Math 220 Final Exam Review

To review, we list below the *Main Topics* covered in this class (this is not a comprehensive list):

(1) Holomorphic functions. Harmonic functions.
(2) Conformal maps. Fractional linear transformations.
(3) Existence of local/global primitives. Logarithm. Winding numbers.
(4) Cauchy's integral formula. Cauchy's estimates.
(5) Taylor and Laurent series.
(6) Zeroes of holomorphic functions, identity principle, open mapping theorem, maximum modulus principle, Liouville's theorem.
(7) Types of singularities. Removable singularities theorem. Meromorphic functions. Residues. Cassorati-Weierstraβ.
(8) Residue theorem. Residues at infinity. Applications to real analysis.
(9) The argument principle. Rouché's theorem.
(10) Sequences of holomorphic functions. Hurwitz's theorem. Weierstraβ convergence theorem.

Additional Practice Problems

Please review the homework problems, and the practice final posted online. In case you need more practice problems, a list is below. There's no need to solve them all before the final; they're here just in case you think you need more practice

1.

(i) Let $x \in \mathbb{C}$. Show that the Laurent expansion

$$\exp\left(\frac{1}{2}x\left(z-\frac{1}{z}\right)\right) = J_0(x) + \sum_{n=1}^{\infty} J_n(x)\left(z^n + \frac{(-1)^n}{z^n}\right)$$

holds for $0 < |z| < \infty$ for some coefficients $J_n(x)$ that depend on x.

Remark: These coefficients J_n are called the Bessel functions of the first kind, and appear for instance in the study of the wave equation.

(ii) Using the expansion of the exponential, show that J_n are entire and

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

(iii) Show that $y = J_n(x)$ is a solution to the Bessel differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

2. Assume that f is entire and $f(z)f(1/z)$ is a bounded function on $\mathbb{C} \setminus \{0\}$. Show that $f(z) = cz^m$ for some $c \in \mathbb{C}$ and an integer $m \geq 0$.

3. Assume that f and g are entire and $f \circ g = 0$. Show that either $f = 0$ or g is constant. This uses a previous homework problem.

4. Compute the following integrals:

(i)
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^a + a^2}\, dx$$

(ii)
$$\int_0^{\infty} \frac{(\log x)^2}{1 + x^2}\, dx$$

(iii)
$$\int_0^{\infty} \frac{x^2}{x^4 + x^2 + 1}\, dx.$$

5. Let $f(z) = \pi^2 z^5 e^{-2z} - 1$. How many roots does f have in $|z| < 1$? How many of these roots are simple?

6. Assume f and g are meromorphic functions on $\mathbb{C}$ such that

$$|f(z) - g(z)| < |g(z)|$$

for all $z \in \mathbb{C}$ which are not poles for f or g. Show that $f = cg$ for some constant c.

7.

(i) Let $A = \{z : |z| \leq R\}$, and let f be a holomorphic function in a neighborhood of A. Explain that for all $\epsilon > 0$, there exists a polynomial p such that

$$\sup_{z \in A} |p(z) - f(z)| < \epsilon.$$

(ii) Assume that $A = \{z : r \leq |z| \leq R\}$ for $R > r > 0$. Show that there exists $\epsilon > 0$ such that for all polynomials p we have

$$\sup_{z \in A} \left| p(z) - \frac{e^z}{z} \right| > \epsilon.$$

That is, show that $\frac{e^z}{z}$ cannot be approximated by polynomials uniformly on A. This is an application of integration.

8.

(i) Show that $C(z) = \frac{z-i}{z+i}$ takes the upper half plane bijectively onto the unit disc; in particular, if $\operatorname{Im} z > 0$ then $|C(z)| < 1$.

(ii) Conclude from (i) that there are no entire functions with $f : \mathbb{C} \to \mathbb{C}$ such that $\operatorname{Im} f(z) > 0$ for all $z \in \mathbb{C}$.

9. Show that if $f = u + iv$ is holomorphic on U and $u + v$ admits a local maximum, then f is constant.

10. Let γ_n be the boundary of the rectangle with corners

$$\pm\left(n+\frac{1}{2}\right)\pm i\left(n+\frac{1}{2}\right)$$

Evalute the integral

$$I_n=\int_{\gamma_n}\frac{1}{z^2\sin\pi z}\,dz.$$

Next, show that $\lim_{n\to\infty} I_n=0$ and deduce from here the identity

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}=-\frac{\pi^2}{12}.$$

11. Find all holomorphic functions f on $\mathbb{C}\setminus\{0\}$ such that there exists a constant $C>0$ with

$$|f(z)|\leq C|z|^2+\frac{C}{|z|^{\frac{1}{2}}}$$

for all $z\neq 0$.

12. Let f be an entire function with $f'(\frac{1}{n})=f(\frac{1}{n})$ and $f(0)=1$. Show that $f(z)=e^z$.

13. Assume that f is entire and N is a positive integer. Assume $|f(z)|\geq|z|^N$, for all z sufficiently large. Show that f is a polynomial.

14. Assume $f_n:U\to\mathbb{C}$ is a sequence of holomorphic functions converging locally uniformly to $f:U\to\mathbb{C}$. Assume $f\not\equiv 0$, but $f(a)=0$ for some $a\in U$. Show there exists a sequence $a_n\in U$ with

(i) $\lim_{n\to\infty} a_n=a$
(ii) $f_n(a_n)=0$ for all $n\geq N$, for some N.

15. Let $f_n(z)=\frac{\sin nz}{\sqrt{n}}$, $f_n:\mathbb{C}\to\mathbb{C}$.

(i) Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$, but that the derivatives $\{f_n'\}$ do not converge even pointwise.
(ii) Does $\{f_n\}$ converge locally uniformly on $\mathbb{C}$? Where does the argument in (i) break down?

16. Show that there are no bijective holomorphic maps $f:\{0<|z|<1\}\to\{1<|z|<2\}$.