Math 220 C - Lecture 4

April 5, 2020

Last time

$u : \bar{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, u(e^{it})\, dt \qquad (\text{Poisson})$$

$a = r e^{i\theta}$

Poisson Kernel

$$P_r(\theta) = \operatorname{Re} \frac{1 + r e^{i\theta}}{1 - r e^{i\theta}}$$

$$= \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

Applications

Harnack Inequality

Schwarz Integral Formula

Schwarz Integral Formula

$u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ

We have seen $u = \operatorname{Re} f$, f holomorphic in Δ.

Question Is there a formula for f?

$$f : \Delta \longrightarrow \mathbb{C}, \quad f(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z+a}{z-a}\, u(z)\, \frac{dz}{z}$$

Claims

(1) f holomorphic in Δ.

(2) $u = \operatorname{Re} f$

Proof of (1)

Key Fact (Math 220A, Homework 3, Problem 7).

$$\boxed{\begin{array}{l} \text{Continuous } \Phi : \{\gamma\} \times U \longrightarrow \mathbb{C} \quad \text{holomorphic in } a \\ \\ \text{then } \quad a \longrightarrow \displaystyle\int_{\gamma} \Phi(z, a)\, dz \quad \text{holomorphic} \end{array}}$$

Apply this to $\Phi : \partial\Delta \times \Delta \longrightarrow \mathbb{C}$, $\Phi(z, a) = \dfrac{z+a}{z-a} \dfrac{u(z)}{z}$.

which is continuous & holomorphic in a to conclude.

$$f(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z+a}{z-a}\, u(z)\, \frac{dz}{z} \quad \text{is holomorphic in } \Delta.$$

Proof of (2)

By definition, we have

$$f(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z+a}{z-a}\, u(z)\, \frac{dz}{z} \qquad \swarrow\ z = e^{it}$$

$$= \frac{1}{2\pi \cancel{i}} \int \frac{1 + a/z}{1 - a/z}\, u(e^{it})\, \cancel{i}\, dt$$

$$\Rightarrow \operatorname{Re} f(a) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{1 + a/z}{1 - a/z} \cdot u(e^{it})\, dt \qquad a/z = r e^{i(\theta - t)}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, u(e^{it})\, dt$$

$$= u(a). \quad \Rightarrow u = \operatorname{Re} f.$$

In the last line we applied Poisson's formula for u.

Hermann Schwarz

1843 - 1921

Doctoral advisor:

Karl Weierstrass
Ernst Kummer

Students:

Lipót Fejér
Paul Koebe
Ernst Zermelo

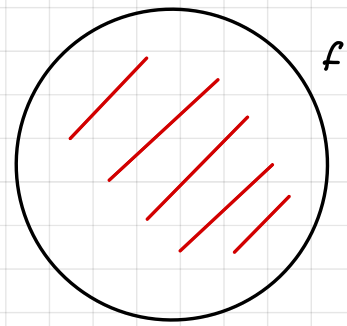
Schwarz lemma
Schwarz integral formula
Schwarz reflection principle
Cauchy–Schwarz inequality.

Dirichlet Problem (for the unit disc)

Given $f : \partial\Delta \longrightarrow \mathbb{R}$ continuous, is there $u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous

(1) u harmonic in Δ

(2) $u\big/_{\partial\Delta} = f$



Answer Yes. Define $u : \overline{\Delta} \longrightarrow \mathbb{R}$ by

$$u(re^{i\theta}) = \begin{cases} f(e^{i\theta}) & , r = 1. \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P_r(\theta - t)\, f(e^{it})\, dt, & r < 1. \end{cases}$$

We need to show

(1) u harmonic in Δ

(2) u continuous in $\overline{\Delta}$

$$\lim_{\substack{r \to 1 \\ \theta \to \theta_0}} \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\, f(e^{it})\, dt = f(e^{i\theta_0}).$$

Johann Peter Gustav Lejeune Dirichlet (1805 – 1859)

It was his father who first went under the name “Lejeune Dirichlet” (meaning “the young Dirichlet”) in order to differentiate from his father, who had the same first name. The name “Dirichlet” (or “Derichelette”) means “from Richelette” after a town in Belgium. '

Proof of (1)

We claim that u is harmonic in Δ. Recall that

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, f(e^{it})\, dt \quad , \quad a \in \Delta$$

Let

$$g(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z+a}{z-a} \cdot f(z) \frac{dz}{z}.$$

We have argued in the proof of Schwarz, g is holomorphic in a & $\operatorname{Re} g = u$. Thus u is harmonic in Δ.

Proof of (2)

Properties of the Poisson kernel

Lemma

[i] $P_r(t) \geq 0$, even in t, 2π-periodic in t.

[ii] $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,dt = 1.$

[iii] $P_r \rightrightarrows 0$ as $r \to 1$, over the domain $\delta \leq |t| \leq \pi$
$\forall\, \delta > 0.$

Proof [i] is clear

[ii] Take $u \equiv 1.$, $a = re^{i\cdot 0}$ in Poisson's formula

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(-t)\,dt\text{, which is what we need.}$$

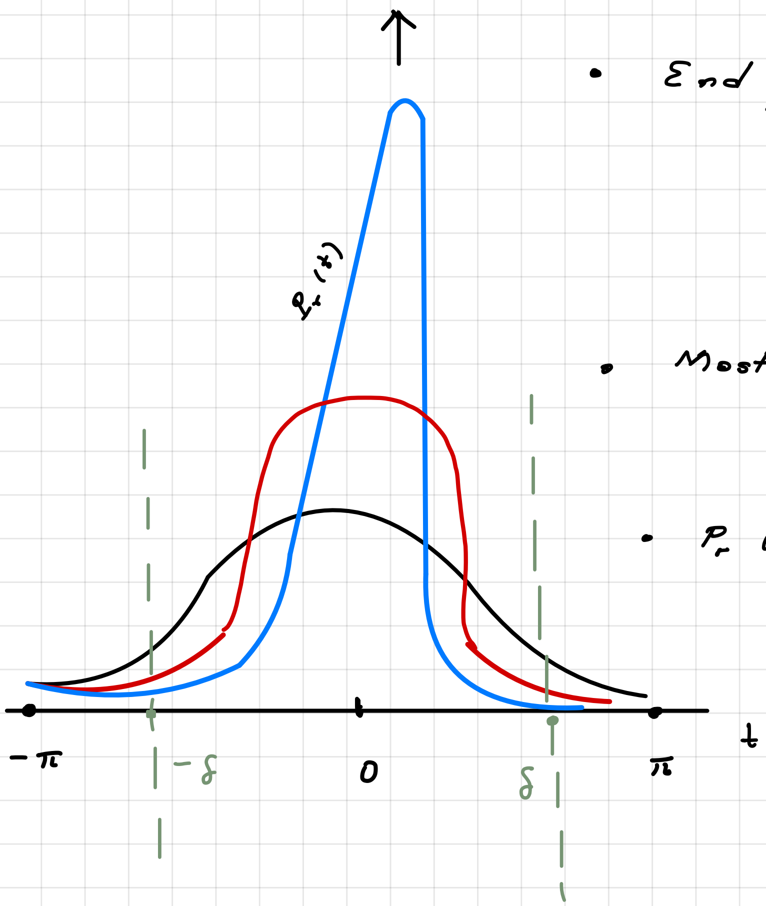
[iii] To prove uniform convergence, we show

$$\sup_{\delta \leq t \leq \pi} |P_r(t)| \longrightarrow 0 \quad \text{as } r \to 1.$$

Note that P_r is decreasing in $t \in [\delta, \pi]$. Then

$$\sup_{\delta \leq t \leq \pi} P_r(t) = P_r(\delta) = \frac{1-r^2}{1-2r\cos\delta + r^2} \longrightarrow 0 \quad \text{as } r \to 1.$$

Heuristics



- Area under the graph: is 1. by $\boxed{\text{"}}$
- End points: $P_r(t) \longrightarrow 0$ as $r \to 1$

  for $t \in [\delta, 1]$.
- Most area concentrated in the middle
- $P_r(0) = \frac{1+r}{1-r} \xrightarrow[r \to 1]{} \infty$.

"Conclusion"

$$\frac{1}{2\pi} P_r(t)\, dt \longrightarrow \delta_0 = \delta - \text{function concentrated at } 0.$$

In our case

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{P_r(\theta - t)}_{\delta_0 \to \theta = t.} f(e^{it})\, dt \qquad r \to 1.$$

$$\text{"}\longrightarrow\text{"} \; f(e^{i\theta}).$$ so we do expect continuity.

We will prove this rigurously next time.

Convolution Product

For functions $g, h : [-\pi, \pi] \longrightarrow \mathbb{R}$ continuous, set

$$g * h(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta - t)\, h(t)\, dt.$$

If we write $u_r(\theta) = u(re^{i\theta})$ and write $f(t)$ instead of $f(e^{it})$, we obtain

$$u_r = P_r * f.$$ Thus we defined the solution to the Dirichlet problem as a convolution.