HW6 - SOLUTIONS

Q1. *Biholomorphic rectangles.*

(i) The map $\ell(w) = a' - w$ is a linear automorphism of $\mathbb{C}$ and

$$\begin{aligned} \ell(0) &= a' & \ell(a') &= 0 \\ \ell(b'i) &= a' - b'i & \ell(a' + b'i) &= -b'i. \end{aligned}$$

Thus ℓ maps $R_{a',b'}$ to $R_{a',-b'}$. Moreover, the first row of the above equation tells us that the horizontal side S' of $R_{a',b'}$ is mapped to the horizontal side S' of $R_{a',-b'}$. Thus composition $\ell \circ f : R_{a,b} \to R_{a',-b'}$ is a biholomorphism mapping S to the horizontal side S' of $R_{a',-b'}$ and satisfies

$$\begin{aligned} \ell \circ f(0) &= \ell(a') = 0 \\ \ell \circ f(a) &= \ell(0) = a'. \end{aligned}$$

The case proved in class gives us $\ell \circ f(z) = \alpha z$, where

$$\alpha = \frac{a'}{a} = \frac{-b'}{b}.$$

(ii) The linear holomorphic maps

$$\ell_1(w) = w - ib' \quad : \quad R_{a',b'} \to R_{a',-b'} \tag{0.1}$$

$$\ell_2(w) = iw \quad : \quad R_{a',b'} \to R_{-b',a'} \tag{0.2}$$

$$\ell_3(w) = i(w - a') \quad : \quad R_{a',b'} \to R_{-b',-a'} \tag{0.3}$$

sends the three sides $[ib', a'+ib']$, $[0, ib']$ and $[a', a'+ib']$ to the horizontal side lying on the real axis of the target rectangles $R_{a',-b'}$, $R_{-b',a'}$ and $R_{-b',-a'}$ respectively.

By considering $\ell_k \circ f$, for appropriate k, we may assume that S maps to S'. For instance, if we use ℓ_3, we obtain that

$$\ell_3 \circ f : R_{a,b} \to R_{-b',-a'}$$

so that via (i) we have

$$\frac{-b'}{-a'} = \frac{a}{b} \implies aa' = bb'.$$

After accounting for all cases, we obtain

$$\frac{a'}{a} = \pm\frac{b'}{b} \quad \text{or} \quad aa' = \pm bb'.$$

Q2. *Schwarz Reflection across arcs.*

(i) Observe that $w \in U = \{z : 1 < |z| < R\}$ if and only if $1/R < |1/\bar{w}| < 1$. Thus

$$U^* = \{z : 1/R < |z| < 1\}.$$

(ii) For any $z \in U$, $|z| > 1$, thus $U^* = \{z : 1/\bar{z} \in U\}$ is a subset of $\Delta \backslash \{0\}$.

Note that the reflection, $z \to \bar{z}$, is an open map (it suffices to check that images of open discs are open discs which is clear). The inverse map, $z \to 1/z$, is a bi-holomorphism from $\mathbb{C}^* \to \mathbb{C}^*$, hence an open map as well. Therefore there composition, $z \to 1/\bar{z}$, from $\mathbb{C}^* \to \mathbb{C}^*$ is an open map. Thus U^* is an open subset of $\Delta \backslash \{0\}$.

(iii) Let $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the Möbius transformation given by

$$\phi(z) = i\frac{z+1}{z-1}, \qquad \phi^{-1}(w) = \frac{w+i}{w-i}.$$

Note that ϕ maps $\Delta(0,1)$ to the lower half plane, and outside of the circle to the upper half plane. Moreover, we observe the following useful relation

$$\overline{\phi(z)} = \phi\left(\frac{1}{\bar{z}}\right).$$

Let $W = \phi(U)$ and $W^* = \phi(U^*)$. For any element $w \in W$, let $z = \phi^{-1}(w) \in U$, then

$$\phi\left(\frac{1}{\bar{z}}\right) = \bar{w}.$$

Note

$$\frac{1}{\bar{z}} \in U^* \implies \bar{w} \in W^*.$$

Thus, W^* is the reflection of W along the real line and we have the following commuting biholomorphisms :

$$\begin{array}{ccc} U & \xrightarrow{\phi} & W \\ \downarrow{\scriptstyle 1/\bar{z}} & & \downarrow{\scriptstyle \bar{w}} \\ U^* & \xrightarrow{\phi} & W^* \end{array}.$$

Define $g = \phi \circ f \circ \phi^{-1} : W \to \mathbb{C}$. By Schwarz reflection principle, we know that the function

$$g^*(w) = \overline{g(\bar{w})}$$

is holomorphic on W^*. Hence, using the useful relation $\phi^{-1}\big(\overline{\phi(\bar{z})}\big) = 1/\bar{z}$ mentioned above, the function

$$\phi^{-1} \circ g^* \circ \phi(z) = \phi^{-1}\left(\overline{\phi \circ f \circ \phi^{-1}\big(\overline{\phi(z)}\big)}\right) \quad (0.4)$$

$$= 1/\overline{f(1/\bar{z})}. \quad (0.5)$$

is holomorphic on U^*.

(iv) An open set V is symmetric with respect to an arc, if $1/\bar{z} \in V$ for any $z \in V$. Define

$$\begin{aligned} U &= V \cap \{|z| > 1\} && (0.6) \\ U^0 &= V \cap \{|z| = 1\} && (0.7) \\ U^* &= V \cap \{|z| < 1\}. && (0.8) \end{aligned}$$

Note that $|z| = 1$ if and only if z is fixed under the reflection (across the arc)

$$z \to \frac{1}{\bar{z}}.$$

(v) Given a holomorphic function f on V^+, such that f extends continuously to U^0 and

$$|f(z)| = 1 \ \forall z \in U^0,$$

define

$$F(z) = \begin{cases} f(z) & \text{if } z \in U \\ f(z) & \text{if } z \in U^0 \\ 1/\overline{f(1/\bar{z})} & \text{if } z \in U^* \end{cases}. \tag{0.9}$$

Theorem 0.1. *The function $V \to \mathbb{C}$ is a holomorphic extension of f beyond the boundary U^0.*

Proof. We will use the notations from part (iii). Let

$$\phi(V) = W \cup W^0 \cup W^*,$$

where W, W^0 and W^* be the intersection of $\phi(V)$ with $\mathfrak{h}^+$, the real axis $\{w : \text{Im } w = 0\}$ and $\mathfrak{h}^-$ respectively.

We see that g is a holomorphic function on W extends continuously to W^0 and $g(w)$ is real for real numbers w: ϕ maps the unit circle to the real line.

By the usual Schwarz reflection principle (along real line), the function

$$G(w) = \begin{cases} g(w) & \text{if } w \in W \\ g(w) & \text{if } w \in W^0 \\ g^*(z) = \overline{g(\bar{w})} & \text{if } w \in W^* \end{cases} \tag{0.10}$$

is holomorphic. We finish the proof by observing that $\phi^{-1} \circ G \circ \phi = F$. ∎

Remark 0.2. We may extend generalize the statement of the above theorem to reflect about an arc in the circle $C = \{z : |z| = r\}$, where the function maps to $C' = \{w : |w| = R\}$. This is simply done by scaling the domain and the range by $1/r$ and $1/R$ on respectively.

We say that V is symmetric about C, if it is closed under the composition of map

$$z \to \frac{z}{r} \to \frac{r}{\bar{z}} \to \frac{r^2}{\bar{z}}.$$

Here, the first and last map is scaling by $1/r$ and r respectively, and the middle one is $z \to 1/\bar{z}$. It is clear that the circle C is fixed under the above composition.

Let U, U^0 and U^* be the intersection of V with outside of C, C and inside of C. Then any holomorphic function, f on U which extends continuously to C and satisfy $|f(z)| = R$ admits the holomorphic extension to V defined by

$$F(z) = \begin{cases} f(z) & \text{if } z \in U \\ f(z) & \text{if } z \in U^0 \\ R^2/\overline{f(r^2/\bar{z})} & \text{if } z \in U^* \end{cases}. \tag{0.11}$$

Q3. *Schwarz Reflection and Conformal Annuli.*

(i) By the open mapping theorem, $f(A_1) \subset A_2$, thus $f^{-1}(\partial A_2) \subset \partial A_1$. The inverse map

$$g = f^{-1} : \overline{A}_2 \to \overline{A}_1$$

is necessarily continuous since the preimages of closed sets in $\overline{A}_1$ (which are automatically compact) under $g = f^{-1}$ are closed in $\overline{A}_2$. (Indeed, these are just the images of compact sets under f which are compact hence closed). Let C be one of the boundary circles of ∂A_2. Then $g(C)$ must be a connected subset of ∂A_1 so it must be contained in one of the boundary circles D. Thus $g = f^{-1} : C \to D$ is an injective continuous maps between circles. Such a map is necessarily surjective $g(C) = D$ by the Lemma below.

In a similar fashion, the other boundary circle C' of ∂A_2 must map bijectively under g to the other boundary circle D' of ∂A_1. This shows

$$g(\partial A_1) = \partial A_2 \iff f(\partial A_2) = \partial A_1.$$

Lemma 0.3. *Any injective continuous map, $\phi : C \to D$, from a circle to a circle is a surjection.*

Proof. Assume $p \notin E = \phi(C)$. The image $E = \phi(C)$ must be connected and compact, hence it must be a closed arc of $D \setminus \{p\}$. The map $\phi^{-1} : E \to C$ is necessarily continuous since preimages of closed (hence compact) sets under ϕ^{-1} are closed and compact, being just images of compact sets under ϕ. This shows that the circle C is homeomorphic to a closed arc of the circle. This is impossible since C is not simply connected, while the arc is. ■

(ii) We shall prove this by successively reflecting across the inner circle of the annulus.

Let $f_1 : \Delta(0; 1/r, 1) \to \Delta(0; 1/R, 1)$ be the defined by reflecting f across $|z| = 1$. Using the defining equation, we note that f_1 extends to a bijective continuous map from $\overline{\Delta}(0; 1/r, 1)$ to$\overline{\Delta}(0; 1/R, 1)$, which is holomorphic in the interior.

Let $r_n = 1/r^{2^n}$ and $R_n = 1/R^{2^n}$. Suppose we have a bijective continuous map

$$f_n : \overline{\Delta}(0; r_n, 1) \to \overline{\Delta}(0; R_n, 1),$$

which is holomorphic in the interior. Using the arguments in part (i), the circle $\{|z| = r_n\}$ maps to $\{|z| = R_n\}$. We now apply the remark in Problem 2-(v) to extend f_n across the circle $\{|z| = r_n\}$.

Observe that if $U = \Delta(0; r_n, 1)$, its reflection

$$U^* = \left\{ z : \frac{r_n^2}{\bar{z}} \in U \right\} = \Delta(0; r_{n+1}, r_n).$$

We may extend f_n to the function $f_{n+1} : \Delta(0; r_{n+1}, 1) \to \Delta(0; R_{n+1}, 1)$ defined by

$$f_{n+1}(z) = \begin{cases} f(z) & \text{if } z \in U \\ f(z) & \text{if } |z| = r_n \\ R_n^2 / \overline{f(r_n^2/\bar{z})} & \text{if } z \in U^* \end{cases}. \tag{0.12}$$

From the equations above, we see that the f_{n+1} is a bijective holomorphic map to $\Delta(0; R_{n+1}, 1)$, that extends continuously and bijectively to the boundary.

We can continue this process for any number of times and the maps agree on the intersection. Thus we obtain a bijective holomorphic (hence biholomorphic) extension $f^+ : \Delta \backslash \{0\} \to \Delta \backslash \{0\}$.

(iii) We have seen in Lecture 16 that the only automorphisms of $\Delta \backslash \{0\}$ are rotations. In particular, f^+ must map the circle $\{|z| = r_0\}$ to the circle $\{|z| = R_0\}$, thus $r_0 = 1/r = 1/R = R_0$. Hence $r = R$.

(iv) Suppose $\{|z| = 1\}$ is mapped to $\{|z| = R\}$. Note that the function $z \to R/z$ is a bijective continuous map from $\overline{A_2} \to \overline{A_2}$, which is holomorphic in the interior and maps $\{|z| = R\} \to \{|z| = 1\}$. Thus the function $g(z) = R/f(z) : \overline{A_1} \to \overline{A_2}$ is a bijective continuous map, which is holomorphic in the interior and it maps $\{|z| = 1\}$ to $\{|z| = 1\}$. From part (iii), we conclude that $r = R$.