# Math 220 A – Lecture 13

November 9, 2020

## Logistics

Wed, Nov 11 – holiday – no lecture

Office Hour – Wed 4–5PM (discuss homework, midterm)

Questions about the midterm?

Last time — Laurent expansion

**Theorem** Let $f : \Delta(a; r, R) \longrightarrow \mathbb{C}$ holomorphic. Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$ can be represented as

Laurent series, converging absolutely & uniformly on compact subsets of $\Delta(a; r, R)$.

**Today** — classification of singularities Conway V. 1.

— characterization of singularities.

# Types of singularities

$f : \Delta^*(a, R) \longrightarrow \mathbb{C}$, holomorphic.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{Laurent series.}$$

**Terminology** $\boxed{i}$ the coefficient of $(z-a)^{-1}$:

$$a_{-1} = \operatorname*{Res}_{z=a} f = \text{residue}$$

$\boxed{ii}$
$$\sum_{n=-\infty}^{-1} a_n (z-a)^n = \text{principal part.}$$

**Three cases**

$\boxed{A}$ $a_k = 0 \ \forall k < 0 \iff$ Taylor expansion

$\iff f$ extends holomorphically across $a$

Removable singularity

$\boxed{B}$ $a_k = 0 \ \forall k < -N, \ a_{-N} \neq 0$

Pole of order $N$.

$\boxed{C}$ $a_k \neq 0$, $k < 0$ happens infinitely often

Essential singularity

## Case A   a removable singularity

**Theorem A** TFAE $f : \Delta^*(a, R) \to \mathbb{C}$ holomorphic

$\boxed{i}$ $f$ extends holomorphically across $a$

$\boxed{ii}$ $f$ extends continuously across $a$

$\boxed{iii}$ $f$ bounded near $a$

$\boxed{iv}$ $\lim\limits_{z \to a} f(z) \cdot (z-a) = 0$.

**Proof** $\boxed{i} \Rightarrow \boxed{ii} \Rightarrow \boxed{iii} \Rightarrow \boxed{iv}$ is obvious

$\boxed{iv} \Rightarrow \boxed{i}$. WLOG $a = 0$, else work with $f(z+a)$.

We show $a_k = 0$ $\forall\, k < 0$. Fix $\varepsilon > 0$. Since

$$\lim_{z \to 0} z f(z) = 0 \Rightarrow |f(z)| < \frac{\varepsilon}{|z|} \text{ if } |z| < \delta.$$

We have for $0 < r < \delta < R$:

$$|a_k| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}} \, dz \right| \leq \frac{1}{2\pi} \cdot \frac{\varepsilon}{r} \cdot \frac{1}{r^{k+1}} \cdot 2\pi r$$

$$= \frac{\varepsilon}{r^{k+1}}.$$

If $k=-1$: $|a_{-1}| < \varepsilon \quad \forall \varepsilon > 0 \Rightarrow a_{-1} = 0$.

If $k < -1$, take $\varepsilon = 1$, $|a_k| < \dfrac{1}{r^{k+1}}$. Make $r \to 0$

to obtain $a_k = 0$. since $k < -1$.

Example $f: U \longrightarrow \mathbb{C}$ holomorphic

$$g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$ is holomorphic.

Indeed $\dfrac{f(z)-f(a)}{z-a}$ has a removable singularity at $a$

by item $\boxed{IV}$. & $g$ is the continuous / holomorphic

extension across $a$.

Case B   $a$ pole of order $N$.

$$f(z) = (z-a)^{-N} g(z) \quad \text{holomorphic}$$

$$g(z) = \sum_{k=0}^{\infty} a_{k-N} (z-a)^k, \qquad g(a) = a_{-N} \neq 0.$$

Zeros versus Poles

$$\frac{1}{f(z)} = (z-a)^N \cdot \frac{1}{g(z)}, \quad \frac{1}{g} \text{ holomorphic near } a$$

$f$ pole of order $N$ at $a \iff \frac{1}{f}$ zero of order $N$ at $a$

Lemma B   $f : \Delta^*(a, R) \to \mathbb{C}$ holomorphic TFAE

[i] $a$ is a pole

[ii] $\lim_{z \to a} f(z) = \infty$.

Proof [i] $\Rightarrow$ [ii]   Write $f(z) = (z-a)^{-N} g(z)$.

Since $g(a) \neq 0 \Rightarrow |g(z)| \geq M > 0$ in $|z-a| < \delta$.

$$\Rightarrow |f(z)| = \frac{|g(z)|}{|z-a|^N} \geq \frac{M}{|z-a|^N}. \quad \text{Make } z \to a \text{ to}$$

conclude $\lim_{z \to a} f(z) = \infty$.

$\boxed{ii} \Rightarrow \boxed{i}$ Note $\lim_{z \to a} f(z) = \infty \Rightarrow \lim_{z \to a} \frac{1}{f(z)} = 0$

$\Rightarrow \frac{1}{f}$ bounded near $a$ $\Rightarrow \frac{1}{f}$ can be extended across $a$ holomorphically. Note the extension vanishes at $a$, say of order $N$ $\Rightarrow f$ has a pole at $a$ of order $N$.

**Definition** $S \subseteq U$ discrete. A function $f$ holomorphic in $U \setminus S$, with at worst poles at $S$ is called meromorphic.

**Example**

polynomials

$\boxed{i}$ $f(z) = \frac{P(z)}{Q(z)}$, $U = \mathbb{C}$ meromorphic.

$\boxed{ii}$ $f(z) = \frac{1}{\sin \frac{1}{z}}$, $U = \mathbb{C}^{\times}$.

Check $z = \frac{1}{n\pi}$, $n \in \mathbb{Z}^{\times}$ are poles. These do not accumulate in $\mathcal{U} = \mathbb{C} \setminus \{0\}$. Thus $f$ meromorphic in $\mathcal{U} = \mathbb{C}^{\times}$.

Case C $\quad f : \Delta^{*}(a, R) \longrightarrow \mathbb{C}$ holomorphic

$a$ essential singularity e.g. $f(z) = e^{\frac{1}{z}}$.

Example $\quad f(z) = e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^{k}} \cdot \Rightarrow$

$\Rightarrow \quad a = 0$ is essential singularity.

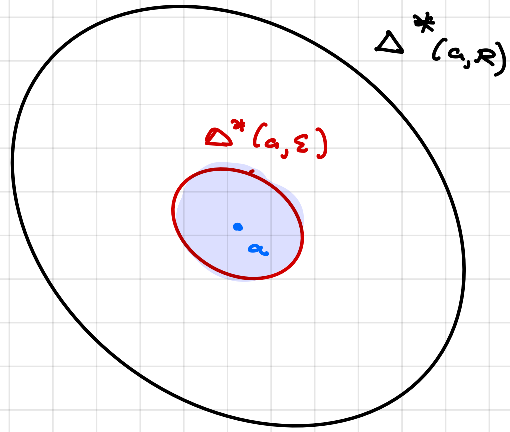
Remark $\quad f$ cannot be bounded or go to $\infty$.

(see cases A & B)

Question $\quad$ How does $f$ behave near $a$?

**Theorem** (Big Picard Theorem) — 220 C.

$\forall\ \Delta^*(a,\varepsilon) \subseteq \Delta^*(a,R)$, $\quad f(\Delta^*(a,\varepsilon)) = \mathbb{C}$ or $\mathbb{C} \setminus \{\text{point}\}$.




**Example** $\quad f(z) = e^{\frac{1}{z}}. \qquad a = 0.$

**Claim** $\quad f(\Delta^*(0,\varepsilon)) = \mathbb{C} \setminus \{0\}. \quad \forall\ \varepsilon > 0$

**Proof** $\quad y \neq 0: \quad y = e^{\frac{1}{z}}, \quad z \in \Delta^*(0,\varepsilon).$

$\iff \quad \dfrac{1}{z} = \log y + 2n\pi i$ for any choice of log

$\iff \quad z = \dfrac{1}{\log y + 2n\pi i} \in \Delta^*(0,\varepsilon)$ if $n >> 0$.

**Theorem** (Casorati – Weierstraß) $\quad f : \Delta^*(a,R) \longrightarrow \mathbb{C}$

$\boxed{\text{i}}$ $f$ has essential singularity at $a$

$\boxed{\text{ii}}$ $\forall\ \Delta^*(a,\varepsilon) \subseteq \Delta^*(a,R)$, $\quad f(\Delta^*(a,\varepsilon))$ is dense in $\mathbb{C}$.

Proof $\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$ Assume for some $\varepsilon > 0$, the set

$f(\Delta^*(a, \varepsilon))$ is not dense in $\mathbb{C}$. Then $\exists\ \Delta(\lambda, R)$

$$(*). \quad f(\Delta^*(a, \varepsilon)) \cap \Delta(\lambda, R) = \phi.$$

Define $g = \dfrac{1}{f - \lambda}$ in $\Delta^*(a, \varepsilon)$. By $(*)$ we know

$$|f - \lambda| \geq R \text{ in } \Delta^*(a, \varepsilon) \Rightarrow |g| \leq \frac{1}{R} \text{ in } \Delta^*(a, \varepsilon)$$

Thm A
$\Rightarrow$ a is removable singularity for $g$. But

$$f = \lambda + \frac{1}{g}. \quad (+)$$

If a is not a zero for $g \Rightarrow \dfrac{1}{g}$ holomorphic $\overset{(+)}{\Rightarrow}$

$f$ extends holomorphically across a $\Rightarrow$ removable singularity.

If a is a zero for $g \Rightarrow \dfrac{1}{g}$ has a pole at a $\overset{(+)}{\Rightarrow}$

$\Rightarrow f$ has pole. at a.

Both cases are impossible.

$\boxed{ii} \Rightarrow \boxed{i}$ Assume $a$ removable singularity $\Rightarrow$

Thm A

$\Rightarrow$ $f$ bounded near $a$ $\Rightarrow$ $\exists\, M>0,\ \varepsilon>0$ with

$$|f(z)| < M \quad \text{in} \quad \Delta^*(a,\varepsilon)$$

$\Rightarrow$ $f(\Delta^*(a,\varepsilon))$ cannot be dense.

Assume $a$ pole $\overset{\text{Lemma B}}{\Rightarrow}$ $\lim\limits_{z\to a} f(z) = \infty$ $\Rightarrow$

$\Rightarrow$ $\exists\ \varepsilon>0$, $|f(z)| \geq 1$ in $\Delta^*(a,\varepsilon)$ $\Rightarrow$

$\Rightarrow$ $f(\Delta^*(a,\varepsilon))$ cannot be dense.

Thus $a$ is essential singularity.


Felice Casorati

1835 – 1890


Karl Weierstraß

1815 – 1897

Émile Picard

1856 – 1941