HW4 - SOLUTIONS

Q1.

(i) Using **Q4** in Homework 1 and noting that $f(z) \in \triangle_1(1)$, there is a complex differentiable logarithm of values $w = f(z)$ in $\Delta_1(1)$. Consider

$$F(z) = \log f(z).$$

Then $F'(z) = \frac{f^{'}(z)}{f(z)}$. It follows that

$$\int_\gamma \frac{f^;}{f} dz = \int_\gamma F'(z)\, dz = 0$$

by the fundamental theorem of calculus.

(ii) $\gamma \sim 0$ where 0 is a constant curve because U is a simply connected domain. Since $\frac{f^{'}}{f}$ is holomorphic over U and we have $\int_\gamma \frac{f^{'}}{f} dz = \int_0 \frac{f^{'}}{f} dz = 0$.

(iii) No. A counter example is $\gamma(t) = e^{it}$ over $t \in [0, 2\pi]$ and $f(z) = z$. Then

$$\int_\gamma \frac{f^{'}}{f} dz = \int_\gamma \frac{1}{z} dz = i \int_0^{2\pi} dt = 2\pi i.$$

Q2. Since $\frac{f^{'}}{f}$ is holomorphic over a simply connected domain U, it follows that $\frac{f^{'}}{f}$ has a primitive function F, i.e. $F^{'} = \frac{f^{'}}{f}$. We compute

$$(f(z)e^{-F(z)})' = f'(z)e^{-F(z)} + f(z)e^{-F(z)}F'(z) = 0.$$

Therefore, $f(z) \cdot e^{-F(z)} = c$. Let $c = e^d$, we have

$$f(z) = e^{F(z)+d}.$$

Define

$$g(z) = e^{\frac{1}{n}(F(z)+d)}.$$

Then

$$g(z)^n = \exp(F(z) + d) = f(z).$$

Q3. Suppose d is the degree of of $p(z)$. Let $a \in \mathbb{C}$. By Cauchy's formula

$$f^{(d+1)}(a) = \frac{(d+1)!}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^{d+2}}\, dz.$$

Since

$$\lim_{z\to\infty} \frac{p(z)}{(z-a)^d} < \infty$$

it follows

$$|p(z)| \le c|z-a|^d$$

for $|z|$ large, for a constant c. For R large, we have therefore

$$|f(z)| \le |p(z)| \le cR^d,$$

for $|z-a| = R$. The integrand is bounded

$$\left|\frac{f(z)}{(z-a)^{d+2}}\right| \le \frac{c}{R^2}.$$

By the basic estimate proved in class

$$|f^{d+1}(a)| \le \frac{(d+1)!}{2\pi}\frac{c}{R^2} \cdot 2\pi R = \frac{c(d+1)!}{R}.$$

Hence, $f^{(d+1)}(a) = 0$ by taking $R \to \infty$ in the above estimate. Hence, $f(z)$ is a polynomial with degree at most d.

Q4.

(i) Suppose $\operatorname{Re}(f)$ is bounded below, then there exists $c \in \mathbb{R}$ such that $\operatorname{Re}(f) \ge c$. Hence, $\left|e^{-f}\right| = e^{-\operatorname{Re}(f)} \le e^{-c}$. By Liouville' s Theorem, it follows that $e^{-f(z)} = e^{-f(0)}$. In other words, for any $z \in \mathbb{C}$, there exists $n_z \in \mathbb{N}$ such that $f(z) - f(0) = 2\pi i n_z$. By continuity of $f(z) - f(0)$, we have n_z is constant, necessarily equal to 0, so $f(z) = f(0)$. Hence, $f(z)$ is a constant. If $\operatorname{Re}(f)$ is bounded above, then$-f$ has its real part bounded below. Hence, $-f$ is constant which implies f is constant.

(ii) Since $\operatorname{Re}((1+i)f) = \operatorname{Re}(f) - \operatorname{Im}(f) \le 0$ and $(1+i)f$ is entire, $(1+i)f$ is constant by part (i) which in turn implies that f is constant.

Q5. Let $g(z) = f(z)^2$. We know $|g(z)| \le (\log(1+r))^3$ when $|z| \le r$. If $|z-a| \le r$, then $|z| \le r + |a|$ so that

$$|g(z)| \le (\log(1+r+|a|))^3.$$

Applying Cauchy's estimates to the circle $|z-a| = r$, we find

$$|g'(a)| \le \frac{1}{r} \cdot (\log(1+r+|a|))^3.$$

Making $r \to \infty$, we conclude $g'(a) = 0$. Thus g must be constant. Setting $z = 0$, we find $|g(0)| \le \log 1$ so $g(0) = 0$. Thus $g \equiv 0$ and $f \equiv 0$ as well.

Q6. Using Cauchy's integral formula, we have

$$\begin{aligned}
|f(w_1) - f(w_2)| &= \frac{1}{2\pi}\left|\int_{|t|=1} \frac{f(t)}{t-w_1}\,dt - \int_{|z|=1} \frac{f(t)}{t-w_2}\,dt\right| \\
&= \frac{1}{2\pi}\left|\int_{t|=1} f(t)\left(\frac{1}{t-w_1} - \frac{1}{t-w_2}\right)dt\right| \\
&= \frac{|w_1-w_2|}{2\pi}\left|\int_{|t|=1} \frac{f(t)}{(t-w_1)(t-w_2)}\,dt\right|
\end{aligned}$$

Note

$$|t-w_1| \geq |t| - |w_1| \geq 1 - \frac{1}{2} = \frac{1}{2}$$

and similarly

$$|t-w_2| \geq \frac{1}{2},$$

while $|f(t)| \leq M$ for $|t| = 1$, we find that the integrand above

$$\left|\frac{f(t)}{(t-w_1)(t-w_2)}\right| \leq 4M$$

Thus by the basic estimate we obtain

$$|f(w_1) - f(w_2)| \leq \frac{|w_1 - w_2|}{2\pi} \cdot 4M \cdot \text{lengh of unit circle} = 4M|w_1 - w_2|.$$

Q7. Any compact set K in $\mathbb{C}$ is contained in a closed ball $\bar{\Delta}(0, R)$ with $R > 1$, so we may assume $K = \bar{\Delta}(0, R)$. Let

$$M = \sup\{|f(w)| : w \in \bar{\Delta}(0, 2R)\}.$$

Let $z \in K$ and $\gamma = \{|w - z| = R\}$. Then $|z| \leq R$, and for all $w \in \gamma$ we have $|w| \leq |w - z| + |z| \leq 2R$ so that

$$|f(w)| \leq M.$$

Thus over the disc $\Delta(z, R)$ we can apply Cauchy's estimate to obtain

$$\left|\frac{f^{(n)}(z)}{n!}\right| \leq \frac{M}{R^n} := M_n.$$

Note that $\sum M_n < \infty$ is the geometric series. Therefore the series

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(z)}{n!}$$

converges uniformly on K by the Weierstrass M-test.

Q8.

(i) We have

$$z = (e^z - 1)\left(B_0 + B_1 z + B_2\frac{z^2}{2} + \dots\right) = \left(z + \frac{z^2}{2} + \frac{z^3}{6} + \dots\right)\left(B_0 + B_1 z + B_2\frac{z^2}{2} + \dots\right)$$

$$= B_0 z + \left(B_1 + \frac{1}{2}\right)z^2 + \left(\frac{B_2}{2} + \frac{B_1}{2} + \frac{1}{6}\right)z^3 + \dots.$$

Thus

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}.$$

(ii) Consider

$$f(z) = \frac{z}{e^z - 1} + \frac{z}{2} = \sum_{k=0}^{\infty} B'_k \frac{z^k}{k!}$$

where $B'_k = B_k$ for $k \neq 1$ and $B'_1 = B_1 + \frac{1}{2}$. We claim that f is an even function, and therefore all the odd powers of z must come with zero coefficients. This will prove that $B_{2k+1} = 0$ for $k \geq 1$. To see that f is even, we compute

$$f(z) - f(-z) = \frac{z}{e^z - 1} + \frac{z}{2} - \frac{-z}{e^{-z} - 1} + \frac{z}{2} = \frac{z}{e^z - 1} - \frac{z}{e^{-z} - 1} + z = 0,$$

where the last equality can be checked by direct computation.

(iii) We consider the expression

$$e^z + e^{2z} + \ldots + e^{Nz} = e^z \cdot (1 + e^z + \ldots + e^{(N-1)z}) = e^z \cdot \frac{e^{Nz} - 1}{e^z - 1} = \frac{e^{Nz} - 1}{1 - e^{-z}}$$
$$= \frac{e^{Nz} - 1}{z} \cdot \frac{z}{1 - e^{-z}}.$$

Note that

$$\frac{e^{Nz} - 1}{z} = \sum_{k=0}^{\infty} N^{k+1} \frac{z^k}{(k+1)!}$$
$$\frac{z}{1 - e^{-z}} = \sum_{j=0}^{\infty} (-z)^j \frac{B_j}{j!}.$$

We look at the coefficient of z^p on the left hand side. It equals

$$\frac{1}{p!}(1^p + \ldots + N^p).$$

The same coefficient on the right hand side equals

$$\sum_{j+k=p} \frac{N^{k+1}}{(k+1)!} \cdot (-1)^j \frac{B_j}{j!} = \sum_{j=0}^{p} N^{p+1-j} (-1)^j B_j \cdot \frac{1}{j!(p+1-j)!}.$$

Matching the two expressions we found gives the result.

The cases $p = 1, 2, 3$ give

$$1 + 2 + \ldots + n = \frac{n^2}{2} + \frac{n}{2}$$
$$1^2 + 2^2 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$
$$1^3 + 2^3 + \ldots + n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$