

Math 220, Problem Set 7.

1. Calculate the following integrals *using the residue theorem*. Carefully explain any estimates you might use.

(i)

$$\int_0^\infty \frac{\log x}{(1+x^2)^2} dx.$$

(ii)

$$\int_0^\infty \frac{x^\alpha}{1+x^n} dx \text{ where } n > 1 + \alpha > 0, n \geq 2 \text{ integer, } \alpha \in \mathbb{R}.$$

2. Assume that f has only finitely many isolated singularities in \mathbb{C} .

(i) Show that for all $R > 0$,

$$\int_{|z|=R} f dz = -2\pi i \sum_j \operatorname{Res}(f(z) dz, a_j)$$

where a_j 's are the singularities of f outside the circle $|z| = R$, including ∞ . (We assume there are no singularities when $|z| = R$.)

(ii) Find the residues of

$$f(z) dz = (z - a)^k dz$$

over $\widehat{\mathbb{C}}$, for k any integer and $a \in \mathbb{C}$.

(iii) Using (i), compute

$$\int_{|z|=5} \frac{z^3 dz}{(z-1)(z-2)(z-3)(z-4)}.$$

3. Show that a function f which is meromorphic over $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ must be rational. By definition, a meromorphic function over $\widehat{\mathbb{C}}$ has isolated poles which could occur at certain points in \mathbb{C} and possibly also at ∞ .

Hint: First note that f must have finitely many zeroes and poles. Construct the rational function $R(z)$ with exactly the same zeroes and poles, and analyze $f(z)/R(z)$.

4. Find the number of zeros of the polynomial $z^4 + 5z + 3$ inside the annulus $1 < |z| < 2$.

5. Show that for all $\lambda > 1$, the equation $z + e^{-z} = \lambda$ has exactly one solution with positive real part.

Hint: The curve γ is the boundary of a half disc contained in the right half plane.

6. Find the number of zeroes of $z^4 + 3z^2 + z + 1$ inside the unit disc.

Hint: The dominant term is not a monomial.

7. Using Rouché's theorem, derive Perron's criterion: a polynomial

$$f(x) = x^n + a_1 x^{n-1} + \dots + a_n \in \mathbb{Z}[x]$$

2

with

$$|a_1| > 1 + |a_2| + \dots + |a_n|, \quad a_n \neq 0$$

is necessarily irreducible over $\mathbb{Z}[x]$.

Hint: Use Rouché to determine how many roots of f are inside the unit disc and how many are outside.