Math 220B — Lecture 21

February 26, 2021

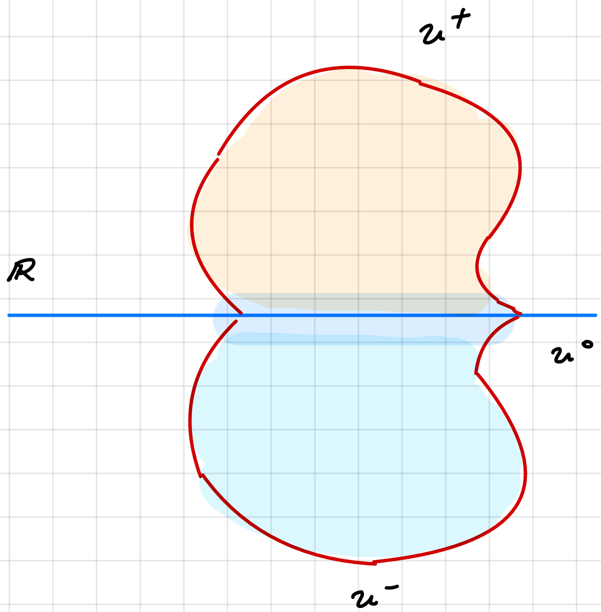
Last time

open $\mathcal{U} \subseteq \mathbb{C}$ symmetric $z \longrightarrow \bar{z}$. $\forall z \in \mathcal{U} \Rightarrow \bar{z} \in \mathcal{U}$.

$\mathcal{U}^+ = \mathcal{U} \cap \mathfrak{h}^+$

$\mathcal{U}^- = \mathcal{U} \cap \mathfrak{h}^-$

$\mathcal{U}^0 = \mathcal{U} \cap \mathbb{R}. = (a, b)$



Given $f: \mathcal{U}^+ \longrightarrow \mathbb{C}$

[i] holomorphic in $\mathcal{U}^+$

[ii] extends continuously to $\mathcal{U}^0$.

[iii] such that the values $f(\mathcal{U}^0) \subseteq \mathbb{R}$.

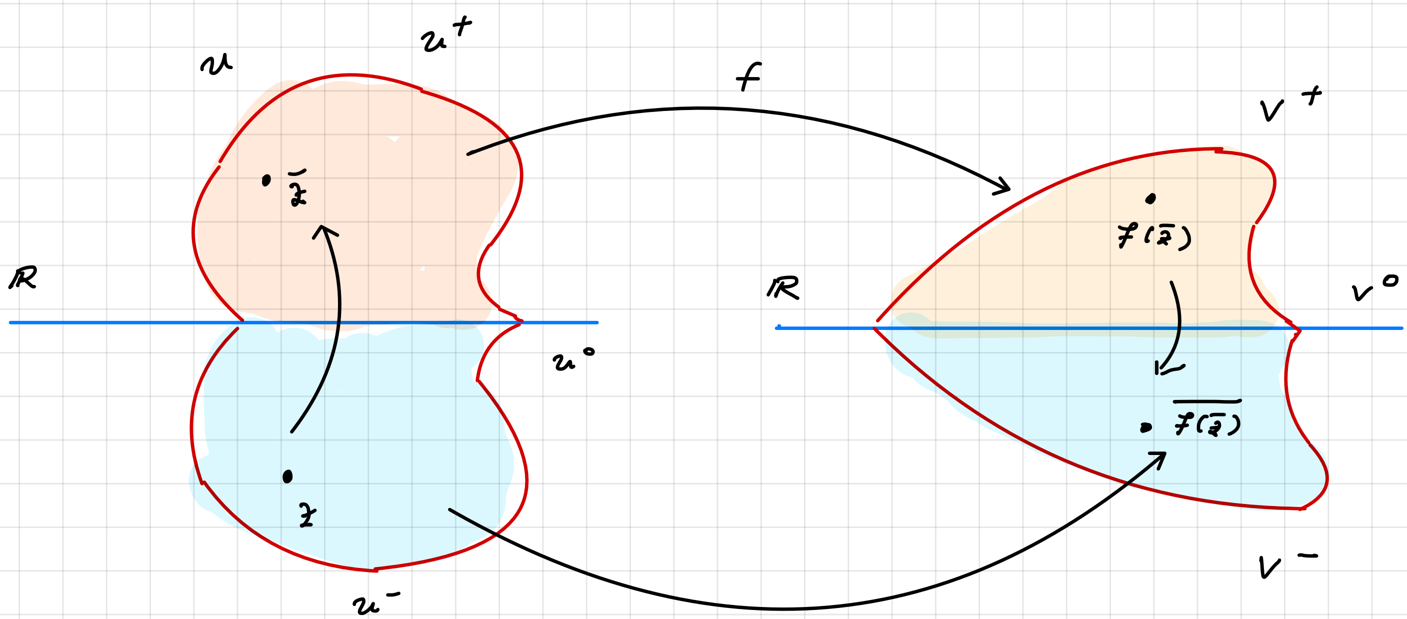
Define

$$F(z) = \begin{cases} f(z) & \text{if } z \in \mathcal{U}^+ \\ f(z) & \text{if } z \in \mathcal{U}^0 \\ \overline{f(\bar{z})} & \text{if } z \in \mathcal{U}^- \end{cases}$$

Theorm The function $F : \mathcal{U} \longrightarrow \mathbb{C}$

is a holomorphic extension of f beyond the boundary.

Visualization



Proof of Schwarz

$\boxed{I}$ F continuous

$\boxed{II}$ F holomorphic in U^+

$\boxed{III}$ F holomorphic in U^-

$\boxed{IV}$ F holomorphic at points of U^0.

Proof of $\boxed{I}$

Let $z_0 \in U_0 \Rightarrow z_0 = \overline{z_0}$

We show $\lim\limits_{\substack{z \to z_0 \\ z \in U^+}} F(z) = \lim\limits_{\substack{z \to z_0 \\ z \in U^-}} F(z)$.

$$\Longleftrightarrow \quad \lim_{\substack{z \to z_0 \\ z \in U^+}} f(z) = \lim_{\substack{z \to z_0 \\ z \in U^-}} \overline{f(\overline{z})}$$

$$\Longleftrightarrow \quad f(z_0) = \overline{f(\overline{z_0})}$$

which holds since $z_0 = \overline{z_0}$ & $f(z_0) = \overline{f(z_0)}$

Proof of $\boxed{iii}$ We show F holomorphic in $\mathcal{U}^-$.

Let $c^- \in \mathcal{U}^-$. Let $c^+ = \overline{c^-} \in \mathcal{U}^+$. Since f is holomorphic at $c^+ \Rightarrow \exists\ \Delta(c^+, r) \subseteq \mathcal{U}^+$. Taylor expand in $\Delta(c^+, r)$:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - c^+)^k, \quad \text{radius of convergence} \geq r.$$

Let $z \in \Delta(c^-, r) = \overline{\Delta(c^+, r)}$. Then

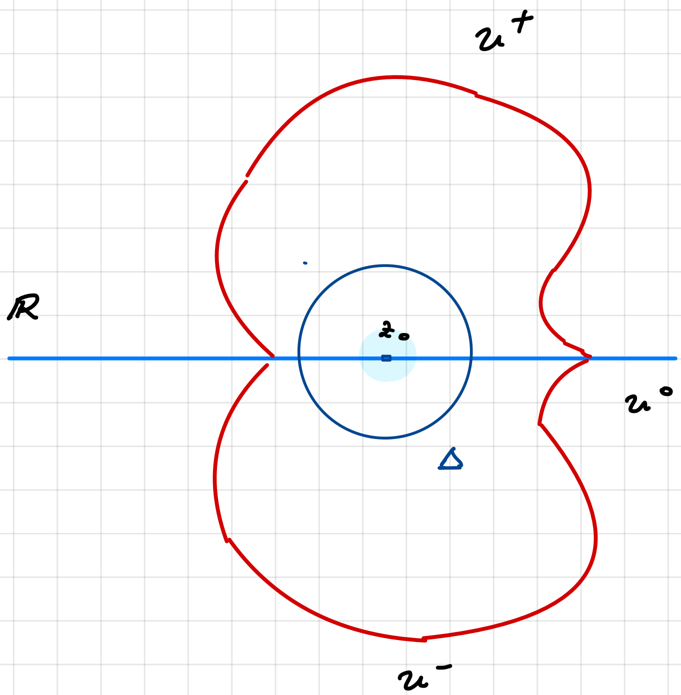
$$F(z) = \overline{f(\bar{z})} = \overline{\sum_{k=0}^{\infty} a_k (\bar{z} - c^+)^k}$$

$$= \sum_{k=0}^{\infty} \overline{a_k} \left(z - \overline{c^+} \right)^k$$

$$= \sum_{k=0}^{\infty} \overline{a_k} (z - c^-)^k, \quad \text{radius of convergence} \geq r.$$

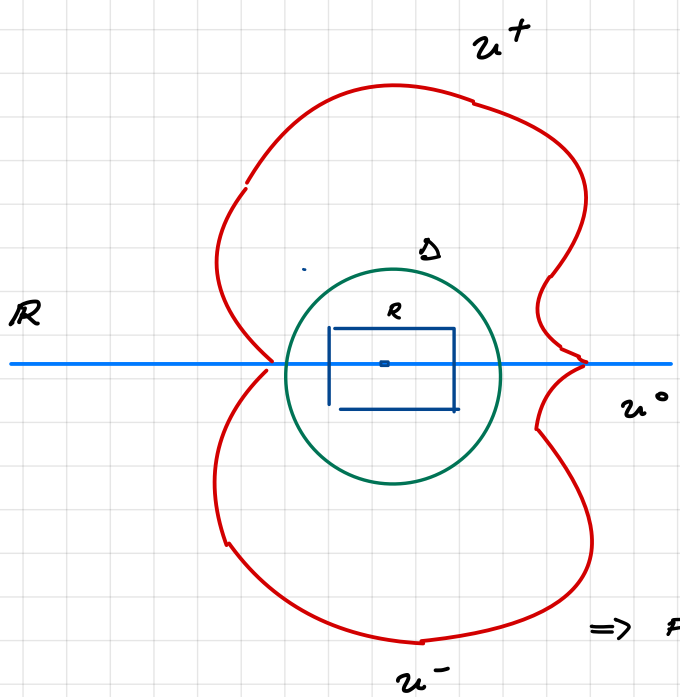
$\Rightarrow F$ holomorphic in $\mathcal{U}^-$.

Proof of $\boxed{IV}$



We show F is holomorphic in discs $z_0 \in \Delta \subseteq \mathcal{U}$ for arbitrary $z_0 \in \mathcal{U}_0$.

This will complete the proof.



Goal $\forall\ \bar{R} \subseteq \Delta$

$$\int_{\partial R} F\, dz = 0$$

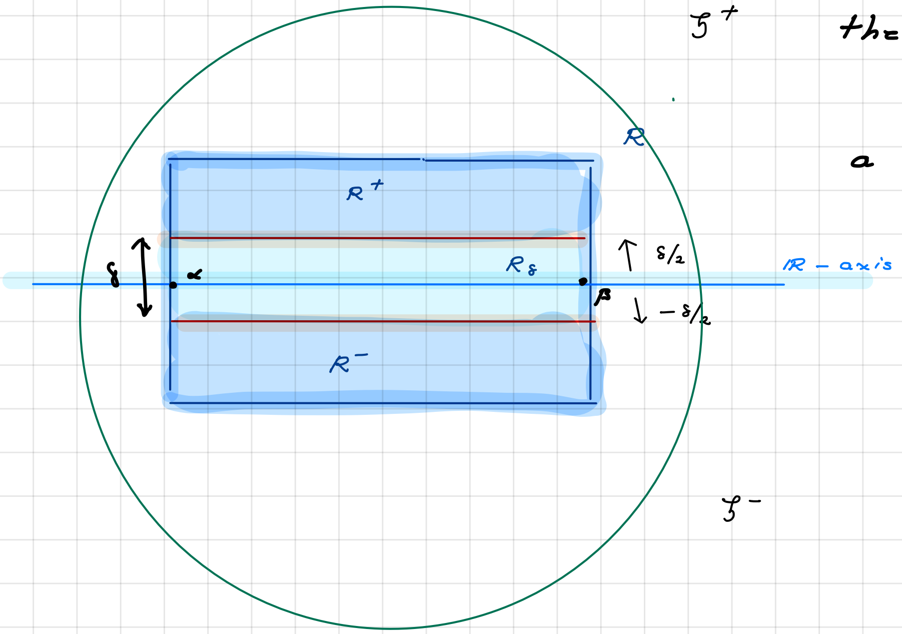
By Math 220A, Lecture 5

$\Rightarrow F = G'$ for some holomorphic G in Δ

$\Rightarrow F$ holomorphic in Δ.

If $\bar{R} \subseteq \mathcal{U}^+$ or $\bar{R} \subseteq \mathcal{U}^-$ this is clear (Goursat / Cauchy).

Assume $\bar{R}$ intersects the real axis. We assume that the intersection is not a side of R. Otherwise the argument is simpler.



We show $\exists\, K > 0$ such that for all $\varepsilon > 0$,

$$\left| \int_{\partial R} F \, dz \right| \leq K \cdot \varepsilon. \implies \int_{\partial R} F \, dz = 0.$$

$\boxed{\text{i}}$ F continuous in $\bar{\Delta}$ $\implies$ $|F(z)| \leq M$ for all $z \in \bar{\Delta}$.

$\boxed{\text{ii}}$ F uniformly continuous in $\bar{\Delta}$ = compact.

$\implies \forall\, \varepsilon \ \exists\, \delta$, $|x - y| \leq \delta \implies |F(x) - F(y)| < \varepsilon$.

We may assume $\delta < \varepsilon$.

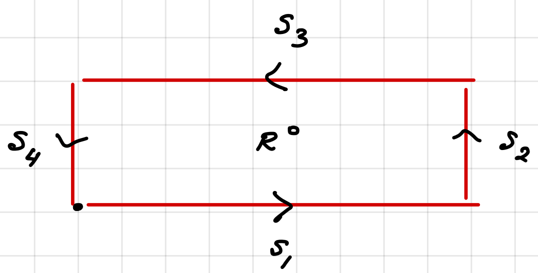
[i] Construct R^+, R^-, R^0 where $R^+ \subseteq \mathcal{U}^+$, $R^- \subseteq \mathcal{U}^-$

$$R^0 = [\alpha, \beta] \times \left[-\frac{\delta}{2}, \frac{\delta}{2}\right].$$

[ii] $\displaystyle\int_{\partial R^+} F\,dz = 0$, $\displaystyle\int_{\partial R^-} F\,dz = 0$ by Goursat.

$$\Longrightarrow \int_{\partial R} F\,dz = \int_{\partial R^0} F\,dz.$$

Estimates:



Sides of R^0: S_1, S_2, S_3, S_4.

(1) $$\left| \int_{S_2} F\,dz + \int_{S_4} F\,dz \right| \leq \left| \int_{S_2} F\,dz \right| + \left| \int_{S_4} F\,dz \right|$$

$$\leq M \cdot \underbrace{\text{length } S_2}_{\delta} + M \cdot \underbrace{\text{length } S_4}_{\delta}$$

$$= 2M\delta < 2M\varepsilon.$$

parametrize

(2) $$\left| \int_{S_1} F\,dz + \int_{S_3} F\,dz \right| \leq \int_{\alpha}^{\beta} \underbrace{\left| F\left(t - \frac{i\delta}{2}\right) - F\left(t + \frac{i\delta}{2}\right) \right|}_{<\varepsilon} dt$$

(uniform continuity).

$$\leq \varepsilon \cdot (\beta - \alpha) \leq \varepsilon \cdot \operatorname{diam}(\Delta)$$

(1)+(2)

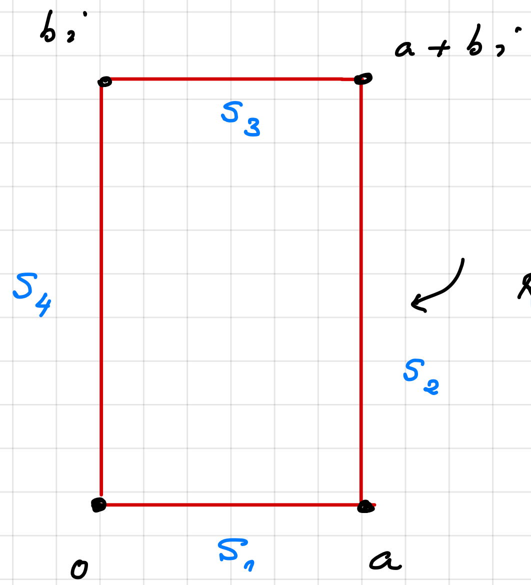
$$\Rightarrow \left| \int_{\partial R^\circ} F\,dz \right| \leq \left| \int_{S_2} F\,dz + \int_{S_4} F\,dz \right| + \left| \int_{S_1} F\,dz + \int_{S_3} F\,dz \right|$$

$$\leq 2M\varepsilon + \varepsilon \cdot \operatorname{diam}(\Delta) = K\varepsilon.$$

This completes the proof.

2. Application

Conformal maps of rectangles



$R_{a,b}$, $a, b \in \mathbb{R}$.

Example

$\exists$ biholomorphism $f: R_{a,b} \longrightarrow R_{a',b'}$ such that

[i] f extends continuously & bijectively to the boundary.

[ii] sending corners to corners & edges to edges.

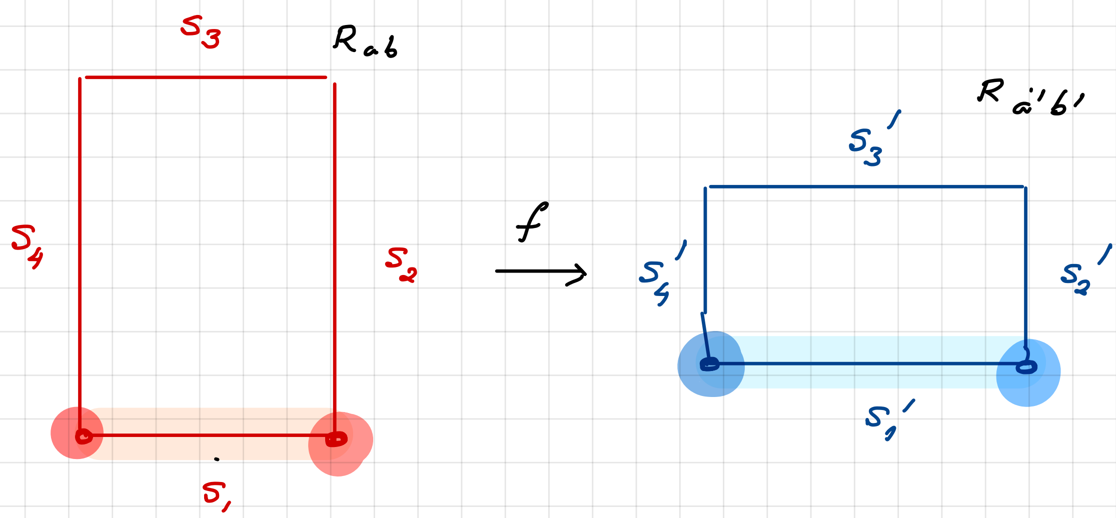
IF AND ONLY IF

$$\frac{a'}{a} = \pm \frac{b'}{b} \quad \text{or} \quad a a' = \pm b b'!$$

Remark Condition $\boxed{II}$ is automatic by Caratheodory, while condition $\boxed{III}$ is really necessary.

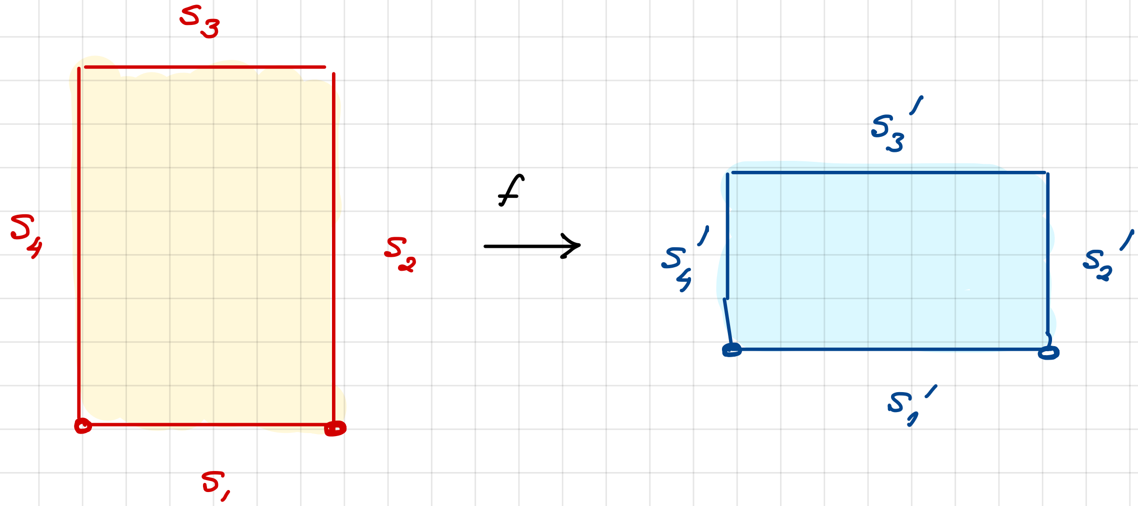
We first assume

$$f : S_1 \longrightarrow S_1', \quad 0 \longrightarrow 0, \quad a \longrightarrow a'.$$

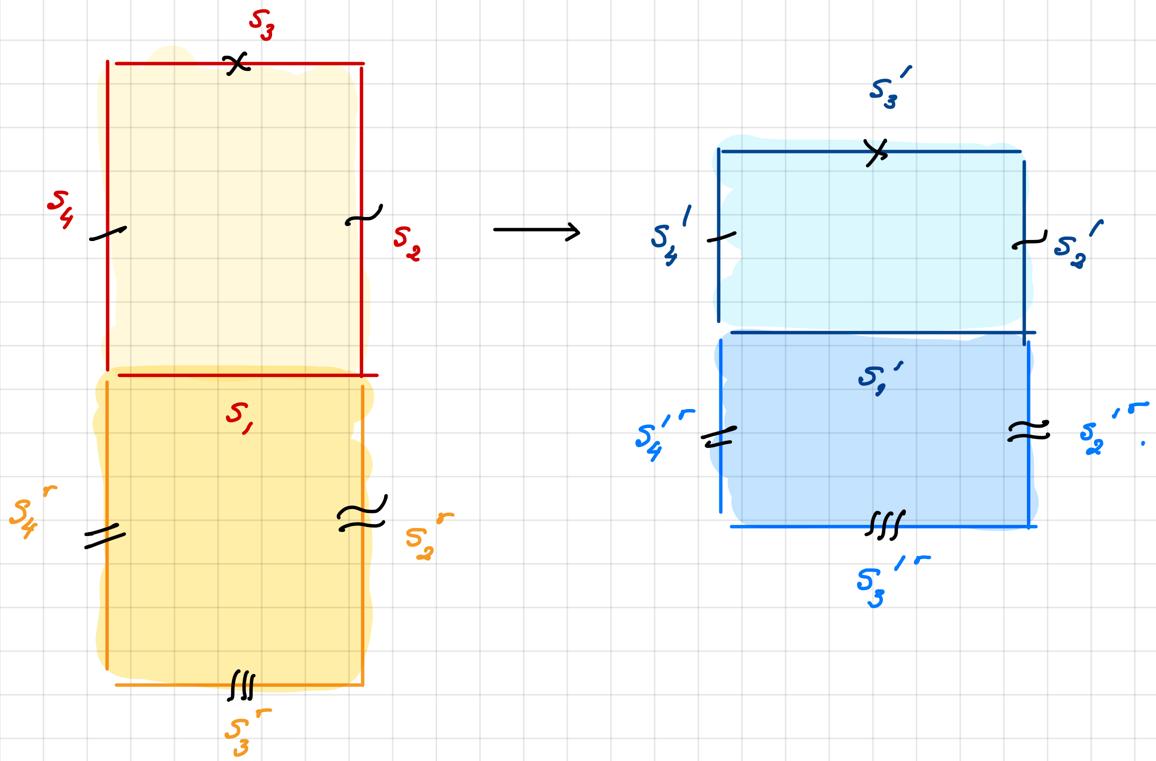


$f(0) = 0, \quad f(a) = a'$

- S_4 is sent to a side containing $f(0) = 0$, hence S_4'
- S_2 is sent to a side containing $f(a) = a'$, hence S_2'
- S_3 is sent to the remaining side S_3'



We use Schwarz Reflection along S_1 & S_1'.



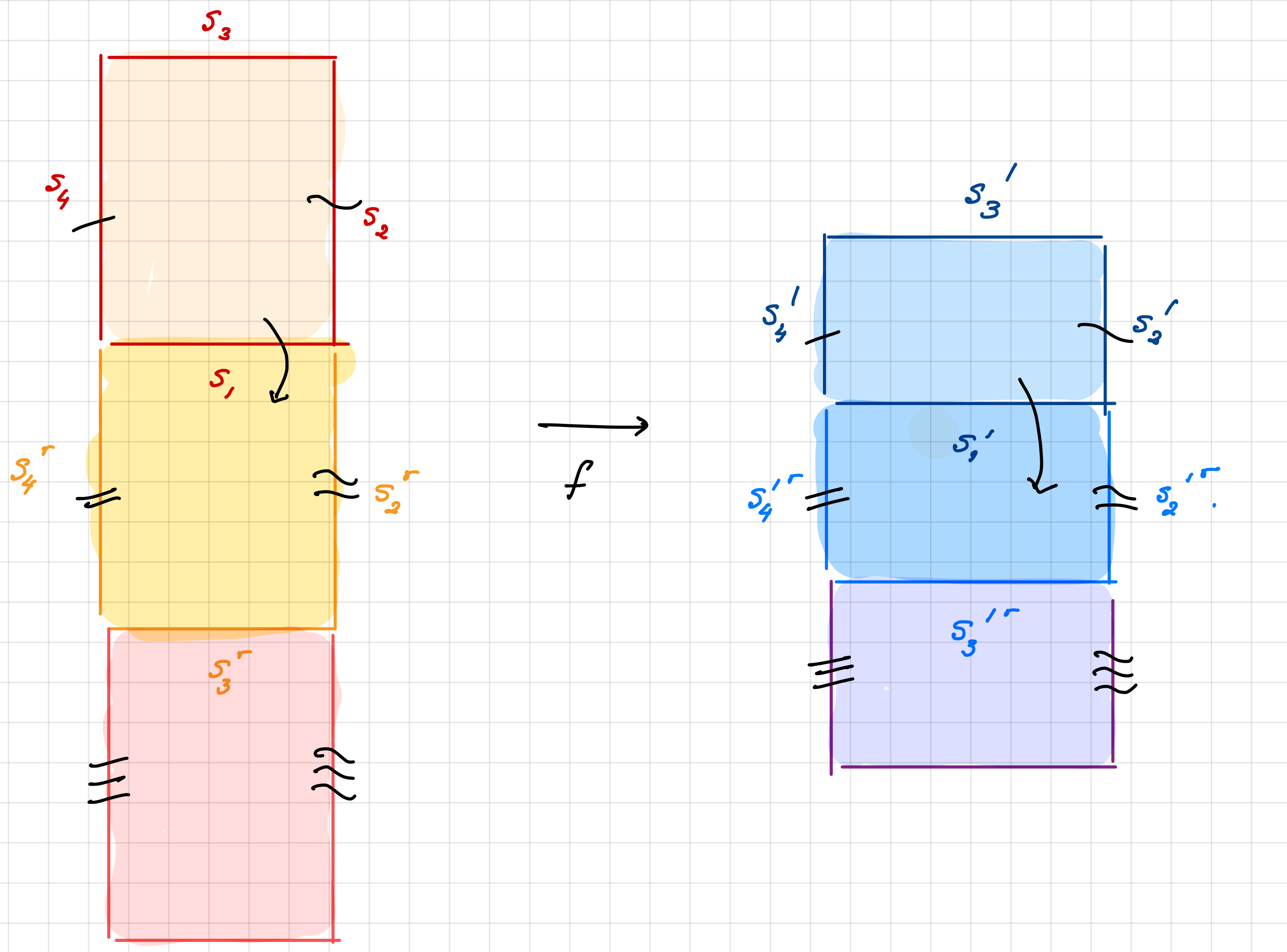
Note

$$S_4^r \longrightarrow S_4'^r, \quad S_2^r \longrightarrow S_2'^r, \quad S_3^r \longrightarrow S_3'^r.$$

from the explicit formula for the extension

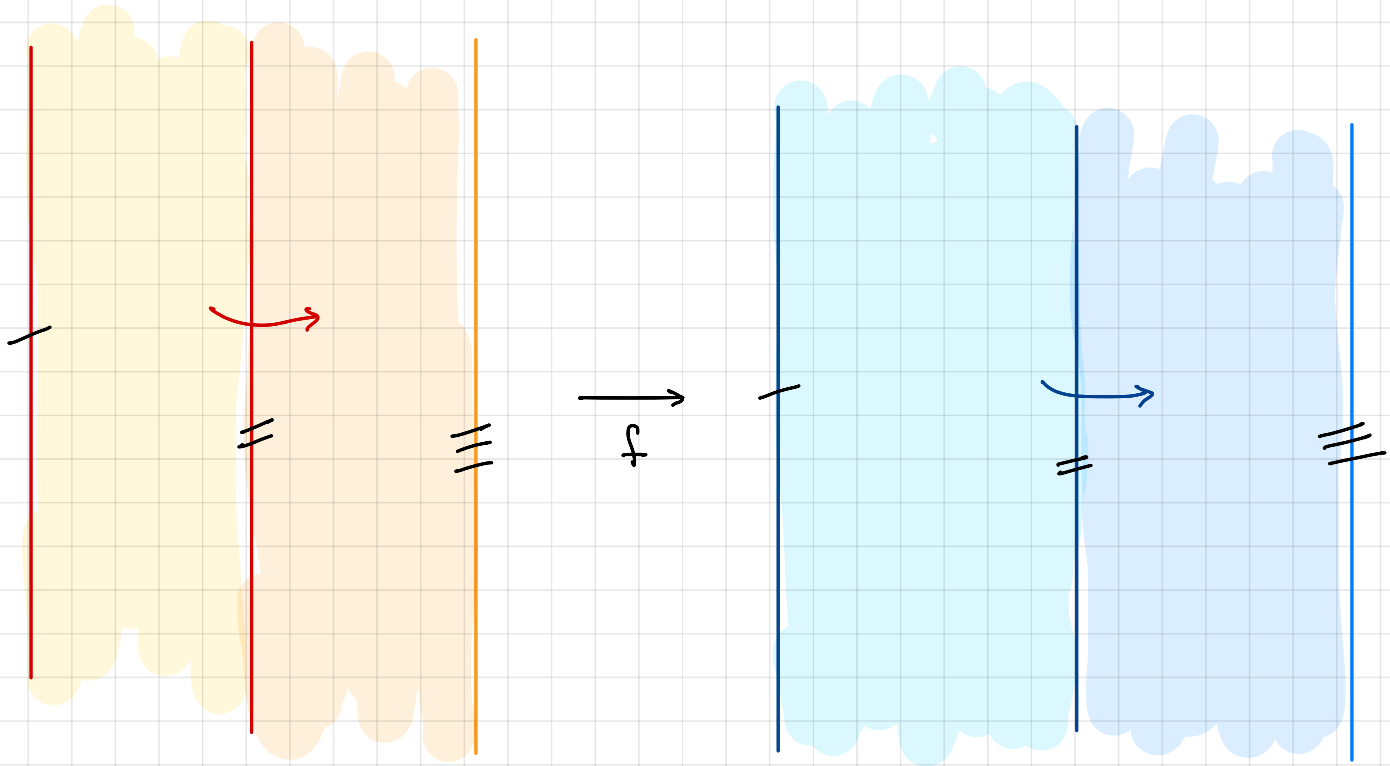
The extension is still bijective. (as the picture shows).

Reflect the new rectangle one more time, across S_3^r & $S_3'^r$.



and continue until we get two strips mapping to each other & their boundaries are mapped to each other.

Now reflect the strips across their sides.



In the end, we obtain $f: \mathbb{C} \longrightarrow \mathbb{C}$ bijective & holomorphic.

We saw in Math 220A, PSet 5 that $f(z) = \alpha z + \beta$.

Since $f(0) = 0 \implies \beta = 0 \implies f(z) = \alpha z$.

$$\left.\begin{array}{l} f(a) = a' \implies \alpha a = a' \\ f(b) = b' \implies \alpha b = b' \end{array}\right\} \implies \frac{a'}{a} = \frac{b'}{b}.$$

The remaining cases are part of Homework 6.