HW 5 - SOLUTIONS

Q1.

(i) Recall (Lecture 15) that any automorphism f of the unit disc is a composition of a rotation with a fractional linear transformation

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z},$$

where $a \in \Delta := \Delta(0,1)$.

Rotation : Observe that

$$d(e^{i\theta}z, e^{i\theta}w) = \left| \frac{e^{i\theta}(z-w)}{1-\overline{(e^{i\theta}z)}e^{i\theta}w} \right| = d(z,w)$$

since $e^{i\theta}$ has absolute value 1 and its conjugate is $e^{-i\theta}$.

Fractional linear transformation : Note that $|a| < 1$ and thus $(1-\bar{a}z)$ and $(1-\bar{a}w)$ are non-zero quantities for $z, w \in \Delta$. We follow the following calculation to get the result.

$$\begin{aligned} d(\phi_a(z), \phi_a(w)) &= \left| \frac{\phi_a(z)-\phi_a(w)}{1-\overline{\phi_a(z)}\phi_a(w)} \right| \\ &= \left| \frac{(z-a)(1-\bar{a}w)-(w-a)(1-\bar{a}z)}{(1-a\bar{z})(1-\bar{a}w)-(\bar{z}-\bar{a})(w-a)} \right| \\ &= \left| \frac{(z-w)(1-|a|^2)}{(1-\bar{z}w)(1-|a|^2)} \right| \\ &= d(z,w). \end{aligned}$$

(ii) We will reduce the problem to the Schwarz lemma. Fix $w \in \Delta$. Let $g = \phi_{f(w)} \circ f \circ \phi_w^{-1}$ and note

$$g(0) = \phi_{f(w)} \circ f \circ \phi_w^{-1}(0) = \phi_{f(w)} \circ f(w) = 0.$$

By Schwarz Lemma

$$d(g(z),0) \leq d(z,0)$$

using that $d(u,0) = |u|$. Then

$$\begin{aligned} d(f(z), f(w)) &= d(\phi_{f(w)} \circ f(z), \phi_{f(w)} \circ (f(w)) \\ &= d(g \circ \phi_w(z), 0) \\ &\leq d(\phi_w(z), 0) \\ &= d(\phi_w(z), \phi_w(w)) = d(z,w). \end{aligned}$$

(iii) We continue with the notations in part (ii). Assume that for a pair (w_0, z_0) we have

$$d(f(z_0), f(w_0)) = d(z_0, w_0).$$

We define $g = \phi_{f(w_0)} \circ f \circ \phi_{w_0}^{-1}$ just as before, and note that in the preceding inequalities we must be equalities throughout. Recall that equality occurs in Schwarz Lemma only if g is a rotation. But if g is a rotation, g is an automorphism of Δ. Then

$$f = \phi_{f(w_0)}^{-1} \circ g \circ \phi_{w_0}$$

is also an automorphism of Δ. By part (i), equality must hold for all pairs z, w.

(iv) Applying ϕ_s to all the arguments and by part (i), it is enough to show that

$$d(z', 0) \le d(w', 0) + d(z', w'),$$

where $z' = \phi_s(z)$ and $w' = \phi_w(s)$. Thus we may assume $s = 0$, and z, w arbitrary.

Note that $d(u, 0) = |u|$. We thus need to show

$$|z| \le |w| + \left|\frac{z - w}{1 - \bar{z}w}\right| \iff |z| - |w| \le \left|\frac{z - w}{1 - \bar{z}w}\right|.$$

There are slick ways of solving this, but a straightforward solution is via polar coordinates. Set

$$z = r_1 e^{it_1}, \quad w = r_2 e^{it_2}.$$

We may rotate z, w to achieve z is positive real. Thus we may take $t_1 = 0$ and $t_2 = t$. We compute

$$\left|\frac{z - w}{1 - \bar{z}w}\right| = \left|\frac{r_1 - r_2 e^{it}}{1 - r_1 r_2 e^{it}}\right|.$$

The inequality to be proven becomes

$$|r_1 - r_2 e^{it}| \ge |1 - r_1 r_2 e^{it}||r_1 - r_2|.$$

This follows by direct calculation. Squaring both sides, we obtain

$$(r_1 - r_2 \cos t)^2 + (r_2 \sin t)^2 \ge (r_1 - r_2)^2((1 - r_1 r_2 \cos t)^2 + (r_1 r_2 \sin t)^2).$$

This in turn becomes

$$r_1^2 + r_2^2 - 2r_1 r_2 \cos t \ge (r_1 - r_2)^2(1 + r_1^2 r_2^2 - 2r_1 r_2 \cos t)$$

Fix r_1, r_2. Regarding this as a linear inequality in $\cos t$ which varies between $[-1, 1]$, we see that it suffices to check only the endpoints, when $\cos t = \pm 1$. (Think of two lines segments – how can you see they do not intersect?) When $\cos t = 1$, the inequality to be proven becomes

$$(r_1 - r_2)^2 \ge (r_1 - r_2)^2(1 - r_1 r_2)^2$$

which is true since

$$0 < 1 - r_1 r_2 \leq 1.$$

When $\cos t = -1$, the inequality to be proven is

$$(r_1 + r_2)^2 \geq (r_1 - r_2)^2 (1 + r_1 r_2)^2.$$

Assuming $r_1 \geq r_2$ for convenience, this becomes

$$r_1 + r_2 \geq (r_1 - r_2)(1 + r_1 r_2) \iff 2r_2 \geq (r_1 - r_2) r_1 r_2 \iff 2 \geq (r_1 - r_2) r_1.$$

This is indeed correct, as $r_1 < 1$ and $0 \leq r_1 - r_2 < 1$.

(v) Letting $z = 1/3$ and $w = 1/2$, we get (by part (ii))

$$\left|\frac{4a-1}{4-a}\right| = d\left(a, \frac{1}{4}\right) \leq d\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{5},$$

where $a = f(1/3)$. Thus

$$|20a - 5| \leq |4 - a|.$$

Using the triangle inequality, we see that

$$20|a| - 5 \leq |20a - 5| \leq |4 - a| \leq 4 + |a| \implies |a| \leq \frac{9}{19}.$$

Similarly,

$$5 - 20|a| \leq |5 - 20a| \leq |4 - a| \leq 4 + |a| \implies \frac{1}{21} \leq |a|.$$

Q2. Recall the Cayley transform which is a biholomorphism $\phi : \Delta \to \mathfrak{h}^+$ given by

$$\phi(w) = i \cdot \frac{1-w}{1+w}.$$

To distinguish the notation, we denote d_Δ and $d_{\mathfrak{h}^+}$ be the distance function defined on Δ (in problem 1) and $\mathfrak{h}^+$ respectively.

Observe that for any $z, w \in \Delta$,

$$\begin{aligned} d_{\mathfrak{h}^+}(\phi(z), \phi(w)) &= \left|\frac{\phi(z) - \phi(w)}{\phi(z) - \overline{\phi(w)}}\right| \\ &= \left|\frac{(1-z)(1+w) - (1-w)(1+z)}{(1-z)(1+\bar{w}) + (1-\bar{w})(1+z)}\right| \\ &= \left|\frac{w-z}{1-\bar{w}z}\right| \\ &= d_\Delta(z, w). \end{aligned}$$

Therefore, the Cayley's transformation ϕ exchanges the two distances.

(i) If f is an automorphism of $\mathfrak{h}^+$, then

$$d_{\mathfrak{h}^+}(z, w) = d_{\mathfrak{h}^+}(f(z), f(w)).$$

(ii) If $f : \mathfrak{h}^+ \to \mathfrak{h}^+$ is a holomorphic function, then

$$d_{\mathfrak{h}^+}(f(z), f(w)) \leq d_{\mathfrak{h}^+}(z, w).$$

The above two are restatement of problem 1 (i) and (ii) : Note that for any automorphism (or holomorphic function) $f : \mathfrak{h}^+ \to \mathfrak{h}^+$, there is an automorphism (or holomorphic function) $g : \Delta \to \Delta$ such that $f = \phi \circ g \circ \phi^{-1}$. Thus the statements follow because the distance function is preserved under ϕ. To be precise, when f in case (ii)

$$\begin{aligned} d_{\mathfrak{h}^+}(f(z), f(w)) &= d_{\mathfrak{h}^+}(\phi \circ g \circ \phi^{-1}(z), \phi \circ g \circ \phi^{-1}(w)) = \\ &= d_\Delta(g \circ \phi^{-1}(z), g \circ \phi^{-1}(w)) \\ &\le d_\Delta(\phi^{-1}(z), \phi^{-1}(w)) \\ &= d_{\mathfrak{h}^+}(z, w). \end{aligned}$$

Q3. Consider the function $g : \Delta \to \mathbb{C}$ given by

$$g(z) = \begin{cases} \frac{f(z)}{z^n} & \text{for } z \neq 0 \\ \frac{f^{(n)}(0)}{n!} & \text{for } z = 0 \end{cases}.$$

The function g is holomorphic i.e. the singularity at the origin is removable. This is guaranteed by the Taylor expansion for f. Indeed, we can write

$$f(z) = \sum_{k=n}^{\infty} a_k z^k$$

with radius of convergence at least 1, so the shifted power series

$$\frac{f(z)}{z^n} = \sum_{k=n}^{\infty} a_k z^{k-n}$$

also has radius of convergence at least 1 by the root test (or the ratio test). The sum must define a holomorphic function in $\Delta(0, 1)$, which is obviously equal to g at $z \neq 0$, while at $z = 0$ we obtain the value

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Now fix $z \in \Delta$ and let $|z| < r < 1$. Note that for all $|w| = r$, we have

$$|g(w)| = \frac{|f(w)|}{|w^n|} \le \frac{1}{r^n}.$$

By the maximum modulus principle applied to $\overline{\Delta}(0, r)$ we obtain

$$|g(z)| \le \sup_{|w| \le r} |g(w)| = \sup_{|w| = r} |g(w)| \le \frac{1}{r^n}.$$

Making $r \to 1$, we conclude

$$|g(z)| \le 1.$$

This means $|f(z)| \le |z|^n$ for all z (including $z = 0$ for trivial reasons), and

$$|g(0)| = \frac{|f^{(n)}(0)|}{n!} \le 1 \implies |f^{(n)}(0)| \le n!.$$

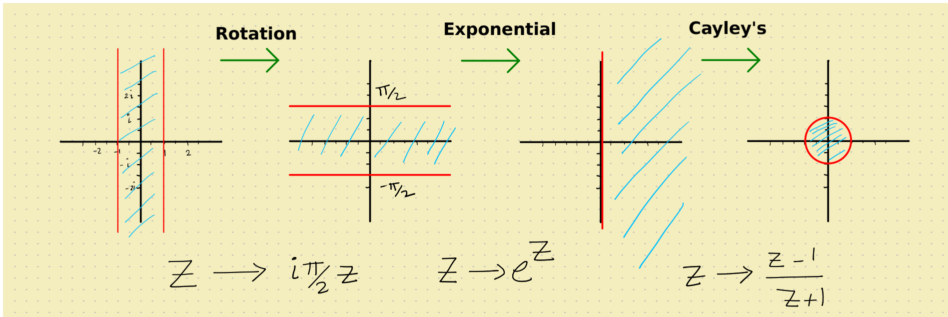

FIGURE 0.1. Biholomorphic map from $\{z : -1 < \mathrm{Re} z < 1\}$ to Δ

Q4.

(a) Suppose p and q are two points in U fixed by f. By Riemann mapping theorem, there exists a biholomorhpism $\psi : U \to \Delta$. Consider the following commuting diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "f"] & \Delta \arrow[d, "g"] \\ U \arrow[r, "\psi"] & \Delta \end{tikzcd}$$

where $g = \psi^{-1} \circ f \circ \psi : \Delta \to \Delta$, and horizontal arrows are biholomorphisms. Since p, q are fixed for f, it follows that $\psi^{-1}(p)$ and $\psi^{-1}(q)$ are fixed for g. Indeed,

$$g(\psi^{-1}(p)) = \psi^{-1} \circ f \circ \psi \circ \psi(p) = \psi^{-1} f(p) = \psi^{-1}(p)$$

and similarly for q. We have shown in class that any automorphism of Δ has at most two fixed points, unless

$$g = \mathbf{1} \implies \psi^{-1} \circ f \circ \psi = \mathbf{1} \implies f = \mathbf{1}.$$

(b) Consider the entire function $f(z) = z^2$. The points 0 and 1 are fixed under f, and f is clearly not the identity map.

(c) Let $U = \mathbb{C} \backslash \{0\}$, it is not simply connected (because integrating $1/z$ around the unit circle defies Cauchy's integral formula). Consider the function

$$f(z) = 1/z.$$

Note that f is a holomorphic map from U to U. Moreover, 1 and -1 are the fixed points since $f(-1) = -1$ and $f(1) = 1$.

Q5. Refer to Figure 0.1. We claim that the function

$$f(z) = \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1}$$

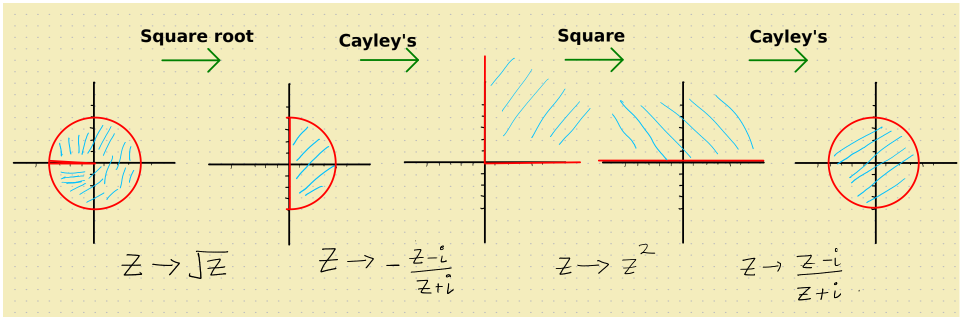

FIGURE 0.2. Biholomorphic map from $\Delta\backslash(-1,0]$ to Δ

is a biholomorphic map from $\{z : -1 < \text{Re}z < 1\}$ to Δ. This follows by noting that f is the composition of $g_1 \circ g_2 \circ g_3$ where the following are biholomorphisms :

$$\begin{aligned} g_1(z) = \frac{z-1}{z+1} &\quad : \quad \{\text{Re}z > 0\} \to \Delta \\ g_2(z) = e^z &\quad : \quad \{\frac{-\pi}{2} < \text{Im}z < \frac{\pi}{2}\} \to \{\text{Re}z > 0\} \\ g_3(z) = \frac{i\pi z}{2} &\quad : \quad \{z : -1 < \text{Re}z < 1\} \to \{\frac{-\pi}{2} < \text{Im}z < \frac{\pi}{2}\}. \end{aligned}$$

Q6. Refer to Figure 0.2. We claim that the function

$$f(z) = \frac{(\sqrt{z}-i)^2 - i(\sqrt{z}+i)^2}{(\sqrt{z}-i)^2 + i(\sqrt{z}+i)^2}$$

is a biholomorphic map from $\Delta\backslash(-1,0]$ to Δ. This follows by noting that f is the composition of $g_1 \circ g_2 \circ g_3 \circ g_4$ where the following are biholomorphisms :

$$\begin{aligned} g_1(z) = \frac{z-i}{z+i} &\quad : \quad \{\text{Im}z > 0\} \to \Delta \\ g_2(z) = z^2 &\quad : \quad \{\text{Im}z > 0, \text{Re}z > 0\} \to \{\text{Im}z > 0\} \\ g_3(z) = -\frac{z-i}{z+i} &\quad : \quad \{|z| < 1, \text{Re}z > 0\} \to \{\text{Im}z > 0, \text{Re}z > 0\} \\ g_4(z) = \sqrt{z} &\quad : \quad \Delta\backslash(-1,0] \to \{|z| < 1, \text{Re}z > 0\}. \end{aligned}$$

The square root map g_4 is defined by taking the standard branch of logarithm and defining

$$\sqrt{z} = e^{\frac{1}{2}\log(z)}.$$

The map g_3 is a biholomorphism because it is the Möbius transformation sending the line $y = 0$ and the circle $|z| = 1$ to the lines $x = 0$ and $y = 0$. The the region enclosed are mapped biholomorphically.