Math 220 B — Lecture 10

January 27, 2021

Last time - Mittag-Leffler Problem

Given
- $a_n \to \infty$ distinct and
- Laurent principal parts g_n

find f meromorphic with poles at a_n & principal parts g_n at a_n

Construction

Step 1 Expand g_n into Taylor series at 0

Step 2 Pick h_n a Taylor polynomial & check

$$|g_n - h_n| < c_n \quad \text{in } \Delta(0, r_n) \quad \text{with} \quad \sum_n c_n < \infty.$$

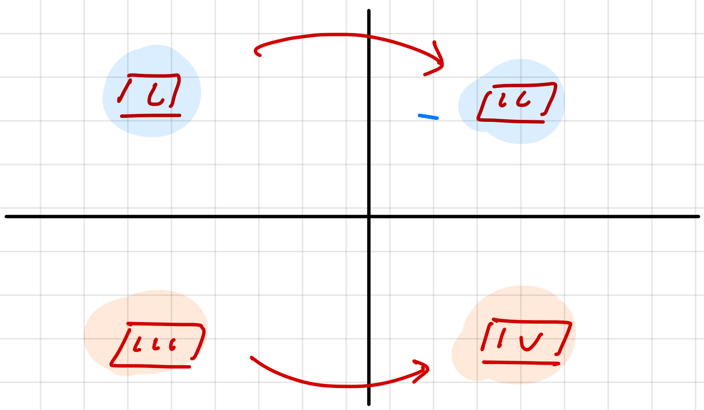
and $r_n < |a_n|$, $r_n \to \infty$

Step 3 Solution

$$f = \sum_{n=1}^{\infty} (g_n - h_n) + \text{add Laurent principal part at } 0.$$

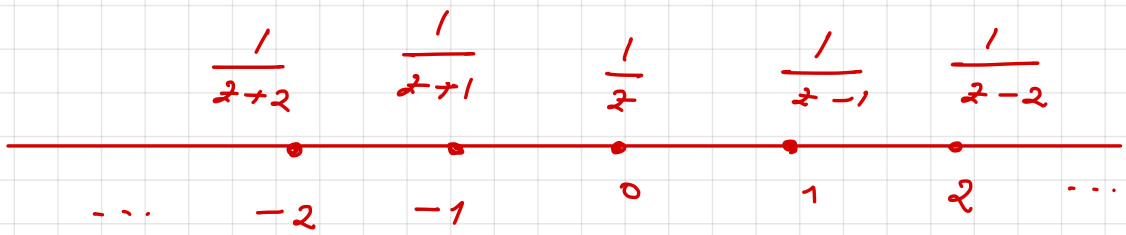
Today – 4 historically important examples

– we group them in pairs of two



poles principal parts

Example $\boxed{1}$ $\left(-n, \dfrac{1}{z+n}\right)$, $n \in \mathbb{Z}$.



Step 1 Taylor expand:

$$g_n = \frac{1}{z+n} = \frac{1}{n} \cdot \frac{1}{1+\frac{z}{n}} = \frac{1}{n}\left(1 - \frac{z}{n} + \frac{z^2}{n^2} - \cdots\right)$$

$$= \frac{1}{n} - \frac{z}{n^2} + \frac{z^2}{n^3} - \cdots$$

$$h_n = \frac{1}{n}, \quad n \neq 0$$

Step 2 Let $r_n = \frac{1}{2}|n|^{1/2}$. If $|z| \leq r_n$:

$$|g_n - h_n| = \left|\frac{1}{z+n} - \frac{1}{n}\right| = \frac{|z|}{|n|\,|n+z|} \leq \frac{r_n}{|n|\,(|n| - r_n)} = c_n$$

Since $\displaystyle\lim_{n\to\infty} \frac{c_n}{|n|^{-3/2}} < \infty$ and $\displaystyle\sum_n \frac{1}{|n|^{3/2}} < \infty$. $\implies \displaystyle\sum_{n=1}^{\infty} c_n < \infty$.

Step 3 Mittag - Leffler solution

$$f = \sum_{n \neq 0} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \frac{1}{z}.$$

Collecting the terms for n & $-n$ we find

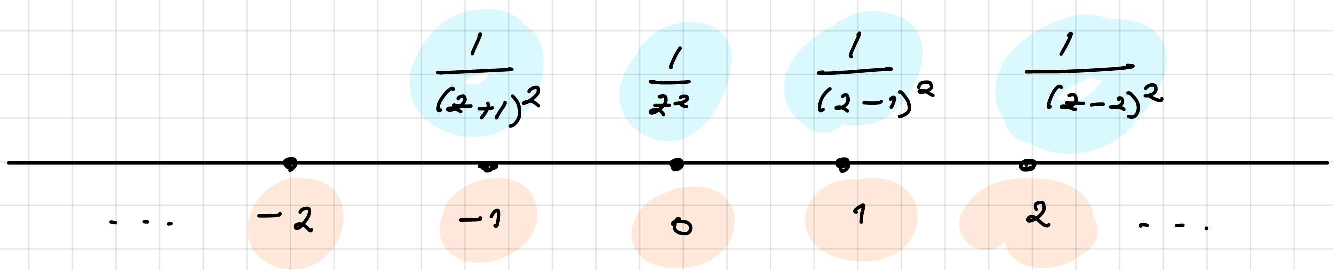
$$f = \sum_{n>0} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) + \frac{1}{z}$$

Math 220A, HWK6.

$$= \sum_{n>0} \frac{2z}{z^2 - n^2} + \frac{1}{z} = \pi \cot \pi z$$

[11] Poles at $-n \in \mathbb{Z}$, principal parts $\dfrac{1}{(z+n)^2}$.

$$\left(-n, \frac{1}{(z+n)^2}\right)$$



Step 1

$$g_n = \frac{1}{(z+n)^2}$$

$$h_n = 0$$

Step 2 $\quad r_n = \frac{1}{2}|n|^{1/2}$. If $|z| \leq r_n$

$$|g_n - h_n| = \left|\frac{1}{(z+n)^2}\right| \leq \frac{1}{(|n| - r_n)^2} =: c_n.$$

$$\lim_{n\to\infty} \frac{c_n}{|n|^{-2}} = 1 \quad \& \quad \sum_{n\neq 0} \frac{1}{n^2} < \infty \implies \sum_{n \neq 0} c_n < \infty$$

Step 3 Mittag - Leffler function

$$f = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}.$$

We have seen $f = \dfrac{\pi^2}{\sin^2 \pi z}$ in Math 220A, HWK6, #7.

6. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2}\, dz$$

via the residue theorem. Making $n \to \infty$, show that

$$\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.$$

7. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners

$$\pm \left(n + \frac{1}{2}\right) \pm ni.$$

Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{(z+a)^2}\, dz$$

via the residue theorem, and use this to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

HWK 6, Math 220A

Remark Compare $\boxed{i}$ & $\boxed{ii}$

$$\left(-n,\ \frac{1}{z+n}\right) \longleftrightarrow \left(-n,\ \frac{1}{(z+n)^2}\right)$$

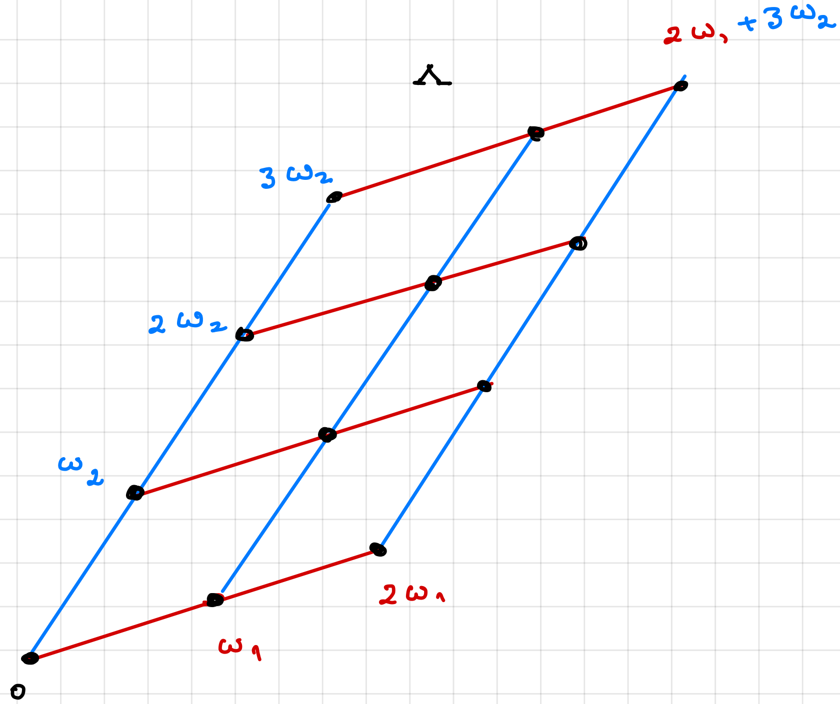
$$\pi \cot \pi z \longleftrightarrow \frac{\pi^2}{\sin^2 \pi z}$$

These are related by differentiation (up to a sign).

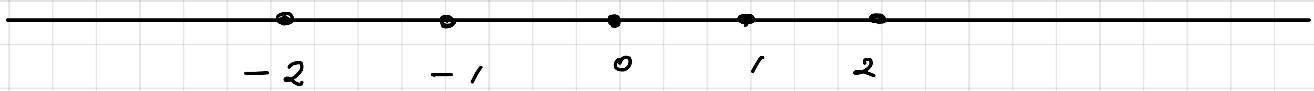
For the next examples, we replace

by the lattice

$$\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2 = \{\, m\,\omega_1 + n\,\omega_2 \;:\; m,n \in \mathbb{Z} \,\}, \qquad \frac{\omega_1}{\omega_2} \notin \mathbb{R}$$



Main Difference



$$\sum_{n \neq 0} \frac{1}{|n|^{\alpha}} \quad \text{converges if } \alpha = 2$$

$$\text{if } \alpha > 1.$$

For the lattice,

$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{|\lambda|^{\alpha}} \quad \text{converges if } \alpha = 3 \quad (\text{HWK } 2)$$

$$\text{if } \alpha > 2.$$

111

Poles at $\lambda \in \Lambda$, principal parts $\frac{1}{z-\lambda}$.

$$\left(\lambda, \frac{1}{z-\lambda}\right)_{\lambda \in \Lambda}.$$

Step 1 $\lambda \neq 0$

Taylor expand

$$g_\lambda = \frac{1}{z-\lambda} = \frac{1}{\lambda} \cdot \frac{-1}{1-\frac{z}{\lambda}}$$

$$= \frac{-1}{\lambda}\left(1+\frac{z}{\lambda}+\frac{z^2}{\lambda^2}+\dots\right)$$

$$= -\frac{1}{\lambda} - \frac{z}{\lambda^2} - \frac{z^2}{\lambda^3} - \dots$$

$$h_\lambda = -\frac{1}{\lambda} - \frac{z}{\lambda^2}$$

Step 2 Let $r_\lambda = \min\left(\frac{1}{2}|\lambda|, |\lambda|^{1/4}\right)$.

If $|z| \leq r_\lambda$ then

$$|g_\lambda - h_\lambda| = \left|\sum_{k=2}^{\infty} \frac{z^k}{\lambda^{k+1}}\right|$$

$$= \frac{|z|^2}{|\lambda|^3} \sum_{k=0}^{\infty} \left|\frac{z}{\lambda}\right|^k \leq \frac{r_\lambda^2}{|\lambda|^3} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} =$$

$$= 2 \cdot \frac{r_\lambda^2}{|\lambda|^3} \leq 2 \cdot \frac{1}{|\lambda|^{5/2}} = c_\lambda .$$

Since $\sum_{\lambda \neq 0} \frac{1}{|\lambda|^{5/2}} < \infty$. we get $\sum_{\lambda \neq 0} c_\lambda < \infty$.

Step 3 Mittag-Leffler solution

$$\zeta = \frac{1}{z} + \sum_{\lambda \neq 0} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right)$$

Weierstraß ζ- function (HWK 3, #3)

$\boxed{IV}$ Poles at $\lambda \in \Lambda$, principal parts $\frac{1}{(z-\lambda)^2}$.

$$\left(\lambda, \frac{1}{(z-\lambda)^2}\right)_{\lambda \in \Lambda}$$

Step 1 $\quad \lambda \neq 0$

$$g_\lambda = \frac{1}{(z-\lambda)^2} = \frac{1}{\lambda^2} \cdot \frac{1}{\left(1 - \frac{z}{\lambda}\right)^2} =$$

$$= \frac{1}{\lambda^2}\left(1 + \frac{2z}{\lambda} + \frac{3z^2}{\lambda^2} + \cdots\right)$$

$$= \frac{1}{\lambda^2} + \frac{2z}{\lambda^3} + \frac{3z^2}{\lambda^3} + \cdots$$

$$\frac{1}{(1-w)^2} = 1 + 2w + 3w^2 + \cdots$$

$$h_\lambda = \frac{1}{\lambda^2}.$$

Step 2 $\quad r_\lambda = \min\left(\frac{|\lambda|}{2}, |\lambda|^{1/4}\right)$

$$|h_\lambda - g_\lambda| = \left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| = \left|\frac{z^2 - 2z\lambda}{\lambda^2 (z-\lambda)^2}\right|$$

$r_\lambda \le \frac{|\lambda|}{2}$

$$\le \frac{r_\lambda^2 + 2r_\lambda|\lambda|}{|\lambda|^2 (|\lambda| - r_\lambda)^2} \le 4 \cdot \frac{r_\lambda^2 + 2r_\lambda|\lambda|}{|\lambda|^4} = c_\lambda.$$

Note

$$\lim_{\lambda\to\infty} \frac{c_\lambda}{|\lambda|^{-5/2}} < \infty \implies \sum c_\lambda \sim \sum \frac{1}{|\lambda|^{5/2}} < \infty .$$

Step 3 The Mittag-Leffler solution

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) = \text{Weierstraß } \wp \text{ function.}$$

Homework 3, #4.

Compare $\boxed{III}$ & $\boxed{IV}$

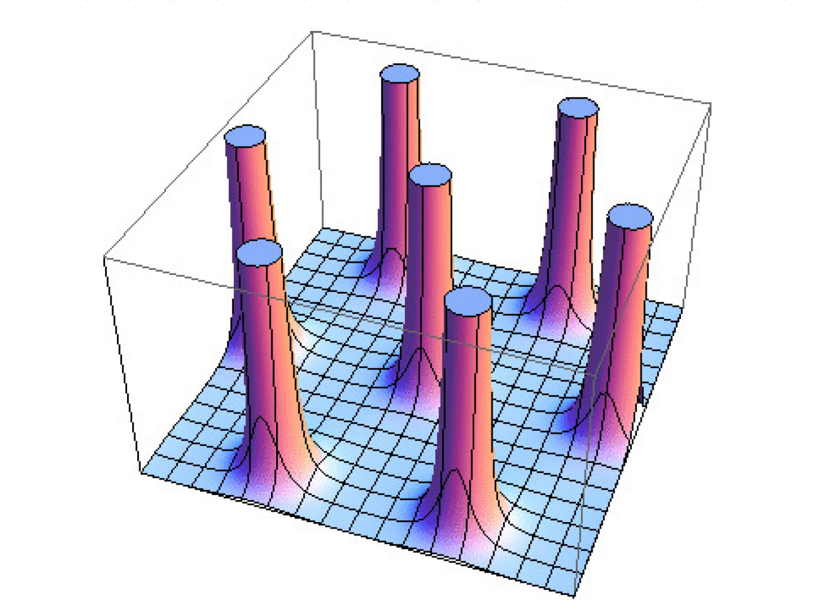
$$\left(\lambda,\ \frac{1}{z-\lambda}\right) \quad \xleftrightarrow{-\text{ derivative}} \quad \left(\lambda,\ \frac{1}{(z-\lambda)^2}\right)$$

$$\zeta \quad \longleftrightarrow \quad \wp = -\zeta'$$

$\boxed{I}$ $\boxed{II}$

$$\left(-n,\ \frac{1}{z+n}\right) \quad \xleftrightarrow{-\text{ derivative}} \quad \left(-n,\ \frac{1}{(z+n)^2}\right)$$

$$\pi \cot \pi z \quad \longleftrightarrow \quad \frac{\pi^2}{\sin^2 \pi z} = -(\pi \cot \pi z)'$$



Weierstraß $\wp$ function

Remark These results also hold for $\mathcal{U} \subseteq \mathbb{C}$.

The proof follows Step 1 & step 2 of Weierstraß problem. We will not give it here.