Math 220B, Problem Set 1.

The day when you can start solving each question is indicated.

1. (*Monday, January 4.*) Directly from the definitions, show that

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}.$$

2. (*Partition Products. Friday, January 8.*) Let $|q|<1$. Euler studied the products

$$Q(z)=\prod_{n=1}^{\infty}(1+q^n z)$$

in connection with the theory of partitions and pentagonal numbers $\frac{n(3n-1)}{2}$. These products thus appear in combinatorics as well as number theory.

Historical remark: For fixed values of z, we can study the Taylor expansion of Q viewed as function of q. Two cases are of special interest. For $z=1$, one easily sees that

$$Q(1)=\sum_{n=0}^{\infty}p(n)q^n$$

where $p(n)$ is the number of partitions into distinct parts. When $z=-1$, Euler's pentagonal number theorem states that

$$Q(-1)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}}.$$

We will not need/prove these statements here. Instead, our point of view will be to regard Q as a function of z, for fixed $|q|<1$.

(i) Show that Q is an entire function in z.

(ii) Show that

$$Q(z)=(1+qz)Q(qz).$$

(iii) Write

$$Q(z)=\sum_{n=0}^{\infty}a_n z^n.$$

Derive the recursion

$$a_n=\frac{q^n}{1-q^n}a_{n-1}$$

and derive the identity

$$Q(z)=1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\dots(1-q^n)}z^n.$$

(iv) Write out the strange looking identitties proved in (iii) for $z=1$ and $z=-1$.

3. (*Factorization of sine. Monday, January 11.*) Show that

$$e^{az} - e^{bz} = (a-b)ze^{\frac{a+b}{2}z} \prod_{n=1}^{\infty} \left(1 + \frac{(a-b)^2 z^2}{4n^2\pi^2}\right).$$

One way to solve this question is to reduce to the factorization of the sine via simple manipulations.

4. (*Towards the* Γ *function. Friday, January 8.*) Write Log for the principal branch of the logarithm.

(i) Possibly using Taylor expansion, show that if $|w| \leq \frac{1}{2}$,

$$|\mathrm{Log}(1+w) - w| \leq 2|w|^2.$$

(ii) Let $a, b \in \mathbb{C}$ have positive real parts. Show that

$$\mathrm{Log}(ab) = \mathrm{Log}(a) + \mathrm{Log}(b).$$

(iii) Let $r > 0$. Show that there exists N such that for all $n \geq N$, $1 + \frac{z}{n}$ and $e^{-\frac{z}{n}}$ have positive real parts for all $z \in \Delta(0, r)$.

(iv) (Possibly taking Logs and arguing that the series of Log's coverges absolutely and locally uniformly), show that the product

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

converges to an entire function.

5. (*Blaschke products. Friday, January 8.*) This is a modified version of Conway VII.5, Problem 4. The point of this question is to characterize certain holomorphic functions on the unit disc.

For $\alpha \in \Delta(0,1) \setminus \{0\}$, define

$$B_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z} \cdot \frac{|\alpha|}{\alpha}.$$

(i) Let $\alpha \in \Delta(0,1) \setminus \{0\}$ and $z \in \overline{\Delta}(0, r)$ for $r < 1$. Prove that

$$\left| \frac{\alpha + |\alpha| z}{(1 - \bar{\alpha}z)\alpha} \right| \leq \frac{1+r}{1-r}.$$

(ii) Let $\alpha_n \in \Delta(0,1) \setminus \{0\}$ be a sequence of nonzero numbers in the unit disc. Assume that

$$\sum_n (1 - |\alpha_n|) < \infty.$$

Using (i), show that the *Blaschke product*

$$B(z) = \prod_{n=1}^{\infty} B_{\alpha_n}(z)$$

converges to a holomorphic function

$$B : \Delta(0,1) \to \mathbb{C}$$

with zeroes only at α_n.

Remark: Conversely, we will show later (Jensen's formula) that the zeroes of a bounded holomorphic function B on $\Delta(0,1)$ satisfy

$$\sum_n (1 - |\alpha_n|) < \infty.$$

Parts (iii) and (iv) concern finite Blaschke products, and can be solved using only the material from Math 220A.

(iii) Show that for $\alpha \in \Delta(0,1) \setminus \{0\}$,

$$B_\alpha : \overline{\Delta}(0,1) \to \overline{\Delta}(0,1)$$

in such a fashion that

$$|B_\alpha(z)| = 1 \text{ for } |z| = 1.$$

(iv) Conversely, let $f : \Delta(0,1) \to \Delta(0,1)$ be a holomorphic function extending continuously to $\overline{\Delta}(0,1)$ such that

$$|f(z)| = 1 \text{ for } |z| = 1.$$

Show that f can be expanded as a finite Blaschke product:

$$f(z) = cz^m \prod_{n=1}^{N} B_{\alpha_n}(z).$$

Hint: Assume first that $f(0) \neq 0$. Show that f can have only finitely many zeroes $\alpha_1, \ldots, \alpha_n \in \Delta(0,1) \setminus \{0\}$. Construct the suitable Blaschke product $B(z)$ and work with the function $g(z) = f(z)/B(z)$. What properties does g have?