Math 220B — Lecture 16

February 10, 2021

0. Midterm Exam

(1) 5 Questions

- Infinite Products, Γ function, sine
- Weierstraß factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz lemma & applications

(2) Available on Friday at noon, due Tuesday at noon.

You can think about the Questions for as long as you wish in this interval.

(3) Closed book / closed notes / no internet / no collaboration

(4) e-mail if questions arise

(5) you may use theorems proved in lecture but no homework problems can be used without proof.

(6) Office hour 4 - 5:30 today.

1. Last time

- if $f(0) = 0$ then
  - we proved Schwarz Lemma
  - we determined $f \in \operatorname{Aut} \Delta$, $f(0) = 0$
- if $f(0) \neq 0$
  - we determined $f \in \operatorname{Aut} \Delta$

Idea: Use φ_a to recenter f so that 0 maps to 0.

Question: Is there a version of Schwarz if $f(0) \neq 0$?

Yes — Schwarz-Pick Lemma.

— we illustrate it for derivatives

Schwarz - Pick $f: \Delta \longrightarrow \Delta$ holomorphic, $\forall\, a \in \Delta = \Delta(0,1)$.

$$\frac{|f'(a)|}{1-|f(a)|^2} \leq \frac{1}{1-|a|^2}.$$

Remark $f(0)=0$, $a=0$ recovers Schwarz $|f'(0)| \leq 1$.

Example Conway VI. 2. 3 $f: \Delta \longrightarrow \Delta$ holomorphic.

If $f\left(\frac{1}{2}\right) = \frac{1}{4}$, find the maximum value of $\left|f'\left(\frac{1}{2}\right)\right|$.

Proof We know this when $a = 0$ & $\alpha = f(a) = 0$.

We use $\mathrm{Aut}(\Delta)$ to reduce to this case

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, shift right, "\varphi_a"'] & \alpha \arrow[d, "\varphi_\alpha"] \\ 0 \arrow[r, "\tilde{f}"'] \arrow[u, shift right, "\varphi_{-a}"'] & 0 \end{tikzcd}$$

Let $f(a) = \alpha$. Let

$$\tilde{f} = \varphi_\alpha \circ f \circ \varphi_{-a} \Rightarrow \tilde{f}(0) = 0$$

as the diagram shows.

By Schwarz, $|\tilde{f}'(0)| \leq 1$. We compute using the chain rule

$$\tilde{f}'(0) = \varphi_\alpha'(f(\varphi_{-a}(0))) \cdot f'(\varphi_{-a}(0)) \cdot \varphi_{-a}'(0)$$

$$= \varphi_\alpha'(\alpha) \cdot f'(a) \cdot \varphi_{-a}'(0)$$

$$= \frac{1}{1-|\alpha|^2} \cdot f'(a) \cdot (1-|a|^2)$$

& $|\tilde{f}'(0)| \leq 1$ gives

$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}$$

as needed.

Remark

Schwarz $f(0)=0$		Schwarz – Pick
$\lvert f'(0)\rvert \leq 1$	$\Longleftarrow$	$\lvert f'(a)\rvert \leq \dfrac{1-\lvert f(a)\rvert^2}{1-\lvert a\rvert^2}$
$\lvert f(z)\rvert \leq \lvert z\rvert$		?

Define $d(z,w) = \left| \dfrac{z-w}{1-\bar{z}w} \right|$ = pseudo hyperbolic distance

Schwarz – Pick Holomorphic maps decrease pseudo hyperbolic distance.

This will be made precise in HWK 5.

2. Further applications of Schwarz

We can use Schwarz to study other domains e.g.

[i] $\mathcal{U} = \Delta^\times = \Delta(0,1) \setminus \{0\}$

[ii] $\mathcal{U} = \mathfrak{H}^+ =$ upper half plane

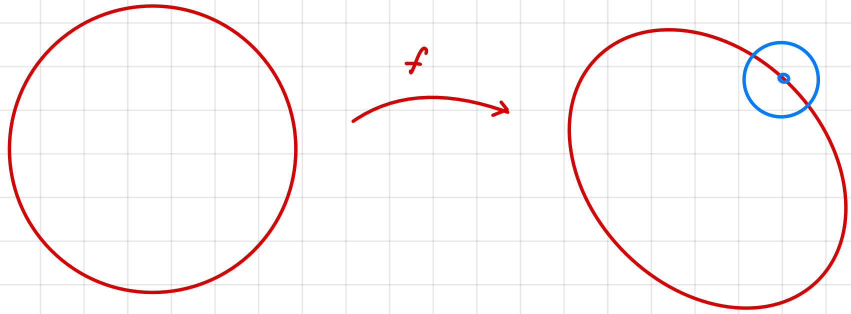
[i] Example: All automorphisms of $\Delta^\times$ are rotations.

Proof: Let $f: \Delta^\times \longrightarrow \Delta^\times$. Since $\operatorname{Im} f$ is bounded $\Rightarrow$

$\Rightarrow f$ can be extended across 0 by the removable singularity theorem. The extension $\tilde{f} : \Delta \longrightarrow \bar{\Delta}$ is holomorphic.

Its image $\operatorname{Im} \tilde{f} \subseteq \Delta$ by the open mapping theorem

(draw picture)



We claim $\tilde{f}(0)=0$. Then $f: \Delta^\times \longrightarrow \Delta^\times$ shows $\tilde{f}$ bijective from $\Delta \longrightarrow \Delta$ hence a biholomorphism preserving 0. Then $\tilde{f}$ is a rotation.

To show $\tilde{f}(0)=0$ assume otherwise $\tilde{f}(0)=\alpha \neq 0$.

Since $\alpha \in \Delta^\times$ we can find $a \in \Delta^\times$: $f(a)=\alpha$.

By the open mapping theorem, we can find small discs $\Delta_0, \Delta_a, \Delta_\alpha$ near $0, a, \alpha$ with $\Delta_0 \cap \Delta_a = \phi$ and.

$$\Delta_\alpha \subseteq \tilde{f}(\Delta_0), \quad \Delta_\alpha \subseteq f(\Delta_a). \quad (\text{why?}).$$

Let $b \in \Delta_\alpha \setminus \{\alpha\} \Rightarrow b \in \tilde{f}(\Delta_0) \Rightarrow b = f(u),\ u \neq 0, u \in \Delta_0$

$$\Rightarrow b \in f(\Delta_a) \Rightarrow b = f(v), \quad v \in \Delta_a$$

$\Rightarrow f(u) = f(v) = b$

$u \neq v$ since $\Delta_0 \cap \Delta_a = \phi$

$\Rightarrow f$ not injective (contradiction).

[II] Upper half plane

Key idea Use $\mathfrak{H}^+ \xrightarrow{c} \Delta$, $c(z) = \frac{z-i}{z+i}$

$$c^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

Questions we can answer:

[I] Aut $(\mathfrak{H}^+)$ next time

Schwarz Lemma for $f : \mathfrak{H}^+ \longrightarrow \mathfrak{H}^+$

Schwarz-Pick for $f : \mathfrak{H}^+ \longrightarrow \mathfrak{H}^+$.

[II] Biholomorphisms $\Delta \longrightarrow \mathfrak{H}^+$

Schwarz Lemma for $f : \Delta \longrightarrow \mathfrak{H}^+$

Schwarz-Pick for $f : \mathfrak{H}^+ \longrightarrow \Delta$

for derivatives or for distance ...

It is impossible to record them all.

Example $f : \Delta \to \mathfrak{H}^+$, $f(0) = i$. Show

$$|f'(0)| \leq 2.$$

Let $\tilde{f} = c \circ f$. Then $\tilde{f}(0) = 0$ since $c(i) = \left.\frac{z-i}{z+i}\right|_{z=i} = 0$

$\Rightarrow |\tilde{f}'(0)| \leq 1$ by Schwarz. We compute

$$|\tilde{f}'(0)| = |c'(f(0)) \cdot f'(0)| = |c'(i) \cdot f'(0)| \leq 1.$$

Since $c'(i) = \frac{1}{2i}$ $\Rightarrow |f'(0)| \leq 2.$

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3. Further discussion of Aut. — Loose ends

$\boxed{i}$ Aut $\mathbb{C}$

$\boxed{ii}$ Aut $\hat{\mathbb{C}}$

$\boxed{iii}$ Aut Δ

$\boxed{iv}$ Aut $\mathfrak{H}^+$

$\boxed{v}$ Aut $\Delta^\times$

next time