HW1 - SOLUTIONS

Q1. We compute the partial products

$$p_k = \prod_{n=2}^{k}\left(1-\frac{1}{n^2}\right) = \prod_{n=2}^{k}\left(\frac{n^2-1}{n^2}\right) = \prod_{n=2}^{k}\frac{n-1}{n}\cdot\frac{n+1}{n} = \prod_{n=2}^{k}\frac{n-1}{n}\prod_{n=2}^{k}\frac{n+1}{n}.$$

After cancellations we find

$$p_k = \frac{1}{k}\cdot\frac{k+1}{2} = \frac{k+1}{2k} \to \frac{1}{2}$$

as $k\to\infty$.

Q2.

(i) Let $f_n(z) = q^n z$. Then $\sum_{n=1}^{\infty} f_n$ converges absolutely locally uniformly. Indeed, if $|z|\le R$, then

$$|f_n| = |q^n z| \le R|q|^n$$

and

$$\sum_{n=1}^{\infty} R|q|^n = \frac{R}{1-|q|},$$

so the uniform convergence over the disc $|z|\le R$ follows by Weierstraβ M-test. By the theorem proved in class, the product Q converges to an entire function.

(ii) By direct calculation, we have

$$Q(qz) = \prod_{n=1}^{\infty}(1+q^n\cdot qz) = \prod_{n=1}^{\infty}(1+q^{n+1}z) = \prod_{n=2}^{\infty}(1+q^n z).$$

This shows

$$Q(z) = (1+qz)Q(qz).$$

(iii) If $Q(z) = \sum_{n=0}^{\infty} a_n z^n$, then $Q(qz) = \sum_{n=0}^{\infty} a_n q^n z^n$. Thus

$$Q(z) = (1+qz)Q(qz) \implies \sum_{n=0}^{\infty} a_n z^n = (1+qz)\sum_{n=0}^{\infty} a_n q^n z^n = a_0 + \sum_{n=1}^{\infty}(a_n q^n + a_{n-1}q^n)z^n.$$

Identifying the coefficients of z^n we find

$$a_n = a_n q^n + a_{n-1}q^n$$

so

$$a_n = a_{n-1}\cdot\frac{q^n}{1-q^n}.$$

Clearly $a_0 = 1$. By induction, the above recursion implies

$$a_n = \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

This means

$$Q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\dots(1-q^n)} z^n.$$

(iv) When $z = 1$, we obtain

$$\prod_{n=1}^{\infty}(1+q^n) = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

When $z = -1$, we obtain

$$\prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot q^{n(n+1)/2}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Q3. We established in class that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Use this for $iz(a-b)/2\pi$ to get

$$\sin\left(iz(a-b)/2\right) = iz(a-b)/2 \cdot \prod_{n=1}^{\infty}\left(1 + \frac{z^2(a-b)^2}{4n^2\pi^2}\right).$$

Since

$$\begin{aligned}\sin\left(iz(a-b)/2\right) &= \frac{1}{2i}\left(\exp(-(a-b)/2z) - \exp((a-b)/2z)\right)\\ &= \frac{i}{2}\left(\exp(az) - \exp(bz)\right)\exp(-(a-b)z/2),\end{aligned}$$

the conclusion follows by rearranging terms.

Q4.

(i) By Taylor expansion in $\Delta(0,1)$, we write

$$\mathrm{Log}(1+w) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} w^k}{k}.$$

We compute

$$|\mathrm{Log}(1+w) - w| = \left|\sum_{k=2}^{\infty} \frac{(-1)^{k-1} w^k}{k}\right| \le \sum_{k=2}^{\infty} \frac{|w|^k}{k} \le \sum_{k=2}^{\infty} |w|^k = \frac{|w|^2}{1-|w|} \le 2|w|^2$$

for $|w| \le 1/2$.

(ii) Write $a = re^{i\alpha}$, $b = Re^{i\beta}$. Since a, b are in the right half plane, we can choose the arguments α, β to lie in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus $ab = rRe^{i(\alpha+\beta)}$ where the argument of the product ab is $\alpha + \beta \in (-\pi, \pi)$. This shows

$$\mathrm{Log}(ab) = \log(rR) + i(\alpha+\beta)$$

(and in particular it is well-defined). Note that

$$\mathrm{Log}(a) = \log r + i\alpha, \quad \mathrm{Log}(b) = \log R + i\beta \implies \mathrm{Log}(ab) = \mathrm{Log}(a) + \mathrm{Log}(b).$$

(iii) Pick $N > 0$ such that $\frac{r}{n} < 1$, for $n \geq N$. Take $z = x + iy \in \Delta(0, r)$. In particular, $|x| \leq r, |y| \leq r$. Therefore, for $n \geq N$ we have

$$\operatorname{Re}\left(1 + \frac{z}{n}\right) = 1 + \frac{x}{n} \geq 1 - \frac{r}{n} > 0$$

and

$$\operatorname{Re}\left(e^{\frac{-z}{n}}\right) = e^{-\frac{x}{n}} \cos\frac{y}{n} > 0,$$

since $\left|\frac{y}{n}\right| \leq \frac{r}{n} < \frac{\pi}{2}$.

(iv) Fix $r > 0$. We show that the series of principal Logs converges absolutely and uniformly over $\Delta(0, r)$

$$\sum_{n=1}^{\infty} \operatorname{Log}\left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right].$$

Pick N sufficiently large so that (iii) holds, and thus we can use (ii) to compute the Logarithm. Assume in addition that $\frac{r}{n} \leq 1/2$ for $n \geq N$. In particular $|\frac{z}{n}| < \frac{r}{n} \leq \frac{1}{2}$. By (i) and (ii) we have

$$\left|\operatorname{Log}\left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right]\right| = \left|\operatorname{Log}\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right| \leq 2\left|\frac{z}{n}\right|^2 \leq \frac{2r^2}{n^2}.$$

Using the Weierstraβ M-test,

$$\sum_{n=1}^{\infty} \operatorname{Log}\left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right]$$

converges absolutely and uniformly on $\Delta(0, r)$ since $\sum 1/n^2 < \infty$. Hence, as shown in class, the product

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

converges to a holomorphic function $G(z)$ in $\Delta(0, r)$ for all r. This establishes that G is entire.

Q5.

(i) Using the triangle inequality, we have

$$\left|\frac{\alpha + |\alpha| z}{(1 - \bar{\alpha} z)\alpha}\right| \leq \frac{|\alpha| + |\alpha||z|}{|1 - \bar{\alpha} z||\alpha|} = \frac{1 + |z|}{|1 - \bar{\alpha} z|} \leq \frac{1 + |z|}{1 - |\bar{\alpha}||z|} \leq \frac{1 + |z|}{1 - |z|} \leq \frac{1 + r}{1 - r}.$$

By direct calculation

$$|1 - B_\alpha(z)| = \left|1 + \frac{z - \alpha}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha}\right| = \left|\frac{\alpha(1 - \bar{\alpha} z) + (z - \alpha)|\alpha|}{\alpha(1 - \bar{\alpha} z)}\right|$$

$$= \left|\frac{\alpha - \alpha\bar{\alpha} z + z|\alpha| - \alpha|\alpha|}{\alpha(1 - \bar{\alpha} z)}\right| = \left|\frac{\alpha - z|\alpha|^2 + z|\alpha| - \alpha|\alpha|}{\alpha(1 - \bar{\alpha} z)}\right| = (1 - |\alpha|) \cdot \left|\frac{\alpha + |\alpha| z}{(1 - \bar{\alpha} z)\alpha}\right| \leq \frac{1 + r}{1 - r}(1 - |\alpha|).$$

(ii) By (i), we have

$$|1 - B_{\alpha_n}(z)| \leq \frac{1 + r}{1 - r}(1 - |\alpha_n|)$$

over $\overline{\Delta}(0,r)$. By the Weierstraβ M-test,

$$\sum_{n=1}^{\infty} |1 - B_{\alpha_n}(z)|$$

converges uniformly over $\overline{\Delta}(0,r)$, hence locally uniformly in $\Delta(0,1)$. By the theorem proved in class

$$\prod_{n=1}^{\infty} B_{\alpha_n}(z)$$

converges to a function B holomorphic in $\Delta(0,1)$. The same result implies B has zeros among the zeros of B_{α_n}, namely at α_n.

(iii) Clearly B_α has a pole at $z = \frac{1}{\bar{\alpha}}$ which is outside the unit disc, so B_α is holomorphic in $\Delta = \Delta(0,1)$ and continuous over $\overline{\Delta}$. We show $|B_\alpha(z)| = 1$ for $|z| = 1$. That is, we show $|z - \alpha| = |1 - \bar{\alpha}z|$ for $|z| = 1$. When $|z| = 1$, we have $\bar{z} = \frac{1}{z}$ so

$$|z - \alpha| = |\bar{z} - \bar{\alpha}| = \left|\frac{1}{z} - \bar{\alpha}\right| = \left|\frac{1 - z\bar{\alpha}}{z}\right| = |1 - z\bar{\alpha}|$$

as claimed. By the maximum modulus principle, B_α achieves its maxmum over the boundary $|z| = 1$, so $|B_\alpha(z)| < 1$ for $|z| < 1$. Thus B_α maps $\Delta(0,1)$ to $\Delta(0,1)$.

(iv) Assume that 0 is a zero of order m for f. Define $f(z) = f(z)/z^m$, so g must have a removable singularity at the origin. Extend g to a holomorphic function over $\Delta(0,1)$. Then g is continuous over $\bar{\Delta}(0,1)$ and

$$|g(z)| = 1 \quad \text{for } |z| = 1.$$

Let $\alpha_1, \ldots, \alpha_n$ be the zeros of g, possibly repeated according to multiplicity. We must have only finitely many zeros since if there are infinitely many, they must accumulate in $\overline{\Delta}(0,1)$. The accumulation point cannot be in $\Delta(0,1)$ since g is holomorphic, so it must lie on the boundary. But by continuity, g must be 0 at this point as well, which is impossible as $|g| = 1$ for $|z| = 1$.

Write $B(z) = \prod_{k=1}^{n} B_{\alpha_k}(z)$. Then B has zeros at α_i just as g. The quotient $h = g/B$ is holomorphic over $\Delta(0,1)$, continuous over $\overline{\Delta}(0,1)$, and it has no zeros. Furthermore, $|B(z)| = 1$ for $|z| = 1$ by part (iii), so $|h(z)| = |g(z)|/|B(z)| = 1$ for $|z| = 1$. It remains to show that h is constant.

We use the maximum modulus principle for h. Since $|h| = 1$ on the boundary, it follows that $|h(z)| \leq 1$ for all $z \in \Delta$. Working with $1/h$ instead (the latter function is holomorphic as h has no zeroes), we have $|1/h(z)| \leq 1$ for all $z \in \Delta$. Thus $|h(z)| = 1$ for all $z \in \Delta$. Either by the maximum modulus or by the open mapping theorem, it follows that h must be constant.