Quantitative Finance Qualifying Exam
2017 Summer

INSTRUCTIONS

You have 4 hours to do this exam.

Reminder: This exam is closed notes and closed books. No electronic devices are permitted. Phones must be turned completely off for the duration of the exam.

PART 1: Do 2 out of problems 1, 2, 3.
PART 2: Do 2 out of problems 4, 5, 6.
PART 3: Do 2 out of problems 7, 8, 9.
PART 4: Do 2 out of problems 10, 11, 12.

All problems are weighted equally.
On this cover page write which eight problems you want graded.

Problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty.
Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number:

Signature

Stony Brook University
Applied Mathematics and Statistics

1. Spot Rate Computation and Applications

You are given the following information. Assume annual compounding throughout.

- A 1-year zero-coupon bond with a face value of \$100,000 sells at a discount of \$98,522.17.
- A 2-year bond with a face value of \$10,000 and an annual coupon of \$300 sells at price of \$10,099.25.
- A 3-year bond with a face value of \$100,000 and an annual coupon of \$3,100 sells at a price of \$100,356.00.

Solve for the following:

a) Spot Curve: Using the market data, above compute the 3-year spot curve.

b) Bond Valuation: Using the spot rates computed above compute the price of a 3-year bond with a face value of \$1,000,000 and annual coupon of \$2,000.

c) Forward Rate: Compute the forward rate $f_{2,3}$.

2. Market Portfolio

Consider the following simple quadratic program representing the portfolios on the Capital Market Line (CML) and its solution to proportionality. The parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, are the returns' mean vector and covariance matrix, respectively, and r_f is the risk-free rate. The value of $\lambda \geq 0$ parameterizes the CML:

$$\min\left\{ \tfrac{1}{2}\mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} - \lambda(\boldsymbol{\mu} - r_f)^T \mathbf{x} \right\}$$

Assume the Capital Asset Pricing Model (CAPM), *i.e.*, at time t for an asset i with return $r_i(t)$, market M with return $r_M(t)$ and risk-free rate r_f, and mean-zero, uncorrelated error terms $\varepsilon_i(t)$, the following expression holds

$$r_i(t) - r_f = \beta_i (r_M(t) - r_f) + \varepsilon_i(t)$$

Further assume that the covariance matrix is invertible. Show that the value of x_i, the allocation of asset i in the market portfolio, is proportional to

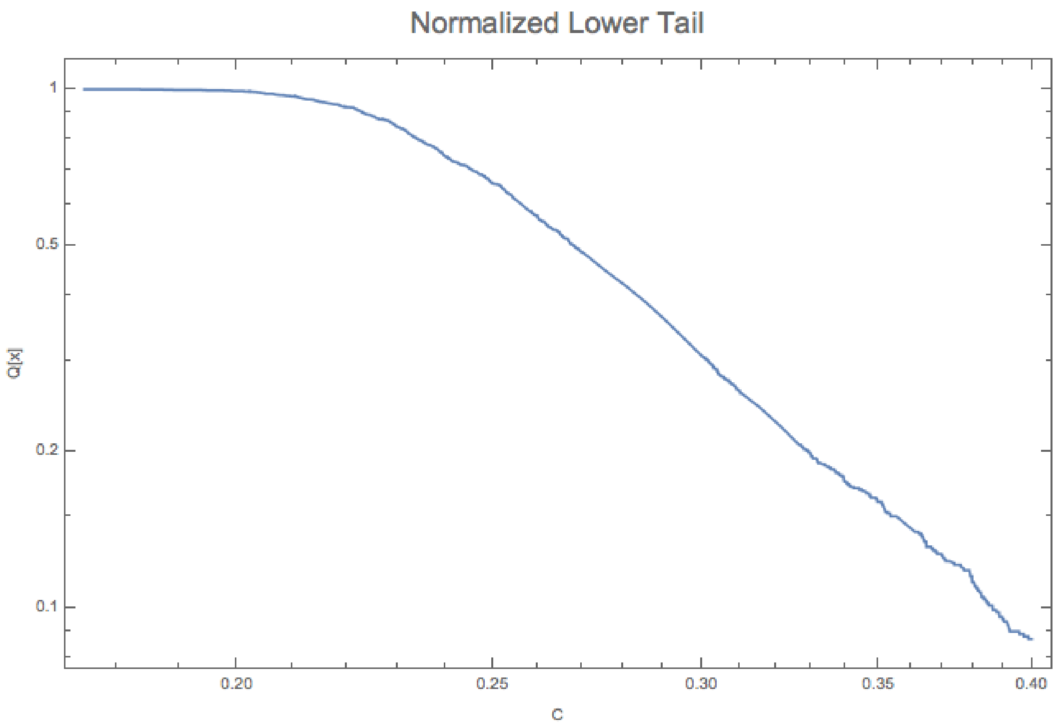
$$x_i \propto \frac{\beta_i}{\mathrm{Var}[\varepsilon_i]}$$

3. Chooser European Option

A standard chooser European option is one in which the option holder has the right to decide if the option is a vanilla European put or a call at some point prior to the expiry of the option. We assume that the strike price K and expiry T are the same for both the put and call. The current time is t and the time the choice must be made is τ with $0 \leq t \leq \tau \leq T$. Write the expression for the price of the chooser $F(t)$ in terms of vanilla European options and, if necessary, any discounted cash-flows.

4. Power Law Model

We wish to investigate the lower tail of a return distribution. Let $Q(x) = \text{Prob}[X \geq x]$ denote the *survival function* of x. A log-log plot of the survival function for $x \geq 0$ is shown below.



a) Does the distribution of x display at any point evidence that the tail of the distribution follows a power law? Explain what you looked for to determine this.

b) If so, at what point does that behavior emerge? Explain your answer.

c) If there is evidence of a power law in the upper tail, estimate its exponent. Employ a simple visual approximation but explain how you accomplished it. If not, hypothesize a reasonable return distribution.

d) Based on your work above, define to proportionality the PDF and CDF of the upper tail in the power law region.

e) What can you say about the existence of the moments of the distribution based on the work above? Explain you answer.

5. Markowitz Portfolio

Assume that returns follow a multivariate Normal distribution with mean vector $\boldsymbol{\mu}$, positive-definite covariance matrix $\boldsymbol{\Sigma}$ and risk-free rate r_f. The mean-variance portfolio optimization with unit capital is the quadratic program below. Note that both long and short positions are allowed in this instance.

$$\mathcal{M} = \min_{\mathbf{x}} \left\{ \frac{1}{2} \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} - \lambda\, (\boldsymbol{\mu}^T - r_f)^{\boldsymbol{T}} \mathbf{x} \mid \mathbf{1}^T \mathbf{x} = 1 \right\}$$

where the risk-reward trade-off is controlled by the parameter $0 \leq \lambda$.

a) Assuming an investor population of mean-variance optimizers, derive an expression for the market (*i.e.*, tangent) portfolio.

b) Given that different investors have different return goals or risk preferences, explain how an investor uses cash and market portfolio to achieve them.

c) Explain why the approach you described in (b) above is superior in mean-variance terms to any other strategy.

6. Copula

Assume all distributions' CDFs in this question are continuous, monotonically increasing functions. Let $F_X(x)$ be the CDF of a multivariate random variable X with marginals $F_{Xi}(x_i)$, $i = \{1, \dots, n\}$ and let Uniform[0, 1] designate the uniform distribution on the unit interval.

a) Show that the random variable $U = F_{Xi}(X_i) \sim$ Uniform[0, 1].

b) Let $H_Z(z)$ be the CDF of a continuous univariate random variable Z. Show that the random variable $H_Z^{-1}(U)$ where U ~ Uniform[0, 1] realizes a random variable with the same distribution as Z.

c) Derive the copula associated with $F_X(x)$, *i.e.*, a function $C_X(u)$, $u = \{u_1, \dots, u_n\}$ where C is a multivariate CDF with Uniform[0, 1] marginals, representing the dependence structure of $F_X(x)$ separate from its marginals.

d) Let $G_{Yi}[y_i]$, $i = \{1, \dots, n\}$ designate the CDFs of continuous univariate random variables Y_i. We wish to construct a multivariate distribution $G_Y[y]$ with marginals $G_{Yi}[y_i]$ and the same dependence structure as $F_X(x)$. Write an expression for $G_Y[y]$ which accomplishes this.

7. VaR of a European Call

Consider a European call option with parameters as follows: current stock price S_0, strike K, risk-free rate r, volatility rate σ, and time to maturity T years.

Assuming a geometric Brownian motion for the stock price process S_t, use the delta-normal valuation to compute the 95% VaR over a horizon of 3 days for a long position of a European call.

8. Kendall Tau

Let $\mathbf{X} = (X_1, X_2)$ be a bivariate Gaussian copula with correlation 0.5 and continuous margins.

Show that the Kendall's τ is:

$$\rho_\tau(X_1, X_2) = ⅓$$

9. VaR of a GARCH Process

Consider the following AR(1)-GARCH(1,1) model for daily return r_t :

$$r_t = \theta r_{t-1} + u_t \qquad u_t = \sigma_t \varepsilon_t \qquad \sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2$$

where $-1 < \theta < 1$; $\theta \neq 0$; $\alpha > 0$; $\beta > 0$; $\omega > 0$ and $\alpha + \beta < 1$.

What is the 99% 2-day VaR of a long position at time t ?

10. Heat Equation

Compute the partial derivatives of the normal Gaussian distribution function g with respect to space and time: $\frac{\partial g}{\partial x}$, $\frac{\partial^2 g}{\partial x^2}$ and $\frac{\partial g}{\partial t}$, where:

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$$

Verify Kolmogorov PDE (a.k.a. heat equation):

$$\frac{\partial^2 g}{\partial x^2}(x,t) = \frac{\partial g}{\partial t}(x,t)$$

11. Shifted Log-Normal Process

Consider a shifted log-normal model for an option on a stock S_t. A shifted log-normal process is given by:

$$dS_t = (S_t + \delta)\alpha_\delta dt + (S_t + \delta)\beta_\delta dW$$

1. Show that, at maturity T, the distribution of $D_T = S_T + \delta$ is log-normal and give its formula, then the formula for the distribution function of S_T.

   Reminder: if $f(x)$ is the pdf of a random variable X, then the pdf of $Y = \exp(X)$ is $yf(\mathrm{Ln}(y))$

   Don't forget the Itô term when computing the distribution of $\mathrm{Ln}(D_T)$

2. Compute the expectation $\mu = \mathrm{E}(S_T)$ and the variance $\sigma = \mathrm{Var}(S_T)$ with respect to S_0, δ, α_δ and β_δ
3. Given S_0, δ, μ and σ, compute α_δ and β_δ
4. Show that, when $d \to +\infty$, with fixed μ and σ, then the distribution of S_T tends to a Gaussian distribution

12. Random Variables of Class L^p

We recall that a random variable X on a probability space $(\Omega, \Im, P)$ is of *class* L^p if $E(|X|^p) < +\infty$.

Show that if X is of class L^2, it is of class L^1.

(hint: Cauchy-Schwarz inequality)

More generally, show that if $0 < q < p$ then if X is of class L^p, it is of class L^q.

(hint: consider the events $A = \{|X| \leq 1\}$ and $B = \{|X| > 1\}$)

Scratch paper 1

Scratch paper 2

Scratch paper 3

Scratch paper 4

Scratch paper 5

Scratch paper 6

Scratch paper 7

Scratch paper 8

Scratch paper 9

Scratch paper 10

Scratch paper 11

Scratch paper 12

Scratch paper 13

Scratch paper 14

Scratch paper 15