Math 220 B — Lecture 17

February 17, 2021

1. Further discussion of Aut. — Loose ends

[i] Aut $\mathbb{C}$ $= \{az+b : a \neq 0,\ b \in \mathbb{C}\} \cong$ Aff.

[ii] Aut $\hat{\mathbb{C}}$ $= PGL_2$

[iii] Aut Δ $\cong SU(1,1)/\pm 1 = PSU(1,1)$

[iv] Aut $\mathfrak{H}^+$ $= SL(2,\mathbb{R})/\pm 1. = PSL(2,\mathbb{R})$

[v] Aut $\Delta^\times$ $\cong$ Rotations

Case [i] $\quad \mathcal{U} = \mathbb{C}$

7. Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire and injective. Show that $f(z) = az + b$. You can solve this problem using the notions introduced in Problem 6 above.

Math 220A, Homework 5.

Case [iii] $\mathcal{U} = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Mobius transforms - Math 220A, Lecture 3.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies h_M(z) = \frac{az+b}{cz+d}, \quad h_M : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

$$h_M = h_N \iff M = \lambda N.$$

$$h_M \, h_N = h_{MN}.$$

h_M bijective $\iff$ M invertible since $h_M \circ h_{M^{-1}} = \mathbb{I}$

Define $PGL_2 = GL_2 / \{\lambda \cdot \mathbb{I}, \lambda \neq 0\}$ = invertible 2×2 matrices up to scaling.

Recall from Math 220 A, Lecture 3, the action of Mobius transforms is transitively on $\hat{\mathbb{C}}$.

Theorem $\operatorname{Aut} \widehat{\mathbb{C}} = PGL_2.$

Proof If $f \in \operatorname{Aut} \widehat{\mathbb{C}}$, $f(\infty) = \infty$ then $f : \mathbb{C} \longrightarrow \mathbb{C}$ is bijective. Thus $f(z) = az + b = f_M$ for the matrix

$$M = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

If $f(\infty) \neq \infty$ then $f(\infty) = \lambda \in \mathbb{C}$. Let

$$g(z) = \frac{1}{f(z) - \lambda} \Rightarrow g(\infty) = \infty \Rightarrow g(z) = az + b$$

$$\Rightarrow f(z) = \lambda + \frac{1}{az+b} = \text{fractional linear transformation,}$$

as needed.

Case $\boxed{iii}$ $\quad$ $\mathrm{Aut}(\Delta)$.

Question What is $\mathrm{Aut}(\Delta)$ as an abstract group?

$f \in \mathrm{Aut}\,\Delta$

$$f(z) = e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z} = \frac{e^{i\theta/2}}{e^{-i\theta/2}} \cdot \frac{z-a}{1-\bar{a}z} = h_M.$$

$$M = \begin{bmatrix} e^{i\theta/2} & -a\,e^{i\theta/2} \\ -\bar{a}\,e^{-i\theta/2} & e^{-i\theta/2} \end{bmatrix} = \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} \text{ invertible.}$$

Note $\det M = 1 - |a|^2 > 0$. Let $\lambda = (1-|a|^2)^{-1/2}$.

Rescale $A \longrightarrow \lambda A$, $\lambda \in \mathbb{R}$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \Longrightarrow \quad A\bar{A} - B\bar{B} = |A|^2 - |B|^2 = 1.$

$\quad\quad\quad\, B \longrightarrow \lambda B$, $\lambda \in \mathbb{R}$.

Conclusion $$\mathrm{Aut}\,\Delta = \left\{ \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix} : |A|^2 - |B|^2 = 1 \right\} \Big/ \pm 1$$

$$= SU(1,1)/\pm 1 = PSU(1,1).$$

Case $\boxed{\mathrm{IV}}$ Aut $\mathfrak{H}^+$

Key idea Use Cayley transform:

$$
\begin{tikzcd}
\mathfrak{H}^+ \arrow[r, shift left, "c"] & \Delta \arrow[l, shift left, "c^{-1}"]
\end{tikzcd}
\qquad
c(z) = \frac{z-i}{z+i}
$$

$$
c^{-1}(z) = i \cdot \frac{1+z}{1-z} .
$$

$$
\begin{tikzcd}
\mathfrak{H}^+ \arrow[r, "c", "\sim"'] \arrow[rrr, bend right, "g"', "\sim"] & \Delta \arrow[r, "f", "\sim"'] & \Delta \arrow[r, "c^{-1}", "\sim"'] & \mathfrak{H}^+
\end{tikzcd}
$$

$g = c^{-1} \circ f \circ c$ is an automorphism

Any $g \in \operatorname{Aut} \mathfrak{H}^+$ is of this form for $f = c\, g\, c^{-1}$.

Compute $C^{-1}\underbrace{\begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix}}_{\text{Aut}\,\Delta} C = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$\alpha = \mathrm{Re}\, A + \mathrm{Re}\, B \qquad \delta = \mathrm{Re}\, A - \mathrm{Re}\, B$

$\beta = \mathrm{Im}\, A - \mathrm{Im}\, B \qquad \gamma = -\mathrm{Im}\, A - \mathrm{Im}\, B$

$\Longrightarrow \alpha, \beta, \gamma, \delta \in \mathbb{R}.$

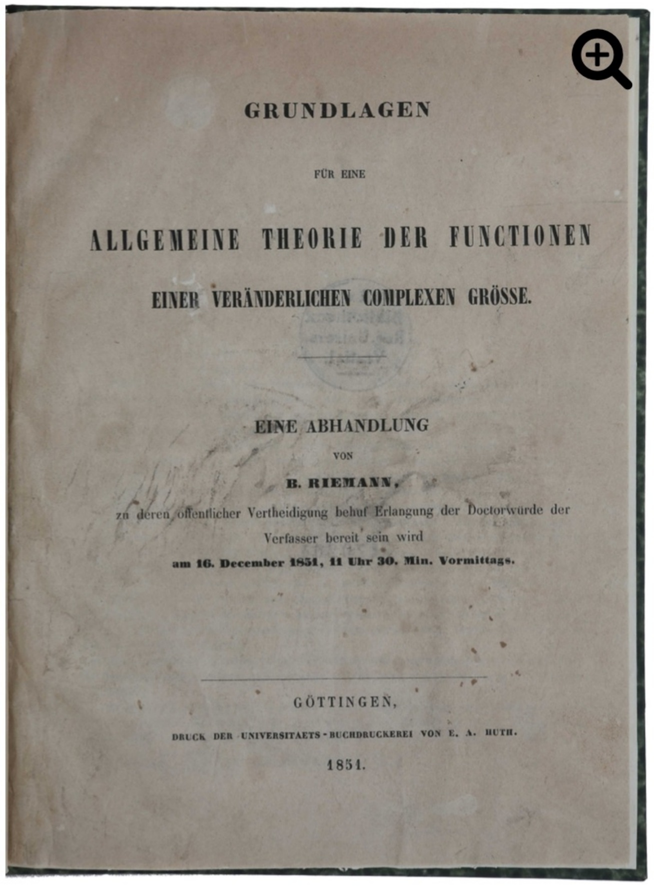
$|A|^2 - |B|^2 = 1 \iff (\mathrm{Re}\, A)^2 + (\mathrm{Im}\, A)^2 - (\mathrm{Re}\, B)^2 - (\mathrm{Im}\, B)^2 = 1.$

$\iff \alpha\delta - \beta\gamma = 1.$

$\Rightarrow \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2, \mathbb{R}).$

Conclusion $\quad \mathrm{Aut}(\mathfrak{h}^+) \cong SL(2,\mathbb{R}) / \{\pm 1\} = PSL(2, \mathbb{R}).$

II Riemann Mapping theorem



GRUNDLAGEN

FÜR EINE

ALLGEMEINE THEORIE DER FUNCTIONEN

EINER VERÄNDERLICHEN COMPLEXEN GRÖSSE.

EINE ABHANDLUNG

VON

B. RIEMANN,

zu deren öffentlicher Vertheidigung behuf Erlangung der Doctorwürde der Verfasser bereit sein wird

am 16. December 1851, 11 Uhr 30. Min. Vormittags.

GÖTTINGEN,

DRUCK DER UNIVERSITAETS-BUCHDRUCKEREI VON E. A. HUTH.

1851.

"Two given simply connected planar surfaces can always be related to each other in such a way that every point of one corresponds to one point of another, which varies continuously with it, and their corresponding smaller parts are similar."

Theorem $\mathcal{U} \neq \mathbb{C}$ simply connected $\Rightarrow \mathcal{U}$ biholomorphic to the unit disc. $\Delta = \Delta(0,1)$.

Remarks $\boxed{I}$ $\mathcal{U} = \mathbb{C}$ is not biholomorphic to Δ.

By Liouville, there cannot exist a holomorphic nonconstant map $\mathbb{C} \longrightarrow \Delta$.

$\boxed{II}$ Implications in topology

$\mathcal{U}$ simply connected, $\mathcal{U} \subseteq \mathbb{C}$. $\Rightarrow \mathcal{U}$ is topologically Δ i.e.

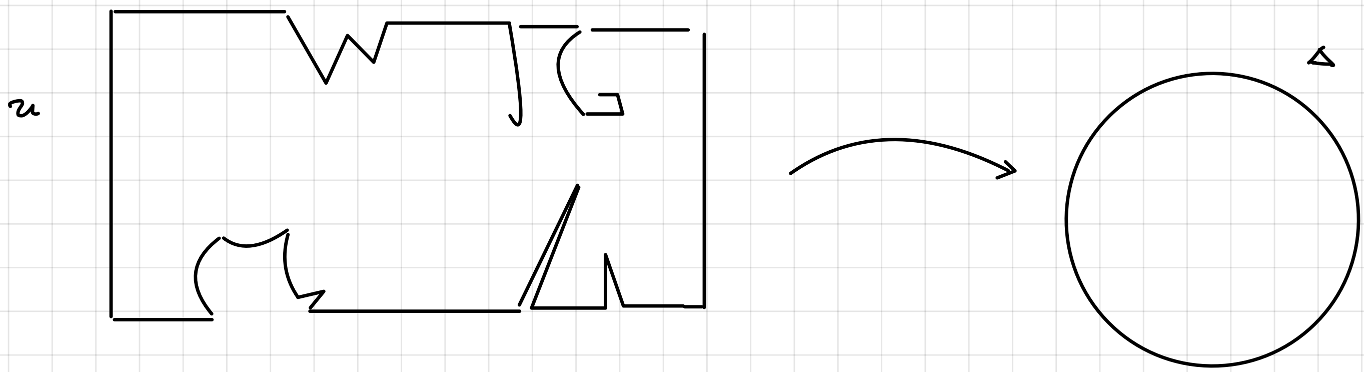
$\exists$ bicontinuous map $\mathcal{U} \longrightarrow \Delta$ (homeomorphism).

This holds even for $\mathcal{U} = \mathbb{C}$ using the map:

not holomorphic.

$$\mathbb{C} \longrightarrow \Delta, \quad z \longrightarrow \frac{z}{\sqrt{1+|z|^2}}$$

Why is the proof difficult? Imagine the domain



It is hard to construct explicit maps (even in the topological category).

Examples

[i] $c : \mathfrak{H}^+ \longrightarrow \Delta$, $\quad c(z) = \dfrac{z-i}{z+i}$.

[ii] biholomorphism between Δ and the slit plane

$\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ (both simply connected).

$\mathbb{C}^-$

We use simple geometric moves:

$$\Delta \longrightarrow \mathfrak{H}^+ \quad \text{via} \quad c^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

$$\mathfrak{H}^+ \longrightarrow \mathbb{C} \setminus \mathbb{R}_{\geq 0} \quad \text{via} \quad w \longrightarrow w^2.$$

$$\mathbb{C} \setminus \mathbb{R}_{\geq 0} \longrightarrow \mathbb{C} \setminus \mathbb{R}_{\leq 0} = \mathbb{C}^- \quad \text{via} \quad s \longrightarrow -s.$$

Composition: $-\left(i \cdot \frac{1+z}{1-z}\right)^2 = \left(\frac{1+z}{1-z}\right)^2 : \Delta \longrightarrow \mathbb{C}^-.$