HOMEWORK 5

DUE MAY 5, 2021 AT 11:59PM

This part of the homework works through the defintion Tor. We are working in the category R-mod for simplicity, but many of these results apply to any abelian category. We will deal mainly with chain complexes, but the corresponding results hold for cochain complexes with essentially the same proofs.

Do problem 1, the three exercises from Atiyah-MacDonald, and any three of the problems 2-8.

1. Prove that every R-module N has a free resolution, and therefore a projective resolution.

2. Prove that a map of complexes $f : A_\bullet \longrightarrow B_\bullet$ induces maps on homology $H_i(A_\bullet) \longrightarrow H_i(B_\bullet)$ for each i. Show furthermore that H_i is a covariant functor from the category of complexes of R-modules to the category of R-modules.

3. We say that two maps of complexes $f, g : C_\bullet \longrightarrow C'_\bullet$ are *homotopic* if there exists a sequence of maps $w : C_i \longrightarrow C'_{i+1}$ such that $f - g = d'w + wd$. (Do work out the indices.) Prove that two homotopic maps give the same maps on homology.

4. (a) Prove that given two R-modules N and N' and free resolutions

$$\mathcal{R} : \ldots \longrightarrow F_i \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

$$\mathcal{R}' \ldots \longrightarrow F'_i \longrightarrow \ldots \longrightarrow F'_1 \longrightarrow F'_0 \longrightarrow N' \longrightarrow 0$$

we can lift any R-module homomorphism $\varphi : N \longrightarrow N'$ to a morphism of the two resolutions

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & F_i & \longrightarrow & \ldots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow\varphi & & \\
\ldots & \longrightarrow & F'_i & \longrightarrow & \ldots & \longrightarrow & F'_1 & \longrightarrow & F'_0 & \longrightarrow & N' & \longrightarrow & 0
\end{array}$$

(b) Show that any two lifts $\mathcal{R} \longrightarrow \mathcal{R}'$ are homotopic.

(c) Prove the same statement for projective resolutions.

(d) Let M be an R-module. Prove that the lift in (a) induces a map between the homology of $M \otimes \mathcal{R}$ and the homology of $M \otimes \mathcal{R}'$, i.e. maps

$$\mathrm{Tor}_i(M,N)_{\mathcal{R}} \longrightarrow \mathrm{Tor}_i(M,N')_{\mathcal{R}'}.$$

We now pull these statements together. (Be sure to digest these completely!)

- We get a map of R-modules $\mathrm{Tor}_i(M,N)_{\mathcal{R}} \longrightarrow \mathrm{Tor}_i(M,N')_{\mathcal{R}'}$ independent of the lift $\mathcal{R} \longrightarrow \mathcal{R}'$.
- Hence for any two resolutions $\mathcal{R}$ and $\mathcal{R}'$ of an R-module N, we get a canonical isomorphism $\mathrm{Tor}_i(M,N)_{\mathcal{R}} \longrightarrow \mathrm{Tor}_i(M,N)_{\mathcal{R}'}$.

  Here's why. Choose lifts $\mathcal{R} \longrightarrow \mathcal{R}'$ and $\mathcal{R}' \longrightarrow \mathcal{R}$ of the identity Id_N . The composition $\mathcal{R} \longrightarrow \mathcal{R}' \longrightarrow \mathcal{R}$ is homotopic to the identity of $\mathcal{R}$. Thus if $\Phi_{\mathcal{R} \longrightarrow \mathcal{R}'}$ is the map induced by the lift $\mathcal{R} \longrightarrow \mathcal{R}'$ and $\Phi_{\mathcal{R}' \longrightarrow \mathcal{R}}$ is the map induced by the lift $\mathcal{R}' \longrightarrow \mathcal{R}$, then

  $\Phi_{\mathcal{R} \longrightarrow \mathcal{R}'} \circ \Phi_{\mathcal{R}' \longrightarrow \mathcal{R}}$ is the identity of $\mathcal{R}'$ and $\Phi_{\mathcal{R}' \longrightarrow \mathcal{R}} \circ \Phi_{\mathcal{R} \longrightarrow \mathcal{R}'}$ is the identity of $\mathcal{R}$.
- The uphsot is that $\mathrm{Tor}_i(M,N)$ does not depend on the choice of resolution. It is a covariant functor R-**mod** $\longrightarrow R$-**mod**.

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5. Prove that $\mathrm{Tor}_0(M,N) \simeq M \otimes_R N$.

6. (Long exact sequence of homology) Prove that a short exact sequence of complexes

$$0_\bullet \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0_\bullet$$

induces a long exact sequence of homology

$$\begin{array}{cccccccc} & & & \dots & \longrightarrow & H_{i+1}(C_\bullet) & \longrightarrow \\ H_i(A_\bullet) & \longrightarrow & H_i(B_\bullet) & \longrightarrow & H_i(C_\bullet) & \longrightarrow \\ H_{i-1}(A_\bullet) & \longrightarrow & \dots \end{array}$$

The maps $H_i(A_\bullet) \longrightarrow H_i(B_\bullet)$ are induced by f and $H_i(B_\bullet) \longrightarrow H_i(C_\bullet)$ are induced by g. The connecting homomorphisms $H_{i+1}(C_\bullet) \longrightarrow H_i(A_\bullet)$ come from the snake lemma applied appropriately.

7. Given an exact sequence of R-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

and free resolutions $F'_\bullet$ of N' and $F''_\bullet$ of N'' show that $F_i = F'_i \oplus F''_i$ form a free resolution of N and we get maps of complexes

$$
\begin{array}{ccccccccccccc}
 & & 0 & & & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & F_i' & \longrightarrow & \dots & \longrightarrow & F_1' & \longrightarrow & F_0' & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & F_i & \longrightarrow & \dots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & F_i'' & \longrightarrow & \dots & \longrightarrow & F_1'' & \longrightarrow & F_0' & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

You do need to show that the middle row above is not just a complex, but in fact it is exact, and therefore forms a resolution of N.

8. Prove that for any short exact sequence of R-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

and any R-module M we get a long exact sequence

$$
\begin{array}{ccccccccc}
 & & & & \dots & \longrightarrow & \mathrm{Tor}_{i+1}(M, N'') & & \\
\longrightarrow & \mathrm{Tor}_i(M, N') & \longrightarrow & \mathrm{Tor}_i(M, N) & \longrightarrow & \mathrm{Tor}_i(M, N'') & \longrightarrow & \dots \\
\vdots & & & & & & & \\
\longrightarrow & \mathrm{Tor}_1(M, N') & \longrightarrow & \mathrm{Tor}_1(M, N) & \longrightarrow & \mathrm{Tor}_1(M, N'') & & \\
\longrightarrow & M \otimes_R N' & \longrightarrow & M \otimes_R N & \longrightarrow & M \otimes_R N'' & \longrightarrow & 0.
\end{array}
$$

From Atiyah-MacDonald:

Chapter 2: 19, 20, 24 (all of them)