Math 220 B — Lecture 4

January 11, 2021

0. Last time

$f_k : U \to \mathbb{C}$ holomorphic

$\sum_{k=1}^{\infty} |f_k|$ converges locally uniformly

$$(1) \qquad h(z) = \prod_{k=1}^{\infty} (1 + f_k(z)) \quad \text{holomorphic}$$

$$(2) \qquad \frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{1 + f_k}$$

The series on RHS converges absolutely locally uniformly on $U \setminus \mathrm{Zero}(h)$.

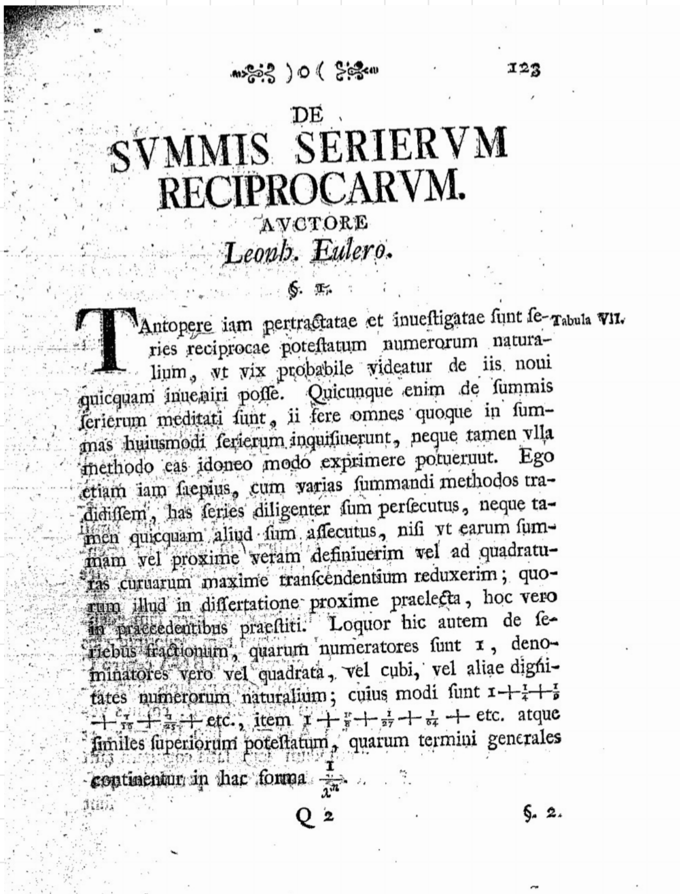
Remark

If $\sum_{k=1}^{\infty} |\mathrm{Log}(1 + f_k)|$ converges locally uniformly

the same conclusions hold.

Today $\boxed{i}$ factorization of sine   Conway VII.6.

Euler, 1734



DE SVMMIS SERIERVM RECIPROCARVM.

AVCTORE *Leonh. Eulero.*

§. 1.

TAntopere iam pertractatae et inueſtigatae ſunt ſeries reciprocae poteſtatum numerorum naturalium, vt vix probabile videatur de iis noui quicquam inueniri poſſe. Quicunque enim de ſummis ſerierum meditati ſunt, ii fere omnes quoque in ſummas huiusmodi ſerierum inquiſiuerunt, neque tamen vlla methodo eas idoneo modo exprimere potueruut. Ego etiam iam ſaepius, cum varias ſummandi methodos tradidiſſem, has ſeries diligenter ſum perſecutus, neque tamen quicquam aliud ſum aſſecutus, niſi vt earum ſummam vel proxime veram definiuerim vel ad quadraturas curuarum maxime tranſcendentium reduxerim; quorum illud in diſſertatione proxime praelecta, hoc vero in praecedentibus praeſtiti. Loquor hic autem de ſeriebus fractionum, quarum numeratores ſunt 1, denominatores vero vel quadrata, vel cubi, vel aliae dignitates numerorum naturalium; cuius modi ſunt $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+$ etc., item $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+$ etc. atque ſimiles ſuperiorum poteſtatum, quarum termini generales continentur in hac forma $\frac{1}{x^n}$. Tabula VII.

$\boxed{ii}$ Γ - function   Conway VII. 7

Euler, Bernoulli, Gauss, Legendre, Weierstraß

These two topics are naturally connected

1. Factorization of sine (Euler, 1734)

Theorem

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

Idea. Both sides have the same zeroes (with multiplicity)

Question When do two entire functions have exactly the same zeroes?

Lemma If $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ entire, with the same zeroes and multiplicities. Then $f = g e^{h}$ for some $h : \mathbb{C} \longrightarrow \mathbb{C}$ entire.

Proof Let $H = \frac{f}{g}$. $\Rightarrow$ H entire without zeroes by hypothesis. We show $H = e^{h}$.

The function $\frac{H'}{H}$ is entire so it admits primitive h.

$\Rightarrow \frac{H'}{H} = h'$. Then

$$(H e^{-h})' = H' e^{-h} - H e^{-h} h' = e^{-h}(H' - H h') = 0$$

$\Rightarrow H e^{-h} = c \neq 0 \Rightarrow H = c e^{h} = e^{\log c + h}$.

Remark The same holds for $f, g : \mathcal{U} \longrightarrow \mathbb{C}$, $\mathcal{U}$ simply connected.

Proof of the sine factorization

(1) convergence:

Note that $\sum_{k=1}^{\infty} \left|\frac{z^2}{k^2}\right|$ converges locally uniformly $\Rightarrow \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$ converges.

(2) location of zeroes:

Both sides $\sin \pi z$ & $\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$ have simple zeroes at the integers & nowhere else.

(3) completing the proof

By the Lemma, $\exists$ h entire

$$\sin \pi z = e^{h} \, \pi z \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$$

We show $h \equiv 0$. Compute logarithmic derivative

$$\frac{\pi \cos \pi z}{\sin \pi z} = \frac{(e^h)'}{e^h} + \frac{\pi}{\pi z} + \sum_{k=1}^{\infty} \frac{-\frac{2z}{k^2}}{1-\frac{z^2}{k^2}}$$

$$\pi \cot \pi z = h' + \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}$$

Recall Math 220, Hwk 6:

6. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2}\, dz$$

via the residue theorem. Making $n \to \infty$, show that

$$\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.$$

Thus $h' \equiv 0 \implies h$ constant. We show $h \equiv 0$.

From

$$\frac{\sin \pi z}{\pi z} = e^{h} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \quad \text{make } z \to 0$$

$$\downarrow$$

$$1 = e^{h(0)} \cdot 1 \implies h(0) = 0 \implies h \equiv 0.$$

This completes the proof.

Remark

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

$\boxed{\mathrm{I}}$ $z = \frac{1}{2}$

$$1 = \frac{\pi}{2} \prod_{k=1}^{\infty} \left(1 - \frac{1}{4k^2}\right) = \frac{\pi}{2} \cdot \prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{(2k)(2k)}$$

$$\Rightarrow \frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k)}{(2k-1)} \frac{(2k)}{(2k+1)}$$

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

Wallis, 1655

$\boxed{\mathrm{II}}$ $z = i$

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2}\right) = \frac{\sin \pi i}{\pi i} = \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

$\boxed{\mathrm{III}}$

$$\cos \pi z = \frac{\sin 2\pi z}{2 \sin \pi z} = \frac{2\pi z \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{k^2}\right)}{2\pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)}$$

Splitting into k even / odd:

$$\cos \pi z = \prod_{\ell=1}^{\infty} \left(1 - \frac{4z^2}{(2\ell-1)^2}\right).$$

2. Γ- function – probability, statistics, combinatorics, ...

"The product $1 . 2 . \ldots \cdot x$ is the function that must be introduced in analysis" (Gauss to Bessel, 1811)

$$\Pi x = \text{"}1 . 2 . 3 \ldots \cdot x\text{"} = \Gamma(x+1)$$

"The theory of analytic factorials does not seem to have the importance some mathematicians used to attribute to it"

Weierstraß 1854

Definition $\quad G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$.

Remark The convergence (absolutely & locally uniformly) of the product is HWK 1, #4. There, you show

$$\sum_{n=1}^{\infty} \left| \operatorname{Log}\left[\left(1+\frac{z}{n}\right) e^{-z/n}\right]\right|$$ converges locally uniformly.

Properties of the function G

$\boxed{i}$ $G(z)\,G(-z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}$

$$= \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right)e^{z/n}e^{-z/n}$$

$$= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi z} \quad \text{by Euler.}$$

$\boxed{ii}$ $G(z-1) = z\,G(z)\,e^{-\gamma}$ where γ is Euler constant.

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right).$$

We will prove this next time.

Definition

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\cdot\frac{1}{G(z)}$$

Remark G has zeroes at $-1, -2, \dots -n, \dots$

$\Rightarrow$ Γ meromorphic in $\mathbb{C}$ with zeroes at $-1, -2, \dots, -n, \dots$