# Math 220B — Lecture 15

February 8, 2021

# Midterm Exam

(1) 4 - 5 Questions

- Infinite Products, $\Gamma$ function, sine
- Weierstraß factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz lemma & applications

(2) Available on Friday at noon, due Tuesday at noon.

You can think about the Questions for as long as you wish in this interval.

# I. Schwarz Lemma – Conway VI. 2.

$\Delta = \Delta(0,1)$

**Theorem** Given $f: \Delta \longrightarrow \Delta$, $f(0) = 0$ then

[i] $|f'(0)| \leq 1$, and

[ii] $|f(z)| \leq |z|$

If either $|f'(0)| = 1$ or $\exists z_0 \neq 0$ with $|f(z_0)| = |z_0|$ then $f$ is a rotation i.e. $f(z) = e^{i\theta} z$

**Proof** Let $g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$. By the removable singularity theorem (Lecture 13, Math 220A), $g$ is holomorphic.

This uses $f(0) = 0$.

Let $0 < r < 1$. Then for $|w| = r$,

$$|g(w)| = \frac{|f(w)|}{|w|} \leq \frac{1}{r} \quad \text{since } \operatorname{Im} f \subseteq \Delta.$$

By maximum modulus principle,

$$\sup_{|w|\le r} |g(w)| = \sup_{|w|=r} |g(w)| \le \frac{1}{r}.$$

In particular, for all $|z| < r < 1$, we have

$$|g(z)| \le \frac{1}{r}$$

Make $r \to 1$ keeping $z$ fixed. Then $|g(z)| \le 1$. In particular

$|g(0)| = |f'(0)| \le 1$ & $|f(z)| \le |z|$.

If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for $z_0 \ne 0$ then either

$|g(0)| = 1$ or $|g(z_0)| = 1$. Since $|g(z)| \le 1$ $\forall z$ then $g$ must be constant by MMP again. Thus $g(z) = e^{i\theta}$. $\Rightarrow f(z) = e^{i\theta} z$.

**Corollary.** $f : \Delta \longrightarrow \Delta$ biholomorphism, $f(0) = 0$ then $f$ is a rotation.

**Proof** Note $f(0) = 0 \Rightarrow f^{-1}(0) = 0$. We apply Schwarz to both $f, f^{-1}$. We obtain

$$|f(z)| \leq z \quad \text{and} \quad |f^{-1}(w)| \leq w.$$

Let $w = f(z)$ to get $|z| \leq |f(z)|$. Therefore $|f(z)| = |z| \ \forall z \Rightarrow f$ rotation.

## II. Automorphisms of the unit disc

$\Delta = \Delta(0,1)$.

**Question** What can we do if we are given

$$f : \Delta \longrightarrow \Delta \quad \text{with} \quad f(0) = a \neq 0, \quad |a| < 1.$$

**Key Idea**

$$\exists \ \varphi_a : \Delta \longrightarrow \Delta \quad \text{with} \quad \varphi_a(a) = 0.$$

We can then recenter $f$ by considering $\tilde{f} = \varphi_a \circ f$.

$$\begin{tikzcd}
0 \arrow[r, bend left, "f"] \arrow[rr, bend right, "\varphi_a \circ f"'] & a \arrow[r, bend left, "\varphi_a"] & 0
\end{tikzcd}$$

Specifically

$$\boxed{\varphi_a(z) = \frac{z - a}{1 - \bar{a} z}}.$$

# Important Properties

[i] $\varphi_a : \Delta \longrightarrow \Delta$, $\varphi_a : \partial\Delta \longrightarrow \partial\Delta$

[ii] $\varphi_a(0) = -a$ , $\varphi_a(a) = 0$

[iii] $\varphi_a, \varphi_{-a}$ are inverses

[iv] $\varphi_a'(0) = \underbrace{1 - |a|^2}_{\text{shrinks } <1.}$, $\varphi_a'(a) = \underbrace{\dfrac{1}{1-|a|^2}}_{\text{expands } >1}$.

**Proof** [ii] – [iv] follow by direct calculation.

[i] Note that $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$ has pole at $\dfrac{1}{\bar{a}}$ but this is not in $\bar{\Delta}$ since $|a|<1$. Thus $\varphi_a$ is holomorphic in $\Delta$, continuous in $\bar{\Delta}$. If we show

(*) $|\varphi_a(z)| = 1$ if $|z| = 1$, by the maximum modulus, it follows $|\varphi_a(z)| < 1$ $\forall$ $|z|<1$ so $\varphi_a : \Delta \longrightarrow \Delta$.

To see (*) we show $|z-a| = |1-\bar{a}z|$ if $|z|=1$.

Note $|1 - \bar{a} z| = |1 - a\bar{z}|$ conjugation

$$= \left|1 - \frac{a}{z}\right| \quad \text{since } z\bar{z} = |z|^2 = 1$$

$$= \frac{|z-a|}{|z|} = |z-a| \quad \text{as needed.}$$

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**Theorem** If $f : \Delta \longrightarrow \Delta$ is biholomorphic then

$$f(z) = e^{i\theta} \cdot \frac{z - a}{1 - \bar{a} z} \quad \text{for } |a| < 1.$$

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**Proof**

$$\begin{tikzcd} 0 \arrow[r, bend left, "\varphi_{-a}"] \arrow[rr, bend right, "f \circ \varphi_{-a}"'] & a \arrow[r, bend left, "f"] & 0 \end{tikzcd}$$

Let $a$ be such that $f(a) = 0$. Let

$$\tilde{f} = f \circ \varphi_{-a} \Rightarrow \tilde{f}(0) = 0.$$

Note $\tilde{f}$ is a biholomorphism. Then $\tilde{f}$ is a rotation

$$\Rightarrow \tilde{f}(w) = e^{i\theta} w \Rightarrow f \circ \varphi_{-a}(w) = e^{i\theta} w \Rightarrow f(z) = e^{i\theta} \varphi_a(z).$$

setting $w = \varphi_a(z)$.

**Remark** We have seen $\varphi_a$'s in HWK 1.

**Blaschke's products**

$$\varphi : \Delta \longrightarrow \Delta, \quad \partial\Delta \longrightarrow \partial\Delta \quad \text{then}$$

$$f(z) = c\, z^m \prod_{k=1}^{N} \varphi_{a_k}, \quad |c| = 1.$$

$a_k$ → zeroes of $f$.

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**Exercise** Assume $f: \Delta \longrightarrow \Delta$, $\partial\Delta \longrightarrow \partial\Delta$

whose only zeroes are at $1/2$ & $1/4$ with multiplicities 2 & 3

Find $|f(0)|$.

**Solution** $f(z) = c\, \varphi_{\frac{1}{2}}^{2}\, \varphi_{\frac{1}{4}}^{3}$. Then

$$f(0) = c \cdot \left(-\frac{1}{2}\right)^2 \cdot \left(-\frac{1}{4}\right)^3, \quad |c| = 1 \implies |f(0)| = \frac{1}{2^8}.$$

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## III. Understanding the action of $\mathrm{Aut}(\Delta)$ on $\Delta$

**Important Remark** The action of $\mathrm{Aut}(\Delta)$ on $\Delta$ is transitive $\forall\, a, b \in \Delta \;\; \exists\, f \in \mathrm{Aut}\,\Delta,\; f(a) = b.$

$$\begin{tikzcd} a \arrow[r, bend left, "\varphi_a"] \arrow[rr, bend right, "f"'] & 0 \arrow[r, bend left, "\varphi_{-b}"] & b \end{tikzcd}$$

Note $f = \varphi_{-b} \circ \varphi_a$ is an automorphism and

$$f(a) = \varphi_{-b}\, \varphi_a(a) = \varphi_{-b}(0) = b.$$

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### Application - Fixed points

Show if $f : \Delta \longrightarrow \Delta$ holomorphic, $f \neq \mathbb{1} \Rightarrow f$ has at most 1 fixed point.

**Proof** Assume $f(a) = a$ & $f(b) = b$. & $a \neq b$.

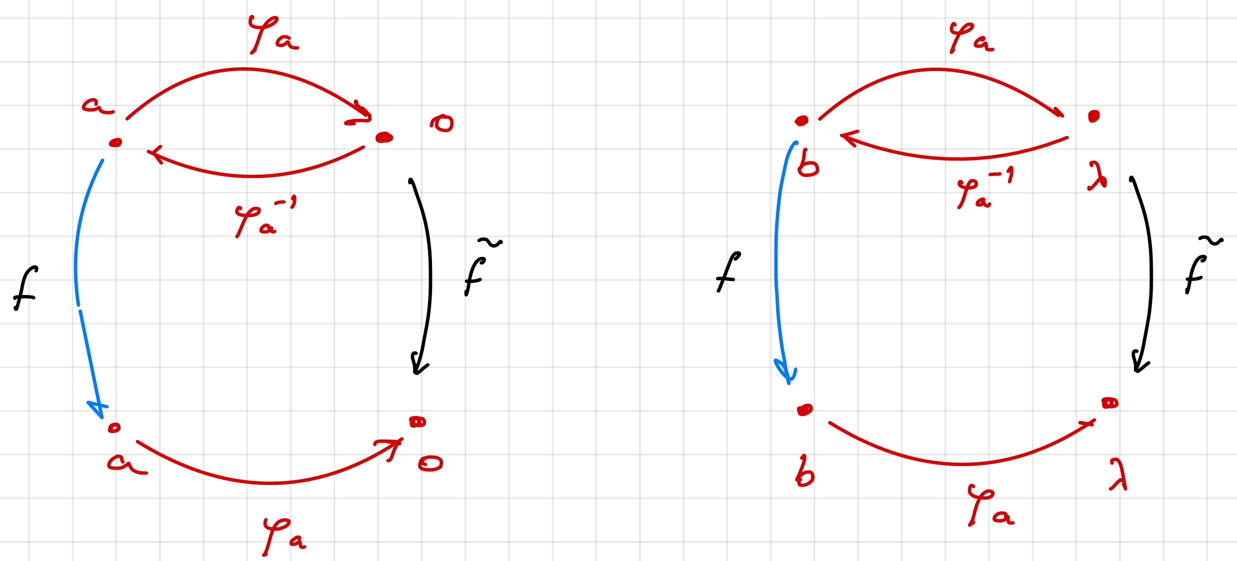
If $a = 0$ then $f(0) = 0$ & $f(b) = b$ $\Rightarrow$ $f$ rotation via

Schwarz $f(z) = e^{i\theta} z$. Using $f(b) = b$ $\Rightarrow$ $e^{i\theta} = 1$ $\Rightarrow$

$\Rightarrow$ $f = \mathbb{1}$ which is disallowed.

For $a \neq 0$, we reduce to this case. Let

$$\tilde{f} = \varphi_a f \varphi_a^{-1} \quad \& \quad \lambda = \varphi_a(b) \neq 0 = \varphi_a(a).$$




Then $\tilde{f}(0) = 0$ and $\tilde{f}(\lambda) = \lambda$ $\Rightarrow$ $\tilde{f} = \mathbb{1}$ $\Rightarrow$

$\Rightarrow$ $\varphi_a f \varphi_a^{-1} = \mathbb{1}$ $\Rightarrow$ $f = \mathbb{1}$, again a contradiction.

Thus $f$ has at most one fixed point.

## Recap

- if $f(0) = 0$ then
  - we proved Schwarz Lemma
  - we determined $f \in \mathrm{Aut}\,\Delta$, $f(0) = 0$
- if $f(0) \neq 0$
  - we determined $f \in \mathrm{Aut}\,\Delta$

**Question** Is there a version of Schwarz if $f(0) \neq 0$?

Yes — Schwarz-Pick Lemma.

- we illustrate it for derivatives

**Proposition** $f : \Delta \longrightarrow \Delta$ holomorphic, $\forall a \in \Delta$

$$\frac{|f'(a)|}{1-|f(a)|^2} \leq \frac{1}{1-|a|^2}.$$

**Remark** If $a = 0$ this gives $|f'(0)| \leq 1 - |f(0)|^2$.

If $f(0) = 0$ this gives $|f'(0)| \leq 1$. Thus the Proposition

generalizes Schwarz Lemma.

The proof will be given next time.

**Remark** This is naturally formulated in hyperbolic

geometry.