## Math 220B Final Exam Review

To review, we list below the *Main Topics* covered in this class (this is not a comprehensive list):

(1) Products of holomorphic functions. Convergence, zeroes, logarithmic derivatives.
(2) Weierstraß elementary factors. Weierstraß factorization. Weierstraß problem in arbitrary regions.
(3) Mittag-Leffler problem in $\mathbb{C}$. Examples.
(4) Factorization of the sine function. The Gamma function.
(5) Normal families. Montel's theorem.
(6) Schwarz's lemma. Automorphisms of the disc. Schwarz-Pick.
(7) Riemann Mapping Theorem.
(8) Schwarz Reflection Principle.
(9) Runge's Theorem. Polynomial and rational approximation.

### Additional Practice Problems

*Please review the course material and the homework problems. In case you need more practice problems, a list is below. There's no need to solve them all before the final; they're here just in case you think you need more practice*

**1.** (*Qualifying Exam 2019.*) Show that there are no bijective holomorphic maps $f : \{0 < |z| < 1\} \to \{1 < |z| < 2\}$.

**2.** (*Conway, Chapter VI.2.6, page 133.*) Assume $f$ is holomorphic in a region containing $\overline{\Delta} = \overline{\Delta}(0,1)$, and that $|f(z)| = 1$ if $|z| = 1$. Assume that $f$ has a simple zero at $z = \frac{1}{4}(1+i)$ and a double zero at $z = \frac{1}{2}$. Can $f(0) = 1/2$?

**3.** (*Conway, Chapter VI.2.8, page 133.*) Is there a holomorphic function $f : \Delta \to \Delta$ such that $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$? Is it unique?

**4.** (*Qualifying Exam 2017.*) Let $f : \mathfrak{h}^+ \to \mathbb{C}$ such that $f(i) = 0$ and $|f(z)| < 1$ for all $z \in \mathfrak{h}^+$. What is the maximum value of $|f(2i)|$?

**5.** Let $f$ be a holomorphic map between the strip $S = \{-1 < \text{Re } z < 1\}$ and the unit disc $\Delta(0,1)$ such that $f(0) = 0$. What is the maximum value of $|f'(0)|$?

**6.** Let $A = \{2 < |z| < 3\}$ and $f(z) = \frac{e^z}{z}$. Can the function $f$ be approximated locally uniformly in $A$ by polynomials? Can it be approximated locally uniformly in $A$ by rational functions with poles only at 1 and 4? By rational functions with poles only at 4?

**7.** (*Conway, Chapter VII.6.1, page 166.*) Show that

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

**8.** Show that $f(z) = -\cos z$ determines a biholomorphism between the half infinite strip $\{z = x + iy : 0 < x < \pi, y > 0\}$ and the upper half plane.

**9.** Let $\mathcal{F}$ be the family of holomorphic functions on $\Delta(0,1)$ such that

$$|f'(z)|(1 - |z|^2) + |f(0)| \leq 1.$$

Is $\mathcal{F}$ normal?

**10.** (*Qualifying Exam 2017.*) Construct a meromorphic function with simple poles at $z = n$ and residues equal to $n\sqrt{n}$ for $n = 1, 2, 3 \ldots$.

**11.** (*Qualifying Exam 2008.*) Prove that there exist a sequence $R_n$ of rational functions whose finite poles are only at $\frac{3}{2}$ such that

$$\lim_{n\to\infty} R_n(z) = 1 \quad \text{for } |z| = 1, \quad \lim_{n\to\infty} R_n(z) = 2 \quad \text{for } 2 \leq |z| \leq 3.$$

**12.** (*Qualifying Exam 2020.*) Let $G \neq \mathbb{C}$ be a connected set such that $\widehat{\mathbb{C}} \setminus G$ is connected. Show that if $f : G \to G$ is holomorphic and admits 2 fixed points then $f$ is the identity.

**13.** (*Conway, Chapter VIII.3.5, page 209.*) Assume that $a_n$ is an infinite sequence such that $|a_n| \to \infty$ and let $A_n \in \mathbb{C}$. Show that there exists an entire function $f$ such that $f(a_n) = A_n$.

In fact, prove the stronger statement that fixing $a_n$, $m_n$, and values $A_{nk}$ for $0 \leq k \leq m_n$, one can construct an entire function $f$ such that

$$f^{(k)}(a_n) = A_{nk}$$

for all $0 \leq k \leq m_n$.

**14.** (*Qualifying Exam 2009.*) Assume that $\alpha_n$ is an infinite sequence such that $|\alpha_n| \to \infty$ and let $\beta_n$ be complex numbers. Show that there exists an entire function such that $f(\alpha_n) = \beta_n$ with multiplicity 2, that is $f - \beta_n$ has a zero of order 2 at $\alpha_n$.

**15.** (*Qualifying Exam 2016.*) Let $a_k = 1 - \frac{1}{k^2}$ for $k \geq 1$. Let $f_n(z) = \prod_{k=1}^{n} \frac{a_k - z}{1 - z a_k}$.

(i) Show that $f_k$ converges to a holomorphic function $f : \Delta(0,1) \to \Delta(0,1)$.

(ii) Show that there does not exist an open set $U \subset \mathbb{C}$ and a holomorphic function $g : U \to \mathbb{C}$ such that $\overline{\Delta}(0,1) \subset U$ and $f(z) = g(z)$ for $z \in \Delta$.