<u>Math 220C</u> - <u>Lecture 14</u>

April 28, 2021

Last time — Strategy for proving Little Picard

Assume $\exists f: \mathbb{C} \longrightarrow \mathbb{C}$ entire, not constant, omits 0 & 1.

Step A produces a function g entire, nonconstant and $\alpha > 0$ with $\Delta \not\subset \operatorname{Im} g$ for all discs Δ of radius α

Step B For any g entire & not constant, $\operatorname{Im} g$ contains a disc of any radius, in particular of radius α.

Step A & Step B are incompatible, showing f does not exist $\Rightarrow$ Little Picard.

§1. Bloch's Theorem Conway XII. 1.

Notation G open & bounded $\Rightarrow \bar{G}$ compact

$\mathcal{O}(\bar{G})$ = set of holomorphic functions in a neighborhood of $\bar{G}$

Theorem (version of Conway XII. 1.4). $\Delta = \Delta(0,1)$.

Given $f \in \mathcal{O}(\bar{\Delta})$, $f'(0) = 1$, then $\operatorname{Im} f$ contains a disc a radius $\beta > 0$. In fact $\beta = \frac{3}{2} - \sqrt{2} \cong .085$ works

Crucially β is a constant independent of the function f.

This is important for Little Picard.

Remark The value of β in Conway is $\beta = \frac{1}{72} \cong .01$

This β is smaller, however Conway proves a little more.

Bloch $\Rightarrow$ Step B   Conway XII. 2.

$g : \mathbb{C} \to \mathbb{C}$ entire, not constant $\Rightarrow$ Im g contains a disc of any radius.

Proof Fix a value r for the radius.

g not constant $\Rightarrow \exists\, a$ with $g'(a) \neq 0$. WLOG $a = 0$ else work with $g^{new}(z) = g(z+a)$.

Let $R > 0$. Define $h(z) = \dfrac{g(Rz)}{R\,g'(0)} \Rightarrow h'(0) = 1$ &

h holomorphic in $\bar{\Delta} \Rightarrow$ Im h contains a disc of radius β

$\Rightarrow$ Im g contains a disc of radius $R\,|g'(0)|\,\beta > r$ if R is chosen large.

Remark This completes the proof of Little Picard.

Remark The proof shows $g \in \mathcal{O}(\bar{\Delta}(a,R)) \Rightarrow$ Im g contains a disc of radius $R\,|g'(a)|\,\beta$.

Remark: Optimal value of β — Conway XII. 1.9

Let $\mathcal{F} = \{ f \in \mathcal{O}(\bar{\Delta}),\ f'(0) = 1 \}$

$\boxed{i}$ $L_f = \{$ largest radius of a disc in $f(\Delta)\}$

$$L = \inf_{f \in \mathcal{F}} L_f = \text{Landau constant}$$

$\boxed{ii}$ $B_f = \{$ largest radius of a biholomorphic disc in $f(\Delta)\}$

$$B = \inf_{f \in \mathcal{F}} B_f = \text{Bloch constant}$$

$\boxed{iii}$ Current knowledge: $.5 < L < .544$

$$.433 < B < .472$$

Conjecturally $$B = \sqrt{\frac{\sqrt{3}-1}{2}} \; \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)} \cong .471$$

We will show $L \geq \beta = \frac{3}{2} - \sqrt{2} \cong .08$.

2. Lemma. — *Die Funktion*

$$w = C\,\zeta\,\frac{\int\limits_0^1 t^{-\frac{1}{2}-\frac{1}{2k}}(1-t)^{-\frac{1}{6}+\frac{1}{2k}}(1-\zeta^3 t)^{-\frac{5}{6}+\frac{1}{2k}}\,dt}{\int\limits_0^1 t^{-\frac{1}{2}-\frac{1}{2k}}(1-t)^{-\frac{5}{6}+\frac{1}{2k}}(1-\zeta^3 t)^{-\frac{1}{6}+\frac{1}{2k}}\,dt}$$

vermittelt die konforme Abbildung des Kreises $|\zeta| < 1$ *auf ein gleichseitiges Kreisbogendreieck mit den Winkeln* π/k $(k > 1)$. *Die Punkte* 1, ε, ε^2 $\left(\varepsilon = \frac{-1+i\sqrt{3}}{2}\right)$ *entsprechen den Eckpunkten.*

Man findet die Abbildungsfunktion am einfachsten, wenn man von der Schwarzschen Beziehung

$$\{w, \zeta\} = \frac{9}{2}\left(1 - \frac{1}{k^2}\right)\frac{\zeta}{(\zeta^3-1)^2}$$

ausgeht und die assoziierte lineare Differentialgleichung

$$\frac{y''}{y} = -\frac{9}{4}\left(1 - \frac{1}{k^2}\right)\frac{\zeta}{(\zeta^3-1)^2}$$

zuerst durch die Substitution $y = (\zeta^3 - 1)^{\frac{1}{2}-\frac{1}{2k}}v$, $\zeta^3 = \xi$, dann durch $y = \zeta(\zeta^3-1)^{\frac{1}{2}-\frac{1}{2k}}u$, $\zeta^3 = \xi$ auf eine hypergeometrische reduziert.

Bestimmt man C so, daß $w'(0) = 1$ wird, so ergibt sich

$$w(1) = \frac{\mathrm{B}\left(\frac{1}{2}-\frac{1}{2k}, \frac{1}{6}+\frac{1}{2k}\right)}{\mathrm{B}\left(\frac{1}{2}-\frac{1}{2k}, \frac{5}{6}+\frac{1}{2k}\right)} = \frac{\Gamma\left(\frac{1}{6}+\frac{1}{2k}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{6}+\frac{1}{2k}\right)\Gamma\left(\frac{2}{3}\right)}.$$

Die uns interessierenden Fälle sind $k = 3$ und $k = 6$. Man findet durch Einsetzung

$$\lambda = \frac{w_2(1)}{w_1(1)} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma(1)\,\Gamma\left(\frac{1}{4}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}. \tag{3}$$

Aus (1), (2) und (3) erhält man jetzt endlich

$$\mathfrak{B}' = \sqrt{\frac{\sqrt{3}-1}{2}}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)} = \sqrt{\pi}\cdot 2^{1/4}\cdot\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{4}\right)}\left(\frac{\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{12}\right)}\right)^{1/2} = 0{,}4719\ldots$$

und es ist somit bewiesen, daß

$$\mathfrak{B} \leqq \mathfrak{B}' < 0{,}472.$$

Früher bekannt war die Landausche Abschätzung[1])

$$0{,}396 < \mathfrak{B} < 0{,}555.$$

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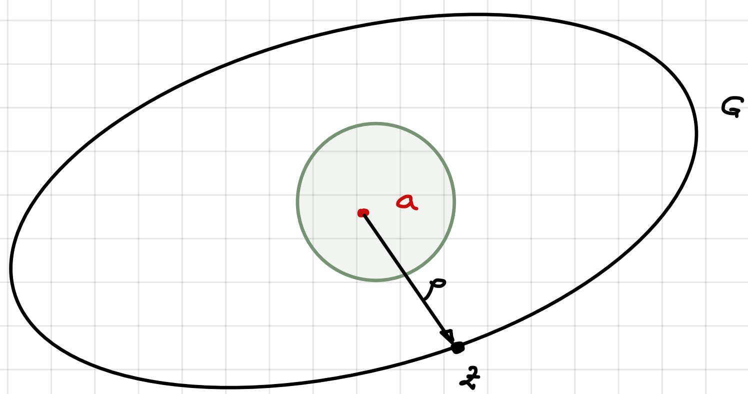
[1]) Landau, Über die Blochsche Konstante und zwei verwandte Weltkonstanten. Mathem. Zeitschr. 30 (1929), 608—634, insbesondere S. 614.

§2. Proof of Bloch's theorem

Question How can we construct a disc in $\operatorname{Im} f$?

Assume G bounded, $a \in G$. Let $\rho = \min_{z \in \partial G} |f(z) - f(a)|$



Lemma A Let $f \in \mathcal{O}(\bar{G})$, $a \in G$, $\rho = \min_{z \in \partial G} |f(z) - f(a)|$.

Then $\operatorname{Im} f$ contains $\Delta(f(a), \rho)$.

Remark This can be viewed as a more precise

Open Mapping theorem.

Proof Let $H = f(G) \subseteq f(\bar{G}) =$ compact since $\bar{G}$ is compact. Then H is bounded $\Rightarrow \partial H$ compact. Let

$$R = d(f(a), \partial H) = \min_{h \in \partial H} |h - f(a)|.$$

$\Rightarrow \Delta(f(a), R) \subseteq H \subseteq \operatorname{Im} f$. We show

$$R \geq \rho$$

$\Rightarrow \Delta(f(a), \rho) \subseteq \operatorname{Im} f$ proving Lemma A.

To prove $R \geq \rho$, let $w \in \partial H$ achieve the minimum R.

Claim $w = f(z)$, $z \in \partial G$.

Then $R = |w - f(a)| = |f(z) - f(a)| \geq \rho$ by definition of ρ as a minimum.

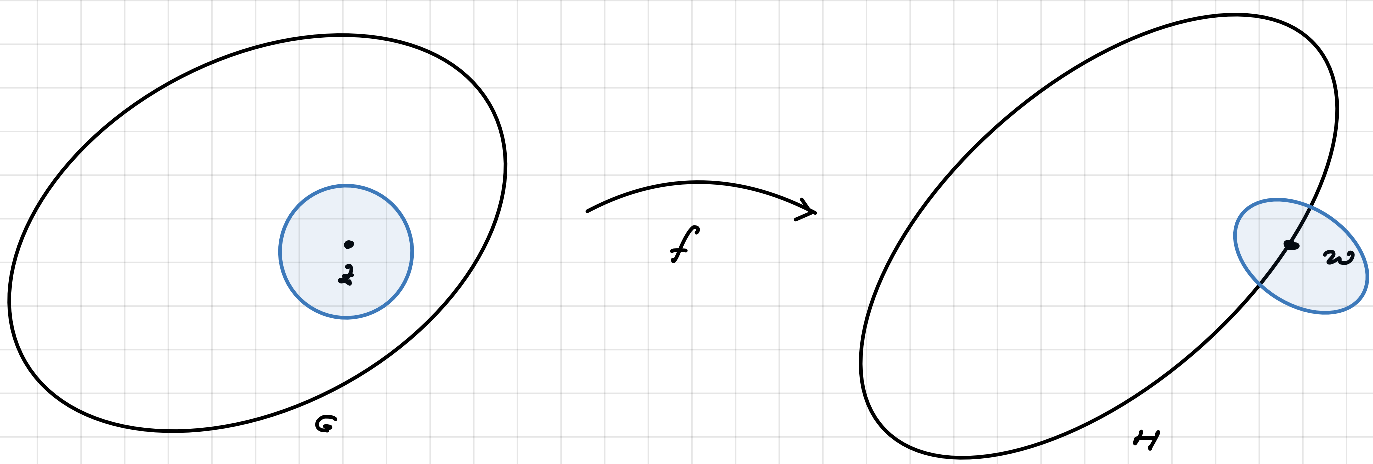
Proof of the Claim Since $w \in \partial H \Rightarrow \exists\, h_n \in H$, $h_n \to w$.

Write $h_n = f(g_n)$, $g_n \in G \subseteq \bar{G}$. Since $\bar{G}$ compact $\Rightarrow$ passing to a subsequence we may assume that $g_n \longrightarrow z \in \overline{G}$. $\Rightarrow$ $f(g_n) \longrightarrow f(z)$. Since

$f(g_n) = h_n \longrightarrow w$, we conclude

$$w = f(z)., \quad z \in \overline{G}$$

If $z \in G$, we contradict Open Mapping theorem.



(Pick a disc near z. its image will be open so it will contain a disc near w but $w \in \partial H$ contradiction).

Thus $z \in \partial G$ proving the Claim & Lemma A.

Strategy for Bloch

Apply Lemma A & show $|f(z) - f(a)| \geq \beta$ for suitable a.

Question: Why is the proof difficult?

Answer: We don't know a. In other words, we don't know where the center of the disc in Bloch should be.

More detailed strategy

$\boxed{\text{I}}$ prove Bloch under Assumption (*)

$\boxed{\text{II}}$ remove Assumption (*)

In Step $\boxed{\text{I}}$ we have control of the center & the radius equals 2β (better than Bloch claims).

In Step $\boxed{\text{II}}$ we lose control of center, radius halves, but we have no assumptions.

Assumption (*)

$f \in \mathcal{O}(\bar{\Delta})$, $|f'(z)| \leq 2|f'(0)|$ $\forall$ $|z| \leq 1$

Lemma B (Bloch assuming (*))

If f satisfies Assumption (*) $\Rightarrow$ $\operatorname{Im} f$ contains a disc with center $f(0)$ & radius $2\beta|f'(0)|$

Remark If $f'(0) = 1$, this implies under Assumption (*)

Lemma C (Bloch without (*)).

For all $f \in \mathcal{O}(\bar{\Delta})$, even in the absence of Assumption (*), $\operatorname{Im} f$ contains a disc of radius $\beta|f'(0)|$.

Note Lemma C $\Rightarrow$ Bloch.

Next time we show Lemma B $\Rightarrow$ Lemma C.