HW 5.

Problem 1. (a) $R(\theta,\theta^*) = E_{\theta^*}\{(\theta^*-\theta)X + \log(1+e^{\theta}) - \log(1+e^{\theta^*})\}$

$= (\theta^*-\theta)\cdot\frac{e^{\theta^*}}{1+e^{\theta^*}} + \log\frac{1+e^{\theta}}{1+e^{\theta^*}}$.

(b) $R(\theta,\theta^*) = E_{\theta^*}\{(\theta^*-\theta)X + e^{\theta} - e^{\theta^*}\}$

$= (\theta^*-\theta)e^{\theta^*} + e^{\theta} - e^{\theta^*}$ since $X\sim \text{Poisson}(e^{\theta^*})$ and $E(X) = e^{\theta^*}$

(c) $p_\theta(X) = (2\pi)^{-\frac{d}{2}}|\Sigma|^{-\frac{1}{2}}\exp\{-\frac{1}{2}(X-\theta)^T\Sigma^{-1}(X-\theta)\}$.

$R(\theta,\theta^*) = E_{\theta^*}\{\frac{1}{2}(X-\theta)^T\Sigma^{-1}(X-\theta) - \frac{1}{2}(X-\theta^*)^T\Sigma^{-1}(X-\theta^*)\}$

$= E_{\theta^*}\left(\frac{1}{2}X^T\Sigma^{-1}X - \theta^T\Sigma^{-1}X + \frac{1}{2}\theta^T\Sigma^{-1}\theta - \frac{1}{2}X^T\Sigma^{-1}X + \theta^{*T}\Sigma^{-1}X - \frac{1}{2}\theta^{*T}\Sigma^{-1}\theta^*\right)$

$= E_{\theta^*}\{(\theta^*-\theta)^T\Sigma^{-1}X + \frac{1}{2}\theta^T\Sigma^{-1}\theta - \frac{1}{2}\theta^{*}\Sigma^{-1}\theta^*\} = (\theta^*-\theta)^T\Sigma^{-1}E_{\theta^*}(X) + \frac{1}{2}\theta^T\Sigma^{-1}\theta - \frac{1}{2}\theta^{*T}\Sigma^{-1}\theta^*$

$= (\theta^*-\theta)^T\Sigma^{-1}\theta^* + \frac{1}{2}(\theta-\theta^*)^T\Sigma^{-1}(\theta+\theta^*) = \frac{1}{2}(\theta-\theta^*)^T\Sigma^{-1}(\theta-\theta^*)$

Problem 2. (a) For 1a: $R(\theta,\theta^*) = (\theta^*-\theta)\cdot\frac{e^{\theta^*}}{1+e^{\theta^*}} + \log\frac{1+e^{\theta}}{1+e^{\theta^*}}$, $\forall\theta\in\Omega$.

$$p_\theta(x_1\cdots x_n) = \prod_{i=1}^n p_\theta(x_i) = \prod_{i=1}^n \frac{e^{\theta x_i}}{1+e^\theta} = \exp(\theta\sum_{i=1}^n x_i)/(1+e^\theta)^n$$

$$l(\theta) = \log p_\theta(x_1\cdots x_n) = \theta\sum_{i=1}^n x_i - n\log(1+e^\theta)$$

$$\Rightarrow l'(\theta) = n\left(\bar{x} - \frac{e^\theta}{1+e^\theta}\right)$$

Let $l'(\hat\theta)=0 \Rightarrow \bar{x} = \frac{e^{\hat\theta}}{1+e^{\hat\theta}} \Rightarrow \hat\theta = \log\frac{\bar{x}}{1-\bar{x}}$

Hence, $E(\hat\theta,\theta^*) = R(\hat\theta,\theta^*) - \inf_{\theta\in\Omega} R(\theta,\theta^*)$

$$= \left(\theta^* - \log\frac{\bar{x}}{1-\bar{x}}\right)\cdot\frac{e^{\theta^*}}{1+e^{\theta^*}} - \log\{(1-\bar{x})(1+e^{\theta^*})\} - \inf_{\theta\in\Omega}\left\{(\theta^*-\theta)\cdot\frac{e^{\theta^*}}{1+e^{\theta^*}} + \log\frac{1+e^\theta}{1+e^{\theta^*}}\right\}.$$

$$= -\log\frac{\bar{x}}{1-\bar{x}}\cdot\frac{e^{\theta^*}}{1+e^{\theta^*}} - \log(1-\bar{x}) - \inf_{\theta\in\Omega}\left\{-\theta\cdot\frac{e^{\theta^*}}{1+e^{\theta^*}} + \log(1+e^\theta)\right\}$$

For 1b: $R(\theta,\theta^*) = (\theta^*-\theta)e^{\theta^*} + e^\theta - e^{\theta^*}$, $\forall\theta\in\Omega$

$$p_\theta(x_1\cdots x_n) = \prod_{i=1}^n p_\theta(x_i) = \exp(\theta\sum_{i=1}^n x_i - ne^\theta)/\prod_{i=1}^n x_i!$$

$$l(\theta) = \log p_\theta(x_1\cdots x_n) = n(\theta\bar{x} - e^\theta) - \log\left(\prod_{i=1}^n x_i!\right)$$

$$\Rightarrow l'(\theta) = n(\bar{x} - e^\theta)$$

Let $l'(\hat\theta)=0 \Rightarrow \bar{x} = e^{\hat\theta} \Rightarrow \hat\theta = \log\bar{x}$

Hence, $E(\hat\theta,\theta^*) = R(\hat\theta,\theta^*) - \inf_{\theta\in\Omega} R(\theta,\theta^*)$

$$= (\theta^* - \log\bar{x})e^{\theta^*} + \bar{x} - e^{\theta^*} - \inf_{\theta\in\Omega}\{(\theta^*-\theta)e^{\theta^*} + e^\theta - e^{\theta^*}\}$$

$$= -e^{\theta^*}\log\bar{x} + \bar{x} - \inf_{\theta\in\Omega}\{-\theta\cdot e^{\theta^*} + e^\theta\}$$

For 1c: $R(\theta, \theta^*) = \frac{1}{2}(\theta - \theta^*)^T \Sigma^{-1} (\theta - \theta^*)$

$p_\theta(x_1 \cdots x_n) = \prod_{i=1}^{n} p_\theta(x_i) = \{(2\pi)^d |\Sigma|\}^{-\frac{n}{2}} \cdot \exp\{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^T \Sigma^{-1}(x_i - \theta)\}$

$\ell(\theta) = \log p_\theta(x_1 \cdots x_n) = -\frac{n}{2}\log\{(2\pi)^d |\Sigma|\} - \frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^T \Sigma^{-1}(x_i - \theta)$

$\Rightarrow \ell'(\theta) = \Sigma^{-1}\sum_{i=1}^{n}(x_i - \theta) = n\Sigma^{-1}(\bar{x} - \theta)$

Let $\ell'(\hat{\theta}) = 0 \Rightarrow \hat{\theta} = \bar{x}$.

Hence, $\mathcal{E}(\hat{\theta}, \theta^*) = R(\hat{\theta}, \theta^*) - \inf_{\theta \in \Omega} R(\theta, \theta^*)$

$= \frac{1}{2}(\bar{x} - \theta^*)^T \Sigma^{-1}(\bar{x} - \theta^*) - \frac{1}{2}\inf_{\theta \in \Omega}(\theta - \theta^*)^T \Sigma^{-1}(\theta - \theta^*)$

(b) Suppose the existance of some $\theta_0 \in \Omega$ st. $R(\theta_0, \theta^*) = \inf_{\theta \in \Omega} R(\theta, \theta^*)$.

Define $T_1 := R(\hat{\theta}, \theta^*) - \hat{R}_n(\hat{\theta}, \theta^*)$, $T_2 := \hat{R}_n(\hat{\theta}, \theta^*) - \hat{R}_n(\theta_0, \theta^*)$, $T_3 := \hat{R}_n(\theta_0, \theta^*) - R(\theta_0, \theta^*)$.

where $\hat{R}_n(\theta, \theta^*) = n^{-1}\sum_{i=1}^{n} L_\theta(x_i)$, with $L_\theta(x) := \log\{\frac{p_{\theta^*}(x)}{p_\theta(x)}\}$. Then, $\mathcal{E}(\hat{\theta}, \theta^*) = R(\hat{\theta}, \theta^*) - R(\theta_0, \theta^*) = T_1 + T_2 + T_3$.

Since $\hat{\theta}$ is the MLE, $\hat{R}_n(\hat{\theta}, \theta^*) \le \hat{R}_n(\theta_0, \theta^*)$ and hence $T_2 \le 0$.

Define $\|\mathbb{P}_n - \mathbb{P}\|_{L(\Omega)} := \sup_{\theta \in \Omega}|n^{-1}\sum_{i=1}^{n} L_\theta(x_i) - E_x\{L_\theta(x)\}|$. $L(\Omega) := \{x \to L_\theta(x) : \theta \in \Omega\}$.

Then, $T_1 \le \|\mathbb{P}_n - \mathbb{P}\|_{L(\Omega)}$ and $T_3 \le \|\mathbb{P}_n - \mathbb{P}\|_{L(\Omega)} \Rightarrow \mathcal{E}(\hat{\theta}, \theta^*) \le 2\|\mathbb{P}_n - \mathbb{P}\|_{L(\Omega)}$

For the sake of simplicity, assume Ω and the support of X are compact.

Then, $\|L_\theta(x)\|_\infty \le b$ for some constant b and by Theorem 4.10, with prob. at least $1 - \exp(-\frac{n\delta^2}{2b^2})$,

$\mathcal{E}(\hat{\theta}, \theta^*) \le 2\|\mathbb{P}_n - \mathbb{P}\|_{L(\Omega)} \le 4R_n(L(\Omega)) + 2\delta$. where $R_n(L(\Omega)) = E_{X,\varepsilon}\{\sup_{\theta \in \Omega}|n^{-1}\sum_{i=1}^{n}\varepsilon_i L_\theta(x_i)|\}$.