Math 220 B — Lecture 18

February 19, 2021

§ 0. Riemann Mapping theorem

Theorem $\mathcal{U} \neq \mathbb{C}$ simply connected $\Rightarrow$ $\mathcal{U}$ biholomorphic to the unit disc. $\Delta = \Delta(0,1)$.

Ingredients in the proof

$\boxed{\text{i}}$ Montel & normal families

$\boxed{\text{ii}}$ Hurwitz's Theorem

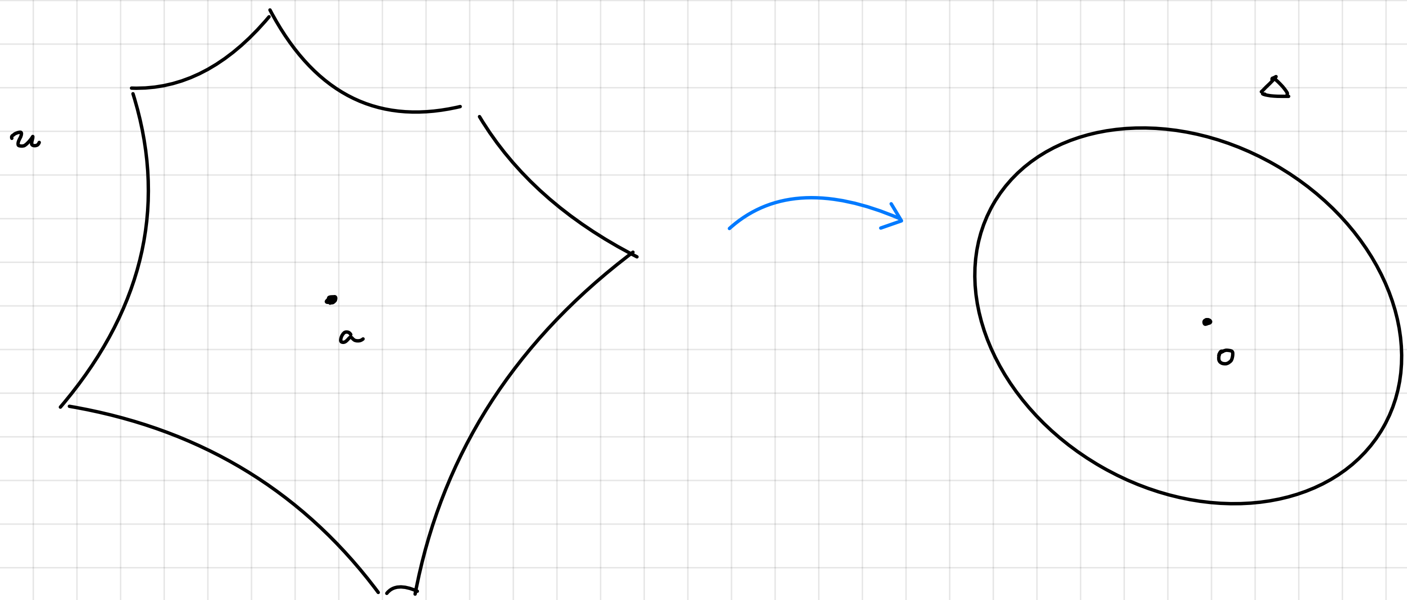
$\boxed{\text{iii}}$ Aut Δ & Schwarz Lemma

$\boxed{\text{iv}}$ Square root trick of Caratheodory – Koebe.

& standard tools: Open Mapping & Weierstraß.

§ 1. Strategy

Fix $a \in \mathcal{U}$



Want $f : \mathcal{U} \longrightarrow \Delta$ & $f(a) = 0$ & f bijective.

Goal #1 First, $f : \mathcal{U} \longrightarrow \Delta$, $f(a) = 0$ &

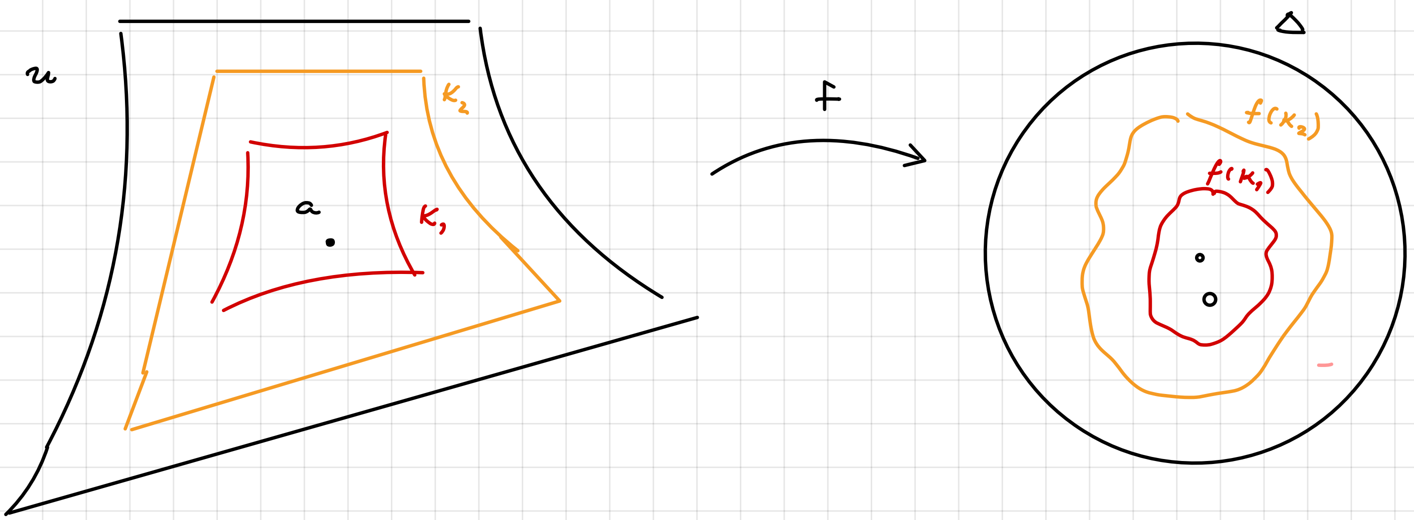
f injective

Main Actor in the Proof Consider the family

$$\mathcal{F} = \{ f : \mathcal{U} \longrightarrow \Delta,\ f(a) = 0,\ f \text{ injective} \}.$$

Want $\mathcal{F} \neq \emptyset$.

Question How to achieve f bijective?



Imagine $\mathcal{U} = \bigcup_n K_n$, $a \in K_n \subseteq \operatorname{Int} K_{n+1}$.

We hope $\bigcup_n f(K_n)$ cover Δ. We expect that this has a chance if $|f'(a)|$ is as large as possible.

Let $M = \sup \{ |f'(a)| : f \in \mathcal{F} \}$.

Goal #2 Show $\exists f \in \mathcal{F}$ with $|f'(a)| = M$.

Goal #3 Show that for this choice, $f : \mathcal{U} \longrightarrow \Delta$ is bijective

Why might this actually work?

Example $U = \Delta$, $a = 0$.

$$\mathcal{F} = \{ f : \Delta \longrightarrow \Delta,\ f(0) = 0,\ f \text{ injective} \}.$$

By Schwarz Lemma, $|f'(0)| \leq 1$. If the maximum value $|f'(0)| = 1$ then f is a rotation so f is bijective.

Remark

We can also consider points $a \in \Delta$, $a \neq 0$. Let

$$\mathcal{F} = \{ f : \Delta \longrightarrow \Delta,\ f(a) = 0,\ f \text{ injective} \}.$$

Schwarz - Pick

$$|f'(a)| \leq \frac{1}{1 - |a|^2} \quad \text{with equality iff}$$

$$f = \mathrm{Rot} \circ \varphi_a \implies f \text{ bijective.}$$

Question How do we use U simply connected?

Answer Math 220A, Homework 4

2. Assume $f : U \to \mathbb{C}$ is a holomorphic function on a simply connected open set U such that $f(z) \neq 0$ for all $z \in U$. Let $n \geq 2$ be an integer. Show that there is a holomorphic function $g : U \to \mathbb{C}$ such that

$$g(z)^n = f(z).$$

Hint: This has something to do with problem 1(ii).

We only need $n = 2$.

U simply connected $\Rightarrow$ any $f: U \longrightarrow \mathbb{C}$ holomorphic, nowhere zero, admits a holomorphic square root $g: U \to \mathbb{C}$

$$f = g^2 \quad (*)$$

"Root domain"

- $U \subseteq \mathbb{C}$ is a root domain if $(*)$ is satisfied.

Remark simply connected $\Rightarrow$ root domain

Remark This turns out to be equivalent to $\mathcal{U}$ simply connected

We will prove: the seemingly stronger form:

Riemann Mapping Theorem

$\mathcal{U} \neq \mathbb{C}$ root domain $\Rightarrow$ $\mathcal{U}$ is biholomorphic to Δ.

§ 2. Study of the family $\mathcal{F}$

Fix $a \in \mathcal{U}$.

$$\mathcal{F} = \{ f : \mathcal{U} \longrightarrow \Delta : f \text{ holomorphic, injective, } f(a) = 0 \}.$$

Step 1 If $\mathcal{U}$ is a root domain, $\mathcal{U} \neq \mathbb{C}$ $\Rightarrow$ $\mathcal{F} \neq \emptyset$

Proof Let $b \notin \mathcal{U}$, which is possible since $\mathcal{U} \neq \mathbb{C}$.

Consider $h(z) = z - b$, $h : \mathcal{U} \longrightarrow \mathbb{C}$. Note $h(z) \neq 0$ for $z \in \mathcal{U}$, since $b \notin \mathcal{U}$. Thus h admits a square root

$g : \mathcal{U} \longrightarrow \mathbb{C}$, $g(z)^2 = z - b$.

Claim 1 g injective.

Indeed, if $g(z_1) = g(z_2) \Rightarrow g(z_1)^2 = g(z_2)^2 \Rightarrow$

$\Rightarrow z_1 - b = z_2 - b \Rightarrow z_1 = z_2$.

Claim 2 $g(\mathcal{U}) \cap (-g)(\mathcal{U}) = \emptyset$.

Indeed, if $\exists\, z_1, z_2 \in \mathcal{U}$ with $g(z_1) = -g(z_2)$

$\Rightarrow g(z_1)^2 = g(z_2)^2 \Rightarrow z_1 - b = z_2 - b \Rightarrow z_1 = z_2$.

But then $g(z_1) = -g(z_2) \Rightarrow g(z_1) = -g(z_1) \Rightarrow g(z_1) = 0$

$\Rightarrow g(z_1)^2 = 0 = z_1 - b \Rightarrow z_1 = b$. But $z_1 \in \mathcal{U}$, $b \notin \mathcal{U}$.

<u>Claim 3</u> $\exists\ c, r$ with $|g(z) - c| > r \quad \forall z \in \mathcal{U}$.

Indeed, by the open mapping theorem, $(-g)(\mathcal{U})$ is open so it contains a disc $\overline{\Delta}(c,r)$. By Claim 2,

$g(\mathcal{U}) \subseteq \mathbb{C} \setminus \overline{\Delta}(c,r) \iff |g(z) - c| > r \quad \forall z \in \mathcal{U}$.

<u>Construction</u> Let $f(z) = \dfrac{r}{g(z) - c}$. $\Rightarrow f$ injective since g is by Claim 1 & $f : \mathcal{U} \longrightarrow \Delta(0,1)$ by Claim 3.

To achive $f(a) = 0$, define $\tilde{f}(z) = \dfrac{f(z) - f(a)}{2}$.

$\Rightarrow \tilde{f}$ injective since f is. & $\tilde{f}(a) = 0$.

Note that since f takes values in Δ, the same is true for $\tilde{f}$

$$|\tilde{f}(z)| \leq \frac{1}{2}\left(|f(z)| + |f(a)|\right) < \frac{1}{2}(1+1) = 1$$

Thus $\tilde{f} \in \mathcal{F} \Rightarrow \mathcal{F} \neq \Phi$.

Step 2 Let $M = \sup\{|f'(a)|, f \in \mathcal{F}\}$

Show: The supremum is achieved by some $f \in \mathcal{F}$.

Proof: Indeed, take $f_n \in \mathcal{F}$ with $|f_n'(a)| \longrightarrow M$. as $n \longrightarrow \infty$

The family $\mathcal{F}$ is bounded by 1 since the functions in $\mathcal{F}$ take values in Δ. $\xRightarrow{\text{Montel}}$ $\mathcal{F}$ normal. $\Rightarrow$

$\Rightarrow$ passing to a subsequence, we may assume

$f_n \rightrightarrows f$ locally uniformly.

Claim 4 f holomorphic, $f(a) = 0$, $|f'(a)| = M$.

Indeed, by Weierstraß convergence, f is holomorphic.

and $f_n' \rightrightarrows f'$ locally uniformly. In particular,

$f_n'(a) \longrightarrow f'(a)$ so $|f'(a)| = M$.

Since $f_n(a) = 0$ & $f_n \rightrightarrows f$ at a, we have

$f(a) = 0$.

Claim 5. $f: U \longrightarrow \Delta$ & f injective.

Indeed, f_n injective & $f_n \overset{l.u.}{\rightrightarrows} f$ shows f is either

injective or f constant by Hurwitz's theorem

(Math 220 A, Lecture 24).

If f = constant $\Rightarrow f'(a) = 0 \Rightarrow M = 0 \Rightarrow$

$\Rightarrow g'(a) = 0 \quad \forall g \in \mathcal{F}$ since M is the supremum.

But if $g \in \mathcal{F}$, g injective and $g'(a) \neq 0$ by

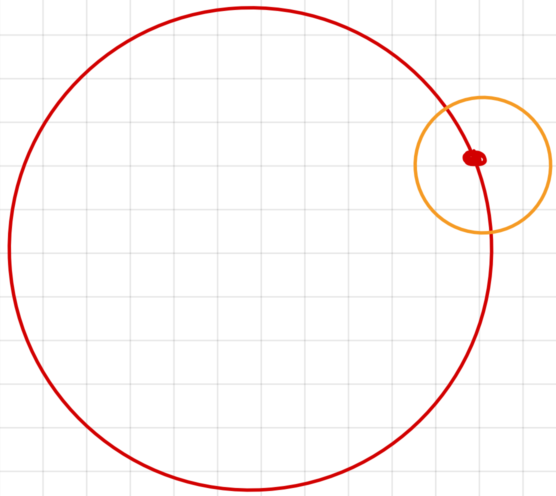
Math 220 A, Final Exam, Problem 7.

Thus f injective.

Note that since $f_n \overset{l.u}{\rightrightarrows} f$ and $f_n : U \longrightarrow \Delta$

shows $f : U \longrightarrow \overline{\Delta}$. By the open mapping theorem,

$f : U \longrightarrow \Delta$ ($f \neq$ not constant).



By Claims 4 & 5, $f \in \mathcal{F}$ and $|f'(a)| = M \Longrightarrow$ Step 2 ✓.

Step 3 For a function $f \in \mathcal{F}$ which achieves the supremum

f is bijective.

Proof : next time.