Math 220A - Fall 2020 - Final

Name: ____________________________________

Student ID: ________________________________

Instructions:

Please print your name and student ID.

You have 180 minutes to complete the test. There is a 15 minute buffer period (6:00-6:15 PST) to upload your answers in Gradescope.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		13
6		10
7		12
Total		75

Problem 1. [*10 points.*]

Consider the function $f(z) = z^2e^{-z} - 4z + 1$. Find the number of zeroes of f inside the disc $\Delta(0, 1)$.

Problem 2. [*10 points.*]

Let $f : \Delta(0,1) \to \mathbb{C}$ be holomorphic and nonconstant, and define $M(r) = \max_{|z|=r} \mathrm{Re} f(z)$ for $0 \leq r < 1$. Show that $M : [0,1) \to \mathbb{R}$ is strictly increasing.

Problem 3. [*10 points.*]

Are there any holomorphic functions $f : \{z : |z| > 4\} \to \mathbb{C}$ such that

$$f'(z) = \frac{z^3 + 2}{z(z-1)(z-3)(2z-7)}?$$

Problem 4. [*10 points.*]

Assume that f is an entire function such that the sequence of derivatives $f, f', f'', \ldots f^{(n)}, \ldots$ converges locally uniformly to a function g with $g(0) = 1$.

Show that there exists N such that the derivatives $f^{(n)}(z) \neq 0$ for all $n \geq N$ and $|z| < 1$.

Hint: You may wish to determine g explicitly.

Problem 5. [*13 points; 5, 3, 5.*]

Let $R(z) = \frac{P(z)}{Q(z)}$ be a rational function such that $\deg P + 2 \leq \deg Q$. Assume that Q has simple zeros at $a_1, \ldots, a_q$, where $a_j \in \mathbb{C} \setminus \mathbb{Z}$.

Show that

$$\sum_{m=-\infty}^{\infty} R(m) = -\pi \sum_{j=1}^{q} \frac{P(a_j)}{Q'(a_j)} \cdot \cot \pi a_j.$$

(i) Let γ_n be the square with corners

$$\pm \left(n + \frac{1}{2}\right) \pm i \left(n + \frac{1}{2}\right).$$

Show that there exist constants $M_1, M_2 > 0$ such that if n is sufficiently large, and z is on the curve γ_n, we have

$$|\pi \cot \pi z| \leq M_1$$

and

$$|R(z)| \leq \frac{M_2}{|z|^2}.$$

(ii) Show that

$$\lim_{n\to\infty} \int_{\gamma_n} R(z)\pi \cot \pi z \, dz = 0.$$

(iii) Show that $\pi \cot \pi z$ has poles at all integers $m \in \mathbb{Z}$ with residue equal to 1. Next, find the poles and residues of $R(z)\pi \cot \pi z$. Conclude the argument.

Problem 6. [*10 points.*]

Let f be a meromorphic function in $\mathbb{C}$. Let $U = \{z \in \mathbb{C} : |z| > 1 \text{ and } z \text{ is not a pole of } f\}$. Assume that for all $z \in U$, we have

$$|f(z)| \leq 1 + |z|.$$

Show that f is a rational function.

Problem 7. [*12 points; 4, 4, 4.*]

Let $U \subset \mathbb{C}$ be an open set containing 0. Let $f : U \to \mathbb{C}$ be an injective holomorphic function.

Show that $f'(0) \neq 0$.

(i) Show that there exists an integer $m > 0$, a disc around the origin $\Delta \subset U$, and a holomorphic function $g : \Delta \to \mathbb{C}$ such that

$$f(z) = f(0) + z^m g(z), \quad g(z) \neq 0 \text{ for all } z \in \Delta.$$

(ii) Show that there exists a holomorphic function $h : \Delta \to \mathbb{C}$ such that

$$f(z) = f(0) + h(z)^m, \quad h(0) = 0, \quad h'(0) \neq 0.$$

(iii) Show that if f is injective then $m = 1$. Conclude that $f'(0) \neq 0$.