Math 220, Problem Set 7.

1. Let $K = \left\{z : \frac{1}{4} \leq |z| \leq \frac{3}{4}\right\}$, $\Delta = \Delta(0,1)$. Show that there exists a function which is holomorphic in K, and which cannot be approximated uniformly in K by holomorphic functions in Δ.

2. Let $f(z) = \frac{1}{(z-2)(z-6)}$.

(i) Does there exists a sequence of rational functions R_n with poles only at 3 and 7 such that

$$\lim_{n\to\infty} \sup_{4\leq|z|\leq 5} |f(z) - R_n(z)| = 0?$$

(ii) Does there exist a sequence R_n of rational functions as above, but with poles only at 7?

3. Show that there exist polynomials p_n such that the pointwise limit

$$\lim_{n\to\infty} p_n(z) = \begin{cases} 1 & \text{if } z \in \mathfrak{h}^+ \\ 0 & \text{if } z \in \mathbb{R} \\ -1 & \text{if } z \in \mathfrak{h}^-. \end{cases}$$

(i) Let

$$K_n = \{z = x + iy : \frac{1}{n} \leq |y| \leq n,\, |x| \leq n\} \cup \{z \in \mathbb{R}, |z| \leq n\}.$$

Note that K_n is compact. Write

$$K_n = K_n^+ \cup K_n^- \cup K_n^0$$

for the intersections with $\mathfrak{h}^+, \mathfrak{h}^-, \mathbb{R}$. Consider the function

$$f_n = \begin{cases} 1 & \text{if } z \in K_n^+ \\ 0 & \text{if } z \in K_n^0 \\ -1 & \text{if } z \in K_n^-. \end{cases}$$

Show that f_n extends holomorphically to a neighborhood of K_n. Show there exist polynomials p_n such that

$$|f_n - p_n| < \frac{1}{2^n} \text{ in } K_n.$$

(ii) Conclude that the polynomials p_n satisfy the above property.

Remark: It is easy to construct sequences of continuous functions whose pointwise limit is discontinuous. It is quite hard to construct sequences of holomorphic functions whose pointwise limit is not holomorphic. An example is provided by this question.

4. Let $U = \left\{z : |z| < 1, \left|z - \frac{1}{4}\right| > \frac{3}{4}\right\}$, $K = \left\{z : |z| \leq 1, \left|z - \frac{1}{4}\right| \geq \frac{3}{4}\right\}$. True or false (please justify):

(i) Every holomorphic function on U can be approximated locally uniformly in U by polynomials.

(ii) Every continuous function in K which is holomorphic in U is uniform limit in K of a sequence of polynomials.

(iii) Every holomorphic function in K can be approximated uniformly in K by Laurent polynomials. A Laurent polynomial is an expression of the form

$$\sum_{n=-N}^{N} a_n z^n.$$