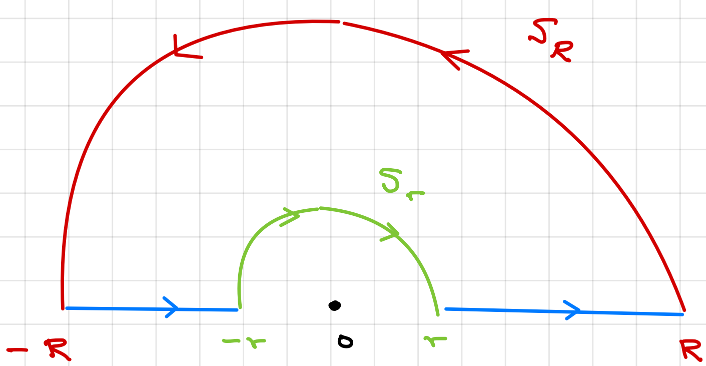
Math 220 A – Lecture 18

November 23, 2020

Fourier Integrals – Part II – Poles on the real axis

Example $I = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$



$f(z) = \frac{e^{iz}}{z}$.

$\gamma = S_R + [-R, -r] + (-S_r) + [r, R]$

$$\underbrace{\int_{S_R} f\,dz}_{\downarrow \; 0} - \underbrace{\int_{S_r} f\,dz}_{\downarrow \; i\pi} + \underbrace{2i \int_r^R \frac{\sin z}{z}\,dz}_{\downarrow \; 2iI} = \int_\gamma f\,dz = 0$$

as $r \to 0$, $R \to \infty$.

Claims $\boxed{a}$ $\lim_{R\to\infty} \int_{S_R} \frac{e^{iz}}{z}\,dz = 0$

$\boxed{b}$ $\lim_{r\to 0} \int_{S_r} \frac{e^{iz}}{z}\,dz = i\pi$

$\Rightarrow I = \frac{\pi}{2}$.

Part $\boxed{a}$ is a consequence of Lemma last time:

for $g(z) = \frac{1}{z}$:

Lemma If $\lim\limits_{\substack{z\to\infty \\ z \in \overline{\mathbb{H}}^+}} |g(z)| = 0$ then

$$\lim_{R\to\infty} \int_{S_R} g(z)\, e^{iz}\, dz = 0$$

Part $\boxed{b}$ uses the next lemma for $g(z) = \frac{1}{z}$

Lemma Let g have simple pole at 0. Then

$$\lim_{r\to 0} \int_{S_r} g(z)\, e^{iz}\, dz = \pi i \operatorname{Res}(g, 0).$$

Proof Since g has a simple pole at 0, write

$$g(z) = \frac{\alpha}{z} + \underbrace{G(z)}_{\text{Taylor series}}$$

$\alpha = \operatorname{Res}(g, 0)$, G holomorphic near 0

$$e^{iz} = 1 + \underbrace{zF(z)}_{\text{Taylor}}, \quad F \text{ holomorphic near } 0$$

$$e^{iz} g(z) = \left(\frac{\alpha}{z} + G\right)\left(1 + zF\right) = \frac{\alpha}{z} + H,$$

$H = G + zFG + \alpha F$ holomorphic near 0

$\Longrightarrow$ H bounded near 0. $\Longrightarrow$ $\exists\, M, \delta$: $|H(z)| \leq M$ if $|z| \leq \delta$.

Compute $\displaystyle\int_{S_r} e^{iz} g(z)\, dz = \int_{S_r} \frac{\alpha}{z} + H \, dz.$

Note $\displaystyle \alpha \int_{S_r} \frac{dz}{z} = \alpha \int_0^{\pi} \frac{d(re^{it})}{re^{it}} = \alpha \int_0^{\pi} i\, dt = \pi i \alpha$

$$\left| \int_{S_r} H\, dz \right| \leq M \cdot \pi r \longrightarrow 0 \quad \text{as } r \longrightarrow 0.$$

Thus $\displaystyle\int_{S_r} e^{iz} g(z)\, dz \longrightarrow \pi i \alpha$ as $r \longrightarrow 0$, as claimed.

Applications of the Residue theorem to real analysis

[a] trigonometric functions

[b] rational functions

[c] Fourier integrals

[d] logarithmic integrals

[e] Mellin transforms

d) Logarithmic integrals

$$\int_0^\infty R(x) \log x \, dx$$

R = even rational function without real poles

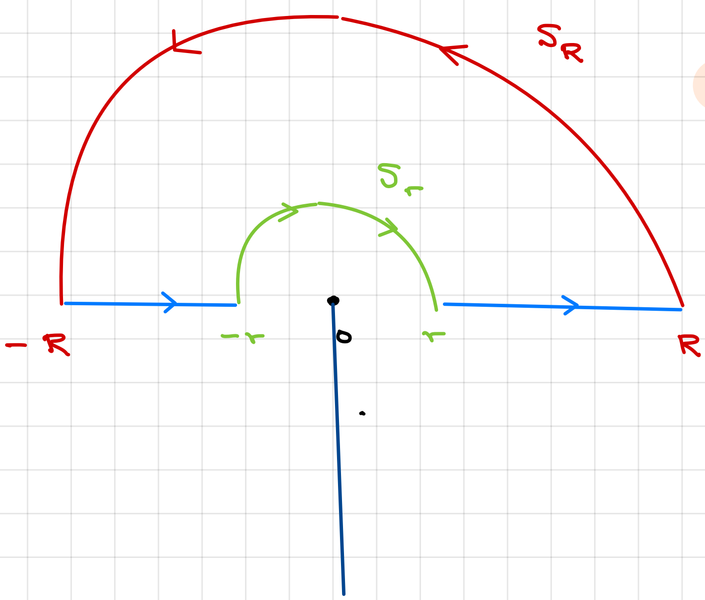
Example $R(x) = \frac{1}{1+x^2} \Longrightarrow \int_0^\infty \frac{\log x}{1+x^2} dx = 0$

HWK $R(x) = \frac{1}{(1+x^2)^2} \Longrightarrow \int_0^\infty \frac{\log x}{(1+x^2)^2} dx.$

Issues:

- logarithm undefined at 0 (use circle S_r)

- holomorphic extension for logarithm



Requires branch cut!

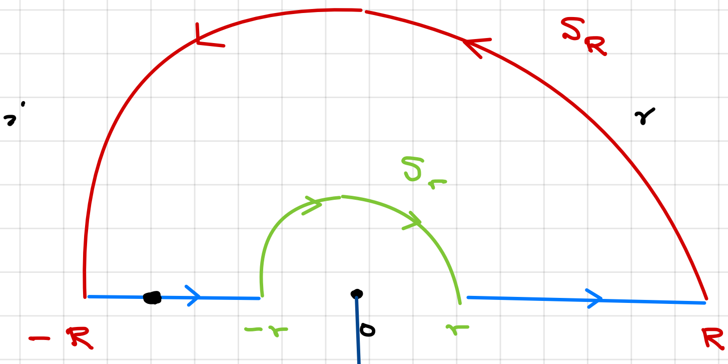
Define for $z = r e^{it}$

$$\ell(z) = \log r + i t$$

$$-\frac{\pi}{2} < t < \frac{3\pi}{2}$$

- $\gamma = S_R + [-R, -r] + (-S_r) + [r, R]$

- $f(z) = \dfrac{\ell(z)}{1+z^2}$ has pole at i



Residue theorem

Method 1

$$\operatorname*{Res}(f, i) = \operatorname*{Res}_{z=i} \frac{\ell(z)}{1+z^2} = \frac{\ell(z)}{2z}\bigg|_{z=i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}.$$

Residue thm: $\displaystyle\int_\gamma f\,dz = 2\pi i \operatorname{Res}(f, i) = i\,\frac{\pi^2}{2}.$

$$\int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R f(x)\,dx + \int_{-R}^{-r} f(x)\,dx. \quad (*).$$

We make $r \to 0$, $R \to \infty$.

Segment integrals

see the definition of ℓ

$$\int_r^R \frac{\ell(x)}{1+x^2}\,dx + \int_{-R}^{-r} \frac{\ell(x)}{1+x^2}\,dx = \int_r^R \frac{\log x}{1+x^2}\,dx + \int_{-R}^{-r} \frac{\log|x| + i\pi}{1+x^2}\,dx$$

$$= 2\int_r^R \frac{\log x}{1+x^2}\,dx + i\pi \int_{-R}^{-r} \frac{dx}{1+x^2}$$

$$\xrightarrow[R\to\infty]{r\to 0} 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi \left.\arctan x\right|_{x=-\pi}^{x=0}$$

$$= 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi\cdot\frac{\pi}{2}$$

Claim

$$\lim_{\substack{\rho\to 0\\ \rho\to\infty}} \int_{S_\rho} \frac{\ell(z)}{1+z^2}\,dz = 0.$$

Conclusion From $(*)$ we get as $r\to 0$, $R\to\infty$:

$$\frac{i\pi^2}{2} = 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi\cdot\frac{\pi}{2}$$

$$\Longrightarrow \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$$

Proof of the claim $\quad z = \rho e^{it}, \quad 0 \le t \le \pi$

$$\left| \int_{S_\rho} \frac{\ell(z)}{1+z^2}\, dz \right| = \left| \int_0^\pi \frac{\log \rho + it}{1+\rho^2 e^{2it}} \cdot \rho e^{it}\, i\, dt \right|$$

$$\le \int_0^\pi \frac{|\log \rho| + \pi}{|1+\rho^2 e^{2it}|} \cdot \rho\, dt$$

$$\le \int_0^\pi \frac{|\log \rho| + \pi}{|\rho^2 - 1|} \cdot \rho\, dt$$

$$= \pi \cdot \frac{\rho\, |\log \rho|}{|\rho^2 - 1|} + \pi^2 \cdot \frac{\rho}{|\rho^2-1|} \longrightarrow 0.$$

As $\rho \to \infty$, $\dfrac{\rho \log \rho}{\rho^2 - 1}$ and $\dfrac{\rho}{\rho^2-1} \longrightarrow 0$.

As $\rho \to 0$, the same is true.

The only term that requires justification is

$$\rho \log \rho = -\frac{w}{e^w} \longrightarrow 0 \text{ as } w \to \infty, \quad \text{where } \rho = e^{-w}, \ \rho \to 0.$$

Applications of the Residue theorem to real analysis

- [a] trigonometric functions
- [b] rational functions
- [c] Fourier integrals
- [d] logarithmic integrals
- [e] Mellin transforms

[e] Mellin transforms

$$\int_0^\infty \frac{R(x)}{x^\alpha}\,dx \quad , \quad 0 < \alpha < 1$$

R = rational function, no poles on positive real axis

Useful in prime counting.

Example $\quad R(x) = \dfrac{1}{x+1} \implies \displaystyle\int_0^\infty \frac{dx}{x^\alpha (x+1)} = \frac{\pi}{\sin \pi\alpha}$

(next time)

Homework $\quad R(x) = \dfrac{1}{x^n+1} \implies \displaystyle\int_0^\infty \frac{dx}{x^\alpha (x^n+1)}$

Remark

[i] Fourier transform

$$f \rightsquigarrow \mathcal{F}f(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x s}\, dx$$

[ii] Laplace transform

$$f \rightsquigarrow \mathcal{L}f(s) = \int_{0}^{\infty} f(x)\, e^{-sx}\, dx$$

[iii] Mellin transform

$$f \rightsquigarrow \mathcal{M}f(s) = \int_{0}^{\infty} f(x)\, \underbrace{x^{s-1}}_{x^{-\alpha} \text{ on previous page}}\, dx$$

Remark (will not use)

The Mellin transform of $f(x) = e^{-x}$ is known as Γ - function

$$\Gamma(s) = \int_{0}^{\infty} e^{-x} x^{s-1}\, dx$$

Hjalmar Mellin (1854 – 1933)

Finnish mathematician