Math 220 A — Lecture 14

November 13, 2020

1. Residues (Conway $\overline{V}$. 2)

a singularity for f

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \quad \text{Laurent series}$$

$$a_{-1} = \operatorname{Res}(f, a) = \text{residue}$$

Problem: Compute $\operatorname{Res}(f, a)$

Method 0 Laurent expansion

Example $f(z) = \dfrac{z}{\sin^4 z}$, $\operatorname{Res}(f, 0) = ?$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)$$

$$\sin^4 z = z^4\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)^4$$

$$= z^4\left(1 - \frac{4z^2}{6} + \dots\right)$$

$$f(z) = \frac{z}{z^4\left(1 - \frac{4z^2}{6} + \dots\right)} = \frac{1}{z^3}\cdot\left(1 + \frac{4z^2}{6} + \dots\right)$$

$$= \frac{1}{z^3} + \frac{4}{6}\cdot\frac{1}{z} + \dots$$

$\Rightarrow \operatorname{Res}(f, 0) = \frac{2}{3}.$

Method 1* $\quad f(z) = \dfrac{g(z)}{h(z)}$, g, h holomorphic

Assume a simple zero for $h \Rightarrow a$ simple pole for f.

$$\operatorname{Res}(f, a) = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to a} (z-a) \frac{g(z)}{h(z) - h(a)}$$

$$= \lim_{z \to a} \frac{g(z)}{\dfrac{h(z)-h(a)}{z-a}} = \frac{g(a)}{h'(a)}$$

Conclusion: $\quad \operatorname{Res}(f, a) = \dfrac{g(a)}{h'(a)}.$

Example $\quad f(z) = \dfrac{z - \sin z}{z^2 \sin z}$

- poles $z = 0, \quad z = n\pi, \ n \neq 0, \ n \in \mathbb{Z}$

$\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dots \implies \dfrac{\sin z}{z} \longrightarrow 1$ as $z \to 0$

- $z = 0$ is removable since $\implies \dfrac{z - \sin z}{z^3} \to \dfrac{1}{3!}$ as $z \to 0$

$$\lim_{z \to 0} \frac{z - \sin z}{z^2 \sin z} = \lim_{z \to 0} \underbrace{\frac{z - \sin z}{z^3}}_{\frac{1}{6}} \cdot \underbrace{\frac{z}{\sin z}}_{1.} = \frac{1}{6}.$$

Since $z = 0$ is removable $\implies \operatorname{Res}(f, 0) = 0.$

- $z = n\pi, \ n \neq 0.$ Take $g(z) = \dfrac{z - \sin z}{z^2}$

$h(z) = \sin z$

$$\implies g(n\pi) = \frac{1}{n\pi}, \quad h'(n\pi) = \cos z \Big|_{z = n\pi} = (-1)^n.$$

$$\implies \operatorname{Res}(f, n\pi) = \frac{g(n\pi)}{h'(n\pi)} = \frac{1}{n\pi} \cdot (-1)^n.$$

Method 2

holomorphic

$$f(z) = \frac{g(z)}{(z-a)^k} \implies \operatorname{Res}(f,a) = \frac{g^{(k-1)}(a)}{(k-1)!}$$

Write $\quad g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(z-a)^n.$

coeff. of $(z-a)^{-1}$ in f $\iff$ coeff. of $(z-a)^{k-1}$ in g.

This equals $\quad \frac{g^{(k-1)}(a)}{(k-1)!}$

Example $\quad f(z) = \frac{z}{(z^2+1)^2} \implies \operatorname{Res}(f,i) = ?$

$$f(z) = \frac{g(z)}{(z-i)^2}, \quad g(z) = \frac{z}{(z+i)^2}. \implies g'(i) = 0 \quad \text{(check)}$$

$$\operatorname{Res}(f,i) = g'(i) = 0.$$

2. Residue Theorem (Conway V.2)

Toy Example $f: \Delta^*(a, R) \longrightarrow \mathbb{C}$, holomorphic.

$$\Rightarrow \boxed{\int_{\gamma_s} f(z)\,dz = 2\pi i \,\mathrm{Res}(f,a)} \text{. where } \gamma_s = \partial \Delta(a,s).$$

Proof Write

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k .$$

This converges uniformly on compact sets, so we can integrate

$$\Rightarrow \int_{\gamma_s} f\,dz = \sum_{k=-\infty}^{\infty} a_k \int_{\gamma_s} (z-a)^k\,dz$$

$k \neq -1$, integral $= 0$

$k = -1$, integral $= 2\pi i$

$$= 2\pi i \, a_{-1} = 2\pi i \,\mathrm{Res}(f,a).$$

$k \neq -1$: $(z-a)^k$ admits a primitive $\dfrac{(z-a)^{k+1}}{k+1}$ $\Rightarrow$ zero integral.

$k = -1$: $\displaystyle\int_{\gamma_s} \frac{dz}{z-a} = 2\pi i \; n(\gamma_s, a) = 2\pi i$

Residue Theorem

$\mathcal{U} \subseteq \mathbb{C}$ open connected, S discrete

- $\gamma \overset{\mathcal{U}}{\sim} 0$, $\{\gamma\} \subseteq \mathcal{U} \setminus S$.
- f holomorphic in $\mathcal{U} \setminus S$, singularities at S.

Then

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$$

Remarks

$\boxed{i}$ $S = \phi \Rightarrow \int_\gamma f \, dz = 0 \Rightarrow$ Cauchy's Theorem

$\boxed{ii}$ $S = \{a\}$, $\gamma = \gamma_r$ = small circle near a $\Rightarrow$ recovers the toy example.

$\boxed{iii}$ $S = \{a\}$, $f(z) = \dfrac{g(z)}{(z-a)^{k+1}}$, g holomorphic, $\gamma \overset{\mathcal{U}}{\sim} 0$

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \frac{1}{2\pi i} \int_\gamma \frac{g(z)}{(z-a)^{k+1}} \, dz = \operatorname{Res}(g, a) \cdot n(\gamma, a)$$

$$= \frac{g^{(k)}(a)}{k!} \cdot n(\gamma, a) \text{ by Method 2.}$$

This recovers CIF for derivatives.

IV The sum in RHS is finite

Claim $\{ s \in S, \ n(\gamma, s) \neq 0 \}$ finite

Proof $W = \{ z \in \mathbb{C} \setminus \gamma : \ n(\gamma, z) \neq 0 \}$.

- W = union of components of $\mathbb{C} \setminus \gamma$ = open (Lecture 7)
- W bounded (Lecture 7)
- $W \subseteq U$. Indeed. if $z \in W$, $z \notin U \Rightarrow n(\gamma, z) \neq 0$. But

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} = 0 \text{ by Cauchy}$$

using $\zeta \to \dfrac{1}{\zeta - z}$ holomorphic in U, $\gamma \overset{U}{\sim} 0$.

$K = W \cup \{\gamma\} \subseteq U$ closed & bounded

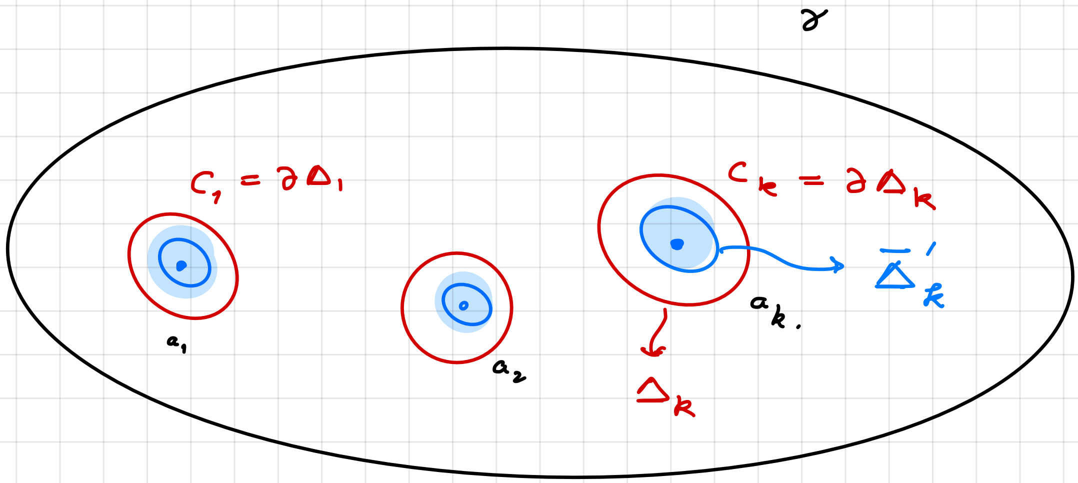
$\Rightarrow K$ compact in U., S discrete in U

$\Rightarrow K \cap S$ = finite.

☑ Naive "proof"

γ simple closed curve

Let $S = \{a_1, \dots, a_k\}$.



Let $C_i = \partial\Delta_i$ be circles centered at a_i, $\Delta_i \subseteq \mathcal{U}$.

Let $\overline{\Delta}'_i \subseteq \Delta_i$. Let $\mathcal{U}' = \mathcal{U} \setminus \bigcup_{i=1}^{k} \overline{\Delta}'_i$ = open.

Let $\eta = \sum_{i=1}^{k} \partial C_i$. Assume we could show

$\gamma \overset{\mathcal{U}'}{\sim} \eta$ and $n(\gamma, a_i) = 1$.

Then by Cauchy applied to $f|_{\mathcal{U}'}$ we'd have

$$\int_\gamma f\,dz = \int_\eta f\,dz = \sum_{i=1}^{k} \int_{C_i} f\,dz$$

$$= 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, a_i)$$ (toy example).

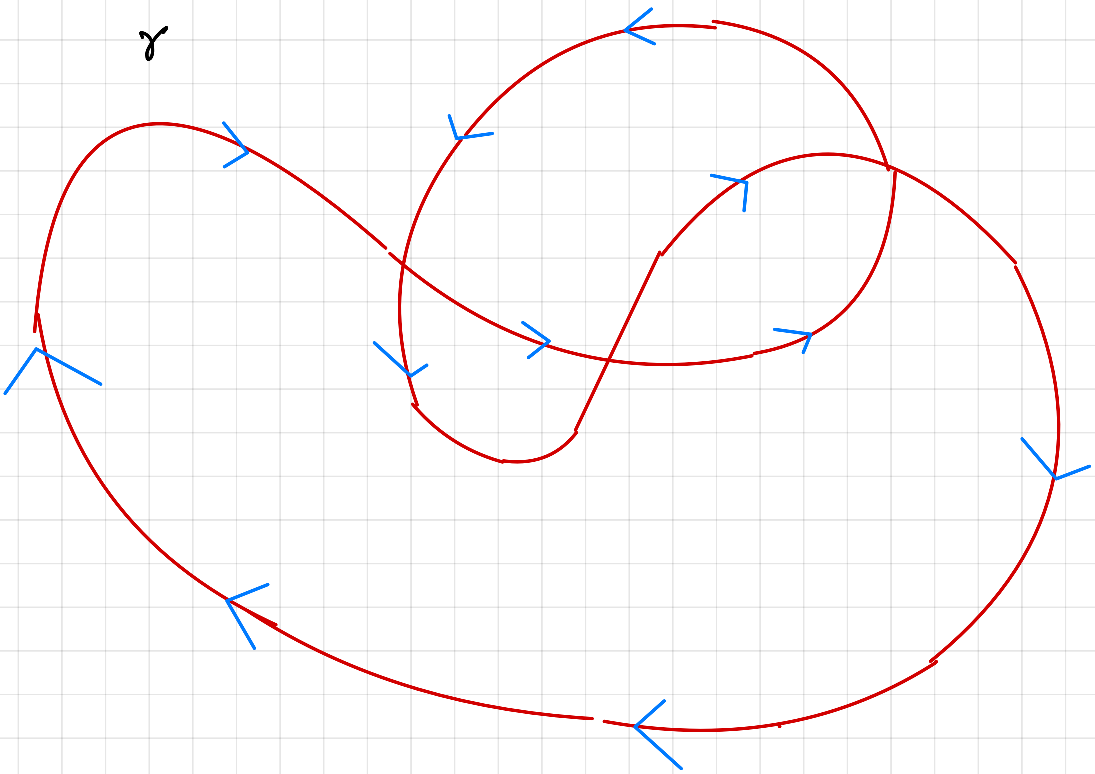
$$= 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, a_i)\, n(\gamma, a_i).$$

Issues :

$\boxed{a}$ η is not a path, but chain

$\boxed{b}$ $\gamma \overset{u'}{\sim} \eta$ and $n(\gamma, a_i) = 1$ need proofs

$\boxed{c}$ how about more complicated curves?



The proof of the residue theorem requires new ideas.

(next time)