Math 220B - Lecture 23

March 3, 2021

Math 220C Survey

- first half: MWF 3-3:50, live
- second half: TBD.

Remaining Topics in Math 220B

- Proof of Runge C (today, Friday)
- Runge O (Monday)
- Summary & Loose Ends (Wednesday)
- Review (Friday)

Runge C (Final) $\Longrightarrow$ Runge C (Almost Final)

Conway VIII.1.7

- rational approximation
- version for $\hat{\mathbb{C}}$

- rational approximation
- poles in each hole

Little Runge C

- polynomial approximation
- K has no holes

Last time

Thm Let $K \subseteq \mathbb{C}$ compact. Let $S \subseteq \hat{\mathbb{C}}$ be a set of points, at least one chosen from each component of $\hat{\mathbb{C}} \setminus K$.

Let f be holomorphic in K. Then

$\boxed{i}$ $\exists\ R_n \rightrightarrows f$ in K

$\boxed{ii}$ R_n are rational with possible poles in S.

in $\hat{\mathbb{C}}$

Strategy

Step 1 Cauchy Integral Formula for compact sets.

Step 2 Approximation without prescribed poles

Step 3 Push the poles to prescribed location.

Step 1

Recall (Math 220A).

f holomorphic in $\mathcal{U}$, $\overline{R} \subseteq \mathcal{U}$, then

$$\frac{1}{2\pi i} \int_{\partial R} \frac{f(z)}{z-a}\, dz = \begin{cases} f(a). & \forall\, a \in R \\ 0 & \forall\, a \notin \overline{R} \end{cases}$$

We wish to do the same for any compact $K \subseteq \mathcal{U}$.

Lemma

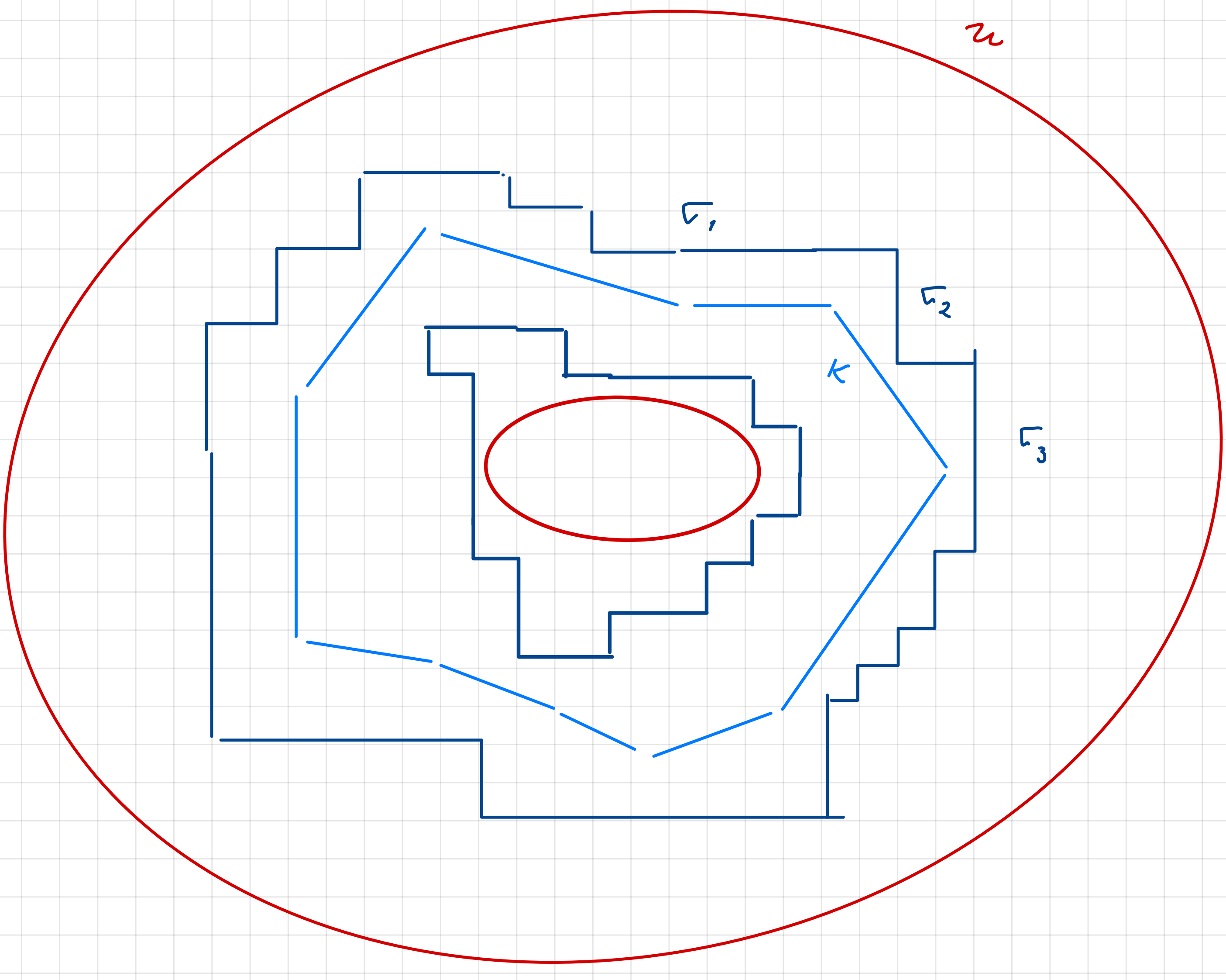
Conway VIII. 1.1.

Let $K \subseteq \mathcal{U}$ compact. There exist segments γ_j such that

$$\gamma = \gamma_1 + \dots + \gamma_n \subseteq \mathcal{U} \setminus K$$

and such that for all functions f holomorphic in $\mathcal{U}$

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(z)}{z-a}\, dz. \qquad \forall\, a \in K.$$



We will construct Γ as a union of closed polygons.

Remark If K has a simple structure this is not so bad. We'd need

$$n(\Gamma, a) = 1 \quad \forall a \in K.$$

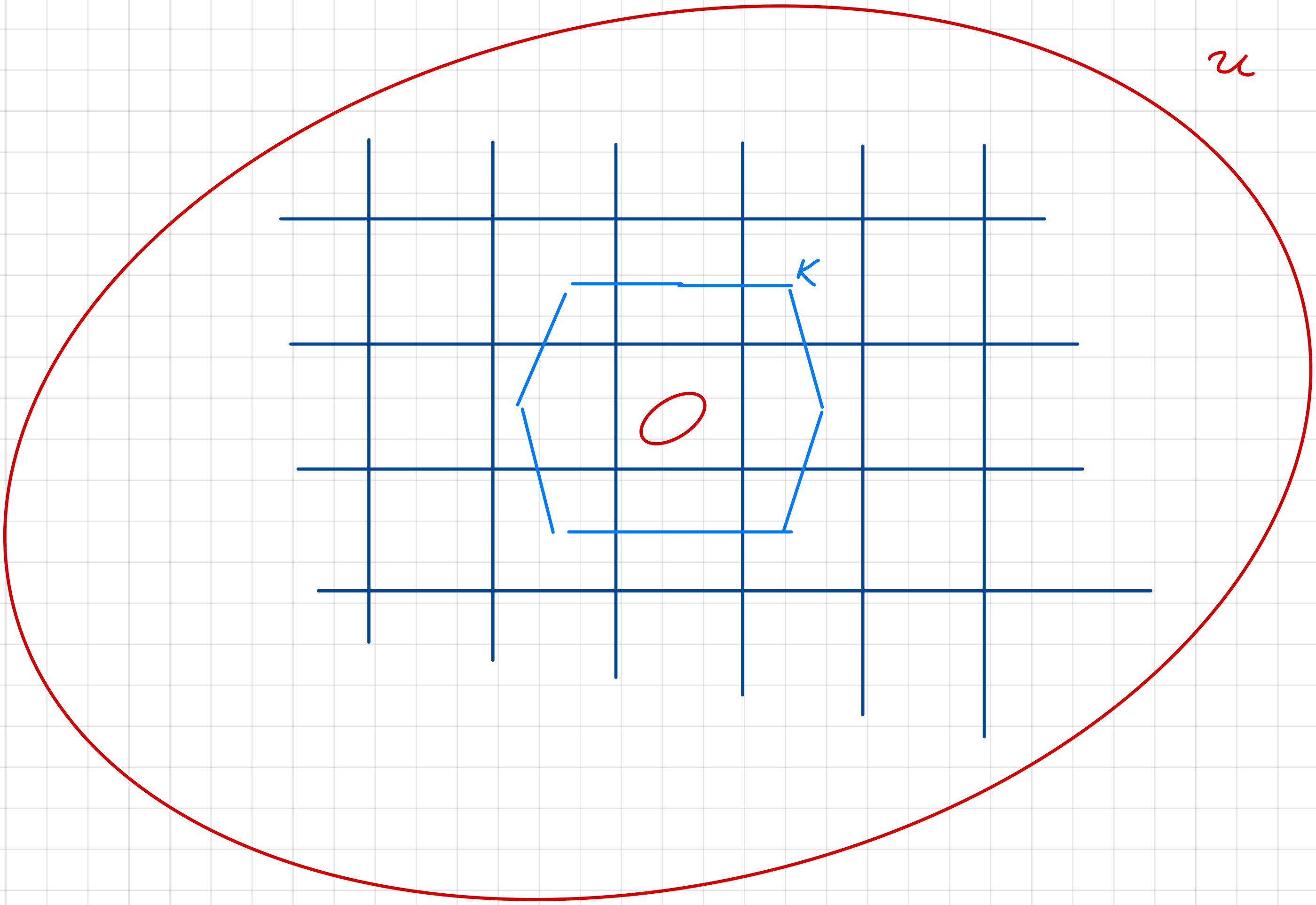
and argue using Cauchy's formula from Math 220A.

The issue is if K has complicated (fractal) structure.

Idea : Lay a grid !.

Proof

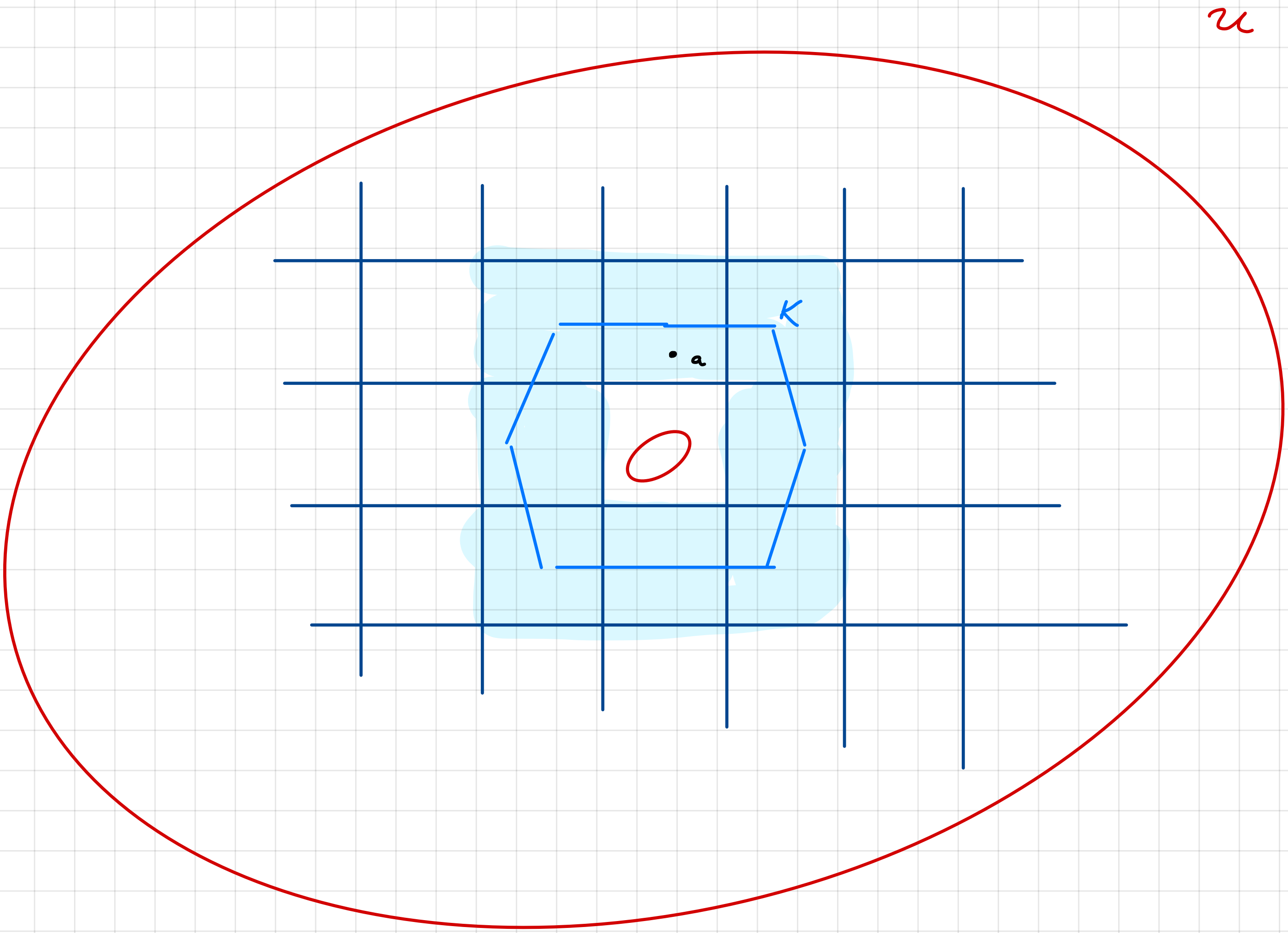
(1) Construction



WLOG $\mathcal{U} \neq \mathbb{C} \Rightarrow \mathbb{C} \setminus \mathcal{U} \neq \emptyset$ is closed. Note

$K \cap (\mathbb{C} \setminus \mathcal{U}) = \emptyset$. Let $d = d(K, \mathbb{C} \setminus \mathcal{U}) > 0$.

Lay a grid of squares of side $< \frac{d}{\sqrt{2}}$.



Consider the closed squares

$Q_1, Q_2 \ldots Q_m$ that intersect K.

There are only finitely many squares since K is compact.

Claim 1 $K \subseteq \bigcup_{j=1}^{m} Q_j \subseteq \mathcal{U}$.

Proof If $k \in K$ then k is contained in a square of the grid. This square intersects K at k so it must be one of the Q_j & $k \in Q_j$. This gives the first inclusion.

For the second inclusion, let $g \in Q_j$ where $Q_j \cap K \neq \emptyset$. Let $k \in Q_j \cap K$. If $g \notin \mathcal{U}$ $\Rightarrow$

$\Rightarrow g \in \mathbb{C} \setminus \mathcal{U}$ and $k \in K$ so

$$d(g, k) \geq d(\mathbb{C} \setminus \mathcal{U}, K) = d.$$

But $g, k \in Q_j$ $\Rightarrow$ $d(g, k) < \operatorname{diam}(Q_j) = d$ contradiction!

Thus $g \in \mathcal{U}$, as needed.

Construction of σ

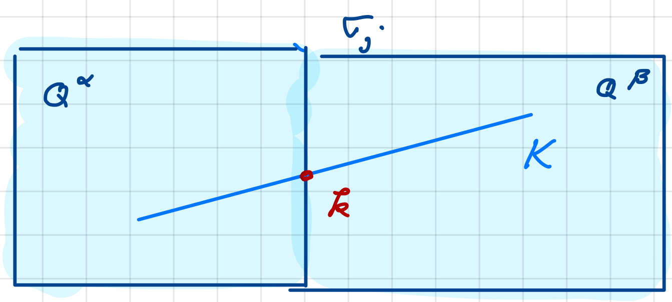
- $\sigma_1, \dots, \sigma_n$ sides of $Q_1, \dots, Q_m$ which are not shared by two squares Q_α, Q_β, $1 \leq \alpha \neq \beta \leq m$.

Claim 2 $\bigcup_{j=1}^{n} \tau_j \subseteq \mathcal{U} \setminus K.$

Proof

Note $\tau_j \subseteq \mathcal{U}$ by Claim 1. Assume $\tau_j \cap K \neq \Phi$.

Let $k \in \tau_j \cap K$. Then τ_j is a side of two squares.



These squares must intersect K necessarily since τ_j does.

These squares must be some of the Q_α, Q_β's,

contradicting the definition of τ_j.

Claim 3 $\forall\, a \in \mathcal{U} \setminus \bigcup_{j=1}^{m} \partial Q_j$ then

$$\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(z)}{z-a}\, dz = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z-a}\, dz.$$

This follows because the common sides of the Q_j's cancel out, leaving only the integral over γ_j's.

Assume $a \in \operatorname{Int} Q_\ell$. By Cauchy for rectangles

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(z)}{z-a}\, dz = f(a) \quad \text{if } j = \ell$$

and 0 otherwise.

$\Rightarrow \forall\, a \in \operatorname{Int} Q_\ell$,

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\gamma_j} \frac{f(z)}{z-a}\, dz \qquad (*)$$

This is almost the Lemma. We have one more step.

Claim 4 $\forall\, a \in K$

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz \qquad (**).$$

Proof The only issue is the case when $a \notin \operatorname{Int} Q_\ell$.

$\Rightarrow$ a must be on a side of some Q_j b/c. $K \subseteq \bigcup_{j=1}^{m} Q_j$

by Claim 1. By Claim 2, $a \notin \Gamma_j$.

Find $a_n \to a$ with a_n in the interior of the

squares Q_s. Both sides of $(**)$ agree at a_n by $(*)$

$$f(a_n) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a_n}\, dz$$

Both sides are continuous in a. This is clear for LHS

& RHS is explained below. Make $n \to \infty$ to conclude

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a}\, dz,$$

proving the Lemma completely.

Continuity of RHS is a consequence of:

Key Fact (Math 220A, Homework 3, Problem 7).

$$\Phi : \overset{z}{\Gamma} \times \overset{a}{\mathcal{U} \setminus \Gamma} \longrightarrow \mathbb{C} \text{ continuous}$$

then $a \longrightarrow \int_{\Gamma} \Phi(z, a)\, dz$ is continuous.

Apply to $\Phi : \Gamma_j \times \mathcal{U} \setminus \Gamma_j \longrightarrow \mathbb{C}$

$$\Phi(z, a) = \frac{f(z)}{z - a}, \quad z \in \Gamma_j, \ a \in \mathcal{U} \setminus \Gamma_j$$

to conclude.

Step 1 is now established. Steps 2 & 3 next time.