Math 220B - Lecture 1

January 4, 2021

0 Logistics

(1) Zoom lectures – MWF 3 – 3:50 PM.

(2) Office Hour – W 4 – 5:30 PM

(3) PSets – due "Fridays," weekly

(4) Grades – 30% HWK

30% midterm

40% Final

(5) Midterm – take home, Feb 12

(6) Final – March 17, 3 – 6 PM

(7) Canvas / Gradescope / Website

math.ucsd.edu/~dopra/220w21.html

(8) Attendance

III Topics to be covered

Part I : Sequences / Series / Products

(1) infinite products of holomorphic functions

Weierstraß Problem

(2) sequences & series of meromorphic functions

Mittag-Leffler Problem

(3) sequences of hol functions, Montel families

Part II : Geometric aspects / Conformal maps

(4) Schwarz Lemma, automorphisms of $\Delta, \mathfrak{H}^+, \Delta^*, \ldots$

(5) Riemann mapping theorem

Part III: Further topics (if time)

(6) Runge's theorem

(7) Schwarz Reflection

(8) harmonic functions

(9) Hadamard factorization

(10) Little & Big Picard.

Some of these will only be covered in Math 220C.

[2.] Three Motivating Questions for Part I

Math 220A, Lecture 10 : $f \not\equiv 0$ entire has countably many zeroes that do not accumulate.

Weierstraß Problem

Given a sequence of distinct $\{a_n\}$, $a_n \to \infty$ and positive integers $\{m_n\}$, is there an entire function with zeroes only at $\{a_n\}$ with order exactly $\{m_n\}$. ?

Weierstraß$^+$ Problem

Given $\{a_n\}, \{m_n\}$ as above, $\{A_{nj}\}_{0 \le j < m_n}$ is there an entire function f with

$$f^{(j)}(a_n) = A_{nj} \quad \forall\ 0 \le j < m_n$$

Mittag - Leffler Problem

Take $\{a_n\}$ as above.

We can always find a meromorphic function f in $\mathbb{C}$ with poles only at a_n. e.g. take g solving Weierstraß at $\{a_n\}$ and set $f = 1/g$.

Mittag - Leffler asks if we can furthermore prescribe the Laurent principal parts.

Given $\{a_n\}$ distinct, $a_n \to \infty$, and polynomials $p_n\left(\frac{1}{z-a_n}\right)$ without constant terms, is there a meromorphic function in $\mathbb{C}$ with poles only at a_n and Laurent expansion

$$f = p_n\left(\frac{1}{z-a_n}\right) + \dots \quad \text{near } a_n.$$

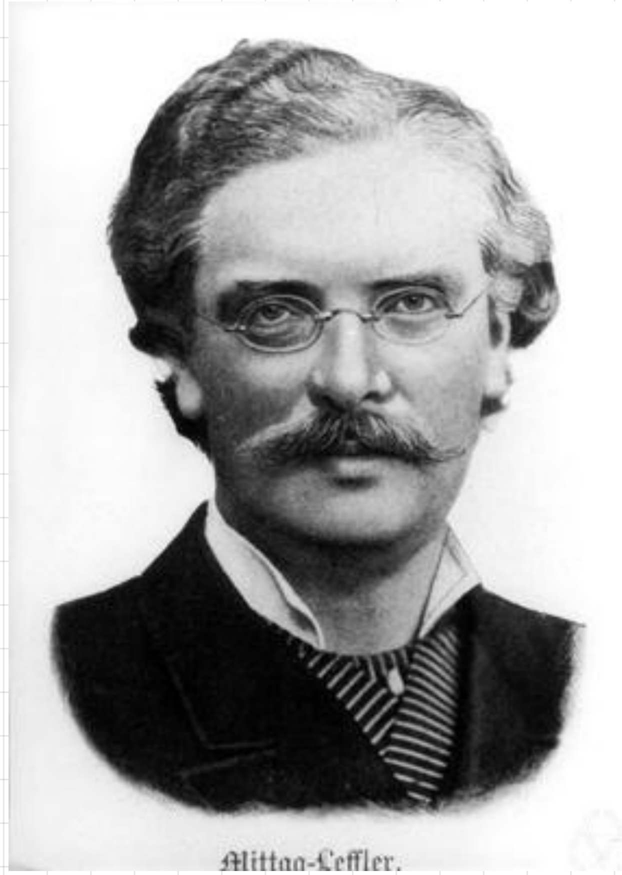
Weierstraß – Poincaré Problem

Is any meromorphic function a quotient of two holomorphic functions?

Remark The three questions above can be asked & answered for all $\mathcal{U} \subseteq \mathbb{C}$ open & connected.

Karl Weierstraß

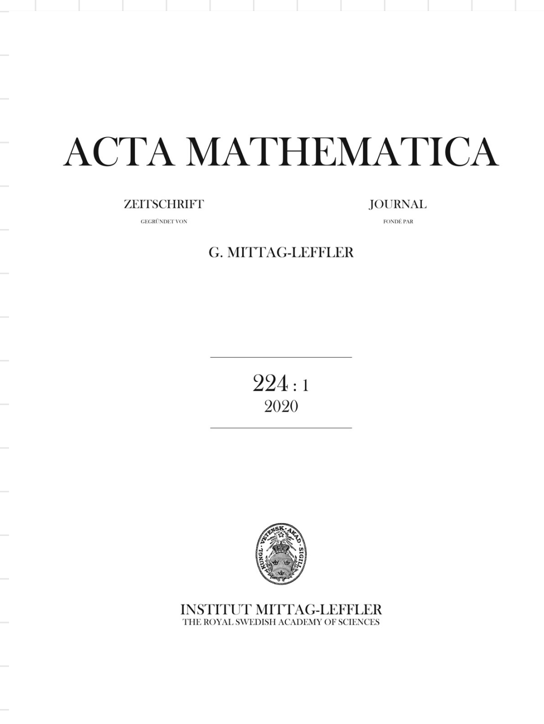
1815 – 1897



Gösta Mittag-Leffler

1846 – 1927

Mittag-Leffler Institut



ACTA MATHEMATICA

ZEITSCHRIFT GEGRÜNDET VON JOURNAL FONDÉ PAR

G. MITTAG-LEFFLER

224 : 1

2020

INSTITUT MITTAG-LEFFLER

THE ROYAL SWEDISH ACADEMY OF SCIENCES

We will also illustrate general theory e.g.

$\boxed{a}$ factorization of sine, Γ- function

$\updownarrow$

Weierstraß problem.

$\boxed{b}$ elliptic functions – Weierstraß $\wp$ function

$\updownarrow$

Mittag - Leffler

Tools — sequences, series, products of

holomorphic & meromorphic functions.

Last quarter [I] sequences

[II] series

of holomorphic functions.

This quarter

[I] Weierstraß requires infinite products of holomorphic functions.

Intuitively, this makes sense. We could try to solve Weierstraß by setting $f(z) = \prod_{n=1}^{\infty} (z - a_n)$ but convergence is an issue

[II] Mittag - Leffler requires infinite sums of meromorphic functions.

[3] Quick Review of the last lectures in Math 220A

Sequences $\{f_n\}$ holomorphic in $\mathcal{U} \subseteq \mathbb{C}$

Recall that the notion of convergence we considered was

local uniform convergence $\Longleftrightarrow$ convergence on compact subsets

$$f_n \overset{l.u.}{\rightrightarrows} f \quad \Longleftrightarrow \quad f_n \overset{c}{\rightrightarrows} f$$

Weierstraß Convergence Theorem

Let $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$. Then

[1] f holomorphic

[2] $f_n^{(k)} \overset{l.u.}{\rightrightarrows} f^{(k)}$

Series $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic. Assume

$(*)$ $\forall\ K \subseteq \mathcal{U}$ compact $\exists\ M_n(K)$, $|f_n| \le M_n(K)$.

over K. & $\sum_{n=1}^{\infty} M_n(K) < \infty$.

M-test

$\Longrightarrow f = \sum_{n=1}^{\infty} f_n$ converges absolutely & uniformly on every K

Weierstraß Thm

$\Longrightarrow f$ holomorphic & $f' = \sum_{n=1}^{\infty} f_n'$

This quarter

$\boxed{I}$ infinite products $\prod_{n=1}^{\infty} f_n(z)$ Weierstraß

$\boxed{II}$ series of meromorphic functions $\sum_{n=1}^{\infty} f_n(z)$

Mittag-Leffler

[4] Infinite Products

Main Question Given $f_k : \mathcal{U} \to \mathbb{C}$ holomorphic, how do we define $f(z) = \prod_{k=1}^{\infty} f_k(z)$? Furthermore,

[i] Is f holomorphic?

[ii] Is it true that $\text{Zero}(f) = \bigcup_k \text{Zero}(f_k)$?

Step back: Given $p_k \in \mathbb{C}$, how to define

$$P = \prod_{k=1}^{\infty} p_k \, ?$$

Wrong answer Form the partial products

$$P_n = \prod_{k=1}^{n} p_k \quad \text{and define} \quad P = \lim_{n \to \infty} P_n$$

Issues [i] If $p_\ell = 0 \Rightarrow P = 0$ no matter what the other p_n's are. Thus one term would determine convergence of the product which is unfair.

[ii] We could have $P = 0$ even though $p_k \neq 0 \ \forall k$

e.g. $\prod_{k=1}^{\infty} \frac{1}{k} = 0$. Thus we have no control over the zeroes of a product of functions.

Question What kind of products will we consider?

Definition $\prod_{k=1}^{\infty} p_k = P$ converges iff $\exists\, M$ such that $\lim_{n \to \infty} \prod_{k=M}^{n} p_k$ exists and equals $\hat{P} \neq 0$. We then set

$$P = p_1 \cdots p_{M-1} \hat{P}$$

Remarks I the value of $\mathcal{P}$ is independent of M (check)

II in the infinite products above only finitely many terms can be 0. ($\hat{P} \neq 0 \Rightarrow p_k \neq 0$ for $k \geq M$).

III With this definition we have control over the zeros.

Indeed

$$P = 0 \iff p_1 \cdots p_{M-1}\, \hat{P} = 0 \quad (\hat{P} \neq 0)$$

$$\iff p_1 = 0 \text{ or } \ldots \text{ or } p_{M-1} = 0$$

$$\iff \exists\, k \text{ with } p_k = 0.$$

Thus this behaves in the same fashion as finite products.