Math 220 A — Lecture 21

December 2, 2020

$\boxed{0}$ Last time   Conway $\overline{\text{V}}$.3.

$f : U \longrightarrow \mathbb{C}$ meromorphic, $U \subseteq \mathbb{C}$, $a \in U$.

Def

$$\operatorname{ord}(f, a) = \begin{cases} n, & a \text{ zero of order } n \\ -n, & a \text{ pole of order } n \\ 0, & \text{otherwise} \end{cases}$$

Remark $\operatorname{ord}(f, a) = k \iff f = (z-a)^k g$

where g holomorphic near a, $g(a) \neq 0$

This definition treats zeros & poles equally.

Question Find poles & residues of $\frac{f'}{f}$

Answer Poles of $\frac{f'}{f}$ come from zeros or poles of f.

Let a be a zero / pole with $\text{ord}(f, a) = k$.

$\Rightarrow f = (z-a)^k g$, g holomorphic, $g(a) \neq 0$.

$$\Rightarrow \frac{f'}{f} = \frac{k(z-a)^{k-1} g + (z-a)^k g'}{(z-a)^k g} = \frac{k}{z-a} + \frac{g'}{g}$$

Since $g \neq 0$ near a $\Rightarrow \frac{g'}{g}$ holomorphic near a

$\Rightarrow \frac{f'}{f}$ has simple pole and

$$\text{Res}\left(\frac{f'}{f}, a\right) = k = \text{ord}(f, a)$$

$\boxed{1.}$ Argument Principle / Conway V. 3

Theorem Given f meromorphic in U, $\gamma \overset{U}{\sim} 0$, avoiding the zeros and poles of f, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_a n(\gamma, a)\ \mathrm{ord}(f, a)$$

This follows by the Residue Theorem & above discussion.

Remarks $\boxed{1}$ In practice, γ is a circle or a simple closed curve with $\mathrm{Int}\,\gamma \subseteq U$. Then

$$n(\gamma, a) = \begin{cases} 1, & a \in \mathrm{Int}\,\gamma \\ 0, & a \in \mathrm{Ext}\,\gamma \end{cases}$$

Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \#\ \text{Zeroes} - \#\ \text{Poles in } \mathrm{Int}\,\gamma.$$

(counted with multiplicity)

[ii] Why is it called "argument principle"?

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int_\gamma d\log f$$

$$= \frac{1}{2\pi i}\,\Delta \log f$$

$$= \frac{1}{2\pi \cancel{i}}\,\Delta\Big(\underbrace{\cancel{\log|f|}}_{0} + \cancel{i}\,\mathrm{Arg} f\Big)$$

$$= \frac{1}{2\pi}\,\Delta\,\mathrm{Arg} f$$

[iii] Enhanced version $g : U \to \mathbb{C}$ holomorphic

f meromorphic in U, $\gamma \overset{U}{\sim} 0$ avoiding the zeros and poles of f,

$$\frac{1}{2\pi i}\int_\gamma g\,\frac{f'}{f}\,dz = \sum_a g(a)\cdot n(\gamma,a)\cdot \mathrm{ord}(f,a)$$

If γ is simple closed, $\operatorname{Int}\gamma \subseteq \mathcal{U}$, then

$$\frac{1}{2\pi i}\int_\gamma g\,\frac{f'}{f}\,dz = \sum g\,(\text{zeros of } f) - g\,(\text{poles of } f)$$

Int γ Int γ

appear with multiplicity

Proof   We apply the Residue theorem.

We show $\operatorname{Res}\left(g\cdot\frac{f'}{f}, a\right) = g(a)\ \operatorname{ord}(f,a)$

Let $\operatorname{ord}(f,a) = k$. We know from page 2:

$$\frac{f'}{f} = \frac{k}{z-a} + F, \quad F, G \text{ holomorphic near } a$$

$$g = g(a) + (z-a)\,G \quad (\text{Taylor expansion})$$

$$\Rightarrow\ g\cdot\frac{f'}{f} = \left(\frac{k}{z-a} + F\right)\left(g(a) + (z-a)\,G\right)$$

$$= \frac{k\ g(a)}{z-a} + H \quad \text{where } H \text{ holomorphic near } a$$

$$\Rightarrow\ \operatorname{Res}\left(g\cdot\frac{f'}{f}, a\right) = k\,g(a) = \operatorname{ord}(f,a)\cdot g(a).$$

2. Applications (Conway V. 3)

Let $f: U \longrightarrow \mathbb{C}$ holomorphic, $\bar{\Delta} \subseteq U$. such that $f|_{\bar{\Delta}}$ injective. Let $V = f(\Delta)$ = open. then

$$f : \Delta \longrightarrow V \quad \text{bijection.}$$

Proposition The following integral formula for the inverse function holds

$$f^{-1}(q) = \frac{1}{2\pi i} \int_{\partial\Delta} z \cdot \frac{f'(z)}{f(z)-q} \, dz \quad \forall q \in V.$$

In particular $f^{-1}: V \longrightarrow \Delta$ is holomorphic.

Proof Apply the enhanced Argument Principle to $f - q$ and $g(z) = z$. Since f injective, $\exists !\ p \in \Delta$ with $f(p) = q$. $\Longrightarrow f^{-1}(q) = p$. But

$$\frac{1}{2\pi i} \int_{\partial\Delta} z \, \underbrace{\frac{f'(z)}{f(z)-q}} \, dz = g(p) = p = f^{-1}(q).$$

no zeros on $\partial\Delta$ since $q \in f(\Delta) \Rightarrow q \notin f(\partial\Delta)$ as $f|_{\bar{\Delta}}$ injective.

Recall from Lecture 16

Key statement $\varphi : U \times \{\gamma\} \longrightarrow \mathbb{C}$

- φ continuous

- $z \longrightarrow \varphi(z, w)$ holomorphic $\forall\, w \in \{\gamma\}$.

Then $g(z) = \int_\gamma \varphi(z, w)\, dw$ holomorphic.

Apply this to $\varphi : \underset{\underset{\large g}{\upsilon}}{\Delta} \times \underset{\underset{\large z}{\upsilon}}{\partial\Delta} \longrightarrow \mathbb{C}$

$$\varphi(g, z) = z \cdot \frac{f'(z)}{f(z) - g} \quad \text{. continuous \&}$$

holomorphic in g $\forall\, z \in \partial\Delta$. Then

$$f^{-1}(g) = \frac{1}{2\pi i} \int_{\partial\Delta} \varphi(g, z)\, dz = \text{holomorphic in } g.$$

Remark

This extends a result from Lecture 11. concerning holomorphicity of the inverse (removes $f' \neq 0$).

3. Further Applications of the Argument Principle

Elliptic functions

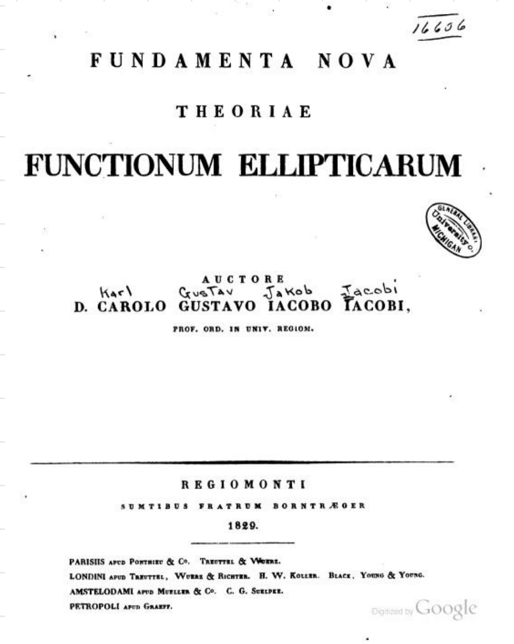
- studied by Abel, Jacobi, Weierstraß
- connected with arclength of ellipse

  elliptic integrals

  elliptic curves

- rich theory
- we will only say a few words about them

(More in Math 220 B & C)



16606

FUNDAMENTA NOVA

THEORIAE

FUNCTIONUM ELLIPTICARUM

GENERAL LIBRARY University of MICHIGAN

AUCTORE

Karl Gustav Jakob Jacobi

D. CAROLO GUSTAVO IACOBO IACOBI,

PROF. ORD. IN UNIV. REGIOM.

REGIOMONTI

SUMTIBUS FRATRUM BORNTRÆGER

1829.

PARISIIS APUD PONTHIEU & Co. TREUTTEL & WÜRTZ.

LONDINI APUD TREUTTEL, WÜRTZ & RICHTER. H. W. KOLLER. BLACK, YOUNG & YOUNG.

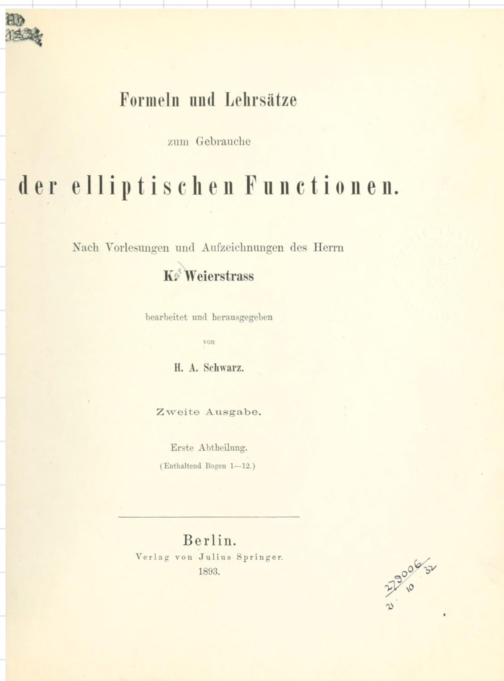
AMSTELODAMI APUD MUELLER & Co. C. G. SULPKE.

PETROPOLI APUD GRAEFF.

Digitized by Google

Carl Gustav Jacob Jacobi (1804 – 1851)

Jacobian, Jacobi symbol, Jacobi identity, symbol ∂



Formeln und Lehrsätze

zum Gebrauche

der elliptischen Functionen.

Nach Vorlesungen und Aufzeichnungen des Herrn

K. Weierstrass

bearbeitet und herausgegeben

von

H. A. Schwarz.

Zweite Ausgabe.

Erste Abtheilung.

(Enthaltend Bogen 1—12.)

Berlin.

Verlag von Julius Springer.

1893.

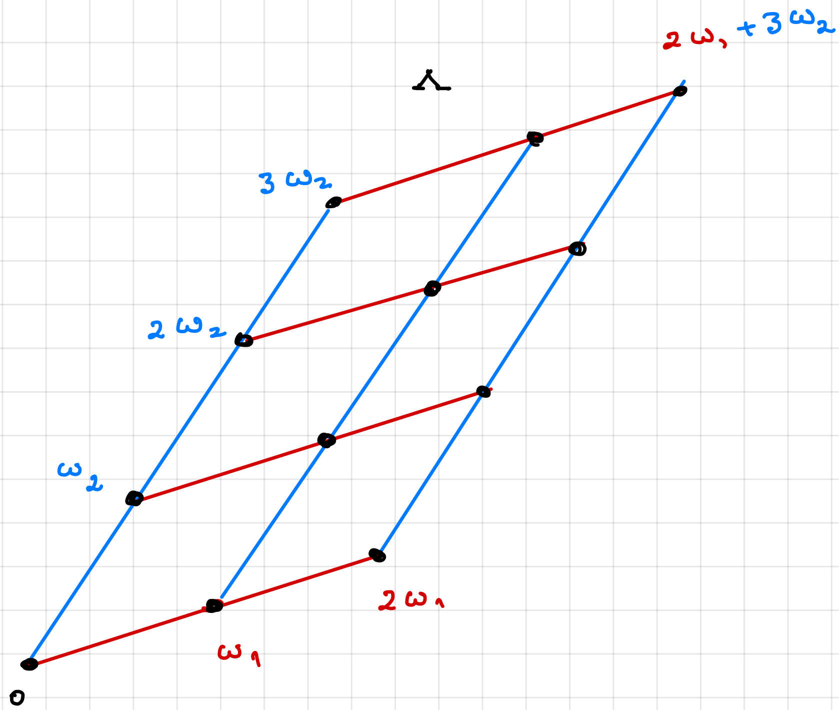
279006

Karl Weierstraß (1815 – 1897)

Definition

Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. Define the lattice

$$\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2 = \{ m\,\omega_1 + n\,\omega_2 \;:\; m,n \in \mathbb{Z} \}$$



Def An elliptic function f satisfies

[i] f meromorphic on $\mathbb{C}$

[ii] f periodic, $f(z) = f(z+\omega_1) = f(z+\omega_2)$

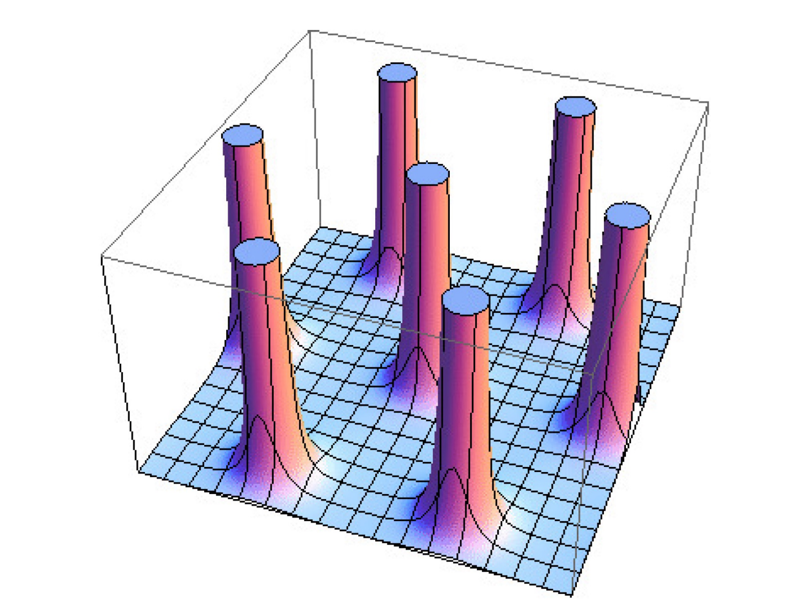
Note that in fact $\forall \lambda \in \Lambda$, $f(z) = f(z+\lambda)$

Remarks $\boxed{i}$ The best-known elliptic function is

Weierstraß

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

We will study this function in detail later in 226 B & C.



$\boxed{ii}$ f elliptic $\Rightarrow$ f' elliptic.

Indeed $f(z) = f(z+\lambda) \Rightarrow f'(z) = f'(z+\lambda)$. $\forall \lambda \in \Lambda$