Math 220B - Winter 2021 - Final Exam

Name: ______________________________

Student ID: ______________________________

Instructions:

There are 7 questions which are worth 60 points.

You may not use any books, notes, internet or collaborate with anybody on this exam. If you use a homework problem you will need to reprove it.

Please upload your answers in Gradescope. There is a 15 minute buffer period to upload the answers in Gradescope. You may not work on the exam during the buffer period.

Question	Score	Maximum
1		5
2		7
3		8
4		10
5		10
6		10
7		10
Total		60

Problem 1. [*5 points.*]

Show that

$$f(z) = \prod_{n=1}^{\infty} \left(1 + n^2 z^n\right)$$

defines a holomorphic function in the unit disc $\Delta(0,1)$.

Problem 2. [*7 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function which takes real values on the real line $f(\mathbb{R}) \subset \mathbb{R}$. Show that

$$\overline{f(z)} = f(\bar{z}).$$

Problem 3. [*8 points.*]

Find a biholomorphism between the strip $S = \{z = x + iy : -\pi < y < \pi\}$ and the first quadrant $Q = \{z = x + iy : x > 0, y > 0\}$.

Problem 4. [*10 points; 3, 3, 4.*]

Let $U \subset \mathbb{C}$ be a simply connected open set. Let $\{a_n\}_{n\geq 1}$ be a sequence in U without limit points in U. Let $\{m_n\}_{n\geq 1}$ be a sequence of positive integers.

Let g be a meromorphic function in U with simple poles only at a_n, and with residues equal to m_n at a_n.

Show that there exists a holomorphic function $h : U \to \mathbb{C}$ such that $h'/h = g$ on $U \setminus \{a_1, a_2, \ldots\}$.

(i) For $f : U \to \mathbb{C}$ holomorphic, with a zero of order m at $a \in U$, show that f'/f has a simple pole at a with residue equal to m.

(ii) Show that there exists $f : U \to \mathbb{C}$ holomorphic such that $f'/f - g$ is holomorphic in U.

(iii) Show that there exists $h : U \to \mathbb{C}$ holomorphic such that $h'/h = g$.

Problem 5. [*10 points.*]

Let U be a simply connected proper subset of $\mathbb{C}$. Let $a \in U$. Let $f : U \to U$ be holomorphic, such that

$$f(a) = a, \quad |f'(a)| = 1.$$

Show that f is a biholomorphism of U.

Problem 6. [*10 points.*]

Let $\mathcal{F}$ be a normal family of holomorphic functions in the unit disc Δ. Show that the family

$$\mathcal{G} = \{f : \Delta \to \mathbb{C} \text{ holomorphic },\ f(0) = 0,\ f' \in \mathcal{F}\}$$

is also normal.

Problem 7. [*10 points.*]

Let f, g be two entire functions. Let A, B be two disjoint nonempty compact sets. Assume $\mathbb{C} \setminus (A \cup B)$ is connected. Show that there exists a polynomial p such that

$$|p(z) - f(z)| < \frac{1}{1000} \text{ for } z \in A$$

and

$$|p(z) - g(z)| > 1000 \text{ for } z \in B.$$