Math 220B — Lecture 25

March 8, 2021

§0. Last time — We established Runge C

Thm [I] $K \subseteq \mathbb{C}$ compact, $S \subseteq \hat{\mathbb{C}} \setminus K$ contains a point from each component of $\hat{\mathbb{C}} \setminus K$.

[II] f holomorphic in K

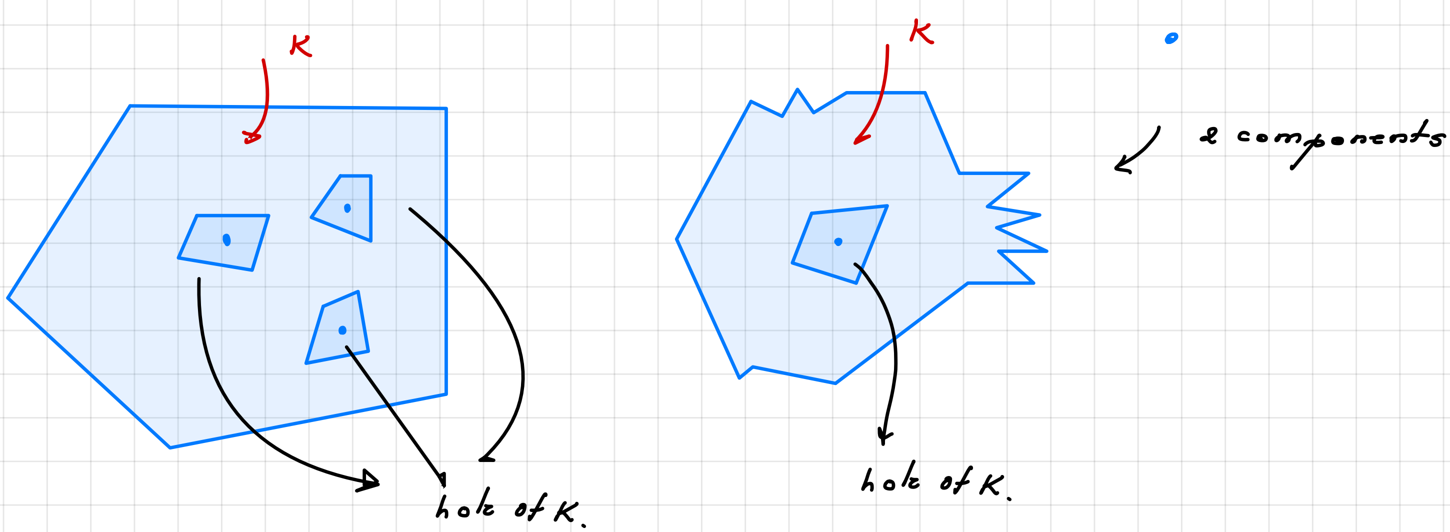
$\Rightarrow \forall \varepsilon \quad \exists R$ rational,

$|f - R| < \varepsilon$ in K and poles $(R) \subseteq S$.

Remark For $\varepsilon = \frac{1}{n} \Rightarrow \exists R_n$ with $|f - R_n| < \frac{1}{n}$ in K

$\Rightarrow R_n \rightrightarrows f$ in K. & poles $(R_n) \subseteq S$.

The set K can be disconnected and quite strange.



Applications

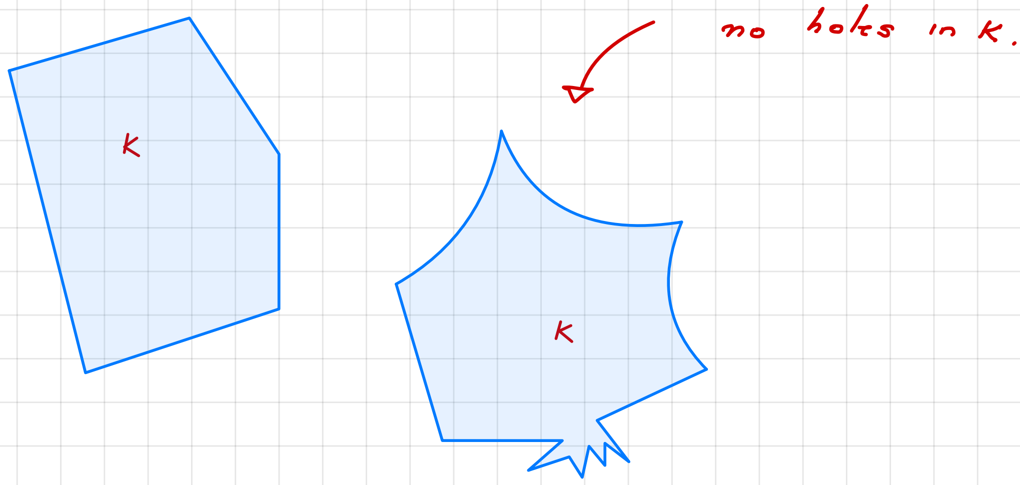
- density in spaces of functions
- new proof of Mittag-Leffler Conway VIII.3.
- polynomial convexity Conway VIII.1.
- generalizations: Mergelyan, ...

Important Special Case – Little Runge C

K has no holes $\Longrightarrow$ $\hat{\mathbb{C}} \setminus K$ has only one unbounded component & we can take $S = \{\infty\}$

$\Downarrow$ $\Downarrow$

All f holomorphic in K can be approximated uniformly in K by polynomials.



The set K can be disconnected

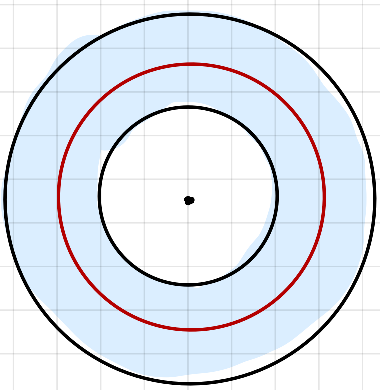
§1. How about the converse?

If K has no holes $\overset{\text{Runge}}{\Rightarrow}$ polynomial approximation holds.

If K has holes $\Rightarrow$ polynomial approximation fails in general

How to see this? Two methods

$\boxed{1}$ (Lecture 22) $K = \{1 \leq |z| \leq 2\}$, $f(z) = \dfrac{1}{z}$.



If $p_n \rightrightarrows f$ in K, p_n polynomials

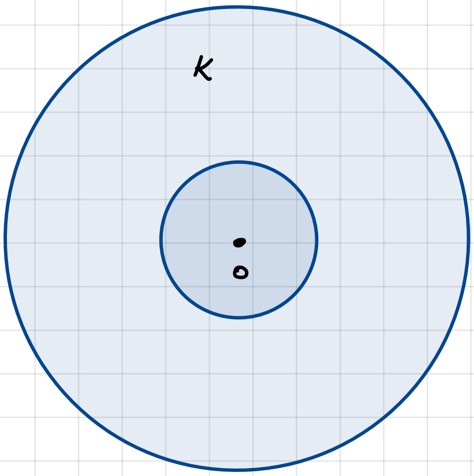
then $\underbrace{\int_{|z|=\frac{3}{2}} p_n \, dz}_{0} \longrightarrow \int_{|z|=\frac{3}{2}} f \, dz = \underbrace{\int_{|z|=\frac{3}{2}} \frac{dz}{z}}_{2\pi i}$

Both integrals follow by the residue theorem, for instance.

This contradiction shows f cannot be approximated uniformly in K by polynomials p_n.

$\boxed{ii}$ (New method).

$$K = \{ 1 \le |z| \le 2 \}, \quad f(z) = \frac{1}{z}$$



Assume $P_n \rightrightarrows f$ in K, P_n polynomials.

$\exists N : \quad |P_N - f| < \frac{1}{4}$ on K

$\iff |P_N - \frac{1}{z}| < \frac{1}{4}$ on K.

$\iff |z P_N - 1| < \frac{|z|}{4}$ on K.

$\Rightarrow |z P_N - 1| < \frac{|z|}{4}$ when $|z| = 1$. $\Rightarrow |z P_N - 1| < \frac{1}{4}$. when $|z| = 1$.

Let $g(z) = 1 - z P_N \Rightarrow g$ entire. Note $|g(0)| = 1$ and

$$|g(z)| < \frac{1}{4} \text{ for } |z| = 1.$$

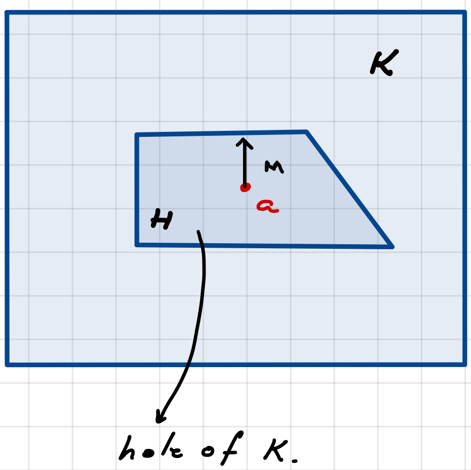
This contradicts Maximum modulus for g in $\bar{\Delta}(0,1)$.

The second method generalizes

Let H be a hole of K. Let $a \in H$., $f(z) = \dfrac{1}{z-a}$

$M = \max\limits_{z \in K} |z-a| > 0$. If $P_n \rightrightarrows f$ in K. find N such that

$$\left| P_N - \frac{1}{z-a} \right| < \frac{1}{2M} \text{ in } K$$



$$\Rightarrow \; |(z-a)P_N - 1| < \frac{|z-a|}{2M} \leq \frac{1}{2}. \quad \text{in } K.$$

$g(z) = 1 - (z-a)P_N$ satisfies

$g(a) = 1$ & $|g(z)| < \frac{1}{2}$ on $\partial H \subseteq K$.

This contradicts maximum modulus for g & the set $\overline{H}$.

Thus f cannot be approximated by polynomials.

Conclusion K has no holes $\Longleftrightarrow$ polynomial approximation holds in K.

§2. Runge for Open Sets ↙ Conway VIII. 1.15.

- We approximate locally uniformly on open sets
- the statement is similar to Runge for compact sets

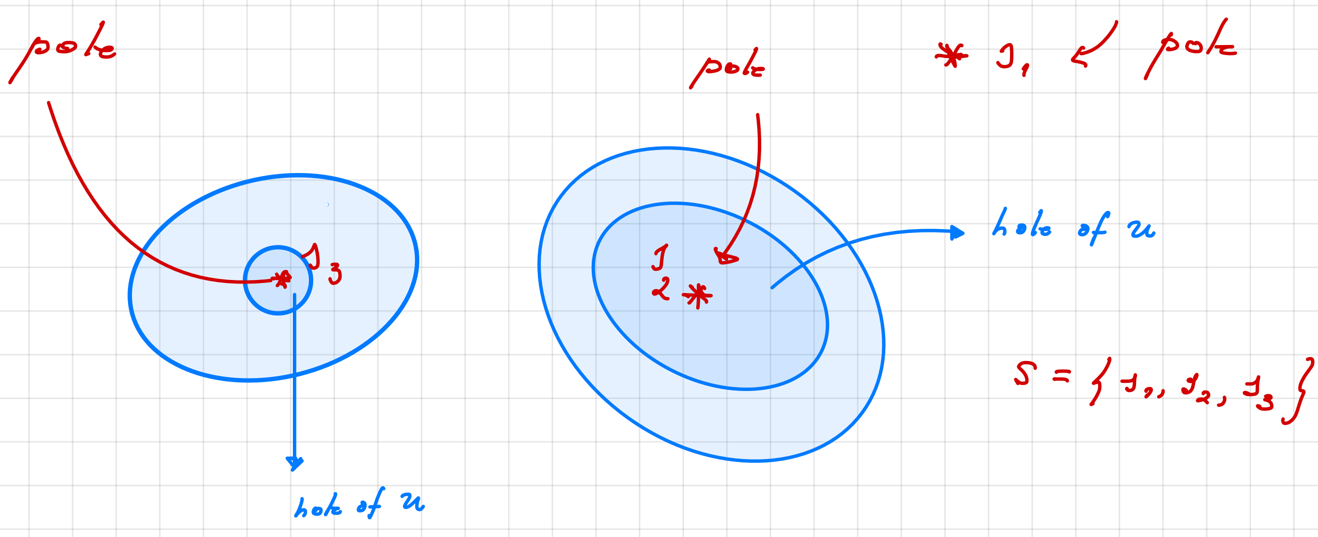
Theorem • $\mathcal{U} \subseteq \mathbb{C}$ possibly disconnected. open set.

• $S \subseteq \hat{\mathbb{C}} \setminus \mathcal{U}$ containing at least a point from each component of $\hat{\mathbb{C}} \setminus \mathcal{U}$.

• $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

Then $\exists\, R_n$ rational functions, poles $(R_n) \subseteq S$ and

$$R_n \overset{\text{l.u.}}{\rightrightarrows} f \quad \text{locally uniformly in } \mathcal{U}.$$



Important Special Case (Little Runge 0)

Let $\mathcal{U} \subseteq \mathbb{C}$, open, $\hat{\mathbb{C}} \setminus \mathcal{U}$ connected.

Any $f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic can be approximated locally uniformly on $\mathcal{U}$ by polynomials.

Indeed, take $S = \{\infty\}$ in Runge 0.

Example Let $\mathcal{U} = \Delta(0,r)$, $f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

We can Taylor expand f in the disc. The Taylor polynomials

$$T_n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k \quad \& \quad T_n \overset{\text{l.u.}}{\rightrightarrows} f \quad (\text{Math } 220A).$$

Little Runge 0 applies to more general sets $\mathcal{U}$.

Proof of Runge Open

Conway VII.1.2.

Topological Lemma For $\mathcal{U} \subseteq \mathbb{C}$ open, we can find $\underbrace{K_n}_{\text{compact}} \subseteq \mathcal{U}$

$$(*) \quad U = \bigcup_{n \geq 1} K_n$$ exhausting compact sets

[i] $K_n \subseteq \operatorname{Int} K_{n+1}$

[ii] $\forall K \subseteq \mathcal{U}$ compact $\Rightarrow \exists n, \quad K \subseteq K_n$.

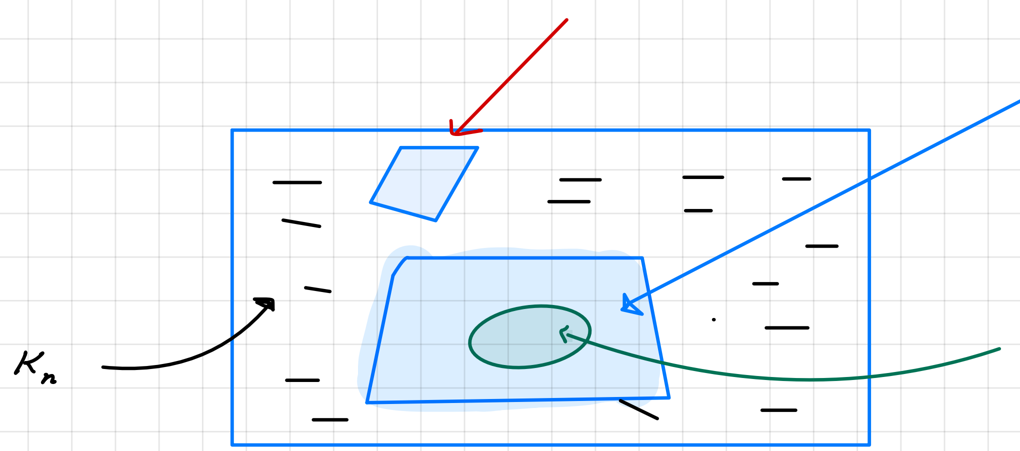
[iii] each component of $\hat{\mathbb{C}} \setminus K_n$ contains a component of $\hat{\mathbb{C}} \setminus \mathcal{U}$.

Remark [iii] means holes of K_n contain holes of $\mathcal{U}$.

Good vs. bad

bad. (hole of K_n, but not of $\mathcal{U}$).

good: hole of K_n containing hole of $\mathcal{U}$



Topological Lemma $\Rightarrow$ Runge 0

Let $f : U \longrightarrow \mathbb{C}$ holomorphic. Let S contain a point from each component of $\hat{\mathbb{C}} \setminus U$. Write

$$U = \bigcup_{n \geq 1} K_n \quad \text{as in the Lemma.}$$

The set S contains a point from each component of $\hat{\mathbb{C}} \setminus K_n$. by $\boxed{iii}$ By Runge C applied to f & K_n, we find.

$$|f - R_n| < \frac{1}{n} \text{ in } K_n, \quad \text{poles}(R_n) \subseteq S.$$

We claim $R_n \overset{\text{l.u.}}{\rightrightarrows} f$. Let K be compact in U. By $\boxed{ii}$

$\Rightarrow K \subseteq K_N$ for some N. For $n \geq N \Rightarrow K \subseteq K_N \subseteq K_n$ by $\boxed{i}$

$\Rightarrow |f - R_n| < \frac{1}{n}$ over $K_n \Rightarrow |f - R_n| < \frac{1}{n}$ in K.

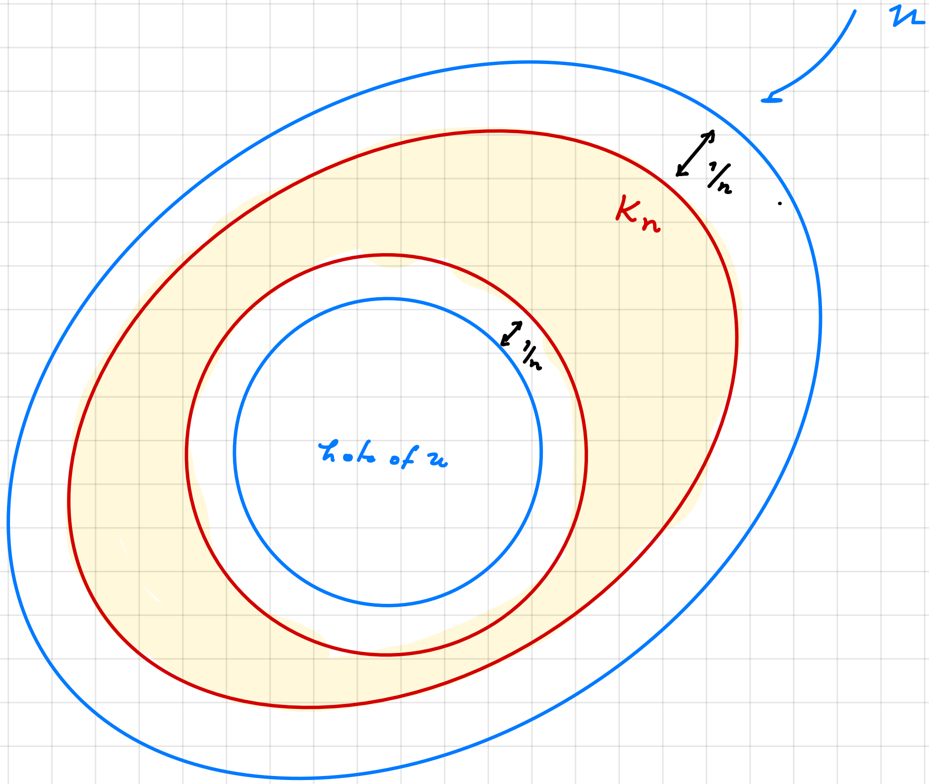
Thus $R_n \rightrightarrows f$ in K, as needed.

Proof of the Topological Lemma

} Conway VII.1.2

WLOG $\mathcal{U} \neq \mathbb{C}$.

Let $K_n = \{ z : |z| \leq n \text{ and } d(z, \underbrace{\mathbb{C} \setminus \mathcal{U}}_{\text{closed.}}) \geq \frac{1}{n} \}$.



It is easy to see $\boxed{\text{i}}$ – $\boxed{\text{iii}}$ hold, using the above pictures.

The technical details follow (see also Conway).

$$K_n = \{ z : |z| \leq n \text{ and } d(z, \mathbb{C} \setminus \mathcal{U}) \geq \tfrac{1}{n} \}.$$

Claim 1 $K_n \subseteq \mathcal{U}$

Proof If $z \in K_n \Rightarrow d(z, \mathbb{C} \setminus \mathcal{U}) \geq \frac{1}{n} \Rightarrow z \notin \mathbb{C} \setminus \mathcal{U} \Rightarrow z \in \mathcal{U}$. Thus $K_n \subseteq \mathcal{U}$.

Claim 2. $\mathcal{U} = \bigcup_{n \geq 1} K_n$

Proof If $z \in \mathcal{U}$ then let n such that $n \geq |z|$ & $d(z, \mathbb{C} \setminus \mathcal{U}) \geq \frac{1}{n}$ which is possible since $d(z, \mathbb{C} \setminus \mathcal{U}) > 0$. Thus $z \in K_n$. $\Rightarrow \mathcal{U} \subseteq \bigcup_n K_n \overset{\text{claim1.}}{\subseteq} \mathcal{U}$

Claim 3 K_n closed & bounded $\Rightarrow K_n$ compact.

Proof K_n is closed since

$$\mathbb{C} \setminus K_n = \{ |z| > n \} \cup \{ z : \exists b \notin \mathcal{U},\ d(z,b) < \tfrac{1}{n} \}$$

$$= \{ |z| > n \} \cup \bigcup_{b \notin \mathcal{U}} \Delta (b, \tfrac{1}{n}). = \text{open}.$$

Claim 4 $K_n \subseteq \operatorname{Int} K_{n+1}$

Proof Let $z \in K_n$. Let $r < \frac{1}{n} - \frac{1}{n+1}$. then

$$\Delta(z, r) \subseteq K_{n+1} \implies z \in \operatorname{Int} K_{n+1} \quad \text{as needed.}$$

To see $\Delta(z,r) \subseteq K_{n+1}$ note for $w \in \Delta(z,r)$

$$|w| \le |z| + |w - z| \le n + r < n+1 \quad \text{and}$$

$$d(w, \mathbb{C} \setminus \mathcal{U}) \ge d(z, \mathbb{C} \setminus \mathcal{U}) - d(z, w) \ge \frac{1}{n} - r > \frac{1}{n+1}.$$

$\implies w \in K_{n+1}$, as needed.

Claim 5 Each compact $K \subseteq \mathcal{U}$ is contained in some K_n.

Proof Let $K \subseteq \mathcal{U} \overset{\text{claim 2}}{=} \bigcup_m K_m \overset{\text{claim 4}}{\subseteq} \bigcup_m \operatorname{Int} K_{m+1}$. Since K is compact we find a finite subcover by $\operatorname{Int} K_j$, $j \le n$.

$$\implies K \subseteq \bigcup_{j \le n} \operatorname{Int} K_j \underbrace{\subseteq}_{\text{claim 4.}} K_n$$

Claim 6 Let $A = \hat{\mathbb{C}} \setminus K_n$, $B = \hat{\mathbb{C}} \setminus \mathcal{U}$ $\Rightarrow A \supseteq B \ni \infty$

(+) Each component of A contains a component of B.

Proof This is a bit more technical. We will use repeatedly:

Easy important fact (by definition)

If $Z \subseteq A$ connected & Z intersects a component A° of A

$\Rightarrow Z \subseteq A^\circ$.

Proof of (+) Let A° be a component of A. By Claim 3 (proof):

$$A = \{z \in \hat{\mathbb{C}} : |z| > n\} \cup \bigcup_{b \in B} \Delta(b, \tfrac{1}{n})$$

contains ∞

$\boxed{\text{I}}$ Note $\infty \in A$. If A° is the component containing ∞, let B° be the component of B containing $\infty \in B$. Note

$$A^\circ \cap B^\circ \neq \emptyset \text{ (contains } \infty) \;\&\; B^\circ \subseteq A \underset{\substack{\text{easy}\\ \text{fact}}}{\Longrightarrow} B^\circ \subseteq A^\circ.$$

This is what we wanted to show.

[ii] If $\infty \notin A^\circ$, then A° cannot be disjoint from all sets $\Delta(b, \frac{1}{n})$.

Why: Else $A^\circ \subseteq \underbrace{\Delta(\infty,n)}_{\text{connected set}} \subseteq A \underset{\text{easy fact.}}{\Rightarrow} \Delta(\infty,n) \subseteq A^\circ \Rightarrow \infty \in A^\circ$.

connected set & intersects A°

false.

Thus $\exists\, b \in B$ with $A^\circ \cap \Delta(b, \frac{1}{n}) \neq \Phi$. Note

$\Delta(b, \frac{1}{n}) \subseteq A$ & intersects A° $\underset{\text{easy fact}}{\Rightarrow}$ $\Delta(b, \frac{1}{n}) \subseteq A^\circ$.

Let $b \in B^\circ$ for some component B°.

Then $B^\circ \cap A^\circ \neq \Phi$ & $B^\circ \subseteq B \subseteq A \underset{\text{easy fact}}{\Rightarrow}$ $B^\circ \subseteq A^\circ$ as needed.