# Math 220 A – Lecture 6

October 16, 2020

Last time $\Delta$ = disc

Proposition C If $f : U \to \mathbb{C}$ holomorphic $\Rightarrow \int_{\partial R} f\,dz = 0$

for all rectangles $\bar{R} \subseteq U$. (Goursat's lemma).

Corollary $f : \Delta \to \mathbb{C}$ holomorphic

$\overset{B+C.}{\Longrightarrow}$ $f$ admits a primitive.

$\overset{A}{\Longrightarrow}$ $\int_\gamma f\,dz = 0 \quad \forall \gamma$ piecewise $C'$ loop

We seek improvements

New assumption.

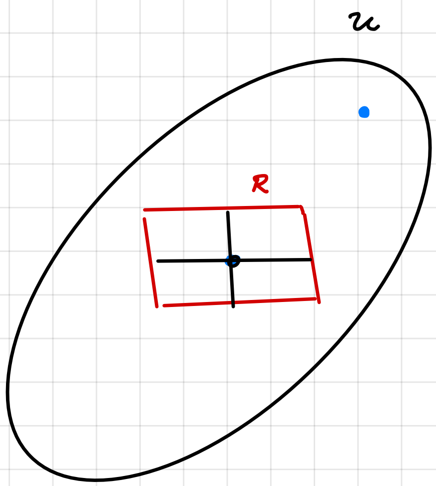
$(*)$ $f : U \to \mathbb{C}$ continuous, holomorphic in $U \setminus \{a\}$.

**Proposition C**$^+$ $f$ satisfies $(*)$ then $\int_{\partial R} f\,dz = 0$.

for all $\overline{R} \subseteq U$.

**Proof** $\boxed{i}$ If $a$ is outside $\overline{R}$, let $U^{new} = U \setminus \{a\}$

& apply Proposition C to $(f, U^{new})$



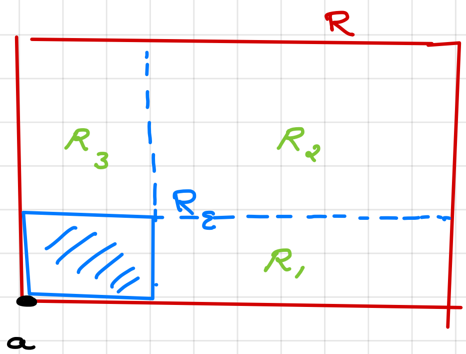

$$\Longrightarrow \int_{\partial R} f\,dz = 0$$

$\boxed{ii}$ If $a \in \overline{R}$, after subdividing $\overline{R}$

we may assume $a$ is a vertex.

$\boxed{iii}$ If $a$ is a vertex, let $R_\varepsilon$ be a

square of side $\varepsilon$ with vertex $a$.




$$\int_{\partial R_j} f\,dz = 0 \Longrightarrow \int_{\partial R} f\,dz = \int_{\partial R_\varepsilon} f\,dz.$$

Suffices $\displaystyle\int_{\partial R_\varepsilon} f\,dz \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Since $f$ cont. at $a$ $\Longrightarrow |f(z)| < |f(a)| + 1$, if $z \in \partial R_\varepsilon$

for small $\varepsilon$ $\displaystyle\Longrightarrow \left|\int_{\partial R_\varepsilon} f\,dz\right| \leq (|f(a)|+1)\,\underbrace{\text{length}(\partial R_\varepsilon)}_{4\varepsilon} \longrightarrow 0$

**Corollary**$^+$ $f : \Delta \to \mathbb{C}$ continuous, holomorphic in $\Delta \setminus \{a\}$.

$\overset{\text{Prop } B+C^+}{\Longrightarrow}$ $f$ admits a primitive.

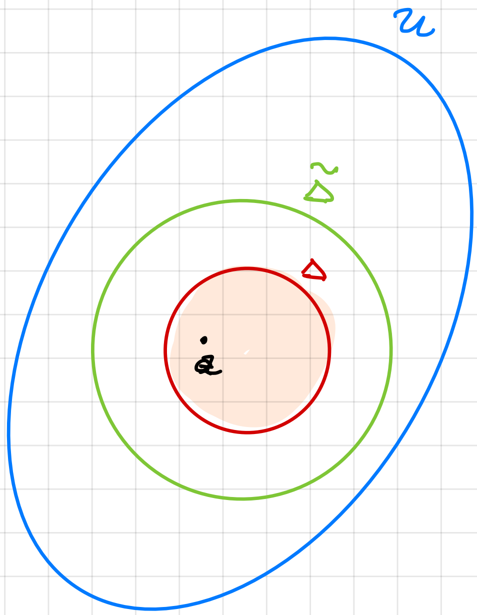
$\overset{\text{Prop } A}{\Longrightarrow}$ $\int_\gamma f \, dz = 0$ $\forall \gamma$ piecewise $C^1$ loop

## Local Cauchy Integral Formula

$f : \mathcal{U} \to \mathbb{C}$ holomorphic, $\overline{\Delta} \subseteq \mathcal{U}$ $\forall a \in \Delta$

$$f(a) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(z)}{z-a} \, dz$$

**Remark** $f|_{\partial\Delta}$ determines $f$ in $\Delta$.



**Proof** Let

$$F(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases}$$

$\Rightarrow$ $F$ continuous on $U$. & holomorphic in $U \setminus \{a\}$.

Let $\tilde{\Delta}$ s.t. $\overline{\Delta} \subseteq \tilde{\Delta} \subseteq \overline{\tilde{\Delta}} \subseteq U$. Apply Corollary$^+$ to $\tilde{\Delta}$ with $\gamma = \partial\Delta$:

$$\Rightarrow \int_{\partial\Delta} F \, dz = 0 \Rightarrow \int_{\partial\Delta} \frac{f(z)}{z-a} - \frac{f(a)}{z-a} \, dz = 0.$$

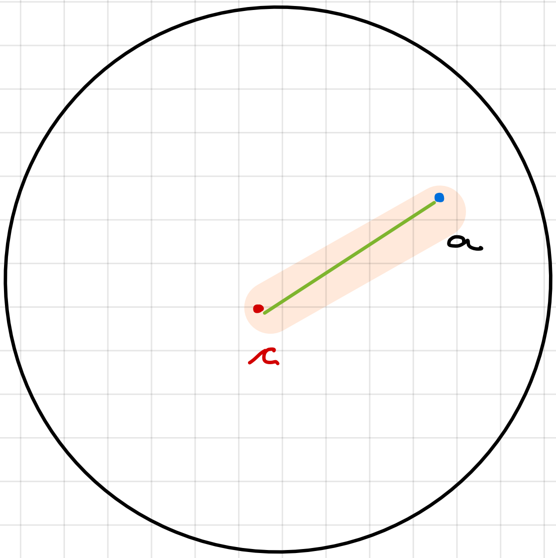
$$\Rightarrow \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(z)}{z-a} \, dz = f(a) \cdot \underbrace{\frac{1}{2\pi i} \int_{\partial\Delta} \frac{dz}{z-a}}_{1.} \quad \text{(next lemma)}$$

$\Rightarrow$ Local Cauchy follows.

**Lemma** If $a \in \Delta \implies \displaystyle\int_{\partial\Delta} \frac{dz}{z-a} = 2\pi i$

**Proof**

$\Delta$ Let $c$ be the center of $\Delta$.




$$\implies \int_{\partial\Delta} \frac{dz}{\underbrace{z-c}_{w}} = 2\pi i$$

$$\iff \int_{\partial\Delta(0,R)} \frac{dw}{w} = 2\pi i$$

which we have seen before.

It suffices to show $\displaystyle\int_{\partial\Delta}\left(\frac{dz}{z-a} - \frac{dz}{z-c}\right) = 0 \iff \int_{\partial\Delta} h\, dz = 0$.

Let $h(z) = \dfrac{1}{z-a} - \dfrac{1}{z-c}$. We show that $h$ admits a primitive in $\mathbb{C} \setminus [a,c]$. Let $\operatorname{Log} \dfrac{z-a}{z-c} = g(z)$ (principal branch)

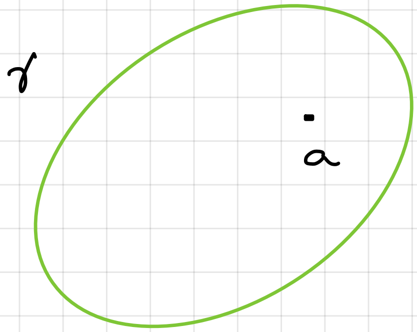
$\implies g' = h$.

**Issue** We need to show $\dfrac{z-a}{z-c} \in \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

$$\frac{z-a}{z-c} = -u,\ u \in \mathbb{R}_{\geq 0} \iff z = a \cdot \frac{1}{u+1} + c \cdot \frac{u}{u+1} \in$$

$\in$ segment from $a$ to $c$. (false)

**Index (winding number)** $a \notin \{\gamma\}$. Define

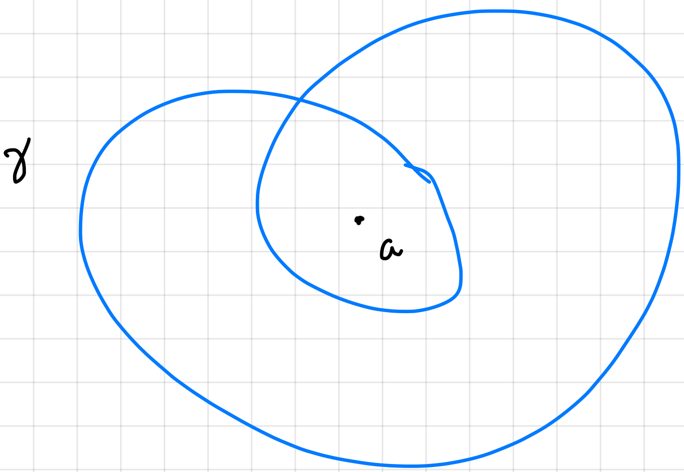



$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

Example A $\gamma$ circle

$n(\gamma, a) = 1$ if $a \in \operatorname{Int} \gamma$.

by the Lemma.




Example B $\gamma_k(t) = e^{2\pi i t k}$, $0 \le t \le 1$.

$\Longrightarrow n(\gamma_k, 0) = k$.

$$n(\gamma_k, 0) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{dz}{z} =$$

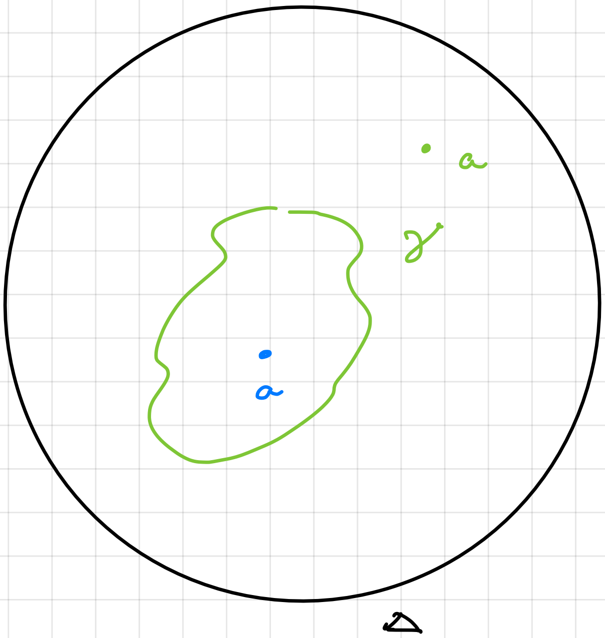
$$= \frac{1}{\cancel{2\pi i}} \int_0^1 \frac{\cancel{e^{2\pi i t k}} \cdot \cancel{2\pi i} k}{\cancel{e^{2\pi i t k}}} \, dt$$

$$= k.$$

Cauchy (revisited) $f : \Delta \to \mathbb{C}$ holomorphic,

$\gamma$ closed $C^1$ loop in $\Delta$, $a \in U \setminus \{\gamma\}$.

$$f(a) \cdot n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$




The proof is identical to the previous proof.

**Lemma** $n(\gamma, a) \in \mathbb{Z}$ , $a \notin \{\gamma\}$.

**Proof** $n(\gamma, a) = \frac{1}{2\pi i} \int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s) - a} ds$ where

$\gamma : [\alpha, \beta] \to U$ is a piecewise $C^1$ loop $\gamma(\alpha) = \gamma(\beta)$.

Consider

$$h(t) = \int_\alpha^t \frac{\gamma'(s)}{\gamma(s) - a} ds \, , \quad h(\alpha) = 0 .$$

Want $h(\beta) \in 2\pi i \mathbb{Z}$.

Compute

$$h'(t) = \frac{\gamma'(t)}{\gamma(t) - a} .$$

$$\Rightarrow \left( e^{-h(t)} (\gamma(t) - a) \right)' = e^{-h(t)} \Big( \underbrace{-h'(t)(\gamma(t) - a) + \gamma'(t)}_{0} \Big)$$

$\Rightarrow e^{-h(t)} (\gamma(t) - a)$ constant. Let $t = \alpha$, $t = \beta$:

$$\underbrace{e^{-h(\alpha)}}_{1} (\gamma(\alpha) - a) = e^{-h(\beta)} (\gamma(\beta) - a).$$

$\Rightarrow e^{-h(\beta)} = 1 \Rightarrow h(\beta) \in 2\pi i \mathbb{Z}$. QED.