Math 220, Problem Set 5.

0. Regarding Math 220C:

(i) do you prefer meeting 2 times a week MW (75 minutes) instead of 3 times a week MWF (50 minutes)? Answers: Yes/No/Don't have a preference.
(ii) can you make MW 3-4:15 if we decide to change the time? Answers: Yes/No.
(iii) can you make MW 1-2:15 if we decide to change the time? Answers: Yes/No.
(iv) do you prefer live lectures/recorded lectures/first half live (material required for the qual) - second half recorded (material on Riemann surfaces)? Answers: live/recorded/half-half.

1. (*Schwarz-Pick for the unit disc.*) The pseudo-hyperbolic distance on the unit disc $\Delta = \Delta(0,1)$ is given by

$$d(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|, \quad z,w \in \Delta.$$

(i) Show, by direct calculation, that if f is an automorphism of Δ then

$$d(z,w) = d(f(z), f(w)).$$

In other words, automorphisms of Δ preserve the pseudo-hyperbolic distance i.e. they are *isometries.*

Hint: You may wish to recall that f is a composition of a rotation with the fractional linear transformation ϕ_a. It thus suffices to check the above equality for f a rotation and for $f = \phi_a$ separately.

(ii) Let $f : \Delta \to \Delta$ be holomorphic. Using (i) to reduce to a familiar case, show that

$$d(f(z), f(w)) \leq d(z,w).$$

Thus holomorphic maps contract the pseudo-hyperbolic distance.

Further hint: Recentering (i.e. considering $\Phi \circ f \circ \Psi$ for suitable automorphisms Φ, Ψ of the disc) you may assume $f(w) = 0$ and $w = 0$.

(iii) Show that if there exist $z, w \in \Delta$ and $f : \Delta \to \Delta$ holomorphic with

$$d(f(z), f(w)) = d(z,w)$$

then f is an automorphism of Δ. Consequently, by (i), the equality

$$d(f(z), f(w)) = d(z,w)$$

holds for all $z, w \in \Delta$.

(iv) Show that d is indeed a distance. That is, show that

$$d(z,s) \leq d(w,s) + d(z,w).$$

You may wish to reduce to the case $s = 0$ using part (i).

Further hint: When $s = 0$, you may first rotate z to make it positive real. Using polar coordinates $z = r_1, w = r_2 e^{it}$ you need to establish a linear inequality in $\cos t$ which only needs to be checked at the endpoints $\cos t = \pm 1$ (why?)

(v) As an application to (ii), assume $f : \Delta \to \Delta$ satisfies $f\left(\frac{1}{2}\right) = \frac{1}{4}$. Show that

$$\frac{1}{21} \leq \left| f\left(\frac{1}{3}\right) \right| \leq \frac{9}{19}.$$

Remark (only if you have seen some differential geometry): The hyperbolic distance is given by $2 \tanh^{-1} d(z, w)$. It comes from the metric

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2}$$

on the unit disc Δ, whose Gaussian curvature equals -1. The Schwarz-Pick lemma can be further generalized with this observation as the starting point, for holomorphic maps between domains/Riemann surfaces with appropriate curvature.

2. (*Schwarz-Pick for the upper half plane.*) On the upper half plane, define

$$d(z, w) = \left| \frac{z - w}{z - \overline{w}} \right|, \quad z, w \in \mathfrak{h}^+.$$

Formulate and briefly justify the analogues of Problem 1(i) and (ii) for $\mathfrak{h}^+$.

Hint: You won't have to redo the proofs in Problem 1 – the shortcut is to check that the Cayley transform exchanges the two distances defined in Problem 1 and Problem 2.

3. (*Generalized Schwarz Lemma.*) Assume $f : \Delta \to \Delta$ is holomorphic with a zero of order n at the origin. Show that

$$|f(z)| \leq |z|^n \quad \forall z \in \Delta, \qquad \text{and} \qquad |f^{(n)}(0)| \leq n!.$$

4. (*Riemann Mapping Theorem.*) Let $U \subset \mathbb{C}$ be simply connected, $U \neq \mathbb{C}$.

(i) Show that any holomorphic map $f : U \to U$, other than the identity, admits at most one fixed point.

(ii) Show that the above statement is false for $U = \mathbb{C}$.

(iii) Exhibit a nonsimply connected set U and a function $f : U \to U$, not equal to the identity, for which (i) fails.

5. (*Riemann Mapping Theorem.*) Exhibit a biholomorphic map from $\{z : -1 < \operatorname{Re} z < 1\}$ to the unit disc.

6. (*Riemann Mapping Theorem.*) Exhibit a biholomorphic map between the slit unit disc $\Delta \setminus (-1, 0]$ and the unit disc.

Hint: In Problems 5 and 6, think of several geometric moves that can take you from the strip or the slit disc to the unit disc using familiar maps.