Math 280A Fall '21

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Administrative

(See Canvas for details)

- Lectures recorded by EVT and podcast system.

  sign up!

  Media Gallery in Canvas

- Piazza for discussion
- Gradescope for Homework

- Reference list at
  math.ucsd.edu/~pfitz
  click on 280A link

- Course Grade based on HW assignments (7 or 8 in total).

Math 280 content

A : Goal is SLLN (Chap. 1–4)

- probability space
- random variable
- expectation (= integration)
- independence
- modes of convergence

B:

- CLT $\bar{X}_n = \mu + \frac{\sigma}{\sqrt{n}} Z + \cdots$
- Martingale (key tool in modern probability)

(chapters 8 ~ 10)

C: Markov Chains

Brownian Motion

Ergodic Theory or Poisson process

. . .

Preview Borel's Strong Law of Large Numbers

$X_1, X_2, \ldots$ $\underbrace{\text{i.i.d.}}_{}$ symmetric Bernoulli r.v.'s

$$P(X_k = 1) = P(X_k = -1) = 1/2$$

$$\therefore \quad E(X_k) = 0$$

$$Var(X_k) = 1$$

Define $\bar{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k$ ("running average")

SLLN : $\bar{X}_n \longrightarrow 0$ a.s. , $n \longrightarrow \infty$

That is

$$P\left(\lim_{n \to \infty} \bar{X}_n = 0 \right) = 1$$

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Notice $E(\bar{X}_n) = 0$

$Var(\bar{X}_n) = 1/n$

$E[\bar{X}_n^4] = \frac{3}{n^2} - \frac{2}{n^3} \le \frac{3}{n^2}$

$$\therefore \sum_{n=1}^{\infty} E(\bar{X}_n^4) \le \sum_{1}^{\infty} \frac{3}{n^2} = \frac{\pi^2}{2} < \infty$$

$$\therefore E\left(\sum_{n=1}^{\infty} \bar{X}_n^4\right) < \infty$$

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If this sum diverges we assign it value $+\infty$

$$\therefore P\left(\sum_{1}^{\infty} \bar{X}_n^4 < \infty\right) = 1$$

$$\therefore P\left(\lim_{n\to\infty} \bar{X}_n^4 = 0\right) = 1$$

$$\therefore P\left(\lim_{n\to\infty} \bar{X}_n = 0\right) = 1 \quad !$$

(Needs only $E(X_k)=0$, $E(X_k^4) < \infty$)

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σ-field (aka σ-algebra)

Ω non-empty set "sample space"

A σ-field is a collection $\mathcal{B} \subset \mathcal{P}(\Omega)$ such that

(1) $\emptyset \in \mathcal{B}$

(2) $B \in \mathcal{B} \Rightarrow B^c \in \mathcal{B}$

(3) $B_1, B_2, \ldots \in \mathcal{B} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$

(2) is "closure under formation of complements"

(3) is "closure under formation of countable unions"

Because of (1), if $B_1, B_2, \ldots, B_n$ are $\in \mathcal{B}$

then so is $\bigcup_{k=1}^{n} B_k$

— use $B_1, B_2, B_3, \ldots, B_n, \emptyset, \emptyset, \ldots$

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Ex. 1

$\Omega = (0, 1]$

$\mathcal{A} = \{$ finite unions of intervals shaped like $(a, b] \}$

$\mathcal{A}$ is a field $\left((a,a] = \emptyset \right)$ but

$\mathcal{A}$ is not a σ-field :

$$\bigcup_{n=1}^{\infty} \left(0, 1 - 1/n\right] = (0, 1) \notin \mathcal{A}$$

Ex. 2 Generators

Ω arbitrary

$\mathcal{C} \subset \mathcal{P}(\Omega)$

$$\sigma(\mathcal{C}) = \bigcap \{ \mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-field}, \ \mathcal{B} \supset \mathcal{C} \}$$

$= \sigma$ - field generated by $\mathcal{C}$

$=$ least σ - field containing $\mathcal{C}$

If $A \subset \Omega$ then

$A \in \sigma(\mathcal{C})$ iff $A \in \mathcal{B}$ for each σ - field $\mathcal{B} \supset \mathcal{C}$

Another way to say this:

$\sigma(\mathcal{C})$ is a σ-field containing $\mathcal{C}$, and if

$\mathcal{B}$ is another σ-field containing $\mathcal{C}$ then

$\sigma(\mathcal{C}) \subset \mathcal{B}$.

Important Special Case

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$\mathcal{A}$ as in ex. 1

$\sigma(\mathcal{A})$ is called the Borel σ-field on $(0,1]$

Similarly

$$\mathcal{B}(\mathbb{R}) = \sigma\left(\{(-\infty, b] : b \in \mathbb{R}\}\right).$$

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