

## Math 220, Problem Set 1.

*You may use that any complex differentiable function admits as many continuous derivatives as you need.*

1. Let  $U \subset \mathbb{C}$  be a connected open set. Find all complex differentiable functions  $f : U \rightarrow \mathbb{C}$  such that

$$(\operatorname{Re} f)^2 + i(\operatorname{Im} f)^2$$

is also complex differentiable.

2. For a complex number  $\alpha \in \mathbb{C}$  and positive integer  $n > 0$ , define the Pochhammer symbol as

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1).$$

By definition  $(\alpha)_0 = 1$ .

Let  $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q$  be complex numbers which are not integers. The generalized hypergeometric series is defined as

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

- (i) Find the radius of convergence of the hypergeometric series.
- (ii) Traditionally, one considers the case  $p = 2$  and  $q = 1$ . Show that the hypergeometric series

$$w = {}_2F_1(a_1, a_2; b; z)$$

satisfies the hypergeometric differential equation

$$z(1-z) \frac{d^2 w}{dz^2} + (b - (1 + a_1 + a_2)z) \frac{dw}{dz} - a_1 a_2 w = 0.$$

- (iii) Write the series  ${}_2F_1(1; 1; 1; z)$  and  ${}_2F_1(1; 2; 1; z)$  in closed form.

3. Let

$$f(z) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Show that the real and imaginary part of  $f$  satisfy the Cauchy Riemann equations at  $z = 0$ , but  $f$  is not complex differentiable at  $z = 0$ . Why doesn't this contradict the results proved in class?

4. Let  $U \subset \mathbb{C}$  be a connected open set. A logarithm function  $\ell : U \rightarrow \mathbb{C}$  is a continuous function such that

$$\exp(\ell(z)) = z \text{ for all } z \in U.$$

We emphasize the continuity assumption in the above definition.

- (i) Show that if  $\ell$  is a complex differentiable function with  $\ell'(z) = \frac{1}{z}$ , then  $\ell$  must be a logarithm function as defined above, possibly up to a constant.

- (ii) Let  $\Delta_1(1)$  be the open disc of radius 1 centered at 1. Using (i), show that the power series

$$L(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z-1)^k$$

defines a logarithm function on  $\Delta_1(1)$ .

- (iii) Let  $a \in \mathbb{C} \setminus \{0\}$  and let  $b$  be any logarithm of  $a$ . Using (ii), derive that

$$L\left(\frac{z}{a}\right) + b$$

is a logarithm function over the ball  $\Delta_{|a|}(a)$  of radius  $|a|$  centered at  $a$ .

- (iv) Show that if  $\ell_1$  and  $\ell_2$  are two logarithm functions defined over a connected open set  $U$  then  $\ell_1 = \ell_2 + 2\pi in$  for some integer  $n$ .

- (v) Using (iii) and (iv), conclude that any logarithm function  $\ell$  is automatically complex differentiable. Show that  $\ell'(z) = \frac{1}{z}$ .

**5.** Let  $U \subset \mathbb{C}$  be open and connected. Assume that  $f_1, \dots, f_n : U \rightarrow \mathbb{C}$  are complex differentiable functions such that

$$|f_1|^2 + \dots + |f_n|^2 = 1.$$

Show that  $f_1, \dots, f_n$  are constant.

*Hint:* You may wish to compute

$$\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} |f|^2.$$