Math 220 A – Lecture 24

December 9, 2020

0 Last time Conway VII.2.

$f_n : \mathcal{U} \longrightarrow \mathbb{C}$, $f : \mathcal{U} \longrightarrow \mathbb{C}$

$$f_n \overset{l.u.}{\rightrightarrows} f \iff f_n \overset{c}{\rightrightarrows} f$$

$$\iff \forall z \in \mathcal{U},\ \exists\ \Delta(z, r_z) \subseteq \mathcal{U},\ f_n \rightrightarrows f \text{ in } \Delta(z, r_z).$$

$$\iff f_n \rightrightarrows f \text{ in } K \quad \forall K \subseteq \mathcal{U}. \text{ compact}$$

1 Weierstraß' Theorem

Let $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$. Then

1 f holomorphic

2 $f_n^{(k)} \overset{l.u.}{\rightrightarrows} f^{(k)}$

Proof 1 Let $\bar{R} \subseteq \mathcal{U}$ closed rectangle, ∂R = compact.

Since $f_n \overset{l.u.}{\rightrightarrows} f \implies \int_{\partial R} f_n \, dz \longrightarrow \int_{\partial R} f \, dz$

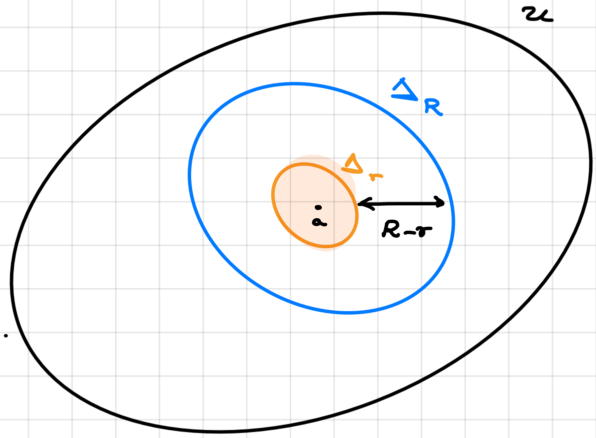
Since f_n holomorphic $\implies \int_{\partial R} f_n \, dz = 0.$ (Lecture 5)

$\implies \int_{\partial R} f \, dz = 0 \overset{\text{Lecture 5}}{\implies} f$ admits a primitive F in any disc in $\mathcal{U}$.

$\implies f = F' =$ holomorphic in any disc $\implies f$ holomorphic.

[ii] By induction, suffices to show

$$f_n' \overset{l.u.}{\rightrightarrows} f' \text{ in } \mathcal{U}.$$



Let $a \in \mathcal{U}$, $\overset{\overline{\Delta}_r}{\overset{\shortparallel}{\overline{\Delta}(a,r)}} \subseteq \overset{\overline{\Delta}_R}{\overset{\shortparallel}{\overline{\Delta}(a,R)}} \subseteq \mathcal{U}$.

$r < R$.

Suffices $f_n' \rightrightarrows f'$ in $\overline{\Delta}_r$.

$|w - z| \geq |w| - |z|$

$\geq R - r.$

We use CIF for $z \in \overline{\Delta}_r$

$$\left| f_n'(z) - f'(z) \right| = \left| \frac{1}{2\pi i} \int_{\partial \Delta_R} \frac{f_n(w) - f(w)}{(w-z)^2}\, dw \right|$$

$$\leq \frac{1}{\cancel{2\pi}} \cdot \sup_{\partial \Delta_R} |f_n - f| \cdot \frac{1}{(R-r)^2} \cdot \cancel{2\pi} R$$

Thus $\displaystyle \sup_{\overline{\Delta}_r} |f_n' - f'| \leq \frac{R}{(R-r)^2} \cdot \sup_{\partial \Delta_R} |f_n - f| \longrightarrow 0$ as $n \to \infty$.

$$\Rightarrow f_n' \overset{l.u.}{\rightrightarrows} f'.$$

Series $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic. Assume

$(*)$ $\forall\ K \subseteq \mathcal{U}$ compact $\exists\ M_n(K)$, $|f_n| \leq M_n(K)$ over K. & $\sum_{n=1}^{\infty} M_n(K) < \infty$.

M-test

$$\Rightarrow f = \sum_{n=1}^{\infty} f_n \text{ converges absolutely \& uniformly on every } K.$$

Weierstraß Thm

$$\Rightarrow f \text{ holomorphic \& } f' = \sum_{n=1}^{\infty} f_n'$$

Remark We have seen a particular case of this for power series. (Lecture 2).

Example (ζ- function)

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ gives a holomorphic function in $\operatorname{Re} s > 1$.

Take $f_n(s) = \frac{1}{n^s}$. holomorphic in s.

Take $K \subseteq \{\operatorname{Re} s > 1\}$. Since $\operatorname{Re} : K \longrightarrow \mathbb{R}$ is continuous,

it achieves a minimum on K $\Rightarrow \operatorname{Re} z \geq \alpha \quad \forall z \in K, \quad \alpha > 1.$

$$|f_n| = \left|\frac{1}{n^s}\right| = \left|\frac{1}{n^x} \cdot \cancel{\frac{1}{n^{iy}}}^{\,1}\right| = \frac{1}{n^x} \cdot 1. \leq \underbrace{\frac{1}{n^\alpha}}_{M_n}. \quad \text{where } s = x+iy$$

$\sum_{n=1}^{\infty} M_n < \infty$. by real analysis $\Longrightarrow$ $\sum_{n=1}^{\infty} f_n$ holomorphic. in s

$\Rightarrow \zeta$ holomorphic in s, $\operatorname{Re} s > 1$.

Remarks $\boxed{i}$ We have seen $\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2\,(2n)!}$ (HWK6)

$\boxed{ii}$ This can be extended holomorphically to $s \neq 1$

(requires work).

[2.] Hurwitz' Theorem

$f_n : \mathcal{U} \to \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$, $\overline{V} \subseteq \mathcal{U}$ compact

If $f|_{\partial V}$ has no zeroes,

$$\#\underset{\overline{V}}{\text{zeroes}}(f_n) = \#\underset{\overline{V}}{\text{zeroes}}(f) \quad \forall n \geq N.$$

Proof

[1] Most useful case (Conway)

$V = \overline{\Delta}(a, R)$

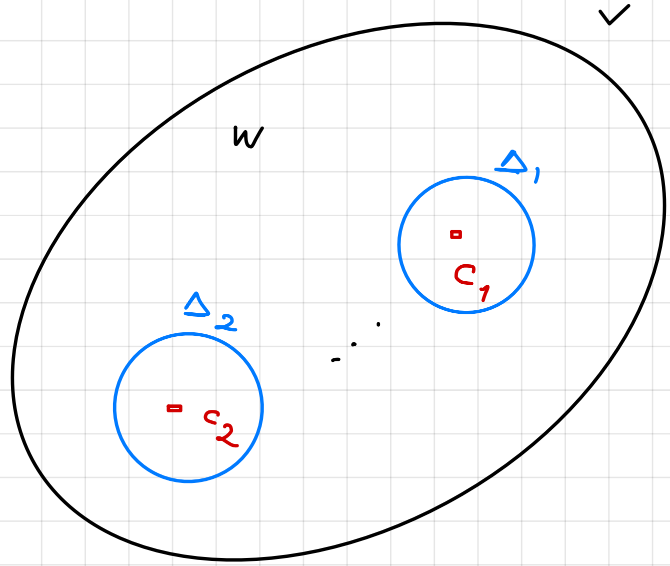
Since $f|_{\partial V}$ has no zeroes $\Rightarrow$ $\varepsilon = \min\limits_{\partial V} |f| > 0$.

Since $f_n \rightrightarrows f$ over ∂V $\Rightarrow$ $\exists N$ s.t. over ∂V $\forall n \geq N$.

$$|f_n - f|_{\partial V} < \varepsilon \leq |f|_{\partial V} \Rightarrow |f_n - f| < |f| \text{ over } \partial V.$$

$\Rightarrow$ Rouché: $\#\text{zeroes}(f) = \#\text{zeroes}(f_n)$ in $\overline{V}$.

ii) General case



$\overline{V}$ compact $\Rightarrow$ f has finitely many zeroes $c_1 \dots c_k$ in $\overline{V}$.

Surround c_j by small disjoint discs Δ_j, $W = V \setminus \bigcup_j \overline{\Delta_j}$

$\Rightarrow$ f has no zeroes in $\overline{W}$. $\Rightarrow$ $\exists N$ s.t. $\forall n \geq N$, f_n has no zeroes in $\overline{W}$. (If $\varepsilon = \min_{\overline{W}} |f| > 0 \Rightarrow \exists N$, s.t. $\forall n \geq N$ $|f_n - f| < \varepsilon$ in $\overline{W}$ $\Rightarrow f_n \neq 0$ in $\overline{W}$ for $n \geq N$.)

$$\Rightarrow \#\underset{\overline{V}}{\text{Zeroes}}(f) = \sum_{j=1}^{k} \#\underset{\overline{\Delta_j}}{\text{Zeroes}}(f) =$$

for n large by 1st case applied to f_n on $\overline{\Delta_j}$.

$$= \sum_{j=1}^{k} \#\text{Zeroes}(f_n) =$$

using f_n has no zeroes in $\overline{W}$

$$= \#\text{Zeroes}(f_n) \text{ for } n \geq N.$$

Corollary A

$f_n \overset{l.u.}{\rightrightarrows} f$, f_n holomorphic in $\mathcal{U}$,

If f_n is zero free $\forall n$ $\Rightarrow$ f zero - free or $f \equiv 0$.

This fails in real analysis, $f_n = x^2 + \frac{1}{n} \rightrightarrows f = x^2$.

Proof Indeed if $f \not\equiv 0$, let a be chosen so that $f(a) = 0$. Let $V = \overline{\Delta}(a, r)$, $f/_{\partial V}$ has no zeroes.

(Argue by contradiction, otherwise zeroes of f would accumulate).

Hurwitz

$\Rightarrow$ $\underbrace{\# \text{ zeroes}_{V} (f_n)}_{0. \text{ assumption.}} = \# \text{ Zeroes}_{V} (f) \geq 1.$ $\forall$ $n \geq N$.

a is a zero - contradiction.

$\Rightarrow$ f is zero - free

Example $\mathcal{U} = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$

- $f_n(z) = z$, $f(z) = z$, $f_n \overset{l.u.}{\rightrightarrows} f$, f zero free.

- $f_n(z) = \frac{z}{n}$, $f(z) = 0$, $f_n \overset{l.u.}{\rightrightarrows} f$, $f \equiv 0$.

Both possibilities occur.

Corollary B $f_n \overset{l.u.}{\rightrightarrows} f$, f_n holomorphic in $\mathcal{U}$,

If f_n are injective $\forall n$ $\Rightarrow$ f injective or f constant.

Proof Assume f not injective, $f(a) = f(b)$, $a \neq b$.

$\tilde{f}_n = f_n - f_n(a)$.

$\tilde{f} = f - f(a)$.

Since $\left.\begin{array}{l} f_n(a) \to f(a) \\ f_n \overset{l.u}{\rightrightarrows} f \end{array}\right\} \Rightarrow \tilde{f}_n \overset{l.u.}{\rightrightarrows} \tilde{f}$.

f_n injective $\Rightarrow$ $\tilde{f}_n$ zero free on $\tilde{\mathcal{U}} = \mathcal{U} \setminus \{a\}$.

Corollary A

$\Rightarrow$ $\tilde{f}$ is zero free on $\tilde{\mathcal{U}}$ or $\tilde{f} \equiv 0$. on $\tilde{\mathcal{U}}$

Note that $\tilde{f}(b) = f(b) - f(a) = 0$ $\Rightarrow$ $\tilde{f}$ is not zero-free

in $\tilde{\mathcal{U}}$. Thus $\tilde{f} \equiv 0$ in $\tilde{\mathcal{U}}$ $\Rightarrow$ f constant.

Adolf Hurwitz (1859 – 1919)