,1)$.

The second method generalizes

Let H be a hole of K. Let $a \in H$., $f(z) = \dfrac{1}{z-a}$

$M = \max_{z \in K} |z-a| > 0.$ If $P_n \rightrightarrows f$ in K. find N such that

$$\left| P_N - \frac{1}{z-a} \right| < \frac{1}{2M} \quad \text{in } K$$



hole of K.

$$\Rightarrow \; |(z-a) P_N - 1| < \frac{|z-a|}{2M} \leq \frac{1}{2}. \quad \text{in } K.$$

$g(z) = 1 - (z-a) P_N$ satisfies

$g(a) = 1$ & $|g(z)| < \frac{1}{2}$ on $\partial H \subseteq K$.

This contradicts maximum modulus for g & the set $\overline{H}$.

Thus f cannot be approximated by polynomials.

Conclusion K has no holes $\Longleftrightarrow$ polynomial approximation holds in K.

§2. Runge for Open Sets ↙ Conway VIII. 1.15.

— We approximate locally uniformly on open sets

— the statement is similar to Runge for compact sets

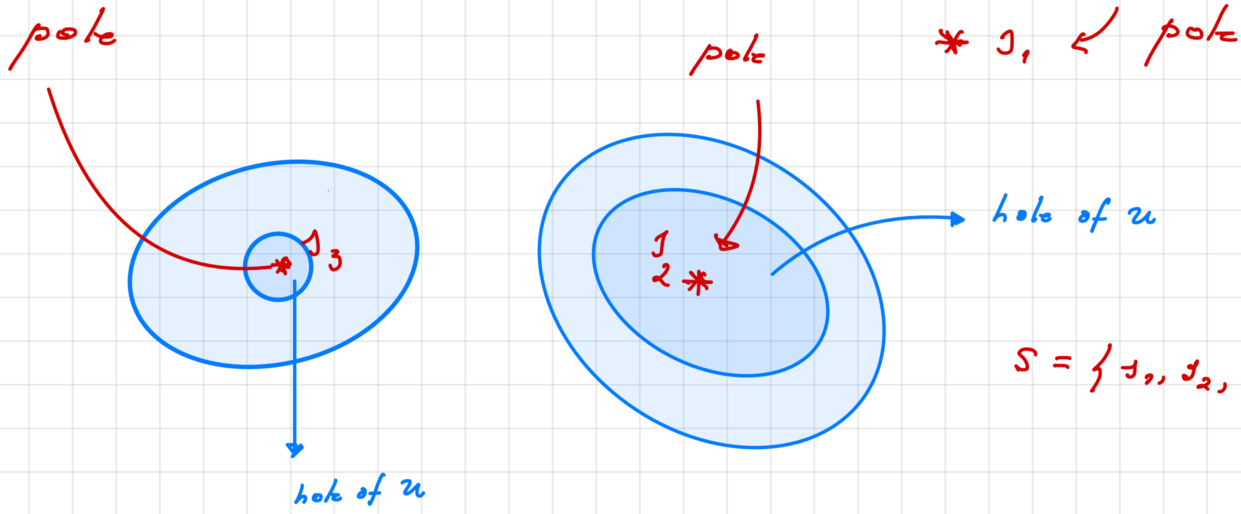
Theorem • $\mathcal{U} \subseteq \mathbb{C}$ possibly disconnected. open set.

• $S \subseteq \hat{\mathbb{C}} \setminus \mathcal{U}$ containing at least a point from each component of $\hat{\mathbb{C}} \setminus \mathcal{U}$.

• $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

Then $\exists\, R_n$ rational functions, $\text{poles}(R_n) \subseteq S$ and

$$R_n \overset{\text{l.u.}}{\rightrightarrows} f \quad \text{locally uniformly in } \mathcal{U}.$$



$S = \{ s_1, s_2, s_3 \}$

Important Special Case (Little Runge 0)

Let $\mathcal{U} \subseteq \mathbb{C}$, open, $\hat{\mathbb{C}} \setminus \mathcal{U}$ connected.

Any $f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic can be approximated locally uniformly on $\mathcal{U}$ by polynomials.

Indeed, take $S = \{\infty\}$ in Runge 0.

Example Let $\mathcal{U} = \Delta(0,r)$, $f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

We can Taylor expand f in the disc. The Taylor polynomials

$$T_n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k \quad \& \quad T_n \overset{l.u.}{\rightrightarrows} f \quad \text{(Math 220A).}$$

Little Runge 0 applies to more general sets $\mathcal{U}$.

Proof of Runge Open

Conway VII.1.2.

Topological Lemma For $\mathcal{U} \subseteq \mathbb{C}$ open, we can find $\underbrace{K_n}_{\text{compact}} \subseteq \mathcal{U}$

$$(*) \quad U = \bigcup_{n \geq 1} K_n$$ exhausting compact sets

[i] $K_n \subseteq \operatorname{Int} K_{n+1}$

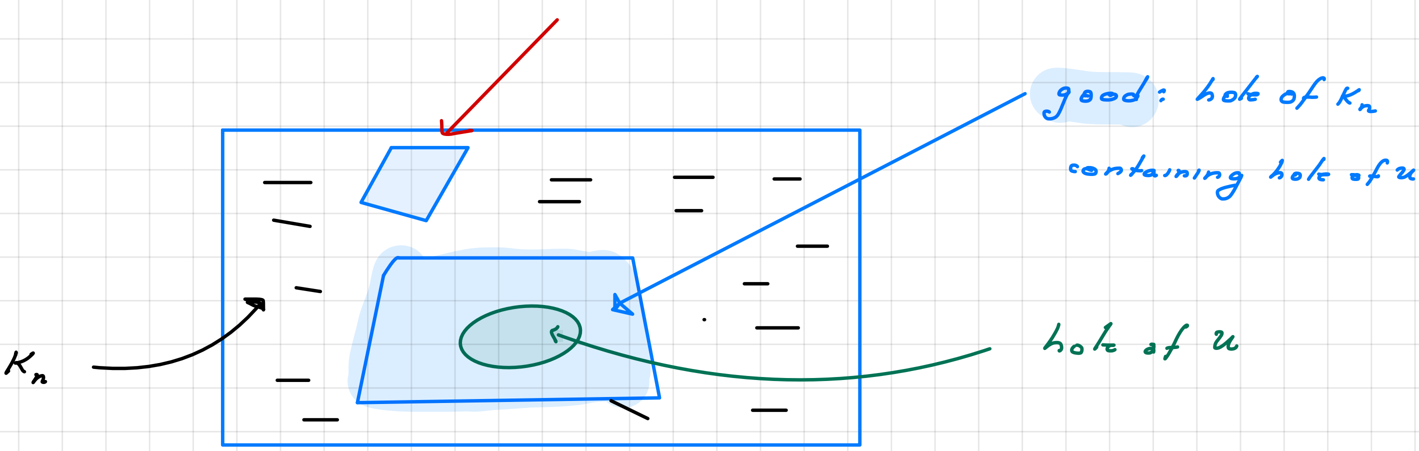
[ii] $\forall\, K \subseteq \mathcal{U}$ compact $\Rightarrow \exists\, n,\quad K \subseteq K_n$.

[iii] each component of $\hat{\mathbb{C}} \setminus K_n$ contains a component of $\hat{\mathbb{C}} \setminus \mathcal{U}$.

Remark [iii] means holes of K_n contain holes of $\mathcal{U}$.

Good vs. bad

bad. (hole of K_n, but not of $\mathcal{U}$).



Topological Lemma $\Rightarrow$ Runge 0

Let $f: \mathcal{U} \to \mathbb{C}$ holomorphic. Let S contain a point from each component of $\hat{\mathbb{C}} \setminus \mathcal{U}$. Write

$$\mathcal{U} = \bigcup_{n \geq 1} K_n \quad \text{as in the Lemma.}$$

The set S contains a point from each component of $\hat{\mathbb{C}} \setminus K_n$. by $\boxed{iii}$ By Runge C applied to f & K_n, we find.

$$|f - R_n| < \frac{1}{n} \text{ in } K_n, \quad \text{poles}(R_n) \subseteq S.$$

We claim $R_n \overset{\text{l.u.}}{\rightrightarrows} f$. Let K be compact in $\mathcal{U}$. By $\boxed{ii}$

$\Rightarrow K \subseteq K_N$ for some N. For $n \geq N \Rightarrow K \subseteq K_N \subseteq K_n$ by $\boxed{i}$

$\Rightarrow |f - R_n| < \frac{1}{n}$ over $K_n \Rightarrow |f - R_n| < \frac{1}{n}$ in K.

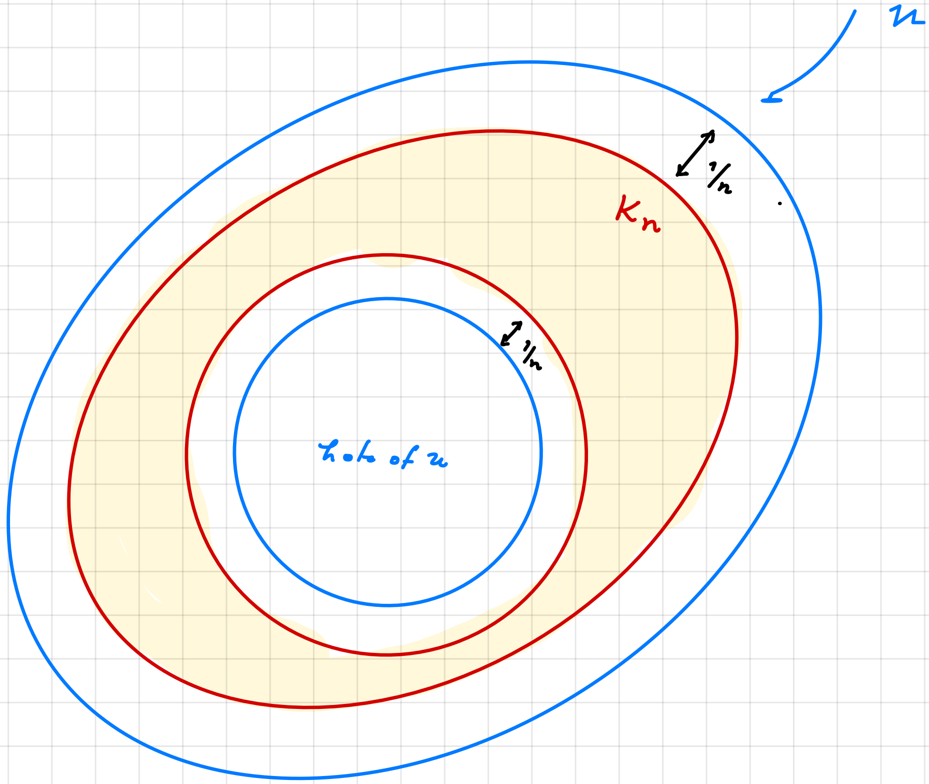
Thus $R_n \rightrightarrows f$ in K, as needed.

Proof of the Topological Lemma

} Conway VII.1.2

WLOG $u \neq \mathbb{C}$.

Let $K_n = \{ z : |z| \leq n \text{ and } d(z, \underbrace{\mathbb{C} \setminus u}_{\text{closed.}}) \geq \frac{1}{n} \}$.



It is easy to see $\boxed{i}$ – $\boxed{iii}$ hold, using the above pictures.

The technical details follow (see also Conway).

$$K_n = \{ z : |z| \leq n \text{ and } d(z, \mathbb{C} \setminus \mathcal{U}) \geq \frac{1}{n} \}.$$

Claim 1 $K_n \subseteq \mathcal{U}$

Proof If $z \in K_n \Rightarrow d(z, \mathbb{C} \setminus \mathcal{U}) \geq \frac{1}{n} \Rightarrow z \notin \mathbb{C} \setminus \mathcal{U} \Rightarrow z \in \mathcal{U}$. Thus $K_n \subseteq \mathcal{U}$.

Claim 2. $\mathcal{U} = \bigcup_{n \geq 1} K_n$

Proof If $z \in \mathcal{U}$ then let n such that $n \geq |z|$ & $d(z, \mathbb{C} \setminus \mathcal{U}) \geq \frac{1}{n}$ which is possible since $d(z, \mathbb{C} \setminus \mathcal{U}) > 0$. Thus $z \in K_n$. $\Rightarrow \mathcal{U} \subseteq \bigcup_n K_n \overset{\text{claim1.}}{\subseteq} \mathcal{U}$

Claim 3 K_n closed & bounded $\Rightarrow K_n$ compact.

Proof K_n is closed since

$$\mathbb{C} \setminus K_n = \{ |z| > n \} \cup \{ z : \exists b \notin \mathcal{U},\ d(z,b) < \frac{1}{n} \}$$

$$= \{ |z| > n \} \cup \bigcup_{b \notin \mathcal{U}} \Delta(b, \tfrac{1}{n}). = \text{open}.$$

Claim 4 $K_n \subseteq \operatorname{Int} K_{n+1}$

Proof Let $z \in K_n$. Let $r < \frac{1}{n} - \frac{1}{n+1}$. then

$$\Delta(z,r) \subseteq K_{n+1} \implies z \in \operatorname{Int} K_{n+1} \quad \text{as needed.}$$

To see $\Delta(z,r) \subseteq K_{n+1}$ note for $w \in \Delta(z,r)$

$$|w| \leq |z| + |w-z| \leq n + r < n+1 \quad \text{and}$$

$$d(w, \mathbb{C} \setminus \mathcal{U}) \geq d(z, \mathbb{C} \setminus \mathcal{U}) - d(z,w) \geq \frac{1}{n} - r > \frac{1}{n+1}.$$

$\implies w \in K_{n+1}$, as needed.

Claim 5 Each compact $K \subseteq \mathcal{U}$ is contained in some K_n.

Proof Let $K \subseteq \mathcal{U} \overset{\text{claim 2}}{=} \bigcup_m K_m \overset{\text{claim 4}}{\subseteq} \bigcup_m \operatorname{Int} K_{m+1}$. Since K is compact we find a finite subcover by $\operatorname{Int} K_j$, $j \leq n$.

$$\implies K \subseteq \bigcup_{j \leq n} \operatorname{Int} K_j \underbrace{\subseteq}_{\text{claim 4.}} K_n$$

Claim 6 Let $A = \hat{\mathbb{C}} \setminus K_n$, $B = \hat{\mathbb{C}} \setminus \mathcal{U}$ $\Rightarrow A \supseteq B \ni \infty$

(+) Each component of A contains a component of B.

Proof This is a bit more technical. We will use repeatedly:

Easy important fact (by definition)

If $Z \subseteq A$ connected & Z intersects a component A° of A

$\Rightarrow Z \subseteq A^\circ$.

Proof of (+) Let A° be a component of A. By Claim 3 (proof):

$$A = \underbrace{\{z \in \hat{\mathbb{C}} : |z| > n\}}_{\text{contains } \infty} \cup \bigcup_{b \in B} \Delta(b, \tfrac{1}{n})$$

[I] Note $\infty \in A$. If A° is the component containing ∞, let B° be the component of B containing $\infty \in B$. Note

$$A^\circ \cap B^\circ \neq \emptyset \text{ (contains } \infty) \text{ \& } B^\circ \subseteq A \underset{\text{easy fact}}{\Longrightarrow} B^\circ \subseteq A^\circ.$$

This is what we wanted to show.

[ii] If $\infty \notin A^\circ$, then A° cannot be disjoint from all sets $\Delta(b, \frac{1}{n})$.

Why Else $A^\circ \subseteq \Delta(\infty, n) \subseteq A \Rightarrow \Delta(\infty, n) \subseteq A^\circ \Rightarrow \infty \in A^\circ$.

connected set & intersects A°

easy fact.

false.

Thus $\exists\, b \in B$ with $A^\circ \cap \Delta(b, \frac{1}{n}) \neq \Phi$. Note

$\Delta(b, \frac{1}{n}) \subseteq A$ & intersects A° $\Rightarrow$ $\Delta(b, \frac{1}{n}) \subseteq A^\circ$.

easy fact

Let $b \in B^\circ$ for some component B°.

Then $B^\circ \cap A^\circ \neq \Phi$ & $B^\circ \subseteq B \subseteq A \Rightarrow$ $B^\circ \subseteq A^\circ$ as needed.

easy fact

Math 220 B – Lecture 26

March 10, 2021

Putting the pieces together   Conway VIII.2.

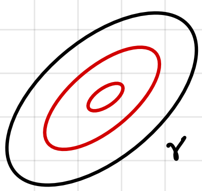
We tie up loose ends from Math 220A & B

Common theme : simply connected regions.

Topology $\longleftrightarrow$ Analysis

Review of Lecture 15, 220A $\mathcal{U} \subseteq \mathbb{C}$ connected

$\boxed{i}$ $\mathcal{U}$ is simply connected iff $\forall \gamma$ closed path in $\mathcal{U}$



$$\gamma \overset{\mathcal{U}}{\sim} 0$$

$\boxed{ii}$ γ piecewise C^1 loop in $\mathcal{U}$, $\gamma \overset{\mathcal{U}}{\approx} 0$ (null homologous) iff

$$\forall a \notin \mathcal{U}, \quad n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = 0$$

Recall

$$\gamma \overset{U}{\sim} 0 \implies \gamma \overset{U}{\approx} 0$$

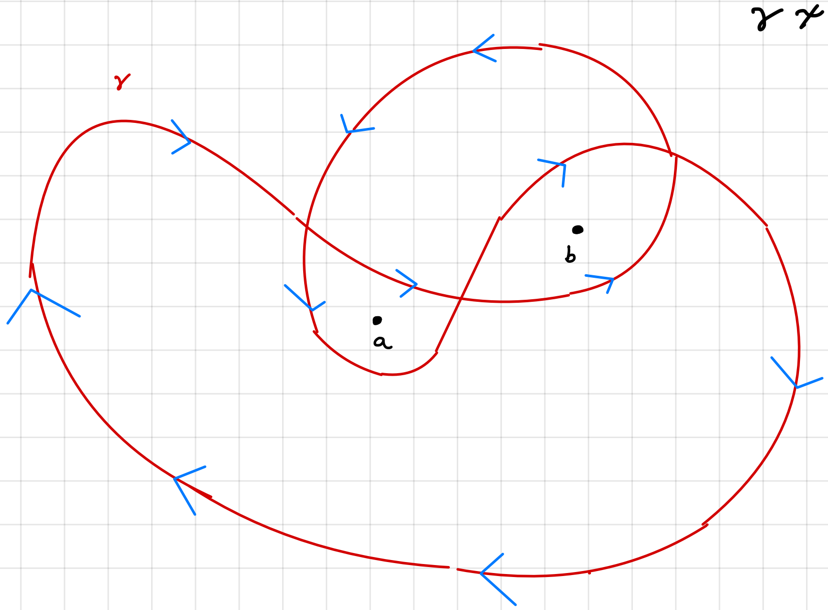
Indeed, $a \notin U$.

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 0$$ by the homotopy form of Cauchy applied to the holomorphic function

$$z \longrightarrow \frac{1}{z-a} \text{ in } U.$$

However the converse is false $U = \mathbb{C} \setminus \{a, b\}$

$\gamma \not\sim 0$, $\gamma \approx 0$.



Theorem Let $\mathcal{U} \subseteq \mathbb{C}$ open, connected. TFAE

Conway VIII.2.

$\boxed{a}$ $\mathcal{U}$ simply connected

$\Downarrow$

$\boxed{b}$ $\forall \gamma$ piecewise C' loop, $\gamma \overset{\mathcal{U}}{\approx} 0$

$\Downarrow$

$\boxed{c}$ $\hat{\mathbb{C}} \setminus \mathcal{U}$ connected.

$\Downarrow$

$\boxed{d}$ polynomial approximation $\forall f$ holomorphic in $\mathcal{U}$ can be approximated $p_n \overset{\ell.u.}{\rightrightarrows} f$ in $\mathcal{U}$

$\Downarrow$

$\boxed{e}$ $\forall \gamma$ piecewise C' loop, f holomorphic in $\mathcal{U}$

$$\int_\gamma f \, dz = 0.$$

$\Downarrow$

$\boxed{f}$ primitives : any holomorphic $f : \mathcal{U} \to \mathbb{C}$ admits a primitive.

$\Downarrow$

$\boxed{g}$ logarithms : $\forall f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, nowhere zero can be written $f = e^{g}$, $g : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

$\Downarrow$

$\boxed{h}$ roots: $\forall f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, nowhere zero can be written $f = h^2$, $h : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

$\Downarrow$

$\boxed{i}$ $\mathcal{U}$ is homeomorphic to $\Delta(0,1)$.

Recall $\mathcal{U}, \mathcal{V} \subseteq \mathbb{C}$ are homeomorphic if $\exists\, f: \mathcal{U} \to \mathcal{V}$

$g: \mathcal{V} \to \mathcal{U}$ continuous & inverse to each other.

Proof

$\boxed{a} \Rightarrow \boxed{b}$ This is the statement $\gamma \overset{\mathcal{U}}{\sim} 0 \Rightarrow \gamma \overset{\mathcal{U}}{\approx} 0.$

that we saw previously.

$\boxed{b} \Rightarrow \boxed{c}$ Assume $\hat{\mathbb{C}} \setminus \mathcal{U} = A \cup B$

$A, B \neq \emptyset$ closed & disjoint. Assume $\infty \in B. \Rightarrow$

$\Rightarrow A$ is closed in $\underbrace{\hat{\mathbb{C}} \setminus \mathcal{U}}_{\text{closed}} \Rightarrow A$ closed in $\hat{\mathbb{C}} \Rightarrow A$ compact.

Let $\underset{\substack{\cancel{\infty} \\ \infty}}{V} = \mathcal{U} \cup A = \hat{\mathbb{C}} \setminus \underbrace{B}_{\text{closed}}. \Rightarrow V$ open subset of $\mathbb{C}$., $A \subseteq V$.

In Lecture 23, we saw Cauchy's formula for compact sets.

A = compact, $A \subseteq V$. $\Rightarrow \exists$ polygons $\Gamma_1 \dots \Gamma_n$ in $V \setminus A = \mathcal{U}$.

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz \quad \forall\, a \in A,\ f \text{ holom. in } V.$$

Take $f \equiv 1$ then $1 = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{dz}{z-a} = \sum_{j=1}^{n} \underbrace{n(\Gamma_j, a)}_{0}$.

However, by assumption $n(\Gamma_j, a) = 0 \ \forall j$ since Γ_j is a piecewise C^1 loop in $\mathcal{U}$ and $a \in A \Rightarrow a \notin \mathcal{U}$. This contradicts

$$\sum_{j=1}^{n} n(\Gamma_j, a) = 1.$$

$\boxed{c} \Rightarrow \boxed{d}$ this is Little Runge 0.

$\boxed{d} \Rightarrow \boxed{e}$ If $p_n \overset{l.u.}{\rightrightarrows} f$ in $\mathcal{U}$ then $\int_\gamma p_n\, dz \longrightarrow \int_\gamma f\, dz$.

However p_n admits a primitive $p_n = g_n'$ so by

Lecture 5, Math 220 A $\int_\gamma p_n\, dz = \int_\gamma g_n'\, dz = 0$

$\Rightarrow \int_\gamma f\, dz = 0$.

$\boxed{e} \Rightarrow \boxed{f}$ This was done in Lecture 5, Math 220 A

$\boxed{f} \Rightarrow \boxed{g}$ Math 220 A, Homework 4. Recall the argument.

Consider $\frac{f'}{f}$ holomorphic in $\mathcal{U}$. Then $\frac{f'}{f} = g'$ for some g by $\boxed{f}$

$\Rightarrow (e^{-g} f)' = 0 \Rightarrow f = c e^{g} = e^{\tilde{g}}$, $\tilde{g} = g + \log c$, $c \neq 0$.

$\boxed{g} \Rightarrow \boxed{h}$ Write $f = e^{g}$ and let $h = e^{g/2}$.

$\boxed{h} \Rightarrow \boxed{b}$ If $\mathcal{U} \neq \mathbb{C}$, Riemann Mapping shows $\mathcal{U}$ and Δ are biholomorphic hence homeomorphic.

If $\mathcal{U} = \mathbb{C}$ then $z \longrightarrow \dfrac{z}{\sqrt{1+|z|^2}}$ is a homeomorphism between $\mathbb{C}$ and Δ.

$\boxed{b} \Rightarrow \boxed{a}$ Let f, g be the two inverse homeomorphisms $\mathcal{U} \underset{g}{\overset{f}{\rightleftarrows}} \Delta$.

Let γ be a loop in $\mathcal{U}$ $\Rightarrow f \circ \gamma \overset{\Delta}{\sim} 0 \Rightarrow g \circ f \circ \gamma \overset{\mathcal{U}}{\sim} g(0). \Rightarrow \gamma \overset{\mathcal{U}}{\sim} g(0)$

$\Rightarrow \mathcal{U}$ simply connected.

Remark The implications $a \Rightarrow b, c, d, e \ldots$ are very useful.

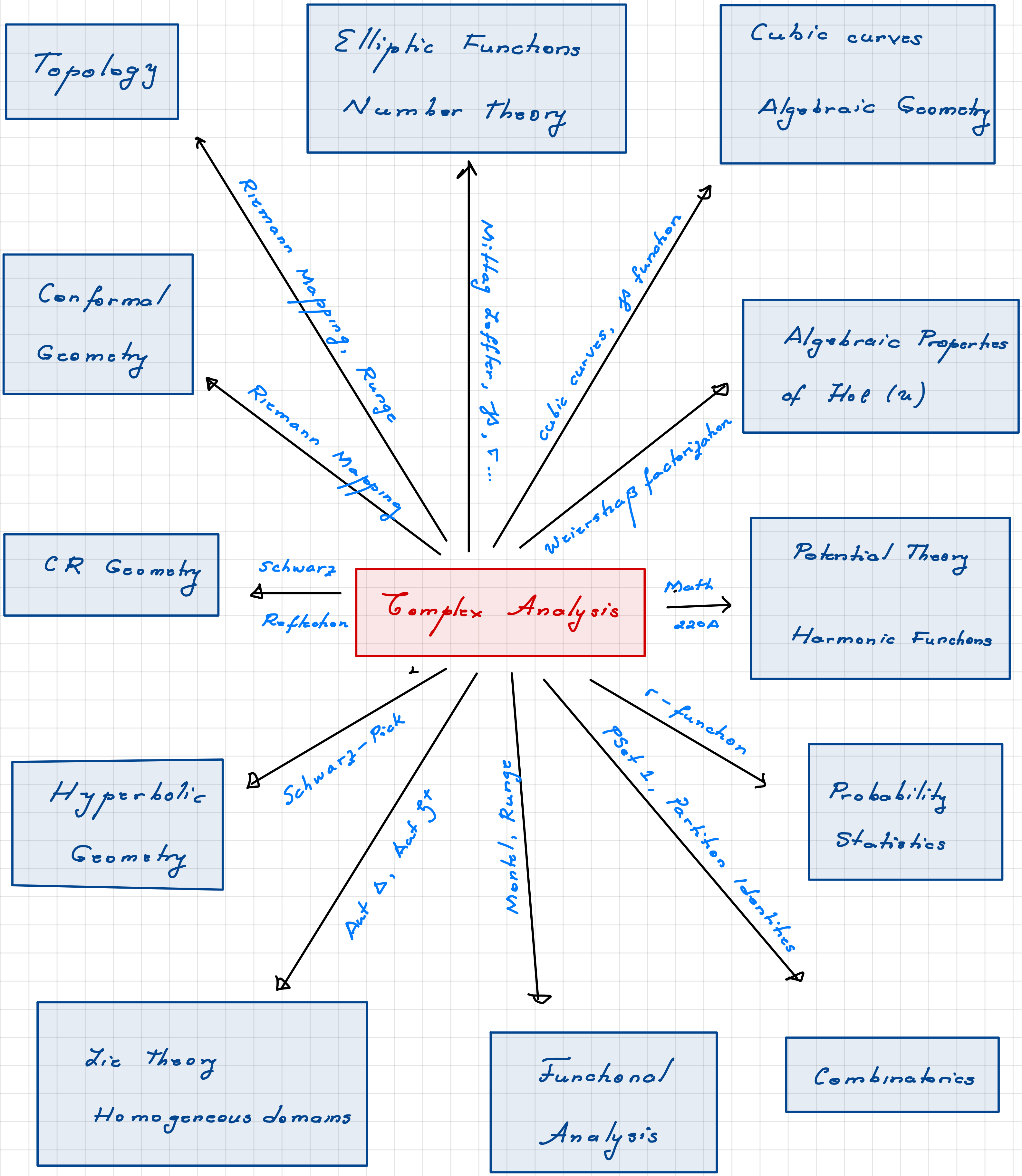
For the converse, $c \Rightarrow a$ is important.

Remark

Topology: $a, c, i, \ldots$

Analysis: $d, e, f, g, \ldots$

Summary of Math 220A – B



Topics for Math 220C

(1) Harmonic Functions — Conway X.

(2) Hadamard Factorization — Conway XI.

(3) Picard's theorems — Conway XII.

Math "220 D"

(4) Introduction to Riemann Surfaces.

Logistics

(1) Office Hours: Today 4 – 5:30 PM

(2) Homework 7 due Friday, 11:59 PM.

No Sunday afternoon extensions.

(3) Final Exam, Wed March 17, 3– 6 PM.

(4) Office Hours:

Tuesday March 16, 2 – 4 PM (Dragos)

Tuesday March 16, 4 – 6 PM (Shubham)

(5) Practice Problems online

(6) Last lecture – Review.