Math 220B — Lecture 14

February 5, 2021

0. Logistics

(1) Poll regarding Math 220C

[i] MW 3 – 4:20

[ii] live / recorded / half live – half recorded?

(2) Midterm – Friday 12 – take home

will cover everything up to and including Monday

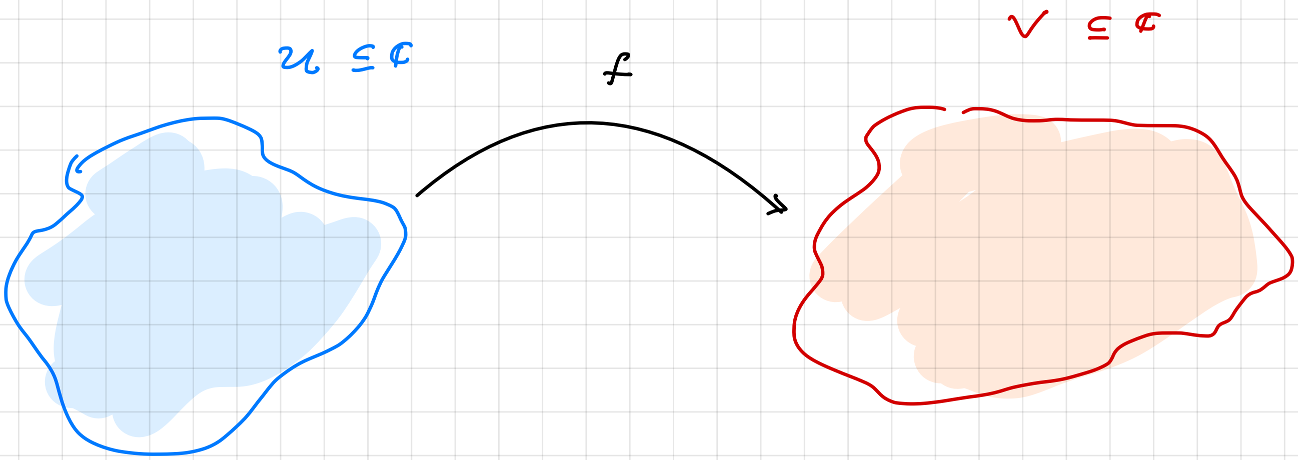
Conflicts?

Topics we covered:

- Infinite Products, Γ function, sine
- Weierstraß factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz lemma & applications

220B Part II — Mapping Theory

The goal is to frame the discussion & formulate guiding questions.



Given $\mathcal{U}, V \subseteq \mathbb{C}$ we wish to study holomorphic

$$f : \mathcal{U} \longrightarrow V.$$

This may be too general. We can ask

$\boxed{i}$ f injective

$\boxed{ii}$ f finite to one

$\boxed{iii}$ f bijective

$\boxed{iv}$ f proper ... etc.

We will focus on bijective holomorphic maps.

Remark

[a] Final Exam, Math 220A, we showed

Let $U \subset \mathbb{C}$ be an open set containing 0. Let $f : U \to \mathbb{C}$ be an injective holomorphic function. Show that $f'(0) \neq 0$.

The same argument works for any U & any point of U:

$f: U \longrightarrow V$ injective holomorphic $\Rightarrow$ f' has no zeroes.

[b] In Math 220A, Lecture 11, we showed

Example $f: U \longrightarrow V$ bijective, holomorphic & $f'(a) \neq 0$ $\forall\, a \in U$. Then f^{-1} holomorphic.

Conclusion $f: U \longrightarrow V$ holomorphic & bijective

$\Rightarrow$ f^{-1} holomorphic

Biholomorphism = holomorphic + bijective

We focus on biholomorphisms

Question A Given $U, V \subseteq \mathbb{C}$ are U, V biholomorphic?

Remark This has implications in topology & differential geometry. In particular U, V are

homeomorphic, diffeomorphic

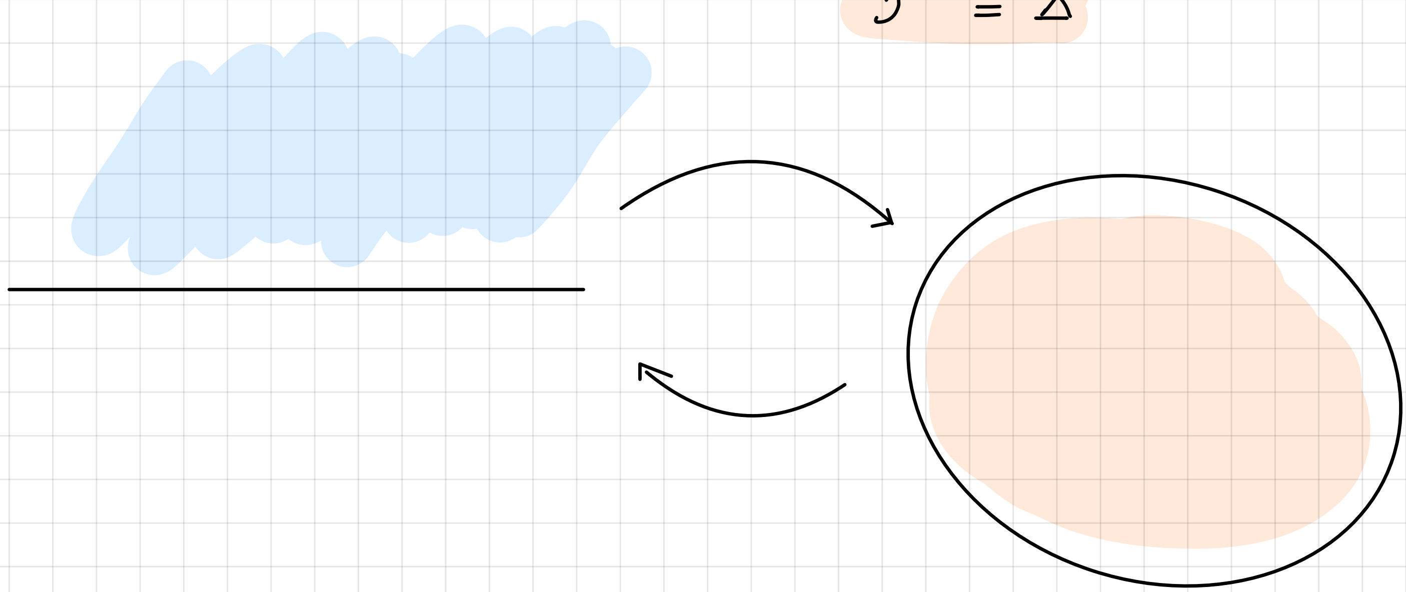
Examples

[i] $U = \mathbb{C}$, $V = \Delta(0,1)$, $U \not\to V$. This follows by Liouville's theorem.

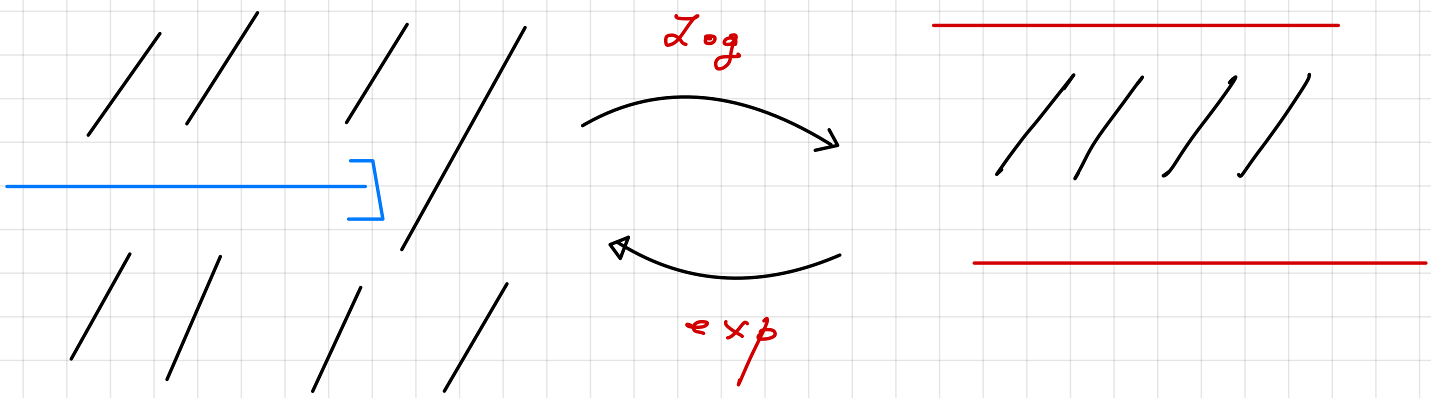
[ii] $U = \mathfrak{H}^+$, $V = \Delta$, $c : \mathfrak{H}^+ \longrightarrow \Delta$ Math 220A

Cayley transform: $c(z) = \dfrac{z-i}{z+i}$, $c^{-1}(w) = i \cdot \dfrac{1-w}{1+w}$.

$\mathfrak{H}^+ \cong \Delta$



$\boxed{III}$ $\mathcal{U} = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $V =$ strip $-\pi < \operatorname{Im} z < \pi$



This is Homework 2, Math 220A.

Very Important Theorem (Riemann Mapping Theorem)

Given $\mathcal{U}, V \neq \mathbb{C}$, $\mathcal{U}, V$ simply connected $\Rightarrow$ $\mathcal{U}, V$ are biholomorphic.

In particular, if $V = \Delta(0,1)$, then any $\mathcal{U} \neq \mathbb{C}$ simply connected then $\mathcal{U}$ is biholomorphic to $\Delta(0,1)$.

— Riemann's dissertation (1851) sketched a proof

— Refereed by Gauss

" The whole is a solid work of high quality, not merely fulfilling the requirements usually set for doctoral thesis, but far surpassing them."

— it took the effort of many great minds

Weierstraß, Caratheodory, Hilbert, Schwarz, Koebe, Fejer, Riesz & others to finalize the proof.

Question B

Given $\mathcal{U}, \mathcal{V} \subseteq \mathbb{C}$ biholomorphic can we construct

$\boxed{i}$ one biholomorphism $\mathcal{U} \longrightarrow \mathcal{V}$ explicitly?

$\boxed{ii}$ all biholomorphism $\mathcal{U} \longrightarrow \mathcal{V}$ explicitly?

Special cases of $\boxed{i}$

We saw some specific examples above e.g. the Cayley transform for $\mathcal{H}^+$ and $\Delta(0,1)$.

When $\mathcal{U} = V$, Question B $\boxed{ii}$ becomes.

Question C

What are all biholomorphisms $f : \mathcal{U} \longrightarrow \mathcal{U}$?

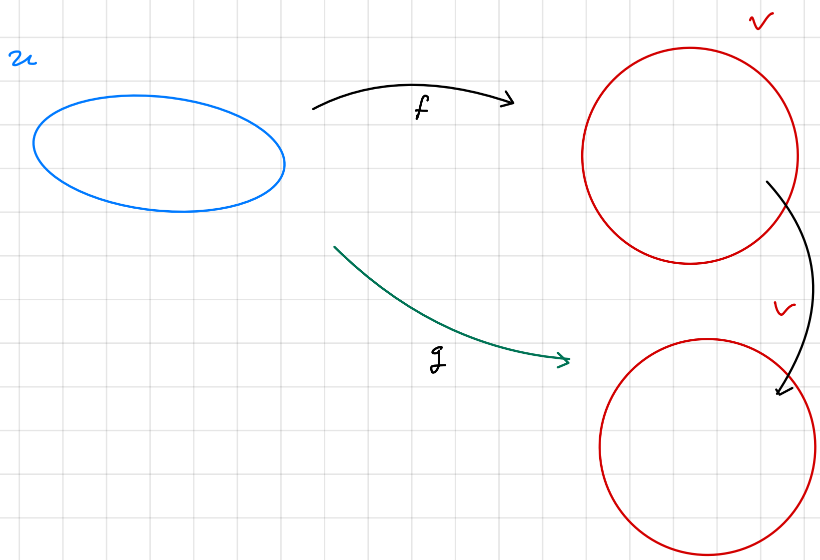
Remarks

$\boxed{i}$ $\mathrm{Aut}(\mathcal{U}) = \{ f : \mathcal{U} \longrightarrow \mathcal{U} : f$ holomorphic & bijective$\}$

is a group. Indeed $f \in \mathrm{Aut}(\mathcal{U}) \Rightarrow f^{-1} \in \mathrm{Aut}(\mathcal{U})$ using that f^{-1} is automatically holomorphic by the above remarks.

$\boxed{ii}$ Examples: We can consider this question for $\mathcal{U} = \Delta, \mathfrak{H}^+, \mathbb{C}, \Delta^\times, \mathbb{C}^\times$ etc ...

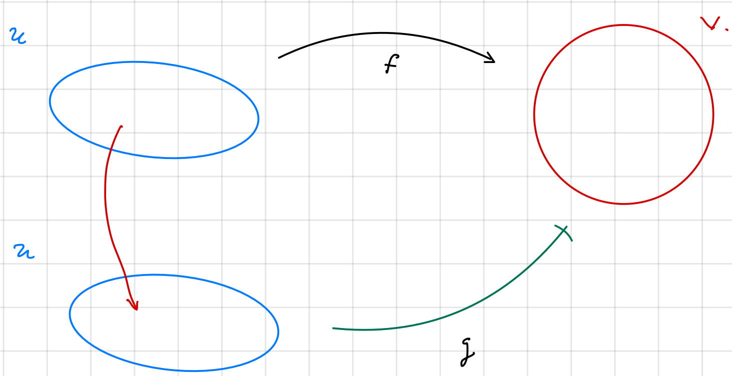
$\boxed{iii}$ If $f, g : U \longrightarrow V$, f = known biholomorphism then any other biholomorphism $g : U \longrightarrow V$ differs from f by automorphisms:



$$g = \Phi \circ f$$

$$\Phi \in Aut(V)$$

Indeed, $\Phi = g \circ f^{-1}$



In the same fashion

$g = f \circ \varphi$ where $\varphi = g \circ f^{-1}$

and $\varphi \in Aut(U)$.

Thus knowledge of Question C helps with aspects of Question B.

Question D

Is the action of $Aut(\mathcal{U})$ on $\mathcal{U}$ transitive i.e.

$\forall\, a, b \in \mathcal{U} \quad \exists\, f \in Aut(\mathcal{U})$ with $f(a) = b$?

Example $\mathcal{U} = \mathbb{C} \cup \{\infty\}$. FLT are automorphisms of $\mathcal{U}$

& action is transitive. (Math 220A)

Question E

Given $a \in \mathcal{U}$, describe $f : \mathcal{U} \longrightarrow \mathcal{U}$

biholomorphism, with $f(a) = a$.

Many other questions can be asked.

We begin the discussion with the case

$$\mathcal{U} = \Delta(0,1) = \Delta.$$

The crucial statement is <u>Schwarz Lemma</u>

Theorem Given $f: \Delta \longrightarrow \Delta$, $\Delta = \Delta(0,1)$ holomorphic, $f(0)=0$.

then [i] $|f'(0)| \leq 1$ and

[ii] $|f(z)| \leq |z|$.

If $|f'(0)| = 1$ or if $|f(z_0)| = |z_0|$ for some $z_0 \in \Delta \setminus \{0\}$ then

f is a rotation, $f(z) = e^{i\alpha} z$. $\forall z \in \Delta$.

Proof – next time.