Math 220B – Lecture 20

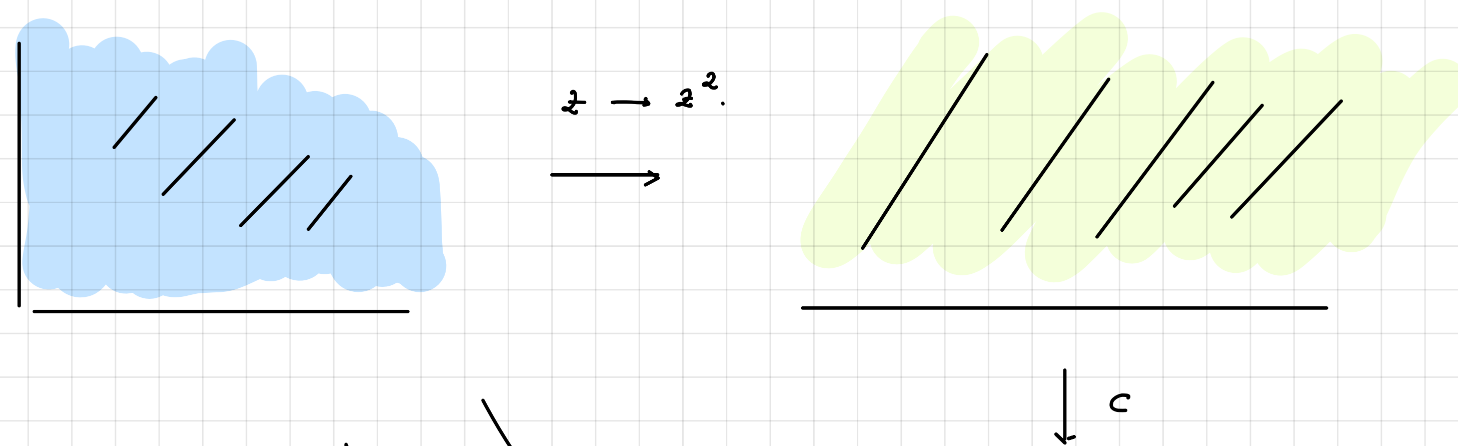
February 24, 2021

Extra hints added to 1 ii. & 1 iv.

Office Hour : 4 – 5:30 today

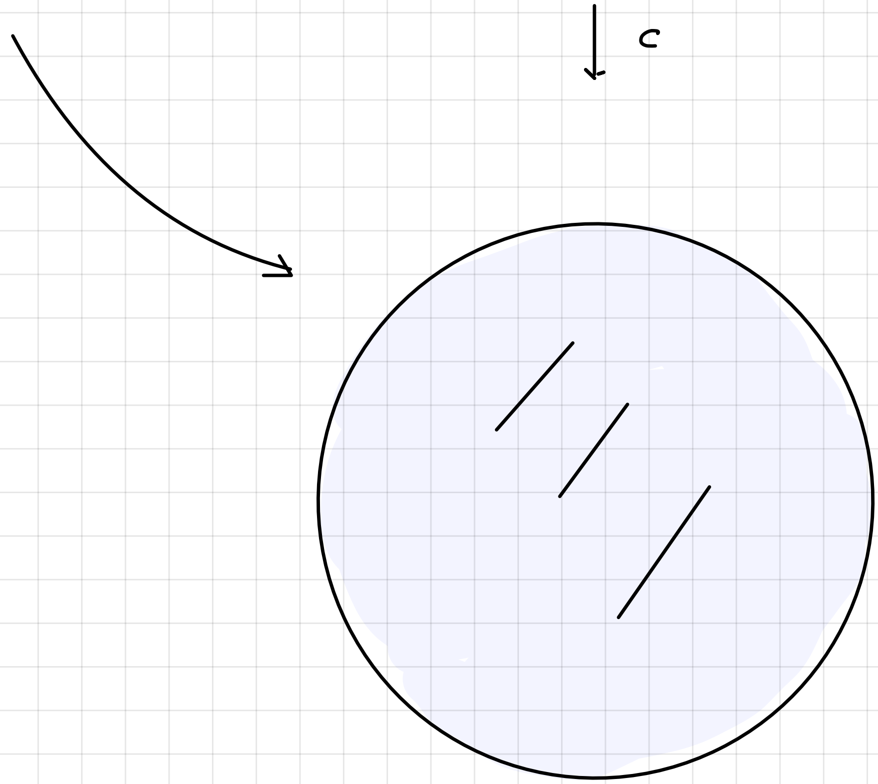
1. More examples of biholomorphisms

Example $\boxed{i}$ Squaring in $\mathfrak{H}^+$ "Half $\mathfrak{H}^+ \longrightarrow \mathfrak{H}^+$"



$$c(z) = \frac{z - i}{z + i}$$

$$c : \mathfrak{H}^+ \longrightarrow \Delta.$$



Example $\boxed{1c}$ $\Delta^+ =$ upper half disc (open)

Question Find $\underset{0}{\underset{\Psi}{\Delta^+}} \xrightarrow{\sim} \underset{0}{\underset{\Psi}{\Delta}}$. $\qquad z \to z^2$

Answer Not done by squaring since $0 \in \Delta$, $0 \notin \Delta^+$.

Instead We use the Cayley transform & work in $\mathfrak{h}^+$.

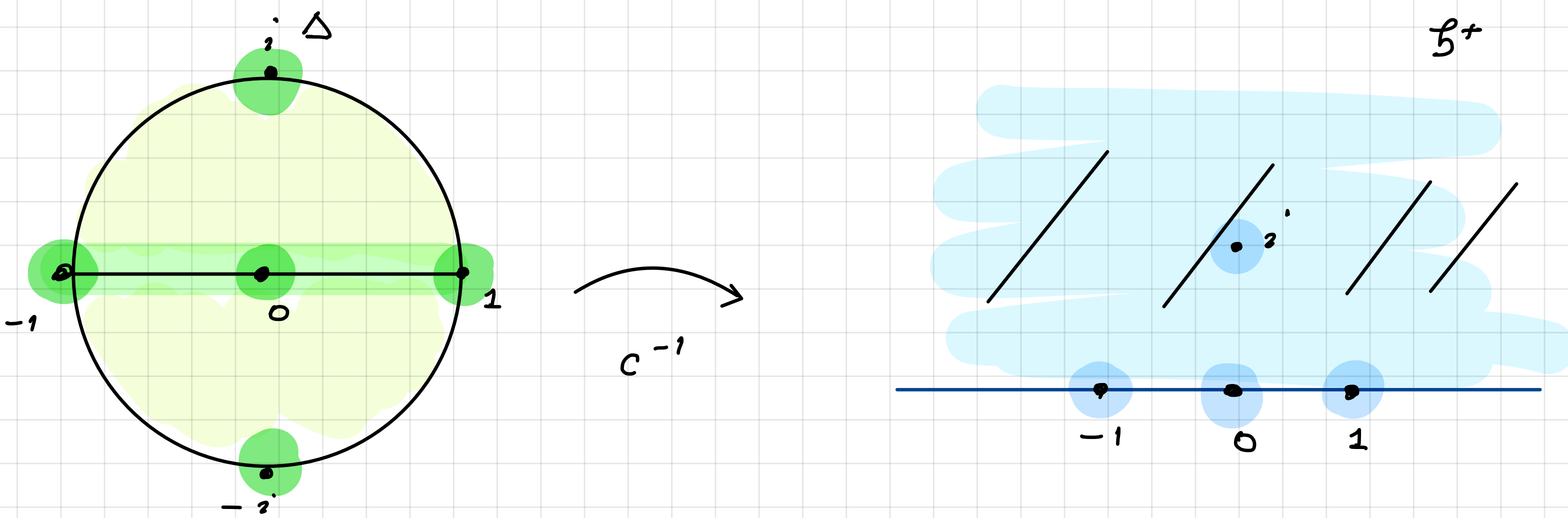
Idea

$$
\begin{tikzcd}
\Delta \arrow[r, "c^{-1}"] & \mathfrak{h}^+ \\
\underbrace{\Delta^+}_{\text{half disc}} \arrow[u, bend left] \arrow[r, "c^{-1}"] & \text{"half } \mathfrak{h}^+\text{"} \arrow[u, bend right, "\text{square}"']
\end{tikzcd}
$$

the map we want

Concretely Consider $c : \mathfrak{h}^+ \to \Delta$. the Cayley transform

$c^{-1} : \Delta \to \mathfrak{h}^+$, $\quad c^{-1}(z) = i \cdot \dfrac{1+z}{1-z}$



Check Under c^{-1}, we map

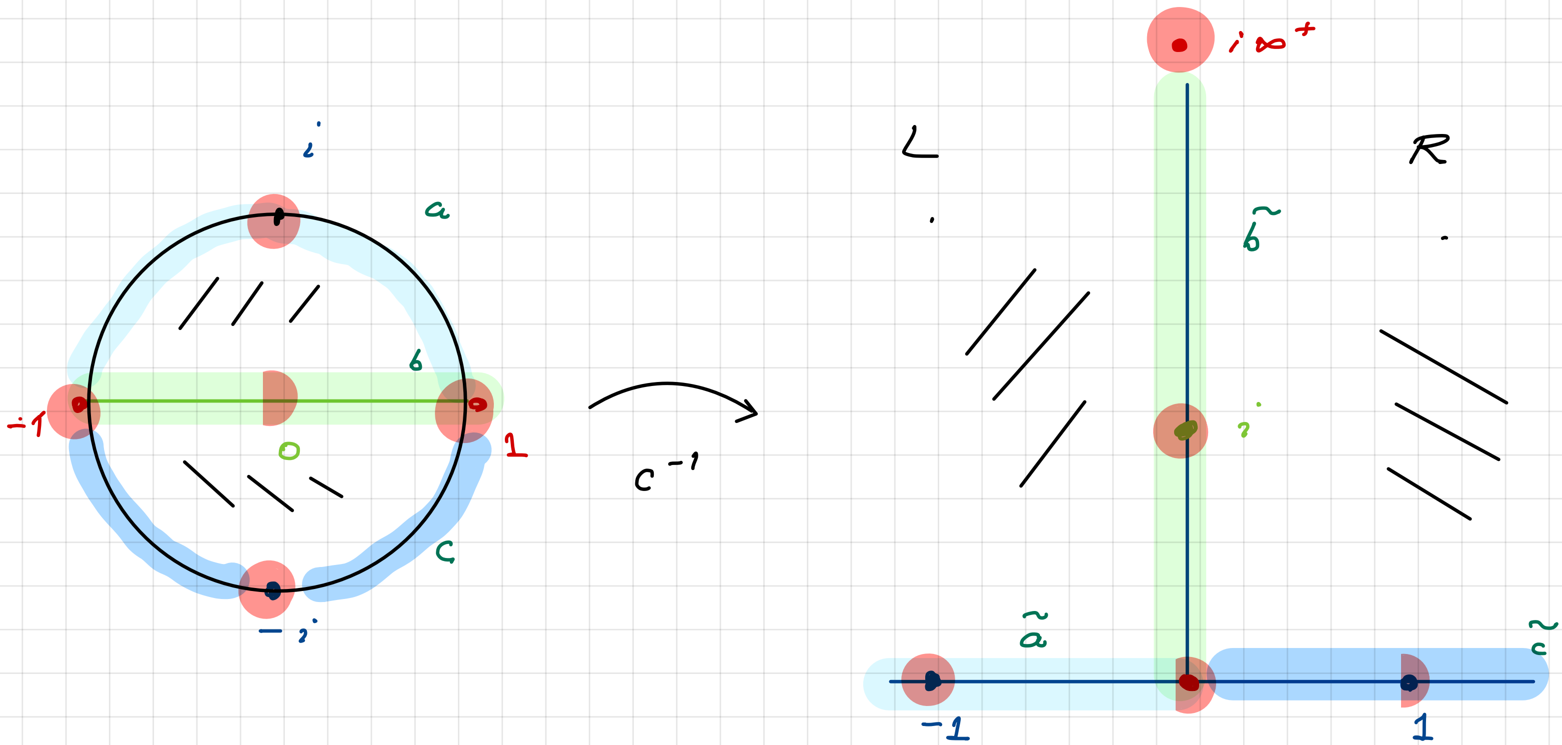
$-1 \longrightarrow 0$

$1 \longrightarrow \infty$

$0 \longrightarrow i$

$i \longrightarrow -1$

$-i \longrightarrow +1.$



Conclusions

[i] diameter b $\longrightarrow$ imaginary axis $\tilde{b}$

arc a $\longrightarrow$ negative real axis $\tilde{a}$

arc c $\longrightarrow$ positive real axis $\tilde{c}$

[ii] $\Delta^+ \xrightarrow{\sim} L$ = 2nd quadrant (left)

[iii] $\Delta^- \xrightarrow{\sim} R$ = 1st quadrant (right)

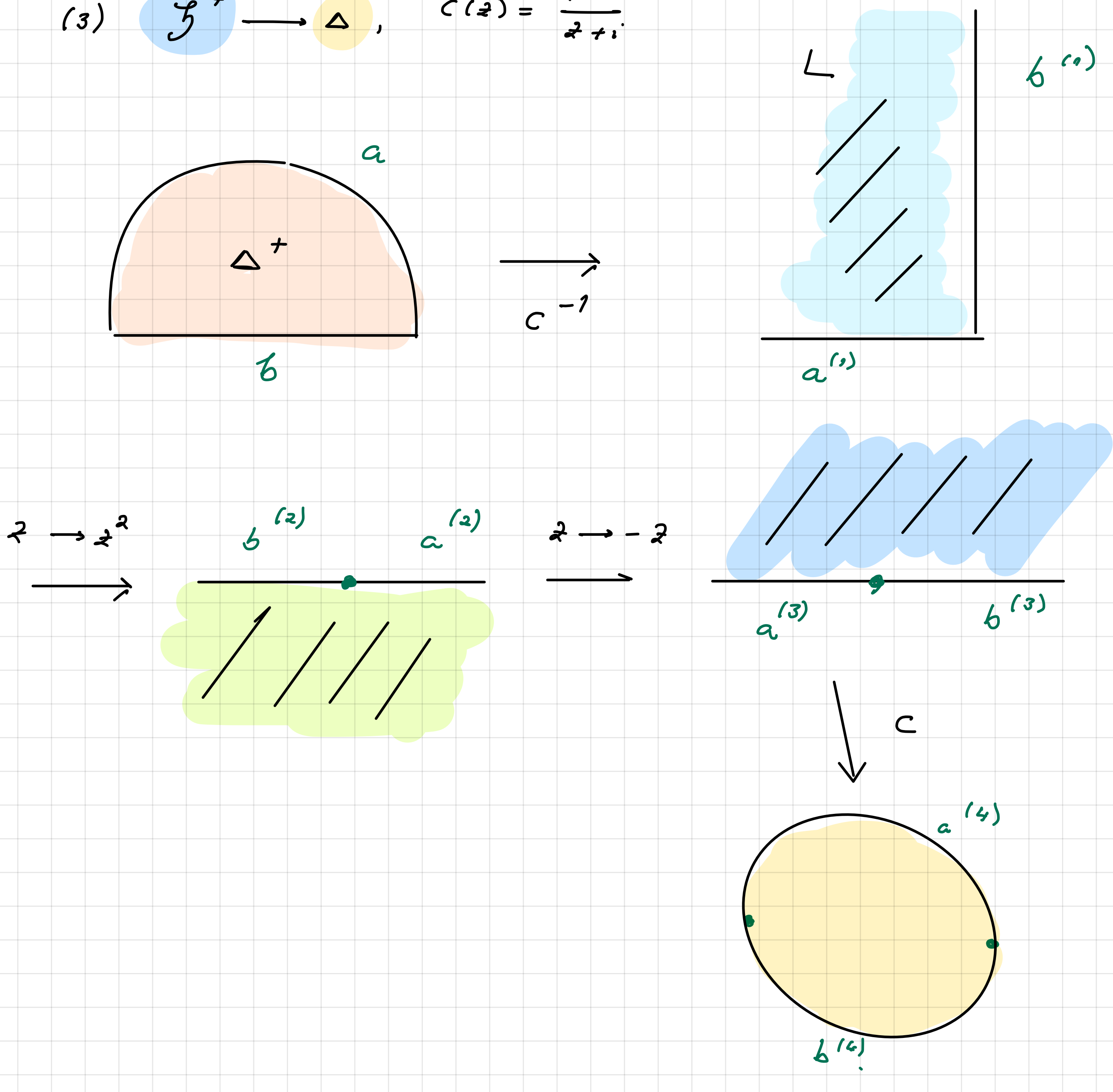
Construction of biholomorphism $\Delta^+ \longrightarrow \Delta$

as a composition of three moves:

(1) $\Delta^+ \xrightarrow{C^{-1}} L$, $\quad z \longrightarrow i \cdot \dfrac{1+z}{1-z}$.

(2) $L \longrightarrow \mathfrak{H}^+$ $\quad z \longrightarrow -z^2$.

(3) $\mathfrak{H}^+ \xrightarrow{C} \Delta$, $\quad C(z) = \dfrac{z-i}{z+i}$



Conclusion

The biholomorphism $\Delta^+ \longrightarrow \Delta$ extends to $\partial\Delta^+ \longrightarrow \partial\Delta$ continuously & bijectively.

(the upper arc a is sent to the upper arc & the diameter b is sent to the lower arc).

2. Extension to the boundary

Question Given $f: U \longrightarrow \Delta$ biholomorphism, does it extend $\bar{f}: \bar{U} \longrightarrow \bar{\Delta}$ bicontinuously?

Answer [i] yes if U bounded & ∂U = simple closed curve.

Carathéodory's theorem

[ii] We will not give the proof in this course.

3. Beyond the boundary

Question Can we extend beyond the boundary?

The easiest instance is provided by

Schwarz Reflection Principle Conway IX. 1.

There are several versions but two stand out:

[i] reflection across line segments (book)

[ii] reflection across circular arcs (HWK6).

Applications

[i] biholomorphic maps between rectangles, annuli

[ii] analytic continuation ...

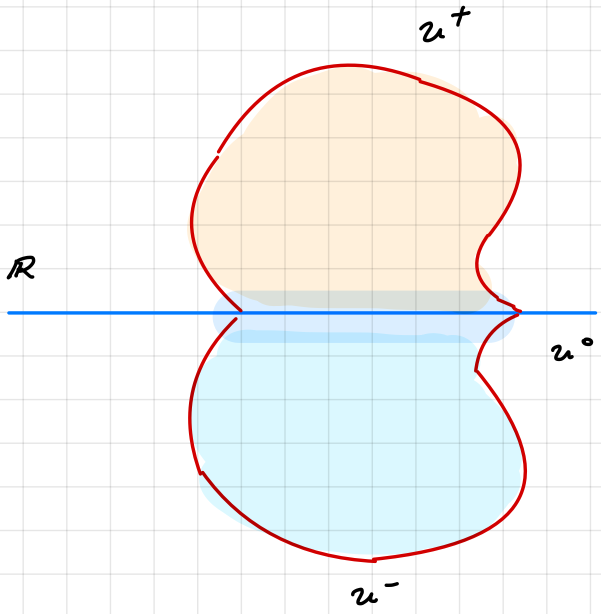
4. Reflection across segments

open $\mathcal{U} \subseteq \mathbb{C}$ symmetric $z \longrightarrow \bar{z}$. $\forall z \in \mathcal{U} \implies \bar{z} \in \mathcal{U}$.

$$\mathcal{U}^+ = \mathcal{U} \cap \mathfrak{H}^+$$

$$\mathcal{U}^- = \mathcal{U} \cap \mathfrak{H}^-$$

$$\mathcal{U}^\circ = \mathcal{U} \cap \mathbb{R} = (a, b)$$



Given $f : \mathcal{U}^+ \longrightarrow \mathbb{C}$

[i] holomorphic in $\mathcal{U}^+$

[ii] extends continuously to $\mathcal{U}^\circ$.

[iii] such that the values $f(\mathcal{U}^\circ) \subseteq \mathbb{R}$.

Define

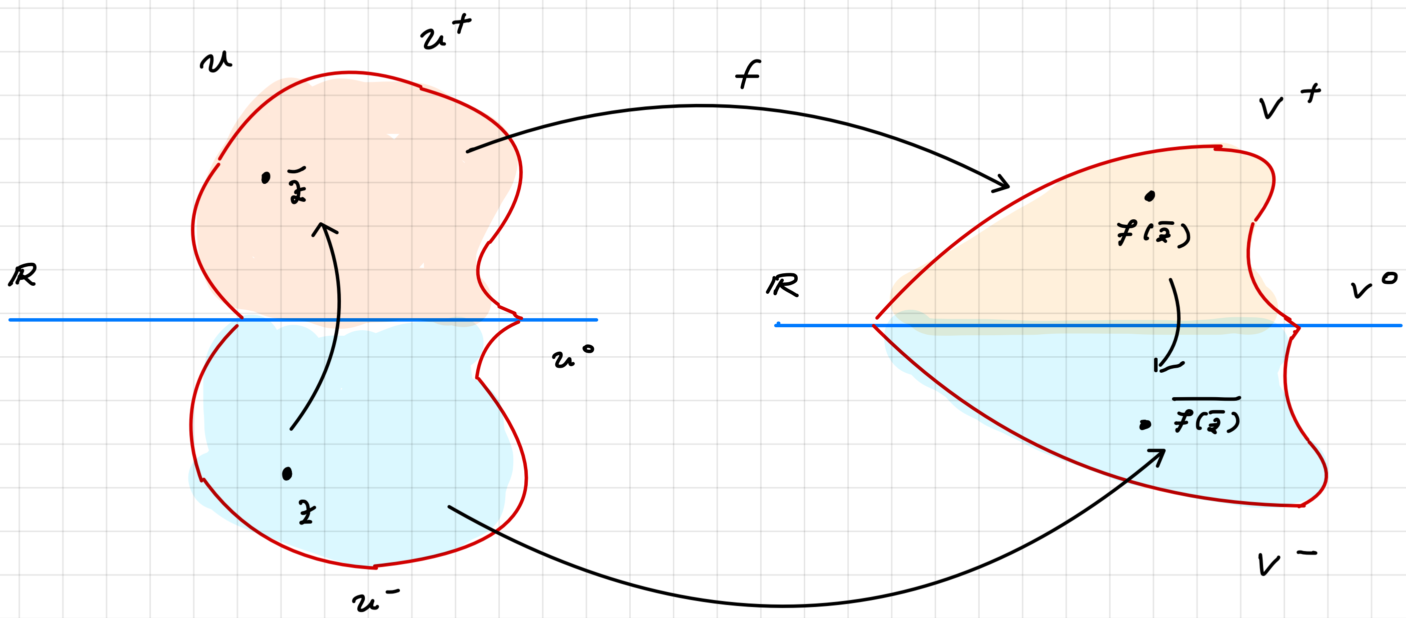
$$F(z) = \begin{cases} f(z) & \text{if } z \in \mathcal{U}^+ \\ f(z) & \text{if } z \in \mathcal{U}^\circ \\ \overline{f(\bar{z})} & \text{if } z \in \mathcal{U}^- \end{cases}$$

Theorem The function $F: U \longrightarrow \mathbb{C}$

is a holomorphic extension of f beyond the boundary.

Remarks

$\boxed{1}$ Visualization

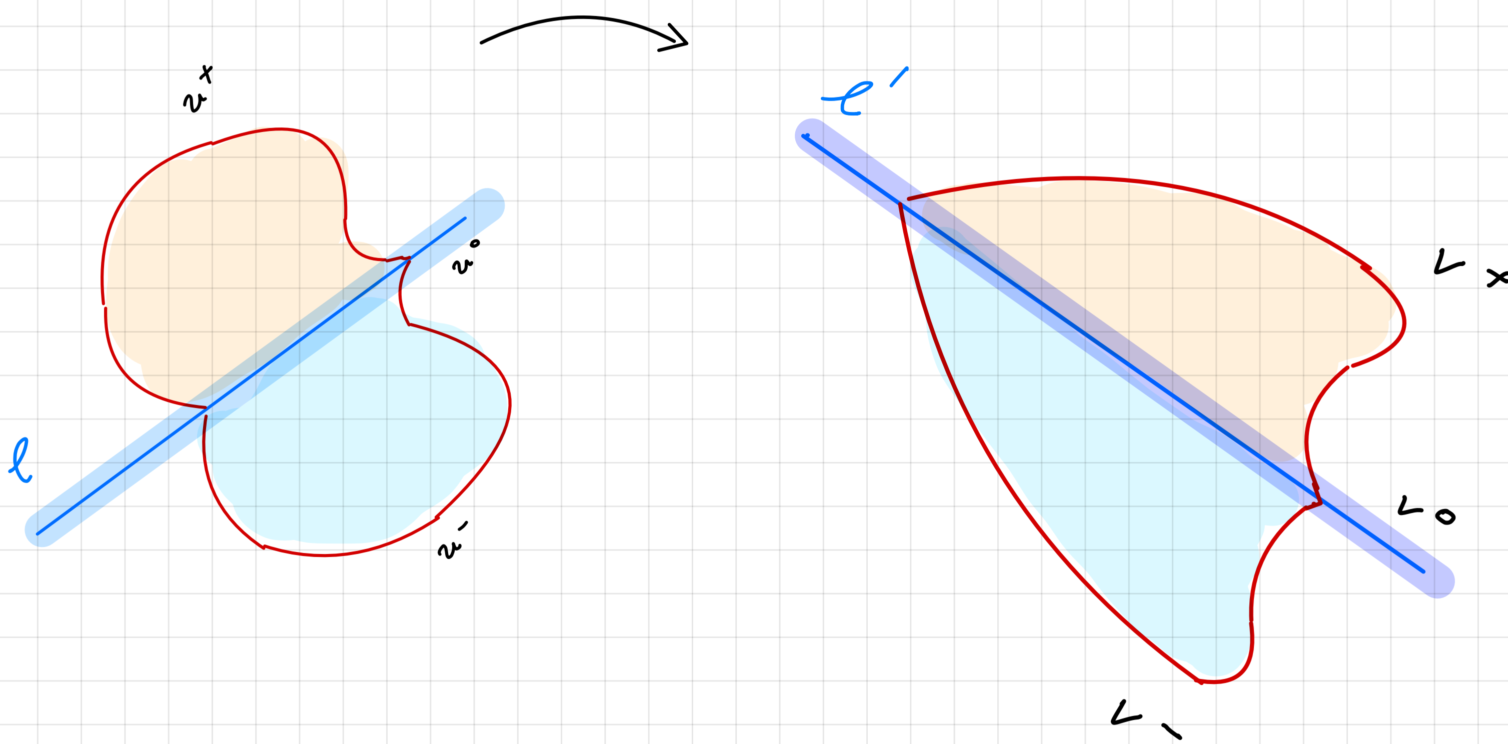


11 The condition

$$f(U_0) \subseteq \mathbb{R}$$

ensures we reflect across real axis on both sides.

More generally, we can reflect across arbitrary lines



This can be deduced via rotations

$\boxed{iii}$ Using the Cayley transform

$$C : \Delta \longrightarrow \mathfrak{H}^+$$

we can also reflect across arcs in the unit disc.

(HWK 6).