Math 281a – Problem Set # 2
Module 3

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Due @ October 16th, 2021

Problem 1

Let X_n be a sequence of independent random variables. Let a_n be a sequence of positive numbers. Let τ and θ be some unknown but fixed real values. Suppose that $a_n(X_n - \theta)$ converges in distribution to a normal random variable $\mathcal{N}(0, \tau^2)$. What can you say about the distribution of $|X_n|$?

Problem 2

The sample skewness of a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n \in \mathbb{R}$ is defined as

$$l_n = \frac{n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^3}{\left(n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2\right)^{3/2}}$$

(a) Show that it converges in probability to the skewness of the underlying distribution, defined as $\lambda = \mu_2/\sigma^3$. What assumptions are needed on the distribution of the data? Recall from the lecture video that μ_3 is the centered third moment whereas σ^3 is the third power of the standard deviation of one observation.

(b) Assume that the underlying data has standard normal distribution. Show the following

$$\sqrt{n}(l_n - \lambda) \xrightarrow{d} \mathcal{N}(0, v^2)$$

for appropriately computed asymptotic variance v^2. Compute v^2.

Problem 3

Consider a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n \in \mathbb{R}$.

Find the limiting distribution of the sample kurtosis

$$k_n = \frac{n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^4}{S^4} - 3$$

where S^2 denotes the sample variance.

Problem 4

Suppose $X_1, X_2, X_3, \cdots$ are i.i.d. $\mathcal{N}(\mu, 1)$ with $-\infty < \mu < \infty$ and $\mu \neq 0$. Find the limiting distribution of $|\bar{X}|$ for $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$. Will this limiting distribution change if $\mu = 0$?

Problem 5

Let X_n, Y_m be independent Poisson with means n, m. Suppose $n, m \geq 1$. Find the limiting distribution of

$$\frac{X_n - Y_m - (n - m)}{\sqrt{X_n + Y_m}}$$

as $n, m \to \infty$.