Math 220C - Lecture 11

April 21, 2021

§0. Last time   Conway XI. 3.

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire of order λ, $f \not\equiv 0$.

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$$

genus $h = \max(p, \deg g)$ if g polynomial or ∞ otherwise.

Hadamard's theorem (1893)

$$h \leq \lambda \leq h+1$$

Remark [i] The theorem doesn't assume h, λ finite.

If one of them is infinite $\Rightarrow$ so is the other.

[ii] These ideas played an important role in

Hadamard's proof of Prime Number Theorem. (1896)

Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann (¹);

PAR M. J. HADAMARD.

1. La décomposition d'une fonction entière $F(x)$ en facteurs primaires, d'après la méthode de M. Weierstrass,

$$F(x) = e^{G(x)} \prod_{p=1}^{\infty} \left(1 - \frac{x}{\xi_p}\right) e^{Q_p(x)} \tag{1}$$

a conduit à la notion du genre de la fonction F.

On dit que F est du genre E si, dans le second membre de l'équation (1), tous les polynômes Q_p sont de degré E, et que la fonction entière $G(x)$ se réduise également à un polynôme de degré E au plus.

Dans un article inséré au *Bulletin de la Société mathématique de France* (²), M. Poincaré a démontré une propriété des fonctions de genre E. L'énoncé auquel il est parvenu est le suivant :

Dans une fonction entière de genre E, *le coefficient de* x^m, *mul-*

(¹) Les principaux résultats contenus dans le présent Mémoire ont été présentés à l'Académie des Sciences dans un travail couronné en 1892 (grand prix des Sciences mathématiques).

(²) Année 1883, pages 136 et suiv.

§1. Applications – Picard's Theorems (weak versions)

To illustrate the power of this result we show:

Application A (Conway 3.6)

> f entire & not constant & finite order
>
> $\Longrightarrow$ f omits at most one value.

Remark Little Picard (next week) removes the assumption the order is finite.

Proof Assume f omits $\alpha \neq \beta$. Define

$$f^{new} = \frac{f-\alpha}{\beta-\alpha} \quad \text{omits } 0 \ \& \ 1.$$

Since f^{new} omits $0 \Rightarrow f^{new} = e^{g}$ & f^{new} omits 1

$\Rightarrow g$ omits 0. Since order (f^{new}) = order $(f) < \infty$

$\Longrightarrow$ genus of f^{new} is finite by Hadamard. $\Rightarrow g$ polynomial.

& g omits 0. $\rightarrow g$ = constant $\Rightarrow f$ constant. False!

Easy Observations (used above)

$\boxed{i}$ $\lambda \geq 0$

We have seen $|f(z)| \leq e^{|z|^{\lambda+\varepsilon}}$ if $|z| \geq R_\varepsilon$ last lecture.

If $\lambda < 0$, let $\varepsilon > 0$ with $\lambda + \varepsilon < 0$. Then $|f(z)| \leq e^{|z|^0} = e$

for $|z| \geq R$ and $|f(z)| \leq M$ for $|z| \leq R$ by continuity. Thus

f bounded $\Rightarrow$ f constant (order 0). Thus $\lambda \geq 0$

$\boxed{ii}$ f & αf have the same order $\forall\, \alpha \neq 0$

HWK

Indeed $\lambda(\alpha f) \leq \max(\lambda(\alpha), \lambda(f)) = \max(0, \lambda(f)) = \lambda(f)$ by $\boxed{i}$

the previous line

Similarly $\lambda(f) = \lambda\left(\alpha f \cdot \frac{1}{\alpha}\right) \leq \lambda(\alpha f)$. Thus $\lambda(f) = \lambda(\alpha f)$.

$\boxed{iii}$ f & $f - \alpha$ have the same order

Same proof as in $\boxed{ii}$ using sums versus products

$\boxed{iv}$ f & Pf have the same order $\forall\, P$ polynomial.

if $\deg P > 0$.

We have $f \leq Pf$ if $|z| >> 0$ $\Rightarrow$ $\lambda(f) \leq \lambda(Pf)$.

HWK 4 $\qquad\qquad\qquad\qquad$ $\boxed{i}$

Also $\lambda(Pf) \leq \max(\lambda(P), \lambda(f)) = \max(0, \lambda(f)) = \lambda(f)$.

Thus $\lambda(Pf) = \lambda(f)$

Application B

f entire of finite order & $\lambda \notin \mathbb{Z}$ $\Rightarrow$ f assumes each of its values infinitely many times.

Remark Great Picard (next week) strengthens this result.

Proof Let α be a value of f. Define $f^{new} = f - \alpha$. We show f^{new} has ∞-many zeroes. Assume f^{new} has finitely many zeroes $a_1, \dots, a_n$. Let $P = \prod_{k=1}^{n} (z - a_k)$. Then f^{new}/P has no zeroes so it equals e^g. $\Rightarrow$

$\Rightarrow f^{new} = P e^g$. Note by previous remarks we have

order f = order f^{new} = order $e^g < \infty$. $\Rightarrow$ genus $< \infty$

$\Rightarrow g$ polynomial & order $(e^g) = \deg g \in \mathbb{Z}$. $\Rightarrow$ order $(f) \in \mathbb{Z}$

contradiction.

Plan for the Proof of Hadamard $\quad h \le \lambda \le h+1$

$\boxed{i}$ $\lambda \le h+1$ (today).

$\boxed{ii}$ $h \le \lambda$ — $p \le \lambda$ (next time)

— $\deg g \le \lambda$ (next time)

§ 2. First half of Hadamard

WTS $\lambda \le h+1$

WLOG h finite, else we're done.

Key Lemma $\log |E_p(w)| \le C_p \, |w|^{p+1}$ for some $C_p > 0$.

Proof Recall $f(z) = z^m e^g \prod_n E_p\left(\frac{z}{a_n}\right)$. WTS $\lambda \le h+1$.

Recall $\text{order}(uv) \le \max(\text{order } u, \text{order } v)$.

Recall $\text{order}(z^m) = 0 \le h+1$

$\text{order}(e^g) = \deg g \le h < h+1$.

We show $\text{order} \prod_n E_p\left(\frac{z}{a_n}\right) \le p+1 \le h+1$.

Note

$$\log\left|\prod_n E_p\left(\frac{z}{a_n}\right)\right| = \sum_n \log\left|E_p\left(\frac{z}{a_n}\right)\right|$$

Lemma $\{$

$$\le C_p \sum_n \left|\frac{z}{a_n}\right|^{p+1} = K|z|^{p+1}$$

where $K = C_p \sum \frac{1}{|a_n|^{p+1}} < \infty$. Thus order $\le p+1$, as needed.

Remark (will not prove/use)

$\text{order} \prod_n E_p\left(\frac{z}{a_n}\right) = \alpha$ (exercise in Conway).

Proof of Lemma

Recall $E_p(w) = (1-w)\exp\left(w + \frac{w^2}{2} + \dots + \frac{w^p}{p}\right).$

We induct on p.

When $p = 0$,

$$\log|1-w| \le \log(1+|w|) \le |w| \quad \text{so take } C_0 = 2.$$

Inductive step

$\boxed{i}$ When $|w| \ge \frac{1}{2}$: Note

$$E_p(w) = E_{p-1}(w)\exp\left(\frac{w^p}{p}\right)$$

$$\Rightarrow \log|E_p(w)| = \log|E_{p-1}(w)| + \log\left|\exp\left(\frac{w^p}{p}\right)\right|$$

$$\le C_{p-1}|w|^p + \log\exp \operatorname{Re}\left(\frac{w^p}{p}\right)$$

$$= C_{p-1}|w|^p + \operatorname{Re}\left(\frac{w^p}{p}\right)$$

$$\le C_{p-1}|w|^p + \left|\frac{w^p}{p}\right| = \left(C_{p-1} + \frac{1}{p}\right)|w|^p$$

$$\le 2\left(C_{p-1} + \frac{1}{p}\right)|w|^{p+1} \quad \text{since } |w| \ge \frac{1}{2}.$$

$\boxed{a}$ When $|w| \leq \frac{1}{2}$. Note

$$E_p(w) = (1-w)\exp\left(w + \frac{w^2}{2} + \cdots + \frac{w^p}{p}\right)$$

$$= \exp\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)$$

using Taylor expansion

$$\log(1-w) = -w - \frac{w^2}{2} - \cdots - \frac{w^k}{k} - \cdots \quad \text{for } |w| < 1.$$

Then

$$\log|E_p(w)| = \log\left|\exp\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)\right|$$

$$= \mathrm{Re}\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)$$

$$\leq \left|-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right|$$

$$\leq \sum_{k \geq p+1} \left|\frac{w^k}{k}\right| = |w|^{p+1} \sum_{k \geq 0} \frac{|w|^k}{p+k+1}$$

$$\leq |w|^{p+1} \sum_{k \geq 0} |w|^k \leq$$

$$\leq |w|^{p+1} \sum_{k \geq 0} \left(\frac{1}{2}\right)^k = 2|w|^{p+1}.$$

Take $c_p = \max\left(2,\ 2\left(c_{p-1} + \frac{1}{p}\right)\right)$. We obtain in both cases

$$\log |E_p(w)| \leq c_p |w|^{p+1}$$ as needed.