HW 2 Solutions

Problem 1. $a_n(X_n-\theta) \xrightarrow{d} N(0,\tau^2)$. Let $\phi(u)=|u|$ $\forall u \in \mathbb{R}$.

Case 1: $\theta \neq 0$. Then, $\phi'(\theta) = \operatorname{sgn}(\theta)$.

By the delta method, $a_n(\phi(X_n)-\phi(\theta)) \xrightarrow{d} N(0, \tau^2 \operatorname{sgn}^2(\theta))$, i.e., $a_n(|X_n|-|\theta|) \xrightarrow{d} N(0, \tau^2)$.

Case 2: $\theta = 0$. Then, $\phi'(0)$ does not exist and we cannot use the delta method.

Note that, ϕ is a continuous function.

By continuous mapping theorem, $a_n|X_n| = |a_n(X_n-\theta)| \xrightarrow{d} |N(0,\tau^2)|$

Problem 2. (a) $n^{-1}\sum_{i=1}^{n}(X_i-\bar{X})^3 = 2Y_1^3 - 3Y_1Y_2 + Y_3$, $n^{-1}\sum_{i=1}^{n}(X_i-\bar{X})^2 = -Y_1^2+Y_2$. where $(\mu := \mathbb{E}X)$

$Y_1 := n^{-1}\sum_{i=1}^{n}(X_i-\mu) \xrightarrow{P} 0$, $Y_2 := n^{-1}\sum_{i=1}^{n}(X_i-\mu)^2 \xrightarrow{P} \sigma^2$. $Y_3 := n^{-1}\sum_{i=1}^{n}(X_i-\mu)^3 \xrightarrow{P} \mu_3$. if $\mathbb{E}|X-\mu|^3<\infty$

Hence. $l_n = \dfrac{2Y_1^3 - 3Y_1Y_2+Y_3}{(-Y_1^2+Y_2)^{3/2}} \to \dfrac{2\cdot 0^3 - 3\cdot 0\cdot \sigma^2 + \mu_3}{(-0^2+\sigma^2)^{3/2}} = \dfrac{\mu_3}{\sigma^3}$ (Slucky's Theorem)

(b) By CLT , when $X_i \overset{iid}{\sim} N(0,1)$,

$$\sqrt{n}\cdot\left\{\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} - \begin{pmatrix} 0 \\ \sigma^2 \\ 0 \end{pmatrix}\right\} \xrightarrow{d} N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 & 3\sigma^4 \\ 0 & 2\sigma^4 & 0 \\ 3\sigma^4 & 0 & 15\sigma^6 \end{pmatrix}\right) \quad (\sigma^2=1).$$

$$l_n = \frac{2Y_1^3 - 3Y_1Y_2+Y_3}{(-Y_1^2+Y_2)^{3/2}} := \phi(Y_1, Y_2, Y_3).$$

$$\phi'(0,\sigma^2,0) = (-3\sigma^{-1}, 0, \sigma^{-3}).$$

By Delta's method,

$$\sqrt{n}\{\phi(Y_1,Y_2,Y_3) - \phi(0,\sigma^2,0)\} \xrightarrow{d} N\left(\langle\phi'(0,\sigma^2,0),(0,0,0)\rangle,\ \phi'(0,\sigma^2,0)\cdot\begin{pmatrix} \sigma^2 & 0 & 3\sigma^4 \\ 0 & 3\sigma^4 & 0 \\ 3\sigma^4 & 0 & 15\sigma^6 \end{pmatrix}\cdot\phi'(0,\sigma^2,0)^T\right)$$

That is ,

$$\sqrt{n}(l_n - 0) \xrightarrow{d} N(0, 6)$$

Problem 3. Let $Y_j = n^{-1}\sum_{i=1}^{n}(X_i-\mu)^j$, $j=1,2,3,4$. ($\mu = \mathbb{E}X$). Then

$$k_n = \frac{-3Y_1^4+6Y_1^2Y_2-4Y_1Y_3+Y_4}{(-Y_1^2+Y_2)^2} - 3 =: \phi(Y_1, Y_2, Y_3, Y_4)$$

By CLT, with $\mu_j = E(X-EX)^j$,

$$\sqrt{n}\left\{\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} - \begin{pmatrix} 0 \\ \sigma^2 \\ \mu_3 \\ \mu_4 \end{pmatrix}\right\} \xrightarrow{d} N_4\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} \mu_2-\mu_1^2 & \mu_3-\mu_1\mu_2 & \mu_4-\mu_1\mu_3 & \mu_5-\mu_1\mu_4 \\ \mu_3-\mu_1\mu_2 & \mu_4-\mu_2^2 & \mu_5-\mu_2\mu_3 & \mu_6-\mu_2\mu_4 \\ \mu_4-\mu_1\mu_3 & \mu_5-\mu_2\mu_3 & \mu_6-\mu_3^2 & \mu_7-\mu_3\mu_4 \\ \mu_5-\mu_1\mu_4 & \mu_6-\mu_2\mu_4 & \mu_7-\mu_3\mu_4 & \mu_8-\mu_4^2 \end{pmatrix}}_{=:\Sigma}\right)$$

$$\phi'(0, \sigma^2, \mu_3, \mu_4) = (-4\mu_3\sigma^{-4}, -2\mu_4\sigma^{-6}, 0, \sigma^{-4})$$

By Delta's Method.

$$\sqrt{n}\left(k_n - \frac{\mu_4}{\sigma^4} + 3\right) \xrightarrow{d} N\left(0, \phi'(0, \sigma^2, \mu_3, \mu_4)\,\Sigma\,\phi'(0, \sigma^2, \mu_3, \mu_4)^T\right).$$

Problem 4. Since $X_i \sim^{iid} N(\mu, 1)$, $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ is a linear combination of multivariate normal random vectors.

Hence, $\bar{X}$ is normally distributed with $E(\bar{X}) = \mu$ and $Var(\bar{X}) = \frac{1}{n}$. That is, $\bar{X} \sim N(\mu, \frac{1}{n})$.

Therefore, $\sqrt{n}(\bar{X} - \mu) \sim N(0, 1)$. The remaining parts follow directly from Problem 1.

Case 1: $\mu \neq 0$. $\sqrt{n}(|\bar{X}| - |\mu|) \xrightarrow{d} N(0,1)$

Case 2: $\mu = 0$. $\sqrt{n}|\bar{X}| \xrightarrow{d} |N(0,1)|$.

Problem 5. Let $U_i, V_i \overset{iid}{\sim}$ Poisson (1). Then, $X_n \sim \sum_{i=1}^{n} U_i$ and $Y_m \sim \sum_{i=1}^{m} V_i$. Note that $E(U_i) = Var(U_i) = 1$.

By CLT, $\tilde{X}_n := \sqrt{n}(n^{-1}X_n - 1) \xrightarrow{d} N(0,1)$ and $\tilde{Y}_m := \sqrt{m}(m^{-1}Y_m - 1) \xrightarrow{d} N(0,1)$.

Note that $\dfrac{X_n - Y_m - (n-m)}{\sqrt{n+m}} = \dfrac{n(n^{-1}X_n - 1) - m(m^{-1}Y_m - 1)}{\sqrt{n+m}} = \sqrt{\dfrac{n}{n+m}}\tilde{X}_n - \sqrt{\dfrac{m}{n+m}}\tilde{Y}_m$,

Let $\tau := \lim_{n,m\to\infty} \dfrac{n}{n+m}$. Then, by Slutsky's theorem,

$$\sqrt{\frac{n}{n+m}}\tilde{X}_n \xrightarrow{d} N(0, \tau) \quad \text{and} \quad \sqrt{\frac{m}{n+m}}\tilde{Y}_m \xrightarrow{d} N(0, 1-\tau).$$

Since $X_n \perp\!\!\!\perp Y_n$, we have $\tilde{X}_n \perp\!\!\!\perp \tilde{Y}_n$ and hence $\begin{pmatrix} \sqrt{\frac{n}{n+m}}\tilde{X}_n \\ \sqrt{\frac{m}{n+m}}\tilde{Y}_m \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau & 0 \\ 0 & 1-\tau \end{pmatrix}\right)$.

Let $\phi(x,y) := x - y$, then $\phi'(x,y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. By delta method,

$$\frac{X_n - Y_m - (n-m)}{\sqrt{n+m}} = \sqrt{\frac{n}{n+m}}\tilde{X}_n - \sqrt{\frac{m}{n+m}}\tilde{Y}_m \xrightarrow{d} N\left((1\ -1)\cdot\begin{pmatrix} 0 \\ 0 \end{pmatrix}, (1\ -1)\begin{pmatrix} \tau & 0 \\ 0 & 1-\tau \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \sim N(0,1). \quad (*)$$

By WLLN, $n^{-1}X_n \xrightarrow{P} 1$ and $m^{-1}Y_n \xrightarrow{P} 1$. Hence, $\dfrac{X_n + Y_m}{n+m} = \dfrac{n}{n+m}\cdot n^{-1}X_n + \dfrac{m}{n+m}m^{-1}Y_m \xrightarrow{P} \tau + 1 - \tau = 1$ and $\dfrac{\sqrt{X_n+Y_m}}{\sqrt{n+m}} \xrightarrow{P} 1$.

Together with $(*)$, by Slutsky's theorem, $\dfrac{X_n - Y_m - (n-m)}{\sqrt{X_n + Y_m}} \xrightarrow{d} N(0,1)$.