Lecture 6 1/15/21

Recall

<u>Thm.</u> R a PID, M a f.g. R-module.

① Then $M \cong R^r \oplus R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_m^{e_m})$

some primes $p_1, \ldots, p_m$.

② $M \cong M'$ if they have same rank r and elementary divisors
$p_1^{e_1}, \ldots, p_m^{e_m}$
(up to order, and associates)

<u>Pf.</u> Let $T = \mathrm{Tors}(M)$. Then

M/T is torsion free.

M/T is f.g., so M/T is free.

Then $\pi : M \longrightarrow M/T$ splits

So $M \cong M/T \oplus T$

and $F = M/T$ is free of finite rank, say $F = R^r = \underbrace{R \oplus \cdots \oplus R}_{r}$

Now T is torsion, f.g. (as a summand of M)

So $T \cong T_{p_1} \oplus \cdots \oplus T_{p_k}$

for some primes $p_1, \ldots, p_k$ (non-associate)

where $T_{p_i} = \{ x \in T \mid (p_i)^n x = 0 \text{ some } n \}$

is a p_i-primary component.

Finally $T_{p_i} \cong R/(p_i^{f_1}) \oplus \cdots \oplus R/(p_i^{f_{n_i}})$

So $M \cong R^r \oplus R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_m^{e_m})$

(2) Say M, M' f.g. modules

$f : M \to M'$ an iso.

$f(\mathrm{Tors}(M)) = \mathrm{Tors}(M')$.

Since for $m \in M$, $f(rm) = r f(m)$

So $rm = 0$ some $r \neq 0$ iff
$r f(m) = 0$ some $r \neq 0$.

f restricts to an iso $T \to T'$.

So f induces an iso $M/T \to M'/T'$
$(m+T) \mapsto (f(m)+T')$.

$$\begin{array}{ccc} F & \cong & F' \\ \| & & \| \\ R^r & & R^{r'} \end{array}$$

By HW, $r = r'$.

Notice $f(T_p) = T'_p$.

Since $f(rm) = r f(m)$

So $p^i m = 0$ some $i \geq 1$ iff
$p^i f(m) = 0$ " ".

So f restricts to isos on all the

p_i primary components. So

$$T \cong T_{p_1} \oplus \cdots \oplus T_{p_k}$$

Then $T' \cong T'_{p_1} \oplus \cdots \oplus T'_{p_k}$.

Last, show if $f: T_p \longrightarrow T'_p$

T_p, T_p' are p primary

and $T_p \cong R/(p^{i_1}) \oplus \cdots \oplus R/(p^{i_s})$

$$T_p' \cong R/(p^{j_1}) \oplus \cdots \oplus R/(p^{j_t})$$

Then $s = t$ and $\underline{i_k = j_k}$ after rearrangement

idea: define $b \geq 1$

$\underline{T_p[b]} = \{ m \in T_p \mid p^b m = 0 \}$

Note $T_p[0] \subseteq T_p[1] \subseteq T_p[2] \subseteq \cdots$

and $\overline{T_p[i+1] / T_p[i]}$ is an $R/(p) = K$ - vector space.

with dimension = the number of i_k which are $\geq \underline{i+1}$.

(full details in notes).

Invariant factor form.

<u>Thm.</u> R a PID, M f.g.

$$M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_n)$$

where $a_1, \ldots, a_n \in R$ nonzero nonunits s.t.

$a_1 | a_2, a_2 | a_3, \ldots, a_{n-1} | a_n$.

a_i are <u>invariant factors</u> and they are uniquely determined by M up to $\cong$ (and associates)

We'll just give examples.

<u>Ex.</u> $R = \mathbb{Z}$ then the fundamental thm says:

Every f.g. Abelian group G is $\cong$ to

$\mathbb{Z}^r \oplus H$ $\quad$ H finite

and

$$H \cong \mathbb{Z}/(p_1^{e_1}) \oplus \cdots \oplus \mathbb{Z}/(p_k^{e_k})$$

or

$$H \cong \mathbb{Z}/(a_1) \oplus \cdots \oplus \mathbb{Z}/(a_n)$$

$a_1 \mid a_2 \mid \cdots \mid a_n.$

Ex.

$$G \cong \mathbb{Z}/(3) \oplus \mathbb{Z}/(15) \oplus \mathbb{Z}/(60) \oplus \mathbb{Z}/(\underline{120})$$

invariant factor form.

$$\mathrm{ann}_{\mathbb{Z}}(G) = (120) = \underline{120\mathbb{Z}}$$

what are elementary divisors?

<u>CRT</u> says

$$\mathbb{Z}/(p_1^{e_1} \cdots p_k^{e_k}) \qquad p_i \text{ pairwise nonassociate}$$

$$\cong \mathbb{Z}/(p_1^{e_1}) \oplus \cdots \oplus \mathbb{Z}/(p_k^{e_k})$$

(true for general R)

Now $3 = 3^1$

$15 = 3^1 \cdot 5^1$

$60 = 2^2 \cdot 3^1 \cdot 5^1$

$120 = 2^3 \cdot 3^1 \cdot 5^1.$

So

$$G \cong \mathbb{Z}/(3) \oplus \mathbb{Z}/(3)$$

$$\oplus \mathbb{Z}/(5) \oplus \mathbb{Z}/(2^2)$$

$$\oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(5)$$

$$\oplus \mathbb{Z}/(2^3) \oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(5).$$

$$H \cong \mathbb{Z}/(3) \oplus \mathbb{Z}/(3^2) \oplus \mathbb{Z}/(3^4) \oplus \mathbb{Z}/(5) \oplus \mathbb{Z}/(5^2) \oplus \mathbb{Z}/(13).$$

$$a_3 = 13 \cdot 5^2 \cdot 3^4$$

$$a_2 = 5 \cdot 3^2$$

$$a_1 = 3$$

are invariant factors

$$H \cong \mathbb{Z}/(a_1) \oplus \mathbb{Z}/(a_2) \oplus \mathbb{Z}/(a_3).$$

Ex. $R = \mathbb{Q}[x]$.

Suppose

$$M \cong \frac{\mathbb{Q}[x]}{(x^3-1)} \oplus \frac{\mathbb{Q}[x]}{((x^6-1)(x-1))}$$

$$(x^7 - x^6 - x + 1)$$

Rmk. a degree 2 poly over $\mathbb{Q}$ is irreducible iff it has no root in $\mathbb{Q}$.

$$x^3 - 1 = (x-1)(x^2+x+1)$$

x^2+x+1 is irreducible in $\mathbb{Q}[x]$.

$(x-1)$

$$x^6 - 1 = (x^3-1)(x^3+1)(x-1)$$

$$= (x-1)^2 (x^2+x+1)$$

$$(x+1)(x^2-x+1)$$

elem. divisors

$(x-1)$, (x^2+x+1)

$(x-1)^2$, (x^2+x+1),

$(x+1)$, (x^2-x+1)