Math 220A - Fall 2020 - Midterm

Problem 1.

Let
$$f(z) = \frac{z}{z^2 - 4}.$$
Expand f into Laurent series around 0 in the two regions $|z| < 2$ and $|z| > 2$ respectively.

`Answer`: *We use the geometric series to find the Laurent expansion of f.*

(i) *When $|z| < 2$, we write*
$$f(z) = \frac{z}{z^2 - 4} = -\frac{z}{4} \cdot \frac{1}{1 - \frac{z^2}{4}} = -\frac{z}{4} \cdot \left(1 + \frac{z^2}{4} + \left(\frac{z^2}{4}\right)^2 + \dots\right) = -\sum_{n=0}^{\infty} \frac{z^{2n+1}}{4^{n+1}}.$$

(ii) *When $|z| > 2$, we write*
$$f(z) = \frac{z}{z^2 - 4} = \frac{z}{z^2} \cdot \frac{1}{1 - \frac{4}{z^2}} = \frac{1}{z} \cdot \left(1 + \frac{4}{z^2} + \left(\frac{4}{z^2}\right)^2 + \dots\right) = \sum_{n=0}^{\infty} \frac{4^n}{z^{2n+1}}.$$

Problem 2.

(i) Show that if $h : U \to \mathbb{C}$ is nonconstant and holomorphic, then Re $h : U \to \mathbb{C}$ is an open map.

(ii) Let $f : U \to \mathbb{C}$ be holomorphic with $f'(z) \neq 0$ for all $z \in U$. Show that

$$\{\mathrm{Re}\ f(z) \cdot \mathrm{Im}\ f(z) : z \in U\}$$

is an open subset of $\mathbb{R}$.

`Answer`:

(i) *Since h is holomorphic and nonconstant, it must be open. Let $\pi : \mathbb{C} \to \mathbb{R}$ be given by*

$$\pi(x + iy) = x.$$

We claim that π is an open map. Then Re $h = \pi \circ h$ is an open map as well.

To see that π is open, it suffices to let $z_0 = x_0 + iy_0 \in W$ be open, and show $\pi(W)$ contains a neighborhood of

$$\pi(z_0) = x_0.$$

Note that $\Delta(z_0, \epsilon) \subset W$ for some $\epsilon > 0$. We claim

$$(\pi(z_0) - \epsilon, \pi(z_0) + \epsilon) \subset \pi(W).$$

Indeed, if

$$\alpha \in (\pi(z_0) - \epsilon, \pi(z_0) + \epsilon) \implies |\alpha - x_0| < \epsilon.$$

Write $\alpha = \pi(w)$ for $w = \alpha + iy_0$. Since

$$|w - z_0| = |\alpha - x_0| < \epsilon \implies w \in \Delta(z_0, \epsilon) \subset W$$

completing the proof that $\alpha \in \pi(W)$.

(ii) *Using (i) for*

$$h(z) = -\frac{i}{2}z^2$$

we conclude that the function

$$g = Reh \implies g(x + iy) = xy$$

is open. Since $f'(z) \neq 0$ it follows f is not constant, hence f is an open map. Then $g \circ f$ is also open so in particular

$$V = \{Re\ f(z) \cdot Im\ f(z) : z \in U\} = (g \circ f)(U)$$

is open.

Problem 3.

Suppose $f : \Delta(0,1) \to \mathbb{C}$ is holomorphic such that for all $z \neq 0$, we have

$$|f(z)| \leq -\log |z|.$$

Show that $f \equiv 0$.

`Answer`: *Fix $a \in \Delta(0,1)$. Let r be so that $|a| < r < 1$. By Cauchy's formula*

$$|f(a)| = \frac{1}{2\pi} \left| \int_{|z|=r} \frac{f(z)}{z-a}\, dz \right| \leq \frac{1}{2\pi} \frac{-\log r}{r - |a|} \cdot 2\pi r.$$

Here, we used the basic estimate, the fact that

$$|f(z)| \leq -\log |z| \text{ and } |z-a| \geq r - |a|.$$

Making $r \to 1$ in the above estimate, we conclude conclude $f(a) = 0$ for all a. Thus $f \equiv 0$.

`Alternate Answer:` *Use Cauchy's estimates at $z = 0$ to get*

$$|f^{(k)}(0)| \leq k! \cdot \frac{M_R}{R^k} \leq k! \cdot \frac{-\log R}{R^k}$$

where $M_R = \sup_{|z|=R} |f(z)|$. Making $R \to 1$ and using $\log R \to 0$, we get

$$f^k(0) = 0 \quad \text{for all} \quad k \geq 0.$$

Using Taylor series around 0 valid in the disc $\Delta(0,1)$ we see that

$$f(z) = \sum \frac{f^{(k)}(0)}{k!} z^k = 0.$$

`Alternate Answer`: *Let $a \in \Delta(0,1)$ and let $\epsilon > 0$. Then $e^{-\epsilon} < 1$. Let $\max(|a|, e^{-\epsilon}) < R < 1$. If $|z| = R$, we have*

$$|f(z)| \leq -\log |z| = -\log R < \epsilon.$$

Apply the maximum modulus principle to f over the closed disc $\overline{\Delta}(0,R)$ which contains a. Thus

$$\max_{\overline{\Delta}(0,R)} |f| = \max_{|z|=R} |f| < \epsilon.$$

Therefore, $|f(a)| < \epsilon$ for all $\epsilon > 0$ hence $f(a) = 0$. Thus $f \equiv 0$.

`Alternate Answer:` *Let $|z_0| = 1$ be a point on the boundary $\partial\Delta(0,1)$. We have*

$$\lim_{z \to z_0} \log |z| = \log |z_0| = \log 1 = 0 \implies \lim_{z \to z_0} f(z) = 0,$$

using the hypothesis. Extend f over $\partial\Delta(0,1)$ to a function g by continuity. The limit computation above shows $g = 0$ on $\partial\Delta(0,1)$. By the MMP, $|g|$ must achieve its maximum on the boundary, where g is 0. Thus $g \equiv 0$. This implies $f \equiv 0$.

Problem 4.

Assume that $f : \overline{\Delta}(0,1) \to \mathbb{C}$ is continuous, and f is holomorphic in $\Delta(0,1)$. Show that if $f(z) = 0$ for all $z = e^{it}$ with $0 \leq t < \pi$ then $f \equiv 0$.

`Answer`: *If the question stated instead that $f(e^{it}) = 0$ for $0 \leq t < 2\pi$, we'd be done by applying the maximum modulus principle: $|f|$ must achieve its maximum on the boundary, but $|f| = 0$ there so $f = 0$ everywhere. We wish to place ourselves in this situation, so we need to consider the lower half-circle as well.*

To this end, let

$$g : \Delta(0,1) \to \mathbb{C}, \quad g(z) = f(z) \cdot f(-z).$$

Note that g is continuous over $\overline{\Delta}(0,1)$ and holomorphic in $\Delta(0,1)$. Observe that

$$g(e^{it}) = 0 \quad \text{for all} \quad 0 \leq t < 2\pi.$$

Indeed, if $0 \leq t < \pi$ then

$$f(e^{it}) = 0 \implies g(e^{it}) = 0$$

while if $\pi \leq t < 2\pi$ we have $e^{it} = -e^{i(t-\pi)}$ so

$$f(-e^{it}) = f(e^{i(t-\pi)}) = 0 \implies g(e^{it}) = 0.$$

By the maximum modulus principle, we have

$$\sup_{\overline{\Delta}(0,1)} |g(z)| = \sup_{|z|=1} |g(z)| = 0.$$

Thus $g(z) = 0$ for all $z \in \Delta(0,1)$. Thus for each $z \in \Delta(0,1)$, either

$$f(z) = 0 \quad or \quad f(-z) = 0.$$

Let

$$U = \{z \in \Delta(0,1) : f(-z) \neq 0\}, \quad V = \{z \in \Delta(0,1) : f(z) = 0\}.$$

If $U = \emptyset$, then

$$f(-z) = 0 \text{ for all } z \implies f \equiv 0.$$

If $U \neq \emptyset$, by the above remarks,

$$z \in U \implies f(-z) \neq 0 \implies f(z) = 0$$

so $U \subset V$. Then V has an accumulation point at points of U. By the identity principle $f \equiv 0$.