Problem 1.

Consider the function $f(z) = z^2e^{-z} - 4z + 1$. Find the number of zeroes of f inside the disc $\Delta(0, 1)$.

`Answer`: *We apply Rouché's theorem to the holomorphic functions*

$$f(z) = z^2e^{-z} - 4z + 1, \quad g(z) = -4z$$

over the curve $|z| = 1$. *We have* $|g| = 4$ *over* $|z| = 1$, *and*

$$|f - g| = |z^2e^{-z} + 1| \leq |z^2e^{-z}| + 1 = |z|^2e^{-Re\ z} + 1 = e^{-Re\ z} + 1 \leq e^1 + 1 < 4 = |g|$$

where we used $Re\ z \geq -1$ *for* $|z| = 1$. *Therefore,* f *and* g *have the same number of zeroes inside the unit disc. Clearly* g *vanishes only at* $z = 0$ *with order* 1, *hence* f *must have only one simple zero in* $\Delta(0, 1)$ *as well.*

Problem 2.

Consider $f : \Delta(0,1) \to \mathbb{C}$ holomorphic and nonconstant, and define $M(r) = \max_{|z|=r} \mathrm{Re} f(z)$ for $0 \le r < 1$. Show that $M : [0,1) \to \mathbb{R}$ is strictly increasing.

`Answer`: *The function $g(z) = e^{f(z)}$ is holomorphic over $\Delta(0,1)$. We let*

$$N(r) = \max_{|z|=r} |g(z)|.$$

Since $|e^w| = e^{Re\,(w)}$, it follows that $N(r) = e^{M(r)}$. Since $M(r) = \log N(r)$, it suffices to show that the function N is strictly increasing.

Let $r_1 < r_2$. Use the maximum principle over the disc $\overline{\Delta}(0,r_2)$. The maximum of $|g|$ over $\overline{\Delta}(0,r_2)$ must be achieved over the boundary, hence

$$N(r_2) = \max_{|z|=r_2} |g(z)| = \max_{|z|\le r_2} |g(z)| \ge N(r_1) = \max_{|z|=r_1} |g(z)|,$$

since the circle $|z| = r_1$ is contained in $\overline{\Delta}(0,r_2)$. If we had equality, then there would be an interior point, namely a point on the circle $|z| = r_1$, which achieves the maximum of $|g|$ over $\overline{\Delta}(0,r_2)$. Therefore, g is constant in $\Delta(0,r_2)$ so that $g(z) = c \neq 0$ over $\Delta(0,r_2)$. This implies

$$f(z) = \log c + 2\pi i n_z$$

for some $n_z \in \mathbb{Z}$ that may depend on z, and for some choice of logarithm. By continuity of f, n_z must be be constant, hence $f = K$ constant in $\Delta(0,r_2)$. The zeros of $f - K$ would then not be isolated in $\Delta(0,1)$, hence $f - K = 0$ in $\Delta(0,1)$. This is however not allowed, as f is not constant. Hence equality cannot occur and $N(r_1) < N(r_2)$. The proof is completed.

Problem 3.

Are there any holomorphic functions $f : \{z : |z| > 4\} \to \mathbb{C}$ such that

$$f'(z) = \frac{z^3+2}{z(z-1)(z-3)(2z-7)}?$$

`Answer`: *We claim no such functions exists. Let γ denote the circle $|z| = 5$. We have*

$$\frac{1}{2\pi i}\int_\gamma f'\,dz = 0$$

by the fundamental theorem of calculus. On the other hand

$$-\frac{1}{2\pi i}\int_\gamma \frac{z^3+2}{z(z-1)(z-3)(2z-7)}\,dz = Res\left(\frac{z^3+2}{z(z-1)(z-3)(2z-7)}\,dz, \infty\right) = -\frac{1}{2}$$

showing that f cannot exist.

To find the residue at ∞, we change variables $z = 1/w$ and compute the residue at 0 of

$$\frac{1/w^3+2}{1/w(1/w-1)(1/w-3)(2/w-7)} \cdot \frac{-dw}{w^2} = -\frac{1+2w^3}{w(1-w)(1-3w)(2-7w)}\,dw.$$

The latter is the coefficient of w^{-1} in

$$-\frac{1+2w^3}{w(1-w)(1-3w)(2-7w)}$$

or alternatively the constant term in

$$-\frac{1+2w^3}{(1-w)(1-3w)(2-7w)} = -\frac{1}{2} + \dots.$$

Problem 4.

Assume that f is an entire function such that the sequence of derivatives $f, f', f'', f''', \ldots$ converges locally uniformly to a function g with $g(0) = 1$.

Show that there exists N such that all derivatives $f^{(n)}(z) \neq 0$ for all $n \geq N$ and $|z| < 1$.

`Answer:` *Since f is entire, all derivatives $f^{(n)}$ are entire functions. Since the sequence $\{f^{(n)}\}$ converges locally uniformly, the limit function g must be entire as well by Weierstraß. Since*

$$f^{(n)} \overset{c}{\rightrightarrows} g,$$

by the second part of Weierstraß we can take derivatives to obtain

$$f^{(n)} \overset{c}{\rightrightarrows} g'.$$

By uniqueness of the limits, we find

$$g' = g \implies (e^{-z}g)' = 0 \implies e^{-z}g = c \implies g = ce^z.$$

Since $g(0) = 1$ it follows $c = 1$ and $g(z) = e^z$. In particular, g is nowhere vanishing.

We invoke Hurwitz's theorem for the disc $V = \Delta(0,1)$. Note that g has no zeroes on the boundary ∂V. Thus by Hurwitz there exists N such that for all $n \geq N$, in $\bar{V}$ we have

$$\#\textit{Zeroes } f^{(n)} = \#\textit{Zeroes } g = 0.$$

This shows that $f^{(n)}(z) \neq 0$ when $|z| < 1$, $n \geq N$.

Problem 5.

Let $R(z) = \frac{P(z)}{Q(z)}$ be a rational function such that $\deg P + 2 \leq \deg Q$. Assume that Q has simple zeros at $a_1, \ldots, a_q$, where $a_j \in \mathbb{C} \setminus \mathbb{Z}$. Show that

$$\sum_{m=-\infty}^{\infty} R(m) = -\pi \sum_{j=1}^{q} \frac{P(a_j)}{Q'(a_j)} \cdot \cot \pi a_j.$$

(i) Let γ_n be the square with corners

$$\pm \left(n + \frac{1}{2}\right) \pm i \left(n + \frac{1}{2}\right).$$

Show that there exist constants $M_1, M_2 > 0$ such that if n is sufficiently large, and z is on the curve γ_n, we have

$$|\pi \cot \pi z| \leq M_1$$

and

$$|R(z)| \leq \frac{M_2}{|z|^2}.$$

(ii) Show that

$$\lim_{n\to\infty} \int_{\gamma_n} R(z)\pi \cot \pi z \, dz = 0.$$

(iii) Show that $\pi \cot \pi z$ has poles at all integers $m \in \mathbb{Z}$ with residue equal to 1. Find the poles and residues of $R(z)\pi \cot \pi z$. Conclude the argument.

`Answer`:

(i) *We have*

$$\lim_{|z|\to\infty} R(z)z^2 = \lim_{|z|\to\infty} \frac{P(z)z^2}{Q(z)} = \alpha$$

where α denotes the quotient of leading terms in P and Q if $\deg P + 2 = \deg Q$ and $\alpha = 0$ otherwise. Thus, for $|z| > \eta$ we have

$$|R(z)z^2| < \alpha + 1 \implies |R(z)| \leq \frac{\alpha + 1}{|z|^2} = \frac{M_2}{|z|^2}.$$

To show the claim about the cotangent, it suffices by the fact that cotangent is odd, to consider only two sides of the square, for instance the sides:

$$y = n + \frac{1}{2}, |x| \leq n + \frac{1}{2} \text{ and } x = n + \frac{1}{2}, |y| \leq n + \frac{1}{2}.$$

We compute

$$|\cot \pi z| = \frac{|e^{\pi i z} + e^{-\pi i z}|}{|e^{\pi i z} - e^{-\pi i z}|} = \left| \frac{e^{2\pi i z} + 1}{e^{-2\pi i z} - 1} \right| = \left| 1 + \frac{2}{e^{-2\pi i z} - 1} \right| \leq 1 + \frac{2}{|e^{-2\pi i z} - 1|}.$$

We will show $|e^{-2\pi i z} - 1| > 1$ over the two sides, thus proving

$$|\cot \pi z| \leq 3.$$

Indeed, over the side $y = n + \frac{1}{2}$*, we have*

$$|e^{-2\pi iz} - 1| = |e^{-2\pi ix}e^{2\pi(n+1/2)} - 1| \geq e^{2\pi(n+\frac{1}{2})} - 1 > 1.$$

Over the side $x = n + \frac{1}{2}$*, we have*

$$|e^{-2\pi iz} - 1| = |e^{-2\pi i(n+1/2)+2\pi y} - 1| = |-e^{2\pi y} - 1| = e^{2\pi y} + 1 > 1.$$

(ii) *Using (i) we have*

$$|R(z)\pi\cot\pi z| \leq \frac{M_1M_2}{|z|^2} \leq \frac{M_1M_2}{(n+1/2)^2}.$$

Thus

$$\left|\int_{\gamma_n} R(z)\pi\cot\pi z\right| \leq \frac{M_1M_2}{(n+1/2)^2} \cdot \textit{length } (\gamma_n) = \frac{M_1M_2}{(n+1/2)^2} \cdot 4(2n+1) \to 0$$

as $n \to \infty$*.*

(iii) *Clearly* $\cot\pi z = \frac{\cos\pi z}{\sin\pi z}$ *has poles whenever* $\sin\pi z = 0$ *so for* $z = m$*,* $m \in \mathbb{Z}$*. These are simple poles since*

$$(\sin\pi z)'|_{z=m} = \pi\cos\pi m = (-1)^m\pi.$$

By the rules of computing residues, we have

$$Res_{z=m}(\pi\cot\pi z) = Res_{z=m}\left(\frac{\pi\cos\pi z}{\sin\pi z}\right) = \frac{\pi\cos\pi z}{(\sin\pi z)'}|_{z=m} = \frac{\pi\cos\pi z}{\pi\cos\pi z}|_{z=m} = 1.$$

The function $R(z)\pi\cot\pi z$ *has poles at* $z = m$ *and at* $z = a_j$*. Since* R *is holomorphic near* m*, using Taylor and Laurent expansion for* R *and* $\pi\cot\pi z$ *respectively, we have*

$$Res_{z=m}(\pi\cot\pi z R(z)) = Res_{z=m}\left(\frac{1}{z-m} + \ldots\right)(R(m) + (z-m)R'(m) + \ldots) = R(m).$$

Similarly, since $\cot\pi z$ *is holomorphic near* a_j*, and* R *has a simple pole at* a_j *with residue* $\frac{P(a_j)}{Q'(a_j)}$*, using Taylor and Laurent expansions, we obtain*

$$Res_{z=a_j}(\pi\cot\pi z R(z)) = Res_{z=a_j}(\pi\cot\pi a_j + (z-a_j)\textit{hol. fn.})\left(\frac{P(a_j)}{Q'(a_j)}\frac{1}{z-a_j} + \textit{hol.fn.}\right)$$

$$= \pi\cot\pi a_j \cdot \frac{P(a_j)}{Q'(a_j)}.$$

Putting everything together via the residue theorem,

$$0 = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\gamma_n} R(z)\pi\cot\pi z\,dz = \sum_{m=-\infty}^{\infty} R(m) + \sum_{j=1}^{q}\pi\cot\pi a_j \cdot \frac{P(a_j)}{Q'(a_j)}.$$

This is what we set out to prove.

Problem 6.

Let f be a meromorphic function on $\mathbb{C}$. Let $U = \{z \in \mathbb{C} : |z| > 1 \text{ and } z \text{ is not a pole of } f\}$. Assume

$$|f(z)| \leq 1 + |z|$$

for all $z \in U$. Show that f is a rational function.

`Answer`: *By the removable singularity theorem, f extends holomorphically to the region $|z| > 1$. Indeed, for each pole α of f with $|\alpha| > 1$, we have*

$$\lim_{z \to \alpha} |f(z)(z - \alpha)| \leq \lim_{z \to \alpha} (1 + |z|) \cdot |z - \alpha| = 0$$

so the removable singularity theorem applies. Extending the function across these removable singularities, we obtain a function f which has all poles in the disc $|z| \leq 1$.

Since the poles f do not accumulate and $|z| \leq 1$ is compact, it follows that there can only be finitely many poles. Enumerate the poles $\alpha_1, \ldots, \alpha_n$ with multiplicity. The function

$$P(z) = f(z)(z - \alpha_1) \cdots (z - \alpha_n)$$

is therefore holomorphic even at α_j, hence P is entire. We claim P must be a polynomial. This implies

$$f(z) = \frac{P(z)}{(z - \alpha_1) \cdots (z - \alpha_n)}$$

is a rational function.

To see P is a polynomial of degree at most $n + 1$, we show that for all a we have

$$P^{(n+2)}(a) = 0.$$

Let $r > |a| + 1$ arbitrary. By Cauchy's estimates

$$|P^{(n+2)}(a)| \leq (n + 2)! \frac{M_r}{r^{n+2}}$$

where M_r is the maximum value of $|P(z)|$ over $|z - a| = r$. Since over this circle

$$1 < r - |a| \leq |z|$$

by the triangle inequality, we see that

$$\begin{aligned} |P(z)| &\leq |f(z)||z - \alpha_1| \cdots |z - \alpha_n| \leq (1 + |z|) \cdot |z - \alpha_1| \cdots |z - \alpha_n| \\ &\leq (1 + |a| + |z - a|) \cdot (|z - a| + |a + \alpha_1|) \cdots (|z - a| + |a + \alpha_n|) \\ &= (1 + |a| + r)(r + |a + \alpha_1|) \cdots (r + |a + \alpha_n|). \end{aligned}$$

Thus

$$|P^{(n+2)}(a)| \leq (n + 2)! \frac{(1 + r + |a|)(r + |a + \alpha_1|) \cdots (r + |a + \alpha_n|)}{r^{n+2}}.$$

Making $r \to \infty$, it follows $P^{(n+2)}(a) = 0$ as needed.

Problem 7.

Let $f : U \to \mathbb{C}$ be an injective holomorphic function, where U is an open neighborhood of 0.

We wish to show that $f'(0) \neq 0$.

(i) Show that there exists an integer $m > 0$, a disc around the origin $\Delta \subset U$, and a holomorphic function $g : \Delta \to \mathbb{C}$ such that
$$f(z) = f(0) + z^m g(z), \quad g(z) \neq 0 \text{ for all } z \in \Delta.$$

(ii) Show that there exists a holomorphic function $h : \Delta \to \mathbb{C}$ such that
$$f(z) = f(0) + h(z)^m, \quad h(0) = 0, \quad h'(0) \neq 0.$$

(iii) Show that if f is injective then $m = 1$. Conclude that $f'(0) \neq 0$.

`Answer:`

(i) *The function $f(z) - f(0)$ is not identically zero, and it vanishes at 0. Let m be the order of the zero 0. We have shown in class that*
$$f(z) - f(0) = z^m g(z)$$
for some g holomorphic in a disc Δ near 0, with $g(0) \neq 0$. We also have for $z \in \Delta \setminus \{0\}$, $g(z) \neq 0$ since otherwise
$$f(z) - f(0) = z^m g(z) = 0 \implies f(z) = f(0)$$
contradicting f injective.

(ii) *In this disc, we claim we can write $g = G^m$ for G holomorphic in Δ. In particular, $G(0) \neq 0$. Setting*
$$h(z) = zG(z) \implies f(z) = f(0) + z^m G^m = f(0) + h(z)^m$$
and
$$h(0) = 0, \quad h'(0) = (zG(z))'|_{z=0} = G(0) \neq 0.$$

The existence of G such that $G^m = g$ was an older homework problem. The argument is as follows. Using $g \neq 0$, it follows $\frac{g'}{g}$ is holomorphic over the simply connected domain Δ. Thus $\frac{g'}{g}$ has a primitive F, i.e. $F' = \frac{g'}{g}$. We compute
$$(g(z)e^{-F(z)})' = g'(z)e^{-F(z)} + g(z)e^{-F(z)}F'(z) = 0.$$
Therefore, $g(z) \cdot e^{-F(z)} = c$. Let $c = C^n$, for some C. Define
$$G(z) = Ce^{\frac{1}{n}(F(z))}.$$
Then
$$G(z)^n = C^n e^{F(z)} = ce^{F(z)} = g(z).$$

(iii) *If h is constant, then f is constant in Δ hence not injective. Thus h is not constant, and by the open mapping theorem, $h(\Delta)$ is open. Since $h(0)=0$, it follows that $h(\Delta)$ contains a disc $\widetilde{\Delta}$ around $h(0)=0$. Let $\zeta = \exp\left(\frac{2\pi i}{m}\right)$. Assume $m \neq 1$, so that $\zeta \neq 1$. Let $a \in \widetilde{\Delta}$, $a \neq 0$. But both $a, a\zeta$ are in $\widetilde{\Delta}$ so they are contained in the image of $h(\Delta)$. Thus for some $z, w \in \Delta$ we have*

$$h(z)=a, \quad h(w)=a\zeta \implies f(z)=f(0)+h(z)^m=f(0)+a^m, \quad f(w)=h(w)^m=f(0)+a^m.$$

Thus $f(z)=f(w)$ whereas $z \neq w$ since $h(z) \neq h(w)$. This shows f is not injective, a contradiction. Thus $m=1$, and thus

$$f(z)=f(0)+h(z) \implies f'(0)=h'(0)\neq 0.$$