Math 220A - Fall 2016 - Final Exam Solutions

Problem 1.

Consider the function $f(z) = ze^{3-z} - 1$. Show that f has exact one zero inside the disc $\Delta(0, 1)$.

`Answer`: *We apply Rouché's theorem to* $f(z) = ze^{3-z} - 1$ *and* $g(z) = ze^{3-z}$ *and the curve* $|z| = 1$. *We have*

$$|f - g| = 1, |g| = |ze^{3-z}| = |z| \cdot e^{3-Re\ z} = e^{3-Re\ z} \geq e^2 > |f - g|$$

where we used $Re\ z \leq 1$ *for* $|z| = 1$. *Therefore,* f *and* g *have the same number of zeroes inside the unit disc. Clearly* g *vanishes only at* $z = 0$, *hence* f *must have only one simple zero in* $\Delta(0, 1)$ *as well.*

Problem 2.

Calculate the integral
$$\int_0^\infty \frac{dx}{x^{2n}+1}, \text{ for } n \geq 2.$$

`Answer`: *We consider γ_R the curve consisting of the segment $[-R,R]$ and the half disc S_R. We assume $R>1$. By the residue theorem, we have*
$$\int_{\gamma_R} \frac{dz}{z^{2n}+1} = 2\pi i \sum_{a^{2n}+1=0} Res\left(\frac{1}{z^{2n}+1}, a\right).$$
The possible values for the poles are
$$a = \zeta_k = \exp\left((2k+1)\cdot\frac{\pi i}{2n}\right), \ \ 0 \leq k \leq n-1.$$
The last inequality $k \leq n-1$ comes from the fact that the poles must be contained in the upper half plane. Using the method of computing residues in class, we have
$$Res\left(\frac{1}{z^{2n}+1}, \zeta_k\right) = \frac{1}{(z^{2n}+1)}|_{z=\zeta_k} = \frac{1}{2n\zeta_k^{2n-1}} = -\frac{\zeta_k}{2n},$$
using that $\zeta_k^{2n} = -1$. Thus
$$\begin{aligned}\int_{\gamma_R} \frac{dz}{z^{2n}+1} &= -2\pi i \sum_{k=0}^{n-1} \frac{\zeta_k}{2n} = -2\pi i \cdot \frac{\exp\left(\frac{\pi i}{2n}\right)}{2n} \cdot \sum_{k=0}^{n-1} \exp\left(\frac{k\pi i}{n}\right) \\ &= -2\pi i \cdot \frac{\exp\left(\frac{\pi i}{2n}\right)}{2n} \cdot \frac{1-\exp(\pi i)}{1-\exp\left(\frac{\pi i}{n}\right)} = -\frac{2\pi i}{n} \cdot \frac{\exp\left(\frac{\pi i}{2n}\right)}{1-\exp\left(\frac{\pi i}{n}\right)} \\ &= -\frac{2\pi i}{n} \cdot \frac{1}{\exp\left(-\frac{\pi i}{2n}\right) - \exp\left(\frac{\pi i}{2n}\right)} = \frac{\pi}{n} \cdot \frac{1}{\sin\left(\frac{\pi}{2n}\right)}.\end{aligned}$$
As a result,
$$\int_{-R}^{R} \frac{dx}{x^{2n}+1} + \int_{S_R} \frac{dz}{z^{2n}+1} = \frac{\pi}{n} \cdot \frac{1}{\sin\left(\frac{\pi}{2n}\right)}.$$
We show
$$\lim_{R\to\infty} \int_{S_R} \frac{dz}{z^{2n}+1} = 0.$$
Indeed, $|z^{2n}+1| \geq R^{2n}-1$, and by the basic estimate
$$\left|\int_{S_R} \frac{dz}{z^{2n}+1}\right| \leq 2\pi R \cdot \frac{1}{R^{2n}-1} \to 0.$$
In consequence,
$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^{2n}+1} = \frac{\pi}{n} \cdot \frac{1}{\sin\left(\frac{\pi}{2n}\right)} \implies 2\int_0^\infty \frac{dx}{x^{2n}+1} = \frac{\pi}{n} \cdot \frac{1}{\sin\left(\frac{\pi}{2n}\right)} \implies \int_0^\infty \frac{dx}{x^{2n}+1} = \frac{\pi}{2n} \cdot \frac{1}{\sin\left(\frac{\pi}{2n}\right)}.$$

Problem 3.

Consider

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_n.$$

Show that there exists c with $|c| = 1$ such that

$$|f(c)| \geq 1.$$

`Answer`: *By Cauchy's estimates, we have*

$$|f^{(n)}(0)| \leq \frac{n!}{r^n} \cdot M(r)$$

for $M(r) = \sup_{|z|=r} |f(z)|$. *Making* $r = 1$ *and noting* $f^{(n)}(0) = n!$ *we obtain*

$$1 \leq M(1) = \sup_{|z|=1} |f(z)| \implies |f(c)| \geq 1 \textit{ for some } |c| \geq 1.$$

`Alternate Answer`: *Let*

$$g(z) = z^n f(1/z) = a_n z^n + \ldots + a_1 z + 1.$$

Note that $g(0) = 1$. *By the maximum modulus principle,*

$$\max_{|z|=1} |g(z)| = \max_{|z|\leq 1} |g(z)| \geq |g(0)| = 1.$$

Thus, there exists $|z_0| = 1$, *with* $|g(z_0)| \geq 1$. *We have*

$$|g(z_0)| = |z_0^n f(1/z_0)| = |f(1/z_0)| \geq 1.$$

Setting $c = 1/z_0$, *we obtain*

$$|f(c)| \geq 1$$

and $|c| = 1$.

`Alternate Answer`: *Assume that for all* $|z| = 1$ *we have* $|f(z)| < 1$. *Let*

$$G(z) = -z^n, \quad F(z) = f(z) + G(z).$$

Then for $|z| = 1$ *we have*

$$|F(z) - G(z)| = |f(z)| < 1 = |z^n| = |G(z)|.$$

By Rouché's theorem, F *and* G *have the same number of zeros inside the unit disc counted with multiplicity. Clearly,* G *has exactly one zero with multiplicity* n. *However,*

$$F(z) = f(z) + G(z) = a_1 z^{n-1} + \ldots + a_n$$

is either identically equal to 0 *or it has at most* $n - 1$ *zeros. The latter case contradicts Rouché. The former case shows* $f(z) = -G(z)$ *and the assertion of the problem is trivial since* $|f(c)| = |G(c)| = |c^n| = 1$ *for* $|c| = 1$.

Problem 4.

Assume that f is entire and $f(z) = f(z+1)$ such that $|f(z)| \leq e^{|z|}$. Show that f is constant.

(i) Consider
$$g(z) = \frac{f(z) - f(0)}{\sin \pi z}.$$
Show that g is periodic and that g can be extended to an entire function.

(ii) By direct calculation, show that g is bounded in the strip $0 \leq \text{Re } z \leq 1$.

(iii) Conclude from (ii) that $g = 0$ hence f is constant.

`Answer`:

(i) *We have*
$$g(z+1) = \frac{f(z+1) - f(0)}{\sin \pi(z+1)} = \frac{f(z) - f(0)}{\sin \pi z} = g(z).$$
To show that g is entire, it suffices to show that g has removable singularities at $z = n$, $n \in \mathbb{Z}$. By periodicity, it suffices to prove that g has removable singularity at $z = 0$. But
$$\lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} \cdot \frac{z}{\sin \pi z} = f'(0) \cdot \frac{1}{\pi}$$
hence g is bounded near 0, so the singularity is removable.

(ii) *Write $z = x + iy$ with $0 \leq x \leq 1$. We have*
$$|g(z)| \leq \frac{|f(z)| + |f(0)|}{|\sin \pi z|} \leq \frac{e^{|z|} + |f(0)|}{|\sin \pi z|}.$$
We have $e^{|z|} \leq e^{|y|+1}$ since $|z| = |x + iy| \leq |x| + |y| \leq 1 + |y|$. Assume $y > 0$, the argument for $y < 0$ being similar. Similarly,
$$|2 \sin \pi z| = |e^{\pi i z} - e^{-\pi i z}| = |e^{\pi i x} e^{-\pi y} - e^{-\pi i x} e^{\pi y}| \geq e^{\pi y} |e^{-\pi i x}| - e^{-\pi y} |e^{\pi i x}| = e^{\pi y} - e^{-\pi y} > 0.$$
Thus
$$|g(z)| \leq \frac{e^{y+1} + |f(0)|}{e^{\pi y} - e^{-\pi y}}.$$
The last expression converges to 0 as $y \to \infty$, so it is bounded. Thus $|g(z)|$ is bounded in the strip $0 \leq Re\ z \leq 1$.

(iii) *Since g is bounded for $0 \leq Re\ z \leq 1$, it follows by periodicity that g is bounded over the complex plane. By Liouville's theorem, g is constant. We have also seen that $\lim_{y \to \infty} g(z) = 0$, hence this constant must vanish. Thus $g(z) = 0 \implies f(z) = f(0)$ showing f is constant.*

Problem 5.

Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function with $\deg P \le \deg Q - 2$ such that Q has no zeros along the non-negative real axis. Show that

$$\int_0^\infty f(x)\,dx = -\sum_{a\in\mathbb{C}\setminus\mathbb{R}_{\ge 0}} \operatorname{Res}_{z=a}(f(z)\log z)$$

where for $z \in \mathbb{C}\setminus\mathbb{R}_{\ge 0}$ we set $\log z = \log r + i\theta$ and $\theta \in (0, 2\pi)$.

`Answer`: *We consider the keyhole contour consisting of a portion of a circle $C_{r^\star}$ of radius $r^\star$, a portion of a circle $C_{R^\star}$ of radius $R^\star > r^\star$, and two line segments L^+ and L^- at distance δ and $-\delta$ away from the positive x-axis, going from $[r, R]$. Here $R^\star = \sqrt{R^2+\delta^2}$, $\quad r^\star = \sqrt{r^2+\delta^2}$. We let*

$$\gamma = L^+ \cup C_{R^\star} \cup (-L^-) \cup (-C_{r^\star}).$$

By the residue theorem

$$\int_\gamma f(z)\log z\,dz = 2\pi i \sum_a Res_{z=a}(f(z)\log z).$$

This gives

$$\int_{C_{R^\star}} R(z)\log z\,dz - \int_{C_{r^\star}} R(z)\log z\,dz + \int_{L^+} R(z)\log z\,dz - \int_{L^-} R(z)\log z\,dz = 2\pi i \sum_a Res_{z=a}(R(z)\log z).$$

We first make $\delta \to 0$, and then we make $r \to 0$, $R \to \infty$. We claim the limits of the integrals over $C_{R^\star}$ and $C_{r^\star}$ equal 0. We first consider $C_{R^\star}$. Indeed, since

$$\deg P \le \deg Q - 2 \implies \lim_{z\to\infty} z^2 f(z) < \infty.$$

Hence

$$|f(z)| \le \frac{M}{|z|^2},$$

for $|z|$ sufficiently large. If $|z| = R^\star$, then $|\log z| = |\log R^\star + i\theta| \le |\log R^\star| + 2\pi$. By the basic estimate, we have

$$\left|\int_{C_R^\star} f(z)\log z\,dz\right| \le 2\pi R^\star \cdot \frac{M}{R^{\star 2}} \cdot (\log R^\star + 2\pi) \to 0.$$

The estimates for $C_{r^\star}$ are similar, but using that $f(z)$ is bounded near the origin by continuity. Then,

$$\left|\int_{C_{r^\star}} f(z)\log z\,dz\right| \le 2\pi r^\star \cdot M' \cdot (|\log r^\star| + 2\pi) \to 0 \ as r^\star \to 0.$$

Here, we used $r^\star \log r^\star \to 0$ which can be seen using the change of variables $r^\star = 1/y$ with $y \to \infty$.

By continuity arguments (f is continuous, and $\log$ is continuous provided we stay in compact regions either above or below the non-negative real axis), we have

$$\lim_{\delta\to 0}\int_{L^+} f(z)\log z\,dz = \int_r^R f(x)\log x\,dx$$

$$\lim_{\delta\to 0}\int_{L^-} f(z)\log z\,dz = \int_r^R f(x)(\log x + 2\pi i)\,dx = \int_r^R f(x)\log x\,dx + 2\pi i\int_r^R f(x)\,dx.$$

Substituting in the residue theorem, after taking limits, first for $\delta \to 0$, and then $r \to 0$, $R \to \infty$, we obtain

$$-2\pi i\int_0^\infty f(x)\,dx = 2\pi i\sum_{a\in\mathbb{C}\setminus R_{\geq 0}} Res_{z=a}(f(z)\log z).$$

This completes the proof.

Problem 6.

Let $a, b \neq 0$ be real numbers and U a connected open set. Let $f : U \to \mathbb{C}$ be a holomorphic function. Show that if $a \operatorname{Re} f + b \operatorname{Im} f$ is constant, then f is constant.

`Answer`: *The image of $f = u + iv$ is contained in the line $au + bv = c$, by assumption. This violates the open mapping theorem, unless f is constant.*

`Alternate Answer`: *Let $f = u + iv$. Then*

$$au + bv = c \implies au_x + bv_x = 0 \text{ and } au_y + bv_y = 0.$$

Using the Cauchy-Riemann equations, we have

$$u_x = v_y, u_y = -v_x$$

hence substituting

$$au_x - bu_y = 0, \;\; au_y + bu_x = 0.$$

The above system has matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with determinant $a^2 + b^2 \neq 0$. Thus $u_x = u_y = 0$. Similarly $v_x = v_y = 0$. This shows that u and v must be constant, hence f must be constant as well.

Problem 7.

Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

`Answer`: *Assume $f(\mathbb{C})$ is not dense in $\mathbb{C}$. In this case, there exists $\lambda \in \mathbb{C}$ and $R > 0$ such that*

$$f(\mathbb{C}) \cap \Delta(\lambda, R) = \emptyset.$$

In other words,

$$|f(z) - \lambda| \geq R$$

for all $z \in \mathbb{C}$. Consider the function

$$g(z) = \frac{1}{f(z) - \lambda}.$$

Clearly, g is holomorphic since $f(z) \neq \lambda$. Furthermore,

$$|g(z)| \leq \frac{1}{R}$$

so g is bounded. By Liouville's theorem, g must be constant. This in turn implies that f is constant, a contradiction.

Problem 8.

Assume that f is continuous in the closed unit disc $\overline{\Delta}$ and holomorphic inside the unit disc Δ. Assume that

$$|f(z)| = 1 \text{ for all } |z| = 1.$$

(i) If f is nonconstant, show that f must have a zero inside Δ.

(ii) Show that if f has a unique simple zero at $z = 0$ then $f(z) = \alpha z$.

`Answer`:

(i) *By the maximum modulus principle, $|f|$ must achieve the maximum on the boundary. If f has no zeros, then the minimum modulus principle holds as shown in class. Thus the minimum of $|f|$ also occurs on the boundary. But $|f| = 1$ on the boundary, so both the min and the max are 1. Thus $|f| = 1$ is constant on Δ. This contradicts the open mapping theorem, since the image of f is then not open unless f is constant.*

(ii) *Consider the function*

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}.$$

Clearly, g is holomorphic in $\Delta \setminus \{0\}$ and continuous at 0, hence bounded near 0. Therefore, g has a removable singularity at $z = 0$, so it can be extended to a holomorphic function across the origin. The function g satisfies

$$|g(z)| = 1 \text{ when } |z| = 1.$$

Furthermore, g has no zeroes on $\Delta \setminus \{0\}$ and

$$g(0) = f'(0) \neq 0$$

since the order of the zero for f equals 1. By part (i), it follows that g is constant $g = \alpha$, hence $f(z) = \alpha z$.