Math 220 B — Lecture 7

January 20, 2021

Last time

$\boxed{i}$ We defined the elementary / primary factors

$$E_p(z) = \begin{cases} 1 - z & , p = 0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right) & , p > 0 \end{cases}$$

$\boxed{ii}$ We saw that given $a_n \longrightarrow \infty$, $a_n \neq 0$,

$$f(z) = z^m e^h \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

are entire with zeroes at a_n.

$\boxed{iii}$ The p_n's are chosen so that

$$\forall\, r > 0 \,, \quad \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n + 1} < \infty$$

Remark [i] Convergence requires the estimate

$$|1 - E_p(z)| \leq |z|^{p+1}$$ if $|z| \leq 1$. (next)

[ii] Analogy: The factorization.

$$f(z) = z^m e^h \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

is reminiscent of the factorization of integers into primes.

primes $\longleftrightarrow$ E_p

units $\longleftrightarrow$ e^h

Difference. Not canonical / uniqueness of p_n's.

We can however ask questions with arithmetic flavor.

Wedderburn: Can we write $1 = Af + Bg$

when f, g have no common zeroes?

Remarks We have freedom in the choice of p_n.

Question Is there a canonical choice?

Assume $\exists\, h \in \mathbb{Z}_{\geq 0}$ with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} < \infty$.

If such h exists, pick the smallest one. This is called genus of the canonical product $\displaystyle\prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right)$

Example

$\boxed{\mathrm{I}}$ $\displaystyle Q(z) = \prod_{k=1}^{\infty}(1+q^k z) = \prod_{k=1}^{\infty} E_0(-q^k z)$

genus 0

$\boxed{\mathrm{II}}$ $\displaystyle G(z) = \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k} = \prod_{k=1}^{\infty} E_1\left(-\frac{z}{k}\right)$

genus 1

$\boxed{\mathrm{III}}$ $\displaystyle \sigma = z \prod_{\lambda \in \Lambda\setminus\{0\}} E_2\left(\frac{z}{\lambda}\right)$ genus 2. (HWK)

Remark

The genus controls the growth of zeroes via the expression $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}}$.

Remarkably, genus controls the growth of entire functions (Hadamard factorization theorem). This will be covered in Math 220C.

Proof of the estimate

$$|1 - E_p(z)| \leq |z|^{p+1} \quad \text{for} \quad |z| \leq 1.$$

where $E_p(z) = (1-z)e^{u}$, $u = z + \frac{z^2}{2} + \dots + \frac{z^p}{p}$

Write $E_p(z) = \sum_{k=0}^{\infty} a_k z^k$.

By definition $E_p(0) = 1 \implies a_0 = 1$.

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k.$$

Claim

[i] $a_1 = a_2 = \dots = a_k = 0$

[ii] a_k real & $a_k \leq 0 \quad \forall k \geq p+1$.

[iii] $\sum_{k=p+1}^{\infty} a_k = -1$.

Assuming the Claim, we compute

$$|E_p(z) - 1| = \left| \sum_{k=1}^{\infty} a_k z^k \right| \overset{[i]}{=} \left| \sum_{k=p+1}^{\infty} a_k z^k \right|$$

$$= |z|^{p+1} \left| \sum_{k=p+1}^{\infty} a_k z^{k-p-1} \right|$$

$$\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k| \, |z|^{k-p-1} \qquad |z| \leq 1$$

$$\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k|$$

$$\overset{\boxed{II}}{=} -|z|^{p+1} \sum_{k=p+1}^{\infty} a_k \overset{\boxed{III}}{=} |z|^{p+1}$$

Proof of the claim

$\boxed{I}$ $\quad E_p(z) = (1-z)\, e^{u}, \qquad u = z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}.$

Note $\quad u' = 1 + z + \ldots + z^{p-1} \implies (1-z)\, u' = 1 - z^p.$

Compute

$$E_p'(z) = \left((1-z)\, e^{u}\right)' =$$

$$= -e^{u} + (1-z)\, u'\, e^{u}$$

$$= -e^{u} + (1-z^p)\, e^{u}$$

$$= -z^p e^{u} \qquad (1)$$

Since

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \implies E_p'(z) = \sum_{k=1}^{\infty} k\, a_k z^{k-1}. \qquad (2)$$

The terms in (1) have powers of z^p.

Comparing with (2) we see $a_k = 0 \quad \forall \; 1 \le k \le p$.

$\boxed{ii}$ Also for $k \ge p+1$,

$$a_k = -\frac{1}{k} \cdot \text{Coefficient of } z^{k-p-1} \text{ in } e^{u}.$$

Since

$$e^{u} = e^{z} \cdot e^{z^2/2} \cdot \ldots \cdot e^{z^p/p} \text{ \& using the}$$

expansion of the exponential, we see that

real number

$$\text{Coefficient of } z^{k-p-1} \text{ in } e^{u} \ge 0 \implies a_k \le 0.$$

$\boxed{iii}$ Set $z=1$:

$$0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} a_k \implies \sum_{k=p+1}^{\infty} a_k = -1.$$

Further remarks – Looking forward (not needed)

[i] A divisor is a formal sum

$$D = \sum_{p \in \mathbb{C}} n_p [p] \quad \text{where} \quad n_p \in \mathbb{Z}$$

We require that this sum be locally finite.

A divisor is non-negative (effective) if $n_p \geq 0 \ \forall p$.

Example $D = 3[a] + 5[b]$ is a divisor.

[ii] Any entire function gives rise to a divisor

Indeed,

$$\operatorname{div}(f) = \sum_{p \text{ zero for } f} \operatorname{ord}(f, p)\,[p]$$

Example

$$f = (z-a)^3 (z-b)^5 \Rightarrow \operatorname{div}(f) = 3[a] + 5[b]$$

[iii] Weierstraß Problem can be rephrased

Every effective divisor is the divisor on an entire function

$$D \geq 0 \ , \quad D = \operatorname{div}(f).$$

[iv] For a meromorphic function f

$$\operatorname{div}(f) = \sum_{\substack{p \text{ zero or} \\ \text{pole}}} \operatorname{ord}(f,p)\,[p]$$

"principal divisor"

Question Is every divisor the divisor of a meromorphic function?

Yes For a general divisor D we can separate

$$D = D_+ - D_- \ , \quad D_+, D_- \text{ nonnegative.}$$

Write $D_+ = \operatorname{div} f_+$, $D_- = \operatorname{div} f_-$ & set $f = f_+/f_-$

Then $\operatorname{div}(f) = \operatorname{div}(f_+) - \operatorname{div}(f_-)$ (check)

$$= D_+ - D_- = D.$$

$\boxed{\text{IV}}$ These questions naturally lead to sheaf cohomology.

(Math 220c).

Next time the Weierstraß problem in $\mathcal{U} \subseteq \mathbb{C}$.

This is a bit more involved.