Math 220C - Lecture 13

April 26, 2021

(1) Homework 6, available on Friday, due May 7

— last homework

(2) We drop the lowest homework

(3) Next 3 lectures - Little Picard.

In Lecture 11

Application A (Conway XI. 3.6)

f entire & not constant & finite order

$\Rightarrow f$ omits at most one value.

Today — Picard's Theorems — Conway XII

Little Picard

$f: \mathbb{C} \longrightarrow \mathbb{C}$ entire, non constant $\Rightarrow f$ omits at most one value.

For example, $f(z) = e^{z}$ only omits the value 0.

Little Picard is a generalization of

Liouville's Theorem $f: \mathbb{C} \longrightarrow \mathbb{C}$ entire, non constant

$\Rightarrow f$ cannot be bounded.

Great Picard

$f: G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

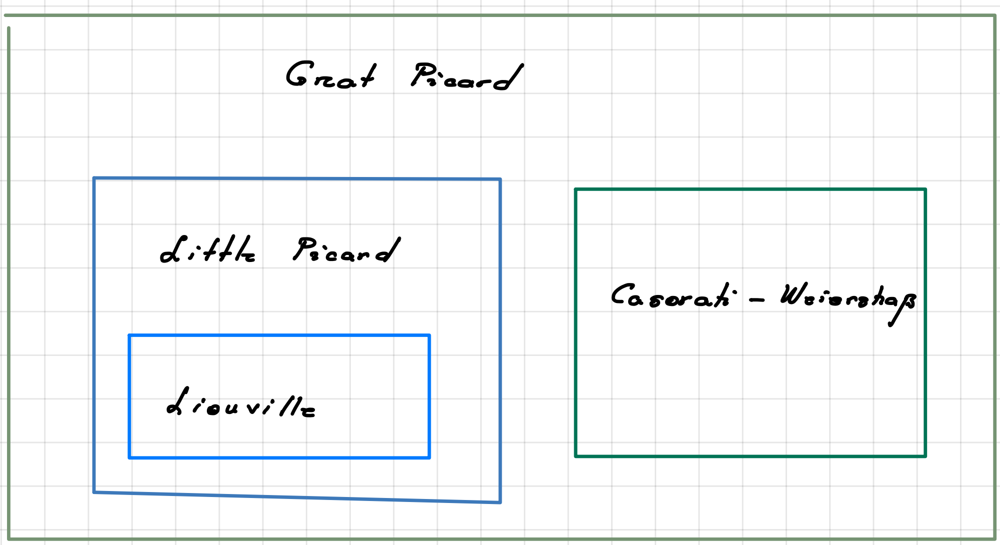
If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f/_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.

Great Picard is a generalization of

Casorati-Weierstraß

$f: G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f/_{\Delta^*(a,r)}$ has dense image in $\mathbb{C}$.



Great Picard > Little Picard   Conway XII. 4.4

Lemma

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, not polynomial.

$\Rightarrow$ f assumes all complex values ∞ - many times, with at most one exception.

Proof Let $g(z) = f\left(\frac{1}{z}\right) : \mathbb{C}^{\times} \longrightarrow \mathbb{C}$. Note that g has an essential singularity at 0. $\Leftrightarrow$ g does not have at worst a pole at 0 $\Leftrightarrow$ f does not have at worst a pole at ∞.

Recall from Math 220 A, Homework 5, Problem 6 that entire functions with poles at ∞ are polynomial, which is not the case for f.

Thus g has essential singularity at 0. Apply Great Picard to conclude.

We showed Great Picard $\Rightarrow$ Lemma $\Rightarrow$ Little Picard.

Examples

$\boxed{I}$ $e^f + e^g = 1$, f, g entire $\Rightarrow$ f, g constant.

Indeed, $h = e^f$ omits 0 & $h = 1 - e^g$ also omits 1.

Little Picard $\Rightarrow$ h constant $\Rightarrow$ f, g constant.

$\boxed{II}$ $f^n + g^n = 1$, $n \geq 3$, f, g entire $\Rightarrow$ f, g constant.

(HWK 6)

Émile Picard (1856 – 1941) .

Known for

Picard group
Picard's Little and Great theorems

Doctoral advisor:

Gaston Darboux

Doctoral students:

Jacques Hadamard
Gaston Julia
Paul Painlevé

Picard made contributions to applied mathematics, telegraphy and elasticity.

He wrote one of the first textbooks on the theory of relativity as well as biographies of several French mathematicians.

Une fonction entiere, qui ne devient jamais ni a a ni a b est necessairement une constante.

(An entire function which is never equal to either a or b must be constant.)

E. Picard, 1879.

§2. Proof of Little Picard

Step A Landau's lemma — Conway XII. 2

Step B due to Bloch — Conway XII. 1

Assume $\exists\ f: \mathbb{C} \longrightarrow \mathbb{C}$ entire, not constant, omits 0 & 1.

Step A produces a function g entire and $\alpha > 0$ with

$\Delta \not\subset \operatorname{Im} g$ for all discs Δ of radius α

Step B For any g entire & not constant, $\operatorname{Im} g$ contains a disc of any radius, in particular of radius α.

Step A & Step B are incompatible, showing f does not exist $\Rightarrow$ Little Picard.

Landau's Lemma

Let $h: G \longrightarrow \mathbb{C}$ holomorphic, G simply connected

Assume h omits -1 & 1. Then $\exists\ F: G \longrightarrow \mathbb{C}$ holomorphic

such that $\boxed{h = \cos F.}$

Proof Note $1 - h^2$ is nowhere zero in G $\Rightarrow$ let g be a square root of $1 - h^2$ $\Rightarrow$ $g^2 + h^2 = 1$ $\Rightarrow$ $(g + ih)(g - ih) = 1.$

Note $g + ih \neq 0$ in G $\Rightarrow$ $\exists$ logarithm for $g + ih$. Write

$$g + ih = e^{iF} \Rightarrow g - ih = \frac{1}{g + ih} = e^{-iF}$$

$$\Rightarrow g = \frac{1}{2}\left(e^{iF} + e^{-iF}\right) = \cos F.$$

Remark In our case f entire, omits 0 & 1 $\Longrightarrow$

$\Longrightarrow$ $2f-1$ omits -1 & 1 $\Longrightarrow$ by Landau

$\Longrightarrow$ $2f-1 = \cos \pi F$ & F entire.

Since $\cos \pi F = 2f-1 \neq \pm 1 \Longrightarrow F$ omits all integers.

Thus F omits -1 & 1 and by Landau again

$\Longrightarrow$ $F = \cos \pi g$ & $\cos \pi g$ is never an integer.

Conclusion

$$f = \frac{1}{2}\left(1 + \cos \pi F\right) = \frac{1}{2}\left(1 + \cos \pi \cos \pi g\right).$$

Define $A = \left\{ m \pm \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right) \; : \; n \in \mathbb{Z}_{>0} \, , \, m \in \mathbb{Z} \right\}$

Let $\alpha^{\pm}_{mn} = m \pm \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right)$. Note

$$e^{i\pi\alpha^{+}_{mn}} = e^{i\pi m} \cdot e^{-\log\left(n+\sqrt{n^2-1}\right)} = (-1)^m \frac{1}{n+\sqrt{n^2-1}} = (-1)^m \left(n - \sqrt{n^2-1}\right).$$

$$e^{-i\pi\alpha^{+}_{mn}} = e^{-i\pi m} \cdot e^{\log\left(n+\sqrt{n^2-1}\right)} = (-1)^m \left(n + \sqrt{n^2-1}\right)$$

$$\Longrightarrow \cos \pi \alpha^{+}_{mn} = \frac{1}{2}\left(e^{i\pi\alpha^{+}_{mn}} + e^{-i\pi\alpha^{+}_{mn}}\right) = (-1)^m n \in \mathbb{Z}.$$

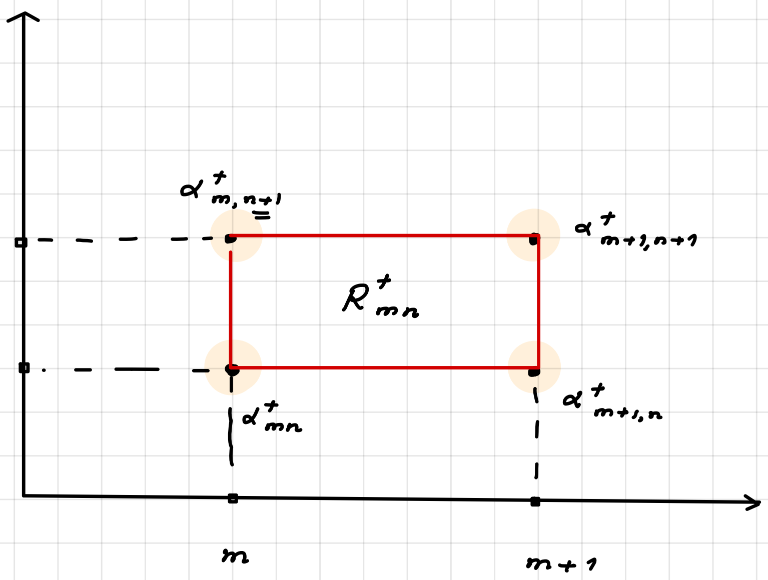
(The same argument works for α^{-}_{mn}.)

But $\cos \pi g$ cannot be an integer.

$$\Downarrow$$

Conclusion $A \cap \operatorname{Img} = \emptyset$.

Visualize A $\quad A = \left\{ m + \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right) : n \in \mathbb{Z}_{>0}, \; m \in \mathbb{Z} \right\}$



The set A gives the vertices of rectangles paving the plane. The upper half plane is paved by rectangles R^+_{mn}

- horizontal side $\quad (m+1) - m = 1$

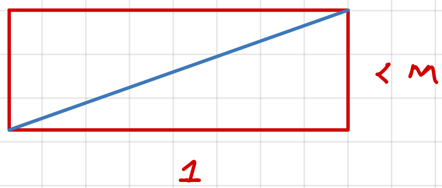
- vertical side $\quad \frac{1}{\pi} \log\left(n+1 + \sqrt{(n+1)^2-1}\right) - \frac{1}{\pi} \log\left(n + \sqrt{n^2-1}\right) =$

$$= \frac{1}{\pi} \log \frac{n+1 + \sqrt{(n+1)^2 - 1}}{n + \sqrt{n^2-1}} < M$$

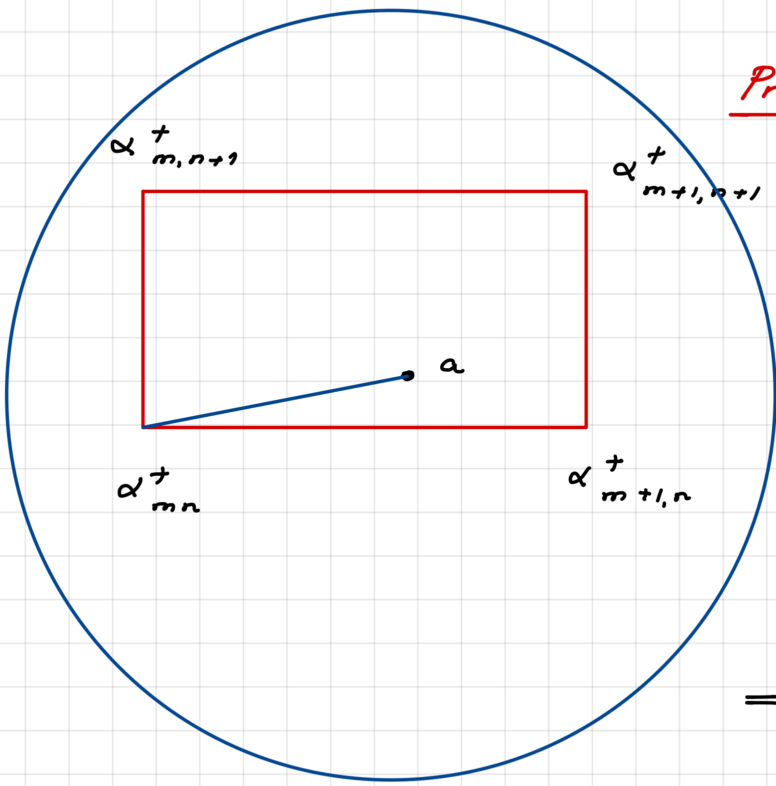
(make $n \to \infty$ to see boundedness).

The α^-_{mn}'s are used to pave the lower half plane.

The diameter of R^+_{mn} is $< \sqrt{1+M^2}$. Let $\alpha = \sqrt{2+M^2}$



Claim If Δ is any disc of radius α then $\Delta \not\subseteq Img$.



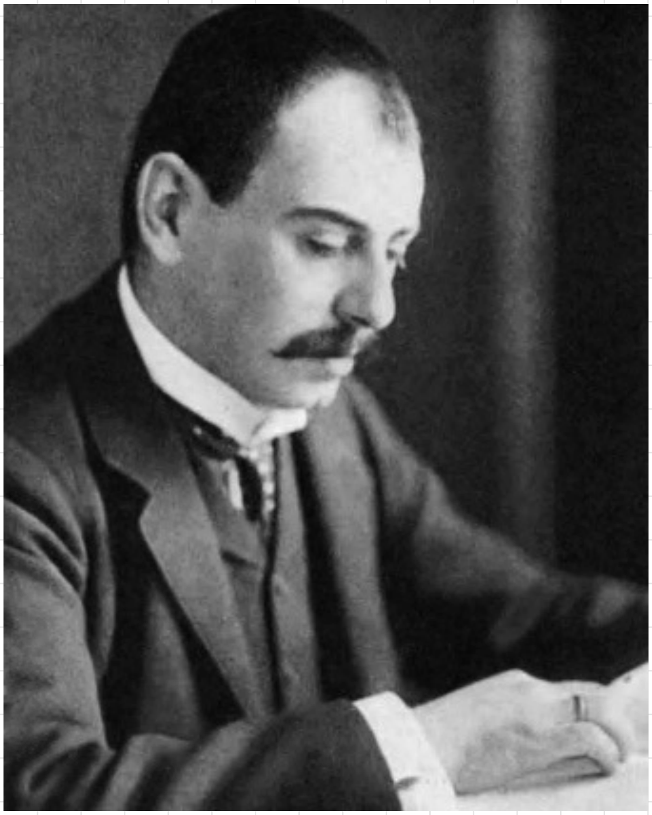
Proof Let a be the center of Δ located say in the upper half plane. Then $a \in R^+_{mn}$.

$\Rightarrow |a - \alpha^+_{mn}| <$ diameter $(R^+_{mn}) < \alpha$

$\Rightarrow \alpha^+_{mn} \in \Delta$.

We have seen $\alpha^+_{mn} \notin Img$. Thus $\Delta \not\subseteq Img$.

This completes the proof of Step A. Step B will be discussed next.



Edmund Landau
(1877 – 1938)

Big O notation

Landau's problems (ICM 1912)

Goldbach's conjecture

Twin prime conjecture

Legendre's conjecture: Does there always exist at least one prime between consecutive perfect squares?

Are there infinitely many primes of the form n^2+ 1?