HW5 - SOLUTIONS

Q1. By the removable singularity theorem, the function f can be extended to an entire function $f : \mathbb{C} \to \mathbb{C}$, which must be bounded. By Liouville's theorem, f must be constant.

Q2. We have $f(z) = \frac{1}{z+1} - \frac{1}{z+2}$. We note that

(i) for $|z| < 1$ we have

$$\frac{1}{z+1} = 1 - z + z^2 - \dots$$

$$\frac{1}{z+2} = \frac{1}{2} \cdot \frac{1}{1+z/2} = \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \dots\right)$$

hence

$$f(z) = \sum_{k=0}^{\infty} z^k \cdot \left((-1)^k - \frac{(-1)^k}{2^{k+1}}\right).$$

(ii) for $1 < |z| < 2$, the first fraction $\frac{1}{z+1}$ needs to be expanded differently. We have

$$\frac{1}{z+1} = \frac{1}{z} \cdot \frac{1}{1+1/z} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \dots\right).$$

The second fraction is expanded the same way. We obtain

$$f(z) = \sum_{k=-\infty}^{-1} (-1)^{k-1} z^k + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2^{k+1}} z^k.$$

(iii) for $|z| > 2$, we use the same expansion for $1/(z+1)$ as in (ii), but the second fraction becomes

$$\frac{1}{z+2} = \frac{1}{z} \cdot \frac{1}{1+2/z} = \frac{1}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \dots\right).$$

Hence

$$f(z) = \sum_{k=-\infty}^{-1} z^k \left((-1)^{k-1} - (-1)^{k-1} 2^{-k-1}\right).$$

Q3. Assume for a contradiction that f is meromorphic near 0 such that f' has a pole of order exactly 1. Without loss of generality we may assume f is holomorphic in $\Delta^\star(0, R)$. By assumption f' must have a nonzero residue at 0. By the formulas for the coefficients in the Laurent expansion we find

$$\mathrm{Res}(f', 0) = \int_{\gamma_r} f'(z)\, dz$$

for any $0 < r < R$. But the latter integral equals 0 by the fundamental theorem of calculus applied over the closed loop γ_r.

Q4. Assume $f(\mathbb{C})$ is not dense in $\mathbb{C}$. In this case, there exists $\lambda \in \mathbb{C}$ and $R > 0$ such that

$$f(\mathbb{C}) \cap \Delta(\lambda, R) = \emptyset.$$

In other words,

$$|f(z) - \lambda| \geq R$$

for all $z \in \mathbb{C}$. Consider the function

$$g(z) = \frac{1}{f(z) - \lambda}.$$

Clearly, g is holomorphic and bounded since

$$|g(z)| \leq \frac{1}{R}.$$

By Liouville's theorem, g must be constant. This in turn implies that f is constant, a contradiction.

Q5. Write

$$P(w) = a_0 w^n + \ldots + a_n.$$

(i) Assume first that a is a removable singularity for f. Then f is bounded in a neighborhood of a, say

$$|f(z)| \leq M$$

for $z \in \Delta(a, \epsilon) \setminus \{a\}$. In this case

$$|P(f(z))| = |a_0 f(z)^n + \ldots + a_n| \leq |a_0| M^n + \ldots + |a_n|$$

is also bounded so the singularity of $P \circ f$ is also removable at a.

(ii) Assume a is a pole for f. We show a is a pole for $P \circ f$ by proving

$$\lim_{z \to a} P(f(z)) = \infty.$$

To this end, let $R > 0$. We seek to show

$$|P(f(z))| > R \text{ for all } z \in \Delta(a, \epsilon) \setminus \{a\}$$

for a suitable ϵ. Note that for w real, we have

$$\lim_{w \to \infty} |a_0| w^n - |a_1| w^{n-1} - \ldots - |a_n| = \infty,$$

we can find $r > 0$ such that if $w > r$, we have

$$|a_0| w^n - |a_1| w^{n-1} - \ldots - |a_n| > R.$$

Since a is a pole, we have $\lim_{z \to a} f(z) = \infty$, so we can find $\epsilon > 0$ such that

$$|f(z)| > r \text{ for all } z \in \Delta(a, \epsilon) \setminus \{a\}.$$

Thus

$$|P(f(z))| = |a_0 f(z)^n + \ldots + a_n| \geq |a_0| |f(z)|^n - \ldots - |a_n| > R$$

where we used the triangle inequality, and the fact that $w = |f(z)| > r$. This shows that a is a pole for $P \circ f$.

(iii) Assume a is an essential singularity for f. Fix α and β two complex numbers such that

$$P(\alpha) \neq P(\beta).$$

We claim that we can find a sequence

$$x_n \to a \text{ such that } f(x_n) \to \alpha.$$

Indeed, by Casoratti-Weierstrass, for n sufficiently large so that $\Delta\left(a, \frac{1}{n}\right) \subset U$, we have $f\left(\Delta\left(a, \frac{1}{n}\right) \setminus \{a\}\right)$ is dense, hence we can find $x_n \in \Delta\left(a, \frac{1}{n}\right) \setminus \{a\}$ such that

$$|f(x_n) - \alpha| < \frac{1}{n}.$$

In a similar fashion we can find

$$y_n \to a \text{ such that } f(y_n) \to \beta.$$

Thus

$$P(f(x_n)) \to P(\alpha), \quad P(f(y_n)) \to P(\beta).$$

If a were removable for $P \circ f$, then $P(\alpha) = P(\beta)$ by continuity. This is however a contradiction. If a were a pole for $P \circ f$, then the two limits would have to be infinite. Thus the only option is that a is an essential singularity.

Q6. Consider $g(z) = f\left(\frac{1}{z}\right)$ which is holomorphic over $\mathbb{C} \setminus \{0\}$.

(i) Assume ∞ is a removable singularity for f. Thus 0 is a removable singularity for g which must be bounded in a neighborhood of 0:

$$|g(z)| \leq M \quad \text{for} \quad |z| \leq R.$$

Thus

$$|f(z)| \leq M \quad \text{for} \quad |z| \geq R^{-1}.$$

Of course, $|f(z)|$ is bounded in $|z| \leq R^{-1}$ by continuity. Thus f is bounded on $\mathbb{C}$, and by Liouville's theorem it must be constant.

(ii) Assume ∞ is a pole for f. Thus 0 is a pole for g, and we may consider the Laurent expansion of g:

$$g(z) = \sum_{n=-N}^{\infty} a_n z^n \implies f(z) = \sum_{n=-N}^{\infty} a_n z^{-n}.$$

Since f is entire, we must have $a_n = 0$ for $n > 0$, hence

$$f(z) = \sum_{n=-N}^{0} a_n z^{-n}.$$

This means f is a polynomial.

Q7. Consider $g(z) = f\left(\frac{1}{z}\right)$ which is holomorphic over $\mathbb{C} \setminus \{0\}$ and also injective.

If ∞ is a pole for f, by **Q6** we know f is a polynomial. Since f is injective it follows that f has degree 1. Otherwise, if $\deg f \geq 2$, then $f(z) - f(a)$ has $z = a$ as its only root by injectivity, hence $z = a$ must be a root of $f - f(a)$ with mutiplicity $\deg f \geq 2$ by the fundamental theorem of algebra. Therefore, the derivative $(f(z) - f(a))'|_{z=a} = 0$ which implies $f'(a) = 0$ for all a, hence f is constant. This is a contradiction.

If $z = \infty$ is an essential singularity for f, then 0 is an essential singularity for g. Then take $r > 0$. By the open mapping theorem $g(\mathbb{C} \setminus \overline{\Delta}(0, r))$ is an open set. By the Caseroti-Weierstrass theorem, $g(\Delta(0, r) \setminus \{0\})$ is dense, hence it must intersect the open set $g(\mathbb{C} \setminus \overline{\Delta}(0, r))$. This however contradicts the fact that g is injective.