Math 220A - Lecture 3

October 9, 2020

Logistics

- 13 votes for MWF 3–3:50
- 5 votes for WF 3–4:15
- 4 votes indifferent

New time : MWF 3–3:50

No lecture : Monday, Oct 26.

Today : Loose ends

[i] power series

[ii] logarithm

[iii] Mobius transformations

Conway $\overline{\underline{III}}$.

I Loose ends from last time

ANALYTIC $\Rightarrow$ HOLOMORPHIC

Theorem Assume that $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence R. Then $\sum_{k=1}^{\infty} k a_k z^{k-1}$ has radius of convergence R as well.

Furthermore, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ then

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}.$$

Proof Radius of convergence for 2nd power series

$$R^{-1} = \limsup_{k\to\infty} \sqrt[k]{k\,|a_k|} = \limsup_{k\to\infty} \sqrt[k]{|a_k|}. \quad \text{since } \sqrt[k]{k} \to 1.$$

Fix $\alpha \in \Delta(0,R)$. We show $f'(\alpha) = g(\alpha)$.

where $g = \sum_{k=1}^{\infty} k a_k z^{k-1}$.

$$\text{Let } S_N = \sum_{k=0}^{N} a_k z^k, \quad R_N = \sum_{k=N+1}^{\infty} a_k z^k.$$

Know $S_N \to f$, $S_N' \to g$.

Fix $\varepsilon > 0$. We wish to find $\delta_\varepsilon > 0$

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} - g(\alpha) \right| < \varepsilon \quad \text{if} \quad z \in \Delta(\alpha, \delta)$$

Let $|\alpha| < \rho < R$. For $z \in \Delta(0, \rho)$ we have

$$(*) = \left| \frac{f(z) - f(\alpha)}{z - \alpha} - g(\alpha) \right| \leq \overbrace{\left| \frac{S_N(z) - S_N(\alpha)}{z - \alpha} - S_N'(\alpha) \right|}^{I.}$$

$$+ \overbrace{\left| S_N'(\alpha) - g(\alpha) \right|}^{II}$$

$$+ \overbrace{\left| \frac{R_N(z) - R_N(\alpha)}{z - \alpha} \right|}^{III.} \overset{?}{<} \varepsilon.$$

We estimate each of these terms. Term $\underline{\overline{III}}$.

$$\left| \frac{R_N(z) - R_N(\alpha)}{z - \alpha.} \right| \leq \sum_{k=N+1}^{\infty} |a_k| \left| \frac{z^k - \alpha^k}{z - \alpha} \right|$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| \left(|z|^{k-1} + \dots + |\alpha|^{k-1} \right)$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| \, k \rho^{k-1} < \varepsilon/3$$

if $N \geq N_1$.

Term $\underline{\overline{II}}$: $\left| s_N'(\alpha) - g(\alpha) \right| < \varepsilon/3$ for $N \geq N_2$.

Fix $N \geq N_1$ & N_2. For this N, find δ such that

Term $\underline{\overline{I}}$: $\left| \frac{s_N(z) - s_N(\alpha)}{z - \alpha} - s_N'(\alpha) \right| < \varepsilon/3$ if $z \in \Delta(\alpha, \delta)$.

Then

$$(*) \leq \underline{\overline{I}} + \underline{\overline{II}} + \underline{\overline{III}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ for}$$

$z \in \Delta(\alpha, \delta) \cap \Delta(0, \rho)$. QED.

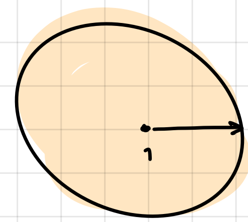
II. Logarithm

$U \subseteq \mathbb{C} \setminus \{0\}$ open & connected

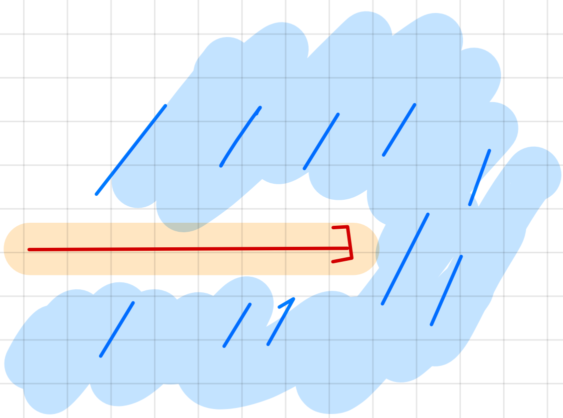
$\ell : U \longrightarrow \mathbb{C}$ continuous & $e^{\ell(z)} = z$.

Example A $\quad U = \Delta(1,1)$

$$\ell(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z-1)$$



Example B $\quad U = \mathbb{C}^{-} = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ slit plane



$z = r e^{i\theta}$

$\operatorname{Log} z = \log r + i\theta$

$\theta \in (-\pi, \pi) \implies e^{\operatorname{Log} z} = z.$

(Principal branch of logarithm).

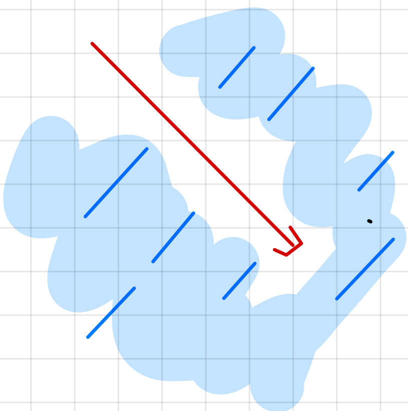
BEWARE $\quad \operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w$

This holds if $\operatorname{Re} z > 0$, $\operatorname{Re} w > 0$.

Example $\quad \mathrm{Log}(1-i) = \log\sqrt{2} + i\left(-\frac{\pi}{4}\right).$

principal branch

Example C $\quad$ Other branches



$$U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}\, e^{i\alpha}.$$

$$z = re^{i\theta}, \quad \theta \in (\alpha, \alpha + 2\pi).$$

$$\mathrm{Log}_\alpha z = \log r + i\theta.$$

Remark $\boxed{a}$ $\quad U = \mathbb{C} \setminus \{0\} \Rightarrow$ impossible to define logarithm

$\boxed{b}$ $\quad U \subseteq \mathbb{C} \setminus \{0\}$ simply connected

$\Rightarrow$ we can define logarithm (later).

Examples A – C are simply connected.

Remark $z^{\alpha} = \exp\left(\alpha\, \ell(z)\right)$ is multi-valued

– differ by $\exp(\alpha \cdot 2\pi i \cdot n)$.

Example Principal value of $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

$$(1-i)^{i} = \exp\left(i \cdot \operatorname{Log}(1-i)\right)$$

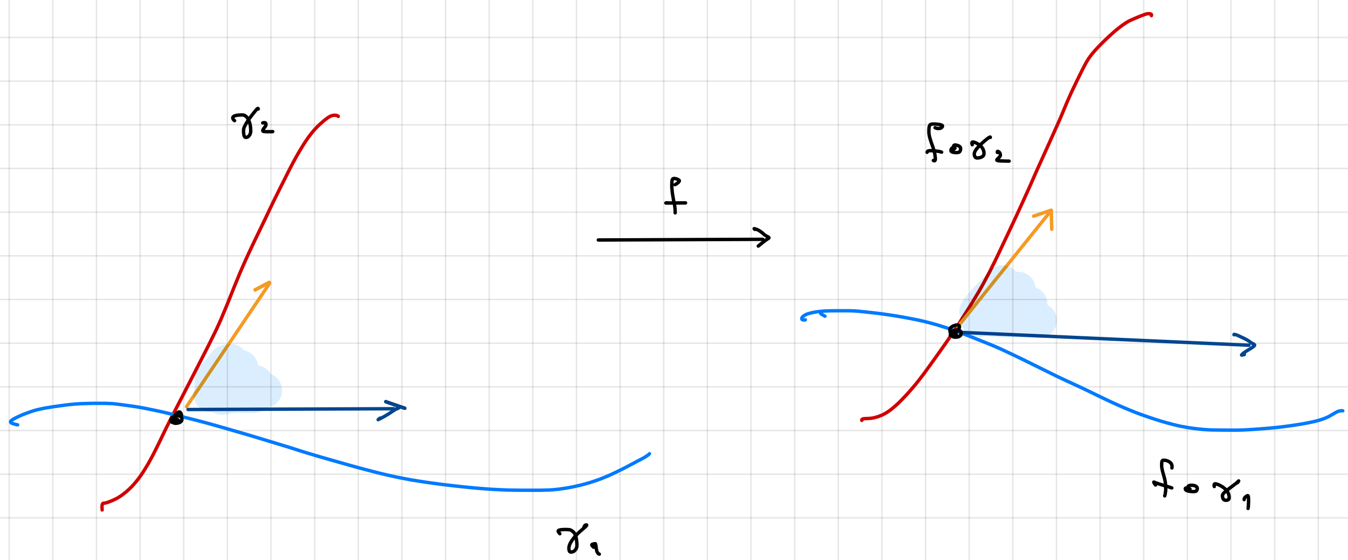
$$= \exp\left(i \cdot \left(\log\sqrt{2} - i\frac{\pi}{4}\right)\right)$$

$$= \exp\left(i \log\sqrt{2} + \frac{\pi}{4}\right)$$

$$= e^{\pi/4}\left(\cos\log\sqrt{2} + i \sin\log\sqrt{2}\right).$$

III. Geometry of holomorphic maps

We have seen holomorphic maps with $f'(z) \neq 0$ preserve angles.



Remark Given $U, V \subseteq \mathbb{C}$, a biholomorphic map $f: U \longrightarrow V$ is

[i] f bijective, holomorphic

[ii] $g = f^{-1}: V \longrightarrow U$ holomorphic.

If $f(p) = q \implies f \circ g\,(z) = z$

$$\implies g'(q) = \frac{1}{f'(p)}, \quad f'(p) \neq 0.$$

Important Question Given $U, V \subseteq \mathbb{C}$, are they biholomorphic?

$$U \underset{f^{-1}}{\overset{f}{\rightleftarrows}} V$$

Today we study a class of transformations which are important for geometric arguments.

Möbius transformations (MT)

Fractional linear transformations (FLT)

Linear fractional transformations (LFT)

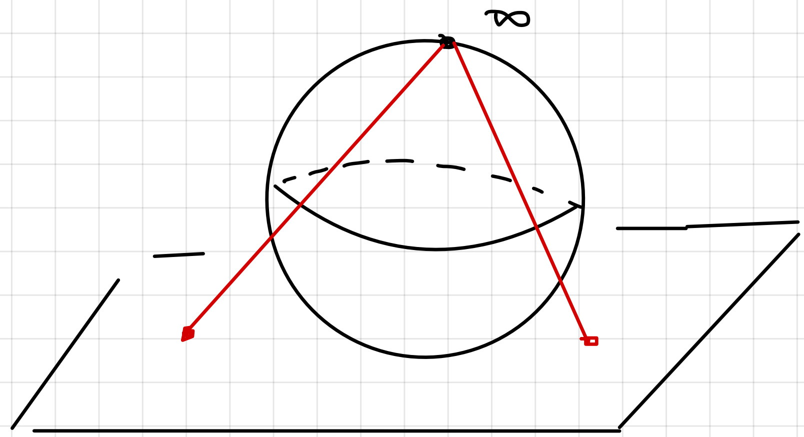
August Ferdinand Möbius (1790 – 1868)

Möbius strip, Möbius inversion, Möbius transform

Möbius published important work in astronomy.

Definition $\mathbb{C}_\infty = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Riemann sphere



Definition Möbius transformations MT.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies h_A : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}} \qquad \frac{1}{0} = \infty.$$

$A \in GL_2.$

$$\begin{cases} z \longrightarrow \dfrac{az+b}{cz+d} \\ \infty \longrightarrow \lim\limits_{z \to \infty} \dfrac{az+b}{cz+d} = \dfrac{a}{c}. \end{cases}$$

biholomorphism $h_A : \mathbb{C} \setminus \{-\frac{d}{c}\} \longrightarrow \mathbb{C} \setminus \{\frac{a}{c}\}.$

Remark

[i] $A = I \implies h_A = 1\!\!1.$

[ii] $A = \lambda B \iff h_A = h_B.$ for $\lambda \neq 0.$

[iii] $h_{AB} = h_A \circ h_B$ If $B = A^{-1}. \implies h_{A^{-1}} = h_A^{-1}.$

Most famous example Cayley transform

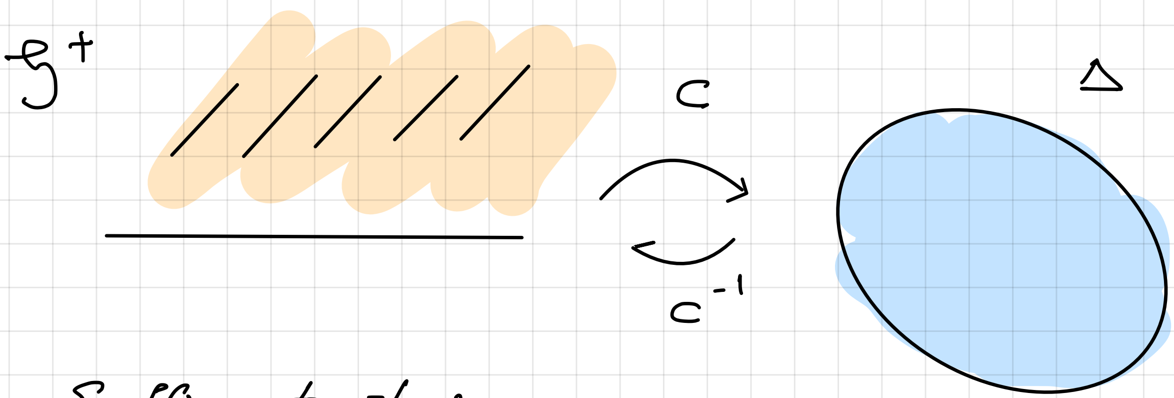
$$c(z) = \frac{z-i}{z+i}, \qquad c^{-1}(w) = i \cdot \frac{1-w}{1+w}.$$

Notation: $\Delta = \Delta(0,1)$

$$\mathfrak{H}^+ = \{ z : \operatorname{Im} z > 0 \}.$$

Claim c is a biholomorphism

$$c : \mathfrak{H}^+ \longrightarrow \Delta.$$



Suffices to show

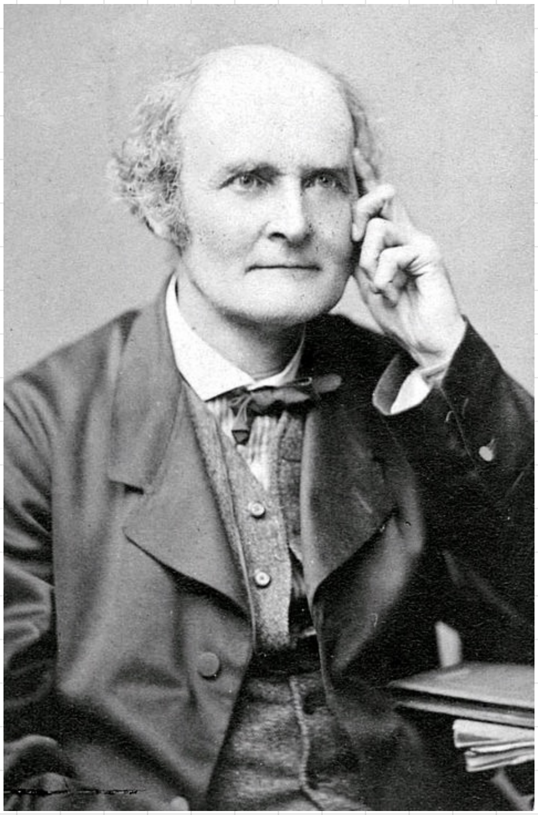
$$z \in \mathfrak{H}^+ \iff c(z) \in \Delta. \qquad \text{Write } z = x+iy$$

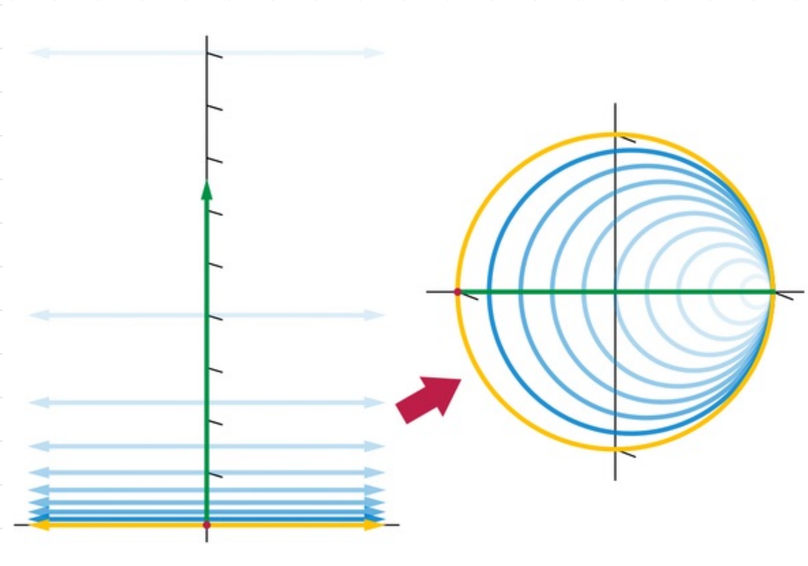
$$\Updownarrow \qquad\qquad \Updownarrow$$

$$y > 0 \qquad\qquad |z-i| < |z+i|$$

$$\Updownarrow$$

$$x^2 + (y-1)^2 < x^2 + (y+1)^2$$





Arthur Cayley (1821 - 1895)

- worked in algebraic geometry, Group theory
- Cayley - Hamilton theorem
- modern definition of a group

Remark

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2}\,\frac{1}{z+\frac{d}{c}} + \frac{a}{c}$$

$c=0$: $$\frac{az+b}{d} = \frac{a}{d}\cdot z + \frac{b}{d}.$$

Types of Mobius transforms

$\boxed{\text{i}}$ translation $Tz = z+\lambda$

$\boxed{\text{ii}}$ rotations $Rz = e^{i\theta}\cdot z$

$\boxed{\text{iii}}$ dilations $Dz = m\,z$, $m \in \mathbb{R}$.

$\boxed{\text{IV}}$ inversion $Sz = \frac{1}{z}$.

Lemma All Möbius transforms are compositions of

$\boxed{\text{I}}$ – $\boxed{\text{IV}}$.

Generalized circles in $\hat{\mathbb{C}}$

$\boxed{\mathrm{i}}$ circles in $\mathbb{C}$

$\boxed{\mathrm{ii}}$ line $L \cup \{\infty\}$ = circle in $\hat{\mathbb{C}}$ through ∞.

Main theorems about Möbius transforms

Theorem A Any Möbius transform maps generalized circles to generalized circles.

Theorem B Given two triples (z_1, z_2, z_3) and (z_1', z_2', z_3') of distinct points in $\hat{\mathbb{C}}$, $\exists!$ Möbius transformation h with

$$h(z_i) = z_i'.$$