Math 220C - Lecture 1

March 29, 2021

0 Logistics

(1) Zoom lectures — MWF 3 – 3:50 PM.

2^{nd} half — TBD.

(2) Office Hour — W 4 – 5:30 PM

(3) PSets — due Fridays, weekly

(4) Grades — HWK & Attendance

(5) Qualifying Exam — TBA

(6) Canvas / Gradescope / Website

math.ucsd.edu/~dopra/220S21.html

(7) Attendance

Topics for Math 220C

(1) Harmonic Functions — Conway X.

(2) Hadamard Factorization — Conway XI.

(3) Picard's theorems — Conway XII.

Math "220D" (if time)

(4) Introduction to Riemann Surfaces.

$\boxed{1.}$ Harmonic Functions

Theme:

Harmonic functions share many properties with holomorphic functions

$\boxed{i}$ mean value property & Integral formulas

$\boxed{ii}$ maximum modulus principle

$\boxed{iii}$ convergence theorems

& others $\rightsquigarrow$ HWK 1.

"Cauchy" estimates, Liouville, Open Mapping Thm.

Convention: $G \subseteq \mathbb{C}$ open & connected. We will assume this from now on.

Recall $G \subseteq \mathbb{C}$ open & connected

$u : G \longrightarrow \mathbb{R}$ harmonic iff $u \in C^2$ and

$u_{xx} + u_{yy} = 0$. (Laplace equation).

Recall (Harmonic conjugates, Math 220A, Lecture 1).

If $f : U \longrightarrow \mathbb{C}$ holomorphic $\Rightarrow$ $u = \operatorname{Re} f$ harmonic.

$v = \operatorname{Im} f$ harmonic

u, v are said to be harmonic conjugates provided

$f = u + iv$ is holomorphic.

(so that $u = \operatorname{Re} f$, $v = \operatorname{Im} f$). Note that u, v satisfy

the Cauchy Riemann equations

$$u_x = v_y$$

$$u_y = -v_x.$$

Lemma Let G be simply connected.

Any $u : G \longrightarrow \mathbb{R}$ harmonic admits a harmonic conjugate v.

e.g. $f = u + iv$ = holomorphic, $u = \mathrm{Re}\, f$.

Proof Let $F = u_x - i\, u_y$.

Claim F holomorphic

Indeed, F is of class C^1 & satisfies CR equations.

$(u_x)_x = (-u_y)_y \iff u_{xx} + u_{yy} = 0$ true

$(u_x)_y = -(-u_y)_x \iff u_{xy} = u_{yx}$. true

$\Rightarrow$ F holomorphic by Math 220, Lecture 2.

Since G is simply connected, F admits a primitive

$\Rightarrow$ $F = f'$ for f holomorphic, $f = \alpha + i\beta$.

$f' = \alpha_x + i\beta_x = F = u_x - i\, u_y$

$\Rightarrow \alpha_x = u_x$

$\Rightarrow \alpha = u + C$.

$\Rightarrow \beta_x = -u_y = -\alpha_y \Rightarrow \alpha_y = u_y$.

Replacing f by $f - c$, we obtain $u = \operatorname{Re} f$ & $v = \operatorname{Im} f$ is the conjugate of u.

Remark Math 220A, HWK 2

3. Show that the function $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ given by

$$u(z) = \log |z|$$

is harmonic, but it is not the real part of a holomorphic function in $\mathbb{C} \setminus \{0\}$.

Thus the Lemma above fails for G not simply connected.

Corollary

u harmonic $\Longrightarrow$ u is of class C^∞.

Proof

Indeed, the statement is local. Let $a \in G$. Let $\overline{\Delta}(a,r) \subseteq G$.

Since $\Delta(a,r)$ simply connected, $u = \operatorname{Re} f$, f holomorphic in $\Delta(a,r)$.

A holomorphic function is ∞-many times complex differentiable

& thus ∞-many times real differentiable. (Math 220A, Lecture 1).

$\Rightarrow u$ is C^∞.

First Properties of Harmonic Functions

[i] mean value property (MVP)

[ii] maximum principle (MP)

[iii] Poisson integral formula (next time)

Def $u : G \longrightarrow \mathbb{R}$ continuous satisfies MVP if

$\forall\, a \in G$, $\bar{\Delta}(a, r) \subseteq G$.

$$\underbrace{u(a)}_{\text{value at center}} = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} u(a + re^{it})\, dt}_{\text{average values over the boundary.}}$$

Theorem $u : G \longrightarrow \mathbb{R}$ harmonic $\Rightarrow$ u satisfies MVP.

Proof Let $\overline{\Delta}(a, r) \subseteq \Delta(a, R) \subseteq G$ Write

$u = \operatorname{Re} f$, f holomorphic in $\Delta(a, R)$.

Cauchy Integral Formula gives

$$f(a) = \frac{1}{2\pi i} \int_{\partial\Delta(a,r)} \frac{f(z)}{z-a}\, dz.$$

$z = a + r e^{it}$

$dz = r i e^{it} dt$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + r e^{it})}{r e^{it}} \cdot r i e^{it}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a + r e^{it})\, dt.$$

Take real part on both sides & conclude.

Maximum Principle

$u : G \longrightarrow \mathbb{R}$, $u \in \mathcal{C}^0(G)$ satisfies MVP. Assume

$\exists\ a \in G$, $u(a) \geq u(z)$ $\forall z \in G$. Then u is constant.

Proof Let $\Omega = \{z : u(z) = u(a)\} \subseteq G$.

(1) $\Omega \neq \emptyset$ because $a \in \Omega$.

(2) Ω is closed, since u is continuous.

(3) Ω is open.

Then G connected $\Rightarrow \Omega = G \Rightarrow u$ constant.

Proof of (3)

Let $z_0 \in \Omega$. Let $\bar{\Delta}(z_0, r) \subseteq G$. We show $\Delta(z_0, r) \subseteq \Omega$.

Let $w \in \Delta(z_0, r)$. $\Rightarrow \rho = |w - z_0|$. Write MVP. for $\partial\Delta(z_0, \rho)$

$$u(a) = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{it})\, dt.$$

$\Longleftrightarrow \quad \frac{1}{2\pi}\int_0^{2\pi} \left(u(z_0+\rho e^{it}) - u(a)\right) dt = 0 \quad \Rightarrow$

Let $f(t) = u(a) - u(z_0+\rho e^{it})$. By assumption, $f(t) \geq 0$ since a is a maximum for u.

Using the Lemma, we have $f \equiv 0$. Since $|w - z_0| = \rho$, write

$$w = z_0 + \rho e^{it_0} \Rightarrow f(t_0) = u(a) - u(w) = 0 \Rightarrow u(a) = u(w)$$

$\Rightarrow w \in \Omega \quad \Rightarrow \quad \Delta(z_0, r) \subseteq \Omega \quad \Rightarrow \Omega$ open.

Lemma $f : [0, 2\pi] \longrightarrow \mathbb{R}$, $f \geq 0$ and f continuous

$$\int_0^{2\pi} f(t)\, dt = 0 \Rightarrow f \equiv 0.$$

Proof If $f(t_0) > 0$, by continuity we can find $\delta > 0$ such that $f(t) > \frac{f(t_0)}{2}$ $\forall\, t \in (t_0 - \delta, t_0 + \delta) \cap [0, 2\pi]$.

Assume $t_0 \neq 0, 2\pi$ since the proof is similar in those cases.

Then $f \geq 0$ gives

$$0 = \int_0^{2\pi} f(t)\,dt \geq \int_{t_0-\delta}^{t_0+\delta} f(t)\,dt > \int_{t_0-\delta}^{t_0+\delta} \frac{f(t_0)}{2}\,dt = \delta f(t_0) > 0.$$

contradiction. Thus $f \equiv 0$.

Remark

[i] u harmonic $\Rightarrow$ u satisfies maximum principle

[ii] u harmonic $\Rightarrow$ $-u$ harmonic

$\Rightarrow$ $-u$ satisfies maximum principle

$\Rightarrow$ u satisfies minimum principle

Georg Friedrich Bernhard Riemann

17 September 1826 – 20 July 1866

Eine harmonische Function u kann nicht in einem Punkt im Innern ein Minimum oder ein Maximum haben, wenn sie nicht uberall constant ist.

(A harmonic function u cannot have either a minimum or a maximum at an interior point unless it is constant.)

"Grundlagen fur eine allgemeine Theorie der Functionen einer veranderlichen complexen Grosse”

Dissertation Gottingen (1851)

Math 220C - Lecture 2

March 31, 2021

Last time

Mean value Property

$$\forall\, a \in G,\ \bar{\Delta}(a,r) \subseteq G,\ u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})\,dt.$$

Maximum Principle

$u : G \longrightarrow \mathbb{R}$ continuous & MVP $\Rightarrow$

u cannot achieve a maximum (minimum) in G.

Notation $\partial_\infty G$ = extended boundary in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

$$\partial_\infty G = \begin{cases} \partial G, & G \text{ bounded} \\ \partial G \cup \{\infty\}, & G \text{ unbounded} \end{cases}$$

A stronger version (MP^+)

(1) $u : G \longrightarrow \mathbb{R}$, u satisfies MVP in G, u continuous

(2) $\forall\, a \in \partial_\infty G$: $\limsup\limits_{z \to a} u(z) \leq 0$.

Then either $u < 0$ in G or $u \equiv 0$ in G.

Proof We will show $u \leq 0$ in G. By the usual MP, u cannot have a maximum in G unless u = constant. This gives the statement we seek. Indeed, if $\exists\, \alpha \in G$ with $u(\alpha) = 0 \Rightarrow \alpha$ maximum in $G \Rightarrow u \equiv 0$. Else $u(\alpha) < 0$, $\forall\, \alpha \in G$.

Thus $u \equiv 0$ or $u < 0$ in G.

To show $u \le 0$, assume that $\exists\, \alpha \in G$ with $u(\alpha) > 0$.

Let $\varepsilon = u(\alpha) > 0$.

Let $K = \{ z \in G : u(z) \ge \varepsilon \}$. Since $\alpha \in K \Rightarrow K \ne \emptyset$.

Claim K is compact.

Assuming this, u cont., u will achieve a maximum in K at z_0. In particular $u(z_0) \ge \varepsilon$. Outside of K, $u < \varepsilon$. Thus z_0 will achieve a maximum for in u in G. This shows u constant

Condition (2) ensures u = constant ≤ 0.

Proof of claim Let $z_n \in K$. We show that passing to a subseq. z_n converges in K. Note $z_n \in \hat{\mathbb{C}}$ & $\hat{\mathbb{C}}$ is compact. Thus WLOG we may assume $z_n \longrightarrow z \in \hat{\mathbb{C}}$ after passing to a subsequence.

Note $u(z_n) \ge \varepsilon$. If $z \in G \Rightarrow u(z) = \lim u(z_n) \ge \varepsilon$. $\Rightarrow z \in K$. as needed. Else $z \in \partial_\infty G$. Then

$$\limsup_{z_n \to z} u(z_n) \ge \varepsilon \quad \text{which contradicts (2).}$$

Thus K is compact.

Corollary G bounded, $u : \overline{G} \longrightarrow \mathbb{R}$ cont., MVP.

$u \equiv 0$ on ∂G $\Rightarrow$ $u \equiv 0$ in G.

Proof We use MP^+. We need to verify condition (2).

G bounded, $\partial_\infty G = \partial G$. If $a \in \partial G$, $\lim_{z \to a} u(z) = \underbrace{u(a)}_{\text{continuity in } \overline{G}} = 0$.

Thus $u < 0$ in G or $u \equiv 0$ in G.

Argue in the same way for $-u$. $\Rightarrow$ either $-u < 0$ in G or $-u \equiv 0$ in G. Thus $u \equiv 0$ in G.

Remark $u, v : \overline{G} \longrightarrow \mathbb{R}$ continuous & harmonic in G. & G bounded. If

$$u/_{\partial G} = v/_{\partial G} \Rightarrow u = v \text{ in } G.$$

Thus $u/_{\partial G} \rightsquigarrow u$ in G uniquely.

§2. Poisson Formula & Dirichlet Problem

Question 1 $u : \overline{G} \longrightarrow \mathbb{R}$ continuous, harmonic in G, G bounded.

$$u\big/_{\partial G} \rightsquigarrow u \quad \text{uniquely in } G.$$

Find a formula for u in G, from the values $u\big/_{\partial G}$.

We will solve this for $G = \Delta(0,1)$. or $\Delta(a, R)$. $\rightsquigarrow$ Poisson Formula

Question 2 Given $f : \partial G \longrightarrow \mathbb{R}$ continuous, is there

$u : \overline{G} \longrightarrow \mathbb{R}$ continuous and

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 \\ u\big/_{\partial G} = f \end{cases}$$

Dirichlet Problem

(boundary value problem)

Harmonic Functions on the unit disc $\Delta = \Delta(0,1)$

Given $u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ,

find a formula for $u(a)$ in terms of $u\big/_{\partial\Delta}$.

Remark $a = 0$ Use MVP over the circle $|z| = r$, $r < 1$.

This smaller circle is contained in Δ, where u satisfies MVP.

then

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})\,dt$$

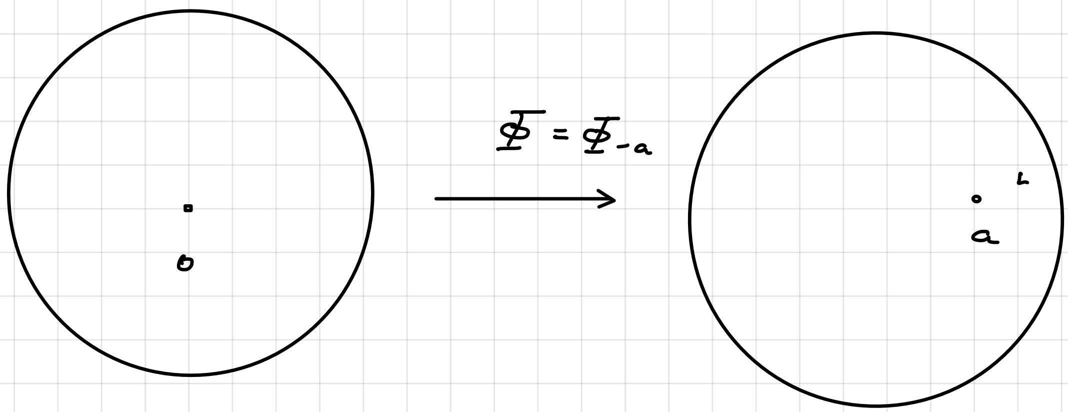
Since u continuous over $\overline{\Delta}$, make $r \longrightarrow 1$. This yields

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\,dt.$$

(To justify the limit use that $u(re^{it}) \longrightarrow u(e^{it})$ uniformly since u is uniformly cont. over $\overline{\Delta}$).

Question: How about the case $a \neq 0$?

General Case



Idea : Recenter! $\Phi : \Delta \longrightarrow \Delta,\ \partial\Delta \longrightarrow \partial\Delta$

$$\Phi(z) = \frac{z+a}{1+\bar{a}z}, \quad \Phi(0) = a.$$

Then $\tilde{u} = u \circ \Phi : \bar{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ (Problem 1, HWK1)

Apply MVP to $\tilde{u}$

$$u(a) = \tilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(e^{is})\, ds = \frac{1}{2\pi}\int_0^{2\pi} u(\Phi(e^{is}))\, ds.$$

Since $\Phi(e^{is}) \in \partial\Delta$ this also shows $u(a)$ is given explicitly in terms of $u|_{\partial\Delta}$.

Next time : We will work out a more explicit expression

$\Longrightarrow$ Poisson Integral Formula

Slogan

MVP + Aut Δ $\Longrightarrow$ Poisson's formula

Math 220C - Lecture 3

April 2, 2021

Last time $u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ.

$\boxed{I}$ $$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{is})\, ds$$ Mean Value Property

$\boxed{II}$ $$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(\Phi(e^{is}))\, ds$$ MVP + Aut Δ to recenter.

where $\Phi : \Delta \longrightarrow \Delta$, $\partial\Delta \longrightarrow \partial\Delta$, $z \longrightarrow \dfrac{z+a}{1+\bar{a}z}$

with inverse $\varphi : \Delta \longrightarrow \Delta$, $\partial\Delta \longrightarrow \partial\Delta$, $z \longrightarrow \dfrac{z-a}{1-\bar{a}z}$

Goal Make formula $\boxed{II}$ even more explicit.

Poisson Kernel

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \cos n\theta, \quad 0 \le r < 1 \quad \text{well defined.}$$

Three additional Formulas

$\boxed{a}$ $$P_r(\theta) = \mathrm{Re}\, \frac{1+z}{1-z}, \quad z = re^{i\theta}.$$

$$\frac{1+z}{1-z} = 1 + \frac{2z}{1-z} = 1 + 2z(1+z+z^2+\dots)$$

$$= 1 + 2\sum_{n=1}^{\infty} z^n = 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta}$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n (\cos n\theta + i \sin n\theta).$$

$$\mathrm{Re}\, \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta$$

$$= 1 + \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta}).$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = P_r(\theta)$$

$\boxed{b}$ $$P_r(\theta) = \frac{1-|z|^2}{|1-z|^2}.$$

$\boxed{a}$

$$P_r(\theta) = \mathrm{Re}\,\frac{1+z}{1-z} = \mathrm{Re}\,\frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})}$$

imaginary

$$= \mathrm{Re}\,\frac{1-z\bar{z}+z-\bar{z}}{|1-z|^2}$$

$$= \frac{1-|z|^2}{|1-z|^2}.$$

$\boxed{c}$ $$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$ very useful.

Indeed use $\boxed{b}$ for $z = re^{i\theta}$:

$$|1-z|^2 = (1-r\cos\theta)^2 + (r\sin\theta)^2$$

$$= 1+r^2-2r\cos\theta \quad \& \quad 1-|z|^2 = 1-r^2.$$

Poisson's Formula

$u : \bar{\Delta} \longrightarrow \mathbb{R}$ continuous & harmonic in Δ , $a = re^{i\theta}$

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, u(e^{it})\, dt.$$

Proof Recall

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(\Phi(e^{is}))\, ds$$

Change of variables $e^{is} = \psi(e^{it})$

Main Claim $ds = P_r(\theta - t)\, dt$

the Poisson kernel arises via change of variables

Assuming this, we obtain

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(\Phi\, \psi(e^{it})) \cdot P_r(\theta - t)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\, P_r(\theta - t)\, dt. \text{, as needed.}$$

Proof of the Main claim

$$ds = \frac{d(e^{is})}{i\, e^{is}} = \frac{d\,\varphi(e^{it})}{i\,\varphi(e^{it})} \overset{\text{chain rule}}{=} \frac{\varphi'(e^{it})\; i\; e^{it}\,dt}{i\;\varphi(e^{it})} = \frac{\varphi'(z)\,z}{\varphi(z)}\,dt$$

Recall $\varphi(z) = \dfrac{z-a}{1-\bar{a}z}$. Taking logarithmic derivatives

$$z \cdot \frac{\varphi'(z)}{\varphi(z)} = \frac{z}{z-a} + \frac{\bar{a}z}{1-\bar{a}z}$$

$$= \frac{z}{z-a} - \frac{1}{2} + \frac{1}{2} + \frac{\bar{a}z}{1-\bar{a}z}$$

$$= \frac{1}{2}\cdot\frac{z+a}{z-a} + \frac{1}{2}\cdot\frac{1+\bar{a}z}{1-\bar{a}z}.$$

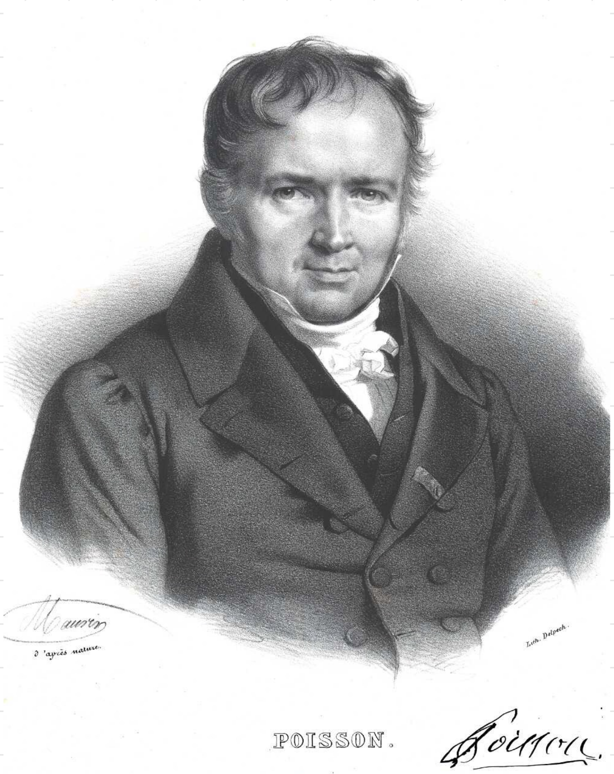
$1 = z\bar{z}$

$$= \frac{1}{2}\cdot\frac{z+a}{z-a} + \frac{1}{2}\cdot\frac{z\bar{z}+\bar{a}z}{z\bar{z}-\bar{a}z}$$

$$= \frac{1}{2}\cdot\frac{z+a}{z-a} + \frac{1}{2}\cdot\frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}$$

$$= \operatorname{Re}\frac{z+a}{z-a} = \operatorname{Re}\frac{1+\frac{a}{z}}{1-\frac{a}{z}} = P_r(\theta-t)$$

using $\boxed{a}$ & $\dfrac{a}{z} = \dfrac{r\,e^{i\theta}}{e^{it}} = r\,e^{i(\theta-t)}$



Siméon Poisson

(1781 – 1840)

Students:

Liouville, Carnot, Dirichlet

Poisson Kernel

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

$$= \mathrm{Re}\,\frac{1+z}{1-z} = \frac{1-|z|^2}{|1-z|^2} \quad \text{for } z = re^{i\theta}.$$

Poisson integral Formula

$u : \bar{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ. Then

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t)\, u(e^{it})\, dt$$

Remark We can dilate & translate to work with any disc $\Delta(a, R)$.

Theorem $u : \overline{\Delta}(a, R) \longrightarrow \mathbb{R}$ continuous & harmonic in $\Delta(a, R)$.

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} \, u(a + Re^{it}) \, dt$$

Proof

$$\tilde{u} : \overline{\Delta} \longrightarrow \mathbb{R}, \quad \tilde{u}(z) = u(a + Rz)$$

We apply the previous result to $\tilde{u}$. Then

$$u(a + re^{i\theta}) = \tilde{u}\left(\frac{r}{R} e^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} P_{\frac{r}{R}}(\theta - t) \, \tilde{u}(e^{it}) \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \left(\frac{r}{R}\right)^2}{1 - 2\frac{r}{R}\cos(\theta - t) + \left(\frac{r}{R}\right)^2} \, u(a + Re^{it}) \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} \, u(a + Re^{it}) \, dt.$$

Two Consequences

[i] Schwarz Integral Formula

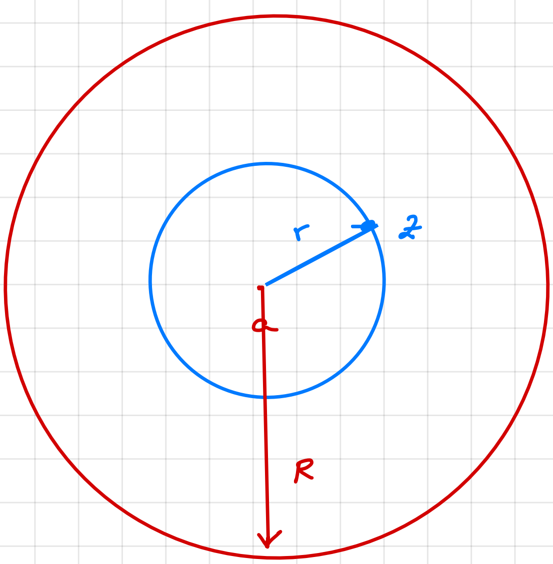
[ii] Harnack Inequality

Harnack's Inequality

$u : \bar{\Delta}(a,R) \longrightarrow \mathbb{R}$ continuous, harmonic in $\Delta(a,R)$. & $u \geq 0$.

If $|z-a| = r \implies$

$$u(a)\cdot\frac{R-r}{R+r} \leq u(z) \leq u(a)\,\frac{R+r}{R-r}$$



Proof

We use $-1 \leq \cos(\theta - t) \leq 1$.

The two inequalities are similar. For instance, 2^{nd} inequality:

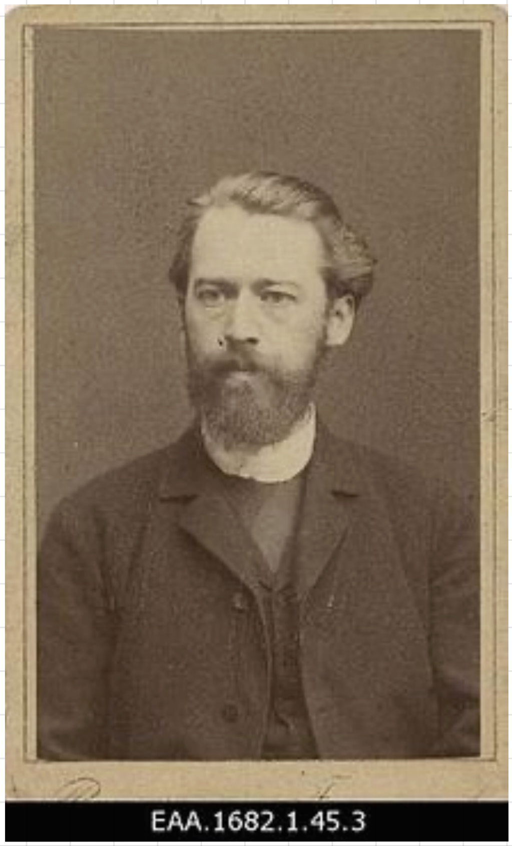
$$u(a + r e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} \, u(a + R e^{it}) \, dt$$

$u \geq 0$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr + r^2} \cdot u(a + R e^{it}) \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\cancel{(R - r)}(R + r)}{(R - r)^{\cancel{2}}} \cdot u(a + R e^{it}) \, dt$$

$$= u(a) \frac{R + r}{R - r}$$ using Mean Value Property.



Axel Harnack (1851-1888) was a Baltic-German mathematician.

He proved Harnack's inequality for harmonic functions & Harnack's curve theorem in real algebraic geometry.

Math 220C - Lecture 4

April 5, 2020

Last time

$u : \bar{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, u(e^{it})\, dt \qquad (\text{Poisson})$$

$a = r e^{i\theta}$

Poisson Kernel

$$P_r(\theta) = \operatorname{Re} \frac{1 + r e^{i\theta}}{1 - r e^{i\theta}}$$

$$= \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

Applications

Harnack Inequality

Schwarz Integral Formula

Schwarz Integral Formula

$u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ

We have seen $u = \operatorname{Re} f$, f holomorphic in Δ.

Question Is there a formula for f?

$$f : \Delta \longrightarrow \mathbb{C}, \quad f(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z+a}{z-a}\, u(z) \frac{dz}{z}$$

Claims

(1) f holomorphic in Δ.

(2) $u = \operatorname{Re} f$

Proof of (1)

Key Fact (Math 220A, Homework 3, Problem 7).

Continuous $\Phi : \{\gamma\} \times U \longrightarrow \mathbb{C}$ holomorphic in a

then $a \longrightarrow \int_\gamma \Phi(z, a)\, dz$ holomorphic

Apply this to $\Phi : \partial\Delta \times \Delta \longrightarrow \mathbb{C}$, $\Phi(z, a) = \dfrac{z+a}{z-a} \dfrac{u(z)}{z}$.

which is continuous & holomorphic in a to conclude.

$$f(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z+a}{z-a}\, u(z) \frac{dz}{z} \quad \text{is holomorphic in } \Delta.$$

Proof of (2)

By definition, we have

$$f(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z+a}{z-a} u(z) \frac{dz}{z} \qquad \swarrow z = e^{it}$$

$$= \frac{1}{2\pi \cancel{i}} \int \frac{1+a/z}{1-a/z} u(e^{it}) \cancel{i} \, dt$$

$$\Rightarrow \operatorname{Re} f(a) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{1+a/z}{1-a/z} \cdot u(e^{it}) \, dt \qquad a/z = r e^{i(\theta - t)}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) \, u(e^{it}) \, dt$$

$$= u(a). \Rightarrow u = \operatorname{Re} f.$$

In the last line we applied Poisson's formula for u.

Hermann Schwarz (1843 - 1921)

Schwarz Lemma, Schwarz Integral Formula

Schwarz Reflection Principle, Cauchy - Schwarz Inequality

Adivor : Weierstraß, Kummer

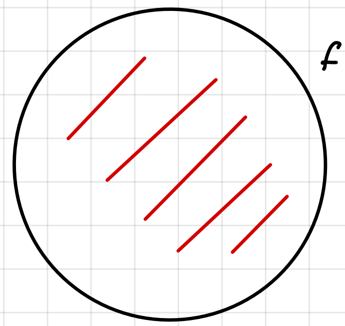
Students: Fejer, Koebe, Zermelo

Dirichlet Problem (for the unit disc)

Given $f : \partial\Delta \longrightarrow \mathbb{R}$ continuous, is there $u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous

(1) u harmonic in Δ

(2) $u\big|_{\partial\Delta} = f$



Answer Yes. Define $u : \overline{\Delta} \longrightarrow \mathbb{R}$ by

$$u(re^{i\theta}) = \begin{cases} f(e^{i\theta}) & , r = 1. \\ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} P_r(\theta - t)\, f(e^{it})\, dt, & r < 1. \end{cases}$$

We need to show

(1) u harmonic in Δ

(2) u continuous in $\overline{\Delta}$

$$\lim_{\substack{r \to 1 \\ \theta \to \theta_0}} \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, f(e^{it})\, dt = f(e^{i\theta_0}).$$

Johann Peter Gustav Lejeune Dirichlet (1805 – 1859)

It was his father who first went under the name “Lejeune Dirichlet” (meaning “the young Dirichlet”) in order to differentiate from his father, who had the same first name.

“Dirichlet” (or “Derichelette”) means “from Richelette” after a town in Belgium.

Proof of (1)

We claim that u is harmonic in Δ. Recall that

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, f(e^{it})\, dt \,, \quad a \in \Delta$$

Let

$$g(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z+a}{z-a} \cdot f(z)\, \frac{dz}{z} .$$

We have argued in the proof of Schwarz, g is holomorphic in Δ & $\mathrm{Re}\, g = u$. Thus u is harmonic in Δ.

Proof of (2)

Properties of the Poisson kernel

Lemma

$\boxed{i}$ $P_r(t) \geq 0$, even in t, 2π-periodic in t.

$\boxed{ii}$ $$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,dt = 1.$$

$\boxed{iii}$ $P_r \rightrightarrows 0$ as $r \to 1$, over the domain $\delta \leq |t| \leq \pi$ $\forall\, \delta > 0$.

Proof $\boxed{i}$ is clear

$\boxed{ii}$ Take $u \equiv 1$, $a = re^{i\cdot 0}$ in Poisson's formula

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(-t)\,dt,$$ which is what we need.

$\boxed{iii}$ To prove uniform convergence, we show

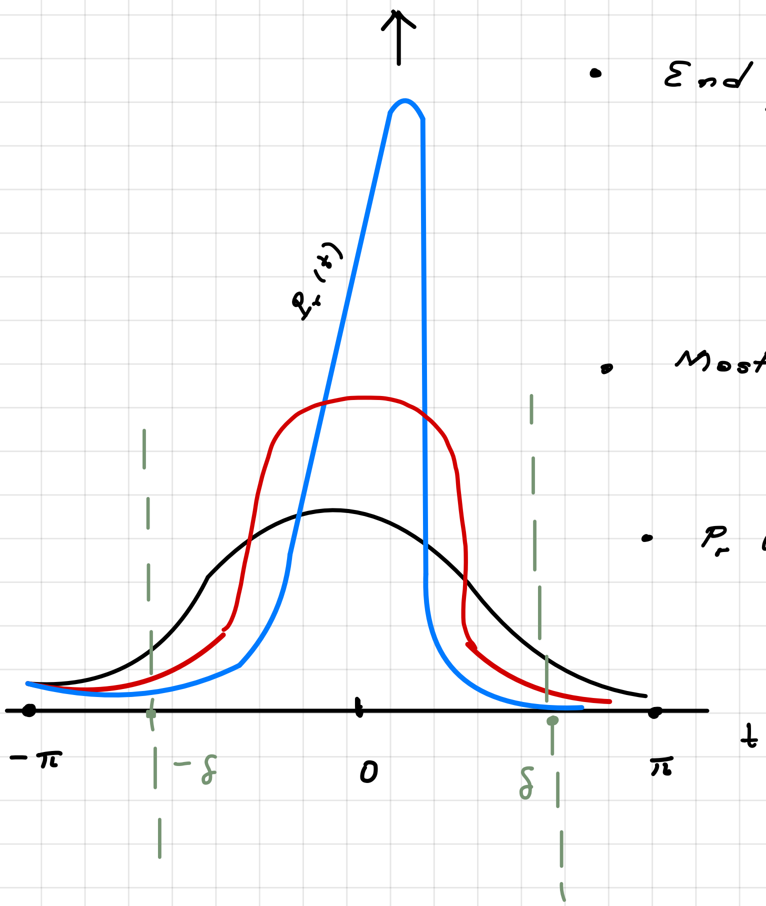
$$\sup_{\delta \leq t \leq \pi} |P_r(t)| \longrightarrow 0 \quad \text{as } r \to 1.$$

Note that P_r is decreasing in $t \in [\delta, \pi]$. Then

$$\sup_{\delta \leq t \leq \pi} P_r(t) = P_r(\delta) = \frac{1-r^2}{1-2r\cos\delta + r^2} \longrightarrow 0 \quad \text{as } r \to 1.$$

Heuristics

- Area under the graph: is 1. by $\boxed{\text{cc}}$



- End points: $P_r(t) \longrightarrow 0$ as $r \to 1$

  for $t \in [\delta, 1]$.

- Most area concentrated in the middle

- $P_r(0) = \dfrac{1+r}{1-r} \xrightarrow[r \to 1]{} \infty$.

"Conclusion"

$$\frac{1}{2\pi} P_r(t)\, dt \longrightarrow \delta_0 = \delta\text{- function concentrated at } 0.$$

In our case

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{P_r(\theta - t)}_{\delta_0 \to \theta = t.} f(e^{it})\, dt \qquad r \to 1.$$

"$\longrightarrow$" $f(e^{i\theta})$. so we do expect continuity.

We will prove this rigurously next time.

Convolution Product

For functions $g, h : [-\pi, \pi] \longrightarrow \mathbb{R}$ continuous, set

$$g * h(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta - t)\, h(t)\, dt.$$

If we write $u_r(\theta) = u(re^{i\theta})$ and write $f(t)$ instead of $f(e^{it})$,

we obtain

$u_r = P_r * f$. Thus we defined the solution to the

Dirichlet problem as a convolution.

Math 220C - Lecture 5

April 7, 2021

Last time (Dirichlet Problem)

Given $f : \partial\Delta \longrightarrow \mathbb{R}$ continuous, define

$$u(re^{i\theta}) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{it})\, dt, & r < 1 \\ f(e^{i\theta}), & r = 1 \end{cases}$$

We have seen u harmonic in Δ & $u\big/_{\partial\Delta} = f$.

We show u continuous in $\bar{\Delta}$.

Conclusion u solves the Dirichlet Problem in $\Delta = \Delta(0,1)$.

Theorem $u : \overline{\Delta} \longrightarrow \mathbb{R}$ is continuous.

Proof The only issue is continuity over $\partial\Delta$ since u is continuous in Δ, being harmonic. We show

$$\lim_{\substack{r \to 1 \\ \theta \to \theta_0}} u(re^{i\theta}) = f(e^{i\theta_0}) \quad \forall\ \theta_0 .$$

Claim WLOG $\theta_0 = 0$

Else, rotate! Let

$\tilde{f}(z) = f(e^{i\theta_0} z)$. Let $\tilde{u}$ be the similar function with $\tilde{f}$ instead of f. By the explicit integral & change of variables

$$\tilde{u}(z) = u(e^{i\theta_0} z).$$

Thus u continuous at $\theta_0 \iff \tilde{u}$ is continuous at 1.

Let $\theta_0 = 0$ from now on.

Fix $\varepsilon > 0$. We show $\exists\, \rho, \delta > 0$ such that

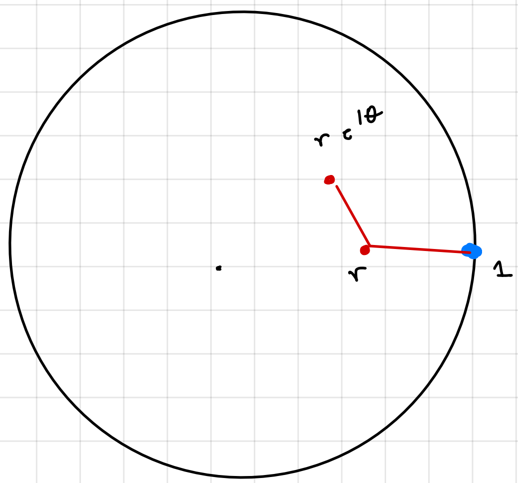
(1) $|u(re^{i\theta}) - u(r)| < \varepsilon$ if $|\theta| < \delta$, all r.

(2) $|u(r) - f(1)| < 2\varepsilon$ if $\rho < r \leq 1$.

Therefore (1) + (2), & triangle inequality gives

$$|u(re^{i\theta}) - f(1)| < 3\varepsilon \quad \forall\ |\theta| < \delta, \quad \rho < r \leq 1.$$

$$\Longrightarrow \lim_{\substack{r \to 1 \\ \theta \to 0}} u(re^{i\theta}) = f(1) \text{ as needed.}$$



Proof of (1)

Since $f : \partial\Delta \longrightarrow \mathbb{R}$ uniformly continuous, let δ such that

$$|x-y|<\delta \implies |f(e^{ix}) - f(e^{iy})| < \varepsilon. \quad (*)$$

We estimate

$$|u(re^{i\theta}) - u(r)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} P_r(\theta-t)\, f(e^{it})\,dt - \int_{-\pi}^{\pi} P_r(-t)\, f(e^{it})\,dt\right|$$

$$= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} P_r(-t)\, f(e^{it+i\theta})\,dt - \int_{-\pi}^{\pi} P_r(-t)\, f(e^{it})\,dt\right|$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(-t)\, \underbrace{|f(e^{it+i\theta}) - f(e^{it})|}_{<\varepsilon \text{ if } |\theta|<\delta \text{ by } (*)}\,dt$$

$$\le \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(-t)\,dt}_{1 \text{ by Lecture 4}} \cdot \varepsilon = \varepsilon$$

Proof of (2)

$$|u(r) - f(1)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t) f(e^{it})\, dt - f(1) \right|$$

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t) f(e^{it})\, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t) f(1)\, dt \right|$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t) \left| f(e^{it}) - f(1) \right| dt$$

If $|t| < \delta \Rightarrow |f(e^{it}) - f(1)| < \varepsilon$. by (*)

$$\frac{1}{2\pi} \int_{-\delta}^{\delta} P_r(-t) \left| f(e^{it}) - f(1) \right| dt \leq \varepsilon \cdot \frac{1}{2\pi} \int_{-\delta}^{\delta} P_r(-t)\, dt$$

$$\leq \varepsilon \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t)\, dt = \varepsilon. \quad \text{(Lecture 4)}$$

If $|t| \geq \delta$. Since f continuous $\Rightarrow |f| \leq M$ over $\partial\Delta$.

$$\frac{1}{2\pi} \int_{|t|>\delta} P_r(-t) \underbrace{\left| f(e^{it}) - f(1) \right|}_{2M} dt \leq \frac{2M}{2\pi} \int_{|t|>\delta} \underbrace{P_r(-t)}_{\frac{\varepsilon}{2M}} dt$$

$$\leq \frac{2M}{2\pi} \cdot \frac{\varepsilon}{2M} \cdot 2\pi = \varepsilon$$

We used that

$P_r(\pm t) \rightrightarrows 0$ as $r \longrightarrow 1$, in $[\delta, \pi]$ by Lecture 4. Thus $\exists\, \rho$

$$P_r(\pm t) < \frac{\varepsilon}{2M} \quad \forall\, t \in [\delta, \pi] \text{ and } \rho \leq r \leq 1.$$

Thus $|u(r) - f(1)| < 2\varepsilon. \quad \forall\, \rho \leq r \leq 1.$

Corollary The Dirichlet Problem can be solved in any disc $\Delta(a, R)$.

Why? This follows via translation & dilation

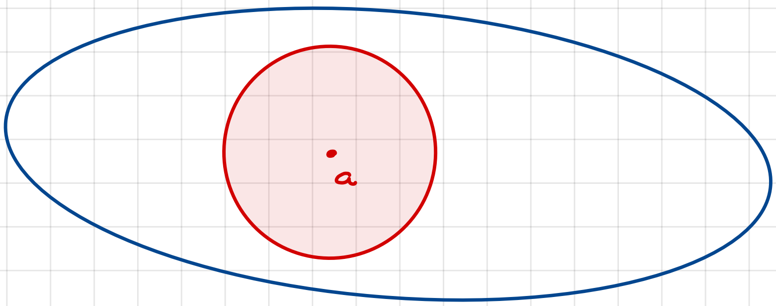
$$z \longrightarrow a + Rz.$$

mapping $\Delta(0,1) \longrightarrow \Delta(a,R)$. We solved the case of $\Delta(0,1)$. above.

Corollary (Converse to MVP)

If $u : G \longrightarrow \mathbb{R}$ continuous & satisfies MVP $\Rightarrow$ u harmonic

Proof



Let $a \in G$. Let $\bar{\Delta}(a, R) \subseteq G$. We show u harmonic in $\Delta(a, R)$.

Let $f = u\big/_{\partial \Delta(a,R)}$. Solve Dirichlet Problem in $\bar{\Delta}(a,R)$.

Thus h harmonic in $\Delta(a,R)$, continuous in $\bar{\Delta}(a,R)$. &

$h\big/_{\partial\Delta(a,R)} = f.$

Let $\Phi = h - u : \bar{\Delta}(a, R) \longrightarrow \mathbb{R}$. $\Rightarrow$ $\Phi\big/_{\partial\Delta(a,R)} = 0$ &

Φ continuous & satisfies MVP (because h, u do). Then $\Phi \equiv 0$

by Corollary to MP^{+} (Lecture 2). Thus $u = h =$ harmonic in $\Delta(a, R)$.

II. Convergence of harmonic functions   Conway X.2.

The natural notion of convergence for harmonic functions is local uniform convergence.

Lemma

If $u_n : G \longrightarrow \mathbb{R}$ harmonic & $u_n \overset{l.u.}{\rightrightarrows} u$ then $u : G \to \mathbb{R}$ harmonic.

Proof Since u_n harmonic $\Rightarrow$ u_n continuous $\Rightarrow$ u continuous.

Since u_n harmonic $\Rightarrow$ u_n satisfies MVP. Let $\overline{\Delta}(a, R) \subseteq G$.

$$u_n(a) = \frac{1}{2\pi} \int_0^{2\pi} u_n(a + Re^{it})\, dt$$

Make $n \longrightarrow \infty$.

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{it})\, dt$$

$\Rightarrow$ u satisfies MVP. $\Rightarrow$ u harmonic.

We have stronger results

Harnack's Theorem Let $u_n : G \longrightarrow \mathbb{R}$ harmonic. and

$u_1 \leq u_2 \leq \dots \leq u_n \leq \dots$ in G. Then either

(1) $u_n \overset{\ell.u.}{\rightrightarrows} u$ & u harmonic. or

(2) $u_n \overset{\ell.u.}{\rightrightarrows} \infty$.

Remark If $a_1 \leq a_2 \leq \dots \leq a_n \leq \dots$ are real numbers, then

(1) $a_n \longrightarrow a$ where $a = \sup\limits_n a_n < \infty$ or .

(2) else $a_n \longrightarrow \infty$

Math 220C - Lecture 6

April 9, 2021

1. Harnack's Theorem Let $u_n : G \longrightarrow \mathbb{R}$ harmonic and

$u_1 \leq u_2 \leq \dots \leq u_n \leq \dots$ in G. Then either

(1) $u_n \overset{\ell.u.}{\rightrightarrows} u$ & u harmonic. or

(2) $u_n \overset{\ell.u.}{\rightrightarrows} \infty$.

Remark If $a_1 \leq a_2 \leq \dots \leq a_n \leq \dots$ are real numbers,

then either

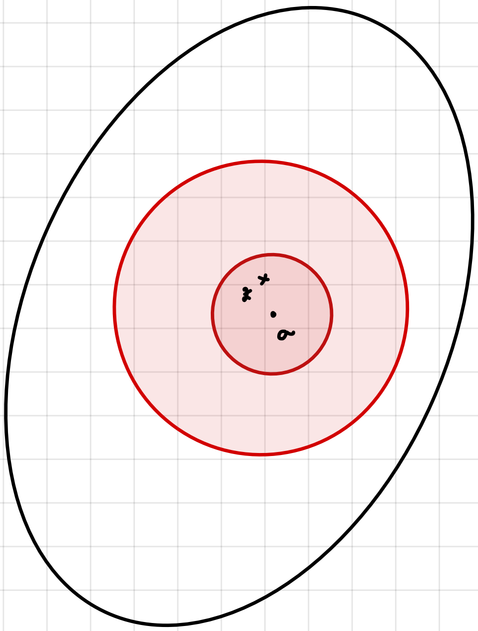
(1) $a_n \longrightarrow a < \infty$

(2) $a_n \longrightarrow \infty$

Remark Harnack Inequality (Lecture 3)

$v : G \longrightarrow \mathbb{R}$, $v \geq 0$, harmonic, $\overline{\Delta}(a, R) \subseteq G$.

If $|z - a| = r < R$, then



$$v(a) \frac{R-r}{R+r} \leq v(z) \leq v(a) \frac{R+r}{R-r}$$

If $r \leq R/2$. $\Rightarrow$ $\frac{R+r}{R-r} \leq 3$, $\frac{1}{3} \leq \frac{R-r}{R+r}$.

If $z \in \Delta\left(a, \frac{R}{2}\right)$ then $\frac{1}{3} v(a) \leq v(z) \leq 3 v(a)$.

Proof of Harnack's theorem

WLOG $u_n \geq 0$. else work with $\tilde{u}_n = u_n - u_1 \geq 0$

Step 1 Pointwise convergence.

Since $\{u_n(z)\}$ is non decreasing $\forall z \in G \Rightarrow$

$\Rightarrow$ either $u_n(z) \longrightarrow \infty$ or $u_n(z) \longrightarrow u(z)$ for some $u(z) < \infty$.

Let $A = \{z \in G : u_n(z) \longrightarrow \infty\}$ $\Rightarrow A \cap B = \emptyset$, $A \cup B = G$.

$B = \{z \in G : u_n(z) \longrightarrow u(z)\}$

It suffices to show A, B open. Since G connected $\Rightarrow$

$A = G$ or $B = G$.

Let $a \in G$. Let $\bar{\Delta}(a, R) \subseteq G$. Let $z \in \bar{\Delta}(a, \frac{R}{2})$.

$$\Rightarrow \frac{1}{3} u_n(a) \leq u_n(z) \leq 3 \, u_n(a).$$

$\boxed{i}$ If $a \in A \Rightarrow u_n(a) \longrightarrow \infty$. $\Rightarrow u_n(z) \longrightarrow \infty$. $\Rightarrow z \in A$

$\Rightarrow \Delta(a, \frac{R}{2}) \subseteq A$. $\Rightarrow A$ open

$\boxed{ii}$ If $a \in B \Rightarrow u_n(a) \longrightarrow u(a) < \infty$. $\Rightarrow u_n(z) \longrightarrow u(z) < \infty \Rightarrow z \in B$.

$\Rightarrow \Delta(a, \frac{R}{2}) \subseteq B$. $\Rightarrow B$ open.

Step 2 Local uniform convergence

Let $a \in G$. Let $\bar{\Delta}(a, R) \subseteq G$. We show uniform convergence in $\Delta(a, \frac{R}{2})$. We have two cases:

$\boxed{i}$ $u_n(a) \longrightarrow \infty$. $\Rightarrow$ $\forall M \; \exists N$: $u_n(a) \geq 3M$ for $n \geq N$

$$\Rightarrow u_n(z) \geq \frac{1}{3} u_n(a) \geq M \quad \forall n \geq N, \; z \in \Delta\left(a, \frac{R}{2}\right).$$

$$\Rightarrow u_n \rightrightarrows \infty \quad \text{in} \quad \Delta\left(a, \frac{R}{2}\right).$$

$\boxed{ii}$ $u_n(a) \longrightarrow u(a)$. Fix $\varepsilon > 0$. Since $\{u_n(a)\}$ Cauchy

$$\Rightarrow \exists N : \; 0 \leq u_n(a) - u_m(a) < \frac{\varepsilon}{3} \quad \forall n \geq m \geq N$$

$$\Rightarrow 0 \leq u_n(z) - u_m(z) < 3\left(u_n(a) - u_m(a)\right) < \varepsilon \quad \forall n \geq m \geq N.$$

Make $n \longrightarrow \infty$ $\Rightarrow$ $0 \leq u(z) - u_m(z) \leq \varepsilon \quad \forall m \geq N, \; z \in \Delta\left(a, \frac{R}{2}\right).$

$$\Rightarrow u_m \rightrightarrows u \quad \text{in} \quad \Delta\left(a, \frac{R}{2}\right).$$

2. Subharmonic Functions — Conway X. 3.

SH functions share many properties with harmonic fns.

Definition $\varphi : G \longrightarrow \mathbb{R}$ continuous, $\forall\, a \in G$, $\exists\, \bar{\Delta}(a, R) \subseteq G$.

$$\varphi(a) \le \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + r e^{it})\, dt \quad \forall\ 0 \le r \le R$$

then φ is called subharmonic.

Superharmonic functions satisfy the opposite inequality.

Remark

[i] φ subharmonic $\Rightarrow -\varphi$ superharmonic

[ii] φ harmonic $\Rightarrow \varphi$ sub/superharmonic

[iii] φ is C^2 & $\Delta \varphi \ge 0 \Rightarrow \varphi$ subharmonic.

This is HWK 2, Problem 1.

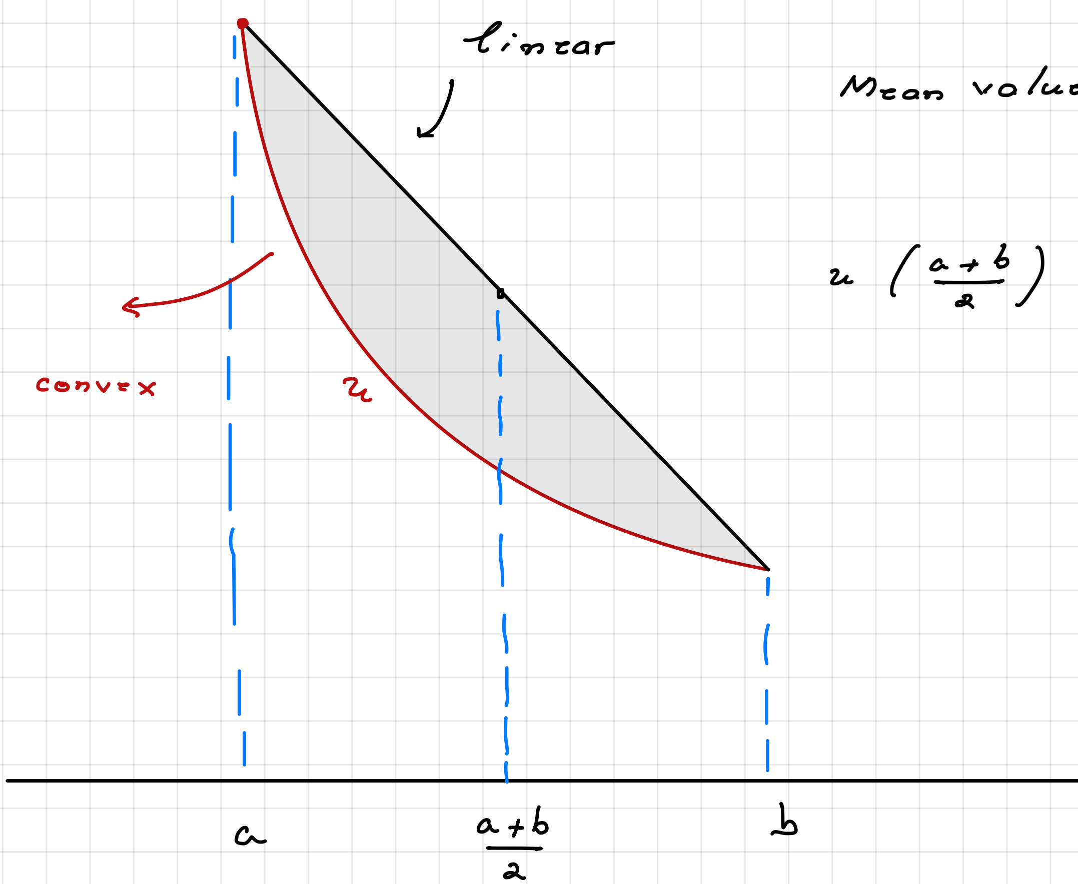
Analogy with 1 real variable

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \longleftrightarrow \frac{\partial^2}{\partial x^2} \quad (1 \text{ variable})$$

"harmonic" $\quad \dfrac{\partial^2 u}{\partial x^2} = 0 \implies u$ linear, $u = ax+b$.

C^2 "subharmonic" $\quad \dfrac{\partial^2 u}{\partial x^2} \geq 0 \implies u$ convex

C^2 "superharmonic" $\quad \dfrac{\partial^2 u}{\partial x^2} \leq 0 \implies u$ concave



Mean value Inequality becomes

$$u\left(\frac{a+b}{2}\right) \leq \frac{u(a)+u(b)}{2}$$

Properties of subharmonic functions

(1) similar to harmonic functions

(2) new properties

---

(1) Maximum principle

$\varphi : G \to \mathbb{R}$ subharmonic

MP: If $\varphi : G \to \mathbb{R}$ SH and achieves a maximum at $a \in G$

$\Rightarrow \varphi$ constant.

MP$^+$: If $\varphi : G \to \mathbb{R}$ SH and $\forall a \in \partial_\infty G$,

$$\limsup_{z \to a} \varphi(z) \leq 0 \Rightarrow \varphi < 0 \text{ or } \varphi \equiv 0 \text{ in } G.$$

Corollary $\varphi : \bar{G} \to \mathbb{R}$, G bounded, φ continuous in $\bar{G}$, subharmonic in G, $\varphi|_{\partial G} \leq 0$. Then $\varphi < 0$ or $\varphi \equiv 0$ in G.

(2) New Property

φ_1, φ_2 subharmonic $\Rightarrow$ $\varphi = \max(\varphi_1, \varphi_2)$ subharmonic.

Proof φ continuous. If $a \in G$, we can find R_1, R_2 with

$$\varphi_1(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi_1(a + re^{it})\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{it})\,dt \quad \forall r < R_1$$

$$\varphi_2(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi_2(a+re^{it})\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{it})\,dt \quad \forall r < R_2$$

For $R = \min(R_1, R_2)$, $r < R$ we have

$$\varphi(a) = \max(\varphi_1(a), \varphi_2(a)) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{it})\,dt$$

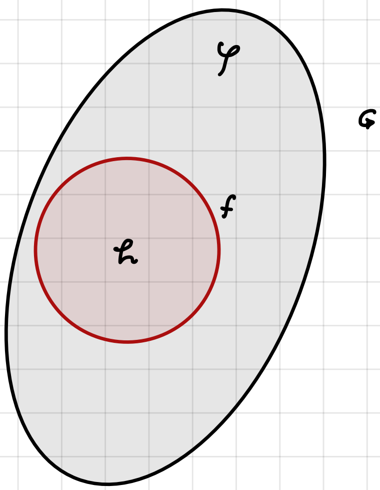
$\Rightarrow$ φ subharmonic.

New Property (Poisson Modification / Bumping)

φ subharmonic $\Rightarrow$ $\tilde{\varphi}$ subharmonic

Construction Let φ be subharmonic.

Let $\overline{\Delta}(a,r) \subseteq G$. Let $f = \varphi|_{\partial\Delta}$.



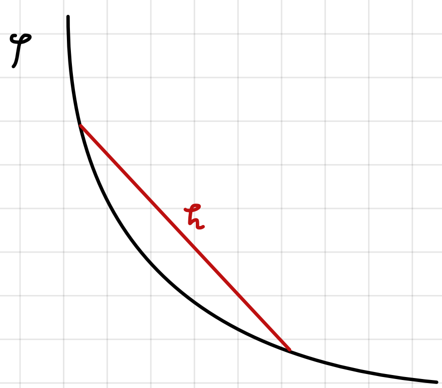
Let h be the solution to Dirichlet in Δ,

boundary value f $\Rightarrow$ $h|_{\partial\Delta} = f$.

Define

$$\tilde{\varphi} = \begin{cases} \varphi & \text{in } G \setminus \overline{\Delta} \\ h & \text{in } \Delta. \end{cases}$$

In one variable



Claims

$\boxed{\text{i}}$ $\varphi \leq \tilde{\varphi}$

$\boxed{\text{ii}}$ $\tilde{\varphi}$ subharmonic

$\boxed{\text{iii}}$ $\varphi \leq \psi \implies \tilde{\varphi} \leq \tilde{\psi}$

We will prove these statements next time.

Math 220C - Lecture 7

April 12, 2021

Homework 3 posted. There are 6 questions.

Questions 2 – 5 are about the Dirichlet Problem.

Last time (Poisson modification / Bumping)

- $\varphi : G \longrightarrow \mathbb{R}$ subharmonic

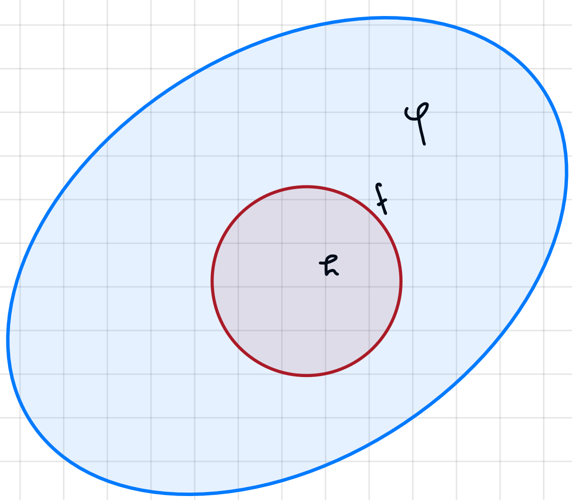
- $\overline{\Delta} \subseteq G$ closed disc

- $f = \varphi \big/_{\partial\Delta}$.

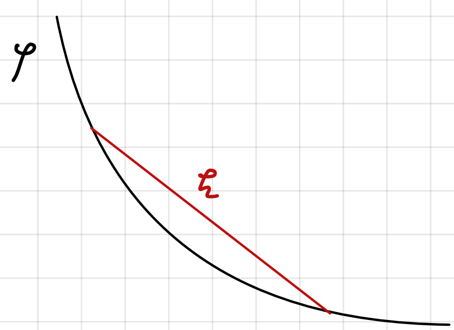
- Solve Dirichlet Problem in $\overline{\Delta}$:

  h continuous in $\overline{\Delta}$, harmonic in Δ, $h \big/_{\partial\Delta} = f$.

- Let $\tilde{\varphi} = \begin{cases} \varphi & \text{in } G \setminus \overline{\Delta} \\ h & \text{in } \overline{\Delta} \end{cases} \implies \tilde{\varphi}$ cont.



in $\mathbb{R}^2$



in $\mathbb{R}$.

Proposition Conway 3.7$^+$

$\boxed{i}$ $\varphi \le \tilde{\varphi}$

$\boxed{ii}$ $\tilde{\varphi}$ subharmonic (HWK 3)

$\boxed{iii}$ $\varphi_1 \le \varphi_2$ subharmonic $\Longrightarrow$ $\tilde{\varphi}_1 \le \tilde{\varphi}_2$.

Proof $\boxed{i}$ Since $\varphi = \tilde{\varphi}$ in $G \setminus \bar{\Delta}$, we only need to prove

$$\varphi \le h \text{ in } \bar{\Delta}.$$

Note that $\varphi - h$ is subharmonic (φ satisfies MV-inequality, h satisfies MV-equality). Note

$$\varphi - h\big|_{\partial\Delta} = f - f = 0.$$

By Maximum Principle $\varphi - h \le 0$ in $\bar{\Delta}$, as needed.

$\boxed{iii}$ Let $f_1 = \varphi_1 \big/_{\partial\Delta}$, $f_2 = \varphi_2 \big/_{\partial\Delta}$.

Let $h_1, h_2 : \overline{\Delta} \longrightarrow \mathbb{R}$ solve Dirichlet Problem with boundary values f_1, f_2.

To show $\tilde{\varphi}_1 \leq \tilde{\varphi}_2$, it suffices to show $h_1 \leq h_2$ in $\overline{\Delta}$.

Note $h_1 - h_2$ harmonic in Δ, continuous in $\overline{\Delta}$

$$h_1 - h_2 \big/_{\partial\Delta} = f_1 - f_2 = \varphi_1\big/_{\partial\Delta} - \varphi_2\big/_{\partial\Delta} \leq 0$$

By Maximum Principle, $h_1 - h_2 \leq 0$ in $\overline{\Delta}$ as needed.

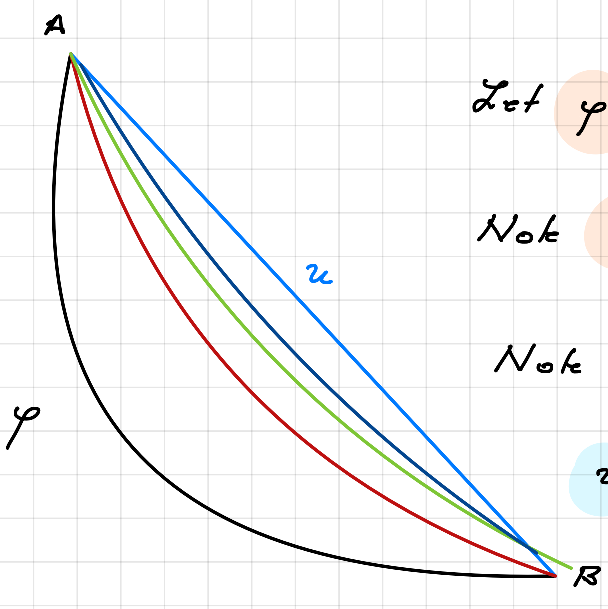
Question How do we construct interesting harmonic functions?

Methods $\boxed{i}$ $u = \mathrm{Re}\, f$, f holomorphic

$\boxed{ii}$ Poisson's formula / Dirichlet Problem, $G = \Delta$

$\boxed{iii}$ Perron method

Idea behind Perron's method – 1 variable



Let φ be a convex path joining A, B.

Note $\varphi \le u$ where u = line.

Note

$$u(z) = \sup\{\varphi(z) : \varphi \text{ convex as above}\}.$$

We wish to extend these observations to $\mathbb{R}^2$.

In $\mathbb{R}^2$: $G \subseteq \mathbb{C}$ bounded, $f : \partial G \longrightarrow \mathbb{R}$ continuous

- Perron family

$$\mathcal{P}(G, f) = \left\{ \varphi : G \longrightarrow \mathbb{R} \text{ subharmonic}, \ \limsup_{z \to a} \varphi(z) \leq f(a) \ \forall \ a \in \partial G. \right\}.$$

- Perron function $u : G \longrightarrow \mathbb{R}$

$$u(z) = \sup \left\{ \varphi(z), \ \varphi \in \mathcal{P}(G, f) \right\}$$

Question Is the Perron function well-defined?

Remarks $\boxed{I}$ $\mathcal{P}(G,f) \neq \emptyset$.

Indeed, ∂G compact, f cont. $\Rightarrow$ $m \leq f \leq M$ in ∂G.

Then $\varphi = m$ is in $\mathcal{P}(G,f)$.

$\boxed{II}$ u is well-defined.

Since $f \leq M$, we have

$$\limsup_{z \to a} \varphi(z) \leq M \quad \forall\, a \in \partial G \Rightarrow \varphi \leq M \text{ by MP}$$

$$\Rightarrow u(z) = \sup\{\varphi(z)\} \leq M.$$

$\boxed{III}$ $\varphi \in \mathcal{P}(G,f) \Rightarrow \tilde{\varphi} \in \mathcal{P}(G,f)$

Indeed, $\tilde{\varphi}$ subharmonic (see Proposition) and $\varphi = \tilde{\varphi}$ near $a \in \partial G$ so

$$\limsup_{z \to a} \tilde{\varphi}(z) = \limsup_{z \to a} \varphi(z) \leq f(a).$$

Theorem Conway 3.11.

The Perron function u is harmonic

Proof Let $x \in G$, $\overline{\Delta} \subseteq G$ a disc around x.

WTS u harmonic in Δ.

Step 1 Find functions $\varphi_n \in \mathcal{P}(G, f)$ with $\varphi_n(x) \longrightarrow u(x)$.

This is possible by the definition of u.

WLOG we may assume $\varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_n \leq \cdots$

Why? Else, define

$$\varphi_n^{new} = \max(\varphi_1, \varphi_2 \cdots \varphi_n).$$

By Lecture 6, Property (2), $\varphi_n^{new} \in \mathcal{P}(G, f)$. Note that

$\varphi_n^{new}(x) \longrightarrow u(x)$ as well and that

$$\varphi_1^{new} \leq \varphi_2^{new} \leq \cdots \leq \varphi_n^{new} \leq \cdots$$

We drop the superscript "new" from now on.

Step 2 **WLOG** We may assume

$$\varphi_1 \le \varphi_2 \le \dots \le \varphi_n \le \dots \quad \text{are harmonic in } \Delta.$$

Why? Indeed,

$$\tilde{\varphi}_1 \le \tilde{\varphi}_2 \le \dots \le \tilde{\varphi}_n \le \dots \quad (\text{see Proposition above})$$

and $\tilde{\varphi}_n \in \mathcal{P}(G, f)$ by Remark [III]. Note $\tilde{\varphi}_n$ are harmonic in Δ.

Furthermore $\tilde{\varphi}_n(x) \longrightarrow u(x)$ still holds. Indeed,

Proposition [I] definition of u as supremum

$$\varphi_n(x) \le \tilde{\varphi}_n(x) \le u(x)$$

Thus $\varphi_n(x) \longrightarrow u(x) \implies \tilde{\varphi}_n(x) \longrightarrow u(x)$

We can work with the functions $\tilde{\varphi}_n$ instead of the φ_n's.

Step 3

By Harnack's convergence

$\tilde{\varphi}_n \overset{l.u.}{\rightrightarrows} U$ in Δ, for U harmonic.

We noted that $\tilde{\varphi}_n(x) \longrightarrow u(x) < \infty$ so the possibility

$\tilde{\varphi}_n \overset{l.u.}{\rightrightarrows} \infty$ in Harnack is not allowed.

Note $U(x) = u(x)$.

Goal We show $U = u$ in Δ. (not only at x).

This will show u is harmonic, as needed

Step 4 Let $y \in \Delta$. We show $U(y) = u(y)$.

Let $\psi_n \in \mathcal{P}$, $\psi_n(y) \to u(y)$, possible by definition of u.

WLOG $\varphi_n \leq \psi_n$

Why? $\psi_n^{new} = \max(\psi_n, \varphi_n) =$ subharmonic (Lecture 6).

We still have $\psi_n^{new} \in \mathcal{P}(G, f)$.

Bonus $\psi_n^{new}(x) \longrightarrow u(x)$ & $\psi_n^{new}(y) \longrightarrow u(y)$.

Why? We know $\varphi_n(x) \to u(x)$

$\varphi_n \leq \psi_n^{new} \leq u$ $\Rightarrow$ $\psi_n^{new}(x) \longrightarrow u(x)$.

def. of ψ_n^{new} definition of u as supremum

The same argument works for y, with ψ_n instead of φ_n.

We run the above Steps for $\psi_1, \psi_2, \dots$

Step 1' $\quad \psi_1 \le \psi_2 \le \dots \le \psi_n \le \dots$

Step 2' $\quad \tilde{\psi}_1 \le \tilde{\psi}_2 \le \dots \le \tilde{\psi}_n \le \dots$ harmonic in Δ

Step 3' $\quad$ Harnack $\quad \tilde{\psi}_n \overset{\text{l.u.}}{\rightrightarrows} V$ in Δ.

Claims

$\boxed{a}$ $$V(y) \overset{\text{Step 3'}}{=} \lim_{n\to\infty} \tilde{\psi}_n(y) \overset{\text{Bonus}}{=} u(y)$$

$\boxed{b}$ $$V(x) \overset{\text{Step 3'}}{=} \lim_{n\to\infty} \tilde{\psi}_n(x) \overset{\text{Bonus}}{=} u(x)$$

$\boxed{c}$ $$U(z) \overset{\text{Step 3}}{=} \lim \tilde{\varphi}_n(z) \le \lim \tilde{\psi}_n(z) \overset{\text{Step 3'}}{=} V(z)$$

using that $\varphi_n \le \psi_n$ and $\tilde{\varphi}_n \le \tilde{\psi}_n$.

Conclusion

We know $U - V \leq 0$ in Δ by $\boxed{c}$

$$U(x) \underset{\text{Step 3}}{=} u(x) \underset{\boxed{b}}{=} V(x) \implies (U - V)(x) = 0.$$

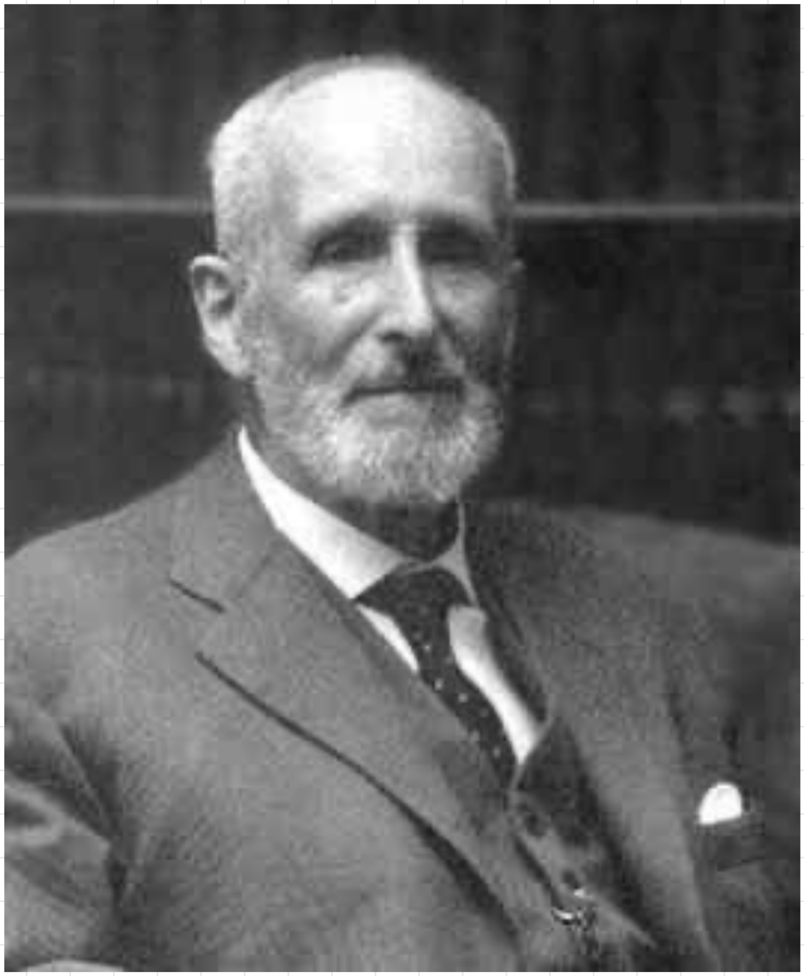
harmonic

By Max. Principle $\implies U - V \equiv 0.$

$\boxed{a}$

In particular, $U(y) = V(y) = u(y).$

Since $y \in \Delta$ is arbitrary $\Rightarrow U \equiv u$, as needed.



Oskar Perron (1880 – 1975) was a German mathematician. He brought contributions to PDE's, known for the Perron method.

Math 220C - Lecture 8

April 14, 2021

Plan — short discussion of Dirichlet Problem

— begin Chp XI — Jensen's formula

Last time G bounded, $f: \partial G \longrightarrow \mathbb{R}$ continuous

- Perron family

$$\mathcal{P}(G, f) = \left\{ \varphi : G \longrightarrow \mathbb{R} \text{ subharmonic}, \ \limsup_{z \to a} \varphi(z) \le f(a) \ \forall \ a \in \partial G. \right\}.$$

- Perron function $u : G \longrightarrow \mathbb{R}$

$$u(z) = \sup \{ \varphi(z), \ \varphi \in \mathcal{P}(G, f) \}$$

- Theorem

The Perron function u is harmonic

Question Does the Perron function solve Dirichlet Problem?

What is the issue?

We know u is harmonic in G.

We need to show $\lim_{z \to a} u(z) = f(a) \quad \forall\, a \in \partial G$.

Answer (HWK 3, # 2) **NO!**

If $G = \Delta(0,1) \setminus \{0\}$, we show that the Dirichlet Problem does not always admit a solution.

Better answer In special cases, it does!

Terminology (differs from Conway $\overline{X}.4$)

Let G be bounded. Let $a \in \partial G$.

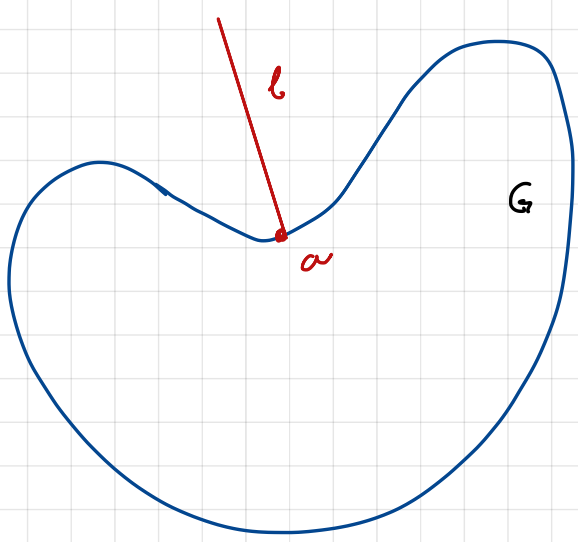
$\omega : \overline{G} \longrightarrow \mathbb{R}$ continuous in $\overline{G}$, harmonic in G,

$\omega(a) = 0$, $\omega > 0$ in $\partial G \setminus \{a\}$

ω is said to be a barrier at a.

The terminology is due to Lebesgue.

Example (HWK 3, # 5) Many reasonable domains satisfy this definition. For instance, if $\exists$ ℓ segment



$\ell \cap \overline{G} = \{a\}$ then there is a barrier at a.

Theorem The Dirichlet Problem can be always be solved in G.

$\iff \forall\, a \in \partial G$, $\exists$ barrier at a.

The Perron function solves the Dirichlet Problem.

Remark „$\Longrightarrow$" HWK 3, #4

„$\Longleftarrow$" A proof is given in the Appendix to the lecture.

& video on Canvas.

§2. Jensen's Formula

$f : G \longrightarrow \mathbb{C}$ holomorphic, f nowhere zero in G, $\overline{\Delta}(0,r) \subseteq G$.

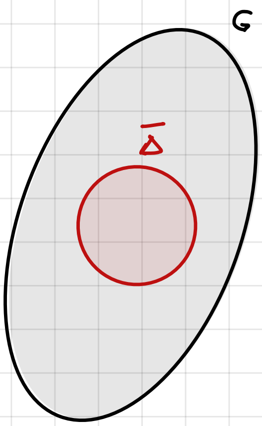
Recall from HWK 1

5. Let $U \subset \mathbb{C}$ be open connected.

(i) Show that if $h : U \to \mathbb{C}$ is holomorphic and nowhere zero in U, then

$$u(z) = \log |h(z)|$$

is harmonic in U.



Mean Value Property for $\log |f|$ gives

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt.$$

Question What if f has zeros?

The zeros of f will give corrections to the formula.

Theorem $f : G \longrightarrow \mathbb{C}$ holomorphic, $\overline{\Delta}(0,r) \subseteq G$, $f(0) \neq 0$.

Let $a_1, \dots, a_k$ be the zeroes of f in $\Delta(0,r)$. Then

$$\log |f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt.$$

Proof Shrinking G, we may assume $G = \Delta(0,R)$

We may assume $r = 1$. Indeed, otherwise let

$$f^{new}(z) = f(rz) \quad \text{defined in } G^{new} = \Delta\left(0, \frac{R}{r}\right) \supseteq \overline{\Delta}(0,1).$$

When f is holomorphic in $\Delta(0,R) \supseteq \overline{\Delta}(0,1)$, we show

$$\log |f(0)| - \sum_{k=1}^{n} \log |a_k| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, dt. \qquad (*)$$

Proof of (*) Let

- $a_1, \dots, a_k$ be zeroes of f in $\Delta = \Delta(0,1)$
- $b_1, \dots, b_m$ be zeroes of f on $\partial\Delta$.

Recall $\varphi_a : \bar{\Delta} \longrightarrow \bar{\Delta}$, $\partial\Delta \longrightarrow \partial\Delta$, $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$.

Let $$F(z) = f(z) \Big/ \prod_{j=1}^{k} \varphi_{a_j}(z) \cdot \prod_{j=1}^{m} \frac{b_j}{b_j - z}$$

Note that F has no zeroes in $\bar{\Delta}$ & in fact in a neighborhood of $\bar{\Delta}$. Note

$$F(0) = f(0) \Big/ \prod_{j=1}^{m} (-a_j)$$

By the previous observation applied to F

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{it})| \, dt. \quad (1).$$

By substitution, we find

$$\log|F(0)| = \log|f(0)| - \sum_{j=1}^{k} \log|a_k| \qquad (2)$$

$$\int_0^{2\pi} \log|F(e^{it})|\,dt = \int_0^{2\pi} \log|f(e^{it})|\,dt$$

0 (see below)

$$- \sum_{j=1}^{k} \int_0^{2\pi} \log|\varphi_{a_j}(e^{it})|\,dt$$

0 (claim)

$$+ \sum_{j=1}^{m} \int_0^{2\pi} \log\left|\frac{b_j}{b_j - e^{it}}\right| dt$$

$$= \int_0^{2\pi} \log|f(e^{it})|\,dt. \qquad (3)$$

Here we used $\varphi_{a_j} : \partial\Delta \longrightarrow \partial\Delta$ so that

$$|\varphi_{a_j}(e^{it})| = 1 \implies \log|\varphi_{a_j}(e^{it})| = 0.$$

Jensen's formula follows from (1), (2), (3).

<u>Claim</u> $\displaystyle\int_0^{2\pi} \log\left|\frac{b}{b - e^{it}}\right| dt = 0 \quad \forall\ |b| = 1.$

Proof of the claim Let $b = e^{i\alpha}$. Then

$$\int_0^{2\pi} \log \left| \frac{b}{b - e^{it}} \right| dt = \int_0^{2\pi} \log \left| \frac{e^{i\alpha}}{e^{i\alpha} - e^{it}} \right| dt$$

$$= \int_0^{2\pi} \log \left| \frac{1}{1 - e^{i(t-\alpha)}} \right| dt \qquad t \to t+\alpha$$

$$= \int_0^{2\pi} \log \frac{1}{|1 - e^{it}|} dt$$

$$= - \int_0^{2\pi} \log |1 - e^{it}| \, dt \overset{?}{=} 0 .$$

We note that

$$|1 - e^{it}|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t = 4 \sin^2 \frac{t}{2} .$$

We need to show

$$\int_0^{2\pi} \log \left| 2 \sin \frac{t}{2} \right| dt = 0 \quad \overset{t = 2u}{\Longleftrightarrow}$$

$$\Longleftrightarrow \quad \int_0^{\pi} \log |2 \sin u| \, du = 0$$

$$\Longleftrightarrow \quad \int_0^{\pi} \log 2 \, du + \int_0^{\pi} \log \sin u \, du = 0$$

$$\Longleftrightarrow \quad \boxed{\int_0^{\pi} \log \sin u \, du = - \pi \log 2.}$$

Calculation $\displaystyle\int_0^{\pi} \log \sin u \, du = -\pi \log 2.$

Convergence

$$\int_0^{\pi} \log \sin u \, du \leq \int_0^{\pi} \log u \, du = u \log u - u \Big|_{u=0}^{u=\pi} < \infty.$$

This uses $\lim\limits_{u \to 0} u \log u = 0.$

Evaluation

$$I = \int_0^{\pi} \log \sin u \, du \overset{u = 2v}{=}$$

$$= 2 \int_0^{\pi/2} \log \sin 2v \, dv = \quad \sin 2v = 2 \sin v \cos v.$$

$$= 2 \int_0^{\pi/2} \log 2 \, dv + 2 \int_0^{\pi/2} \log \sin v \, dv + 2 \int_0^{\pi/2} \log \cos v \, dv$$

$$= \pi \log 2 + 2 \int_0^{\pi/2} \log \sin v \, dv + 2 \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} + v\right) dv$$

$$= \pi \log 2 + 2 \int_0^{\pi} \log \sin v \, dv$$

$$= \pi \log 2 + 2I \implies I = -\pi \log 2.$$

SUR UN NOUVEL ET IMPORTANT THÉORÈME DE LA THÉORIE DES FONCTIONS

PAR

J. L. W. V. JENSEN.

Monsieur le Professeur,

Lors de votre dernier séjour à Copenhague j'ai eu honneur de vous entretenir au sujet d'une intégrale définie appelée, si je ne me trompe, à jouer un rôle dans la théorie des fonctions analytiques. Comme il me parut que cette question vous interéssa vivement, je profiterai de cette occasion — l'envoi des deux petits mémoires[1] destinés à votre Journal — pour vous communiquer le développement détaillé de mon théorème.

Soit $z = re^{\theta i}$ une variable complexe, et α un nombre complexe différent de zéro, on a pour $r < |\alpha|$,

$$l\left(1 - \frac{z}{\alpha}\right) = -\sum_{\nu=1}^{\infty} \frac{1}{\nu}\left(\frac{z}{\alpha}\right)^{\nu}$$

où l désigne la valeur principale du logarithme. En prenant les parties réelles des deux membres et en observant que l'on a $\Re(a) = \frac{1}{2}(a + \dot{a})$,[2] on trouve

$$(1) \qquad l\left|1 - \frac{z}{\alpha}\right| = -\sum_{\nu=1}^{\infty} \frac{r^{\nu}}{2\nu}\left(\frac{e^{\nu\theta i}}{\alpha^{\nu}} + \frac{e^{-\nu\theta i}}{\dot{\alpha}^{\nu}}\right), \qquad r = |z| < |\alpha|.$$

[1] (1) *Sur les fonctions entières.*

(2) *Note sur une condition nécessaire et suffisante pour que tous les zéros d'une fonction entière soient réels.*

[2] Ici et dans la suite je désigne toujours par $\Re(a)$ la partie réelle et par $\dot{a}$ la valeur conjuguée de a.

Acta mathematica. 22. Imprimé le 6 mars 1899.

Acta Math 1899, volume 22

Johan Jensen (1859–1925) was a Danish mathematician. He pursued mathematics while worked as a telephone engineer.

Jensen found his formula while unsuccessfully trying to prove the Riemann hypothesis.

He is also known for Jensen's inequality (about convex functions).

Appendix to Lecture 8

April 14, 2021

Let G be bounded and assume each $a \in \partial G$ is a barrier. Let $f : \partial G \longrightarrow \mathbb{R}$ be continuous.

Theorem The Perron function u for (G, f) satisfies

$$\lim_{z \to a} u(z) = f(a)$$

Corollary The Perron function solves the Dirichlet Problem under the above assumptions.

We let ω be a barrier at a. Thus

- $\omega : \overline{G} \longrightarrow \mathbb{R}$, ω cont in $\overline{G}$, ω harmonic in G
- $\omega(a) = 0$, $\omega > 0$ in $\partial G \setminus \{a\}$.

Proof WLOG $f(a) = 0$. Let $\varepsilon > 0$. We show

[i] $\limsup_{z \to a} u(z) \leq \varepsilon$

[ii] $\liminf_{z \to a} u(z) \geq -\varepsilon$

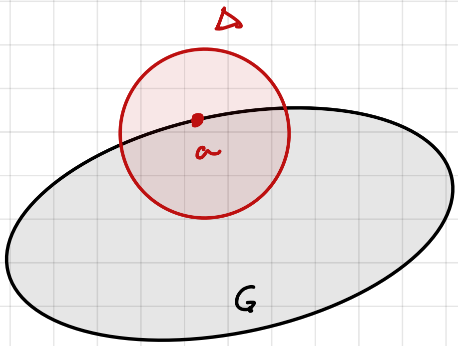
Then $\lim_{z \to a} u(z) = 0 = f(a)$, as needed.

Let Δ be a disc with

$$-\varepsilon < f < \varepsilon \quad \text{in } \partial G \cap \Delta \text{ and } a \in \Delta. \qquad (1)$$

Let $M = \sup_{\partial G} |f| \implies -M \leq f \leq M$ in ∂G. (2)

∂G = compact



Since $\bar{G} \setminus \Delta$ is compact, let

$$\omega_0 = \min_{\bar{G} \setminus \Delta} \omega > 0$$

Why? By Minimum Principle^{+}, either $\omega \equiv 0$ in G (not true as $\omega|_{\partial G} \not\equiv 0$) or else $\omega > 0$ in G. But $\omega > 0$ in $\partial G \setminus \{a\}$. Thus $\omega > 0$ in $\bar{G} \setminus \{a\}$. Since $\bar{G} \setminus \Delta \subseteq \bar{G} \setminus \{a\}$, we get the claim.

Proof of $\boxed{II}$

Let $V(z) = -\varepsilon - \dfrac{\omega(z)}{\omega_0} \cdot M$. = harmonic in G.

cont in $\bar{G}$

Claim 1 $V \le f$ over ∂G

Proof Let $z \in \partial G$.

$\omega \ge 0$ on ∂G .

- $z \in \partial G \cap \Delta$: $V(z) \le -\varepsilon \overset{(1)}{<} f(z)$
- $z \in \partial G \setminus \Delta$: $V(z) < -M \overset{(2)}{\le} f(z)$

$\omega \ge \omega_0$ in $\bar{G} \setminus \Delta$

Claim 2 $V \in \mathcal{P}(G, f)$.

Proof We know V harmonic. For $\zeta \in \partial G$,

$$\lim_{z \to \zeta} V(z) = V(\zeta) \le f(\zeta) \text{ by Claim 1.}$$

Since u is defined as a supremum over $\mathcal{P}(G, f)$ & $V \in \mathcal{P}(G,f)$

$\Longrightarrow$ $u(z) \ge V(z) \quad \forall z \in G$

$\Longrightarrow$ $\liminf\limits_{z \to a} u(z) \ge V(a) = -\varepsilon$ as needed.

$\hookrightarrow \omega(a) = 0$.

Proof of [ii] Let

$$W(z) = \varepsilon + \frac{\omega(z)}{\omega_0} \cdot M = \text{harmonic in } G, \text{ cont. in } \bar{G}.$$

Claim 1' $W \geq f$ over ∂G.

Proof

$\omega \geq 0$ in ∂G

- $z \in \partial G \cap \Delta$, $W(z) \overset{(1)}{\geq} \varepsilon > f(z)$

- $z \in \partial G \setminus \Delta$, $W(z) > M \overset{(2)}{\geq} f(z)$

$\omega \geq \omega_0$ in $\bar{G} \setminus \Delta$

We do not know $W \in \mathcal{P}$, but we can compare W to any $\varphi \in \mathcal{P}$

Claim 2' $W(z) \geq \varphi(z)$ $\forall \varphi \in \mathcal{P}$ $\forall z \in G$.

Proof Let $\zeta \in \partial G$. Then

definition of $\mathcal{P}$ — claim 1'

$$\limsup_{z \to \zeta} \varphi(z) \leq f(\zeta) < W(\zeta) = \lim_{z \to \zeta} W(z)$$

$\Rightarrow \varphi(z) \leq W(z)$ $\forall z \in G$ by MP^+ applied to the function $\varphi - W$.

Since $u(z) = \sup\{\varphi(z) : \varphi \in \mathcal{P}\} \Rightarrow u(z) \leq W(z)$ by

Claim 2 $\forall z \in G$. Then

$$\limsup_{z \to a} u(z) \leq \lim_{z \to a} W(z) = W(a) = \varepsilon,$$ as needed.

Math 220C - Lecture 9

April 16, 2021

Today — Poisson – Jensen formula

— Application of Jensen

— Order of entire functions

Last time

- $f : G \longrightarrow \mathbb{R}$ holomorphic, $f(0) \neq 0$, $\bar{\Delta}(0,r) \subseteq G$
- $a_1, \dots, a_k$ all zeroes of f in $\Delta(0,r)$. w/ multiplicities

$$\log|f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|\, dt.$$

Question How about values not at the center?

§1. Poisson – Jensen formula

We generalize both

– Jensen's formula & Poisson's formula

Theorem Let $f: G \longrightarrow \mathbb{C}$ holomorphic, $\bar{\Delta}(0,r) \subseteq G$, $z_0 \in \bar{\Delta}(0,r)$.

$f(z_0) \neq 0$. Let $a_1, \dots, a_n$ be the zeroes of f in $\Delta(0,r)$. Then

$$\log|f(z_0)| + \underbrace{\sum_{k=1}^{n} \log\left|\frac{r^2 - \bar{a}_k z_0}{r(z_0 - a_k)}\right|}_{\text{contribution from zeroes}} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\operatorname{Re}\frac{re^{it} + z_0}{re^{it} - z_0}}_{\text{Poisson Kernel (Lectures 3 \& 4)}} \cdot \log|f(re^{it})|\,dt$$

Remark [I] When $z_0 = 0$, we recover Jensen's formula.

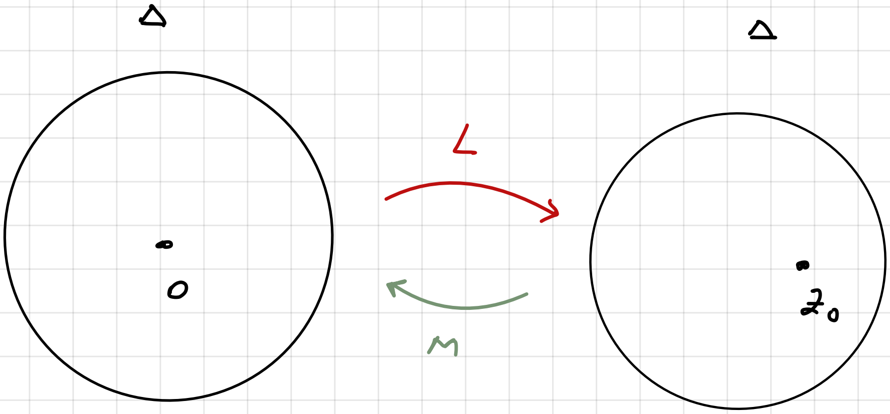
[II] If f has no zeroes, this becomes Poisson's formula

for the function $\log|f|$, which is harmonic in this case.

Proof WLOG $r=1$. Let $\Delta = \Delta(0,1)$. **WTS**

$$\log|f(z_0)| + \sum_{k=1}^{n} \log\left|\frac{1-\bar{a}_k z_0}{z_0 - a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\frac{e^{it}+z_0}{e^{it}-z_0} \cdot \log|f(e^{it})|\,dt$$

Idea of the proof Recenter z_0 to 0 using $\mathrm{Aut}\,\Delta$.



$L, M : \Delta \longrightarrow \Delta$

$$L(w) = \frac{w + z_0}{1 + w\bar{z}_0}$$

$$M(w) = \frac{w - z_0}{1 - w\bar{z}_0}$$

Note L, M are inverses & $L(0) = w_0$.

Let $\tilde{f} = f \circ L$. Apply Jensen to $\tilde{f}$.

Claim Zeroes of $\tilde{f}$ in Δ are $M(a_1), \dots M(a_n)$

Proof $\tilde{f}(z) = 0 \iff f\,L(z) = 0 \iff L(z) = a_k$

$\iff z = ML(z) = M(a_k)$.

By Jensen for $\tilde{f}$:

$$\log|\tilde{f}(0)| + \sum_{k=1}^{n} \log \frac{1}{M(a_k)} = \frac{1}{2\pi}\int_0^{2\pi} \log|\tilde{f}(e^{is})|\,ds$$

$$\Longleftrightarrow \quad \log|f(z_0)| + \sum_{k=1}^{n} \log\left|\frac{1-a_k\bar{z}_0}{a_k - z_0}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(L(e^{is}))|\,ds \quad (1)$$

We change variables $e^{is} = M(e^{it})$. Then

$$f(L(e^{is})) = f\,L\,M(e^{it}) = f(e^{it}).$$

Furthermore,

$$ds = \text{Poisson Kernel} \cdot dt.$$

This was proven in Lecture 3, Main Claim.

Thus RHS of (1) $= \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})| \cdot \text{Poisson Kernel} \cdot dt.$

With this observation, (1) yields Poisson – Jensen.

§2. Applications of Jensen

$f: \mathbb{C} \longrightarrow \mathbb{C}$ entire, $f(0) = 1$.

- $M(R) = \sup_{|z|=R} |f(z)|$ = growth of f
- $N(R) = \#$ zeroes of f in $\Delta(0,R)$ with multiplicities

Apply Jensen in $\Delta(0, 3R)$:

$$\underbrace{\log|f(0)|}_{0} + \sum_{|a_k|<3R} \log\left|\frac{3R}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(3Re^{it})|\,dt$$

$$\leq \log M(3R)$$

$$\Rightarrow \log M(3R) \geq \sum_{|a_k|<R} \log\left|\frac{3R}{a_k}\right| + \sum_{R\leq|a_k|<3R} \log\left|\frac{3R}{a_k}\right|$$

$$\geq \sum_{k=1}^{N(R)} \log 3 + \sum_{R\leq|a_k|<3R} \log 1 = N(R)\log 3 > N(R).$$

Conclusion: $N(R) < \log M(3R)$.

What do we learn from this? $\exists$ correlation between

- growth of entire functions $M(R)$
- distribution of their zeroes $N(R)$

The higher the N, the higher the M (at R & $3R$).

Prototypical Example f polynomial, $\deg f = d$

- $N(R) = d$ if $R >> 0$ by Fundamental Thm Algebra
- $M(R) \sim R^d$.

Thus $\dfrac{\log M(R)}{\log R} \longrightarrow d.$ as $R \longrightarrow \infty$

The converse is also true. If $\lim\limits_{R\to\infty} \dfrac{\log M(R)}{\log R} = d \Rightarrow$

$\Rightarrow \log M(R) < (d+1) \log R$ for $R >> 0$

$\Rightarrow |f(z)| < |z|^{d+1}$ for $|z| >> 0 \Rightarrow f$ polynomial by

Generalized Liouville. (Math 220A, HWK 4, Problem 3)

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§3. Order of entire functions

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire. We first consider

$$M(R) = \sup_{|z|=R} |f(z)|$$ growth of f

Goal We want to measure growth of entire functions such as

1. polynomials

2. $e^{z}, \; e^{z^2}, \; e^{z^3}, \; \dots,$

3. $e^{e^{z}}, \; e^{e^{e^{z}}}, \; e^{e^{e^{e^{z}}}} \dots$

Case 1 We have seen $\dfrac{\log M(R)}{\log R} \longrightarrow d$ & conversely.

This quantity is a good measure of growth but only in this case.

Case 2 The examples in 2 roughly speaking "grow like $e^{\text{polynomial}}$". For those, we need one log to get the exponent, and one additional log to use the measure in $\boxed{\mathrm{I}}$.

Case $\boxed{\mathrm{III}}$ These examples grow very fast, and we will have less to say about them.

Case $\boxed{\mathrm{II}}$ motivates the following:

Definition (Conway XI. 2.15)

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be entire. The order of f is

$$\lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}.$$

This may be infinite.

Examples

$\boxed{i}$ We have $\lambda(fg) \leq \max(\lambda(f), \lambda(g))$ (HWK 4)

$$\lambda(f+g) \leq \max(\lambda(f), \lambda(g)).$$

$\boxed{ii}$ $f = e^{P}$, $\deg P = d \implies \text{order}(f) = d = \deg P$

(exercise).

$\boxed{iii}$ $f(z) = e^{e^{z}} \implies \text{order}(f) = \infty.$ (exercise)

$\boxed{iv}$ $f(z) = \cos z$, $\sin z$ have order 1

(HWK 4).

$f(z) = \cos\sqrt{z}$ has order $\frac{1}{2}$

Math 220C - Lecture 10

April 19, 2021

§0. Last time

$f: \mathbb{C} \longrightarrow \mathbb{C}$ entire function

Main Question Establish relationship between

$$\{\text{Growth of } f\} \longleftrightarrow \{\text{Distributions of zeros}\}$$

Subquestion: How do we interpret the two sides mathematically?

---

§1. Left hand side

Order Recall $M(R) = \sup_{|z|=R} |f(z)|$. & we defined

$$\lambda(f) = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}$$

Intuitively, " $f(z) \sim e^{|z|^{\lambda}}$ ".

Question How to prove a function f has order λ ?

We need to show two statements:

[I] $\forall\, \varepsilon > 0 \ \exists\, r$ such that $|f(z)| < e^{|z|^{\lambda+\varepsilon}} \ \forall\, |z| > r$

This shows $M(R) < e^{R^{\lambda+\varepsilon}} \ \forall\, R > r$ &

$$\lambda(f) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \leq \lambda + \varepsilon \ \forall\, \varepsilon. \overset{\varepsilon\to 0}{\Longrightarrow} \lambda(f) \leq \lambda$$

[II] $\forall\, \varepsilon > 0 \ \exists\, z_n \longrightarrow \infty$ with $|f(z_n)| > e^{|z_n|^{\lambda-\varepsilon}}$

This shows

$$\lambda(f) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \geq \limsup_{n\to\infty} \frac{\log\log|f(z_n)|}{\log|z_n|} \geq \lambda - \varepsilon$$

$$\overset{\varepsilon\to 0}{\Longrightarrow} \lambda(f) \geq \lambda.$$

Properties

$\boxed{i}$ $\lambda(z^m) = 0$, $M(R) = R^m \Rightarrow \lambda = 0$.

$\boxed{ii}$ $\lambda(e^P) = \deg P$ for P = polynomial (check)

$\boxed{iii}$ $\lambda(fg) \leq \max(\lambda(f), \lambda(g))$ (HWK 4)

§2. Right hand side & Distribution (growth) of zeroes

Assume f has zeroes at

$$|a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots \quad . \quad a_n \longrightarrow \infty \,, \quad a_n \neq 0$$

Several quantities attached to growth of zeroes:

$\boxed{\text{i}}$ rank $= p$

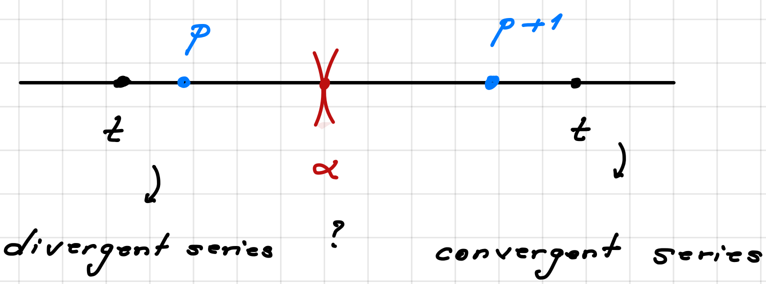
The smallest integer p such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$.

If such a p doesn't exist, $p = \infty$.

$\boxed{\text{ii}}$ critical exponent (HWK 4, #5)

$\alpha = \inf \left\{ t > 0 : \sum \frac{1}{|a_n|^t} < \infty \right\}$ may not be an integer

By the homework



Thus by definition $p \le \alpha \le p+1$.

If $\alpha \notin \mathbb{Z}$ then α determines p uniquely.

[iii] $N(R)$ = # zeroes in $\Delta(0,R)$ with multiplicity

Fact* (we will not use/prove)

$$\alpha = \limsup_{R\to\infty} \frac{\log N(R)}{\log R}$$

Example* Let $a_n = n^3$, $n>0$. then

$$N(R) = \#\{n : n^3 < R\} \sim R^{1/3} \Rightarrow \frac{\log N(R)}{\log R} \to \frac{1}{3}$$

Note

$$\sum \frac{1}{n^{3t}} < \infty \iff 3t > 1 \iff t > \frac{1}{3} \quad \text{so} \quad \alpha = \frac{1}{3}.$$

harmonic series

Upshot We have defined the following quantities measuring growth/distribution of zeroes

$N(R)$, α, p.

Note $N(R)$ determines α, α determines p if $\alpha \notin \mathbb{Z}$.

Best for us: p (or h to be defined next).

Small variation — Genus of an entire function

Let f has zeroes at $a_1, a_2 \dots, a_n, \dots, a_k \neq 0$.

where $\{a_n\}$ has rank p. $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$

Recall Weierstraß Factorization

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right).$$

Recall

$$E_p(z) = \begin{cases} 1-z, & p=0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right), & p>0 \end{cases}$$

Define

$$h = \text{genus}(f) = \begin{cases} \max(p, q) & \text{if } g \text{ polynomial of degree } q \\ \infty & \text{if } g \text{ not polynomial or } p = \infty. \end{cases}$$

If the exponential e^g doesn't appear then $h = p$.

In general $p \leq h$.

Example (Math 220B)

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$ factorization of sine.

Rewrite this as

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} \left(1 + \frac{z}{n\pi}\right) e^{-z/n\pi}$$

$$= z \prod_{n=1}^{\infty} E_1\left(\frac{z}{n\pi}\right) E_1\left(-\frac{z}{n\pi}\right)$$

$\Rightarrow$ g doesn't appear. Thus genus $h = p$.

The zeroes are at $n\pi$, $n \in \mathbb{Z}$. We want

$$\sum_{n \neq 0} \frac{1}{|n\pi|^{p+1}} < \infty \iff p+1 > 1 \iff p > 0.$$ Thus the

harmonic series

smallest p equals 1.

The genus of $z \longrightarrow \sin z$ equals 1.

§3. Revisiting the Main Question (now made precise)

Establish relationship between

$$\{\text{Growth of } f\} \longleftrightarrow \{\text{Growth of zeroes}\}$$

measured by λ measured by h = genus.

Answer **Theorem** (Hadamard)

$$h \le \lambda \le h+1$$

Remarks $\boxed{i}$ If $\lambda \notin \mathbb{Z}$ then λ determines h uniquely.

$\boxed{ii}$ If e^g doesn't appear then $h = p$ so in this case.

$$p \le \lambda \le p+1$$

$\boxed{iii}$ We have $p \le h \le \lambda$ so the order bounds the p in the Weierstrass Factorization. The statement that we can take $p \le \lambda$ is called Hadamard Factorization.

Conclusion $\exists$ connections between

- $M(R)$ and λ by definition $\lambda = \limsup_{R \to \infty} \dfrac{\log \log M(R)}{\log R}$
- $N(R)$, α, p as we saw above
- λ and $h = \max(p, q)$ via Hadamard $h \le \lambda \le h+1$

Next

- proof that $\lambda \le h+1$
- proof that $h \le \lambda$
- Applications

Jacques Hadamard (1865 – 1963)

Proved the Prime Number Theorem.

Advisor: Émile Picard.

Students: Maurice Fréchet, André Weil

Math 220C - Lecture 11

April 21, 2021

§0. Last time Conway XI. 3.

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire of order λ, $f \not\equiv 0$.

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$$

genus h = max $(p, \deg g)$ if g polynomial or ∞ otherwise.

Hadamard's theorem (1893)

$$h \leq \lambda \leq h+1$$

Remark [i] The theorem doesn't assume h, λ finite.

If one of them is infinite $\Rightarrow$ so is the other.

[ii] These ideas played an important role in

Hadamard's proof of Prime Number Theorem. (1896)

Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann [1];

PAR M. J. HADAMARD.

1. La décomposition d'une fonction entière $F(x)$ en facteurs primaires, d'après la méthode de M. Weierstrass,

$$F(x) = e^{G(x)} \prod_{p=1}^{\infty} \left(1 - \frac{x}{\xi_p}\right) e^{Q_p(x)} \tag{1}$$

a conduit à la notion du genre de la fonction F.

On dit que F est du genre E si, dans le second membre de l'équation (1), tous les polynômes Q_p sont de degré E, et que la fonction entière $G(x)$ se réduise également à un polynôme de degré E au plus.

Dans un article inséré au *Bulletin de la Société mathématique de France* [2], M. Poincaré a démontré une propriété des fonctions de genre E. L'énoncé auquel il est parvenu est le suivant :

Dans une fonction entière de genre E, *le coefficient de* x^m, *mul-*

[1] Les principaux résultats contenus dans le présent Mémoire ont été présentés à l'Académie des Sciences dans un travail couronné en 1892 (grand prix des Sciences mathématiques).

[2] Année 1883, pages 136 et suiv.

Journal de Mathematiques Pures et Appliquées

(1893)

§1. Applications – Picard's Theorems (weak versions)

To illustrate the power of this result we show:

Application A (Conway 3.6)

> f entire & not constant & finite order
>
> $\Rightarrow$ f omits at most one value.

Remark Little Picard (next week) removes the assumption the order is finite.

Proof Assume f omits $\alpha \neq \beta$. Define

$$f^{new} = \frac{f-\alpha}{\beta-\alpha} \quad \text{omits } 0 \ \& \ 1.$$

Since f^{new} omits 0 $\Rightarrow$ $f^{new} = e^{g}$ & f^{new} omits 1

$\Rightarrow$ g omits 0 Since order (f^{new}) = order $(f) < \infty$

$\Rightarrow$ genus of f^{new} is finite by Hadamard. $\Rightarrow$ g polynomial.

& g omits 0. $\to$ g = constant $\Rightarrow$ f constant. False!

Easy Observations (used above)

[i] $\lambda \geq 0$

We have seen $|f(z)| \leq e^{|z|^{\lambda+\varepsilon}}$ if $|z| \geq R_\varepsilon$ last lecture.

If $\lambda < 0$, let $\varepsilon > 0$ with $\lambda + \varepsilon < 0$. Then $|f(z)| \leq e^{|z|^0} = e$

for $|z| \geq R$ and $|f(z)| \leq M$ for $|z| \leq R$ by continuity. Thus

f bounded $\Rightarrow f$ constant (order 0). Thus $\lambda \geq 0$

[ii] f & αf have the same order $\forall \alpha \neq 0$

Indeed $\lambda(\alpha f) \overset{\text{HWK}}{\leq} \max(\lambda(\alpha), \lambda(f)) = \max(0, \lambda(f)) = \lambda(f)$ by [i]

Similarly $\lambda(f) = \lambda\left(\alpha f \cdot \frac{1}{\alpha}\right) \overset{\text{the previous line}}{\leq} \lambda(\alpha f)$. Thus $\lambda(f) = \lambda(\alpha f)$.

$\boxed{iii}$ f & $f-\alpha$ have the same order

Same proof as in $\boxed{ii}$ using sums versus products

$\boxed{iv}$ f & $P f$ have the same order $\forall P$ polynomial.

if $\deg P > 0$.

We have $f \leq Pf$ if $|z| >> 0$ $\Rightarrow$ $\lambda(f) \leq \lambda(Pf)$.

HWK 4

Also $\lambda(Pf) \leq \max(\lambda(P), \lambda(f)) = \max(0, \lambda(f)) = \lambda(f)$. $\boxed{i}$

Thus $\lambda(Pf) = \lambda(f)$

Application B

f entire of finite order & $\lambda \notin \mathbb{Z}$ $\Rightarrow$ f assumes each of its values infinitely many times.

Remark Great Picard (next week) strengthens this result.

Proof Let α be a value of f. Define $f^{new} = f - \alpha$. We show f^{new} has ∞-many zeroes. Assume f^{new} has finitely many zeroes $a_1, \dots, a_n$. Let $P = \prod_{k=1}^{n} (z - a_k)$. Then f^{new}/P has no zeroes so it equals e^g. $\Rightarrow$

$\Rightarrow$ $f^{new} = P e^g$. Note by previous remarks we have

order f = order f^{new} = order $e^g < \infty$. $\Rightarrow$ genus $< \infty$

$\Rightarrow$ g polynomial & order(e^g) = deg $g \in \mathbb{Z}$. $\Rightarrow$ order$(f) \in \mathbb{Z}$

contradiction.

Plan for the Proof of Hadamard $\quad h \le \lambda \le h+1$

[i] $\lambda \le h+1$ (today).

[ii] $h \le \lambda$ — $p \le \lambda$ (next time)

— $\deg g \le \lambda$ (next time)

§2. First half of Hadamard

WTS $\quad \lambda \le h+1$

WLOG $\quad h$ finite, else we're done.

Key Lemma $\quad \log|E_p(w)| \le C_p\, |w|^{p+1}$ for some $C_p > 0$.

Proof Recall $f(z) = z^m e^g \prod_n E_p\left(\frac{z}{a_n}\right)$. WTS $\lambda \le h+1$.

Recall order $(uv) \le \max(\text{order } u, \text{order } v)$.

Recall order $(z^m) = 0 \le h+1$

order $(e^g) = \deg g \le h < h+1$.

We show order $\prod_n E_p\left(\frac{z}{a_n}\right) \le p+1 \le h+1$.

Note

$$\log\left|\prod_n E_p\left(\frac{z}{a_n}\right)\right| = \sum_n \log\left|E_p\left(\frac{z}{a_n}\right)\right|$$

Lemma $\{$

$$\le C_p \sum_n \left|\frac{z}{a_n}\right|^{p+1} = K\,|z|^{p+1}$$

where $K = C_p \sum \frac{1}{|a_n|^{p+1}} < \infty$. Thus order $\le p+1$, as needed.

Remark (will not prove/use)

order $\prod_n E_p\left(\frac{z}{a_n}\right) = \alpha$ (exercise in Conway).

Proof of Lemma

Recall $E_p(w) = (1-w)\exp\left(w + \frac{w^2}{2} + \dots + \frac{w^p}{p}\right)$.

We induct on p.

When $p = 0$,

$$\log|1-w| \le \log(1+|w|) \le |w| \quad \text{so take } c_0 = 2.$$

Inductive step

$\boxed{i}$ When $|w| \ge \frac{1}{2}$: Note

$$E_p(w) = E_{p-1}(w)\exp\left(\frac{w^p}{p}\right)$$

$$\Longrightarrow \log|E_p(w)| = \log|E_{p-1}(w)| + \log\left|\exp\left(\frac{w^p}{p}\right)\right|$$

$$\le c_{p-1}|w|^p + \log\exp \operatorname{Re}\left(\frac{w^p}{p}\right)$$

$$= c_{p-1}|w|^p + \operatorname{Re}\left(\frac{w^p}{p}\right)$$

$$\le c_{p-1}|w|^p + \left|\frac{w^p}{p}\right| = \left(c_{p-1} + \frac{1}{p}\right)|w|^p$$

$$\le 2\left(c_{p-1} + \frac{1}{p}\right)|w|^{p+1} \quad \text{since } |w| \ge \frac{1}{2}.$$

$\boxed{a}$ When $|w| \le \frac{1}{2}$. Note

$$E_p(w) = (1-w)\exp\left(w + \frac{w^2}{2} + \cdots + \frac{w^p}{p}\right)$$

$$= \exp\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)$$

using Taylor expansion

$$\log(1-w) = -w - \frac{w^2}{2} - \cdots - \frac{w^k}{k} - \cdots \quad \text{for } |w| < 1.$$

Then

$$\log|E_p(w)| = \log\left|\exp\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)\right|$$

$$= \mathrm{Re}\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)$$

$$\le \left|-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right|$$

$$\le \sum_{k \ge p+1} \left|\frac{w^k}{k}\right| = |w|^{p+1} \sum_{k \ge 0} \frac{|w|^k}{p+k+1}$$

$$\le |w|^{p+1} \sum_{k \ge 0} |w|^k \le$$

$$\le |w|^{p+1} \sum_{k \ge 0} \left(\frac{1}{2}\right)^k = 2|w|^{p+1}.$$

Take $c_p = \max\left(2, \ 2\left(c_{p-1} + \frac{1}{p}\right)\right)$. We obtain in both cases

$$\log |E_p(w)| \leq c_p |w|^{p+1}$$ as needed.

Math 220C - Lecture 12

April 23, 2021

Theorem $f: \mathbb{C} \to \mathbb{C}$, $f \not\equiv 0$ entire. Then

$$h \leq \lambda \leq h+1.$$

Conway XI. 3

We already know $\lambda \leq h+1$. We show $h \leq \lambda$.

WLOG λ finite & $f(0) = 1$

Indeed, write $f(z) = c z^m \tilde{f}(z)$ with $\tilde{f}(0) = 1$. Note

order $\tilde{f}$ = order f & genus $\tilde{f}$ = genus f.

WTS If λ is finite, then

$\boxed{i}$ $p \leq \lambda$

$\boxed{ii}$ g polynomial of degree $\leq \lambda$.

where $f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$.

Proof of $\boxed{\text{I}}$ By HWK 4, Problem 5:

$\alpha \leq \lambda$ and by Lecture 10, $\rho \leq \alpha$. Thus $\rho \leq \lambda$.

Proof of $\boxed{\text{II}}$ is technical.

Let $m \leq \lambda < m+1$.

Write $f(z) = e^{g(z)} P(z)$ where $P(z) = \prod_n E_p\left(\frac{z}{a_n}\right)$

We will prove

$$D^{m+1} g = 0 \quad \text{in } \mathbb{C} \setminus \{a_1, a_2 \dots a_n \dots\}.$$

This will show $D^{m+1} g = 0$ in $\mathbb{C}$, say by identity principle. $\Rightarrow$ g polynomial of degree $\leq m$.

Here D = derivative $= \frac{\partial}{\partial z}$

Take logarithmic derivatives

$$f = e^{g} P \implies \frac{f'}{f} = g' + \frac{P'}{P}$$

Take m usual derivatives next to get

$$D^{m} \frac{f'}{f} = D^{m+1} g + D^{m} \frac{P'}{P}.$$

We will show $\boxed{D^{m} \frac{f'}{f} = D^{m} \frac{P'}{P}} \Rightarrow \boxed{D^{m+1} g = 0}$ as needed.

Claim $\quad D^{m} \frac{P'}{P} = -m! \sum_{n} \frac{1}{(a_n - z)^{m+1}}$

Proof Recall $E_p(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right)$.

$$\Rightarrow \frac{E_p'(z)}{E_p(z)} = -\frac{1}{1-z} + 1 + z + \dots + z^{p-1}$$

$$\Rightarrow \boxed{D^{m} \frac{E_p'(z)}{E_p(z)} = -\frac{m!}{(1-z)^{m+1}}} + 0 \quad \text{since } p \le \lambda < m+1 \text{ by.}$$

Part $\boxed{1}$

Recall If $u = \prod_n u_n$

converges absolutely & locally uniformly then

$$\frac{u'}{u} = \sum_n \frac{u_n'}{u_n}$$

absolutely & locally uniformly away from zeroes. (Math 220B)

In our case

$P(z) = \prod_n E_p\left(\frac{z}{a_n}\right)$ converges absolutely & locally unif.

$$\Rightarrow \frac{P'}{P} = \sum_n \frac{\left(E_p\left(\frac{z}{a_n}\right)\right)'}{E_p\left(\frac{z}{a_n}\right)}$$

$$\Rightarrow D^m \frac{P'}{P} = \sum_n D^m \frac{\left(E_p\left(\frac{z}{a_n}\right)\right)'}{E_p\left(\frac{z}{a_n}\right)}$$

(switching differentiation & summation by Weierstraß convergence thm).

$$= -\sum_n \frac{m!}{(a_n - z)^{m+1}}$$ as needed.

Lemma f entire, $f(0) = 1$, $m+1 > \lambda$

$$D^m \frac{f'}{f} = -m! \sum_n \frac{1}{(a_n - z)^{m+1}}$$

The Lemma & above computation shows $D^m \frac{f'}{f} = D^m \frac{P'}{P}$ as claimed.

Proof By Poisson - Jensen formula in $\Delta(0,R)$, $z \neq a_k$:

$$\log|f(z)| + \sum_{k=1}^{N(R)} \log\left|\frac{R^2 - \bar{a}_k z}{R(z - a_k)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{Re^{it} + z}{Re^{it} - z} \cdot \log|f(Re^{it})|\, dt$$

Idea [i] differentiate

[ii] make $R \longrightarrow \infty$.

The Lemma will follow.

Remark

$$2\frac{\partial}{\partial z}\log|f| = \frac{\partial}{\partial z}\log|f(z)|^2$$

$$= \frac{\partial}{\partial z}\log(f\cdot\bar{f})$$

$$= \frac{\partial}{\partial z}\log f + \frac{\partial}{\partial z}\log\bar{f}$$

$$= \frac{f'}{f} + \overline{\frac{\partial}{\partial \bar{z}}\log f}$$

$$= \frac{f'}{f} + 0.$$

This follows because $\log f$ is locally, away from zeroes, a holomorphic function and thus $\frac{\partial}{\partial \bar{z}}\log f = 0$ by Cauchy–Riemann equations (Math 220 A, Lecture 1).

Step 1 : Apply $2\frac{\partial}{\partial z}$ to Poisson-Jensen

$$\log|f(z)| + \sum_{k=1}^{n} \log\left|\frac{R^2 - \bar{a}_k z}{R(z-a_k)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{Re^{it}+z}{Re^{it}-z} \cdot \log|f(Re^{it})|\,dt$$

Compute

$$\left(\frac{Re^{it}+z}{Re^{it}-z}\right)' = \left(-1 + \frac{2Re^{it}}{Re^{it}-z}\right)' = \frac{2Re^{it}}{(Re^{it}-z)^2}.$$

Differentiating, we obtain

$$\frac{f'}{f} = \sum_{k=1}^{N(R)} \frac{\bar{a}_k}{R^2-\bar{a}_k z} - \sum_{k=1}^{N(R)} \frac{1}{a_k - z} + \frac{1}{\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{2Re^{it}}{(Re^{it}-z)^2} \log|f(Re^{it})|\,dt$$

Step 2 : Differentiate m-more times

By direct computation, we have

$$D^m \frac{f'}{f} = -\,m! \sum_{k=1}^{N(R)} \frac{1}{(a_k - z)^{m+1}} + \underbrace{m! \sum_{k=1}^{N(R)} \frac{\bar{a}_k^{\,m+1}}{(R^2 - \bar{a}_k z)^{m+1}}}_{\text{term I}} + \underbrace{\text{Integral term}}_{\text{term II}}$$

where the integral term is

$$\frac{(m+1)!}{\pi} \int_0^{2\pi} \operatorname{Re} \frac{2Re^{it}}{(Re^{it} - z)^{m+2}} \log|f(Re^{it})|\, dt.$$

We show Term I & Term II converge to 0 as $R \to \infty$, yielding the Lemma. This will be achieved in the last two steps.

Step 3: Estimate term I.

$$\sum_{k=1}^{N(R)} \frac{\bar{a}_k^{\,m+1}}{(R^2-\bar{a}_k z)^{m+1}} \longrightarrow 0 \quad \text{as } R \to \infty.$$

Let $R > 2|z|$. Since $\lambda < m+1$, we can pick ε with $\lambda+\varepsilon < m+1$

Note

$$|R^2 - \bar{a}_k z| \geq R^2 - |a_k|.|z| > R^2 - R.\frac{R}{2} = \frac{R^2}{2}.$$

$$\Rightarrow \left|\frac{\bar{a}_k}{R^2-\bar{a}_k z}\right|^{m+1} \leq \frac{R^{m+1}}{|(R^2/2)|^{m+1}} = \frac{2^{m+1}}{R^{m+1}}$$

$$\Rightarrow \left|\sum_{k=1}^{N(R)} \left(\frac{\bar{a}_k}{R^2-\bar{a}_k z}\right)^{m+1}\right| \leq \sum_{k=1}^{N(R)} \left|\frac{\bar{a}_k}{R^2-\bar{a}_k z}\right|^{m+1}$$

$$\leq N(R).\frac{2^{m+1}}{R^{m+1}} \leq (3R)^{\lambda+\varepsilon}.\frac{2^{m+1}}{R^{m+1}} \longrightarrow 0$$

since $m+1 > \lambda+\varepsilon$.

Here, we used

$$N(R) < \log M(3R) < \log e^{(3R)^{\lambda+\varepsilon}} = (3R)^{\lambda+\varepsilon}.$$

Step 4 – Estimate the Integral (term II).

$$\int_0^{2\pi} \mathrm{Re}\, \frac{2Re^{it}}{(Re^{it}-z)^{m+2}} \log|f(Re^{it})|\, dt. \longrightarrow 0 \quad \text{as} \quad R \to \infty$$

Claim

$$\int_0^{2\pi} \frac{2Re^{it}}{(Re^{it}-z)^{m+2}}\, dt \overset{w = Re^{it}}{=} \int_{|w|=R} \frac{2w}{(w-z)^{m+2}} \frac{dw}{iw} = 0$$

because the integrand admits an antiderivative.

Rewrite

$$\text{Term } \mathrm{II} = \int_0^{2\pi} 2Re^{it} \frac{1}{(Re^{it}-z)^{m+2}} \log|f(Re^{it})|\, dt$$

$$\overset{\text{Claim}}{=} \int_0^{2\pi} 2Re^{it} \cdot \frac{1}{(Re^{it}-z)^{m+2}} \left(\log|f(Re^{it})| - \log M(R)\right) dt$$

$$\left|\text{Term } \underline{\overline{II}}\right| \le \int_0^{2\pi} 2R \cdot \frac{1}{|Re^{it}-z|^{m+2}} \left(\log M(R) - \log|f(Re^{it})|\right) dt$$

using $|z| < R/2$

$$\le \int_0^{2\pi} 2R \cdot \frac{1}{(R/2)^{m+2}} \left(\log M(R) - \log|f(Re^{it})|\right) dt$$

$$= \frac{2^{m+3}}{R^{m+1}} \cdot 2\pi \cdot \log M(R) - \frac{2^{m+3}}{R^{m+1}} \cdot \int_0^{2\pi} \log|f(Re^{it})|\, dt$$

Jensen's formula

$$\le \frac{2^{m+3}}{R^{m+1}} \cdot 2\pi \cdot R^{\lambda+\varepsilon} - \frac{2^{m+3}}{R^{m+1}} \Bigg(\underbrace{\sum_{|a_k|<R} \log\left|\frac{R}{a_k}\right|}_{\text{positive contributions}} + \underbrace{\log|f(0)|}_{0} \Bigg)$$

$$\le \frac{2^{m+3}}{R^{m+1}} \cdot 2\pi \cdot R^{\lambda+\varepsilon} \longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

This completes the proof.

Math 220C - Lecture 13

April 26, 2021

(1) Homework 6, available on Friday, due May 7

— last homework

(2) We drop the lowest homework

(3) Next 3 lectures – Little Picard.

In Lecture 11

Application A (Conway XI. 3.6)

f entire & not constant & finite order

$\Rightarrow f$ omits at most one value.

Today – Picard's Theorems – Conway XII

Little Picard

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, non constant $\Rightarrow f$ omits at most one value.

For example, $f(z) = e^z$ only omits the value 0.

Little Picard is a generalization of

Liouville's Theorem $f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, non constant

$\Rightarrow f$ cannot be bounded.

Great Picard

$f : G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

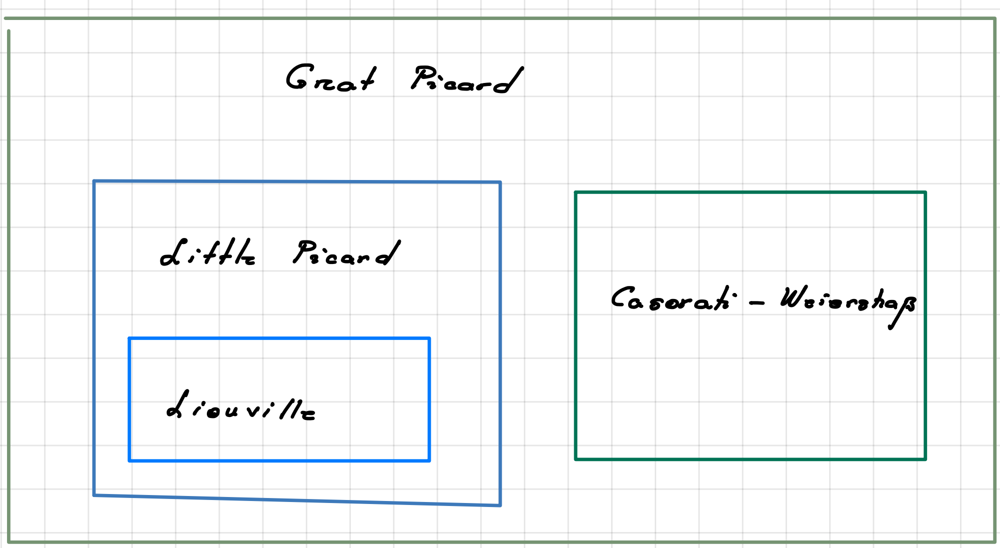
If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f|_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.

Great Picard is a generalization of

Casorati-Weierstraß

$f : G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f|_{\Delta^*(a,r)}$ has dense image in $\mathbb{C}$.



Great Picard $>$ Little Picard

Conway XII. 4.4

Lemma

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, not polynomial.

$\Rightarrow f$ assumes all complex values ∞ - many times, with at most one exception.

Proof Let $g(z) = f\left(\frac{1}{z}\right) : \mathbb{C}^{\times} \longrightarrow \mathbb{C}$. Note that g has an essential singularity at 0. $\Leftrightarrow$ g does not have at worst a pole at 0 $\Leftrightarrow$ f does not have at worst a pole at ∞.

Recall from Math 220 A, Homework 5, Problem 6 that entire functions with poles at ∞ are polynomial, which is not the case for f.

Thus g has essential singularity at 0. Apply Great Picard to conclude.

We showed Great Picard $\Rightarrow$ Lemma $\Rightarrow$ Little Picard.

Examples

$\boxed{I}$ $e^f + e^g = 1$, f, g entire $\Rightarrow$ f, g constant.

Indeed, $h = e^f$ omits 0 & $h = 1 - e^g$ also omits 1.

Little Picard $\Rightarrow$ h constant $\Rightarrow$ f, g constant.

$\boxed{II}$ $f^n + g^n = 1$, $n \geq 3$, f, g entire $\Rightarrow$ f, g constant.

(HWK 6)

Emile Picard (1856 - 1941).

"Une fonction entiere, qui ne devient jamais ni a ni b est necessairment une constante" (Picard, 1879)

§ 2. Proof of Little Picard

Step A Landau's lemma — Conway XII. 2

Step B due to Bloch — Conway XII. 1

Assume $\exists\, f: \mathbb{C} \longrightarrow \mathbb{C}$ entire, not constant, omits 0 & 1.

Step A produces a function g entire and $\alpha > 0$ with

$\Delta \not\subset \operatorname{Im} g$ for all discs Δ of radius α

Step B For any g entire & not constant, $\operatorname{Im} g$ contains a disc of any radius, in particular of radius α.

Step A & Step B are incompatible, showing f does not exist $\Longrightarrow$ Little Picard.

Landau's Lemma

Let $h : G \longrightarrow \mathbb{C}$ holomorphic, G simply connected

Assume h omits -1 & 1. Then $\exists\ F : G \longrightarrow \mathbb{C}$ holomorphic

such that $\boxed{h = \cos F.}$

Proof Note $1 - h^2$ is nowhere zero in G $\Rightarrow$ let g be a

square root of $1 - h^2$ $\Rightarrow$ $g^2 + h^2 = 1$ $\Rightarrow$ $(g + ih)(g - ih) = 1.$

Note $g + ih \neq 0$ in G $\Rightarrow$ $\exists$ logarithm for $g + ih$. Write

$$g + ih = e^{iF} \Rightarrow g - ih = \frac{1}{g + ih} = e^{-iF}$$

$$\Rightarrow g = \frac{1}{2}\left(e^{iF} + e^{-iF}\right) = \cos F.$$

Remark In our case f entire, omits 0 & 1 $\Rightarrow$

$\Rightarrow$ $2f-1$ omits -1 & 1 $\Rightarrow$ by Landau

$\Rightarrow$ $2f-1 = \cos \pi F$ & F entire.

Since $\cos \pi F = 2f-1 \neq \pm 1$ $\Rightarrow$ F omits all integers.

Thus F omits -1 & 1 and by Landau again

$\Rightarrow$ $F = \cos \pi g$ & $\cos \pi g$ is never an integer.

Conclusion

$$f = \frac{1}{2}(1+\cos \pi F) = \frac{1}{2}(1 + \cos \pi \cos \pi g).$$

Define $A = \left\{ m \pm \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right) : n \in \mathbb{Z}_{>0},\ m \in \mathbb{Z} \right\}$

Let $\alpha^{\pm}_{mn} = m \pm \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right)$. Note

$$e^{i\pi\alpha^{+}_{mn}} = e^{i\pi m} \cdot e^{-\log\left(n+\sqrt{n^2-1}\right)} = (-1)^m \frac{1}{n+\sqrt{n^2-1}} = (-1)^m \left(n - \sqrt{n^2-1}\right).$$

$$e^{-i\pi\alpha^{+}_{mn}} = e^{-i\pi m} \cdot e^{\log\left(n+\sqrt{n^2-1}\right)} = (-1)^m \left(n + \sqrt{n^2-1}\right)$$

$$\Longrightarrow \cos \pi \alpha^{+}_{mn} = \frac{1}{2}\left(e^{i\pi\alpha^{+}_{mn}} + e^{-i\pi\alpha^{+}_{mn}}\right) = (-1)^m n \in \mathbb{Z}.$$

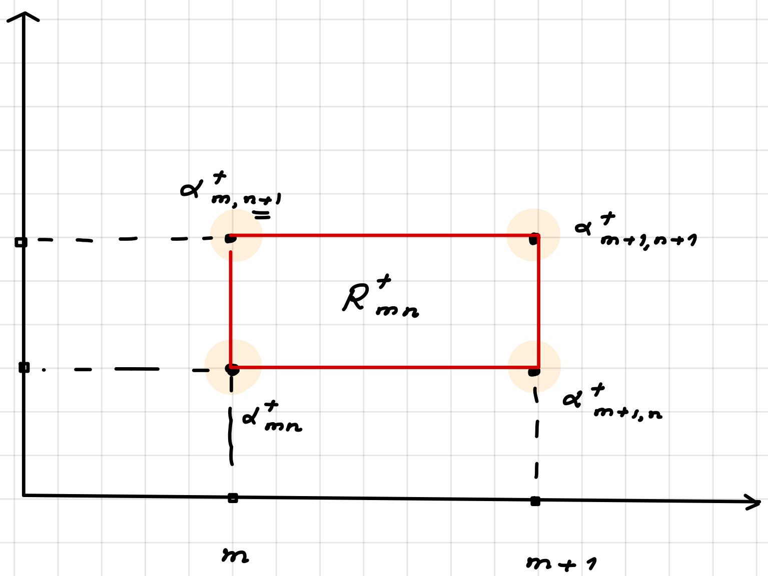
(The same argument works for α^{-}_{mn}.)

But $\cos \pi g$ cannot be an integer.

$$\Downarrow$$

Conclusion $\boxed{A \cap \operatorname{Img} = \emptyset.}$

Visualize A $\quad A = \left\{ m + \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right) : n \in \mathbb{Z}_{>0}, \; m \in \mathbb{Z} \right\}$



The set A gives the vertices of rectangles paving the plane. the upper half plane is paved by rectangles R^+_{mn}

- horizontal side $\quad (m+1) - m = 1$

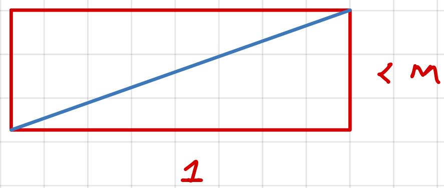
- vertical side $\quad \frac{1}{\pi} \log\left(n+1 + \sqrt{(n+1)^2-1}\right) - \frac{1}{\pi} \log\left(n + \sqrt{n^2-1}\right) =$

$$= \frac{1}{\pi} \log \frac{n+1 + \sqrt{(n+1)^2 - 1}}{n + \sqrt{n^2-1}} < M$$

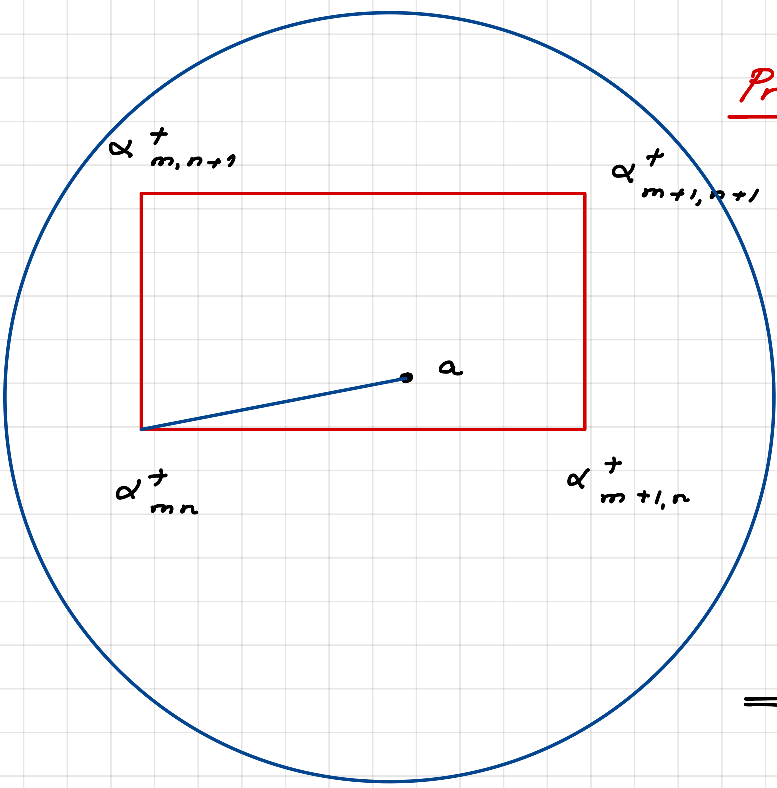
(make $n \longrightarrow \infty$ to see boundedness).

The α^-_{mn}'s are used to pave the lower half plane.

The diameter of R^+_{mn} is $< \sqrt{1+M^2}$. Let $\alpha = \sqrt{2+M^2}$



<u>Claim</u> If Δ is any disc of radius α then $\Delta \not\subseteq \mathrm{Img}$.



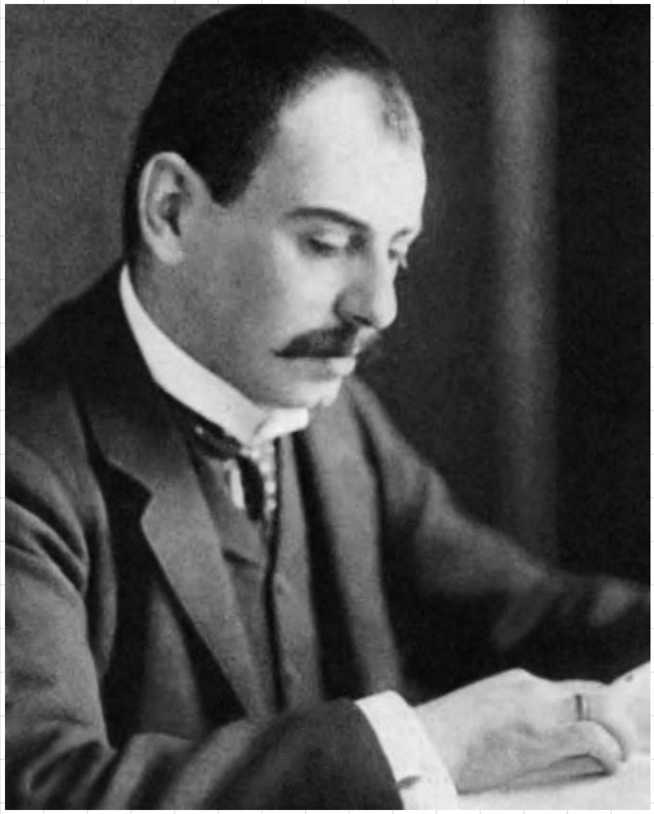
<u>Proof</u> Let a be the center of Δ located say in the upper half plane. Then $a \in R^+_{mn}$.

$\Rightarrow |a - \alpha^+_{mn}| < \text{diameter}(R^+_{mn}) < \alpha$

$\Rightarrow \alpha^+_{mn} \in \Delta$.

We have seen $\alpha^+_{mn} \notin \mathrm{Img}$. Thus $\Delta \not\subseteq \mathrm{Img}$.

This completes the proof of Step A. Step B will be discussed next.



Edmund Landau (1877 – 1938)

Big O - notation

Landau's Problems (ICM 1912)

- Goldbach's conjecture
- Twin prime conjecture
- Primes of the form n^2+1
- Primes between 2 consecutive perfect squares

Math 220C - Lecture 14

April 28, 2021

Last time — Strategy for proving Little Picard

Assume $\exists f: \mathbb{C} \longrightarrow \mathbb{C}$ entire, not constant, omits 0 & 1.

Step A produces a function g entire, nonconstant and $\alpha > 0$ with $\Delta \not\subset \operatorname{Im} g$ for all discs Δ of radius α

Step B For any g entire & not constant, $\operatorname{Im} g$ contains a disc of any radius, in particular of radius α.

Step A & Step B are incompatible, showing f does not exist $\Rightarrow$ Little Picard.

§1. Bloch's Theorem   Conway XII. 1.

Notation G open & bounded $\Rightarrow \bar{G}$ compact

$\mathcal{O}(\bar{G})$ = set of holomorphic functions in a neighborhood of $\bar{G}$

Theorem (version of Conway XII. 1.4). $\Delta = \Delta(0,1)$.

Given $f \in \mathcal{O}(\bar{\Delta})$, $f'(0) = 1$, then $\operatorname{Im} f$ contains a disc a radius $\beta > 0$. In fact $\beta = \frac{3}{2} - \sqrt{2} \cong .085$ works

Crucially β is a constant independent of the function f.

This is important for Little Picard.

Remark The value of β in Conway is $\beta = \frac{1}{72} \cong .01$

This β is smaller, however Conway proves a little more.

Bloch $\Rightarrow$ Step B   Conway XII. 2.

$g : \mathbb{C} \longrightarrow \mathbb{C}$ entire, not constant $\Rightarrow$ Im g contains a disc of any radius.

Proof Fix a value r for the radius.

g not constant $\Rightarrow \exists\, a$ with $g'(a) \neq 0$. WLOG $a = 0$ else work with $g^{new}(z) = g(z+a)$.

Let $R > 0$. Define $h(z) = \dfrac{g(Rz)}{R g'(0)} \Rightarrow h'(0) = 1$ &

h holomorphic in $\bar{\Delta} \Rightarrow$ Im h contains a disc of radius β

$\Rightarrow$ Im g contains a disc of radius $R\,|g'(0)|\,\beta > r$ if R is chosen large.

Remark This completes the proof of Little Picard.

Remark The proof shows $g \in \mathcal{O}(\bar{\Delta}(a,R)) \Rightarrow$ Im g contains a disc of radius $R|g'(a)|\beta$.

Remark : Optimal value of β Conway XII. 1. 9

Let $\mathcal{F} = \{ f \in \mathcal{O}(\bar{\Delta}),\ f'(0) = 1 \}$

$\boxed{i}$ $L_f = \{$ largest radius of a disc in $f(\Delta) \}$

$$L = \inf_{f \in \mathcal{F}} L_f = \text{Landau constant}$$

$\boxed{ii}$ $B_f = \{$ largest radius of a biholomorphic disc in $f(\Delta) \}$

$$B = \inf_{f \in \mathcal{F}} B_f = \text{Bloch constant}$$

$\boxed{iii}$ Current knowledge: $.5 < L < .544$

$$.433 < B < .472$$

Conjecturally $$B = \sqrt{\frac{\sqrt{3}-1}{2}} \; \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)} \cong .471$$

We will show $L \geq \beta = \frac{3}{2} - \sqrt{2} \cong .08$.

2. Lemma. — *Die Funktion*

$$w = C\,\zeta\,\frac{\int\limits_0^1 t^{-\frac{1}{2}-\frac{1}{2k}}(1-t)^{-\frac{1}{6}+\frac{1}{2k}}(1-\zeta^3 t)^{-\frac{5}{6}+\frac{1}{2k}}\,dt}{\int\limits_0^1 t^{-\frac{1}{2}-\frac{1}{2k}}(1-t)^{-\frac{5}{6}+\frac{1}{2k}}(1-\zeta^3 t)^{-\frac{1}{6}+\frac{1}{2k}}\,dt}$$

vermittelt die konforme Abbildung des Kreises $|\zeta| < 1$ *auf ein gleichseitiges Kreisbogendreieck mit den Winkeln* π/k $(k > 1)$. *Die Punkte* 1, ε, ε^2 $\left(\varepsilon = \frac{-1+i\sqrt{3}}{2}\right)$ *entsprechen den Eckpunkten.*

Man findet die Abbildungsfunktion am einfachsten, wenn man von der Schwarzschen Beziehung

$$\{w, \zeta\} = \frac{9}{2}\left(1 - \frac{1}{k^2}\right)\frac{\zeta}{(\zeta^3-1)^2}$$

ausgeht und die assoziierte lineare Differentialgleichung

$$\frac{y''}{y} = -\frac{9}{4}\left(1 - \frac{1}{k^2}\right)\frac{\zeta}{(\zeta^3-1)^2}$$

zuerst durch die Substitution $y = (\zeta^3-1)^{\frac{1}{2}-\frac{1}{2k}}v$, $\zeta^3 = \xi$, dann durch $y = \zeta(\zeta^3-1)^{\frac{1}{2}-\frac{1}{2k}}u$, $\zeta^3 = \xi$ auf eine hypergeometrische reduziert.

Bestimmt man C so, daß $w'(0) = 1$ wird, so ergibt sich

$$w(1) = \frac{\mathrm{B}\left(\frac{1}{2}-\frac{1}{2k},\ \frac{1}{6}+\frac{1}{2k}\right)}{\mathrm{B}\left(\frac{1}{2}-\frac{1}{2k},\ \frac{5}{6}+\frac{1}{2k}\right)} = \frac{\Gamma\left(\frac{1}{6}+\frac{1}{2k}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{6}+\frac{1}{2k}\right)\Gamma\left(\frac{2}{3}\right)}.$$

Die uns interessierenden Fälle sind $k = 3$ und $k = 6$. Man findet durch Einsetzung

$$\lambda = \frac{w_2(1)}{w_1(1)} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma(1)\,\Gamma\left(\frac{1}{4}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}. \tag{3}$$

Aus (1), (2) und (3) erhält man jetzt endlich

$$\mathfrak{B}' = \sqrt{\frac{\sqrt{3}-1}{2}}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)} = \sqrt{\pi}\cdot 2^{1/4}\cdot\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{4}\right)}\left(\frac{\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{12}\right)}\right)^{1/2} = 0{,}4719\ldots$$

und es ist somit bewiesen, daß

$$\mathfrak{B} \leqq \mathfrak{B}' < 0{,}472.$$

Früher bekannt war die Landausche Abschätzung[1])

$$0{,}396 < \mathfrak{B} < 0{,}555.$$

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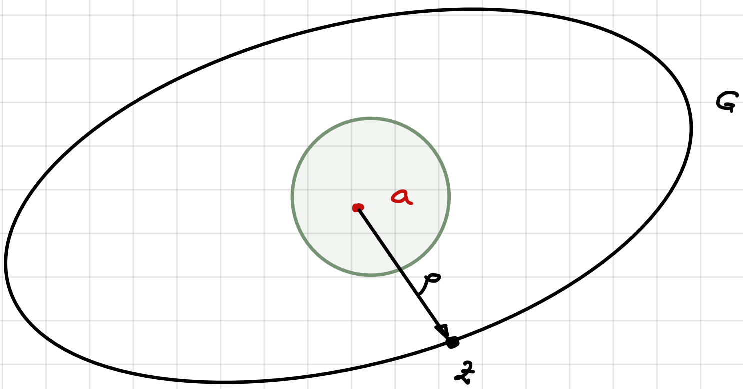
[1]) Landau, Über die Blochsche Konstante und zwei verwandte Weltkonstanten. Mathem. Zeitschr. 30 (1929), 608—634, insbesondere S. 614.

Ahlfors - Grunski, Math. Zeitscrift, 42 (1937)

§2. Proof of Bloch's theorem

Question How can we construct a disc in $\operatorname{Im} f$?

Assume G bounded, $a \in G$. Let $\rho = \min_{z \in \partial G} |f(z) - f(a)|$



Lemma A Let $f \in \mathcal{O}(\bar{G})$, $a \in G$, $\rho = \min_{z \in \partial G} |f(z) - f(a)|$.

Then $\operatorname{Im} f$ contains $\Delta(f(a), \rho)$.

Remark This can be viewed as a more precise Open Mapping theorem.

Proof Let $H = f(G) \subseteq f(\bar{G}) =$ compact since $\bar{G}$ is compact. Then H is bounded $\Rightarrow$ ∂H compact. Let

$$R = d(f(a), \partial H) = \min_{h \in \partial H} |h - f(a)|.$$

$\Rightarrow$ $\Delta(f(a), R) \subseteq H \subseteq \operatorname{Im} f$. We show

$$R \geq \rho$$

$\Rightarrow$ $\Delta(f(a), \rho) \subseteq \operatorname{Im} f$ proving Lemma A.

To prove $R \geq \rho$, let $w \in \partial H$ achieve the minimum R.

Claim $w = f(z)$, $z \in \partial G$.

Then $R = |w - f(a)| = |f(z) - f(a)| \geq \rho$ by definition of ρ as a minimum.

Proof of the Claim Since $w \in \partial H \Rightarrow \exists\, h_n \in H$, $h_n \to w$.

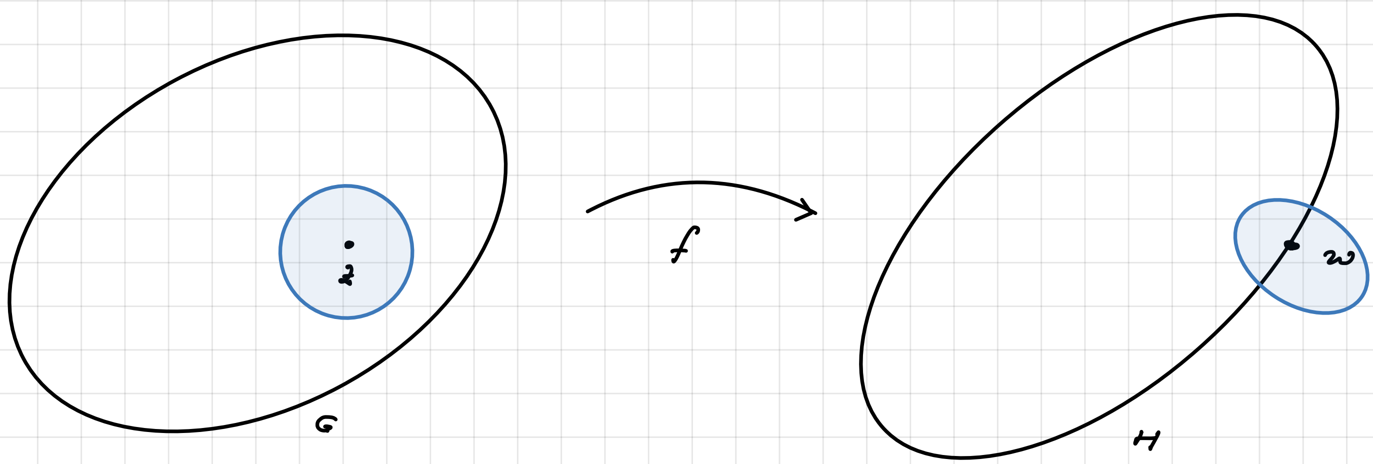
Write $h_n = f(g_n)$, $g_n \in G \subseteq \bar{G}$. Since $\bar{G}$ compact

$\Rightarrow$ passing to a subsequence we may assume that $g_n \longrightarrow z \in \overline{G}$. $\Rightarrow$ $f(g_n) \longrightarrow f(z)$. Since

$f(g_n) = h_n \longrightarrow w$, we conclude

$$w = f(z).,\quad z \in \overline{G}$$

If $z \in G$, we contradict Open Mapping Theorem.



(Pick a disc near z. its image will be open so it will contain a disc near w but $w \in \partial H$ contradiction).

Thus $z \in \partial G$ proving the Claim & Lemma A.

Strategy for Bloch

Apply Lemma A & show $|f(z) - f(a)| \geq \beta$ for suitable a.

Question: Why is the proof difficult?

Answer: We don't know a. In other words, we don't know where the center of the disc in Bloch should be.

More detailed strategy

$\boxed{I}$ prove Bloch under Assumption (*)

$\boxed{II}$ remove Assumption (*)

In Step $\boxed{I}$ we have control of the center & the radius equals 2β (better than Bloch claims).

In Step $\boxed{II}$ we lose control of center, radius halves, but we have no assumptions.

Assumption (*)

$$f \in \mathcal{O}(\bar{\Delta}), \quad |f'(z)| \le 2|f'(0)| \quad \forall\ |z| \le 1$$

Lemma B (Bloch assuming (*))

If f satisfies Assumption (*) $\Rightarrow$ $\operatorname{Im} f$ contains a disc with center $f(0)$ & radius $2\beta|f'(0)|$

Remark If $f'(0) = 1$, this implies under Assumption (*)

Lemma C (Bloch without (*)).

For all $f \in \mathcal{O}(\bar{\Delta})$, even in the absence of Assumption (*), $\operatorname{Im} f$ contains a disc of radius $\beta|f'(0)|$.

Note Lemma C $\Rightarrow$ Bloch.

Next time we show Lemma B $\Rightarrow$ Lemma C.

Math 220C - Lecture 15

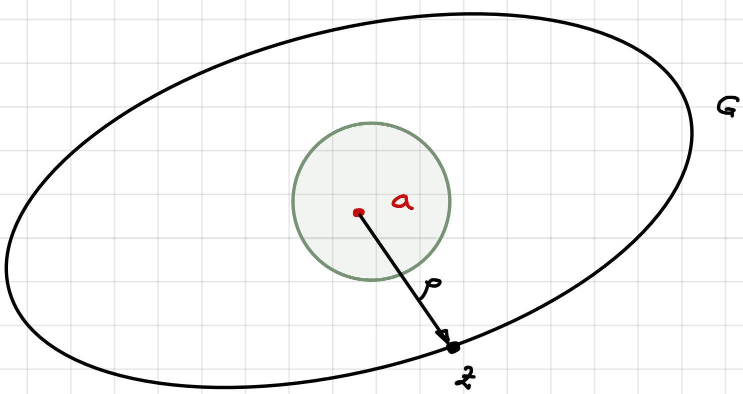
April 30, 2021

. Last time

Theorem (version of Conway XII. 1.4). $\Delta = \Delta(0,1)$.

Given $f \in \mathcal{O}(\bar{\Delta})$, $f'(0) = 1$, then $\operatorname{Im} f$ contains a disc a radius $\beta > 0$. In fact $\beta = \frac{3}{2} - \sqrt{2} \cong .085$ works

Main Tool — Sharper Open Mapping Theorem



Lemma A Let $f \in \mathcal{O}(\bar{G})$, $a \in G$, $\rho = \min_{z \in \partial G} |f(z) - f(a)|$.

Then $\operatorname{Im} f$ contains $\Delta(f(a), \rho)$.

Lemma B (Stronger form of Bloch but with assumptions)

If $f \in \mathcal{O}(\bar{\Delta}(a,R))$ and $|f'(z)| \leq 2|f'(a)|$ in $\bar{\Delta}(a,R)$, then $\operatorname{Im} f$ contains a disc of radius $2\beta\,|f'(a)|\,R$.

Remark $R=1$, $a=0$, $f'(0)=1$ is Bloch under the assumption $|f'(z)| \leq 2$. We get a disc of radius 2β!

Remark We state this for all centers a since we don't know where our center will end up.

Proof WLOG $R=1$ & $a=0$, else rescale & translate.

WLOG $f(0)=0$ else work with $f-f(0)$.

Hypothesis $|f'(z)| \leq 2|f'(0)|$ for $|z| \leq 1$.

Goal Disc of radius $2\beta\,|f'(0)|$.

Plan Estimate $\rho = \min |f(z) - f(0)|$ when $|z| = r$.

& Apply **Lemma A** $\Rightarrow \operatorname{Im} f \supseteq \Delta(f(0), \rho)$.

Estimate for ρ

Let $F(z) = f(z) - z f'(0) =$

$$= \int_0^z (f'(w) - f'(0))\, dw$$

$w = zt,\ 0 \le t \le 1$

$$= \int_0^1 (f'(tz) - f'(0))\, z\, dt$$

Apply **Cauchy Integral Formula**

$$f'(tz) - f'(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f'(\zeta)}{\zeta - tz}\, d\zeta - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f'(\zeta)}{\zeta}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{|\zeta|=1} f'(\zeta) \left(\frac{1}{\zeta - tz} - \frac{1}{\zeta} \right) d\zeta$$

$$= \frac{1}{2\pi i} \int_{|\zeta|=1} f'(\zeta) \cdot \frac{tz}{\zeta(\zeta - tz)}\, d\zeta$$

Substituting,

$$F(z) = \frac{1}{2\pi i} \int_0^1 \int_{|\zeta|=1} f'(\zeta) \cdot \frac{tz^2}{\zeta(\zeta - tz)} \cdot d\zeta\, dt$$

Take absolute values. Note

Hypothesis

$$\left| f'(\zeta) \cdot \frac{tz}{\zeta(\zeta - tz)} \right| \le 2|f'(0)| \cdot \frac{t|z|}{1 \cdot (1-|z|)}$$

since $|\zeta - tz| \ge |\zeta| - |tz| = 1 - t|z| \ge 1 - |z|$.

Therefore

$$|F(z)| \le \frac{1}{2\pi} \cdot \int_0^1 2|f'(0)| \cdot \frac{t|z|^2}{1-|z|} \cdot dt \cdot \underbrace{\text{length}(|\zeta| = 1)}_{2\pi}$$

$$= 2|f'(0)| \cdot \frac{r^2}{1-r} \cdot \underbrace{\int_0^1 t\, dt}_{1/2}$$

$$= |f'(0)| \cdot \frac{r^2}{1-r}. \qquad \text{for } |z| = r. \qquad (1)$$

On the other hand, by triangle inequality

$$|F(z)| = |z f'(0) - f(z)| \ge |z f'(0)| - |f(z)|$$

$$= r|f'(0)| - |f(z)| \qquad (2)$$

Using (1) & (2) we find

$$r|f'(0)| - |f(z)| \le |f'(0)| \cdot \frac{r^2}{1-r}$$

$$\Rightarrow \quad |f(z)| \geq |f'(0)| \cdot \left(r - \frac{r^2}{1-r} \right). \qquad \text{for } |z| = r.$$

We haven't specified r yet. In any case, from Lemma A applied to $f|_{\bar{\Delta}(0,r)}$, the image of f contains a disc of radius

$$|f'(0)| \left(r - \frac{r^2}{1-r} \right).$$

To get the best radius, we maximize

$$r - \frac{r^2}{1-r}.$$

The critical point is $r_0 = 1 - \frac{1}{\sqrt{2}}$, maximum value equals 2β.

We obtain a disc of radius $2\beta\,|f'(0)|$.

Lemma B $\Rightarrow$ Bloch We show

For all $f \in \mathcal{O}(\overline{\Delta})$, $\operatorname{Im} f$ contains a disc of radius $\beta |f'(0)|$.

When $f'(0) = 1$, this is exactly Bloch's theorem.

Proof Let $h(z) = |f'(z)| (1 - |z|)$ continuous in $\overline{\Delta}$.

Let M be the maximum of h achieved at p.

Let $1 - |p| = 2t$ $\Rightarrow$

$$M = h(p) = |f'(p)| (1 - |p|) = 2t\, |f'(p)|.$$

If $z \in \overline{\Delta}(p, t)$ then $|z - p| \le t$, $|p| = 1 - 2t$ $\Rightarrow$ $|z| \le |z - p| + |p| \le 1 - t$

$\Rightarrow$ $1 - |z| \ge t$. Therefore since p is a maximum,

$$\underbrace{(1 - |z|)}_{\ge t} |f'(z)| \le \underbrace{(1 - |p|)}_{2t} |f'(p)| \Rightarrow |f'(z)| \le 2 |f'(p)|.$$

in $\overline{\Delta}(p, t)$.

Apply Lemma B to $f|_{\bar{\Delta}(p,t)}$ $\Rightarrow$ the image of f

contains a disc of center $f(p)$ and radius

$$2\beta\,|f'(p)|\,t = \beta M.$$

Note $M = \max_{|z|\le 1} h \ge h(0) = |f'(0)|$ $\Rightarrow$ the disc we constructed

has radius $\beta M. \ge \beta\,|f'(0)|.$

This completes the proof of Bloch & Little Picard along with it.

Remark* (will not use)

The reason for our choice of h is not transparent

The choice of center is also mysterious. We motivate

these choices below.

Question What is the most natural h ?

Answer We seek to achieve $|f'(z)| \leq 2|f'(0)|$ & use Lemma B.

We have a better chance if we maximize $|f'(0)|$

What happens if we replace f by $f \circ \varphi_{-\alpha}$ where $\varphi_{-\alpha} \in \mathrm{Aut}\,\Delta$. ?

Note

Math 220B

$$|(f \circ \varphi_{-\alpha})'(0)| = |f'(\varphi_{-\alpha}(0)) \cdot \varphi_{-\alpha}'(0)| = |f'(\alpha)|\,(1-|\alpha|^2).$$

Thus to get a larger derivative, we are led to maximizing

$$\tilde{h}(z) = |f'(z)|\,(1-|z|^2)$$

which is similar to what we used. This also suggests the new center is $\varphi_{-\alpha}(0) = \alpha$, which is also what we used.

Exercise Run the above argument using $\tilde{h}$ instead of h.

Remark* (will not use) Original proof of Little Picard.

Sketch:

Construct $\lambda : \mathfrak{H}^+ \longrightarrow \mathbb{C}\setminus\{0,1\}$ universal cover

In the diagram

$$\begin{tikzcd}
 & \mathfrak{H}^+ \arrow[r, "\overset{\text{Cayley}}{\sim}", "c"'] \arrow[d, "\lambda"] & \Delta \\
\mathbb{C} \arrow[r, "f"] \arrow[ur, dashed, "\tilde f"] & \mathbb{C}\setminus\{0,1\} &
\end{tikzcd}$$

show we can lift f to a holomorphic function $\tilde f$.

Then $c\circ\tilde f : \mathbb{C}\longrightarrow\Delta$ is entire & bounded $\Rightarrow$ $c\circ\tilde f$ is constant $\Rightarrow$ $\tilde f$ constant $\Rightarrow$ $f=\lambda\circ\tilde f$ is constant. QED.

The crux of the matter is the construction of λ

(1) λ holomorphic

(2) λ Γ-invariant, Γ = Deck transformations.

λ is a modular function for the group $\Gamma(2)$.

(1) End of material for Qualifying Exam.

(2) Qualifying Exam – May 18, 5 – 8 PM.

(3) closed book, notes, internet, via Gradescope

(4) covers Math 220 AB & Math 220C up to and including Lecture 15

(5) Past Qualifying Exams are linked on website

(6) Review – closer to the date (May 14? May 17?)

(7) Last homework – Homework 6.

What is next?

(1) Great Picard – Conway XII.3, XII.4.

(2) An introduction to Riemann Surfaces.

Math 220C - Lecture 16

May 3, 2021

Lecture 14 – Proof of Little Picard (summary)

Step A $\quad f : G \longrightarrow \mathbb{C} \setminus \{0,1\}$, G simply connected

(1) Write $f = \frac{1}{2}(1 + \cos \pi \cos \pi g)$.

(2) Im g contains no disc of radius α.

(3) $h(z) = \dfrac{g(Rz)}{R\, g'(0)}$, $h \in \mathcal{O}(\bar{\Delta})$, $h'(0) = 1$.

If $R >> 0$, we showed h contradicts Bloch.

Step B (Bloch – Lecture 15)

$h \in \mathcal{O}(\bar{\Delta})$, $h'(0) = 1 \implies$ Im h contains a disc of radius β

Road map to Great Picard

$f : G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f|_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.

$$\begin{tikzcd}
 & \text{Bloch \& Landau} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Schottky (today)} \arrow[d, Rightarrow] & & \text{Little Picard} \\
\text{Strong Montel (next time)} \arrow[d, Rightarrow] & & \\
\text{Great Picard (next time)} & &
\end{tikzcd}$$

The broad goal is to study the family

$$\mathcal{F} = \{ f : G \longrightarrow \mathbb{C} \setminus \{0,1\} \text{ holomorphic} \} .$$

When $G = \mathbb{C}$, $\mathcal{F}$ consists of constant functions.

by Little Picard

When $G = \Delta^*(a,r)$ this is relevant for Great Picard

Question Is $\mathcal{F}$ normal?

Remark To answer this question we need uniform bounds on $|f(z)|$ in small discs.

Schottky's theorem

$\exists$ function $c(a,b)$ for $0 < a < \infty$, $0 < b < 1$, increasing in each variable so that

$\forall f \in \mathcal{O}(\bar{\Delta})$ omitting 0 & 1, $|f(0)| = a$, then

$$|f(z)| \le c(a,b) \quad \text{if } |z| \le b$$

Remark The theorem controls the growth of $f \in \mathcal{F}$ in a universal fashion provided $|f(0)| =$ fixed.

Remark We will show that

$$c(a,b) = \frac{1}{2} + \frac{1}{2} \exp \pi \exp \pi \left(3 + 2a + \frac{\alpha}{\beta} \cdot \frac{b}{1-b} \right).$$

Key Claim

For each $z \in \mathbb{C}$, the equation $\cos \pi a = z$ admits a solution

$$|a| \leq 1 + |z|.$$

Proof It is easy to check that $\cos \pi a = z$ admits a solution a by converting into a quadratic equation in $w = e^{\pi i a}$ using

$$\cos \pi a = \frac{w + w^{-1}}{2} = z$$

Note that if a is a solution, $a + 2$ is also a solution.

Thus we may assume $\operatorname{Re} a \in [-1, 1] \Rightarrow |\operatorname{Re} a| \leq 1$.

Then

$$|a| \leq |\operatorname{Re} a| + |\operatorname{Im} a| \overset{(*)}{\leq} 1 + |\cos \pi a| = 1 + |z|.$$

Inequality $(*)$ $\quad |\operatorname{Im} a| \leq |\cos \pi a|$

Proof $\quad a = x + iy$

$$\cos \pi a = \frac{e^{\pi a i} + e^{-\pi a i}}{2} =$$

$$= \frac{e^{\pi x i} e^{-\pi y} + e^{-\pi x i} \cdot e^{\pi y}}{2}$$

$$= \cos \pi x \left(\frac{e^{\pi y} + e^{-\pi y}}{2} \right) + i \sin \pi x \left(\frac{e^{-\pi y} - e^{\pi y}}{2} \right)$$

$$\Rightarrow |\cos \pi a|^2 = \cos^2 \pi x \left(\frac{e^{\pi y} + e^{-\pi y}}{2} \right)^2 + \sin^2 \pi x \left(\frac{e^{-\pi y} - e^{\pi y}}{2} \right)^2$$

$$= \left(\frac{e^{-\pi y} - e^{\pi y}}{2} \right)^2 + \cos^2 \pi x$$

$$= \sinh^2 \pi y + \cos^2 \pi x$$

$$\geq \sinh^2 \pi y \geq (\pi y)^2 > y^2 = |\mathrm{Im}\, a|^2.$$

This completes the proof.

Proof of Schottky's theorem

Step 1 $\boxed{\text{Revisit Landau's Lemma}}$

Let $f \in \mathcal{O}(\bar{\Delta})$ omitting 0 & 1 $\Rightarrow$ $2f-1$ omits -1 & 1.

By Landau

$$2f-1 = \cos \pi F \Rightarrow 2f(0)-1 = \cos \pi F(0).$$

By Key Claim, we may assume

$$\boxed{|F(0)| \leq 1 + |2f(0)-1|}$$

By Lecture 13, F omits ± 1. We write

$$F = \cos \pi g \Rightarrow F(0) = \cos \pi g(0).$$

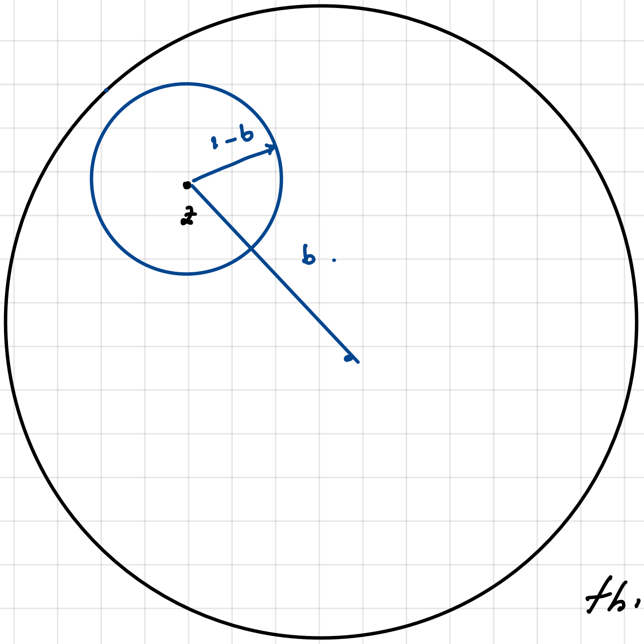
By Key Claim, we may assume

$$\boxed{|g(0)| \leq 1 + |F(0)|.} \leq 1 + 1 + |2f(0)-1| \leq 3 + 2|f(0)|.$$

Conclusion $f = \frac{1}{2}(1 + \cos \pi \cos \pi g)$ &

$$\boxed{|g(0)| \leq 3 + 2a} \quad \text{if } |f(0)| = a.$$

Step 2 Bounding g'



Let $|z| \le b$. $\Rightarrow \bar{\Delta}(z, 1-b) \subseteq \bar{\Delta}$.

Define

$$h(w) = \frac{g(z + (1-b)w)}{(1-b)\, g'(z)}$$

(recenter & rescale). Compare this to item (3) Step A in Little Picard.

$\Rightarrow h \in \mathcal{O}(\bar{\Delta})$, $h'(0) = 1$. By Bloch

$\Rightarrow$ Im h contains a disc of radius β

$\Rightarrow$ Im g contains a disc of radius $\beta(1-b)|g'(z)|$

We showed in Lecture 13, Im g contains no disc of radius α

$\Rightarrow \alpha \ge \beta(1-b)|g'(z)|$

$$\Rightarrow |g'(z)| \le \frac{\alpha}{\beta} \cdot \frac{1}{1-b}. \quad \forall\ |z| \le b.$$

Step 3   Bounding g and f

We have shown $|g'(z)| \leq \frac{\alpha}{\beta} \cdot \frac{1}{1-b}$ if $|z| \leq b$.

Note

$$|g(z)| = \left| g(0) + \int_0^z g'(w)\, dw \right|$$

$$\leq |g(0)| + \left| \int_0^z g'(w)\, dw \right|$$

$$\leq \underbrace{(3+2a)}_{\text{step 1}} + \underbrace{\left(\frac{\alpha}{\beta} \cdot \frac{1}{1-b} \right)}_{\text{step 2}} |z|$$

$$\leq (3+2a) + \frac{\alpha}{\beta} \cdot \frac{b}{1-b} \qquad \forall\ |z| = b,\ |f(0)| = a.$$

To bound f, we need

Claim $\quad |\cos w| \leq \exp|w|$

Proof $\quad |\cos w| = \left| \frac{e^{iw} + e^{-iw}}{2} \right| \leq \frac{|e^{iw}| + |e^{-iw}|}{2}$

$$= \frac{e^{\operatorname{Re}(iw)} + e^{\operatorname{Re}(-iw)}}{2}$$

$$\leq \frac{e^{|iw|} + e^{|-iw|}}{2} = e^{|w|}.$$

Now we can finish the argument

$$|f(z)| = \left|\frac{1}{2} + \frac{1}{2}\cos\pi\cos\pi g(z)\right|$$

$$\leq \frac{1}{2} + \frac{1}{2}\left|\cos\pi\cos\pi g(z)\right|$$

Claim

$$\leq \frac{1}{2} + \frac{1}{2}\exp\pi\left|\cos\pi g(z)\right|$$

Claim

$$\leq \frac{1}{2} + \frac{1}{2}\exp\pi\exp\pi|g(z)|$$

$$\leq \frac{1}{2} + \frac{1}{2}\exp\pi\exp\pi\left(3 + 2a + \frac{\alpha}{\beta}\cdot\frac{b}{1-b}\right)$$

$$= c(a,b) \quad \text{if} \quad |f(0)| = a, \quad |z| = b.$$

Friedrich Schottky

(1851 - 1935)

Academic advisors

Karl Weierstrass

Worked on elliptic, abelian, and theta functions.

Schottky problem:

Characterization of Jacobian varieties amongst abelian varieties.

The author is of a clumsy appearance, unprepossessing, a dreamer, but if I'm not completely wrong, he possesses an important mathematical talent. [...] As rector I had to cancel his name from the register because neither had he attended lectures nor were his whereabouts in Berlin known. (Weierstrass.)

https://mathshistory.st-andrews.ac.uk/Biographies/Schottky/

Math 220C - Lecture 17

May 5, 2021

Last time — Schottky's Theorem

$\exists$ function $c(a,b)$ for $0 < a < \infty$, $0 < b < 1$, increasing in each variable so that

$\forall f \in \mathcal{O}(\overline{\Delta}(0,R))$ omitting 0 & 1, $|f(0)| = a$, then

$$|f(z)| \leq c(a,b) \text{ if } |z| \leq bR.$$

Remark Last time, $R = 1$. The above statement follows by rescaling

$$f^{new}(z) = f(Rz).$$

§1. Strong Montel — Conway XII.4

$$\mathcal{F} = \{ f \text{ holomorphic in } G, \text{ omitting } 0 \,\&\, 1 \}.$$

Question Is $\mathcal{F}$ normal?

Answer No! If $G = \mathbb{C}$, $\mathcal{F}$ consists of constants by Little Picard.

If $f_n = c_n = \text{constant}$, it may happen that $c_n \to \infty$.

We modify the definition of a normal family:

Definition A family $\mathcal{F}$ is normal in the extended sense if every sequence in $\mathcal{F}$ admits a subsequence converging locally uniformly to a function or converging locally uniformly to ∞.

Strong Montel Theorem

The family

$$\mathcal{F} = \{ f \text{ holomorphic in } G, \text{ omitting } 0 \,\&\, 1 \}.$$

is normal in the extended sense.

Remark This is clear for $G = \mathbb{C}$ by Little Picard.

Remark In Math 220B, we showed

$\mathcal{H}$ normal $\iff$ $\mathcal{H}$ locally bounded.

Proof Fix $z_0 \in G$. Define

$$\mathcal{F}^+ = \{ f \in \mathcal{F}, \ |f(z_0)| \leq 1 \}$$

$$\mathcal{F}^- = \{ f \in \mathcal{F}, \ |f(z_0)| \geq 1 \}$$

$$\implies \mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-.$$

Claim $\mathcal{F}^+$ locally bounded.

Thus $\mathcal{F}^+$ is normal.

Let $f_n \in \mathcal{F}$ be a sequence. Either

$\boxed{i}$ ∞ - many terms of f_n are in $\mathcal{F}^+$

$\boxed{ii}$ ∞ - many terms of f_n are in $\mathcal{F}^-$

In case $\boxed{i}$, we may assume $f_n \in \mathcal{F}^+$ after relabelling.

$\mathcal{F}^+$ normal shows $\{f_n\}$ has a convergent subsequence as needed.

In case $\boxed{ii}$, we may assume $f_n \in \mathcal{F}^-$ after relabelling.

$\Rightarrow$ $1/f_n \in \mathcal{F}^+$. Passing to a subsequence, we may assume $1/f_n \overset{l.u.}{\rightrightarrows} \varphi$ since $\mathcal{F}^+$ is normal. Since $1/f_n$ is zero-free

$\Rightarrow$ φ is zero-free or $\varphi \equiv 0$ by Hurwitz's theorem.

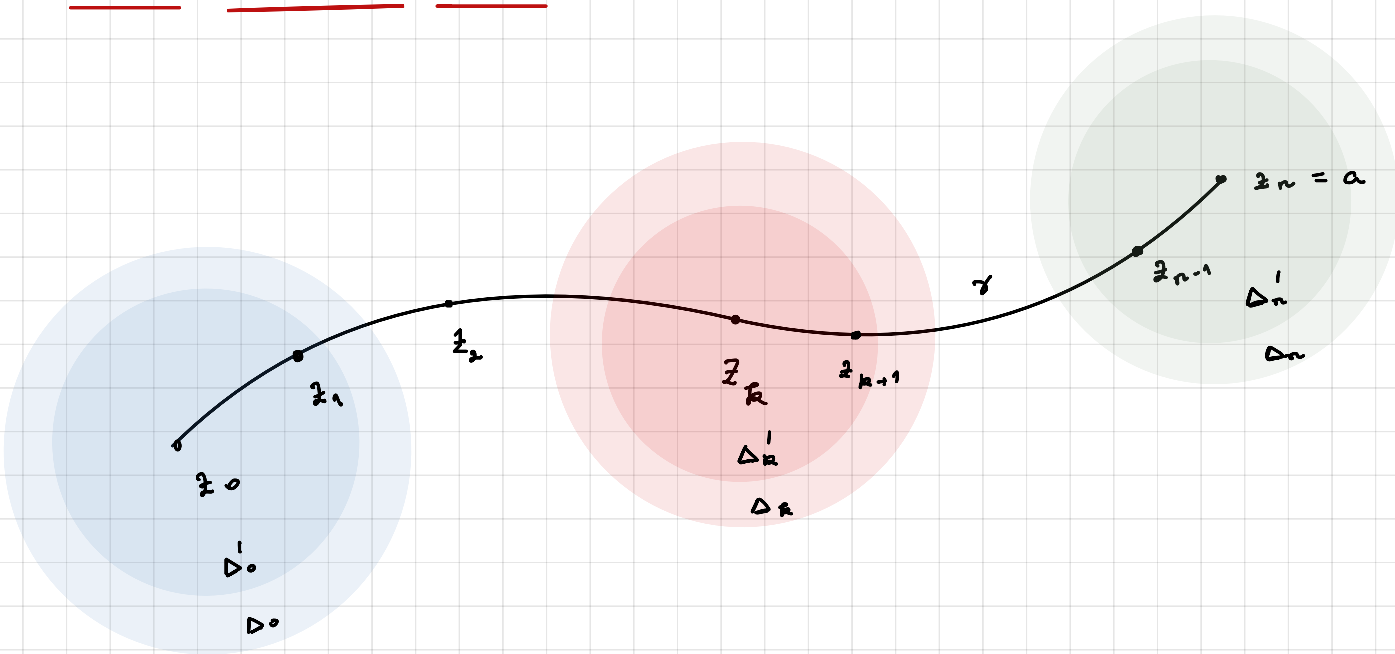
If φ zero-free, $f_n \overset{l.u.}{\rightrightarrows} 1/\varphi$. If $\varphi \equiv 0$, $1/f_n \overset{l.u.}{\rightrightarrows} 0$ so $f_n \overset{l.u.}{\rightrightarrows} \infty$. This is what we needed.

Math 220 A - Lecture 24

Hurwitz $\quad f_n \overset{l.u.}{\rightrightarrows} f$, f_n holomorphic in $\mathcal{U}$,

If f_n is zero free $\forall n \implies f$ zero-free or $f \equiv 0$.

Proof of the claim



We know $|f(z_0)| \leq 1$ for $f \in \mathcal{F}^+$. We seek to bound f near any $a \in G$.

Let γ be a path joining z_0 to a. Cover γ by

- discs $\Delta_0', \Delta_1', \dots, \Delta_n'$ of centers $z_0, \dots, z_n = a$
- such that $\bar{\Delta}_k' \subseteq \Delta_k \subseteq G \quad \forall k$.
- $z_{k+1} \in \Delta_k' \quad \forall k$

This can be done via a compactness argument.

Apply Schottky's Theorem in $\Delta_0' = \Delta_0'(z_0, R_0)$. Note $f \in \mathcal{O}(\bar{\Delta}_0')$. Then $z_1 \in \Delta_0'$ so

$$|f(z_1)| \le C\left(|f(z_0)|, \frac{|z_1 - z_0|}{R_0}\right) \le C\left(1, \frac{|z_1 - z_0|}{R_0}\right) := C_1$$

Apply Schottky's Theorem in $\Delta_1' = \Delta_1'(z_1, R_1)$.

$$|f(z_2)| \le C\left(|f(z_1)|, \frac{|z_2 - z_1|}{R_1}\right) \le C\left(C_1, \frac{|z_2 - z_1|}{R_1}\right) := C_2$$

Continue in this fashion. We obtain

$$|f(z_n)| \le C_n. \implies |f(a)| \le C_n \text{ since } z_n = a.$$

Apply Schottky one more time in $\overline{\Delta}(a, \frac{R_n}{2})$.

$$|f(z)| \le C\left(|f(z_n)|, \frac{|z_n - z|}{R_n}\right) \le C\left(c_n, \frac{1}{2}\right) := M.$$

Thus $|f(z)| \le M \quad \forall f \in \mathcal{F}$, in the disc $\overline{\Delta}(a, R/2)$.

This proves the claim & Strong Montel.

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§2. Great Picard

$f : G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f|_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.

Remark The proof is very similar to Math 220B, Midterm 2

Problem 2. [*10 points.*]

Let $f : \Delta(0,1) \setminus \{0\} \to \mathbb{C}$ be a holomorphic function on the punctured unit disc. Let

$$f_n : \Delta(0,1) \setminus \{0\} \to \mathbb{C}, \quad f_n(z) = f\left(\frac{z}{n}\right).$$

Show that the family $\mathcal{F} = \{f_n : n \geq 1\}$ is normal iff f has a removable singularity at the origin.

Proof We show $f\big|_{\Delta^*(a,r)}$ omits at most one value.

WLOG $a = 0$. Write $\Delta^* = \Delta(0,r) \setminus \{0\}$.

Assume $f\big|_{\Delta^*}$ omits two values, say 0 & 1.

Let $\mathcal{F} = \{ f : \Delta^* \longrightarrow \mathbb{C},\ f \text{ omits } 0 \,\&\, 1 \}$. $\Rightarrow$ $\mathcal{F}$ normal in the extended sense. Let

$$f_n(z) = f\left(\frac{z}{n}\right) \Rightarrow f_n \in \mathcal{F}.$$

Thus $\{f_n\}$ is normal in the extended sense being a subfamily of $\mathcal{F}$.

Thus $f_{n_k} \overset{\text{l.u.}}{\rightrightarrows} \varphi$ or $f_{n_k} \overset{\text{l.u.}}{\rightrightarrows} \infty$.

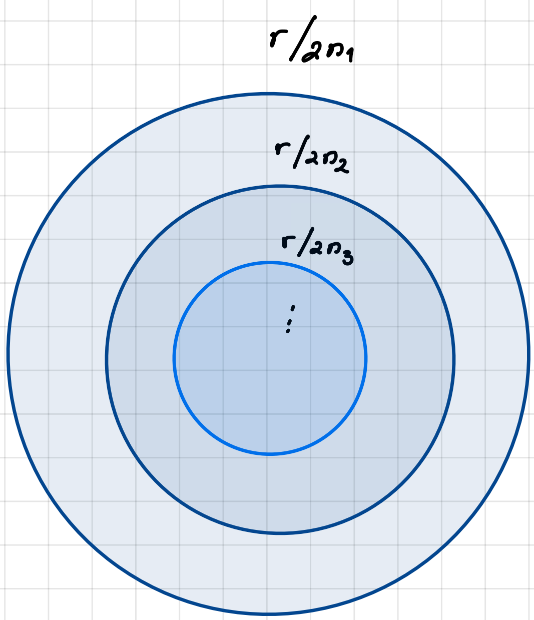
In the first case we show f has a removable singularity.

This follows from the midterm problem whose argument we recall. Since φ cont. for $|z| = \frac{r}{2}$ $\Rightarrow$ $|\varphi(z)| \le M$ for $|z| = \frac{r}{2}$

Since $f_{n_k} \rightrightarrows \varphi$ for $|z| = \frac{r}{2}$ $\Rightarrow$ $|f_{n_k}(z) - \varphi(z)| \le M$ for $|z| = \frac{r}{2}$

$\Rightarrow$ $|f_{n_k}(z)| \le |f_{n_k}(z) - \varphi(z)| + |\varphi(z)| \le M + M = 2M$ $\forall\ |z| = \frac{r}{2}$

$\Rightarrow$ $|f(w)| \le 2M$ $\forall\ |w| = \frac{r}{2n_k}$.



By maximum principle,

$|f(w)| \le 2M$ for $\frac{r}{2n_{k+1}} \le |w| \le \frac{r}{2n_k}$.

Since $\bigcup \bar{\Delta}\left(0; \frac{r}{2n_{k+1}}, \frac{r}{2n_k}\right)$ cover a punctured neighborhood of 0, say $\tilde{\Delta}^*$, we have

$$|f(w)| \leq 2M \quad \text{in } \tilde{\Delta}^* \implies f \text{ has a removable}$$

singularity at 0. This contradicts the fact that the singularity is essential.

If $f_{n_k} \overset{l.u.}{\rightrightarrows} \infty$ then $1/f_{n_k} \overset{l.u.}{\rightrightarrows} 0$. By the argument above $1/f$ has a removable singularity $\Rightarrow$ f has at worst a pole, a contradiction.

Conclusion $f|_{\Delta^*(a,r)}$ omits at most one value $\forall r$

If 2 values are achieved finitely many times, shrink r & note that in $\Delta^*(a, r^{new})$ two values are omitted.

Conclusion $f|_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.

Math 220C - Lecture 18

May 7, 2021

Announcements

(1) Review — Friday, May 14

(2) Office Hours:

Wed, May 12, 4–5

Fri, May 14, 1–2

Mon, May 17, 3–4

(3) Qualifying Exam — Tuesday, May 18, 5–8

(4) No lecture/office hours on Wednesday, May 19

(5) Plenty of Practice Exams linked on website

(6) Gradescope

10 lectures left – Minicourse on Riemann Surfaces

First Goal – Introduction & basic properties

– So far, we have done complex analysis for domains $G \subseteq \mathbb{C}$ & studied holomorphic functions

– Many results carry over if we replace $G \subseteq \mathbb{C}$ by Riemann surfaces.

– The subject merges ideas from Complex Analysis with Geometry & Topology

– Connections w/ many fields

topology

differential geometry

algebraic geometry

arithmetic geometry

number theory

dynamics

....

Historically, Riemann Surfaces arose from attempts to understand analytic continuation of multi-valued functions

e.g. log ; algebraic functions

See Conway $\underline{\overline{IX}}$.

Riemann Surfaces – first defined by Riemann in his dissertation 1851

– the same dissertation considered the Riemann Mapping Theorem (Math 220B).

Bernhard Riemann (1826 – 1866).

Riemann surfaces were introduced by Riemann in his dissertation at Göttingen (1851). This transformed complex analysis, merging it with topology & algebraic geometry.

"We restrict the variables x, y to a finite domain by considering as the locus of the point O no longer the plane A but a surface T spread over the plane"

"We admit the possibility ... of covering the same part of the plane several times. However in such a case, we assume that those parts of the surface lying on top of one another are not connected by a line. Thus a fold or a splitting of parts of the surface cannot occur."

Translation by. R. Remmert,

"From Riemann surfaces to complex spaces"

Soc. Math. France, Congr 3 (1998)

- Klein : "Riemann's methods were regarded almost with distrust by other mathematicians".

- Ahlfors : "Riemann's writings are full of almost cryptic messages to the future".

§ 1. Sheaves

Sheaves in agriculture – a collection of stalks bundled together

Sheaves in mathematics

– we seek to formalize the concept of "function-like objects" e.g. holomorphic functions on Riemann surfaces

– the most elegant way of doing so is via sheaf theory

Definition Let X be a topological space. A *presheaf* of sets, abelian groups, rings ... is an assignment

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U})$$

of sets, abelian groups, rings... for all $\mathcal{U} \subseteq X$ open.

& restriction maps

$$\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

which should be homomorphisms of ... . We require

[i] $\rho_{UU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is the identity

[ii] $\forall\ W \subseteq V \subseteq \mathcal{U}$ we have

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV} : \begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"'] \arrow[rr, bend right, "\rho_{UW}"'] & \mathcal{F}(V) \arrow[r, "\rho_{VW}"'] & \mathcal{F}(W) \end{tikzcd}$$

Terminology

[i] elements $s \in \mathcal{F}(\mathcal{U})$ are called sections.

[ii] restriction maps $\rho_{UV}(s) = s/_V$.

Definition A presheaf $\mathcal{F} \to X$ is a **sheaf** provided

$\forall$ $\mathcal{U} = \bigcup_i \mathcal{U}_i$ open cover, $s_i \in \mathcal{F}(\mathcal{U}_i)$ with

$$s_i\big|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_j\big|_{\mathcal{U}_i \cap \mathcal{U}_j}$$

$\Rightarrow$ $\exists!$ $s \in \mathcal{F}(\mathcal{U})$ such that $s\big|_{\mathcal{U}_i} = s_i$.

Examples

$\boxed{\text{I}}$ X topological space, $\mathcal{F} = \mathcal{C}$ is the sheaf:

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U}) = \{ f : \mathcal{U} \to \mathbb{C} \text{ continuous} \}.$$

with the usual restriction maps $\mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(V)$, $f \to f\big|_V$.

$\boxed{\text{II}}$ $X \subseteq \mathbb{R}^n$ open, $\mathcal{F} = \mathcal{C}^k$, $0 \le k \le \infty$, $k = \omega$

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U}) = \{ f : \mathcal{U} \to \mathbb{C} \text{ of class } \mathcal{C}^k \}$$

is a sheaf.

III $G \subseteq \mathbb{C}$ open, $\mathcal{F} = \mathcal{O}_G$, $U \subseteq G$ open

$$\mathcal{O}_G(U) = \{f : U \longrightarrow \mathbb{C} \text{ holomorphic}\} \text{ is a sheaf.}$$

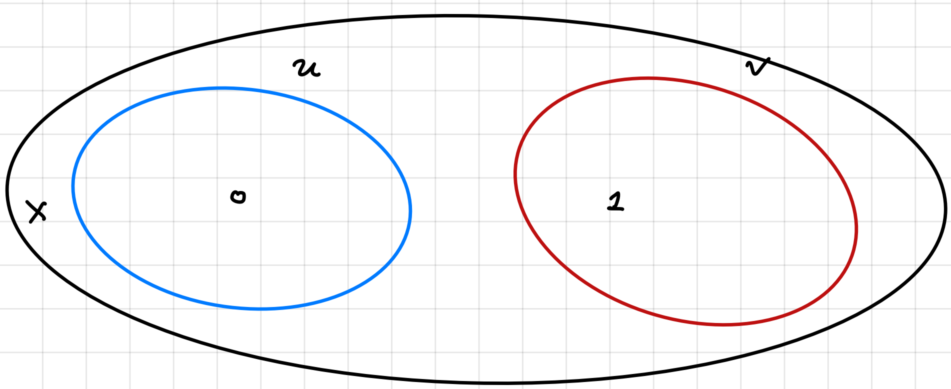
IV $p \in X$ topological space. The skyscraper sheaf

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

V The constant presheaf over X = top. space

$$\underline{\mathbb{C}}(U) := \{f : U \longrightarrow \mathbb{C} \text{ constant}\} \text{ is not a sheaf.}$$

<u>Why?</u> Assume $U, V \subseteq X$



$U \cap V = \emptyset$

$f_1 \equiv 0$ on U

$f_2 \equiv 1$ on V

$\Rightarrow f_1|_{U \cap V} = f_2|_{U \cap V}.$

Let $W = U \cup V$. Gluing fails.

However

$$\underline{\mathbb{C}}^{sh}: \mathcal{U} \longrightarrow \{f: \mathcal{U} \longrightarrow \mathbb{C} \text{ locally constant}\}$$ is a sheaf.

VII Restriction of sheaves to open sets

$\mathcal{F} \longrightarrow X$ sheaf, $\mathcal{U} \subseteq X$ open

Define $\mathcal{F}/_{\mathcal{U}}$ a sheaf over $\mathcal{U}$ via

$$\mathcal{F}/_{\mathcal{U}}(V) = \mathcal{F}(V)$$ for $V \subseteq \mathcal{U}$ open. Note that

$V \subseteq X$ is also open since $\mathcal{U} \subseteq X$ is open, so the above makes sense.

Sheaves were discovered by Leray in the 40s as POW.

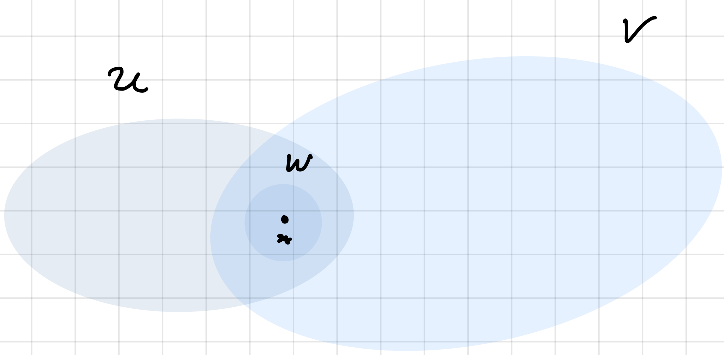
His papers were sent to Hopf in Zürich for publication.

Stalks & Germs

$\mathcal{F} \longrightarrow X$ presheaf, $x \in X$

Consider pairs (U, s), consisting of $x \in U \subseteq X$ open and $s \in \mathcal{F}(U)$ a section.

$(U, s) \sim (V, t)$ provided $\exists\ x \in W \subseteq U \cap V$ open with $s|_W = t|_W$.



This is an equivalence relation.

The stalk of $\mathcal{F}_x$ is the set of equivalence classes.

An equivalence class is called a germ.

We have $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$

Math 220C - Lecture 19

May 10, 2021

Goals

I. Define Riemann Surfaces

II. Define holomorphic functions

III. examples

Aside (Point Set Topology) X Hausdorff

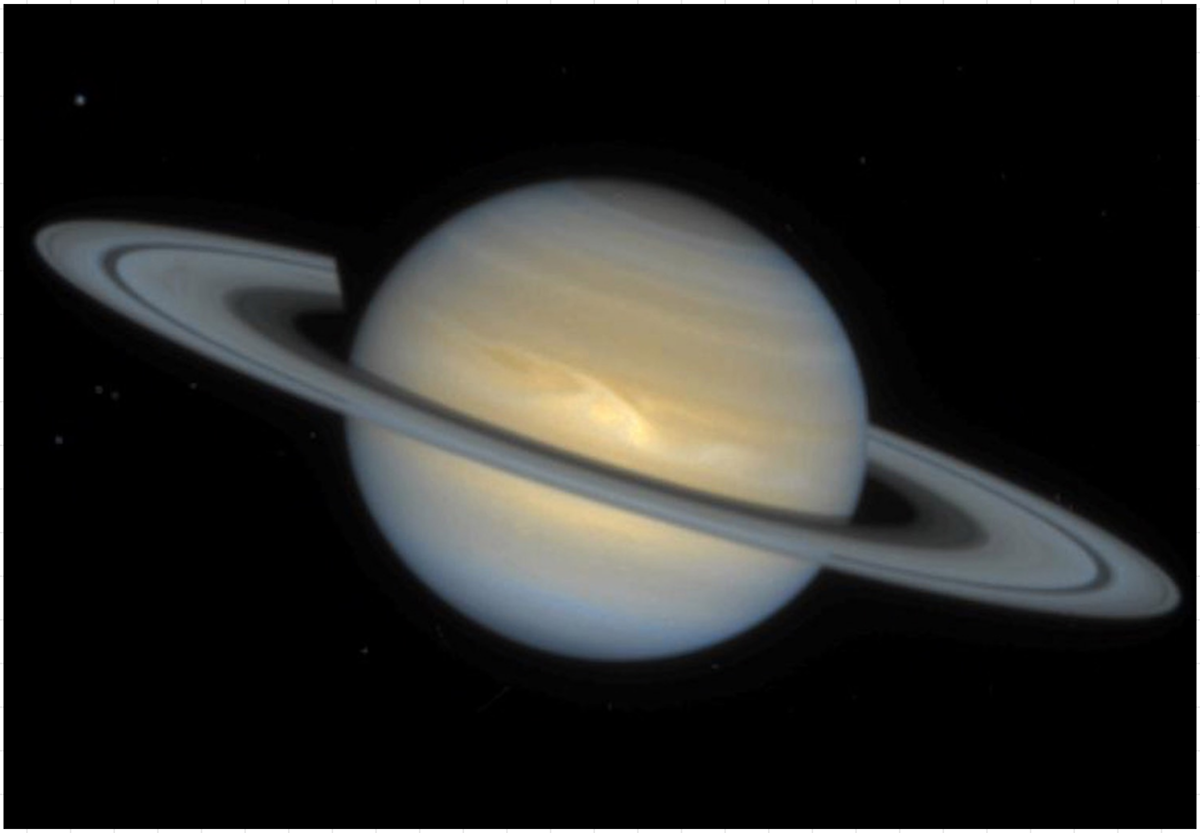
$\boxed{i}$ X is 2^{nd} countable if X admits a countable base for its topology

$\boxed{ii}$ X is paracompact if all open covers admit a locally finite subcover

$\boxed{iii}$ $X = \bigcup \mathcal{U}_\alpha$ open cover. A partition of unity $f_\alpha : X \longrightarrow \mathbb{R}$ continuous satisfies

- $\operatorname{supp} f_\alpha \subseteq \mathcal{U}_\alpha$ & $\operatorname{supp} f_\alpha$ is locally finite
- $\sum f_\alpha = 1$, $0 \leq f_\alpha \leq 1$.

In general $\boxed{ii} \iff \boxed{iii}$. $\boxed{i} \iff \boxed{ii}$ for manifolds.



Ringed spaces

A ringed space $(X, \mathcal{O}_X)$ is the datum of

[i] X topological space

[ii] sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras of complex valued continuous functions. ("regular functions")

Morphisms

$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces

[i] f continuous

[ii] $\forall\, U \subseteq Y,\ \varphi \in \mathcal{O}_Y(U)$. the pullback $\varphi \circ f : f^{-1}(U) \longrightarrow \mathbb{C}$ is a section of $\mathcal{O}_X(f^{-1}U)$.

Remark By [i], $f^{-1}U$ is open which is needed for [ii] to make sense.

Example $G, G' \subseteq \mathbb{C}$

$f : (G, \mathcal{O}_G) \longrightarrow (G', \mathcal{O}_{G'})$ is a morphism of ringed spaces

$\iff$ f holomorphic.

Why? $\Leftarrow$ If φ holomorphic in $U \subseteq G'$ & f holomorphic then $\varphi \circ f$ is holomorphic in $f^{-1}(U)$.

$\Rightarrow$ If f morphism, let $\varphi(z) = z$ holomorphic in $U = G'$

Then $\varphi \circ f = f$ is holomorphic by condition $\boxed{ii}$.

Remark We have the notion of an isomorphism.

Remark If X ringed space, $(X, \mathcal{O}_X)$.

$U \subseteq X$ open $\Rightarrow$ $(U, \mathcal{O}_X|_U)$ is a ringed space.

Definition A $\mathcal{C}^k$-manifold ($k \geq 0$, $k = \infty$, $k = \omega$) of dim. n.

[i] X Hausdorff, connected, 2^{nd} countable

[ii] $\exists$ open cover $X = \bigcup \mathcal{U}_\alpha$ and open subsets $G_\alpha \subseteq \mathbb{R}^n$ such that $(\mathcal{U}_\alpha, \mathcal{O}_X/_{\mathcal{U}_\alpha})$ is isomorphic as a ringed space to $(G_\alpha, \mathcal{C}^k)$.

Definition A Riemann surface $(X, \mathcal{O}_X)$ is

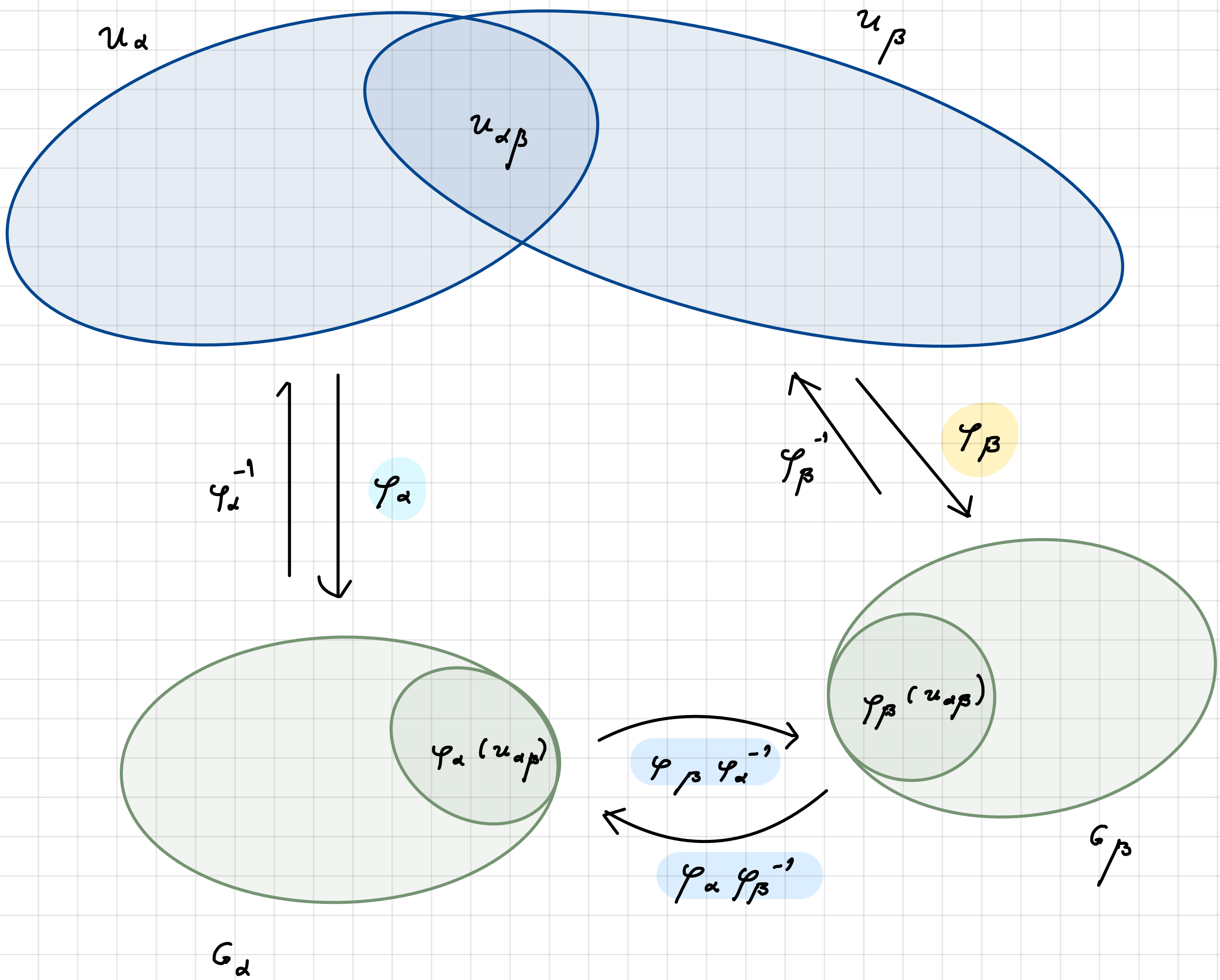
[i] X Hausdorff, connected, 2^{nd} countable top space

[ii] $\exists$ open cover $X = \bigcup \mathcal{U}_\alpha$ and open subsets $G_\alpha \subseteq \mathbb{C}$ such that $(\mathcal{U}_\alpha, \mathcal{O}_X/_{\mathcal{U}_\alpha})$ is isomorphic$_{\mathbb{C}}$ as a ringed space to $(G_\alpha, \mathcal{O}_{G_\alpha})$.

Any Riemann surface is a $\mathcal{C}^k$-manifold of real dimension 2. $\forall\, k$.

In concrete terms Let X Riemann surface. Let $X = \bigcup_\alpha \mathcal{U}_\alpha$

s.t. $(\mathcal{U}_\alpha, \mathcal{O}_X|_{\mathcal{U}_\alpha}) \cong (G_\alpha, \mathcal{O}_{G_\alpha})$ via isomorphism φ_α.



Let $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Note $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_{\alpha\beta}) \longrightarrow \varphi_\beta(\mathcal{U}_{\alpha\beta})$.

must be an isomorphism of ringed spaces.

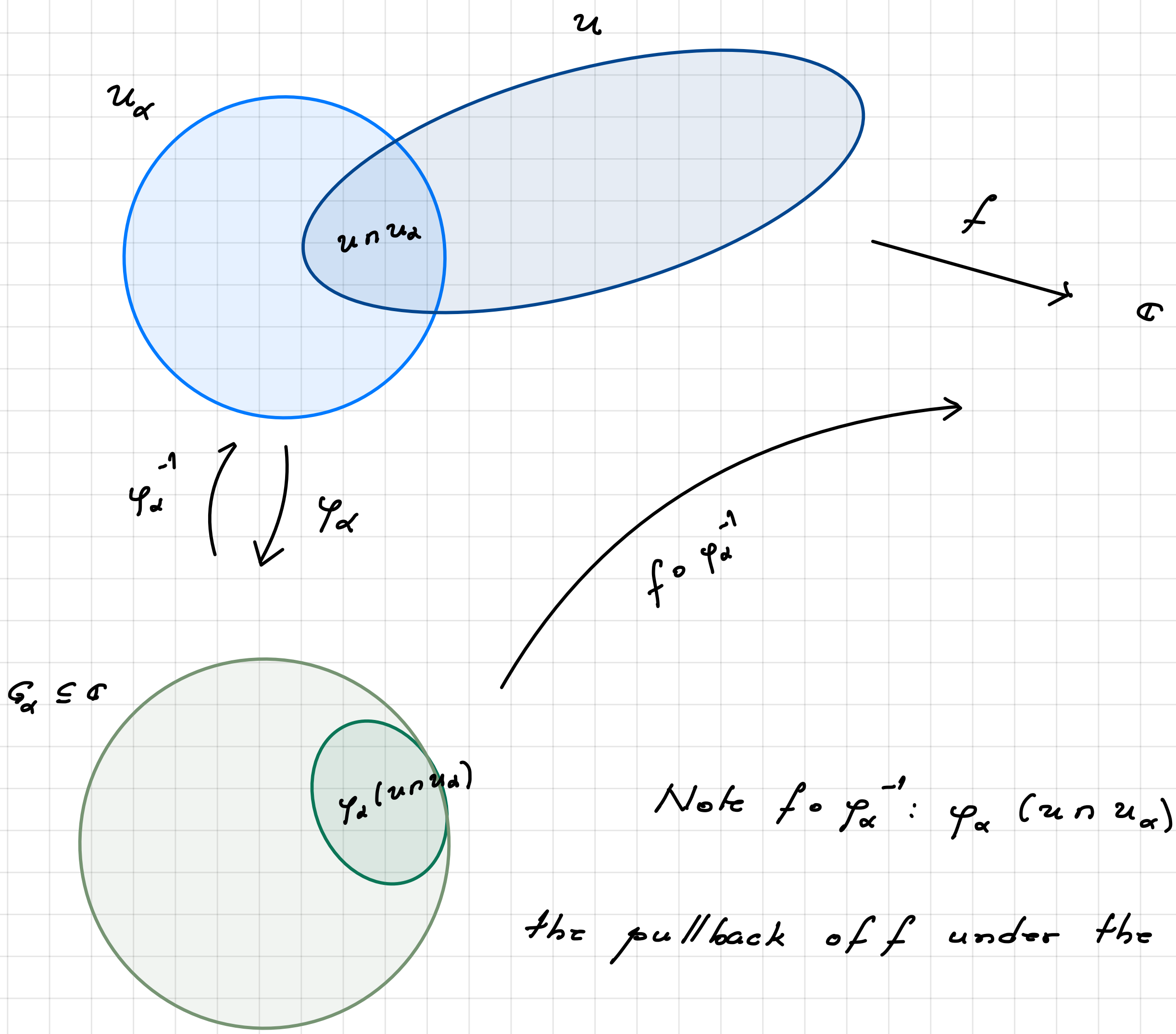
Thus $\varphi_\beta\varphi_\alpha^{-1}$ is a biholomorphism between open subsets of $\mathbb{C}$.

Holomorphic functions

Let X be a Riemann surface & $U \subseteq X$ open.

A holomorphic function on U is a section of $\mathcal{O}_X(U)$.

Concretely



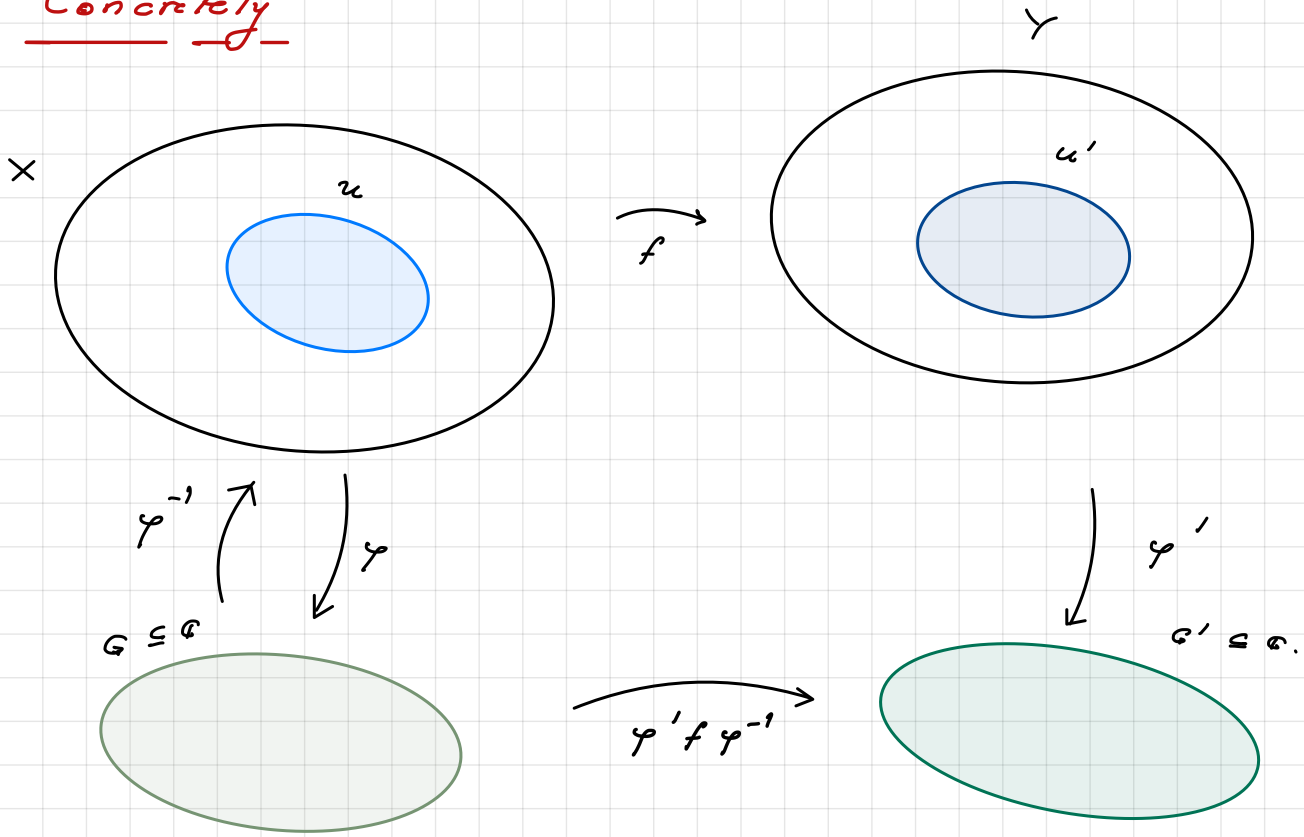
Note $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap U_\alpha) \longrightarrow \mathbb{C}$ is the pullback of f under the morphism φ_α^{-1}.

Therefore $f \circ \varphi_\alpha^{-1}$ is holomorphic in the set $\varphi_\alpha(U \cap U_\alpha) \subseteq \mathbb{C}$.

Holomorphic maps between Riemann Surfaces

$f: X \longrightarrow Y$ holomorphic iff f is a morphism of ringed spaces.

Concretely



If (u, G, φ) and (u', G', φ') are coordinate charts with $f(u) \subseteq u'$ we have

$$\varphi' f \varphi^{-1} : \varphi(u) \longrightarrow \varphi'(u') \text{ is a morphism of ringed}$$

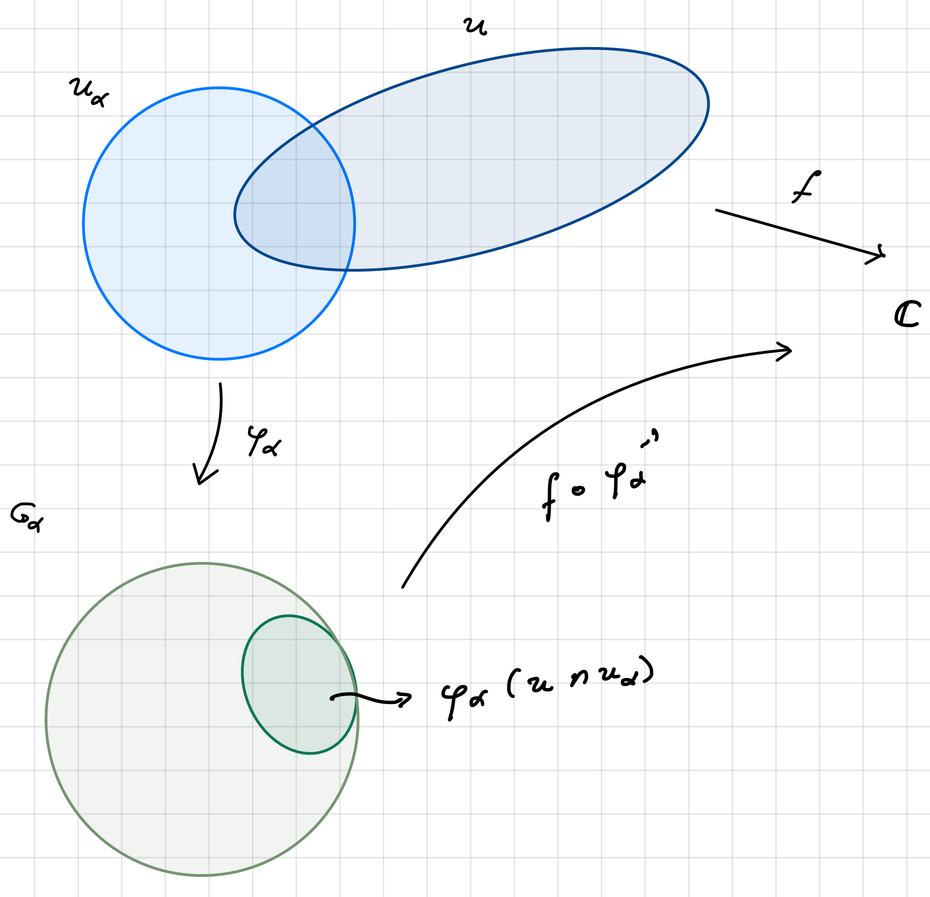
spaces $\Rightarrow$ $\varphi' f \varphi^{-1}$ is holomorphic as a map between subsets of $\mathbb{C}$.

Math 220C - Lecture 20

May 12, 2021

Holomorphic functions

Let X be a Riemann surface, $(U_\alpha, G_\alpha, \varphi_\alpha)$ coordinate charts.



We showed last time that

f holomorphic iff $f \circ \varphi_\alpha^{-1}$ is holomorphic in $\varphi_\alpha(U \cap U_\alpha)$ $\forall \alpha$.

Remark We can also turn this discussion around.

Let X be a topological space (Hausdorff, 2^{nd} countable)

$X = \bigcup_\alpha \mathcal{U}_\alpha$ open cover. Assume we are given

- $\varphi_\alpha : \mathcal{U}_\alpha \longrightarrow G_\alpha$ homeomorphisms, $G_\alpha \subseteq \mathbb{C}$ such that

- $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \longrightarrow \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ biholomorphism

These are called compatible coordinate charts

Then X becomes a Riemann surface.

Issue Define the sheaf $\mathcal{O}_X$.

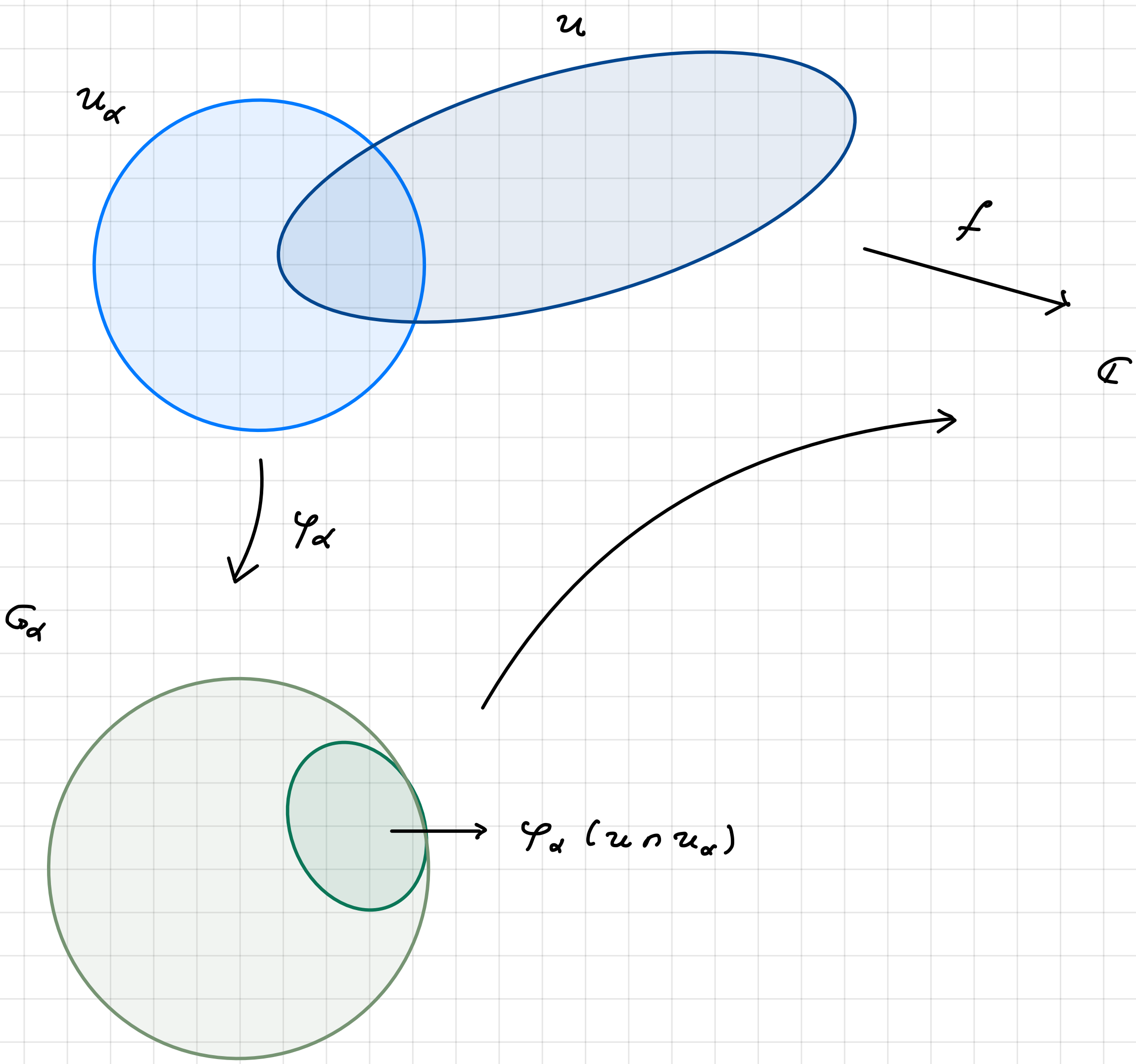
Note $\mathcal{U}$ open $\iff \mathcal{U} \cap \mathcal{U}_\alpha$ open $\iff \varphi_\alpha(\mathcal{U} \cap \mathcal{U}_\alpha)$ open in G_α.

Declare $f : \mathcal{U} \longrightarrow \mathbb{C}$ to be a section of $\mathcal{O}_X$ provided.

$$f \in \mathcal{O}_X(\mathcal{U}) \iff f \circ \varphi_\alpha^{-1} \text{ holomorphic in } \varphi_\alpha(\mathcal{U} \cap \mathcal{U}_\alpha).\ \forall \alpha.$$

Check $\mathcal{O}_X$ is indeed a sheaf. & $(X, \mathcal{O}_X)$ Riemann surface

Meromorphic functions



Definition

f meromorphic in $\mathcal{U}$ provided $f\varphi_\alpha^{-1}$ meromorphic in $\varphi_\alpha(\mathcal{U} \cap \mathcal{U}_\alpha)$ $\forall \alpha$

Note there exists a sheaf $\mathcal{M}$ of meromorphic functions

$$\mathcal{U} \longrightarrow \text{meromorphic functions in } \mathcal{U}$$

Zeroes, poles, order

We define the order of a pole or a zero for f to be the order of a pole or a zero for $f\varphi_\alpha^{-1}$ at $\varphi_\alpha(p)$ for $p \in \mathcal{U}_\alpha$

Claim This is independent of choice of α.

Subclaim

Let g be meromorphic in $\mathcal{U}$, $a \in \mathcal{U}$. Let $T : V \longrightarrow \mathcal{U}$ be a biholomorphism with $T(b) = a$, $b \in V$. Then

g has order m at a $\Rightarrow$ $g \circ T$ has order m at b.

We use this for $g = f\varphi_\alpha^{-1}$, $a = \varphi_\alpha(p)$

$T = \varphi_\alpha \varphi_\beta^{-1}$, $b = \varphi_\beta(p)$. $\Rightarrow g \circ T = f\varphi_\beta^{-1}$.

The subclaim shows that the order thus defined is independent of the choice of α.

Proof of the Subclaim

WLOG $a = b = 0$, else we can translate.

Write $g(z) = z^m G(z)$, $G(0) \neq 0$.

Since $T(0) = 0$ & $T'(0) \neq 0$ since T is biholomorphism, we have $T(z) = z\, S(z)$, $S(0) \neq 0$.

Note $g \circ T(z) = T(z)^m G(T(z))$

$$= z^m S(z)^m G(T(z)).$$

Since $S(z)^m G(T(z))\Big|_{z=0} = S(0)^m G(0) \neq 0 \Rightarrow$

$\Rightarrow$ order $g \circ T$ at $z = 0$ equals m. as needed.

Remarks Essential singularities can be defined similarly.

Aside – Divisors on Riemann surfaces

Definition A divisor on a Riemann surface X is a formal sum

$$D = \sum_{p \in X} n_p [p] \quad \text{with } n_p \in \mathbb{Z} \text{ such that}$$

$S = \{ p : n_p \neq 0 \}$ is locally finite.

Examples

[I] $X = \hat{\mathbb{C}}$, $D = 2[0] + 3[\infty] - 5[1]$ divisor on X

[II] D is said to be effective if $n_p \geq 0 \;\; \forall p \in X$

[III] Divisors can be formally added & subtracted

$$D = \sum n_p [p], \quad E = \sum m_p [p]$$

$$\Longrightarrow D \pm E = \sum (n_p \pm m_p) [p] \text{ is a divisor}$$

[IV] restrictions, $U \subseteq X$ open. If

$$D = \sum_{p \in X} n_p [p] \Longrightarrow D/_U = \sum_{p \in U} n_p [p]$$

IV $\exists$ sheaf of divisors $\underline{\mathcal{D}iv}$.

$$\mathcal{U} \longrightarrow \{\text{divisors in } \mathcal{U}\}$$

VI degree. If X is compact, any divisor is a finite sum.

$$D = \sum n_p [p], \quad n_p \in \mathbb{Z}. \Rightarrow \deg D := \sum_p n_p.$$

Principal divisors If f meromorphic in X, define

I $\operatorname{div} f = \sum_{z \in X} \operatorname{ord}(f, z)[z]$

$$= \sum_{z \text{ zero}} \operatorname{mult}_z(f)[z] - \sum_{p \text{ pole}} \operatorname{mult}_p(f)[p]$$

II Check: $\operatorname{div}(fg) = \operatorname{div} f + \operatorname{div} g$.

Example $X = \hat{\mathbb{C}}$. $f = \dfrac{\prod_{i=1}^{m}(z-a_i)}{\prod_{i=1}^{n}(z-b_i)}$ meromorphic function in $\hat{\mathbb{C}}$

$a_i, b_i \in \mathbb{C}$.

$$\operatorname{div} f = \sum_{i=1}^{m}[a_i] - \sum_{i=1}^{n}[b_i] + (n-m)[\infty]$$

$$\Rightarrow \deg \operatorname{div} f = \sum_{i=1}^{m} 1 - \sum_{i=1}^{n} 1 + (n-m) = 0.$$

Examples of Riemann surfaces

$\boxed{\text{I}}$ not compact

$\boxed{\text{II}}$ compact

Non - compact examples

$\boxed{a}$ $G \subseteq \mathbb{C}$ open subset is a Riemann surface

$\boxed{b}$ $X \subseteq \mathbb{C}^2$, $X = \{ (x,y) \in \mathbb{C}^2 : f(x,y) = 0 \} \subseteq \mathbb{C}^2$.

Assume $\forall p \in X$,

$$f_x (p) \neq 0 \quad \text{or} \quad f_y (p) \neq 0.$$

Claim X is a Riemann surface

Proof We construct charts & show they are compatible.

Let $p \in X$.

- if $f_y (p) \neq 0 \implies$ by implicit function theorem,

$\exists \; p \in \mathcal{U} \subseteq X$ open such that

$y = g(x)$ for $(x,y) \in \mathcal{U}$ where $g : V \longrightarrow \mathbb{C}$ is holomorphic.

Then $\mathcal{U} \longrightarrow G$, $(x, y) \longrightarrow x$ has inverse

$x \longrightarrow (x, g(x))$. $\Rightarrow$ $\mathcal{U}$ is a chart

- If $f_x(p) \neq 0$, we similarly have

$x = h(y)$ for $(x, y) \in \mathcal{U}$, $h : H \longrightarrow \mathbb{C}$ holomorphic

Then $\mathcal{U} \longrightarrow H$ is a chart $(x, y) \longrightarrow y$ with inverse

$y \longrightarrow (h(y), y)$. $\Rightarrow$ $\mathcal{U}$ is a chart

<u>Compatibility</u> Charts of the first type are clearly compatible. Same for charts of 2^{nd} type.

We check compatibility between charts of different types.

WLOG we may assume we are around a point p with

$f_x(p) \neq 0$ & $f_y(p) \neq 0$.

Then the change of coordinates is

$$\begin{tikzcd} & U \arrow[r] \arrow[dl] & G \arrow[dll, "T"'] \\ H & & \end{tikzcd}
\qquad
\begin{tikzcd} & (x,y) \arrow[r] \arrow[dl] & x \arrow[dll, "T"'] \\ y & & \end{tikzcd}$$

$$T : G \longrightarrow H$$
$$x \longrightarrow y = g(x)$$
$$T^{-1} : H \longrightarrow G$$
$$y \longrightarrow x = h(y)$$

Both T & T^{-1} are holomorphic, as needed.

Math 220C - Lecture 21

May 21, 2021

Last time

- noncompact Riemann surfaces

$X \subseteq \mathbb{C}$ or $X = \{ f(z,w) = 0 \} \subseteq \mathbb{C}^2$

Require: $\partial_z f(p) \neq 0$ or $\partial_w f(p) \neq 0$ if $p \in X$.

---

§1. Compact Riemann Surfaces

$\boxed{a}$ $X = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

We construct charts U_0, U_∞, $X = U_0 \cup U_\infty$

$$U_0 = \{ z : z \neq \infty \} \xrightarrow{\phi_0} \mathbb{C}, \quad z \longrightarrow z$$

$$U_\infty = \{ z : z \neq 0 \} \xrightarrow{\phi_\infty} \mathbb{C}, \quad z \longrightarrow \frac{1}{z}$$

These two charts are compatible. The transition map is

$$T = \phi_\infty \phi_0^{-1} : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times, \quad z \longrightarrow 1/z \quad \text{biholomorphic}$$

$\Rightarrow X$ Riemann surface

Projective curves

Let $\mathbb{P}^2 = \{ [x : y : z] \,,\; (x, y, z) \neq (0,0,0) \,,\; x, y, z \in \mathbb{C} \} / \sim$

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad \text{if } \lambda \in \mathbb{C}^\times.$$

$$U_1 = \{ x \neq 0 \} \xrightarrow{\phi_1} \mathbb{C}^2 \qquad [x : y : z] \longrightarrow \left(\frac{y}{x}, \frac{z}{x} \right).$$

$$U_2 = \{ y \neq 0 \} \xrightarrow{\phi_2} \mathbb{C}^2 \qquad [x : y : z] \longrightarrow \left(\frac{x}{y}, \frac{z}{y} \right).$$

$$U_3 = \{ z \neq 0 \} \xrightarrow{\phi_3} \mathbb{C}^2 \qquad [x : y : z] \longrightarrow \left(\frac{x}{z}, \frac{y}{z} \right).$$

Let f homogeneous of degree d. in variables x, y, z.

(*) if $p \in \mathbb{P}^2$, $f(p) = 0$ then $f_x(p) \neq 0$ or $f_y(p) \neq 0$ or $f_z(p) \neq 0$.

Then

$$X = \{ [x : y : z] : f(x, y, z) = 0 \} \hookrightarrow \mathbb{P}^2$$

is a Riemann Surface (check).

$\boxed{c}$ torus : $\omega_1, \omega_2 \neq 0$, $\omega_1/\omega_2 \notin \mathbb{R}$

Let. $\Lambda = \{ m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \} \hookrightarrow \mathbb{C}$

Let $X = \mathbb{C}/\Lambda$ where Λ acts on $\mathbb{C}$ by translations

Let $\pi : \mathbb{C} \longrightarrow X$

$\boxed{i}$ X has the quotient topology, $\mathcal{U} \subseteq X$:

$\mathcal{U}$ open $\Longleftrightarrow$ $\pi^{-1}\mathcal{U}$ open

$\boxed{ii}$ π continuous & open

$\mathcal{U}$ open, $\pi(\mathcal{U})$ open since $\pi^{-1}\pi\,\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{U} + \lambda$ = open.

$\boxed{iii}$ coordinate charts. Let $\varepsilon < \frac{1}{2} \min_{\lambda \in \Lambda} |\lambda|$.

Let $x \in X$, $\pi(z) = x$, $z \in \mathbb{C}$

Let $\pi : \Delta(z, \varepsilon) \longrightarrow \bar{D}_{z,\varepsilon} = \pi(\Delta(z, \varepsilon))$.

Claim $D_{z,\varepsilon}$ is a chart

π surjective, injective, continuous, open hence a homeomorphism

Claim The charts $D_{z,\varepsilon}$ are compatible

$\varphi_1 : D_{z_1,\varepsilon} \longrightarrow \Delta(z_1,\varepsilon)$

$\varphi_2 : D_{z_2,\varepsilon} \longrightarrow \Delta(z_2,\varepsilon)$

$u = D_{z_1,\varepsilon} \cap D_{z_2,\varepsilon}$

$T = \varphi_2\,\varphi_1^{-1}$ is given by $z \longrightarrow z+\lambda$. on $\varphi_1(u)$.

Indeed $\pi\,T(z) = \pi\,\varphi_2\,\varphi_1^{-1}(z) = \pi\,\varphi_2\,\pi(z) = \pi(z)$

$\Rightarrow T(z) = z+\lambda.$ biholomorphic

Conclusion Give X the complex structure determined by these charts. X is a Riemann surface.

Remark Meromorphic functions on $X = \mathbb{C}/\Lambda$ are meromorphic functions in $\mathbb{C}$, Λ-periodic

$\rightsquigarrow$ elliptic functions e.g. $\wp, \wp', \wp'', \ldots$

§2. Basic Results

[a] Identity Theorem

$f, g : X \longrightarrow Y$ holomorphic maps between Riemann Surf.

$S = \{x : f(x) = g(x)\}$ has a limit point in X.

Then $f \equiv g$.

[b] Open Mapping theorem

$f : X \longrightarrow Y$ holomorphic, non constant $\Rightarrow f$ is open

[c] Maximum Modulus

$f : X \longrightarrow \mathbb{C}$ holomorphic & $|f|$ has a maximum at $p \in X$

$\Rightarrow f$ constant.

Corollary

$f : X \longrightarrow \mathbb{C}$, X compact $\Rightarrow f$ constant

Proof of Maximum Principle

Let $p \in U_\alpha$. Let $\varphi_\alpha : U_\alpha \longrightarrow G_\alpha$ be a chart. Let $f \circ \varphi_\alpha^{-1} : G_\alpha \longrightarrow \mathbb{C}$, $G_\alpha \subseteq \mathbb{C}$. Then $|f \circ \varphi_\alpha^{-1}|$ has a maximum at $\varphi_\alpha(p)$. By the usual maximum principle for G_α

$\Rightarrow f \circ \varphi_\alpha^{-1} = \text{constant}$ in G_α $\Rightarrow f = \text{constant}$ in U_α $\Rightarrow$

$\Rightarrow f = \text{constant}$ by the identity theorem.

Rephrasing in terms of sheaves

- $\mathcal{F} \longrightarrow X$, $H^0(X, \mathcal{F}) := \mathcal{F}(X)$

- X compact $\Rightarrow H^0(X, \mathcal{O}_X) = \mathbb{C}$.

Proof of Identity Principle

$$\Omega = \{ x \in X, \ f = g \text{ in a neighborhood of } x \}$$

Claims

$\boxed{i}$ $\quad \Omega \neq \emptyset$

$\boxed{ii}$ $\quad \Omega$ open $\qquad \Rightarrow \Omega = X \Rightarrow f \equiv g.$

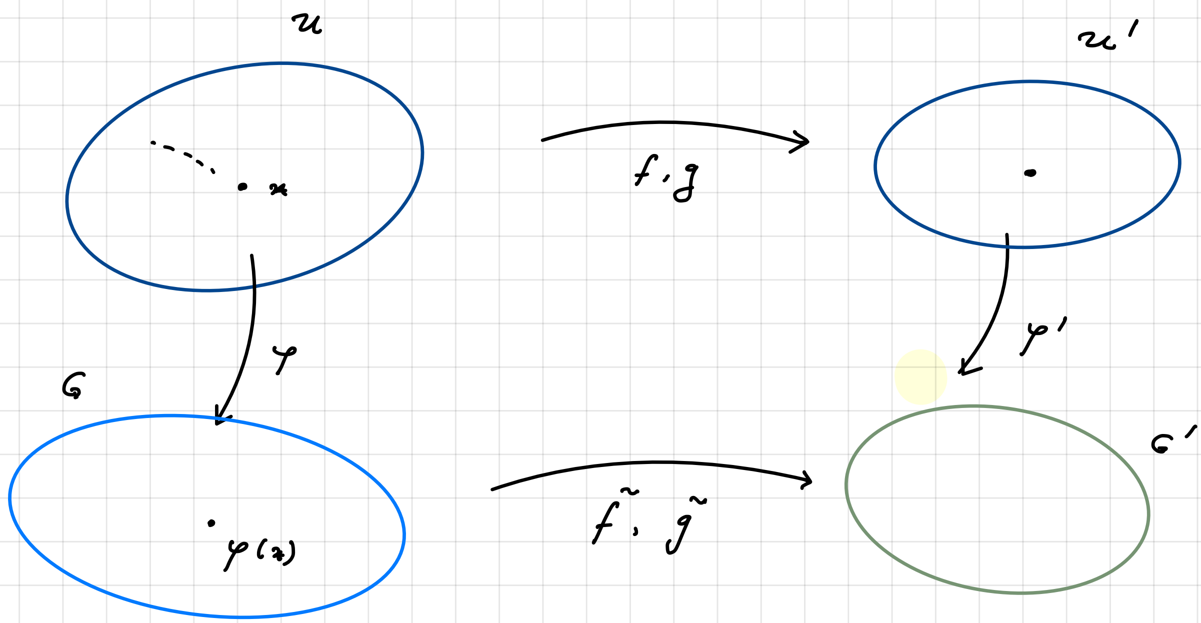
$\boxed{iii}$ $\quad \Omega$ closed

Proof of $\boxed{i}$

Let $S = \{ s : f(s) = g(s) \} \subseteq X$ have a limit point x. We show $x \in \Omega$.

Let U be a chart near x, U' a chart in Y near $f(x) = g(x) = y$. Shrinking if needed we may assume

$f(U) \subseteq U'$, $\ g(U) \subseteq U'$, $\ U$ connected.



Let $\varphi : \mathcal{U} \longrightarrow G$, $\varphi' : \mathcal{U}' \longrightarrow G'$ be coordinate charts, $G, G' \subseteq \mathbb{C}$. Let $\tilde{f} = \varphi' f \varphi^{-1}$; $\tilde{g} = \varphi' g \varphi^{-1}$. Let $\tilde{S} = \{ \tilde{s} \in G : \tilde{f}(\tilde{s}) = \tilde{g}(\tilde{s}) \}$. Note $\tilde{S} \supseteq \varphi(S)$ so $\tilde{S}$ has a limit point $\varphi(x) \Rightarrow \tilde{f} \equiv \tilde{g}$ in $G \Rightarrow f = g$ in $\mathcal{U} \Rightarrow$

$\Rightarrow x \in \Omega \Rightarrow \Omega \neq \emptyset$.

Part $\boxed{II}$ is clear by definition. Part $\boxed{III}$ is a repetition of the above argument. (check it!)

§ 3. Questions about functions on Riemann surfaces

Question A Is every divisor $D = \sum_{p \in X} n_p [p]$ the divisor of a meromorphic function?

Answer depends on X.

$\boxed{1}$ non-compact $X \subseteq \mathbb{C}$ open

If $D \geq 0$, $n_p \geq 0 \ \forall p$, the question is equivalent to the Weierstraß Problem.

In general write $D = D_+ - D_-$, D_+, D_- effective.

Write $D_+ = \operatorname{div} f_+$, $D_- = \operatorname{div} f_-$, $f = f_+/f_-$. Then

$$D = \operatorname{div} f_+ - \operatorname{div} f_- = \operatorname{div} f_+/f_- = \operatorname{div} f.$$

$\boxed{12}$ compact X

Example $X = \hat{\mathbb{C}}$. We need $\deg D = 0$ since we already noted $\deg \operatorname{div} f = 0$.

Conversely if $\deg D = 0$, $D = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

If $a_i, b_i \in \mathbb{C}$, let $f = \dfrac{\prod_{i=1}^{n}(z-a_i)}{\prod_{i=1}^{n}(z-b_i)} \Rightarrow \operatorname{div} f = D.$

If one of the a_i's or b_i's equals ∞, use first a FLT to reduce to the previous case. Thus

$$D \text{ principal} \iff \deg D = 0 \quad \text{if } X = \hat{\mathbb{C}}.$$

Example $X = \mathbb{C}/\Lambda$

Let $D = \sum_i z_i - \sum_i p_i$ be a divisor on X.

We allow repetitions among the z_i, p_i's. . . .

Meromorphic functions on X are elliptic functions

We want $\operatorname{div} f = D \iff f$ has zeroes/poles at z_i/p_i.

We have seen in Math 220A, Lecture 22 that

(1) $\#$ zeros $(f) = \#$ poles (f) $\iff$ $\deg D = 0$

(2) $\sum$ zeros of f $- \sum$ poles of $D \in \Lambda$

$\rightsquigarrow$ new condition

for any elliptic functions.

These were consequences of the argument principle.

Conversely, recall

$$\sigma(z) = z \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} E_2\left(\frac{z}{\lambda}\right)$$ – Math 220B, HWK 2

$$f(z) = \frac{\prod_i \sigma(z - z_i)}{\prod_i \sigma(z - p_i)}$$

Issue (Check!) Using (1) & (2) one checks that f is Λ-periodic $\Rightarrow$ f elliptic function $\Rightarrow$ f is meromorphic function on X.

Note $\operatorname{div} f = \sum z_i - \sum p_i = D$.

Thus Question A $\Longleftrightarrow$ Conditions (1) & (2) for $X = \mathbb{C}/\Lambda$.

Math 220C - Lecture 22

May 24, 2021

Let X be a Riemann surface

Let $D = \sum_{p \in X} n_p [p]$ divisor on X.

Question A

Is every divisor D the divisor of a meromorphic function?

Answer depends on X.

[i] non-compact $X \subseteq \mathbb{C}$ open

Yes! Weierstraß Problem

If X compact — additional conditions are needed.

[ii] $X = \hat{\mathbb{C}}$ we need $\deg D = 0$

[iii] $X = \mathbb{C}/\Lambda$ we need $\deg D = 0$ & another condition

What is the issue?

Cover X by coordinate charts $\mathcal{U}_\alpha$ with $\mathcal{U}_\alpha \overset{\varphi_\alpha}{\cong} G_\alpha \subseteq \mathbb{C}$.

Since we can solve the Weierstraß problem in $\mathcal{U}_\alpha$, we have

$$D\big/_{\mathcal{U}_\alpha} = \operatorname{div} f_\alpha \,, \quad f_\alpha \text{ meromorphic in } \mathcal{U}_\alpha .$$

Compatibility

$$\operatorname{div} f_\alpha \big/_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} = \operatorname{div} f_\beta \big/_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} = D\big/_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \Rightarrow$$

$$\Rightarrow \operatorname{div} \frac{f_\alpha}{f_\beta} = 0 \text{ in } \mathcal{U}_\alpha \cap \mathcal{U}_\beta$$

$\Rightarrow$ f_α / f_β is nowhere zero holomorphic in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

Want

f meromorphic in X, $\operatorname{div} f = D$

$$\Leftrightarrow \operatorname{div} f\big/_{\mathcal{U}_\alpha} = D\big/_{\mathcal{U}_\alpha} = \operatorname{div} f_\alpha$$

$$\Leftrightarrow \operatorname{div} \frac{f}{f_\alpha} = 0 \text{ in } \mathcal{U}_\alpha$$

$\Leftrightarrow$ f / f_α is nowhere zero holomorphic in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

Question A (rephrased) Given

- open cover $X = \bigcup \mathcal{U}_\alpha$, $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$
- $f_\alpha \in \mathcal{M}^*(\mathcal{U}_\alpha)$ with $f_\alpha / f_\beta \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$

we want $f \in \mathcal{M}^*(X)$ with

$$f / f_\alpha \in \mathcal{O}^*(\mathcal{U}_\alpha).$$

Notation

- $\mathcal{O}$ = sheaf of holomorphic functions
- $\mathcal{O}^*$ = sheaf of holomorphic nonvanishing fns.
- $\mathcal{M}$ = sheaf of meromorphic functions
- $\mathcal{M}^*$ = nonzero meromorphic functions

Aside : There is a similar additive question.

Question B

Given

$$X = \bigcup \mathcal{U}_\alpha \ , \ f_\alpha \text{ meromorphic in } \mathcal{U}_\alpha$$

such that

$$f_\alpha - f_\beta \in \mathcal{O}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

Want f meromorphic in X with

$$f - f_\alpha \in \mathcal{O}(\mathcal{U}_\alpha).$$

Special case

- $X \subseteq \mathbb{C}$ open, $\mathcal{U}_\alpha$ open near p_α, $p_\beta \notin \mathcal{U}_\alpha$ for $\alpha \neq \beta$
- f_α = Laurent principal part near p_α.

This recovers Mittag-Leffler.

Rephrasing in terms of sheaves

- $\mathcal{M}^*$ = sheaf of meromorphic functions $\neq 0$
- $\underline{\mathcal{D}iv}$ = sheaf of divisors

Recall $H^0(X, \mathcal{F}) = \mathcal{F}(X)$.

Rephrasing Question A

Is $H^0(X, \mathcal{M}^*) \longrightarrow H^0(X, \underline{\mathcal{D}iv})$ surjective?

$$f \longmapsto \operatorname{div} f$$

Strategy for Answering Question A

We rephrase the problem using sheaf cohomology.

Goals

I Cohomology

For all $\mathcal{F} \to X$, we will define $H^p(X, \mathcal{F})$, $p \geq 0$. We already have seen $H^0(X, \mathcal{F})$.

II Exact sequences of sheaves

We will define
- morphisms of sheaves
- exact sequences of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

III Short exact sequences & cohomology

Given $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ we will show

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \longrightarrow$$
$$\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \longrightarrow \cdots$$

We assume this for now & give the details starting next time.

How is this relevant for us?

[a] We will show that

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \xrightarrow{div} \underline{\mathcal{D}iv} \longrightarrow 0 \quad \text{is exact.}$$

[b] If $H^1(X, \mathcal{O}^*) = 0$ $\Rightarrow$ Question A YES

Indeed, $H^0(X, \mathcal{M}^*) \xrightarrow{div} H^0(X, \underline{\mathcal{D}iv}) \longrightarrow \underbrace{H^1(X, \mathcal{O}^*)}_{0}$

$\Rightarrow$ div is surjective

[c] In the additive case, $\mathcal{O}^*$ gets replaced by $\mathcal{O}$.

If $H^1(X, \mathcal{O}) = 0$ $\Rightarrow$ Question B YES

Question C Given

- $z_1 \dots z_n \in X$, $p_1 \dots p_m \in X$
- $\mu_1 \dots \mu_n \geq 0$, $\nu_1, \dots \nu_m \geq 0$ integers

Want f meromorphic in X

- f has zeroes at z_i of order $\geq \mu_i$
- f has poles at p_i of order $\leq \nu_i$

Other zeroes are allowed, but no other poles.

Let $D = -\sum_i \mu_i [z_i] + \sum_i \nu_i [p_i]$

Want $\operatorname{div} f - \sum_i \mu_i [z_i] + \sum_i \nu_i [p_i] \geq 0$

$\iff D + \operatorname{div} f \geq 0$ (non-negative coefficients).

Sheaves associated to divisors

Given a divisor D, we form a sheaf $\mathcal{O}_X(D)$

via the assignment

$$U \longrightarrow \{ f \text{ meromorphic} \neq 0,\ \operatorname{div} f + D|_U \geq 0 \} \cup \{0\}.$$

if U connected.

Conclusion Question C is asking to describe

$$V_D = H^0(X, \mathcal{O}_X(D)).$$

Gustav Roch (1839 – 1866)

While in Göttingen, Roch attended lectures of Riemann.

Example $X = \mathbb{C}/\Lambda$, $D = d\,[0]$

$$V_d = \{ f \text{ elliptic}, \ f \text{ has pole at } 0 \text{ of order } \le d \}$$

Claim $\dim V_d = d$.

Proof Recall the Weierstraß $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \cdots$$

$\Longrightarrow$ $\wp$ has pole of order 2 at 0

$\Longrightarrow$ $\wp'$ has pole of order 3 at 0

$\Longrightarrow$ $\wp^{(k)}$ has pole of order $k+2$ at 0.

$\Longrightarrow$ $1, \wp, \wp', \ldots, \wp^{(d-2)}$ are in V_d.

Claim $1, \wp, \wp', \dots, \wp^{(d-2)}$ are independent.

Proof If

$$\underbrace{\wp^{(k)}}_{\text{order } k+2} = \underbrace{\sum_{j=0}^{k-1} a_j \wp^{(j)}}_{\text{order} \le j+2 \le k+1}$$

we inspect the order of the pole at 0.

Contradiction!

Conclusion $\dim V_d \ge d$.

We will show the opposite inequality.

Claim: $\dim V_d \leq d.$

Let f be an elliptic function in V_d. Laurent expand

$$f = \frac{a_{-d}}{z^d} + \cdots + \frac{a_{-1}}{z} + a_0 + \cdots$$

Note $a_{-1} = \frac{1}{2\pi i} \int_{\partial P} f \, dz = 0$. by the Residue Theorem

& periodicity of f.

The coefficients $(a_{-d}, \ldots, a_{-2}, a_0)$ determine f

at most uniquely $\Rightarrow \dim V_d \leq d.$

Indeed, assume f_1, f_2 are elliptic and have the same Laurent coefficients a_{-k}. Then

$f_1 - f_2$ = holomorphic at 0 & everywhere else

(by definition of V_d) & Λ-periodic

= constant by Liouville. & vanishing at 0

$\Rightarrow f_1 - f_2 \equiv 0 \Rightarrow f_1 \equiv f_2.$

Sheaves on Riemann Surfaces

- $\mathcal{O}$ = sheaf of holomorphic functions
- $\mathcal{O}^*$ = sheaf of holomorphic nonvanishing fns.
- $\mathcal{M}$ = sheaf of meromorphic functions
- $\mathcal{M}^*$ = nonzero meromorphic functions
- $\underline{\mathbb{C}}$ = sheaf of locally constant functions
- $\mathcal{C}^\infty$ = sheaf of smooth functions
- $\underline{\mathcal{D}iv}$ = sheaf of divisors
- $\mathcal{O}_X(D)$ = sheaf associated to divisor D.

[I] These sheaves solve different problems

[II] These sheaves interact with each other

Math 220C - Lecture 23

May 26, 2021

Homological methods & sheaves

$\boxed{I}$ define morphisms

$\boxed{II}$ define exact sequences

Morphisms of sheaves

$\mathcal{F}, \mathcal{G} \to X$ sheaves on a topological space

A morphism of sheaves

$\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ consists in homomorphisms

$\alpha_u : \mathcal{F}(u) \longrightarrow \mathcal{G}(u)$ $\forall$ $u \subseteq X$ open.

We require that

$$\begin{array}{ccc} \mathcal{F}(u) & \xrightarrow{\alpha_u} & \mathcal{G}(u) \\ \downarrow{\scriptstyle \rho^{\mathcal{F}}_{uv}} & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{uv}} \\ \mathcal{F}(v) & \xrightarrow{\alpha_v} & \mathcal{G}(v) \end{array}$$

$\forall$ $v \subseteq u$ open

Remark Given $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ we obtain $\forall\, x \in X$

$$\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

Why? Let $f_x \in \mathcal{F}_x$. Represent f_x by (f, U), $x \in U$, $f \in \mathcal{F}(U)$. Define

$$\alpha_x(f_x) = \alpha(f)_x = \text{germ of } \alpha_U(f) \text{ at } x.$$

Since α is compatible with restrictions, the definition is independent of choices.

Exact sequences

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0 \quad \text{is exact iff}$$

$$\forall\, x \in X, \quad 0 \longrightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \longrightarrow 0 \quad \text{is exact.}$$

Lemma If $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$ exact then

$\forall\ U \subseteq X$, open $\quad 0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ exact.

Proof WLOG $U = X$. Else work with the sheaves

$\mathcal{F}/_U$, $\mathcal{G}/_U$, $\mathcal{H}/_U$ noting that

$$0 \longrightarrow \mathcal{F}/_U \longrightarrow \mathcal{G}/_U \longrightarrow \mathcal{H}/_U \longrightarrow 0$$

since the stalks at $x \in U$ do not change by restriction.

Remark $f = 0 \iff f_x = 0$ for f section of a sheaf $\mathcal{F}$

Proof "$\Leftarrow$" Since $f_x = 0 \overset{\text{def}}{\implies} f = 0$ in $W_x \ni x$. open.

Since $X = \bigcup_x W_x$, it follows $f = 0$ in X by

uniqueness of gluing.

---

(1) $\alpha : \mathcal{F}(X) \longrightarrow \mathcal{G}(X)$ injective.

Assume $\alpha(f) = 0$. For $x \in X \Rightarrow \alpha(f)_x = 0 \Rightarrow$

$\Rightarrow \alpha_x(f_x) = 0$

α_x injective

$\Rightarrow f_x = 0 \overset{\text{Remark}}{\Rightarrow} f = 0$.

(2) $\beta \circ \alpha = 0$ over X

Let $f \in \mathcal{F}(X)$. Note

$$(\beta \circ \alpha)(f)_x = \beta_x \, \alpha_x(f_x) = 0 \quad \text{since}$$

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \longrightarrow 0 \quad \text{is exact.}$$

By the Remark we see $(\beta \circ \alpha)(f) = 0 \Rightarrow \beta \circ \alpha = 0$.

(3) $\operatorname{Ker} \beta_X \subseteq \operatorname{Im} \alpha_X$

Let $g \in \mathcal{G}(X)$, $\beta(g) = 0$. Then $\beta_x(g_x) = 0 \quad \forall x \in X$

$\Rightarrow \exists f_x \in \mathcal{F}_x$ with $g_x = \alpha_x(f_x)$ by exactness of

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \longrightarrow 0$$

Represent the germ f_x by a section (f^x, U^x) with

$$g = \alpha(f^x) \text{ in } U^x.$$

Note $\alpha(f^x|_{U^x \cap U^y}) = \alpha(f^y|_{U^x \cap U^y}) = g|_{U^x \cap U^y}$. We

proved α is injective in Step (1) so

$$f^x|_{U^x \cap U^y} = f^y|_{U^x \cap U^y}.$$

By gluing, we can find $f \in \mathcal{F}(X)$ with

$$f|_{U^x} = f^x.$$

Then $\alpha(f)|_{U^x} = \alpha(f^x) = g|_{U^x} \Rightarrow \alpha(f) = \beta$ by

sheaf axioms. This is what we needed.

Remark Assume we are given

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0 \quad \text{such that}$$

$\forall\, U \subseteq X$ open $\quad 0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact.

Then $\quad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

Why We argue $0 \longrightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \longrightarrow 0$ exact.

We need to show

$\mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ is surjective. The rest is covered by the arguments above.

Take $h_x \in \mathcal{H}_x$, represent it by (h, U). Write

$h = \beta(g)$ since $\beta : \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$ surjective.

Then $h_x = \beta_x(g_x)$ with $g_x \in \mathcal{G}_x$, as needed.

Conclusion

(1) $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact $\Longrightarrow$

$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$ exact $\forall\, U \subseteq X$ open

Exactness on the right may fail.

(2) If $0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact

for a basis of neighborhoods $\{U\}$ in X $\Longrightarrow$

$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$. exact.

This follows from the argument on previous page

Three Examples - Exponential sequence

Let X be a Riemann surface.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathcal{O} \xrightarrow{\beta} \mathcal{O}^* \longrightarrow 1 \quad \text{exact}$$

The morphisms α and β

$\alpha(1) = 1$, $\beta(f) = e^{2\pi i f}$.

Why exact β is surjective on a basis consisting of simply connected coordinate charts. This follows since log's of nowhere zero functions are defined by Math 220 A.

β is not surjective on global sections

$X = \mathbb{C}^\times$, $\beta : f \longrightarrow e^{2\pi i f}$. Note $\operatorname{Im} \beta_X$ does not contain the function z since $\log z$ is not defined in $\mathbb{C}^\times$.

$\Rightarrow$ $\mathcal{O}(X) \longrightarrow \mathcal{O}^*(X)$ not surjective.

Example

$$0 \longrightarrow \mathcal{O}^* \xrightarrow{\alpha} \mathcal{M}^* \xrightarrow{\beta} \underline{\mathcal{D}iv} \longrightarrow 0 \quad \text{exact.}$$

The morphisms α and β

$\alpha(f) = f$

$\Rightarrow \beta \circ \alpha = 0$

$\beta(g) = \operatorname{div} g$

Why exact

We check β is surjective on a basis consisting of coordinate charts. By Weierstraß Problem, in such a chart, every divisor is the divisor of a meromorphic function proving surjectivity of $\beta : \mathcal{M}^* \longrightarrow \underline{\mathcal{D}iv}$.

Example Let $D = \sum n_j [p_j]$, $n_j \geq 0$. Then

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X(D) \xrightarrow{\beta} \prod \underbrace{\mathbb{C}_{p_j}}^{\oplus n_j} \longrightarrow 0 \quad \text{exact}$$

skyscraper sheaf at p_j

The morphisms α and β

- $\alpha(f) = f \Rightarrow \operatorname{div} f + D \geq 0$ since $D \geq 0$, $\operatorname{div} f \geq 0$

  $\Rightarrow \alpha$ is well-defined

- $\beta(f) = \prod_j \left(c_{-n_j}^{(p_j)}, \ldots, c_{-1}^{(p_j)}\right) \in \prod_j \mathbb{C}_{p_j}^{\oplus n_j}$

where the c's are the Laurent coefficients of f near p_j:

$$f = \frac{c_{-n_j}}{(z-p_j)^{n_j}} + \cdots + \frac{c_{-1}}{z-p_j} + \cdots$$

Why exact

β is surjective on a basis consisting of coordinate charts, by Mittag-Leffler in open subsets of $\mathbb{C}$.

Math 220C - Lecture 24

May 28, 2021

§ 1. Homological Methods & Sheaves (Part II)

Given $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves, we define

[i] the sheaf $\operatorname{Ker} \alpha$

[ii] the sheaf $\operatorname{Coker} \alpha$

Kernel When $U \subseteq X$ open, set

$$\operatorname{Ker} \alpha\,(U) = \operatorname{Ker}\left\{ \alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \right\}$$

Restriction maps

$\operatorname{Ker} \alpha\,(U) \longrightarrow \operatorname{Ker} \alpha\,(V)$ are naturally defined.

Check [i] $\operatorname{Ker} \alpha$ is a sheaf

[ii] $(\operatorname{Ker} \alpha)_x = \operatorname{Ker}\left\{ \alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x \right\}$.

Cokernel Presheaf

For $U \subseteq X$ open, define

$$\widetilde{\operatorname{Coker}\,\alpha}\,(U) = \operatorname{Coker}\left\{ \alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \right\}$$

Check

[I] $\widetilde{\operatorname{Coker}\,\alpha}$ is a presheaf

[II] $\left(\widetilde{\operatorname{Coker}\,\alpha}\right)_x = \operatorname{Coker}\left(\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x\right)$

Beware! $\widetilde{\operatorname{Coker}\,\alpha}$ is not always a sheaf

Example $X = \mathbb{C}^\times$, $\mathcal{O} \xrightarrow{\alpha} \mathcal{O}^*$, $\alpha(f) = e^{2\pi i f}$

Let $\boxed{\mathcal{F} = \widetilde{\mathrm{Coker}\,\alpha}}$

- $\boxed{U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}, \quad \mathcal{F}(U) = 0}$

Indeed, U is simply connected, so logarithms make sense $\Rightarrow$ α is surjective in U $\Rightarrow$ $\widetilde{\mathrm{Coker}\,\alpha_U} = 0$.

- $\boxed{V = \mathbb{C} \setminus \mathbb{R}_{\geq 0} \Rightarrow \mathcal{F}(V) = 0}$

- $\boxed{X = U \cup V = \mathbb{C}^\times.}$ Note α is not surjective in X since $\log z$ is not defined in $\mathbb{C}^\times$. $\Rightarrow$ $\boxed{\mathcal{F}(X) \neq 0.}$

Any $s \in \mathcal{F}(X)$ restricts $s|_U = 0$, $s|_V = 0$ since $\mathcal{F}(U) = 0$, $\mathcal{F}(V) = 0$. If $s \neq 0$ this contradicts uniqueness of gluing $\Rightarrow$ $\boxed{\mathcal{F} \text{ not a sheaf.}}$

Sheafification

Goal

Given $\mathcal{F} \longrightarrow X$ a presheaf, we define a sheaf $\mathcal{F}^{\#}$

& morphism of presheaves

$$i : \mathcal{F} \longrightarrow \mathcal{F}^{\#}.$$

Remark In addition,

[i] $\mathcal{F}$ sheaf $\Rightarrow$ $\mathcal{F} = \mathcal{F}^{\#}$

[ii] $\mathcal{F}_x = \mathcal{F}^{\#}_x \quad \forall x$

[iii] Given $\mathcal{F} \longrightarrow \mathcal{G}$ we obtain $\mathcal{F}^{\#} \longrightarrow \mathcal{G}^{\#}$ with

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{F}^{\#} & \longrightarrow & \mathcal{G}^{\#} \end{array}$$

commutative.

Definition

$$\mathcal{F}^{\#}(U) = \Big\{ (f_x)_{x\in U} \in \mathcal{F}_x, \text{ "locally compatible germs" i.e. }$$
$$\forall\, x \in U \ \ \exists\, x \in V \subseteq U,\ s \in \mathcal{F}(V) \text{ with } s_y = f_y \ \ \forall\, y \in V \Big\}$$

Example

$\mathcal{F}$ = presheaf of constant functions

$\mathcal{F}^{\#}$ = sheaf of locally constant functions

Remark

We define $\mathcal{F} \longrightarrow \mathcal{F}^{\#}$ via

$$\mathcal{F}(U) \ni f \longrightarrow (f_x)_x \in \mathcal{F}^{\#}(U).$$

Check

$\mathcal{F}^{\#}$ is a sheaf & $\mathcal{F}_x = \mathcal{F}^{\#}_x$.

Conclusion — Cokernel sheaf

Given $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$, we define the cokernel sheaf:

(1) $\widetilde{\operatorname{Coker}\alpha}$ presheaf

(2) sheafify $\operatorname{Coker}\alpha := \widetilde{\operatorname{Coker}\alpha}^{\#}$.

Why does it work?

Assume $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$. We have by definition

$$\mathcal{G} \longrightarrow \widetilde{\operatorname{Coker}\alpha}$$

This gives

$$\mathcal{G} = \mathcal{G}^{\#} \longrightarrow \widetilde{\operatorname{Coker}\alpha}^{\#}.$$

Then

$$\boxed{0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \widetilde{\operatorname{Coker}\alpha}^{\#} \longrightarrow 0} \quad \text{exact}$$

as needed.

Exactness can be checked on stalks. We note that

$$\left(\widetilde{\operatorname{Coker}\alpha}^{\#}\right)_x = \left(\widetilde{\operatorname{Coker}\alpha}\right)_x$$

$$= \operatorname{Coker}(\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x).$$

$$\Rightarrow \quad 0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \left(\widetilde{\operatorname{Coker}\alpha}^{\#}\right)_x \longrightarrow 0$$

exact, as needed.

§ 2. Flabby sheaves

$\mathcal{F}$ is flabby provided $\forall\ V \subseteq U \subseteq X$ open,

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \text{ is surjective.}$$

Remarks $\boxed{i}$ $\mathcal{F}$ flabby $\Rightarrow$ $\mathcal{F}/_U$ flabby $\forall\ U \subseteq X$ open

Indeed, for $V \subseteq W \subseteq U$,

$$\mathcal{F}/_U\,(W) = \mathcal{F}(W) \longrightarrow \mathcal{F}(V) = \mathcal{F}/_U\,(V) \quad \text{surjective.}$$

$\boxed{ii}$ Suffices to check $\forall\ U \subseteq X$ open

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U) \quad \text{surjective}$$

Indeed,
$$\begin{tikzcd} \mathcal{F}(X) \arrow[rr] \arrow[dr] & & \mathcal{F}(U) \\ & \mathcal{F}(V) \arrow[ur] & \end{tikzcd}$$
shows $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ also surjective for $U \subseteq V$.

Key Lemma

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact, $\mathcal{F}$ flabby then

[i] $0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact, $\forall\, U \subseteq X$ open

[ii] $\mathcal{F}, \mathcal{G}$ flabby $\Rightarrow$ $\mathcal{H}$ flabby

Proof [i] $\Rightarrow$ [ii]

Let $V \subseteq U$ open. Compare the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(U) \arrow[r] \arrow[d, "\text{surj}"] & \mathcal{G}(U) \arrow[r] \arrow[d, "\text{surj}"] \arrow[r, phantom, shift right=4, "\Rightarrow"] & \mathcal{H}(U) \arrow[r] \arrow[d, "\text{surj}"] & 0 \\
0 \arrow[r] & \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \arrow[r] & \mathcal{H}(V) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $\mathcal{H}$ flabby.

Proof of $\overline{\text{II}}$

WLOG $\mathcal{U} = X$ else work the restrictions:

$$0 \longrightarrow \mathcal{F}/_{u} \longrightarrow \mathcal{G}/_{u} \longrightarrow \mathcal{H}/_{u} \longrightarrow 0 \quad \text{exact \& } \mathcal{F}/_{u} \text{ flabby}$$

Suffices

$$\beta : \mathcal{G}(X) \longrightarrow \mathcal{H}(X) \quad \text{surjective if } \mathcal{F} \text{ flabby}$$

Proof Let $h \in \mathcal{H}(X)$. Define

$$\mathcal{A} = \{ (g, u) : g \in \mathcal{G}(u) \text{ and } \beta(g) = h/_{u} \}.$$

Define

$$(g, u) \geq (g', u') \text{ if } u \supseteq u' \text{ \& } g/_{u'} = g'$$

Remark Every linearly ordered chain admits an upper bound (take the union).

$\overset{\text{Zorn}}{\Longrightarrow}$ $\mathcal{A}$ admits a maximal element $(g, \mathcal{U})$.

Claim $\mathcal{U} = X$ for the maximal $(g, \mathcal{U})$.

This gives $\beta : \mathcal{G}(X) \longrightarrow \mathcal{H}(X)$ surjective.

Proof of the claim

(1) $\mathcal{U} \neq X$. We obtain a contradiction. Let $p \in X \setminus \mathcal{U}$.

(2) $\beta_p : \mathcal{G}_p \longrightarrow \mathcal{H}_p$ surjective $\Longrightarrow$

$\Longrightarrow \exists\ \tilde{g}_p$ with $\beta_p(\tilde{g}_p) = h_p$

$\Longrightarrow \exists\ V\ \exists\ \tilde{g} \in \mathcal{G}(V),\ \beta(\tilde{g}) = h|_V$.

(3) Overlaps: $U \cap V$

$$\beta(\tilde{g}|_{U\cap V}) = \beta(g|_{U\cap V}) = h|_{U\cap V}$$

$$\Rightarrow \boxed{\tilde{g}|_{U\cap V} - g|_{U\cap V} = \alpha(f)}, \quad f \in \mathcal{F}(U\cap V).$$

This uses $0 \longrightarrow \mathcal{F}(U\cap V) \longrightarrow \mathcal{G}(U\cap V) \longrightarrow \mathcal{H}(U\cap V)$

exact, as proved in Lecture 23.

(4) $\mathcal{F}$ flabby $\Rightarrow$ extend f to X.

(5) Define $W = U \cup V$ and

$$\tilde{\tilde{g}} = \begin{cases} g & \text{in } \mathcal{U} \\ \tilde{g} - \alpha(f) & \text{in } V \end{cases} \quad \text{well-defined}$$

Note $\beta(\tilde{\tilde{g}}) = h \Rightarrow (\tilde{\tilde{g}}, W) \in \mathcal{A}$.

This element contradicts maximality of $(g, \mathcal{U})$.

Math 220C - Lecture 25

June 2, 2021

Last time

$\mathcal{F} \longrightarrow X$ flabby if $\forall\, V \subseteq U \subseteq X$ open,

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \text{ surjective}$$

Key Lemma

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact, $\mathcal{F}$ flabby then

$\boxed{\text{i}}$ $0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact, $\forall\, U \subseteq X$ open

$\boxed{\text{ii}}$ $\mathcal{F}, \mathcal{G}$ flabby $\Rightarrow$ $\mathcal{H}$ flabby

§1. Main Theorem of Sheaf Cohomology

$\exists$ functors

$$H^p(X, -) : \text{Sheaves on } X \longrightarrow \text{Abelian Groups}$$

such that

$\boxed{a}$ $H^0(X, \mathcal{F}) = \mathcal{F}(X)$

$\boxed{b}$ $\mathcal{F}$ flabby $\Rightarrow H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1$

$\boxed{c}$ Given $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

$\exists$ connecting homomorphisms

$$\delta_p : H^p(X, \mathcal{H}) \longrightarrow H^{p+1}(X, \mathcal{F})$$

functorial in exact sequences such that

$$\begin{aligned} 0 &\longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \xrightarrow{\delta_0} \\ &\hookrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \xrightarrow{\delta_1} \\ &\hookrightarrow \dots \end{aligned}$$

exact.

These requirements determine the functors uniquely.

Aside from Homological Algebra

(1) Given a complex $d \circ d = 0$

$$\longrightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \longrightarrow A^n \longrightarrow \cdots$$

we define $$H^p(A^\bullet) = \frac{\operatorname{Ker} A^p \xrightarrow{d} A^{p+1}}{\operatorname{Im} A^{p-1} \xrightarrow{d} A^p}.$$

(2) Given complexes $A^\bullet, B^\bullet, C^\bullet$ such that

$$0 \longrightarrow A^p \longrightarrow B^p \longrightarrow C^p \longrightarrow 0 \quad \text{exact} \quad \forall p$$

we write $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$. exact

(3) If $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ exact then

$$\hookrightarrow H^p(A^\bullet) \longrightarrow H^p(B^\bullet) \longrightarrow H^p(C^\bullet)$$

$$\hookrightarrow H^{p+1}(A^\bullet) \longrightarrow H^{p+1}(B^\bullet) \longrightarrow H^{p+1}(C^\bullet)$$

$$\hookrightarrow \cdots$$

exact.

Outline of the argument

[i] every sheaf $\mathcal{F}$ admits a canonical flabby resolution

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \quad , \quad \mathcal{F}^p \text{ flabby}$$

[ii] Take global sections. We obtain the complex

$$\mathcal{F}^0(X) \to \mathcal{F}^1(X) \to \mathcal{F}^2(X) \to \cdots$$

[iii] Define

$$H^p(X, \mathcal{F}) = \frac{\operatorname{Ker} \; \mathcal{F}^p(X) \to \mathcal{F}^{p+1}(X)}{\operatorname{Im} \; \mathcal{F}^{p-1}(X) \to \mathcal{F}^p(X)} .$$

[iv] Show this works.

§2. Preparation for the proof — The Godement Sheaf

Definition Given $\mathcal{F} \to X$, define the sheaf $\Phi \mathcal{F}$ via

$$\Phi \mathcal{F}(U) = \prod_{x \in U} \mathcal{F}_x .$$

This is called the sheaf of totally discontinuous sections.

Remarks

[1] We define $\mathcal{F} \to \Phi \mathcal{F}$ sending

$$f \longrightarrow (f_x)_{x \in U} \quad \text{for } f \in \mathcal{F}(U).$$

We have $\mathcal{F} \to \Phi \mathcal{F}$ injective. Indeed,

$$f = 0 \iff f_x = 0 \quad \forall x \in U,$$ see Lecture 23

$\boxed{ii}$ $\Phi\mathcal{F}$ is flabby. Indeed, for $U \supseteq V$,

$$\phi\mathcal{F}(U) \longrightarrow \phi\mathcal{F}(V) \quad \text{surjective}$$

$$\Longleftrightarrow \quad \prod_{x\in U} \mathcal{F}_x \longrightarrow \prod_{x\in V} \mathcal{F}_x \quad \text{surjective}$$, which is clear

$\boxed{iii}$ If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact then

$$\Downarrow$$

$$0 \longrightarrow \phi\mathcal{F} \longrightarrow \phi\mathcal{G} \longrightarrow \phi\mathcal{H} \longrightarrow 0 \quad \text{exact}$$

Why? For all $U \subseteq X$ open, we have

$$0 \longrightarrow \phi\mathcal{F}(U) \longrightarrow \phi\mathcal{G}(U) \longrightarrow \phi\mathcal{H}(U) \longrightarrow 0 \quad \text{exact.}$$

Indeed, this is because

$$0 \longrightarrow \prod_{x\in U} \mathcal{F}_x \longrightarrow \prod_{x\in U} \mathcal{G}_x \longrightarrow \prod_{x\in U} \mathcal{H}_x \longrightarrow 0 \quad \text{exact}$$

which follows since $0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$ exact.

[IV] Assume $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact

exact   exact   exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \quad \text{exact} \\
0 \arrow[r] & \phi\mathcal{F} \arrow[r] \arrow[d] & \phi\mathcal{G} \arrow[r] \arrow[d] & \phi\mathcal{H} \arrow[r] \arrow[d] & 0 \quad \text{exact by [III]} \\
0 \arrow[r] & \phi\mathcal{F}/\mathcal{F} \arrow[r] \arrow[d] & \phi\mathcal{G}/\mathcal{G} \arrow[r] \arrow[d] & \phi\mathcal{H}/\mathcal{H} \arrow[r] \arrow[d] & 0 \quad \text{exact} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\Downarrow$

The last row is exact. This can be checked on stalks.

In the diagram of stalks, the columns & first 2 rows are exact $\Rightarrow$ 3rd row is also exact.

The canonical flabby resolution

Given $\mathcal{F} \longrightarrow X$ we construct a resolution

$$(*) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots$$

where $\mathcal{F}^p$ are flabby $\forall\, p \geq 0$.

How? Form the exact sequences:

$$(0) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \phi\, \mathcal{F} \longrightarrow \widetilde{\mathcal{F}}^1 = \phi\,\mathcal{F}/\mathcal{F} \longrightarrow 0$$

$$(1) \qquad 0 \longrightarrow \widetilde{\mathcal{F}}^1 \longrightarrow \phi\, \widetilde{\mathcal{F}}^1 \longrightarrow \widetilde{\mathcal{F}}^2 = \phi\, \widetilde{\mathcal{F}}^1/\widetilde{\mathcal{F}}^1 \longrightarrow 0$$

$$\vdots$$

$$(p) \qquad 0 \longrightarrow \widetilde{\mathcal{F}}^p \longrightarrow \phi\, \widetilde{\mathcal{F}}^p \longrightarrow \widetilde{\mathcal{F}}^{p+1} = \phi\, \widetilde{\mathcal{F}}^p/\widetilde{\mathcal{F}}^p \longrightarrow 0$$

where we define

- $\widetilde{\mathcal{F}}^0 = \mathcal{F}$
- $\mathcal{F}^p = \phi\, \widetilde{\mathcal{F}}^p$ = flabby
- $\widetilde{\mathcal{F}}^{p+1} = \phi\, \widetilde{\mathcal{F}}_p / \widetilde{\mathcal{F}}_p$

The resolution (*) follows by concatenating the above exact sequences.

Functoriality of the Godement resolution

Assume $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

e.g. $0 \longrightarrow \widetilde{\mathcal{F}}^0 \longrightarrow \widetilde{\mathcal{G}}^0 \longrightarrow \widetilde{\mathcal{H}}^0 \longrightarrow 0$ exact.

We show $0 \longrightarrow \widetilde{\mathcal{F}}^p \longrightarrow \widetilde{\mathcal{G}}^p \longrightarrow \widetilde{\mathcal{H}}^p \longrightarrow 0$ exact $\forall p$.

We use induction on p. The case $p=0$ is clear.

For the inductive step, we use $\boxed{\text{IV}}$ & diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{\mathcal{F}}^p \arrow[r] \arrow[d] & \tilde{\mathcal{G}}^p \arrow[r] \arrow[d] & \tilde{\mathcal{H}}^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}^p \arrow[r] \arrow[d] & \mathcal{G}^p \arrow[r] \arrow[d] & \mathcal{H}^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{\mathcal{F}}^{p+1} \arrow[r] \arrow[d] & \tilde{\mathcal{G}}^{p+1} \arrow[r] \arrow[d] & \tilde{\mathcal{H}}^{p+1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

exact

exact by $\boxed{\text{III}}$

$\Downarrow$

exact

This also shows $0 \longrightarrow \mathcal{F}^p \longrightarrow \mathcal{G}^p \longrightarrow \mathcal{H}^p \longrightarrow 0$ exact.

Conclusion

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact

$\Downarrow$

$0 \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet \longrightarrow 0$ exact.

$\Downarrow$ Key Lemma

$0 \longrightarrow \mathcal{F}^\bullet(X) \longrightarrow \mathcal{G}^\bullet(X) \longrightarrow \mathcal{H}^\bullet(X) \longrightarrow 0$ exact

Proof of the theorem

[i] Godement resolution

$$0 \to \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots \quad , \quad \mathcal{F}^p \text{ flabby}$$

[ii] Global sections

$$\mathcal{F}^0(X) \longrightarrow \mathcal{F}^1(X) \longrightarrow \mathcal{F}^2(X) \longrightarrow \cdots$$

[iii] Define

$$H^p(X, \mathcal{F}) = \frac{\operatorname{Ker} \; \mathcal{F}^p(X) \longrightarrow \mathcal{F}^{p+1}(X)}{\operatorname{Im} \; \mathcal{F}^{p-1}(X) \longrightarrow \mathcal{F}^p(X)} .$$

We verify it works!

Property $\boxed{a}$

WTS $\quad H^0(X, \mathcal{F}) = \operatorname{Ker}\left(\mathcal{F}^0(X) \xrightarrow{\gamma\beta} \mathcal{F}^1(X)\right) \overset{?}{=} \mathcal{F}(X).$

Recall the sequences

(0) $$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\alpha} \mathcal{F}^0(X) \xrightarrow{\beta} \tilde{\mathcal{F}}^1(X)$$

(1) $$0 \longrightarrow \tilde{\mathcal{F}}^1(X) \xrightarrow{\gamma} \mathcal{F}^1(X) \longrightarrow \tilde{\mathcal{F}}^2(X)$$

This shows:

$$\operatorname{Ker} \gamma\beta \overset{\gamma \text{ injective}}{=} \operatorname{Ker} \beta \quad \text{by (1)}$$

$$= \operatorname{Im} \alpha \cong \mathcal{F}(X) \quad \text{by (0).}$$

Property [6]

WTS : $\mathcal{F}$ flabby $\Longrightarrow H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1.$

- $\mathcal{F}^p = \phi \tilde{\mathcal{F}}^p$ flabby $\forall p \geq 0$ by $\boxed{III}$ above
- $\tilde{\mathcal{F}}^p$ flabby $\forall p \geq 0$, $\tilde{\mathcal{F}}^0 = \mathcal{F}$ = flabby (given)

Why? Induct on p. & use the sequence.

$$0 \longrightarrow \tilde{\mathcal{F}}^p \longrightarrow \mathcal{F}^p \longrightarrow \tilde{\mathcal{F}}^{p+1} \longrightarrow 0 \quad \text{exact}$$

The key Lemma shows $\tilde{\mathcal{F}}^p$ flabby $\Rightarrow \tilde{\mathcal{F}}^{p+1}$ flabby.

Also by the Key Lemma, we have

- $0 \longrightarrow \tilde{\mathcal{F}}^p(X) \longrightarrow \mathcal{F}^p(X) \longrightarrow \tilde{\mathcal{F}}^{p+1}(X) \longrightarrow 0$ exact

$\Downarrow$

$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}^0(X) \longrightarrow \mathcal{F}^1(X) \longrightarrow \cdots$ exact

$\Longrightarrow$ no cohomology $H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1.$

Property $\boxed{c}$

Assume $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact

$\Downarrow$

$0 \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet \longrightarrow 0$ exact

$\Downarrow$

$0 \longrightarrow \mathcal{F}^\bullet(X) \longrightarrow \mathcal{G}^\bullet(X) \longrightarrow \mathcal{H}^\bullet(X) \longrightarrow 0$ exact

$$\Rightarrow \quad \hookrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{G}) \longrightarrow H^p(X, \mathcal{H})$$
$$\hookrightarrow H^{p+1}(X, \mathcal{F}) \longrightarrow \cdots$$

using the facts from homological algebra reviewed above for

$$A^\bullet = \mathcal{F}^\bullet(X), \quad B^\bullet = \mathcal{G}^\bullet(X), \quad C^\bullet = \mathcal{H}^\bullet(X).$$

Math 220C - Lecture 26

June 4, 2021

Sheaves of modules

Let $\mathcal{R}$ be a sheaf of rings on X = Riemann surface.

- $\mathcal{R} = \mathcal{O}_X$ = sheaf of holomorphic functions
- $\mathcal{R} = \mathcal{C}^\infty$ = sheaf of smooth functions

Definition We say $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules if

- $\mathcal{F}(U)$ is an $\mathcal{R}(U)$-module $\forall\, U \subseteq X$ open
- $\forall\, U \supseteq V$ open, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{R}(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

Remark We can speak about sheaves of $\mathcal{C}^\infty$-modules or sheaves of $\mathcal{O}_X$-modules.

Remark If $\mathcal{F}$ is a sheaf of $\mathcal{C}^\infty$-modules, it can be shown

$$H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1.$$

This fact uses the existence of partitions of unity & applies to a class of sheaves called fine.

We next consider sheaves of $\mathcal{O}_X$ -modules, also called just $\mathcal{O}_X$ -modules.

Example $\mathcal{O}_X(D)$ is $\mathcal{O}_X$ -module for all divisors D.

Indeed, let $f \in \mathcal{O}_X(D)(U)$ and $g \in \mathcal{O}_X(U)$. We have

$$\operatorname{div} f + D/_U \geq 0.$$

We wish to show $gf \in \mathcal{O}_X(D)(U)$ that is

$$\operatorname{div}(gf) + D/_U \geq 0. \Longleftrightarrow$$

$$\Longleftrightarrow \underbrace{\operatorname{div} g}_{\geq 0} + \underbrace{(\operatorname{div} f + D/_U)}_{\geq 0} \geq 0$$ which is true since

g is holomorphic so $\operatorname{div} g \geq 0$.

Definition $\mathcal{F}$ is locally free of rank r provided

$\exists$ open cover $X = \bigcup U_\alpha$ such that

$$\mathcal{F}|_{U_\alpha} \cong \underbrace{\mathcal{O}_X|_{U_\alpha} \oplus \cdots \oplus \mathcal{O}_X|_{U_\alpha}}_{r \text{ copies}} \quad \text{as } \mathcal{O}_X|_{U_\alpha}\text{-modules}$$

Remark $\mathcal{O}_X(D)$ is locally free of rank 1.

Indeed, let $X = \bigcup_\alpha U_\alpha$ be coordinate charts. Write

$$D = \operatorname{div} f_\alpha \text{ in } U_\alpha.$$

Then g is a section of $\mathcal{O}_X(D)|_{U_\alpha}$ over $U \subseteq U_\alpha$ if

$$\begin{aligned} \operatorname{div} g + D|_U \geq 0 &\iff \operatorname{div} g + \operatorname{div} f_\alpha \geq 0 \text{ in } U \\ &\iff \operatorname{div} g f_\alpha \geq 0 \text{ in } U \\ &\iff g f_\alpha \in \mathcal{O}_X(U). \end{aligned}$$

Thus $\mathcal{O}_X(D)|_{U_\alpha} \longrightarrow \mathcal{O}_X|_{U_\alpha}$ is an isomorphism.

$$g \longrightarrow g f_\alpha$$

Remark If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}(X)$ is a module over $\mathcal{O}_X(X)$. In particular, $H^0(X, \mathcal{F}) = \mathcal{F}(X)$ is a $\mathbb{C}$ - vector space

In fact $H^p(X, \mathcal{F})$ is a $\mathbb{C}$ - vector space.

Coherent sheaves

$\mathcal{F} \longrightarrow X$ is coherent provided every point $x \in X$ admits a neighborhood $U \subseteq X$ and an exact sequence

$$\mathcal{O}_X|_U^{\oplus s} \longrightarrow \mathcal{O}_X|_U^{\oplus r} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

for some integers $r, s \geq 0$.

In particular $\mathcal{F} = \mathcal{O}_X(D)$ is coherent since it is locally free. (take $r = 1$, $s = 0$)

Theorem X compact Riemann surface, $\mathcal{F} \to X$ coherent $\mathcal{O}_X$-
-module. Then

$$\dim_{\mathbb{C}} H^p(X, \mathcal{F}) < \infty \quad \text{for } p = 0 \ \& \ 1.$$

Furthermore $H^p(X, \mathcal{F}) = 0$ for $p \neq 0, 1$.

Example X compact Riemann surface. Define

$$g = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = \text{arithmetic genus} < \infty.$$

The theorem allows us to define

$$\chi(X, \mathcal{F}) = \dim_{\mathbb{C}} H^0(X, \mathcal{F}) - \dim_{\mathbb{C}} H^1(X, \mathcal{F}) < \infty.$$

Example [i] $\chi(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$

$$= 1 - g.$$

[ii] If $\mathcal{F}$ = skyscraper sheaf, $H^0(X, \mathcal{F}) = \mathbb{C}$, $H^1(X, \mathcal{F}) = 0$

$$\Rightarrow \chi(X, \mathcal{F}) = 1.$$

Remark If $0 \to U \to V \to W \to 0$ is an exact sequence of vector spaces then

$$\dim V = \dim U + \dim W$$

More generally, if

$$0 \to V_0 \to V_1 \to \cdots \to V_n \to 0 \quad \text{exact}$$

then $\displaystyle\sum_{k=0}^{n} (-1)^k \dim V_k = 0.$

Remark If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact

then
$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H})$$
$$\hookrightarrow H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \cdots$$

and the previous remark gives

$$\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$$

The Riemann - Roch Theorem

Question Given compact Riemann surface X

- $z_1 \dots z_n \in X$, $p_1 \dots p_m \in X$
- $\mu_1 \dots \mu_n \geq 0$, $\nu_1, \dots \nu_m \geq 0$ integers

Want f meromorphic in X

- f has zeroes at z_i of order $\geq \mu_i$
- f has poles at p_i of order $\leq \nu_i$

Other zeroes are allowed, but no other poles.

Let $D = -\sum_i \mu_i [z_i] + \sum_i \nu_i [p_i]$

We thus want to understand $H^0(X, \mathcal{O}_X(D))$.

Theorem (Riemann – Roch)

$$\chi(X, \mathcal{O}_X(D)) = 1 - g + \deg D.$$

Remark We obtain a lower bound

$$\dim H^0(X, \mathcal{O}_X(D)) \geq \chi(X, \mathcal{O}_X(D)) = 1 - g + \deg D.$$

Remark Serre duality can be used to conclude that we have an equality in certain cases e.g. $\deg D \geq 2g - 1$.

14.

Theorie der *Abel'*schen Functionen.

(Von Herrn *B. Riemann.*)

In der folgenden Abhandlung habe ich die *Abel'*schen Functionen nach einer Methode behandelt, deren Principien in meiner Inauguraldissertation *) aufgestellt und in einer etwas veränderten Form in den drei vorhergehenden Aufsätzen dargestellt worden sind. Zur Erleichterung der Uebersicht schicke ich eine kurze Inhaltsangabe vorauf.

Die erste Abtheilung enthält die Theorie eines Systems von gleichverzweigten algebraischen Functionen und ihren Integralen, soweit für dieselbe nicht die Betrachtung von ϑ-Reihen mafsgebend ist, und handelt im §. 1—5 von der Bestimmung dieser Functionen durch ihre Verzweigungsart und ihre Unstetigkeiten, im §. 6—10 von den rationalen Ausdrücken derselben in zwei durch eine algebraische Gleichung verknüpfte veränderliche Gröfsen, und im §. 11—13 von der Transformation dieser Ausdrücke durch rationale Substitutionen. Der bei dieser Untersuchung sich darbietende Begriff einer *Klasse* von algebraischen Gleichungen, welche sich durch rationale Substitutionen in einander transformiren lassen, dürfte auch für andere Untersuchungen wichtig und die Transformation einer solchen Gleichung in Gleichungen niedrigsten Grades ihrer Klasse (§. 13) auch bei anderen Gelegenheiten von Nutzen sein. Diese Abtheilung behandelt endlich im §. 14—16 zur Vorbereitung der folgenden die Anwendung des *Abel'*schen Additionstheorems für ein beliebiges System allenthalben endlicher Integrale von gleichverzweigten algebraischen Functionen zur Integration eines Systems von Differentialgleichungen.

In der zweiten Abtheilung werden für ein beliebiges System von immer endlichen Integralen gleichverzweigter, algebraischer, $\overline{2p+1}$fach zusammenhangender Functionen die *Jacobi'*schen Umkehrungsfunctionen von p veränderlichen Gröfsen durch $p$fach unendliche ϑ-Reihen ausgedrückt, d. h. durch Reihen von der Form

*) Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Gröfse. Göttingen 1851.

Crelle's Journal 54 (1857)

Crelle's Journal 64 (1864)

Ueber die Anzahl der willkürlichen Constanten in algebraischen Functionen.

(Von Herrn *G. Roch* in Halle.)

Ist s eine durch die Gleichung $F(s, z) = 0$ definirte algebraische Function von z, so kann nach *Riemann* (s. dessen Abhandlung über *Abel*sche Functionen, Band 54. dieses Journals) jede wie s verzweigte algebraische Function s' von z rational durch s und z ausgedrückt werden. Wird die Function s' in m Punkten der Fläche T, welche die Verzweigungsart angiebt, unendlich erster Ordnung, so enthält dieselbe nach §. 5. der erwähnten Abhandlung $m-p+1$ willkürliche Constanten. Schon die a. a. O. untersuchte Bedingung der Existenz von Functionen, die in weniger als $p+1$ Punkten unendlich werden, zeigt, dass die Anzahl der wirklich vorhandenen Constanten eine grössere sein kann. Dies kann aber auch Statt finden, wenn m grösser als p ist. Ist z. B. s' der Quotient zweier Functionen φ, so wird s' in den $2p-2$ Punkten unendlich, in denen der Nenner gleich Null ist (s. §. 10. der citirten Abhandlung), und enthält so viele willkürliche Constanten, als die den Zähler bildende Function, nämlich p, während die Zahl $m-p+1$ im vorliegenden Falle gleich $p-1$ ist. Sind dagegen φ_1, φ_2, ψ_1, ψ_2 solche Functionen φ, welche in $p-1$ Punkten unendlich klein zweiter Ordnung werden, deren Quadratwurzeln *Riemann* Abelsche Functionen nennt, so giebt es eine gewisse Anzahl von Ausdrücken $\sqrt{\frac{\varphi_1\psi_1}{\varphi_2\psi_2}}$, welche rationale Functionen von s und z sind; diese enthalten in der That $p-1$ Constanten in linearer Weise, ein Satz, der von *Riemann* herrührt und für welchen ein Beweis in der folgenden genauen Bestimmung der Constanten-Anzahl mit enthalten ist.

Der allgemeinste Ausdruck eines Integrals zweiter Gattung, welches in m Punkten ε unendlich erster Ordnung wird, ist

$$v = \beta_1 t_1 + \cdots + \beta_m t_m + \alpha_1 w_1 + \cdots + \alpha_p w_p + \text{const.}$$

(Vergl. §. 5. der *Riemann*schen Abhandlung.) Hierbei sind unter $t_1 \ldots t_m$ specielle Integrale zweiter Gattung zu verstehen, welche beziehlich in den Punkten $\varepsilon_1 \ldots \varepsilon_m$ unendlich werden wie $\frac{1}{\sigma_1} \cdots \frac{1}{\sigma_m}$, wenn σ_k eine Grösse be-

Quick proof when $D \geq 0$

Recall from Lecture 23 that

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \underbrace{\prod_j \mathbb{C}_{p_j}^{\oplus n_j}}_{\text{sky scraper}} \longrightarrow 0$$

where $D = \sum n_j [p_j]$.

Then $\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X) + \chi(X, \prod_j \mathbb{C}_{p_j}^{\oplus n_j})$

$$= \chi(X, \mathcal{O}_X) + \sum n_j \, \chi(X, \mathbb{C}_{p_j})$$

$$= 1 - g + \sum n_j \cdot 1$$

previous examples

$$= 1 - g + \deg D$$

General Proof is very similar.

Claim $\forall p \in X$, we have an exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D+p) \longrightarrow \underbrace{\mathbb{C}_p}_{\text{skyscraper sheaf}} \longrightarrow 0$$

Assuming this, let

$$f(D) = \chi(\mathcal{O}_X(D)) - (1-g+\deg D)$$

$\boxed{i}$ $f(0) = \chi(\mathcal{O}_X) - (1-g) = 0$

$\boxed{ii}$ $f(D) = f(D+p)$ since the exact sequence gives

$$\chi(\mathcal{O}_X(D+p)) = \chi(\mathcal{O}_X(D)) + 1 \quad \text{and}$$

$$\deg(D+p) = \deg D + 1.$$

$\boxed{iii}$ $f(D) = f(D+E)$ for all divisors $E \geq 0$.

This follows from $\boxed{ii}$.

This shows f must be constant. Indeed, declare

$$D_1 \geq D_2 \quad \text{if} \quad D_1 - D_2 \geq 0.$$

If D_1, D_2 are two divisors, we can find D_3 with

$$D_3 \geq D_1 \quad \text{and} \quad D_3 \geq D_2.$$

By $\boxed{(a)}$, we have

$$\left.\begin{aligned} f(D_3) &= f(D_1) \\ f(D_3) &= f(D_2) \end{aligned}\right\} \Rightarrow f(D_1) = f(D_2).$$

$$\Rightarrow f \text{ constant} \overset{\boxed{i}}{\Rightarrow} f \equiv 0 \Rightarrow \text{Riemann – Roch.}$$

Proof of the exact sequence (*)

Write $D = E + n[p]$, $p \notin \operatorname{Supp} E$. We show

$$0 \longrightarrow \mathcal{O}_X(E + n[p]) \xrightarrow{\alpha} \mathcal{O}_X(E + (n+1)[p]) \xrightarrow{\beta} \mathbb{C}_p \longrightarrow 0$$

The map α is the natural inclusion.

To define β, take f with

$$\operatorname{div} f + E + (n+1)[p] \geq 0.$$

In local coordinate near p, write

$$f(z) = \frac{a_{-n-1}}{(z-p)^{n+1}} + \cdots$$ Laurent expansion.

Define $\beta(f) = a_{-n-1}$.

Why exact If $\beta(f) = 0$ then $a_{-n-1} = 0 \Rightarrow f$ has pole of order $\leq n$ at p hence f is a section of $\mathcal{O}_X(E + n[p])$

Where to go from here?

(1) sheaf cohomology in more detail

(2) discussion of genus

- arithmetic genus $\quad g = \dim H^1(X, \mathcal{O}_X)$
- topological genus $\quad 2g = \dim H^1(X, \mathbb{C})$
- geometric genus via 1-forms $\quad g = \dim H^0(X, \Omega^1_X)$

(3) Serre duality

$$H^0(X, \mathcal{F}) \cong H^1(X, \mathcal{F}^\vee \otimes \Omega^1_X)^\vee$$

(4) line bundles & the Jacobian

(5) projective embeddings, ample, very ample

line bundles $\quad X \longrightarrow \mathbb{P}^N$

(6) moduli of curves $\mathcal{M}_g$, $\overline{\mathcal{M}_g}$

Many directions are possible!