

Math 220, Problem Set 6.

This is a longer problem set so make sure you start early. The following remarks should help you plan ahead.

Problems 1 – 4 only require the statement of the residue theorem given on Friday, Nov 13.

Problems 5 – 7 are slightly more involved though they only require the statement of the residue theorem and estimates over the contour of integration. In principle, there should be no obstacles to starting these questions on Friday, Nov 13 if you wish.

By contrast, the last question concerns applications to real analysis which we will cover only after we give the proof of the residue theorem. This question should be approached last.

1. Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function, let $a \in U$, and assume that $f(a) = 0$, $f'(a) \neq 0$. Show that if r is sufficiently small then

$$\int_{|z-a|=r} \frac{dz}{f(z)} = \frac{2\pi i}{f'(a)}.$$

2. Calculate the integral

$$\int_{|z|=4} \frac{e^z}{(z-1)^2(z-3)^2} dz.$$

3. Assume f is holomorphic in U except for simple poles a_1, \dots, a_n and g is holomorphic in U . Show that

$$\frac{1}{2\pi i} \int_{\gamma} f g dz = \sum_k n(\gamma, a_k) g(a_k) \operatorname{Res}(f, a_k)$$

for all γ nullhomotopic in U avoiding a_i .

4. Calculate the following integral using the residue theorem:

$$\int_0^{2\pi} \cos^{2n} t dt.$$

You may wish to first write this integral in terms of $z = e^{it} dt$.

5. Consider the power series expansion

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \cdot \frac{z^k}{k!}.$$

The expansion holds for $|z| < 2\pi$, and the coefficients B_k are called Bernoulli numbers.

Consider the function

$$f(z) = \frac{1}{z^{2n}(e^z - 1)}.$$

Let γ_m denote the boundary of the rectangle with corners

$$\pm(2m+1)\pi \pm (2m+1)\pi i.$$

Using the residue theorem, compute the integral

$$\int_{\gamma_m} f(z) dz$$

in terms of Bernoulli numbers. Using a suitable estimate of the function f along γ_m , show furthermore that

$$\lim_{m \rightarrow \infty} \int_{\gamma_m} f(z) dz = 0.$$

Use this to confirm the following formula for the values of the Riemann zeta function

$$\sum_{p=1}^{\infty} \frac{1}{p^{2n}} = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!}.$$

6. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2} dz$$

via the residue theorem. Making $n \rightarrow \infty$, show that

$$\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.$$

7. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners

$$\pm \left(n + \frac{1}{2} \right) \pm ni.$$

Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{(z+a)^2} dz$$

via the residue theorem, and use this to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

8. Calculate the following integrals *using the residue theorem*. Carefully explain any estimates you might use.

(i)

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx.$$

Hint: It may be easier to consider the function $\frac{1-e^{2ix}}{x^2}$.

(ii) (updated: will be part of the next problem set)

$$\int_0^\infty \frac{\log x}{(1+x^2)^2} dx.$$

(iii) (updated: will be part of the next problem set)

$$\int_0^\infty \frac{x^\alpha}{1+x^n} dx \text{ where } n > 1 + \alpha > 0, n \geq 2 \text{ integer, } \alpha \in \mathbb{R}.$$