MATH200A MIDTERM SOLUTION

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Problem 1. By assumption we have $n_3 = 4$. Then one sees easily that $\mathrm{N}_G(P) = P$ for any $P \in \mathrm{Syl}_3(G)$. Now consider the action of G on $\mathrm{Syl}_3(G)$ by conjugation. This gives a group homomorphism $\varphi : G \to S_4$. For any P we must have $\ker(\varphi) \subseteq \mathrm{N}_G(P) = P$ because an element in the kernel must in particular normalize P. By assumption we must have $\ker(\varphi) = \{e\}$, so the map is injective. Our goal is to show that $\varphi(G) = A_4$, and it suffices to show that $\varphi(G)$ has no odd permutations. In the following we identify G and $\varphi(G)$. Let $K = A_4 \cap G$. We know from textbook that $[G : K] = 2$ and $K \triangleleft G$ if G has an odd permutation. Now K has a unique Sylow-3 subgroup P. Hence we have $P \operatorname{char} K \triangleleft G$ which implies $P \triangleleft G$ by Lemma 2.16. This is a contradiction.

Problem 2. Assume we have $\mathrm{Inn}(L) \cong G$. Let $\varphi : L \to \mathrm{Inn}(L) \cong G$ be the natural homomorphism with $\ker(\varphi) = Z(L)$. Denote $\hat{H} = \varphi^{-1}(H)$ and $\hat{K} = \varphi^{-1}(K)$. By the correspondence theorem we have $\hat{H}\hat{K} = L$, and by the first isomorphism theorem we have that $\hat{H}/Z(L)$ and $\hat{K}/Z(L)$ are cyclic. Note $Z(L) \subseteq Z(\hat{H})$ so $\hat{H}/Z(\hat{H})$ is a quotient of $\hat{H}/Z(L)$. This shows that $\hat{H}/Z(\hat{H})$ is cyclic, hence $\hat{H}$ must be abelian, by a previous exercise. Same holds for $\hat{K}$. Then, $K \cap H$ is not trivial implies that there is an element $l \in \hat{K} \cap \hat{H} - Z(L)$. But then l commutes with all elements in $\hat{H}$ and $\hat{K}$ and hence all elements in L. Thus $l \in Z(L)$. This is a contradiction.

Problem 3. The divisors of 2020 are $1, 2, 4, 5, 101, 10, 20, 202, 404, 505, 1010$ and 2020. We want to show that the proposition "G has a subgroup of order k" is true for all the k's above. We have the following points:

(1) k=1, 2020. Trivial.
(2) $k = 2$. Cauchy's theorem.
(3) $k = 4, 5, 101$. Sylow's theorem.
(4) $k = 202, 404, 505$. Note that $n_{101} = 1$. Thus the unique Sylow-101 subgroup P_{101} is normal. For any subgroup $K \neq P_{101}$ in items (2) and (3), KP_{101} is a subgroup of G, and $|KP_{101}| = |K| \cdot 101$. This gives subgroups with desired orders.
(5) $k = 1010$. A same reasoning as above proves this point if we can show that there is a subgroup of order 10.
(6) $k = 10, 20$. Note that n_5 is either 1 or 101. If $n_5 = 1$, then the unique Sylow-5 subgroup is normal; we go over a same reasoning as in (4). If $n_5 = 101$, then by the formula $n_p = |G|/|\,\mathrm{N}_G(P)|$, we know that $|\,\mathrm{N}_G(P)| = 20$ for any Sylow 5-subgroup P. Now $P \triangleleft \mathrm{N}_G(P)$ and by Cauchy's theorem there is an element $l \in \mathrm{N}_G(P)$ of order 2. Then $P\langle l\rangle$ is a subgroup of order 10 in $\mathrm{N}_G(P) \subseteq G$.