Math 220 B — Lecture 3

January 8, 2021

Last time   Conway VII. 5.

$f_k : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $\mathcal{U} \subseteq \mathbb{C}$

Assumption $\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly

Then

(*) $$F(z) = \prod_{k=1}^{\infty} (1 + f_k(z)).$$

converges absolutely for all $z \in \mathcal{U}$.

Remark [i] By Cauchy's criterion (Math 1408).

$\forall\, K \subseteq \mathcal{U}$ compact $\forall\, \varepsilon > 0$ $\exists\, N_{K,\varepsilon}$ if $m > n > N$

$$\Longrightarrow \sum_{k=n}^{m} |f_k(z)| < \varepsilon \quad \text{for } z \in K$$

[ii] In practice, instead of Assumption above we might check.

$$\sum_{k=n}^{m} \sup_{z \in K} |f_k(z)| < \varepsilon. \Longleftrightarrow \sum_{k=1}^{\infty} \sup_{K} |f_k| < \infty.$$

"normal convergence"

This is simply the Weierstraß M-test, with $M_k(K) = \sup_K |f_k(z)|$

so $\boxed{ii} \Rightarrow \boxed{i}$.

Proposition Assume $\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly.

$\boxed{i}$ the partial products of $(*)$ converge locally uniformly. to F

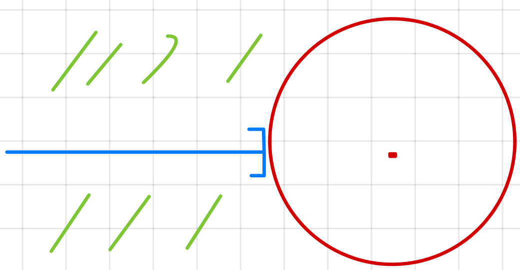
$\boxed{ii}$ F is holomorphic

$\boxed{iii}$ $F(z_0) = 0 \iff \exists\, k$ with $1 + f_k(z_0) = 0$

Furthermore,

$$\operatorname{ord}(F, z_0) = \sum_{k=1}^{\infty} \operatorname{ord}(1 + f_k, z_0).$$

Proof Recall from last time



$\log(1+z)$ is continuous in $\Delta(0,1)$

Important inequality: $\exists\, \rho < 1$ such that

$$|\log(1+z)| \leq \frac{3}{2} |z| \qquad \text{if } |z| \leq \rho$$

Proof of $\boxed{c}$ Let $K \subseteq \mathcal{U}$ compact. By Remark $\boxed{c}$

$\exists N$ such that if $k \geq N \Rightarrow |f_k(z)| < \rho$ for $z \in K$

$\Longrightarrow$ by important inequality

$$\left|\log(1+f_k(z))\right| \leq \frac{3}{2}|f_k(z)| \text{ for } z \in K, \ k \geq N.$$

Since $\sum_{k=N}^{\infty} |f_k|$ converges uniformly by assumption,

comparison test $\Longrightarrow$ $\sum_{k=N} \log(1+f_k(z))$ converges (absolutely). uniformly on K.

Write $G_n = \sum_{k=N}^{n} \log(1+f_k(z)) \underset{K}{\rightrightarrows} G$.

Note that G_n is continuous since $\log(1+w)$ is continuous

for $|w| < \rho$. Thus G is also continuous.

Since $G_n \underset{K}{\rightrightarrows} G$, by the claim $\boxed{a}$ below

$$e^{G_n} \underset{K}{\rightrightarrows} e^{G} \Rightarrow \prod_{k=N}^{n}(1+f_k(z)) \underset{K}{\rightrightarrows} e^{G}.$$

$$\Longrightarrow \prod_{k=1}^{n}(1+f_k(z)) \underset{K}{\rightrightarrows} e^{G}(1+f_1(z)) \cdots (1+f_{N-1}(z)). \quad (+)$$

(Uniform convergence after multiplication uses claim $\boxed{b}$ below)

Thus

$F = e^{G} (1+f_1) \dots (1+f_{N-1})$ in K. & the convergence is

uniform, in K, completing the proof.

$\boxed{ii}$ F holomorphic by $\boxed{i}$ (local uniform convergence)

& Weierstraß Convergence theorem.

$\boxed{iii}$ Recall from last time that

$$F(z_0) = 0 \iff \exists\, k \text{ with } 1 + f_k(z_0) = 0.$$

To prove the assertion about orders, consider (+)

in $K = \bar{\Delta}$, Δ neighb. of z_0

$$F(z_0) = \underbrace{e^{G(z_0)}}_{\neq 0} (1+f_1(z_0)) \dots (1+f_{N-1}(z_0))$$

$$\Rightarrow \operatorname{ord}(F, z_0) = \sum_{k=1}^{N-1} \operatorname{ord}(1+f_k, z_0)$$

$$= \sum_{k=1}^{\infty} \operatorname{ord}(1+f_k, z_0).$$

using that $1+f_k \neq 0$ for $k \geq N$ (because $|f_k| < p < 1$ in K.)

Remark Analyzing the proof, we see the argument only requires

$$\sum_k \left| \operatorname{Log}(1+f_k) \right| \text{ converges locally uniformly.}$$

The following standard claims were used in the proof:

Claim $\boxed{a}$ Let u_n be continuous, $u_n \underset{K}{\rightrightarrows} u$. Then $e^{u_n} \underset{K}{\rightrightarrows} e^u$.

$\boxed{b}$ If $u_n \underset{K}{\rightrightarrows} u$, $v_n \underset{K}{\rightrightarrows} v$ (u_n, v_n continuous). Then

$$u_n v_n \underset{K}{\rightrightarrows} uv.$$

Proof $\boxed{a}$ Suffices to show $\sup_K |e^{u_n} - e^u| \longrightarrow 0$.

Compute

$$\sup_K |e^{u_n} - e^{u}| = \sup_K |e^{u}| \cdot |e^{u_n - u} - 1|$$

$$\leq \sup_K |e^{u}| \cdot \sup_K |e^{u_n - u} - 1|$$

$$= M \cdot \sup_K |e^{u_n - u} - 1| < \varepsilon M \quad \text{for } n \geq N.$$

Why? By continuity, $\exists\, \delta > 0$: $|e^{w} - 1| < \varepsilon$ if $|w| < \delta$.

Since $u_n \underset{K}{\rightrightarrows} u \Rightarrow \exists\, N$ with $|u_n - u| < \delta$ on K

$$\Rightarrow |e^{u_n - u} - 1| < \varepsilon$$

Proof of [b] We show $\sup_K |u_n v_n - uv| \longrightarrow 0.$

Indeed by triangle inequality

$$\sup_K |u_n v_n - uv| \leq \sup_K |(u_n - u)(v_n - v)| + \sup_K |u(v_n - v)| + \sup_K |v(u_n - u)|$$

$$\leq \sup_K |u_n - u| \cdot \sup_K |v_n - v| + \sup_K |u| \cdot \sup_K |v_n - v| + \sup_K |v| \cdot \sup_K |u_n - u|$$

$\longrightarrow 0$ since $\sup_K |u_n - u| \longrightarrow 0$ and $\sup_K |v_n - v| \longrightarrow 0.$

Logarithmic derivative

Taking derivatives of products is messy. It is easier to take logarithmic derivatives

h holomorphic $\Rightarrow$ $\frac{h'}{h}$ = logarithmic derivative

= holomorphic away from Zero(h)

Addition formula

$$h = fg \Rightarrow \frac{h'}{h} = \frac{f'}{f} + \frac{g'}{g}.$$

$$h' = f'g + fg' \Rightarrow \frac{h'}{h} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

Inductively $h = f_1 \cdots f_s \Rightarrow \frac{h'}{h} = \frac{f_1'}{f_1} + \cdots + \frac{f_s'}{f_s}$

We prove the same for infinite products.

$\boxed{i}$ $g_k : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $f_k = 1 + g_k$.

$\boxed{ii}$ $\sum_{k=1}^{\infty} |g_k|$ converges locally uniformly in $\mathcal{U}$

Proposition $h = \prod_{k=1}^{\infty} f_k$. Away from $Zero(h)$:

$$\frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

The RHS converges locally uniformly on $\mathcal{U} \setminus Zero(h)$.

Proof Recall from (+) in the previous Proof that

for $K = \bar{\Delta} \subseteq \mathcal{U}$, Δ neighborhood of an arbitrary point

$\exists$ N with

$$F_n = \prod_{k=N}^{n} f_k \xrightarrow[\Delta]{l.u.} F = e^{G} \quad \text{on } \Delta$$

$$G = \sum_{k=N}^{\infty} \log(1+g_k)$$

Note $h = f_1 \cdots f_{N-1} \prod_{k=N}^{\infty} f_k = f_1 \cdots f_{N-1}\, e^G$.

finite case
$$\Rightarrow \frac{h'}{h} = \frac{f_1'}{f_1} + \cdots + \frac{f_{N-1}'}{f_{N-1}} + \frac{(e^G)'}{e^G}.$$

We need to show $\displaystyle\sum_{k=N}^{n} \frac{f_k'}{f_k} \overset{\ell.u.}{\underset{\Delta}{\rightrightarrows}} \frac{(e^G)'}{e^G}$.

$$\Rightarrow \frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

To see this, by the finite case again

$$\sum_{k=N}^{n} \frac{f_k'}{f_k} = \frac{F_n'}{F_n}.$$ so we show $\displaystyle\frac{F_n'}{F_n} \overset{\ell.u.}{\underset{\Delta}{\rightrightarrows}} \frac{(e^G)'}{e^G}$.

Note $F_n \overset{\ell.u.}{\underset{\Delta}{\rightrightarrows}} e^G \neq 0$ so by Weierstraß $F_n' \overset{\ell.u.}{\underset{\Delta}{\rightrightarrows}} (e^G)'$.

We finish using Claim $\boxed{b}$ above (products) &

Claim $\boxed{c}$ below.

Claim $\boxed{c}$ If $u_n \underset{K}{\rightrightarrows} u$, u_n continuous, u nowhere zero

$$\Rightarrow \frac{1}{u_n} \underset{K}{\rightrightarrows} \frac{1}{u}.$$

Proof We show $\sup_K \left| \frac{1}{u_n} - \frac{1}{u} \right| \to 0$. for $n \to \infty$.

Compute $\sup_K \left| \frac{1}{u_n} - \frac{1}{u} \right| = \sup_K \frac{|u_n - u|}{|u| \cdot |u_n|}$

$$\leq \sup_K |u_n - u| \cdot \frac{1}{\inf_K |u|} \cdot \frac{1}{\inf_K |u_n|} \longrightarrow 0$$

Note $\inf_K |u_n| \geq \inf_K |u| - \sup_K |u - u_n| \longrightarrow \inf_K |u| > 0$

where we used $\sup_K |u_n - u| \to 0$.