Math 220B — Lecture 19

February 22, 2021

Homework now due Friday, due to Office Hrs. on Wed.

Last time

Conway VII. 4.

$\mathcal{U} \neq \mathbb{C}$ root domain $\Leftarrow$ $\mathcal{U}$ simply connected

Let $a \in \mathcal{U}$. Wish $\mathcal{U} \cong \Delta$ biholomorphically.

$$\mathcal{F} = \{ f : \mathcal{U} \longrightarrow \Delta, \ f(a) = 0, \ f \text{ injective} \}$$

Step 1 $\quad \mathcal{U} \neq \mathbb{C}$ root domain $\Longrightarrow$ $\mathcal{F} \neq \emptyset$.

Step 2 $\quad$ Let $M = \sup \{ |f'(a)|, \ f \in \mathcal{F} \}$.

Then M is achieved by some function $f \in \mathcal{F}$.

Today

Step 3 For the extremal function f in Step 2

we show f surjective. Then f biholomorphism

If $f : U \longrightarrow \Delta$ not surjective then we show $\exists\, \tilde{f} \in \mathcal{F}$

with $|\tilde{f}'(a)| > |f'(a)|$ contradicting maximality of $|f'(a)|$.

Strategy

We will in fact show that if $f : U \longrightarrow \Delta$ not surjective

then $\exists\, \tilde{f} : U \longrightarrow \Delta$, $F : \Delta \longrightarrow \Delta$, $f = F \circ \tilde{f}$

$\tilde{f} \in \mathcal{F}$, $F(0) = 0$, $F \notin \operatorname{Aut} \Delta$.

Assume this can be done. The proof is then completed.

Indeed, by Schwarz Lemma $\Rightarrow |F'(0)| < 1$. (The inequality is strict since F is not a rotation as $F \notin \mathrm{Aut}\,\Delta$).

Then we indeed contradict maximality since

$$|f'(a)| = |F'(0)| \cdot |\tilde{f}'(a)| < |\tilde{f}'(a)|.$$

How do we execute the above Strategy?

Assume $f: U \longrightarrow \Delta$ is not surjective.

Let $\alpha \in \Delta \setminus f(U)$.

Construction of the function $\tilde{f}$ "square root trick".

We carry out the following moves:

(1) recenter.

The function $\varphi_\alpha \circ f : \mathcal{U} \longrightarrow \Delta$ omits the value $\varphi_\alpha(\alpha) = 0$ since f omits α. & $\varphi_\alpha \in \operatorname{Aut} \Delta$.

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(2) square root. Since $\mathcal{U}$ is a root domain & $\varphi_\alpha \circ f$ is nowhere zero, we can find $g : \mathcal{U} \longrightarrow \Delta$ holomorphic with $g^2(z) = \varphi_\alpha \circ f$.

Claim g injective.

Indeed $g(z) = g(w) \Rightarrow g(z)^2 = g(w)^2 \Rightarrow \varphi_\alpha \circ f(z) = \varphi_\alpha \circ f(w)$

$\Rightarrow f(z) = f(w) \Rightarrow z = w$ since $f \in \mathcal{F}$ injective.

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(3) recenter. Let $\beta = g(a)$. We define

$$\tilde{f} = \varphi_\beta \circ g \Rightarrow \tilde{f}(a) = \varphi_\beta\, g(a) = \varphi_\beta(\beta) = 0.$$

& $\tilde{f} : \mathcal{U} \longrightarrow \Delta$ injective. Then $\tilde{f} \in \mathcal{F}$.

Outcome

$$g^2 = \varphi_\alpha \circ f, \quad \tilde{f} = \varphi_\beta \circ g, \quad \tilde{f} \in \mathcal{F}.$$

Comparison $g^2 = \varphi_\alpha \circ f \Rightarrow f = \varphi_{-\alpha} \circ g^2.$

Let $s : \Delta \longrightarrow \Delta$, $s(w) = w^2 \Rightarrow f = \varphi_{-\alpha} \circ s \circ g.$

$$\Downarrow$$

$\tilde{f} = \varphi_\beta \circ g \Rightarrow g = \varphi_{-\beta} \circ \tilde{f} \Rightarrow f = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \circ \tilde{f}$

Let $F : \Delta \longrightarrow \Delta$, $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}. \Rightarrow f = F \circ \tilde{f}$

Claim $F \notin \operatorname{Aut} \Delta$, $F(0) = 0.$

Indeed, if $F \in \operatorname{Aut} \Delta$, $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \in \operatorname{Aut} \Delta$

$\Rightarrow s \in \operatorname{Aut} \Delta$. But s is not even injective as $s(z) = s(-z)$.

To see $F(0) = 0$ we compute

$$F(0) = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}(0) = \varphi_{-\alpha} \circ s(\beta) = \varphi_{-\alpha}(\beta^2) = \varphi_{-\alpha}(-\alpha) = 0$$

where we used

$$\beta^2 = g(a)^2 = \varphi_\alpha \circ f(a) = \varphi_\alpha(0) = -\alpha.$$

This is exactly what we needed to complete the proof of Step 3 & the proof of Riemann Mapping.

Remarks

$\boxed{1}$ Uniqueness of the biholomorphism. Take two biholom.

$$f, g : U \longrightarrow \Delta, \quad f(a) = g(a) = 0 \quad \text{then}$$

consider $\Delta \xrightarrow[f^{-1}]{} U \xrightarrow[g]{} \Delta$, $gf^{-1}(0) = 0$, $gf^{-1} \in \mathrm{Aut}\,\Delta$.

Then

$$gf^{-1} = \mathrm{Rot} \implies g = \mathrm{Rot} \circ f.$$

Thus the biholomorphisms we constructed are unique up to rotations.

$\boxed{ii}$ The extremal function f we constructed maximizes the derivatives at a of ALL functions $g : \mathcal{U} \longrightarrow \Delta$, $g(a) = 0$ not only the INJECTIVE ones.

Indeed if $f : \mathcal{U} \longrightarrow \Delta$ is the function we constructed, then $\forall g : \mathcal{U} \longrightarrow \Delta$, $g(a) = 0$,

$$\Delta \xrightarrow[f^{-1}]{} \mathcal{U} \xrightarrow{g} \Delta \ , \quad F = g \circ f^{-1} : \Delta \longrightarrow \Delta .$$

$$F(0) = 0 .$$

Then $g = F \circ f \Rightarrow |g'(a)| = |F'(0)| \, |f'(a)| \leq |f'(a)|$

where we used $|F'(0)| \leq 1$ by Schwarz.

$\boxed{iii}$ $\mathcal{U}, V$ simply connected, $\mathcal{U}, V \neq \mathbb{C} \Rightarrow \mathcal{U}, V$ are biholomorphic. ($\mathcal{U} \cong \Delta \cong V$ transitive)

Loose ends

TFAE

[i] $\mathcal{U}$ simply connected

[ii] $\mathcal{U}$ is a "logarithm domain"

[iii] $\mathcal{U}$ is a root domain

A "logarithm domain" is a domain where $\forall\, f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, f nowhere zero, we can define

$$\log f: \mathcal{U} \longrightarrow \mathbb{C} \text{ holomorphic.}$$

Proof

[i] $\Rightarrow$ [ii]. Math 220A, PSet 4

[ii] $\Rightarrow$ [iii]. Define $\sqrt{f} = \operatorname{Exp}\left(\frac{1}{2} \log f\right)$ for all $f: \mathcal{U} \to \mathbb{C}$ nowhere zero.

[iii] $\Rightarrow$ [i]. If $\mathcal{U} = \mathbb{C} \Rightarrow \mathcal{U}$ simply connected

Let $U \neq \mathbb{C}$ $\Rightarrow$ let $f : U \longrightarrow \Delta$, $g : \Delta \longrightarrow U$ inverse biholomorphisms. If γ is a loop in U, then

$f \circ \gamma$ loop in Δ = simply connected $\Rightarrow$ $f \circ \gamma \overset{\Delta}{\sim} 0$

$\Rightarrow g \circ f \circ \gamma \overset{U}{\sim} g(0) \Rightarrow \gamma \overset{U}{\sim} g(0) \Rightarrow$ γ null homotopic.

<u>Question</u> How do we construct biholomorphism

$$f : U \longrightarrow \Delta \quad \text{explicitly?}$$

<u>Answer</u> : depends on U.

<u>Some examples worth knowing</u>

$\boxed{a}$ Lecture 17 :

$$\mathbb{C}^- \longrightarrow \Delta, \quad z \longrightarrow \left(\frac{1+z}{1-z}\right)^2.$$

We will give more examples next time.

Next : More on boundary behaviour &

Schwarz Reflection (Conway IX. 1)

After : Runge's Theorem. (Conway VIII. 1)