Math 280A Fall '21

Lecture 2

Sections 1.6 - 1.8
of the text

Generators: As at end of last class

$\mathcal{C} \subset \mathcal{P}(\Omega)$.

Define $\mathbb{F}(\mathcal{C}) := \{ \mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-field}, \mathcal{F} \supset \mathcal{C} \}$

$$\sigma(\mathcal{C}) := \bigcap \{ \mathcal{B} : \mathcal{B} \in \mathbb{F}(\mathcal{C}) \}$$

(the σ-field generated by $\mathcal{C}$)

Theorem

(a) $\sigma(\mathcal{C})$ is a σ-field and $\sigma(\mathcal{C}) \supset \mathcal{C}$

(b) If $\mathcal{B}$ is a σ-field with $\mathcal{C} \subset \mathcal{B}$
then $\sigma(\mathcal{C}) \subset \mathcal{B}$

($\sigma(\mathcal{C})$ is the minimal (in the sense of $\subset$) σ-field containing $\mathcal{C}$)

Silly example: $\mathcal{C} = \emptyset \implies \sigma(\mathcal{C}) = \{\emptyset, \Omega\}$ 2

Less silly ex. : $\mathcal{C} = \{A\}$ where $A \subset \Omega$.

$$\sigma(\mathcal{C}) = \{\emptyset, A, A^c, \Omega\}$$

Proof. (a) Suppose $A_1, A_2, \cdots$ are elements of $\sigma(\mathcal{C})$.

Fix $\mathcal{F} \in \mathbb{F}(\mathcal{C})$. Then $A_n \in \mathcal{F}$, $\forall n$.

So $\bigcup_n A_n \in \mathcal{F}$ because $\mathcal{F}$ is a σ-field.

As $\mathcal{F} \in \mathbb{F}(\mathcal{C})$ was arbitrary, this means

that $\bigcup_n A_n \in \bigcap \{\mathcal{F} : \mathcal{F} \in \mathbb{F}(\mathcal{C})\}$

i.e. $\bigcup_n A_n \in \sigma(\mathcal{C})$

Similar arguments show $\sigma(\mathcal{C})$ contains $\emptyset$ and $\sigma(\mathcal{C})$ is closed under complementation.

(b) If $\mathcal{B}$ is a σ-field with $\mathcal{C} \subset \mathcal{B}$, then $\mathcal{B} \in \mathbb{F}(\mathcal{C})$.

$$\therefore \mathcal{B} \supset \bigcap \{ \mathcal{F} : \mathcal{F} \in \mathbb{F}(\mathcal{C}) \} = \sigma(\mathcal{C})$$

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4

Ex. Borel subsets of $\mathbb{R}$

$$
\begin{aligned}
\mathcal{B}(\mathbb{R}) &:= \sigma(\text{ open subsets of } \mathbb{R}) \\
&\overset{①}{=} \sigma(\{(a,b) : a<b,\ a\in\mathbb{R},\ b\in\mathbb{R}\}) \\
&\overset{②}{=} \sigma(\{(a,b) : a<b,\ a\in\mathbb{Q},\ b\in\mathbb{Q}\}) \\
&= \sigma(\{(-\infty,b) : b\in\mathbb{R}\}) \\
&= \sigma(\{(-\infty,b] : b\in\mathbb{R}\}) \\
&\ \vdots \\
&= \sigma(\text{ closed subsets of } \mathbb{R})
\end{aligned}
$$

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For example: 1 because any open subset of $\mathbb{R}$ can be written as a countable union of open intervals

2 If $a<b$ are real, choose rationals a_n, b_n with $a < a_n < b_n < b$ and $a_n \downarrow a$, $b_n \uparrow b$.

Then $(a,b) = \bigcup_n (a_n, b_n)$

Trace σ-field

Context : $(\Omega, \mathcal{B})$ some measurable space

$$\Omega_0 \subset \Omega \qquad \text{(non-void)}$$

$$\mathcal{B}_0 := \mathcal{B} \cap \Omega_0 := \{ B \cap \Omega_0 : B \in \mathcal{B} \}$$

Check : $\mathcal{B}_0$ is a σ-field of subsets of Ω_0.

Theorem (trace vs. generators)

$$\sigma(\mathcal{C} \cap \Omega_0) = \sigma(\mathcal{C}) \cap \Omega_0$$

Proof. (i) $\mathcal{C} \subset \sigma(\mathcal{C})$ (by definition)

$$\therefore \mathcal{C} \cap \Omega_0 \subset \underbrace{\sigma(\mathcal{C}) \cap \Omega_0}_{\text{a } \sigma\text{-field on } \Omega_0}$$

$$\therefore \sigma(\mathcal{C} \cap \Omega_0) \subset \sigma(\mathcal{C}) \cap \Omega_0$$

(minimality of $\sigma(\mathcal{C} \cap \Omega_0)$)

8

(ii) $\supset$ needs a new technique:
"good sets principle"

Define: $\mathcal{G} := \{ A \subset \Omega : A \cap \Omega_0 \in \sigma(\mathcal{C} \cap \Omega_0) \}$

- $\mathcal{C} \subset \mathcal{G}$ because if $B \in \mathcal{C}$ then $B \cap \Omega_0$ is an element of $\mathcal{C} \cap \Omega_0 \subset \sigma(\mathcal{C} \cap \Omega_0)$ so $B \in \mathcal{G}$

- $\mathcal{G}$ is a σ-field of subsets of Ω. (Check!)

$\therefore \boxed{\mathcal{G} \supset \sigma(\mathcal{C}) \quad \text{by minimality of } \sigma(\mathcal{C})}$

i.e. $A \in \sigma(\mathcal{C}) \Rightarrow A \cap \Omega_0 \in \sigma(\mathcal{C} \cap \Omega_0)$

so $\sigma(\mathcal{C}) \cap \Omega_0 \subset \sigma(\mathcal{C} \cap \Omega_0)$. //

Ex. (More Borel sets)

If (S, d) is a metric space, we have an a associated notion of "open set". Let $\mathcal{T}$ be the collection of all open subsets of S.

$$\mathcal{B}(S) := \sigma(\mathcal{T}) \qquad \text{(Borel subsets of } S\text{)}$$

Ex. $\mathcal{B}((0,1]) = \mathcal{B}(\mathbb{R}) \cap (0,1]$