Math 220, Problem Set 4.

You should have the knowledge to solve Problems 1-5 on Friday, January 29. Some questions may require the statement of Montel's theorem. Problem 6 requires material from Monday, February 1.

1. Let f be an entire function and let $\mathcal{F}$ denote the family of functions $f(kz)$ for $k \in \mathbb{Z}_{>0}$ defined in the annulus $r_1 < |z| < r_2$ for $r_1, r_2 > 0$. Show that $\mathcal{F}$ is normal iff f is constant.

2. Let $\mathcal{F}$ be the family of holomorphic functions in $\Delta(0,1)$ with $f(0) = 1$ and $\mathrm{Re}\, f > 0$. Show that $\mathcal{F}$ is normal.

Hint: You may wish to remember the Cayley transform from Math 220A.

3. (*Vitali's theorem.*) Prove Vitali's theorem, Conway VII.2.4, page 154. The statement is as follows.

Let $\{f_n\}$ be a locally bounded sequence of holomorphic functions in $U \subset \mathbb{C}$, and let f be holomorphic in U. If

$$A = \{z \in U : f_n(z) \to f(z)\}$$

has a limit point in U, then f_n converges locally uniformly to f in U.

4. (*Normal families under composition.*) Solve Conway VII.2.7, page 154.

Let $U, \Omega \subset \mathbb{C}$ be open connected. Let $g : \Omega \to \mathbb{C}$ be a holomorphic function which is bounded on bounded sets. Let $\mathcal{F}$ be a normal family of holomorphic functions $f : U \to \Omega$. Show that the family $\{g \circ f : f \in \mathcal{F}\}$ is also normal.

5. Solve the following version of Conway VII.2.8, page 154. The statement is as follows.

Let $\mathcal{F}$ be a family of holomorphic functions in $\Delta(0,1)$. Assume there exist $M_n > 0$ constants with

$$\limsup M_n^{\frac{1}{n}} \leq 1$$

such that for all $f \in \mathcal{F}$,

$$\frac{|f^{(n)}(0)|}{n!} \leq M_n.$$

Show that $\mathcal{F}$ is normal. Show that the converse is also true.

6. (*Optional, do* **not** *hand in.*) Let $\mathcal{F}$ be a family of continuous functions $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is open. Show that the following are equivalent:

(i) $\mathcal{F}$ is equicontinuous on all compacts subsets of U
(ii) $\mathcal{F}$ is locally equicontinuous on U
(iii) $\mathcal{F}$ is equicontinuous at all points $z \in U$, in the sense of definition VII.1.21, page 148.

The harder implication (iii) $\implies$ (i) is proven in Conway, Chapter VII.1.