Math 220 B — Lecture 26

March 10, 2021

Putting the pieces together   Conway VIII.2.

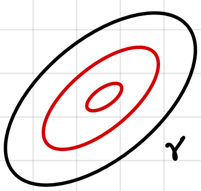
We tie up loose ends from Math 220A & B

Common theme : simply connected regions.

Topology $\longleftrightarrow$ Analysis

Review of Lecture 15, 220A $\mathcal{U} \subseteq \mathbb{C}$ connected

$\boxed{i}$ $\mathcal{U}$ is simply connected iff $\forall \gamma$ closed path in $\mathcal{U}$



$\gamma \overset{\mathcal{U}}{\sim} 0$

$\boxed{ii}$ γ piecewise C^1 loop in $\mathcal{U}$, $\gamma \overset{\mathcal{U}}{\approx} 0$ (null homologous) iff

$$\forall\, a \notin \mathcal{U}, \quad n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = 0$$

Recall

$$\gamma \overset{\mathcal{U}}{\sim} 0 \implies \gamma \overset{\mathcal{U}}{\approx} 0$$

Indeed, $a \notin \mathcal{U}$.

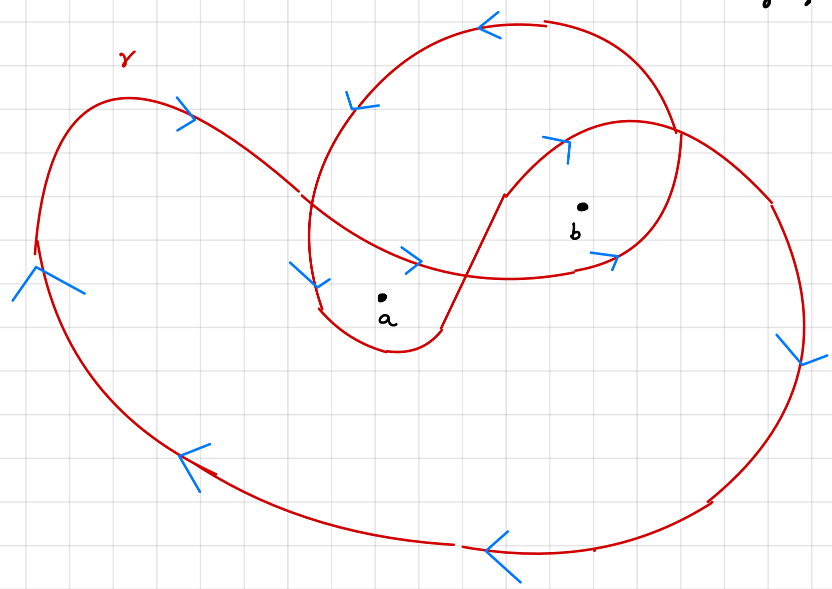
$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = 0$$ by the homotopy form

of Cauchy applied to the holomorphic function

$$z \longrightarrow \frac{1}{z-a} \text{ in } \mathcal{U}.$$

However the converse is false $\mathcal{U} = \mathbb{C} \setminus \{a, b\}$

$\gamma \not\sim 0$, $\gamma \approx 0$.



Theorem Let $\mathcal{U} \subseteq \mathbb{C}$ open, connected. TFAE

Conway VIII.2.

$\boxed{a}$ $\mathcal{U}$ simply connected

$\Downarrow$

$\boxed{b}$ $\forall \gamma$ piecewise C^1 loop, $\gamma \overset{\mathcal{U}}{\approx} 0$

$\Downarrow$

$\boxed{c}$ $\hat{\mathbb{C}} \setminus \mathcal{U}$ connected.

$\Downarrow$

$\boxed{d}$ polynomial approximation $\forall f$ holomorphic in $\mathcal{U}$ can be approximated $p_n \overset{\ell.u.}{\rightrightarrows} f$ in $\mathcal{U}$

$\Downarrow$

$\boxed{e}$ $\forall \gamma$ piecewise C^1 loop, f holomorphic in $\mathcal{U}$

$$\int_\gamma f \, dz = 0.$$

$\Downarrow$

$\boxed{f}$ primitives : any holomorphic $f : \mathcal{U} \to \mathbb{C}$ admits a primitive.

$\Downarrow$

$\boxed{g}$ logarithms : $\forall f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, nowhere zero can be written $f = e^g$, $g : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

$\Downarrow$

$\boxed{h}$ roots : $\forall f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, nowhere zero can be written $f = h^2$, $h : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic.

$\Downarrow$

$\boxed{i}$ $\mathcal{U}$ is homeomorphic to $\Delta(0,1)$.

Recall $\mathcal{U}, \mathcal{V} \subseteq \mathbb{C}$ are homeomorphic if $\exists\, f: \mathcal{U} \longrightarrow V$

$g: V \longrightarrow \mathcal{U}$ continuous & inverse to each other.

Proof

$\boxed{a} \Rightarrow \boxed{b}$ This is the statement $\gamma \overset{\mathcal{U}}{\sim} 0 \Rightarrow \gamma \overset{\mathcal{U}}{\approx} 0$.

that we saw previously.

$\boxed{b} \Rightarrow \boxed{c}$ Assume $\hat{\mathbb{C}} \setminus \mathcal{U} = A \cup B$

$A, B \neq \emptyset$ closed & disjoint. Assume $\infty \in B$. $\Rightarrow$

$\Rightarrow$ A is closed in $\underbrace{\hat{\mathbb{C}} \setminus \mathcal{U}}_{\text{closed}}$ $\Rightarrow$ A closed in $\hat{\mathbb{C}}$ $\Rightarrow$ A compact.

Let $\underset{\not\ni\,\infty}{V} = \mathcal{U} \cup A = \hat{\mathbb{C}} \setminus \underbrace{B}_{\text{closed}}.$ $\Rightarrow$ V open subset of $\mathbb{C}$., $A \subseteq V$.

In Lecture 23, we saw Cauchy's formula for compact sets.

A = compact, $A \subseteq V$. $\Rightarrow \exists$ polygons $\gamma_1 \dots \gamma_n$ in $V \setminus A = \mathcal{U}$.

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(z)}{z-a}\, dz \quad \forall\, a \in A,\ f \text{ holom. in } V.$$

Take $f \equiv 1$ then $1 = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{dz}{z-a} = \sum_{j=1}^{n} \underbrace{n(\gamma_j, a)}_{0}.$

However, by assumption $n(\gamma_j, a) = 0 \ \forall j$ since γ_j is a piecewise C^1 loop in $\mathcal{U}$ and $a \in A \Rightarrow a \notin \mathcal{U}$. This contradicts

$$\sum_{j=1}^{n} n(\gamma_j, a) = 1.$$

$\boxed{c} \Rightarrow \boxed{d}$ this is Little Runge 0.

$\boxed{d} \Longrightarrow \boxed{e}$ If $p_n \overset{l.u.}{\rightrightarrows} f$ in U then $\int_\gamma p_n \, dz \longrightarrow \int_\gamma f \, dz$.

However p_n admits a primitive $p_n = g_n'$ so by Lecture 5, Math 220 A $\int_\gamma p_n \, dz = \int_\gamma g_n' \, dz = 0$

$\Rightarrow \int_\gamma f \, dz = 0$.

$\boxed{e} \Rightarrow \boxed{f}$ This was done in Lecture 5, Math 220 A

$\boxed{f} \Rightarrow \boxed{g}$ Math 220 A, Homework 4. Recall the argument.

Consider $\frac{f'}{f}$ holomorphic in U. Then $\frac{f'}{f} = g'$ for some g by $\boxed{f}$

$\Rightarrow (e^{-g} f)' = 0 \Rightarrow f = c e^g = e^{\tilde{g}}$, $\tilde{g} = g + \log c$, $c \neq 0$.

$\boxed{g} \Rightarrow \boxed{h}$ Write $f = e^g$ and let $h = e^{g/2}$.

$\boxed{h} \Rightarrow \boxed{b}$ If $\mathcal{U} \neq \mathbb{C}$, Riemann Mapping shows $\mathcal{U}$ and Δ are biholomorphic hence homeomorphic.

If $\mathcal{U} = \mathbb{C}$ then $z \longrightarrow \dfrac{z}{\sqrt{1+|z|^2}}$ is a homeomorphism between $\mathbb{C}$ and Δ.

$\boxed{b} \Rightarrow \boxed{a}$ Let f, g be the two inverse homeomorphisms $\mathcal{U} \underset{g}{\overset{f}{\rightleftarrows}} \Delta$.

Let γ be a loop in $\mathcal{U}$ $\Rightarrow f \circ \gamma \overset{\Delta}{\sim} 0 \Rightarrow g \circ f \circ \gamma \overset{\mathcal{U}}{\sim} g(0). \Rightarrow \gamma \overset{\mathcal{U}}{\sim} g(0)$

$\Rightarrow \mathcal{U}$ simply connected.

Remark The implications $a \Rightarrow b, c, d, e \ldots$ are very useful.

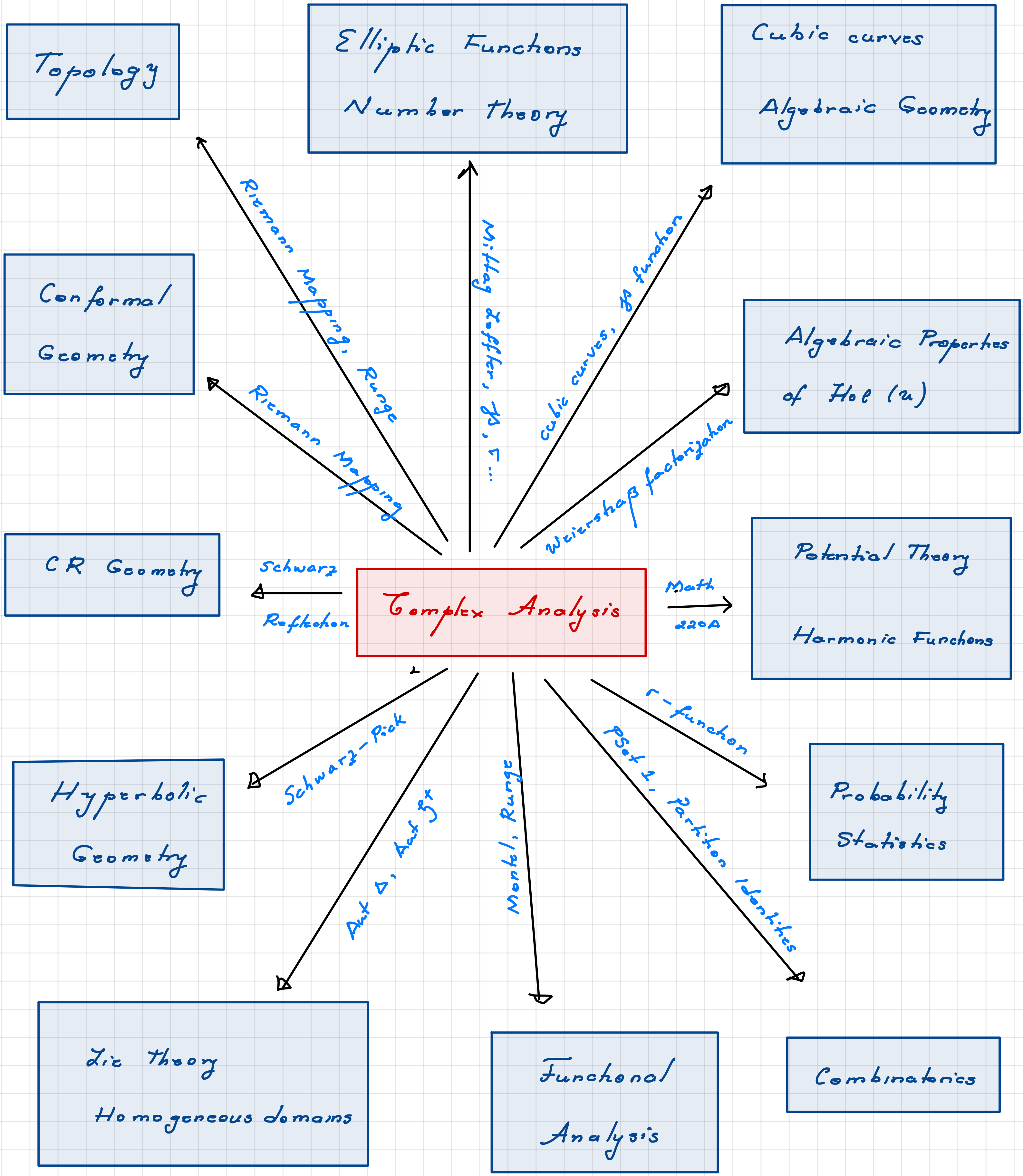
For the converse, $c \Rightarrow a$ is important.

Remark

Topology : a, c, i ...

Analysis : d, e, f, g, ...

Summary of Math 220A – B



Topics for Math 220C

(1) Harmonic Functions — Conway X.

(2) Hadamard Factorization — Conway XI.

(3) Picard's theorems — Conway XII.

Math "220 D"

(4) Introduction to Riemann Surfaces.

Logistics

(1) Office Hours: Today 4 − 5:30 PM

(2) Homework 7 due Friday, 11:59 PM.

No Sunday afternoon extensions.

(3) Final Exam, Wed March 17, 3 − 6 PM.

(4) Office Hours:

Tuesday March 16, 2 − 4 PM (Dragos)

Tuesday March 16, 4 − 6 PM (Shubham)

(5) Practice Problems online

(6) Last lecture − Review.