Math 220 A - Lecture 10

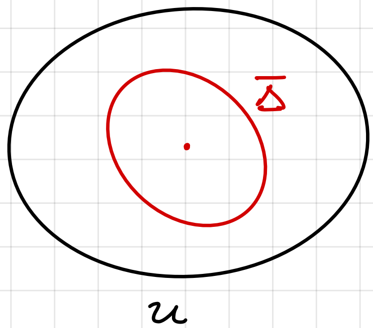
October 28, 2020

Recall — Midterm next Friday

0 Last time (Cauchy's Estimate)

$f: U \longrightarrow \mathbb{C}$ holomorphic, $\overline{\Delta}(a, R) \subseteq U$

$$|f^{(k)}(a)| \leq k! \frac{M_R}{R^k}, \quad M_R = \sup_{|z-a|=R} |f(z)|.$$

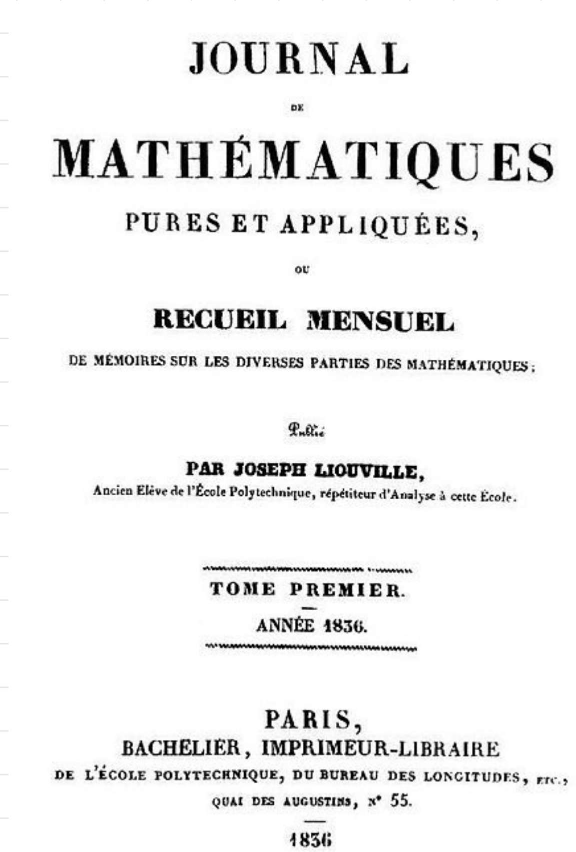


Remark $k = 0$:

$$|f(a)| \leq \sup_{z \in \partial\Delta} |f(z)|$$

1. Liouville's theorem

If $f: \mathbb{C} \longrightarrow \mathbb{C}$ entire & bounded $\Rightarrow f$ constant.



JOURNAL

DE

MATHÉMATIQUES

PURES ET APPLIQUÉES,

OU

RECUEIL MENSUEL

DE MÉMOIRES SUR LES DIVERSES PARTIES DES MATHÉMATIQUES;

Publié

PAR JOSEPH LIOUVILLE,

Ancien Elève de l'École Polytechnique, répétiteur d'Analyse à cette École.

TOME PREMIER.

ANNÉE 1836.

PARIS,

BACHELIER, IMPRIMEUR-LIBRAIRE

DE L'ÉCOLE POLYTECHNIQUE, DU BUREAU DES LONGITUDES, ETC.,

QUAI DES AUGUSTINS, N° 55.

1836

Joseph Liouville

1809 – 1882

Journal de Liouville

Known for: Liouville's theorem

Sturm-Liouville theory

Liouville numbers

Liouville function

...

Proof: f is bounded by M, $|f(z)| \leq M$ $\forall z \in \mathbb{C}$.

Cauchy's estimate for $k=1$. Take $\bar{\Delta}(a, R) \subseteq \mathbb{C}$.

$$|f'(a)| \leq \frac{M_R}{R} \leq \frac{M}{R}.$$

Take $R \longrightarrow \infty$.

Thus $f'(a) = 0$. $\forall a \implies f$ constant.

Fundamental Theorem of Algebra

Any nonconstant polynomial $f \in \mathbb{C}[z]$ has at least one complex root.

Proof: WLOG f monic

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n.$$

Assume

$$f \text{ has no roots} \implies f(z) \neq 0 \quad \forall z.$$

Let $g = \frac{1}{f}$. $\implies g$ is entire. We show g bounded $\implies$

$\implies g$ constant. $\implies f$ constant. This is a contradiction.

We show g bounded. If $|z| = R$

$$|f(z)| = \left| z^n + a_1 z^{n-1} + \dots + a_n \right| \geq |z|^n - \sum_{k=1}^{n-1} |a_k| \, |z|^{n-k}$$

$$= R^n - \sum_{k=1}^{n-1} |a_k| \, R^{n-k} \longrightarrow \infty \quad \text{as} \quad R \longrightarrow \infty.$$

$$\left.\begin{array}{l} \text{If } R \geq R_0 \implies |f(z)| \geq 1. \implies |g(z)| \leq 1. \\ \text{If } R \leq R_0 \implies \text{by continuity of } g: \ |g(z)| \leq K. \end{array}\right\} \implies$$

$$\implies |g(z)| \leq M = \max(1, K), \quad \forall z$$

[2] Zeros of holomorphic functions — Conway IV. 3.

$f : U \longrightarrow \mathbb{C}$ holomorphic, $f \not\equiv 0$, U connected.

$a \in U$ is a zero of order N if

$$f(a) = 0, \quad f'(a) = 0, \quad \dots, \quad f^{(N-1)}(a) = 0, \quad f^{(N)}(a) \neq 0.$$

$\implies$ Taylor expansion in $\Delta(a, R) \subseteq U$

$$f(z) = \sum_{k \geq N} \frac{f^{(k)}(a)}{k!} \cdot (z-a)^k = (z-a)^N g(z) \qquad (*)$$

where g is a power series in $\Delta(a, R)$.

$$g(a) = \frac{f^{(N)}(a)}{N!} \neq 0.$$

We need to rule out the case $N = \infty$.

Lemma $f : U \longrightarrow \mathbb{C}$, U connected. TFAE

$\boxed{i}$ $f \equiv 0$

$\boxed{ii}$ $\exists\, a \in U$, $f^{(k)}(a) = 0 \quad \forall k$

$\boxed{iii}$ $S = \{ z : f(z) = 0 \}$ has a limit point in U.

Proof $\boxed{i} \Rightarrow \boxed{ii}$, $\boxed{i} \Rightarrow \boxed{iii}$

$\boxed{iii} \Rightarrow \boxed{ii}$ Let a be a limit point for S, $a \in U$.

Clearly $f(a) = 0$. Let us assume a has finite order N.

By $(*)$, $f(z) = (z-a)^N g(z)$ in $\Delta(a, R)$. with

g power series, $g(a) \neq 0$. By continuity of g, $g(z) \neq 0$ in

some $\Delta(a, r) \subseteq \Delta(a, R)$. Then

$$S \cap \Delta(a, r) = \{ z : (z-a)^N g(z) = 0 \} = \{a\}.$$

contradiction with a being a limit point.

Thus $N = \infty$. $\Rightarrow \boxed{ii}$.

$\boxed{iii} \Rightarrow \boxed{ii}$. Let $A = \{a : f^{(k)}(a) = 0 \ \forall k\} \subseteq \mathcal{U}$.

By assumption $A \neq \Phi$. We show A is closed & open.

Thus $A = \mathcal{U}$. $\Rightarrow f \equiv 0$.

- A closed. Indeed $A = \bigcap_{k=0}^{\infty} \left(f^{(k)}\right)^{-1}(0) =$ closed.

Since $f^{(k)}$ is continuous $\Rightarrow f^{(k)^{-1}}(0)$ is closed $\Rightarrow A$ closed

- A open. Let $a \in A$. By Taylor if $\Delta(a, R) \subseteq \mathcal{U}$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k = 0 \quad \text{since } f^{(k)}(a) = 0.$$

Since $f = 0$ in $\Delta(a, R) \Rightarrow f^{(k)} = 0$ in $\Delta(a, R) \Rightarrow$

$\Rightarrow \Delta(a, R) \subseteq A \Rightarrow A$ open.

Corollary (Identity principle) Let $f, g : \mathcal{U} \longrightarrow \mathbb{C}$. holomorphic.

If

$S = \{z : f(z) = g(z)\}$ has a limit point $\Rightarrow f = g$.

Proof Work with $h = f - g$. Apply Lemma above.

Remarks

$\boxed{i}$ The zeros of $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic cannot have a limit point in $\mathcal{U}$.

$\boxed{ii}$ $f(z) = \sin \frac{1+z}{1-z}$ holomorphic in $\underbrace{\mathbb{C} \setminus \{1\}}_{\mathcal{U}}$.

Zeros $\quad \frac{1+z}{1-z} = n\pi \iff z = \frac{-1+n\pi}{1+n\pi} \longrightarrow 1.$

Thus the zeros can accumulate to $\partial \mathcal{U}$.

$\boxed{iii}$ This fails for C^{∞}- functions

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-\frac{1}{x^2}} \sin \frac{1}{x}, & x \neq 0. \end{cases}$$

Check f is C^{∞}. Also f has zeros at $\frac{1}{n\pi} \longrightarrow 0$. which has a limit point.

$\boxed{\text{IV}}$ $f \not\equiv 0$ has at most countably many zeros.

Let $U = \bigcup_{n=1}^{\infty} K_n$ where K_n compact. In each compact set K_n, f can only have finitely many zeros.

(indeed this is because $\mathrm{zero}(f)$ can't accumulate in K_n)

$$\Rightarrow \quad \mathrm{Zero}(f) = \bigcup_n \underbrace{\mathrm{Zero}(f) \cap K_n}_{\text{finite}} = \text{countable}.$$

Aufgaben und Lehrsätze,

erstere aufzulösen, letztere zu beweisen.

1.

(Von Herrn *N. H. Abel.*)

49. **Theorème.** Si la somme de la série infinie

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_m x^m + \ldots\ldots$$

est égale à zéro pour toutes les valeurs de x entre deux limites réelles α et β; on aura nécessairement

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0 \ldots\ldots a_m = 0 \ldots\ldots$$

en vertu de ce que la somme de la série s'évanouira pour une valeur quelconque de x.

Identity theorem : Crelle's Journal 1827