Math 220C, Problem Set 4. Due Friday, April 23.

In this problem set, for a holomorphic function $f : G \to \mathbb{C}$ *with* $\overline{\Delta}(0, r) \subset G$ *we write*

- $M(r) = \sup\{|f(z)| : |z| = r\}$
- $N(r)$ *is the number of zeros of* f *in* $\Delta(0, r)$ *counted with multiplicity.*

1. (*Jensen's inequality.*) Let $f : G \to \mathbb{C}$ be holomorphic, and let $\overline{\Delta}(0, r) \subset G$. Assume that $f(0) \neq 0$. Assume $z_1, \ldots, z_k$ are zeros of f in $\Delta(0, r)$.

(Note: these are not necessarily assumed to be all zeros of f in $\Delta(0, r)$; repetitions up to the multiplicities are allowed.)

Using Jensen's formula, show that

$$|f(0)| \leq |z_1 \ldots z_k| \cdot \frac{M(r)}{r^k}.$$

Example: Assume $f : G \to \mathbb{C}$ is holomorphic, $\overline{\Delta}(0, 1) \subset G$ and $|f(z)| \leq 1$ for all $z \in G$. Assume

$$f\left(\frac{1}{2}\right) = f\left(\frac{i}{2}\right) = 0.$$

Then $|f(0)| \leq \frac{1}{4}$.

2. (*Jensen and Blaschke.*) Assume f is a bounded holomorphic function

$$f : \Delta(0, 1) \to \mathbb{C}$$

with zeros $a_1, a_2, \ldots$ listed with multiplicity. Show that

$$\sum_n (1 - |a_n|) < \infty.$$

Remark: Recall that in Math 220B, Homework 1, Problem 5, we used Blaschke's products to construct holomorphic functions in the disc with zeros only at the a_n's under the assumption

$$\sum_n (1 - |a_n|) < \infty.$$

We made a remark at that time that this condition is needed, and now we can prove it.

Hint: When $f(0) \neq 0$, make $r \to 1$ in Problem 1 to show first that $\sum_n \log |a_n|$ converges.

When $f(0) = 0$, write $f(z) = z^m g(z)$, $g(0) \neq 0$, and work with g instead. You will need to show g is bounded.

3. (*Jensen's formula.*) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, $f(0) = 1$.

(i) Show that

$$N(r) \log 2 \leq \log M(2r).$$

(ii) Assume that

$$|f(z)| \le \exp(A|z|^k)$$

for $A > 0$ and k natural number. Show that

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} \le k.$$

4. (*Order.*) If f, g are entire functions of orders λ_1, λ_2, show that fg has order $\le \lambda = \max(\lambda_1, \lambda_2)$.

Remark: The same proof should work for $f + g$, but do not hand it in.

5. (*Exponent of convergence.*) Let

$$|a_1| \le |a_2| \le \dots$$

be a sequence of non-zero complex numbers converging to ∞, and let

$$\alpha = \inf \left\{ t : \sum_n \frac{1}{|a_n|^t} < \infty \right\}.$$

The exponent of convergence α is a measure of the growth of the sequence $\{a_n\}$.

Assume f is an entire function with zeros only at $a_1, a_2 \dots$. Let λ be the order of f.

(i) Show that for any $\epsilon > 0$,

$$\sum_n \frac{1}{|a_n|^{\alpha+\epsilon}} < \infty, \quad \sum_n \frac{1}{|a_n|^{\alpha-\epsilon}} = \infty.$$

(ii) Show that $\alpha \le \lambda$.

This establishes a connection between the growth of zeros (measured by α) and the growth of f (measured by λ).

Hint: Fix $\epsilon > 0$ and show that $\lambda + 2\epsilon > \alpha$. To this end, use Problem 3 to derive

$$n - 1 \le N(r = |a_n|) \le \log M(2|a_n|)/\log 2.$$

On the other hand, use $\log M(r) \le r^{\lambda+\epsilon}$ for r sufficiently large. Find a bound on $|a_n|$ and conclude.

6. (*Qualifying Exam, Spring 2017.*) Show that $f(z) = \cos\sqrt{z}$ is an entire function of order $\frac{1}{2}$.

Also show that the genus of f equals 0. You can find the genus using Hadamard's theorem, but it may be more instructive to recall the factorization of the cosine from Math 220B.

Remark: One may use the same method to find the order and genus of $\cos z$ and $\sin z$. You may try it for yourself, but do not hand it in.