Math 220B – Lecture 9

January 25, 2021

The Mittag-Leffler Problem

Conway VIII.3 simplified.

Weierstraß Problem

Given (i) $\{a_n\}$ distinct, $a_n \longrightarrow \infty$.

(ii) $\{m_n\}$ positive integers

Find entire functions f with zeroes only at a_n of order m_n.

Answer We can always solve the Weierstraß Problem. & we even have a factorization of the solution.

Remark The function $1/f$ is meromorphic & its poles are only at a_n & their order equals m_n.

The Mittag-Leffler Problem asks a sharper question.

The Mittag-Leffler (ML) Problem for $\mathbb{C}$

Given [i] $\{a_n\}$ distinct, $a_n \longrightarrow \infty$.

[ii] Laurent principal parts (singular parts)

$$g_n(z) = \frac{A_{nm_n}}{(z-a_n)^{m_n}} + \frac{A_{nm_n-1}}{(z-a_n)^{m_n-1}} + \cdots + \frac{A_{n1}}{z-a_n}$$

Main Theorem We can always find meromorphic function f with poles only at a_n & Laurent principal parts g_n near a_n.

Remark If f_1, f_2 are two solutions $\Rightarrow$ $f_1 - f_2$ = entire since the singular parts at a_n cancel out

$$f_1 = f_2 + h$$

Remark This makes sense for $U \subseteq \mathbb{C}$.

- student of Hermite & Weierstraß
- Nobel Prize committee
- founder of Acta Math.

Gösta Mittag-Leffler

1846 – 1927

SUR LA REPRÉSENTATION ANALYTIQUE

DES

FONCTIONS MONOGÈNES UNIFORMES

D'UNE VARIABLE INDÉPENDANTE

PAR

G. MITTAG-LEFFLER

À STOCKHOLM.

Les recherches dont je vais exposer ici l'ensemble, ont été publiées auparavant, quant à leurs traits les plus essentiels, dans le Bulletin (Öfversigt) des travaux de l'Académie royale des sciences de Suède, ainsi que dans les Comptes-rendus hebdomadaires de l'Académie des sciences à Paris. Leur but est de faire parvenir, dans un certain sens, la théorie des fonctions analytiques uniformes d'une variable, à ce degré d'achèvement auquel la théorie des fonctions rationnelles est arrivée depuis longtemps.

Soit x une grandeur variable complexe à variabilité illimitée, et x' un point donné fini(1) dans le domaine de la variable x. Soit enfin R une quantité positive donnée. Je dis que l'ensemble des points x remplissant la condition $|x - x'| < R$, constitue le *voisinage* ou *l'entourage* ou les *environs du point* x'(2) *correspondant à* R. Chacun de ces points est dit *appartenir au voisinage* ou *à l'entourage* ou *aux environs* R, ou être

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(1) C'est-à-dire représentant une valeur donnée finie.

(2) Cf.: *Zur Functionenlehre, von* K. Weierstrass. Monatsbericht der Königl. Akademie der Wissenschaften zu Berlin, August 1880, pag. 4.

Remarks Connections:

$$
\begin{tikzcd}
 & & \text{Generalized Weierstraß} \arrow[d, Rightarrow, "\text{Clear}"] \\
\text{Mittag Leffler} \arrow[urr, Rightarrow, "\text{HWK3}"] \arrow[rr, Rightarrow, "f \rightsquigarrow 1/f"'] & & \text{Weierstraß Existence Problem} \\
 & & \text{Weierstraß Factorization} \arrow[u, Rightarrow, "\text{Previous Lectures}"']
\end{tikzcd}
$$

HWK3, Problem 2

2. (*Generalized Weierstraß problem. Monday, January 25.*) Let $\{a_n\}$ be distinct complex numbers with $a_n \to \infty$. Fix complex numbers $\{A_n\}$. Show that there exists an entire function f such that

$$f(a_n) = A_n.$$

Further Connections

In HWK3, Problem 3 we will see that we can derive Mittag-Leffler for simple poles from Weierstraß factorization.

Discussion of the proof

Given $\{a_n\}$, $a_n \to \infty$, g_n = Laurent principal parts

we try $f = \sum_{n=1}^{\infty} g_n$ as solution to Mittag-Leffler

Issue: As usual, this may not converge

New idea: Pick h_n entire functions & argue

$$f = \sum_{n=1}^{\infty} (g_n - h_n) \text{ converges}$$

Since h_n are entire, we are not changing the Laurent principal parts.

Compare this to Weierstraß

$$\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) \quad \text{vs.} \quad \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{h_n}$$

may not converge $\qquad$ could converge.

Terminology

$$\sum_{n=1}^{\infty} (g_n - h_n) = \text{Mittag-Leffler series}$$

$$h_n = \text{convergence enhancing corrections}$$

The h_n's are not unique!

Remark WLOG $a_n \neq 0.$ $\forall n.$

The contributions of the poles at 0 are added at the end:

$$\frac{A_m}{z^m} + \cdots + \frac{A_1}{z} + \text{Solution with } a_n \neq 0.$$

Proof The proof is part of the theorem. Conway VIII. 3.

Fix $\boxed{I}$ $r_n \to \infty$, $r_n < |a_n|$

$\boxed{II}$ c_n, $\sum_{n=1}^{\infty} c_n < \infty$

e.g. $c_n = \frac{1}{2^n}$, $c_n = \frac{1}{n^2}$, ...

Consider
$$g_n(z) = \frac{A_{nm_n}}{(z-a_n)^{m_n}} + \frac{A_{nm_n-1}}{(z-a_n)^{m_n-1}} + \cdots + \frac{A_{n1}}{z-a_n}$$

Since $a_n \neq 0$, g_n is holomorphic at $z=0$ in $\Delta(0, |a_n|)$

We can Taylor expand g_n in $\Delta(0, |a_n|)$ around 0.

Since $\bar{\Delta}(0, r_n) \subseteq \Delta(0, |a_n|)$, the Taylor series of g_n

converges uniformly in $\bar{\Delta}(0, r_n)$. We can pick a

Taylor polynomial h_n such that

$$|g_n - h_n| < c_n \text{ in } \bar{\Delta}(0, r_n).$$

Let $f = \sum_{k=1}^{\infty} (g_k - h_k)$

We show

Claim f meromorphic with poles only at a_k & principal parts g_k near a_k. $\Rightarrow$ f solves Mittag-Leffler.

Proof Let $r > 0$.

Since $r_k \longrightarrow \infty$, $\Rightarrow$ $r_k > r$ if $k \geq N$. Then

$|g_k - h_k| < c_k$ in $\overline{\Delta}(0,r) \subseteq \Delta(0, r_k)$ if $k \geq N$..

By Weierstraß M-test $\sum_{k=N}^{\infty} (g_k - h_k)$ converges uniformly in $\overline{\Delta}(0,r)$. Note that since $|a_k| > r_k > r$

$\Rightarrow$ $g_k - h_k$ holomorphic in $\Delta(0,r)$. Thus the sum

polynomial

the pole a_k is not in $\Delta(0,r)$

$$\sum_{k=N}^{\infty} (g_k - h_k)$$

is holomorphic in $\Delta(0,r)$.

The sum $\sum_{k=1}^{N-1} (g_k - h_k)$ is meromorphic as a finite sum of meromorphic functions in $\Delta(0,r)$. The poles are only at those a_j's with $|a_j| < r$ and the Laurent principal parts are g_j. This is because h_k are polynomials. so they do not contribute to the Laurent principal parts.

$$\text{Thus } f = \sum_{k=1}^{N-1} \underbrace{(g_k - h_k)}_{\text{meromorphic}} + \sum_{k=N}^{\infty} \underbrace{(g_k - h_k)}_{\text{holomorphic}}$$

is meromorphic with poles at $|a_j| < r$ for all $\Delta(0,r)$.

Varying r we get the claim & finish the proof.

Summary of the proof

Step 1 Pick $r_n \longrightarrow \infty$, $|a_n| > r_n$

c_n, $\sum c_n < \infty$

Step 2 Taylor expand g_n near 0

Pick Taylor polynomial h_n with

$|g_n - h_n| < c_n$ in $\Delta(0, r_n)$

Step 3 $f = \sum_{n=1}^{\infty} (g_n - h_n)$

Examples (will be repeated next time)

$\boxed{\text{I}}$ Poles at $-n \in \mathbb{Z}$, principal parts $\dfrac{1}{z+n}$.

For $n \neq 0$, we expand $\dfrac{1}{z+n}$ at $z=0$.

$$g_n = \frac{1}{z+n} = \frac{1}{n}\cdot\frac{1}{1+\frac{z}{n}} = \frac{1}{n}\left(1-\frac{z}{n}+\frac{z^2}{n^2}-\cdots\right)$$

$$= \frac{1}{n} - \frac{z}{n^2} + \frac{z^2}{n^3} - \cdots$$

Let $h_n = \dfrac{1}{n} \implies g_n - h_n = \dfrac{1}{z+n} - \dfrac{1}{n} = \dfrac{z}{n(z+n)}$.

Let $r_n = \sqrt{|n|}$ If $|z| \leq r_n$ then

$$\implies |g_n - h_n| = \frac{|z|}{n\,|z+n|} \leq \frac{\sqrt{n}}{n\,(n-\sqrt{n})} = c_n \quad \text{if } n>0$$

Note $\displaystyle\lim_{n\to\infty} \frac{c_n}{n^{-3/2}} = 1$ & $\displaystyle\sum_{n=}^{\infty} n^{-3/2} < \infty$. Thus $\displaystyle\sum_{n=1}^{\infty} c_n < \infty$.

A similar argument works for $n < \infty$.

$$f = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) + \frac{1}{z}$$ is the solution

we need to add this at the end for $n=0$.

to the Mittag - Leffler Problem.

Remark

Note that the n & $-n$ terms can be collected

$$f = \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) + \frac{1}{z}$$

$$= \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} + \frac{1}{z} = \pi \cot \pi z$$ by

HWK 6 in Math 220A.