HW1 - SOLUTIONS

Q1. Write $f = u + iv$. Since $u^2 + iv^2$ is twice complex differentiable, u^2 is harmonic. We find

$$(u^2)_{xx} = (2uu_x)_x = 2uu_{xx} + 2u_x^2,\ (u^2)_{yy} = 2uu_{yy} + 2u_y^2$$

$$\implies (u^2)_{xx} + (u^2)_{yy} = 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2).$$

Using that u is harmonic, we find $u_{xx} + u_{yy} = 0$. We derive

$$u_x^2 + u_y^2 = 0 \implies u_x = u_y = 0.$$

Therefore, u is constant on U. In a similar fashion, v is constant on U, hence f is constant.

Q2. (i) We find the radius of convergence using the ratio test. Let

$$c_n = \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{1}{n!}.$$

The ratio of two consecutive coefficients is

$$\frac{c_{n+1}}{c_n} = \prod_{j=1}^{p} \frac{(a_j)_{n+1}}{(a_j)_n} \prod_{j=1}^{q} \frac{(b_j)_n}{(b_j)_{n+1}} \frac{n!}{(n+1)!} = \prod_{j=1}^{p} (a_j + n) \prod_{j=1}^{q} \frac{1}{b_j + n} \frac{1}{n+1}$$

$$= n^{p-q-1} \prod_{j=1}^{p} \left(\frac{a_j}{n} + 1\right) \prod_{j=1}^{q} \left(\frac{b_j}{n} + 1\right)^{-1} \frac{1}{1 + \frac{1}{n}}.$$

The last product terms converge to 1. The behavior of the limit is dictated by n^{p-q-1}. Indeed,

- If $p - q < 1$, the above ratio converges to 0, hence $R^{-1} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0 \implies R = \infty$.
- If $p - q = 1$, the above ratio converges to 1, hence $R = 1$.
- If $p - q > 1$, the above ratio converges to ∞, hence $R = 0$.

(ii) We differentiate term by term within the radius of convergence $R = 1$ to find

$$\frac{dw}{dz} = \sum_{n=1}^{\infty} \frac{(a_1)_n (a_2)_n}{(b)_n} \frac{z^{n-1}}{(n-1)!}$$

$$\frac{d^2w}{dz^2} = \sum_{n=2}^{\infty} \frac{(a_1)_n (a_2)_n}{(b)_n} \frac{z^{n-2}}{(n-2)!}.$$

The identity

$$z(1-z)\frac{d^2w}{dz^2} + (b - (1 + a_1 + a_2)z)\frac{dw}{dz} - a_1 a_2 w = 0$$

follows by direct computation, by explicitly computing the coefficient of z^n on the left hand side to be 0. We will leave the routine verification to the reader.

(iii) We have $(1)_n = n!$, $(2)_n = (n+1)!$. By direct compuation

$$ {}_2F_1(1,1;1;z) = \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(1)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}. $$

Similarly,

$$ {}_2F_1(1,2;1;z) = \sum_{n=0}^{\infty} \frac{(1)_n(2)_n}{(1)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} (n+1) z^n = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^{n+1}\right) = \frac{d}{dz}\left(\frac{z}{1-z}\right) $$
$$ = \frac{1}{(1-z)^2}. $$

Q3. We write $f(x,y) = u(x,y) + iv(x,y)$ where

$$ u(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \text{ and } v(x,y) = \begin{cases} \frac{x^2y^3}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}. $$

We claim

$$ u_x(0,0) = v_x(0,0) = u_y(0,0) = v_y(0,0) = 0. $$

For instance

$$ u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0. $$

The other derivatives are found in a similar fashion. Hence, the Cauchy-Riemann equations are trivially satisfied at $z = 0$.

However, let $z_t = ct^2 + it$. Then,

$$ \lim_{t\to 0} \frac{f(z_t) - f(0)}{z_t} = \lim_{t\to 0} \frac{c^2t^2t^2\left(ct^2+it\right)}{c^2t^4+t^4} \frac{1}{(ct^2+it)} = \frac{c^2}{c^2+1}. $$

Hence, the limit depends on the value of c which mean f is not differentiable at 0.

This does not contradict with the result proved in a class because partial derivatives of u and v are not continous at $z = 0$. For instance, direct calculation shows that

$$ u_x\left(t^2, t\right) \to \frac{1}{2} \neq u_x(0,0) = 0 $$

as $t \to 0$.

Q4. (i) By the quotient rule we find

$$ \frac{d}{dz}\left(\frac{\exp(\ell(z))}{z}\right) = \frac{\ell' z \exp(\ell(z)) - \exp(\ell(z))}{z^2} = 0. $$

It follows that $\exp(\ell(z)) = cz$ for some constant z. Note $c \neq 0$ since otherwise $\exp(\ell(z)) = 0$ which is impossible. Let $d \in \mathbb{C}$ be a constant such that

$$ \exp(d) = \frac{1}{c}. $$

Hence, we have

$$\exp\left(\ell\left(z\right)+d\right)=\exp(\ell(z))\cdot\exp(d)=cz\cdot\frac{1}{c}=z.$$

(ii) The radius of convergence of the power series is

$$R=\lim_{k\to\infty}\sqrt[k]{k}=1.$$

For $z\in\triangle_1\left(1\right),$ we can differentiate term by term within the radius of convergence

$$L^{'}\left(z\right)=\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k-1}}{k}k\left(z-1\right)^{k-1}=\sum_{k=0}^{\infty}\left(-1\right)^{k}\left(z-1\right)^{k}=\frac{1}{1+\left(z-1\right)}=\frac{1}{z}.$$

Note that the third equality is because the series is geometric. Also, $L\left(1\right)=0.$ Hence, by (i), $L\left(z\right)$ is a logartihm function on $\triangle_1\left(1\right).$

(iii) Note if $z\in\triangle_{|a|}\left(z\right)$ then $\left|\frac{z}{a}-1\right|<1$. Therefore, we are in the regime where part (ii) applies. We clearly have that $L\left(\frac{z}{a}\right)+b$ is continuous. Furthermore, we compute using (ii) that

$$\exp\left(L\left(\frac{z}{a}\right)+b\right)=\exp\left(L\left(\frac{z}{a}\right)\right)\cdot\exp(b)=\frac{z}{a}\cdot a=z.$$

This proves that $L\left(\frac{z}{a}\right)+b$ is a logarithm function in $\Delta_{|a|}(a)$.

(iv) Let ℓ_1,ℓ_2 be two logarithm functions. Then

$$e^{\ell_1(z)-\ell_2(z)}=1$$

for $z\in U$. Therefore, for any $z\in U$, there exists $n_z\in\mathbb{Z}$ such that

$$\ell_1\left(z\right)-\ell_2\left(z\right)=2n_z i\pi.$$

However, as $\frac{1}{2\pi}\left(\ell_1\left(z\right)-\ell_2\left(z\right)\right)$ is a continuous function on U which is integer valued, it must be locally constant, hence constant since U is connected. Therefore,

$$\ell_1-\ell_2=2\pi in$$

for some $n\in\mathbb{Z}$.

(v) Let $U\subseteq\mathbb{C}-\{0\}$ be a connected open set. Suppose $a\in U$ and pick $\epsilon<1$ sufficiently small so that $\epsilon\triangle_{|a|}\left(a\right)\subseteq U$. By (iii) and (iv), there exists $n\in\mathbb{Z}$ such that

$$\ell\left(z\right)=L\left(\frac{z}{a}\right)+b_a+2\pi in\text{ for }z\in\epsilon\triangle_{|a|}\left(a\right).$$

It is clear from this expression that ℓ is complex differentiable at a. As a is arbitrary in U, ℓ is differentiable everywhere on U.

Q5. Let f be complex differentiable. For simplicity, we write ∂ and $\bar{\partial}$ for the two derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$. The Cauchy Riemann equations give

$$\bar{\partial} f = 0.$$

Taking complex conjugates, we have

$$\partial \bar{f} = 0.$$

We compute

$$\partial\bar{\partial}|f|^2 = \partial\bar{\partial}(f \cdot \bar{f}) = \partial(f \cdot \partial\bar{f} + f \cdot \bar{\partial}\bar{f}) = \partial(f \cdot \bar{\partial}\bar{f}).$$

Here, we used the product rule and the fact that $\partial\bar{f} = 0$ in the last step. Applying the product rule again, we find

$$\partial\bar{\partial}|f|^2 = \partial f \cdot \bar{\partial}\bar{f} + f \cdot \partial\bar{\partial}\bar{f} = \partial f \cdot \overline{\partial f} + \bar{\partial}\partial\bar{f} = |\partial f|^2 + 0 = |\partial f|^2 = |f'(z)|^2.$$

For the second equality, we used that ∂ and $\bar{\partial}$ commute, and for the next equality, we used that $\partial\bar{f} = 0$ one more time.

Assuming $f_1, \ldots, f_m$ are complex differentiable such that

$$|f_1(z)|^2 + \ldots + |f_m(z)|^2 = 1,$$

we obtain

$$\partial\bar{\partial}\left(|f_1(z)|^2 + \ldots + |f_m(z)|^2\right) = 0 \implies |f_1'(z)|^2 + \ldots + |f_m'(z)|^2 = 0$$
$$\implies f_k'(z) = 0 \implies f_k = \text{constant}$$

for all $1 \leq k \leq m$.