Math 220 A - Lecture 4

October 12, 2020

Last time

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $h_A : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, $z \longrightarrow \dfrac{az+b}{cz+d}$

Generalized circles in $\hat{\mathbb{C}}$

$\boxed{\text{i}}$ circles in $\mathbb{C}$

$\boxed{\text{ii}}$ line $L \cup \{\infty\}$

Theorem A Any Möbius transform maps generalized circles to generalized circles.

Theorem B PGL_2 acts triply transitively on $\hat{\mathbb{C}}$.

Given (z_1, z_2, z_3), (z_1', z_2', z_3') triples of distinct elts in $\hat{\mathbb{C}}$, $\exists!\ h$ with $h(z_i) = z_i'$

Proof of Thm A Suffices to consider the cases

[i] translation $z \longrightarrow z+\lambda$ ✓

[ii] rotation $z \longrightarrow e^{i\alpha} z$ ✓

[iii] dilation $z \longrightarrow m z$ ✓

[IV] inversion $z \longrightarrow 1/z$.

Claim A generalized circle is given by

(*) $A z\bar{z} + Bz + C\bar{z} + D = 0.$ where $A, D \in \mathbb{R}.$

and B, C are conjugates.

Proof A circle in $\mathbb{C}$ is given by

$$|z - z_0| = r \iff (z - z_0)\cdot(\bar{z} - \bar{z}_0) = r^2$$

$$\iff z\bar{z} - \bar{z}_0 z - z_0\bar{z} + (z_0\bar{z}_0 - r^2) = 0$$

$\implies$ (*) for $A = 1$, $D = z_0\bar{z}_0 - r^2$, $B = -\bar{z}_0$, $C = -z_0$

Conversely, if $A \neq 0$, (*) can be brought into this form.

When $A = 0$: $\underbrace{Bz + C\bar{z}}_{\text{linear}} + D = 0 \iff$ line.

Proof $\boxed{\text{IV}}$ preserves generalized circles.

$$A z\bar{z} + Bz + C\bar{z} + D = 0.$$

Let $w = \frac{1}{z} \Rightarrow A \cdot \frac{1}{w\bar{w}} + \frac{B}{w} + \frac{C}{\bar{w}} + D = 0$

$$\Rightarrow A + B\bar{w} + Cw + Dw\bar{w} = 0.$$

$\Rightarrow$ generalized circle. $\Rightarrow$ Thm A.

In the case of lines $L \cup \infty$, 0 and ∞ correspond under $\boxed{\text{IV}}$.

Proof of thm B   **Uniqueness**   Assume $\exists\, h, h'$

$$z_1 \xrightarrow[h']{h} z_1'$$

$$z_2 \xrightarrow[h']{h} z_2'$$

$$z_3 \xrightarrow[h']{h} z_3'$$

Let $T = h'^{-1} \circ h \implies T(z_i) = z_i$,

$\iff \dfrac{az+b}{cz+d} = z$ has 3 roots z_1, z_2, z_3

$\iff az+b = cz^2 + dz$ has 3 roots

$\implies a = d,\ b = c \implies T = \mathbb{1} \implies h = h'!$

Existence   Suffices: $\exists\, h$ with

$$\begin{tikzcd} z_1 \arrow[r, "h"] \arrow[rr, bend right, "T"'] & 0 & z_1' \arrow[l, "h'"'] \end{tikzcd}$$

$$\begin{tikzcd} z_2 \arrow[r, "h"] \arrow[rr, bend right, "T"'] & 1 & z_2' \arrow[l, "h'"'] \end{tikzcd}$$

$$\begin{tikzcd} z_3 \arrow[r, "h"] \arrow[rr, bend right, "T"'] & \infty. & z_3' \arrow[l, "h'"'] \end{tikzcd}$$

$h(z_1) = 0$

$h(z_2) = 1$

$h(z_3) = \infty.$

If (z_1', z_2', z_3') is another triple, find h' with

$$h'(z_1') = 0, \quad h'(z_2) = 1, \quad h'(z_3) = \infty.$$

Define $T = h'^{-1} \circ h \implies T(z_i) = z_i'$ as needed.

To deal with (z_1, z_2, z_3) and $(0, 1, \infty)$.

Cross ratio If $z_1, z_2, z_3 \neq \infty$,

$$h(z) = \frac{z - z_1}{z - z_3} \Big/ \frac{z_2 - z_1}{z_3 - z_2}$$

This is sometimes denoted $[z : z_1 : z_2 : z_3]$.

Check $h(z_1) = 0$

$h(z_2) = 1$

$h(z_3) = \infty$.

There are 3 remaining ease $z_1 = \infty$, $z_2 = \infty$ or $z_3 = \infty$.

For example, when $z_1 = \infty$. the above expression is

$$h(z) = \frac{z_2 - z_3}{z - z_3}, \quad h(z_1) = 0, \quad h(z_2) = 1, \quad h(z_3) = \infty.$$

II. Cauchy theory & Integration (Conway IV)

The theory of integration is crucial to complex analysis. Many important results have as starting point Cauchy's integral formula.

§1. Complex integration

[a] $U \subseteq \mathbb{C}$ open & connected

$\gamma : [a,b] \longrightarrow U$ C^1- path

[i] length $(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

[ii] C^1-reparametrization $\hat{\gamma} : [\hat{a}, \hat{b}] \longrightarrow U$

$$\hat{\gamma} = \gamma \circ \Phi \,, \quad \Phi : [\hat{a}, \hat{b}] \longrightarrow [a,b]$$

Orientation preserving: $\phi' > 0$.

[b] A piecewise C^1-path

$$\gamma = \gamma_1 + \dots + \gamma_n, \quad \gamma_i \text{ of class } C^1.$$

if $\exists$ $a = a_0 < a_1 < \dots < a_n = b$

$$\gamma\big/_{[a_{i-1},\, a_i]} = \gamma_i$$

[c] $f : U \longrightarrow \mathbb{C}$ continuous, Define

$$\int_\gamma f\, dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

substitute

$z = \gamma(t)$

$dz = \gamma'(t)\, dt$

This is independent of orientation preserving reparametrization

$$\int_a^b f(\gamma(t))\, \gamma'(t)\, dt \xrightarrow{t = \phi(s).} \int_{\hat a}^{\hat b} f(\hat\gamma(s)) \cdot \hat\gamma'(s)\, ds$$

This is change of variables: $f(\gamma(t)) = f(\hat\gamma(s))$

$$\gamma'(t)\, dt = \hat\gamma'(s)\, ds.$$

Remark $\displaystyle\int_{-\gamma} f\,dz = -\int_{\gamma} f\,dz$ after changing orientation

Remark The definition extends to piecewise C^1 paths

$$\int_{\gamma} f\,dz = \int_{\gamma_1} f\,dz + \dots + \int_{\gamma_n} f\,dz.$$

In particular, we can define $\displaystyle\int_{\partial R} f\,dz$, R rectangle.

Remark Conway works with rectifiable paths.

In the elementary theory of analytic functions it is seldom necessary to consider arcs which are rectifiable, but not piecewise differentiable. However, the notion of rectifiable arc is one that every mathematician should know.

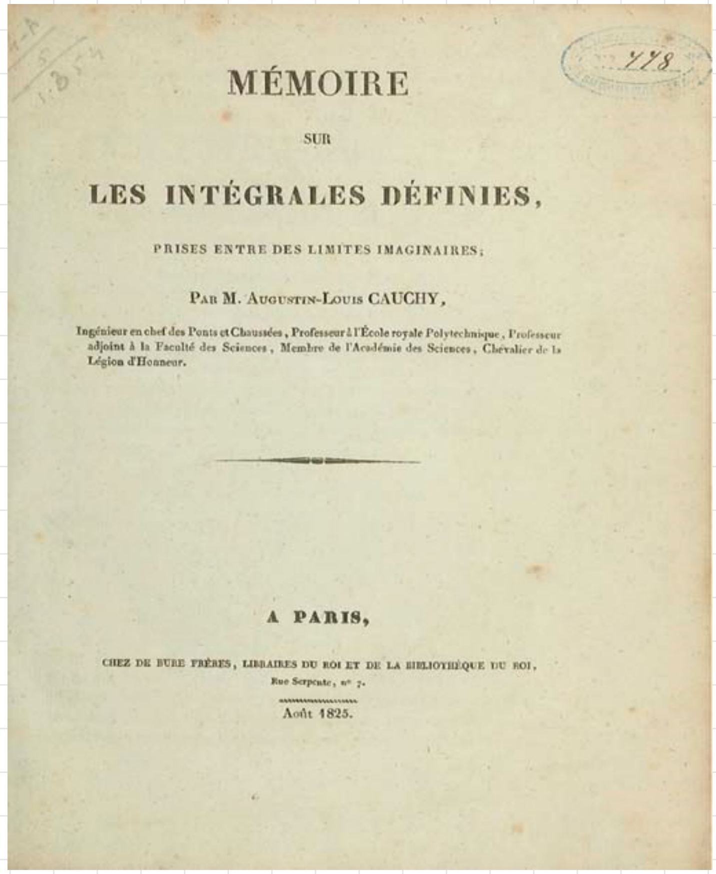
(Ahlfors – Complex Analysis)

Fundamental estimate Assume $|f| \leq M$ along γ

$$\Rightarrow \left| \int_{\gamma} f\,dz \right| \leq \text{length}(\gamma) \cdot M.$$

Proof $\displaystyle\left| \int_{\gamma} f\,dz \right| = \left| \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt \right|$

$$\leq M \int_a^b |\gamma'(t)|\,dt = M \cdot \text{length}(\gamma)$$



MÉMOIRE

SUR

LES INTÉGRALES DÉFINIES,

PRISES ENTRE DES LIMITES IMAGINAIRES;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur en chef des Ponts et Chaussées, Professeur à l'École royale Polytechnique, Professeur adjoint à la Faculté des Sciences, Membre de l'Académie des Sciences, Chevalier de la Légion d'Honneur.

A PARIS,

CHEZ DE BURE FRÈRES, LIBRAIRES DU ROI ET DE LA BIBLIOTHÈQUE DU ROI, Rue Serpente, n° 7.

Août 1825.

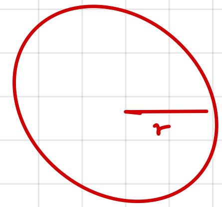
Baron Augustin-Louis Cauchy (1789 - 1857) was a French mathematician who made contributions to several branches of mathematics. He almost singlehandedly founded complex analysis.

Cauchy was a prolific writer: 800 research articles and 5 textbooks.

His name is one of the 72 names inscribed on the Eiffel Tower.

Example A

γ = circle of radius r, $\gamma(t) = r e^{it}$



$z = r e^{it}$

$$\int_\gamma z^n \, dz = \int_0^{2\pi} r^n e^{int} \cdot r e^{it} i \, dt$$

$$= \int_0^{2\pi} r^{n+1} e^{i(n+1)t} i \, dt$$

$$= r^{n+1} \cdot \frac{e^{i(n+1)t}}{i(n+1)} \cdot i \Bigg|_{t=0}^{t=2\pi} = 0. \quad , n \neq -1.$$

When $n = -1$

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{r e^{it} i}{r e^{it}} \, dt = \int_0^{2\pi} i \, dt = 2\pi i$$

Example B

f admits primitive F, $f = F'$.

$$\int_\gamma f \, dz = \int_a^b F'(\gamma(t)) \cdot \gamma'(t) \, dt$$

$$= \int_a^b \left(F(\gamma(t)) \right)' dt = F(\gamma(b)) - F(\gamma(a)).$$

Path independence!

III. Existence of primitives

$U \subseteq \mathbb{C}$ open connected, f continuous. We show three results.

Proposition A TFAE

[i] f admits a primitive

[ii] $\int_\gamma f\, dz = 0$ $\forall\, \gamma$ piecewise C^1 loop.

Remark [i] $\Rightarrow$ [ii] is clear by Example B.

Remark $\frac{1}{z}$ doesn't admit a primitive in $U = \mathbb{C}^\times$.

since $\int_\gamma \frac{dz}{z} = 2\pi i$ by Example A.

$\Rightarrow$ $\nexists$ no logarithm in $U = \mathbb{C}^\times$.

Proposition B If $U = \Delta =$ disc. TFAE

[i] f admits primitive

[ii] $\int_{\partial R} f\, dz = 0$ for all rectangles $\overline{R} \subseteq U$.

Compare:

Prop. A	Prop B.
$U \subseteq \mathbb{C}$	$U = \Delta$
γ piecewise C^1	$\gamma = \partial R$

Proposition C If $f: U \to \mathbb{C}$ holomorphic $\Longrightarrow \boxed{\int_{\partial R} f \, dz = 0}$ for all rectangles $\bar{R} \subseteq U$.

Corollary $f: \Delta \to \mathbb{C}$ holomorphic $\xRightarrow{B+C.}$ f admits a primitive.