Math 220A - Lecture 12

November 2, 2020

[I] Main theorems

[i] Identity Principle

[ii] Open Mapping Theorem

[iii] Maximum Modulus Principle

Theorem $f : U \longrightarrow \mathbb{C}$ holomorphic, non constant $\Rightarrow$

$|f|$ cannot have local maxima.

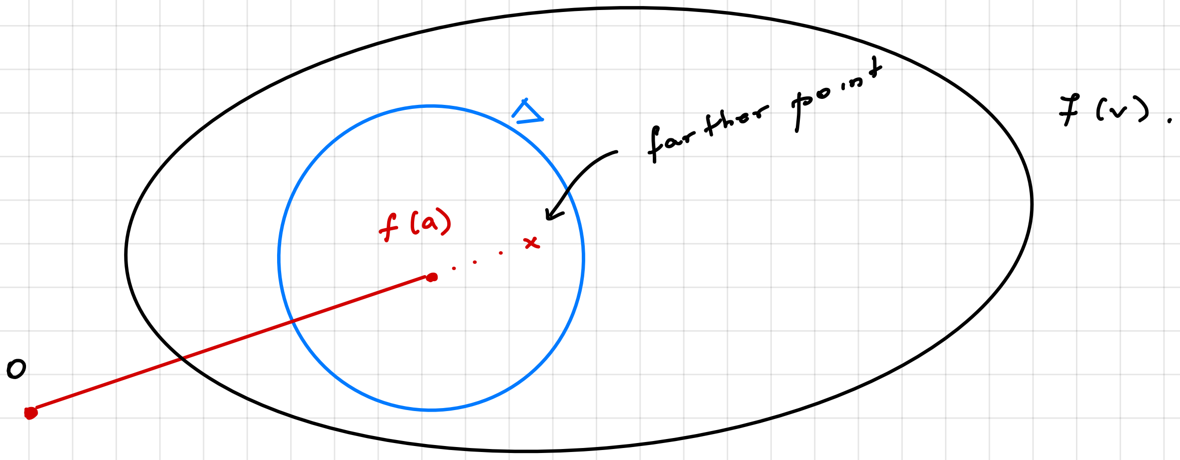
Proof Assume that $|f|$ achieves a local maximum at a.

$\Rightarrow \exists\, V \ni a,\ V \subseteq U$, $|f|$ has a maximum at a.

By OMT, $f(V)$ is open. $\Rightarrow \exists$ disc Δ centered at $f(a)$

$\Delta \subseteq f(V)$. Note that $|f|$ measures distance from the origin. The disc Δ has points farther from 0 than $f(a)$

contradicting the assumption $|f|$ has maximum at a. (in V).



Remarks $\boxed{i}$ Minimum modulus principle

$f : U \to \mathbb{C}$ holomorphic, not constant, f has no zeros in U.

$\Rightarrow$ $|f|$ has no local minimum

Proof Let $g = \frac{1}{f} : U \to \mathbb{C}$. holomorphic. Apply the maximum modulus to the function g & conclude.

$\boxed{ii}$ U bounded, $f : \overline{U} \to \mathbb{C}$ continuous, holomorphic in U

$\Rightarrow$ $\max_{\overline{U}} |f| = \max_{\partial U} |f|$ $\quad (*)$.

Proof Since U bounded $\Rightarrow$ $\overline{U}, \partial U$ compact so f achieves maxima on these sets. Let f achieve maximum in $\overline{U}$ at $a \in \overline{U}$.

If $a \in U \Rightarrow f|_U$ has a maximum at $a \xrightarrow{\text{MMP}}$

$\Rightarrow$ $f = \text{constant}$ & there's nothing to prove.

Otherwise $a \in \partial U$ proving $(*)$.

2. Laurent Series & Functions in annular regions (Conway V.1)

We have seen $f : \Delta(a,r) \longrightarrow \mathbb{C}$ then

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$ — Taylor series

We consider Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

Convergence of Laurent series

$$f^+(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

$$f^-(z) = \sum_{k=-\infty}^{-1} a_k (z-a)^k = \sum_{k=1}^{\infty} a_{-k} (z-a)^{-k}$$

$$f(z) = f^+(z) + f^-(z).$$

Def f converges absolutely & uniformly provided f^+, f^- do so.

Remark

radius of convergence

f^+ converges if $|z-a| < R$.

f^- converges if $|z-a|^{-1} < r^{-1} \iff |z-a| > r$.

For power series, convergence is absolute & uniform on compact subsets.

Define $\Delta(a; r, R) = \{z : r < |z-a| < R\}$, $0 \le r < R \le \infty$.

Theorem Let $f: \Delta(a; r, R) \longrightarrow \mathbb{C}$ holomorphic. Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$ can be expanded into

Laurent series, converging absolutely & uniformly on compact sets in $\Delta(a; r, R)$. Furthermore,

$$a_k = \frac{1}{2\pi i} \int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{k+1}}\, dw. \quad \forall\ r < \rho < R.$$

Remark An important case is $r = 0$. Then

$$\Delta^*(a, R) = \Delta(a, R) \setminus \{a\} = \text{punctured disc.}$$

$f: \Delta^*(a, R) \longrightarrow \mathbb{C}$ holomorphic $\Rightarrow$ $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$

Compare this to Taylor expansion.

Pierre Alphonse Laurent

1813 – 1854

(engineer in the army).

The original work on Laurent series was not published.

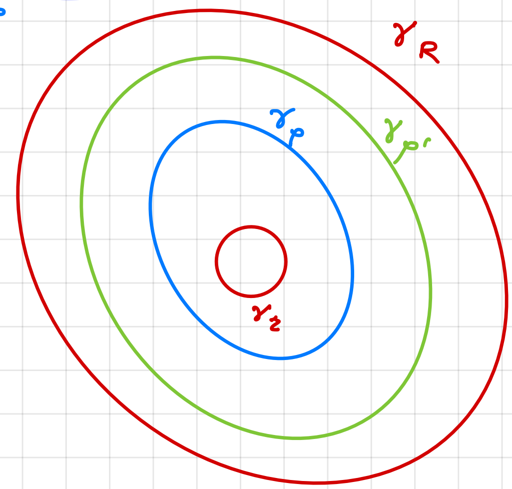
Cauchy writes:

"L'extension donnée par M. Laurent ... nous paraît digne de remarque."

Proof (of Laurent expansion) $A = \Delta(a\,;\, r, R)$.

$\boxed{a}$ WLOG $a = 0$; else work with $f(z+a)$.

$\boxed{b}$ the expression $a_k = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_\rho} \dfrac{f(w)}{w^{k+1}}\, dw$



is independent of ρ. Indeed

$\gamma_\rho \overset{A}{\sim} \gamma_{\rho'}$ and use

Cauchy Homotopy Theorem.

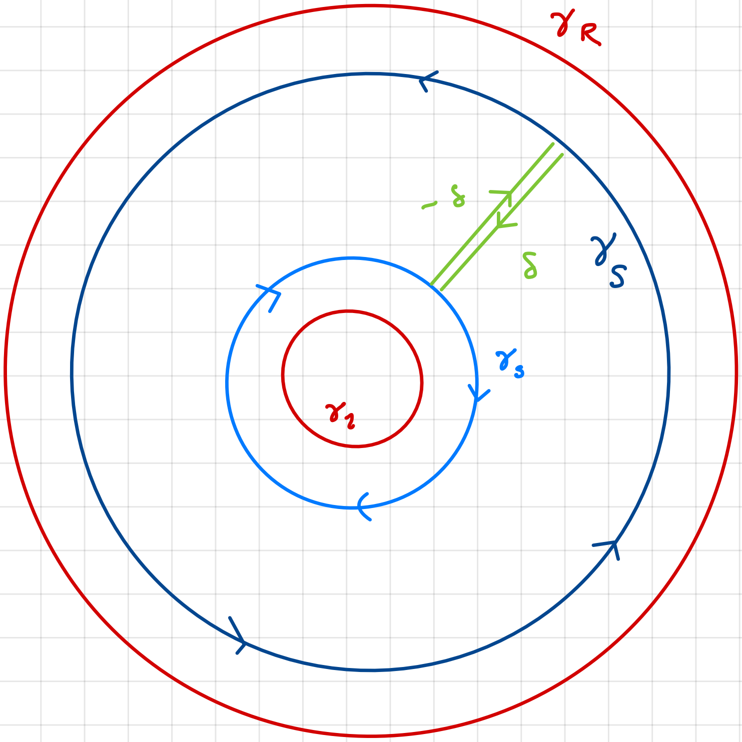
$\boxed{c}$ suffices to prove pointwise convergence in z.

Indeed, convergence of f $\iff$ convergence of f^+ & f^- in $r < |z| < R$.

But then f^+ converges in $|z| < R$ (power series have discs of converge) & we remarked convergence is absolute & uniform on compacts. Same for f^-.

Pointwise convergence

Let $r < s < |z| < S < R$



Let δ be a segment joining γ_s, γ_S avoiding z.

Let

$$\gamma = \gamma_S + \delta + \gamma_s + (-\delta).$$

Note $\gamma \sim 0$. This can be seen by continuously shrinking $\delta \to 0$.

Also $n(\gamma, z) = 1$ since $n(\gamma_s, z) = 0$ as z is outside and $n(\gamma_S, z) = 1$ as z is interior to γ_S. $\Rightarrow n(\gamma, z) = 1$.

CIF:

$$(+) \quad f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z}\, dw.$$

$$= \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w-z}\, dw$$

(cancelling the contribution of $\delta, -\delta$).

The two terms will give the positive/negative parts of Laurent series.

Key expansions (Remember them) $s < |z| < S$.

$\boxed{i}$ over γ_S : $\frac{1}{w - z} = \frac{1}{w} \cdot \frac{1}{1 - \frac{z}{w}} = \sum_{k=0}^{\infty} \frac{1}{w} \left(\frac{z}{w}\right)^k$

$$= \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}. \quad (1)$$

The convergence is uniform in w since $\left|\frac{z}{w}\right| = \frac{|z|}{S} < 1$. We can define $M_k = \frac{|z|^k}{S^{k+1}}$, $f_k(z) = \frac{z^k}{w^{k+1}}$ and invoke Weierstraß M-test to conclude uniform convergence.

We can multiply by $f(w)$. Uniform convergence still holds. (Use $M_k = \frac{|z|^k}{S^{k+1}} \cdot \sup_{\gamma_S} |f|$.)

We can then integrate term by term. (Rudin). Thus

$$\frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w - z} dw \overset{(1)}{=} \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w^{k+1}} dw \cdot z^k$$

$$= \sum_{k=0}^{\infty} a_k z^k. \quad (*)$$

$\boxed{ii}$ over γ_s, we use a different expansion

$$\frac{1}{w - z} = -\frac{1}{z} \cdot \frac{1}{1 - \frac{w}{z}} = \sum_{k=0}^{\infty} -\frac{1}{z} \left(\frac{w}{z}\right)^k$$

$$= \sum_{k=0}^{\infty} -\frac{w^k}{z^{k+1}}. \quad (2)$$

Here $\left|\frac{w}{z}\right| = \frac{r}{|z|} < 1$. By the same arguments

$$-\frac{1}{2\pi i}\int_{\gamma_s} \frac{f(w)}{w-z}\,dw \overset{(2)}{=} \sum_{k=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_r} f(w)\,w^k\,dw \cdot z^{-k-1}$$

$$= \sum_{k=0}^{\infty} a_{-k-1}\, z^{-k-1}$$

$$= \sum_{k=-\infty}^{-1} a_k\, z^k. \quad (**)$$

(+), (*), (**). imply the Theorem.