Math 220C, Problem Set 5. Due Friday, April 30.

1. Let f, g be two entire functions of finite order λ. Assume $f(a_n) = g(a_n)$ for a sequence $\{a_n\}_{n\geq 0}$ of non-zero complex numbers with

$$\sum_{n=0}^{\infty} \frac{1}{|a_n|^{\lambda+1}} = \infty.$$

Show that $f = g$.

2.

(i) Find all entire functions f of finite order such that $f(\log n) = n$ for all integers $n \geq 1$.

(ii) (*Uses only material from Math 220B.*) Give an example of an entire function f with zeroes only at $\log n$ for integers $n \geq 1$.

3. If f is an entire function of finite order λ, show that f' also has order λ.

Hint: You need to prove that order $f' \leq$ order f and vice-versa. For the first inequality, use Cauchy's estimates. For the second inequality, use that

$$f(z) - f(0) = z \int_0^1 f'(tz)dt$$

4. (*Qualifying Exam, Spring 2020.*) Let f be entire, $|f'(z)| \leq e^{|z|}$ and

$$f(\sqrt{n}) = 0 \quad \text{ for all } n \in \mathbb{Z}_{>0}.$$

Show that $f = 0$.

5. (*Qualifying Exam, Fall 2020.*) Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z - \sin z$.

(i) Show that f is an odd entire function of order less or equal to 1.

(ii) Using (i), show that f can be represented as a product

$$f(z) = \frac{z^3}{6} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

where $\{a_n\}$ is a sequence of non-zero complex numbers with

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

6. (*Extra Credit.*) In this question, we show how to compute the order from the coefficients of the Taylor expansion.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function of order λ. Let

$$\mu = \limsup_{n\to\infty} \frac{n \log n}{-\log |c_n|} > 0.$$

Show that $\lambda = \mu$.

(i) First show that $\lambda \geq \mu$ by showing that for all $\epsilon > 0$ we have $\lambda > \mu - \epsilon$.

Hint: By definition

$$n \log n \geq -(\mu - \epsilon) \log |c_n|$$

for infinitely many n. Use Cauchy's estimate for $|c_n|$ to conclude that

$$\log M(R) \geq n \log R - \frac{1}{\mu - \epsilon} n \log n$$

for all R. Use this for

$$R = (en)^{\frac{1}{\mu-\epsilon}} \implies \log M(R) \geq \frac{n}{\mu - \epsilon} = \frac{R^{\mu-\epsilon}}{e(\mu - \epsilon)}.$$

Conclude that $\lambda \geq \mu - \epsilon$.

(ii) Conversely, show that $\lambda \leq \mu$ by showing that $\lambda < \mu + \epsilon$ for all $\epsilon > 0$.

Hint: If n is sufficiently large, $|c_n| \leq n^{-\frac{n}{\mu+\epsilon}}$. Conclude that

$$M(R) \leq \sum_n R^n n^{-\frac{n}{\mu+\epsilon}},$$

up to a constant. To estimate this series, break the sum into two pieces S_1, S_2 corresponding to $n \leq (2R)^{\mu+\epsilon}$ and $n > (2R)^{\mu+\epsilon}$. Show

$$S_2 = \sum_{n>(2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{\mu+\epsilon}} < 1.$$

Show

$$S_1 = \sum_{n\leq(2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{\mu+\epsilon}} \leq R^{(2R)^{\mu+\epsilon}} \sum_{n\geq 1} n^{-\frac{n}{\mu+\epsilon}} \leq CR^{(2R)^{\mu+\epsilon}}.$$

Conclude.

(iii) Let $a > 0$. As an application, find the order of the function

$$f(z) = \sum_n \frac{z^n}{n^{an}}.$$