HOMEWORK 1

DUE APRIL 7, 2021 AT 11:59PM

1. Let $\mathcal{C}$ be a category, and let U_1 and U_2 be objects in $\mathcal{C}$. Suppose U_1 and U_2 are both universally attracting/final objects. Show that there is a unique isomorphism $i : U_1 \longrightarrow U_2$. (For future reference, the same is true if they're both universally repelling/initial objects, with the same proof.)

2. What are the initial and final objects in the following categories, if they exist?

(a) The category of sets.
(b) The category of rings with unit.
(c) The category of commutative rings with unit.
(d) The category of topological spaces.
(e) The category of open sets in a topological space X.

3. Let R be a commutative ring (with $1 \neq 0$) and $\{M_i\}_{i\in I}$ be a collection of R-modules, that is objects in the category R-**mod** of R-modules.

(a) Show that arbitrary direct sums and arbitrary direct products exist in the category of abelian groups.

(b) Consider $\bigoplus_{i\in I} M_i$ and $\prod_{i\in I} M_i$ in the category of abelian groups. Prove that they become R-modules with $r \cdot (m_i)_{i\in I} = (r \cdot m_i)_{i\in I}$.

(c) Show that $\bigoplus_{i\in I} M_i$ is the direct sum in R-**mod** of $\{M_i\}_{i\in I}$ and $\prod_{i\in I} M_i$ is the direct product in R-**mod** of $\{M_i\}_{i\in I}$.

(d) Show that, for every R-module N,

$$\operatorname{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \simeq \prod_{i\in I} \operatorname{Hom}_R(M_i, N)$$

and

$$\operatorname{Hom}_R\left(N, \prod_{i\in I} M_i\right) \simeq \prod_{i\in I} \operatorname{Hom}_R(N, M_i).$$

(e) Show that, for every R-module N,

$$N \otimes_R \left(\bigoplus_{i\in I} M_i\right) \simeq \bigoplus_{i\in I} \left(N \otimes_R M_i\right).$$

(f) Does the tensor product also commute with direct products? Prove or give a counterexample.

(g) Is the tensor product of two free R-modules also free as an R-module? Prove or give a counterexample.

4. Let $\mathcal{R}$ be the category of rings with $1 \neq 0$ (but not necessarily commutative). If R_1 and R_2 are rings, then let $R_1 \times R_2$ be their set-theoretic product, which can also be given the natural structure of a ring.

(a) Show that $R_1 \times R_2$ is the product of R_1 and R_2 in $\mathcal{R}$.

(b) Show that $R_1 \times R_2 \simeq R_1 \oplus R_2$ is not the coproduct of R_1 and R_2 in $\mathcal{R}$.

Note: We'll see later that coproducts do exist in the category $\mathcal{R}_{\text{comm}}$ of commutative rings; they're called tensor products.

5. Let $\mathcal{C}$ be a category. Let X, Y, S be objects in $\mathcal{C}$ and $f : X \longrightarrow S, g : Y \longrightarrow S$ be morphisms in $\mathcal{C}$. A *fiber(ed) product* of f and g in $\mathcal{C}$ (or by abuse of terminology, fiber product of X and Y over S) is an object Z in $\mathcal{C}$ together with morphisms $\pi_1 : Z \longrightarrow X$ and $\pi_2 : Z \longrightarrow Y$ such that

(i) $g \circ \pi_2 = f \circ \pi_1$;

(ii) for any object A in $\mathcal{C}$ and any morphisms $q_1 : A \longrightarrow X, q_2 : A \longrightarrow Y$ such that $g \circ q_2 = f \circ q_1$ there exists a unique morphism $u : A \longrightarrow Z$ such that the diagram

$$\begin{tikzcd}
A \arrow[dr, dotted, "u"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\
& Z \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & S
\end{tikzcd}$$

is commutative.

Show that, if it exists, the fiber product Z of f and g is unique up to isomorphism. The fiber product is denoted $X \times_S Y$ and the diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & S
\end{tikzcd}$$

is called a *fibered/pullback/Cartesian diagram.*

6. (a) Show in the category of sets

$$X \times_S Y = \{(x, y) \in X \times Y; f(x) = g(y)\}.$$

That is, show that the right hand side equipped with the obvious maps to X and Y satisfies the universal property of the fibered products.

(b) If X is a topological space show that fibered products always exist in the category of open sets of X by describing what a fibered product is. (Should be a one-word description.)