Math 220B - Winter 2021 - Final Exam

Problem 1.

Show that

$$f(z) = \prod_{n=1}^{\infty} \left(1 + n^2 z^n\right)$$

defines a holomorphic function on the unit disc $\Delta(0,1)$.

`Solution:` *By a theorem proved in class, it suffices to show that the series*

$$\sum_{n=1}^{\infty} n^2 z^n$$

converges absolutely and locally uniformly on $\Delta(0,1)$. Let K be a compact set in Δ. We show uniform convergence in K. Let $r < 1$ such that $K \subset \overline{\Delta}(0,r)$. Let $M_n = n^2 r^n$. We have

$$|n^2 z^n| \leq M_n$$

for $z \in K$. We also have

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} n^2 r^n < \infty$$

for all $r < 1$, since the radius of convergence R of the series $\sum_{n=1}^{\infty} n^2 r^n$ is given by

$$R^{-1} = \limsup_{n \to \infty} (n^2)^{\frac{1}{n}} = 1.$$

The proof is completed invoking the Weierstraß M-test.

Problem 2.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function which takes real values on the real line $f(\mathbb{R}) \subset \mathbb{R}$. Show that

$$\overline{f(z)} = f(\bar{z}).$$

`Solution:` *Let $f^* : \mathbb{C} \to \mathbb{C}$ be defined as*

$$f^*(z) = \begin{cases} f(z) & \textit{if } z \in \mathfrak{h}^+ \\ f(z) & \textit{if } z \in \mathbb{R} \\ \overline{f(\overline{z})} & \textit{if } z \in \mathfrak{h}^-. \end{cases}$$

The function f^* is holomorphic by the Schwarz reflection principle, hence f^* is entire. Since

$$f = f^*$$

on $\mathfrak{h}^+$ by definition, it follows that $f = f^*$ on $\mathbb{C}$ by the identity principle. Thus for all $z \in \mathfrak{h}^-$ we have

$$f(z) = \overline{f(\overline{z})} \iff \overline{f(z)} = f(\bar{z}).$$

The same identity is also true for $z \in \mathfrak{h}^+$, which can be seen by substituting $\mathfrak{h}^+ \ni z \mapsto \bar{z} \in \mathfrak{h}^-$. For $z \in \mathbb{R}$, we have $f(z) \in \mathbb{R}$ so we also obtain

$$\overline{f(z)} = f(\bar{z}).$$

Problem 3.

Find a biholomorphism between the strip $S = \{z = x + iy : -\pi < y < \pi\}$ and the first quadrant $Q = \{z = x + iy : x > 0, y > 0\}$.

`Solution:` *We find the answer as a composition of three transformations:*

(i) $f_1 : S \to \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $z \mapsto \exp(z)$. *This is biholomorphic with inverse given by the principal branch of the logarithm.*

(ii) $f_2 : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, $z \mapsto -z$. *This is a biholomorphism with inverse given by* $z \mapsto -z$.

(iii) $f_3 : \mathbb{C} \setminus \mathbb{R}_{\geq 0} \to Q$, $z \mapsto z^{1/4}$ *where for* $z = r\exp(i\theta)$ *for* $0 < \theta < 2\pi$, *we set*

$$z^{\frac{1}{4}} = r^{\frac{1}{4}} \exp\left(i\frac{\theta}{4}\right).$$

This has inverse $z \mapsto z^4$.

The composition

$$f = f_3 \circ f_2 \circ f_1 : S \to Q$$

is the biholomorphism we seek. We have

$$f(z) = (-\exp(z))^{\frac{1}{4}} = \exp\left(\frac{\pi i}{4} + \frac{z}{4}\right).$$

Problem 4.

Let $U \subset \mathbb{C}$ be a simply connected open set. Let $\{a_n\}_{n\geq 1}$ be a sequence U without limit points in U. Let $\{m_n\}_{n\geq 1}$ be a sequence of positive integers.

Let g be a meromorphic function in U with simple poles only at a_n, and with residues equal to m_n at a_n.

Show that there exists a holomorphic function $h : U \to \mathbb{C}$ such that $h'/h = g$.

(i) For $f : U \to \mathbb{C}$ holomorphic, with a zero of order m at a, show that f'/f has a simple pole at a with residue equal to m.

(ii) Using (i), show that there exists $f : U \to \mathbb{C}$ holomorphic such that $f'/f - g$ is holomorphic in U.

(iii) Using (ii), show that there exists $h : U \to \mathbb{C}$ holomorphic such that $h'/h = g$.

`Solution:`

(i) *Indeed, if the order of f at a equals m, we can write*

$$f(z) = (z-a)^m f_1(z), \quad g(a) \neq 0$$

in a neighborhood of a. We have

$$\frac{f'}{f} = \frac{m}{z-a} + \frac{f_1'}{f_1}.$$

Since $f_1(a) \neq 0$, it follows that $f_1 \neq 0$ in some neighborhood of a, hence the function f_1'/f_1 is holomorphic near a. This shows that f'/f is meromorphic at a and the residue equals m.

(ii) *We consider the solution $f : U \to \mathbb{C}$ to the Weierstraß problem in U, such that f has zeroes at a_n of order exactly m_n. By (i), we have*

$$Res\left(\frac{f'}{f}, a_n\right) = Ord(f, a_n) = m_n.$$

The function $f'/f - g$ could potentially have simple poles at a_n, but the residues equal 0 at a_n. Thus

$$f'/f - g$$

can be extended to a holomorphic function in U.

(iii) *Let F be a primitive of $f'/f - g$ which exists since U is simply connected. Set*

$$h = fe^{-F}.$$

Then

$$h'/h = f'/f - F' = g.$$

This completes the proof.

Problem 5.

Let U be a simply connected proper subset of $\mathbb{C}$. Let $a \in U$. Let $f : U \to U$ be holomorphic, such that

$$f(a) = a \text{ and } |f'(a)| = 1.$$

Show that f is a biholomorphism of U.

`Solution:` *Since $U \neq \mathbb{C}$, we have that U is biholomorphic to the unit disc Δ by the Riemann mapping theorem. Let $\phi : \Delta \to U$ be the biholomorphism, and let $b \in \Delta$ be such that $\phi(b) = a$. Using the automorphism ϕ_{-b} of Δ, we set*

$$\psi = \phi \circ \phi_{-b}.$$

Thus

$$\psi : \Delta \to U \text{ is a biholomorphism}$$

and furthermore

$$\psi(0) = \phi\phi_{-b}(0) = \phi(b) = a.$$

Set

$$F = \psi^{-1} \circ f \circ \psi : \Delta \to \Delta.$$

We show that F is a rotation. This will imply F is biholomorphism, and the same will be true about

$$f = \psi \circ F \circ \psi^{-1}.$$

To this end, note

$$F(0) = \psi^{-1} f \psi(0) = \psi^{-1} f(a) = \psi^{-1}(a) = 0.$$

Furthermore,

$$\psi \circ F = f \circ \psi.$$

Differentiating at 0 we obtain

$$\psi'(F(0)) \cdot F'(0) = f'(\psi(0)) \cdot \psi'(0) \implies \psi'(0) \cdot F'(0) = f'(a) \cdot \psi'(0).$$

Since ψ is a biholomorphism, we have $\psi'(0) \neq 0$, so

$$F'(0) = f'(a) \implies |F'(0)| = |f'(a)| = 1.$$

By Schwarz Lemma, F must be a rotation hence a biholomorphism.

Problem 6.

Let $\mathcal{F}$ be a normal family of holomorphic functions in Δ. Show that the family

$$\mathcal{G} = \{f : \Delta \to \mathbb{C} \text{ holomorphic }, \ f(0) = 0, \ f' \in \mathcal{F}\}$$

is also normal.

`Solution:` *Let $\{g_n\}$ be a sequence in $\mathcal{G}$. By definition,*

$$g_n' \in \mathcal{F} \text{ and } g_n(0) = 0.$$

Since $\mathcal{F}$ is normal, there exists a subsequence of g_n' which converges locally uniformly to a holomorphic function f. By relabelling, we may thus assume

$$g_n' \overset{l.u.}{\rightrightarrows} f.$$

Since $\mathbb{C}$ is simply connected, f admits a primitive g so that $g' = f$. We may assume $g(0) = 0$ since else we could replace g by $g - g(0)$. We claim

$$g_n \overset{l.u.}{\rightrightarrows} g.$$

Let $K \subset \Delta$ be compact. In particular, $K \subset \overline{\Delta}(0, R)$ for some $R > 0$. We may thus take $K = \overline{\Delta}(0, R)$ and establish uniform convergence on K.

Let $\epsilon > 0$. Since $g_n' \rightrightarrows f$ on $\overline{\Delta}(0, R)$, we can find N such that for all $n \geq N$ we have

$$|g_n'(w) - f(w)| < \epsilon/R$$

for all $w \in \overline{\Delta}(0, R)$. For $z \in \overline{\Delta}(0, R)$, we have

$$g_n(z) = g_n(0) + \int_{\gamma_z} g_n'(w)\, dw = \int_0^z g_n'(w)\, dw$$

where we integrate over the straight line segment from 0 to z. Similarly,

$$g(z) = g(0) + \int_0^z g'(w)\, dw = \int_0^z f(w)\, dw.$$

Then

$$|g_n(z) - g(z)| = \left| \int_0^z g_n'(w) - f(w)\, dw \right|.$$

For all w on the straight line segment from 0 to z, we have $w \in \overline{\Delta}(0, R)$, hence for all $n \geq N$ we have

$$|g_n'(w) - f(w)| < \epsilon/R.$$

We obtain

$$|g_n(z) - g(z)| = \left| \int_0^z g_n'(w) - f(w)\, dw \right| \leq \frac{\epsilon}{R} \cdot \textit{length of the path} < \frac{\epsilon}{R} \cdot |z| < \frac{\epsilon}{R} \cdot R = \epsilon.$$

This proves

$$g_n \rightrightarrows f$$

on $\overline{\Delta}(0, R)$ as needed.

Problem 7.

Let f, g be two entire functions. Let A, B be two disjoint nonempty compact sets. Assume $\mathbb{C} \setminus (A \cup B)$ is connected. Show that there exists a polynomial p such that

$$|p(z) - f(z)| < \frac{1}{1000} \text{ for } z \in A$$

and

$$|p(z) - g(z)| > 1000 \text{ for } z \in B.$$

`Solution:` *We show first that there exist U, V open and disjoint, such that $A \subset U$ and $B \subset V$. Let $d = d(A, B) > 0$. We let*

$$U =: \bigcup_{a \in A} \Delta(a, d/3), \quad V := \bigcup_{b \in B} \Delta(b, d/3).$$

Clearly, U, V are open and note that

$$A \subset U, \quad B \subset V.$$

Furthermore, if $w \in U \cap V$, we can find $a \in A, b \in B$ such that

$$w \in \Delta(a, d/3), \quad w \in \Delta(b, d/3) \implies d(a, w) < d/3, \quad d(b, w) < d/3.$$

Thus

$$d(a, b) \leq d(a, w) + d(b, w) < d/3 + d/3 = 2d/3.$$

This contradicts the fact that $d(A, B) = d$ so $d(a, b) \geq d$ for all $a \in A, b \in B$.

Let $\alpha = 1000 + \frac{1}{1000}$. Consider the function

$$h(z) = \begin{cases} f(z) & \text{if } z \in A \\ g(z) + \alpha & \text{if } z \in B. \end{cases}$$

The function h is holomorphic in the set $K = A \cup B$ which is compact. Indeed, h extends holomorphically to the open set $W = U \cup V$ via

$$H(z) = \begin{cases} f(z) & \text{if } z \in U \\ g(z) + \alpha & \text{if } z \in V. \end{cases}$$

Since $\mathbb{C} \setminus K$ is connected, by Little Runge's theorem, we obtain that there exists a polynomial p with

$$|h(z) - p(z)| < \frac{1}{1000}$$

for $z \in K$. In particular, for $z \in A$, we obtain

$$|f(z) - p(z)| < \frac{1}{1000},$$

while for $z \in B$, we obtain

$$|g(z) + \alpha - p(z)| < \frac{1}{1000}.$$

Using the triangle inequality,

$$|g(z) - p(z)| \geq \alpha - |g(z) + \alpha - p(z)| > \alpha - \frac{1}{1000} = 1000,$$

as needed.