Math 220 A – Lecture 16

November 18, 2020

1. Last time

The proof of Residue thm / Enhanced Cauchy requires:

Theorem (enhanced CIF / Conway IV. 5)

$\gamma \overset{U}{\approx} 0$, $f : U \longrightarrow \mathbb{C}$ holomorphic, $a \in U \setminus \{\gamma\}$

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - a} \, dw = n(\gamma, a) \, f(a).$$

Rewriting.

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - a} \, dw = \frac{1}{2\pi i} \int_\gamma \frac{f(a)}{w - a} \, dw$$

$$\iff \int_\gamma \underbrace{\frac{f(w) - f(a)}{w - a}}_{\varphi(a, w)} \, dw = 0$$

Auxiliary function $\varphi : U \times U \longrightarrow \mathbb{C}$

$$\varphi(z, w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w}, & z \neq w. \\[2ex] f'(z), & z = w. \end{cases}$$

Want: $\displaystyle\int_\gamma \varphi(z, w)\, dw = 0 \quad \forall z \in U \quad (*)$

Apply $(*)$ to $z = a \in U \setminus \{\gamma\}$ to conclude enhanced CIF.

Claims

[I] φ continuous in $U \times U$

[II] $z \longrightarrow \varphi(z, w)$ holomorphic $\forall w \in U$. fixed.

Proof of [II] This was explained in Lecture 13 as an application of Removable Singularity Theorem.

Proof of $\boxed{\text{II}}$ φ continuous in $\mathcal{U} \times \mathcal{U}$. Recall

$$\varphi(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w}, & z \neq w. \\ f'(z), & z = w. \end{cases}$$

- Continuity is clear at points where $z \neq w$.
- We show continuity at (a,a). We have

$$|\varphi(z,w) - \varphi(a,a)| = \left| \frac{1}{w-z} \int_z^w f'(t)\,dt - f'(a) \right|$$

$$= \frac{1}{|w-z|} \left| \int_z^w f'(t) - f'(a)\, dt \right|$$

$$\leq \sup_{t \in [z,w]} |f'(t) - f'(a)| < \varepsilon$$

if $z, w \in \Delta(a, \delta)$.

This holds in $\Delta(a,\delta)$ for some δ, because f' is continuous (in fact holomorphic).

Proof of (*) Want $\int_\gamma \varphi(z,w)\,dw = 0$ if $\gamma \overset{U}{\approx} 0$.

Question : How do we make use of $\gamma \overset{U}{\approx} 0$?

Answer. Define

$$V = \{ z \in \mathbb{C} \setminus \gamma ,\ n(\gamma, z) = 0 \}.$$

- $U \cup V = \mathbb{C}$. (this is the only place where $\gamma \overset{U}{\approx} 0$ is used).

Indeed if $z \notin U \Rightarrow n(\gamma, z) = 0$ since $\gamma \overset{U}{\approx} 0$. Also $z \in \mathbb{C} \setminus \{\gamma\}$.

- V open. Indeed, by Lecture 7, V is union of components of $\mathbb{C} \setminus \{\gamma\}$ = open $\Rightarrow$ V open.

- V unbounded. In fact, by Lecture 7, $\exists\, R \gg 0$ with $\{ |z| > R \} \subseteq V$.

Define $h : \mathbb{C} \longrightarrow \mathbb{C}$

$$h(z) = \begin{cases} \int_\gamma \varphi(z,w)\, dw, & z \in U. \\ \int_\gamma \frac{f(w)}{w-z}\, dw, & z \in V \end{cases}$$

Claims $\boxed{a}$ h well-defined

$\boxed{b}$ h bounded, $\lim_{z \to \infty} h(z) = 0$

$\boxed{c}$ h entire

Conclusion By Liouville $\Rightarrow$ h constant $\overset{\boxed{b}}{\Rightarrow}$ $h \equiv 0$.

Thus if $z \in U$ $\Rightarrow$ $h(z) = \int_\gamma \varphi(z,w)\, dw = 0$ $\Rightarrow$ (*).

QED.

Proof of $\boxed{a}$ h well-defined. Take $z \in U \cap V$. We show

$$\int_\gamma \varphi(z,w)\, dw = \int_\gamma \frac{f(w)}{w-z}\, dw.$$

$$\Longleftrightarrow \int_\gamma \frac{f(z)}{w-z}\, dw = 0 \Longleftrightarrow f(z)\, n(\gamma, z) = 0$$ which is

true since $n(\gamma, z) = 0$ for $z \in V$.

Proof of $\boxed{b}$

Let $K > 0$ such that $\{\gamma\} \subseteq \Delta(0, K)$ by compactness.

We have $|w - z| \geq |z| - |w| \geq |z| - K$ if $w \in \{\gamma\}$.

If $R >> 0$, $|z| \geq R \Rightarrow z \in V$. then

$$|h(z)| = \left| \int_\gamma \frac{f(w)}{w - z} \, dw \right| \leq \underbrace{\text{length}(\gamma) \cdot \sup_{\{\gamma\}} |f| \cdot \frac{1}{|z| - K}}_{\downarrow \atop 0 \text{ as } z \to \infty}$$

Since h is continuous by $\boxed{c}$ $\Rightarrow$ h bounded.

why?

- $\lim_{z \to \infty} h(z) = 0 \Rightarrow \exists \alpha$, $|h(z)| \leq 1$ if $|z| \geq \alpha$
- h continuous $\Rightarrow \exists M$, $|h(z)| \leq M$ if $|z| \leq \alpha$

$\Rightarrow$

$\Rightarrow |h| \leq \max(1, M)$.

Proof of (c) h entire

Recall Conway Exercise IV. 2. 2. / HWK 3 # 7.

Key statement $\varphi : U \times \{\gamma\} \to \mathbb{C}$

- φ continuous
- $z \to \varphi(z, w)$ holomorphic $\forall\, w \in \{\gamma\}$.

Then $g(z) = \int_\gamma \varphi(z, w)\, dw$ holomorphic.

Proof See Solution Set 3.

Alternatively, let $\bar{R} \subseteq U$. Then

$$\int_{\partial R} g\, dz = \int_{\partial R} \int_\gamma \varphi(z, w)\, dw\, dz$$

Fubini's theorem
φ continuous

$$= \int_\gamma \underbrace{\int_{\partial R} \varphi(z, w)\, dz}_{0}\, dw$$

Goursat's lemma or Cauchy

$$= \int_\gamma 0\, dw = 0$$

$\Rightarrow g$ admits a primitive in any disc $\Delta \subseteq U$, $g = G'$

$\Rightarrow g$ holomorphic (G = holomorphic = ∞-many times differentiable)

Back to $\boxed{C}$. Apply Key Statement to

- the set $\mathcal{U}$, for $\varphi = \phi \Rightarrow h$ holomorphic in $\mathcal{U}$
- the set V, for $\varphi(z, w) = \dfrac{f(w)}{w - z} : V \times \{\gamma\} \longrightarrow \mathbb{C}$

  $\Rightarrow h$ holomorphic in V.

Thus h is entire. QED.

2. Applications of the Residue theorem to real analysis

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_{s\in S} \operatorname{Res}(f,s)\cdot n(\gamma,s)\,, \quad \gamma \overset{u.}{\approx} 0$$

Applications

- $\boxed{a}$ trigonometric functions
- $\boxed{b}$ rational functions
- $\boxed{c}$ Fourier integrals
- $\boxed{d}$ logarithmic integrals
- $\boxed{e}$ Mellin transforms

Poisson : "Je n'ai remarqué aucune intégrale qui ne fût pas déjà connue"

$\boxed{a}$ Trigonometric integrals

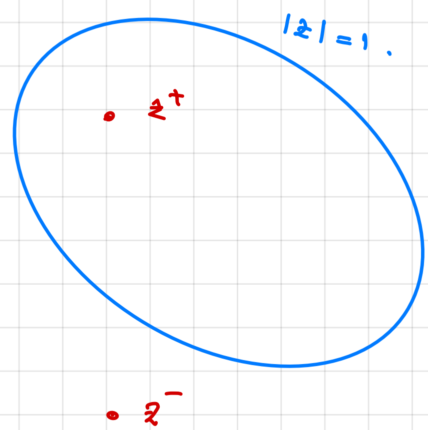
Example $a > 1$, $a \in \mathbb{R}$. $\quad I = \displaystyle\int_0^{2\pi} \frac{dt}{a + \sin t}$.

$$z = e^{it} \implies \frac{dz}{iz} = dt$$

$$\sin t = \frac{z - z^{-1}}{2i}.$$

By substitution, we find

$$I = \int_{|z|=1} \frac{2\,dz}{z^2 - 1 + 2aiz}.$$



poles $z^2 - 1 + 2aiz = 0$

$$\implies z^{\pm} = -ai \pm i\sqrt{a^2-1}$$

Note $|z^+| < 1$, $|z^-| > 1$. Thus

Method 1

$$I = 2\pi i \operatorname{Res}(f, z^+) = 2\pi i \cdot \frac{2}{(z^2 - 1 + 2aiz)'}\Bigg|_{z = z^+},$$

$$= 2\pi i \cdot \frac{2}{2z + 2ai}\Bigg|_{z = z^+} = \boxed{\frac{2\pi}{\sqrt{a^2-1}}}.$$