EXAM **MATH 294**

Monday, February 28, 2022 **Prof. R. Williams**

No books or other notes may be used. Make sure to justify your answers, i.e., explain your reasoning and show your work. You may use theorems proved in class. If you use a theorem, please say which theorem you are using. Remember that part of each problem is to set it up and to arrive at the answer by a progression of logical steps. Please start each problem on a new page, write legibly, and put your name on the first page of your solutions.

There are 4 problems for a total of 100 points.

1. (35 pts) Consider a binomial model with $t = 0, 1, 2$. Assume that the initial stock price $S_0 = \$20$ and the two possible values for the stock price at time one are \$60 and \$10. Suppose that the risk free interest rate is $r = \frac{1}{2}$. Consider an associated American put option with a strike price of $K = \$25$.

(a) Determine the parameters u and d associated with the binomial model. Determine the risk neutral probability p^*.

(b) Draw a binary tree to illustrate all of the possible paths followed by the stock price.

(c) Find the (random) payoffs Y_t at each time $t = 0, 1, 2$ for the American put option and indicate these values in your tree picture.

(d) For $t = 0, 1, 2$, let U_t denote the minimum amount of wealth needed by the seller at time t in order to cover the possibility that the buyer might cash in the American put at time t or later. Find the values of the random variables U_t for each t and indicate their values in your tree picture.

(e) What is the arbitrage free initial price for the American put option?

(f) What is an optimal stopping time for the buyer of the American put?

(g) Let $\{\mathcal{F}_t, t = 0, 1, 2\}$ denote the filtration generated by the stock price process. For the above example, is $\{U_t, \mathcal{F}_t, t = 0, 1, 2\}$ a martingale under the risk neutral probability? Give reasoning to support your answer.

2. (40 pts) Consider a finite market model with $t = 0, 1, 2$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $P(\{\omega_i\}) > 0$ for $i = 1, 2, 3$. Suppose there are two assets, a riskless asset with price process $S^0 = \{S^0_t, t = 0, 1, 2\}$ where $S^0_t = 1$ for $t = 0, 1, 2$ and a risky asset with price process $S^1 = \{S^1_t, t = 0, 1, 2\}$ where

$$\begin{aligned}
S^1_0(\omega_1) &= 5, \quad S^1_1(\omega_1) = 8, \quad S^1_2(\omega_1) = 8 \\
S^1_0(\omega_2) &= 5, \quad S^1_1(\omega_2) = 4, \quad S^1_2(\omega_2) = 5 \\
S^1_0(\omega_3) &= 5, \quad S^1_1(\omega_3) = 4, \quad S^1_2(\omega_3) = 3.
\end{aligned}$$

Let $\{\mathcal{F}_t, t = 0, 1, 2\}$ be the filtration generated by S^1.

(a) Draw a tree to indicate the possible "paths" followed by the risky asset price process S^1.

(b) For each of $t = 0, 1, 2$, describe the partition $\mathcal{P}_t$ of Ω that generates $\mathcal{F}_t$.

(c) Find all of the equivalent martingale measures for this model.

(d) Does this model have any arbitrage opportunities? Give a reason to support your answer.

(e) Is this model complete? Give a reason to support your answer.

(f) Describe the geometric relationship between

$$L = \{G^*_2(\phi) : \phi \text{ is a self-financing trading strategy and } V_0(\phi) = 0\}$$

and the set found in (d), when you view the elements of the set in (d) as probability vectors. (You do not need to find a concrete expression for L.)

3. (15 pts) Consider a single period finite market model with no arbitrage opportunities for which there is a European contingent claim X_1 that is replicable and one European contingent claim X_2 that is NOT replicable. For each of the following questions, your answer should be in the range: $0, 1, 2, ..., \infty$, where ∞ means infinitely many. Make sure to give a reason for each of your answers.

(a) How many equivalent martingale measures (EMMs) are there?

(b) How many arbitrage free initial prices are there for X_1?

(c) How many arbitrage free initial prices are there for X_2?

4. (10 pts) Fix a finite continuous time interval $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, P)$.

(a) Give the definition of a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$, defined over the time interval $[0, T]$.

(b) Given a filtration $\{\mathcal{F}_t, t \in [0, T]\}$ of sub-sigma-algebras of $\mathcal{F}$, give the definition of a local martingale relative to this filtration.