Math 220B – Lecture 5

January 13, 2021

The Γ - function (Conway VII. 7)

Definition $$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Properties of the function G

$\boxed{\text{I}}$ $$G(z)\, G(-z) = \frac{\sin \pi z}{\pi z} \quad \text{(Euler)}$$

$\boxed{\text{II}}$ $$G(z-1) = z\, G(z)\, e^{\gamma}$$ for some constant γ.

We inspect zeroes of both sides.

Zeroes G : $-1, -2, \ldots, -n, \ldots$

$$\left.\begin{aligned} zG(z) &: 0, -1, -2, \ldots, -n, \ldots \\ G(z-1) &: 0, -1, -2, \ldots, -n \end{aligned}\right\} \text{ have the same zeroes}$$

$\Longrightarrow$ $$G(z-1) = z\, G(z)\, e^{\gamma(z)}$$ for some function $\gamma(z)$.

We need $\gamma(z) = \text{constant}$. We verify $\gamma' = 0$.

Take logarithmic derivatives

$$\frac{G'(z-1)}{G(z-1)} = \frac{1}{z} + \frac{G'(z)}{G(z)} + \gamma' \quad (*)$$

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Since logarithmic derivative turns products into sums

$$\frac{G'(z)}{G(z)} = \sum_{n=1}^{\infty} \left(\frac{\frac{1}{n}}{1 + \frac{z}{n}} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right)$$

$$\Longrightarrow \frac{G'(z-1)}{G(z-1)} = \sum_{n=1}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n} \right) = \left(\frac{1}{z} - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n+1} \right)$$

$n \geq 2 \quad n \longrightarrow n+1$

$$= \left(\frac{1}{z} - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)}$$

$$= \left(\frac{1}{z} - \cancel{1} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \cancel{1}$$

$$= \frac{1}{z} + \frac{G'(z)}{G(z)} \quad \overset{(*)}{\Rightarrow} \gamma'(z) = 0 \Rightarrow \gamma(z) = \gamma. = \text{constant}$$

$\boxed{iii}$ The above constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log(n+1)\right) \qquad \text{(Euler constant)}$$

Indeed, $G(0) = 1$ by definition of the function G.

By $\boxed{ii} \implies G(z-1) = z\,G(z)\,e^{\gamma} \overset{z=1}{\implies} G(1) = e^{-\gamma}$.

Using the definition

$$G(1) = \prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)e^{-1/k} =$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-1/k} =$$

$$= \lim_{n\to\infty} \frac{n+1}{\cancel{n}}\cdot\frac{\cancel{n}}{\cancel{n-1}}\cdot\cancel{\ }\cdots\frac{\cancel{2}}{1}\cdot e^{-1-1/2-\dots-1/n}$$

$$= \lim_{n\to\infty} e^{-\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log(n+1)\right)} = e^{-\gamma}.$$

$$\implies \gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log(n+1)\right)$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right)$$

Definition $\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)} = \Gamma$ - function

Properties of Γ

$\boxed{i}$ $\Gamma(1) = \frac{e^{-\gamma}}{G(1)} = 1$ using $G(1) = e^{-\gamma}$ from above.

$\boxed{ii}$ $\Gamma(z+1) = z\,\Gamma(z) \rightsquigarrow$ " Γ behaves like a factorial "

In particular, by induction

$$\Gamma(n) = (n-1)! \qquad \forall\, n > 0,\ n \in \mathbb{Z}.$$

This follows by direct computation

$$\Gamma(z+1) = \frac{e^{-\gamma z-\gamma}}{(z+1)} \cdot \frac{1}{G(z+1)} \overset{???}{=} \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)} \cdot z = z\,\Gamma(z)$$

$\Longleftrightarrow \quad G(z) = (z+1)\,G(z+1)\,e^{\gamma}$ which is true.

(see above)

$\boxed{iii}$ $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$. In particular $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

To prove this, we use $\boxed{ii}$ + Euler:

$$\Gamma(z)\,\Gamma(1-z) \overset{\boxed{ii}}{=} \Gamma(z)\,(-z)\,\Gamma(-z) =$$

definition $\longrightarrow$
$$= \frac{e^{-\gamma z}}{z\,G(z)} \cdot \cancel{(-z)} \cdot \frac{e^{\gamma z}}{\cancel{(-z)}\,G(-z)}$$

$$= \frac{1}{z\,G(z)\,G(-z)} = \frac{\pi}{\sin \pi z} \quad \text{(see above)}$$

$\boxed{IV}$ $\Gamma(z) = \lim\limits_{n\to\infty} \dfrac{n!}{z\,(z+1)\cdots(z+n)}\, n^{z}$ (Gauss' definition)

We use the definition

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)} =$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z}}{z} \cdot \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} e^{z/k}$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z}}{z} \cdot \prod_{k=1}^{n} \frac{k}{z+k} \cdot e^{z\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n!}{z\,(z+1)\cdots(z+n)} \cdot e^{z\,\underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n-\gamma\right)}_{\downarrow\, 0}} \cdot n^{z}$$

$$= \lim_{n\to\infty} \frac{n!}{z\,(z+1)\cdots(z+n)} \cdot n^{z}.$$

Exercise (Conway VII. 7. 3)

Legendre duplication formula.

Use Gauss' definition to check

$$\sqrt{\pi}\ \Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$$

Residues

Note that $\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \frac{1}{G(z)}$ is meromorphic with poles at $0, -1, -2, \ldots$ since G has zeroes at $-1, -2, \ldots$

What are the residues?

$$\mathrm{Res}(\Gamma, -n) = \lim_{z \to -n} (z+n)\, \Gamma(z) =$$

$$\boxed{11}$$

$$= \lim_{z \to -n} \cancel{(z+n)} \frac{\Gamma(z+n+1)}{z(z+1)\cdots\cancel{(z+n)}}$$

$$= \lim_{z \to -n} \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)}$$

$$= \frac{\Gamma(1)}{(-n)\cdots(-1)} = \frac{1}{(-1)^n n!} = \frac{(-1)^n}{n!}$$

Remark It can be shown

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$ for $\operatorname{Re} z > 0$.

Mellin transform of e^{-t}.

Step 1 Convergence of RHS. Fix z, $\operatorname{Re} z > 0$.

$$\left| \int_0^1 e^{-t} t^{z-1}\, dt \right| \le \int_0^1 |e^{-t} t^{z-1}|\, dt$$ since $|e^{-t}| \le 1$

$$\le \int_0^1 t^{\operatorname{Re} z - 1}\, dt$$

$$= \frac{t^{\operatorname{Re} z}}{\operatorname{Re} z} \Big|_{t=0}^{t=1} = \frac{1}{\operatorname{Re} z}$$ using $\operatorname{Re} z > 0$.

Pick A with $|t^{z-1}| \le e^{t/2}$ when $|t| > A$

$$\left| \int_A^\infty e^{-t} t^{z-1}\, dt \right| \le \int_A^\infty |e^{-t} t^{z-1}|\, dt \le \int_A^\infty e^{-t} e^{t/2}\, dt$$

$$= \int_A^\infty e^{-t/2}\, dt$$

$$= 2 e^{-A/2} < \infty.$$

$\int_1^A e^{-t} t^{z-1}\, dt < \infty$ by continuity of $e^{-t} t^{z-1}$ in t.

Step 2 Using integration by parts, one easily shows

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = \frac{n^z \quad n!}{z\,(z+1)\cdots(z+n)}$$

Exercise – check the details.

Step 3 Make $n \to \infty$. From real analysis

$$\left(1-\frac{t}{n}\right)^n \longrightarrow e^{-t} \quad \text{as } n \to \infty.$$ This will also be explained below. We will argue that

(1) $$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt \xrightarrow{n\to\infty} \int_0^\infty e^{-t}\, t^{z-1}\,dt$$

$\parallel$ Step $\boxed{ii}$

$$\frac{n^z \quad n!}{z\,(z+1)\cdots(z+n)} \xrightarrow{n\to\infty} \Gamma(z)$$ by Gauss' formula

This shows $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\,dt.$

Rigorous justification of convergence in (1)

Claim $0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \le \frac{e^{-t} t^2}{n}$ if $0 \le t \le n$.

Assuming the claim, we prove Step 3. Compute

$$\int_0^\infty e^{-t} t^{z-1} dt - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt =$$

$$= \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dt \to 0$$

We claim both terms converge to 0 as $n \to \infty$.

term II. $\int_n^\infty e^{-t} t^{z-1} dt \to 0$ as $n \to \infty$ because

$\int_0^\infty e^{-t} t^{z-1} dt$ converges by Step 1.

term I $\quad \left| \int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n \right) t^{z-1} dt \right| \overset{\text{claim}}{\leq}$

$$\leq \int_0^n \frac{1}{n} t^2 e^{-t} \cdot t^{z-1} dt < \int_0^\infty \frac{1}{n} \cdot t^2 e^{-t} t^{z+1} dt$$

$$= \frac{1}{n} \underbrace{\int_0^\infty e^{-t} t^{z+1} dt}_{\text{converges by step 1.}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Proof of claim (only in the notes)

(a) first inequality.

Use $1 - y \leq e^{-y}$ for $y \geq 0$.

Take

$$y = \frac{t}{n} \Rightarrow 1 - \frac{t}{n} \leq e^{-t/n} \Rightarrow \left(1-\frac{t}{n}\right)^n \leq e^{-t}.$$

To see $1-y \leq e^{-y}$, let $f(y) = e^{-y} - (1-y)$,

$f(0) = 0$, $f' = -e^{-y} + 1 > 0 \Rightarrow f \nearrow \Rightarrow f(y) \geq f(0) = 0$

(b) second inequality.

The inequality to prove is

$$e^{-t} - \left(1-\frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n} \iff$$

$$\iff 1 - e^{t}\left(1-\frac{t}{n}\right)^n \leq \frac{t^2}{n}. \quad (*)$$

Use $e^{y} \geq 1+y$ for $y \geq 0$ proven just as above. Take $y = \frac{t}{n}$

Since $e^{t} = \left(e^{t/n}\right)^n \geq \left(1+\frac{t}{n}\right)^n$, to show $(*)$ we show

$$1 - \left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n \leq \frac{t^2}{n} \iff$$

$$\iff \left(1-\frac{t^2}{n}\right) \leq \left(1-\frac{t^2}{n^2}\right)^n. \text{ This is true.}$$

Indeed, use $(1-y)^n \geq 1-ny$ for $y = \frac{t^2}{n^2}$.

The last inequality can be proved by induction on n.