Math 220, Problem Set 2.

1. Recall from the previous problem set the generalized hypergeometric series

$$ {}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. $$

Many common functions can be expressed as generalized hypergeometric series. Evaluate in closed form the following expressions

$$ {}_0F_0(;;z), \quad z \cdot {}_0F_1\left(;\frac{3}{2}; -\frac{z^2}{4}\right), \quad {}_0F_1\left(;\frac{1}{2}; -\frac{z^2}{4}\right), \quad z \cdot {}_2F_1\left(1, 1; 2; -z\right). $$

2. The dilogarithm is defined as

$$ \mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}. $$

The name comes from the analogy with the expansion

$$ -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}. $$

(i) Show that Li_2 has radius of convergence $R = 1$. Show that Li_2 can be expressed as a generalized hypergeometric series

$$ \mathrm{Li}_2 = z \cdot {}_3F_2(1, 1, 1; 2, 2; z). $$

(ii) Show that Li_2 is injective in $\Delta\left(0, \frac{2}{3}\right)$.

Hint: Use that $z^n - w^n = (z-w)(z^{n-1} + \ldots + w^{n-1})$.

3. Show that the function $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ given by

$$ u(z) = \log |z| $$

is harmonic, but it is not the real part of a holomorphic function in $\mathbb{C} \setminus \{0\}$.

4. Let $f : \mathbb{C}^\star \to \mathbb{C}^\star$ be the inversion $f(z) = \frac{1}{z}$. We have seen in class that f sends generalized circles to generalized circles. Prove the following more precise version of the result in class.

(i) Let C be the circle of center 0 and radius r. Show that $f(C)$ is a circle of center 0 and radius $\frac{1}{r}$.

(ii) Let C be the circle of center $z_0 \neq 0$ and radius $r \neq |z_0|$. Show that $f(C)$ is a circle of center $\frac{\bar{z}_0}{|z_0|^2 - r^2}$ and radius $\frac{r}{||z_0|^2 - r^2|}$.

(iii) Let C be the circle of center $z_0 \neq 0$ and radius $r = |z_0|$. Show that $f(C)$ is the line $\{w : \mathrm{Re}(wz_0) = \frac{1}{2}\} \cup \{\infty\}$.

5. For $a \in (-1, 1)$, let $D_a = \{z\colon |z| < 1,\ \operatorname{Im} z > a\}$. For each such a, either find a Möbius transformation of D_a onto the first quadrant Q, or show that such a transformation cannot exist.

6. Give an example of a biholomorphism between the strip $\{z : -\pi < \operatorname{Im} z < \pi\}$ and the slit complex plane $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

7. The arctangent is defined by the power series

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots$$

with radius of convergence $R = 1$. Show that

$$\arctan z = \frac{1}{2i} \operatorname{Log} \frac{1 + iz}{1 - iz}$$

for $z \in \Delta(0, 1)$. Here Log is the principal branch of the logarithm. You will need to first verify that both sides are well defined. You may wish to take derivatives.