HW3 - SOLUTIONS

Q1. Let $\gamma(t) = e^{it}$ for $t \in [-\pi/2, \pi/2]$. The given function $f(z) = ze^{iz}$ is holomorphic on the entire plane and admits a primitive $F(z) = (e^{iz} - ize^{iz})$, hence

$$\begin{aligned}\int_\gamma ze^{iz}dz &= F(e^{i\pi/2}) - F(e^{-i\pi/2})\\ &= F(i) - F(-i)\\ &= 2e^{-1}.\end{aligned}$$

Q2. The unit half circle is centered at 1. We write $z - 1 = e^{it}$ with t going from $\pi/2$ to $-\pi/2$. By the definition in class we have

$$\sqrt{z-1} = \exp\left(\frac{1}{2}Log(z-1)\right)$$

where Log is the principal branch of the logarithm. We therefore have

$$\sqrt{z-1} = \exp(it/2), \;\; dz = ie^{it}\,dt$$

and the integral becomes

$$\int_C \sqrt{z-1}\,dz = \int_{\pi/2}^{-\pi/2} e^{it/2} \cdot ie^{it}\,dt = -i\int_{-\pi/2}^{\pi/2} e^{3it/2}\,dt = -\frac{2}{3}(e^{3\pi i/4} - e^{-3\pi i/4})$$

$$= -\frac{4i}{3} \cdot \sin\frac{3\pi}{4} = -\frac{2\sqrt{2}i}{3}.$$

Q3. If $h = fg$ then by direct computation we see that

$$\frac{h'}{h} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

Applying this repeatedly to

$$f(z) = c\prod_{\ell=1}^{k}(z - a_\ell)^{m_\ell}$$

we find

$$\frac{f'}{f} = \sum_{\ell=1}^{k}\frac{m_\ell}{z - a_\ell}.$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{\ell}\frac{1}{2\pi i}m_\ell \cdot \int_\gamma \frac{dz}{z - a_k} = \sum_{\ell=1}^{k} m_\ell \cdot n(\gamma, a_\ell).$$

If f is a polynomial with roots $a_1, \ldots, a_\ell$ and $R > \max(|a_\ell|)$ we see that letting γ be the circle $|z| = R$ we have $n(\gamma, a_\ell) = 1$. Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{\ell=1}^{k} m_\ell = \deg f.$$

Q4.

(i) The integrand has only singularity at 1 inside $|z| = 2$ and the residue at 1 is $\frac{e}{4}$. Hence, the integration is $2\pi i \left(\frac{e}{4}\right) = \frac{\pi i e}{2}$.

(ii) The integrand has only singularity at $-i$ inside $|z| = 2$ and the residue at $-i$ is $-\sin i$. Hence, the integration is $-2\pi i \sin i = \pi \left(e - e^{-1}\right)$.

(iii) The integrand is holomorphic inside $|z| = 2$ and hence the integration is 0.

(iv) Note that $\left|z^5 - iz - 4\right| \geq 4 - |z|^5 - |z| \geq 2$ on $|z| = 1$. Hence, the integrand is holomorphic inside $|z| = 1$ and the integration is 0.

Q5. Write $f = u + iv$ and $dz = dx + idy$. Thus

$$f dz = (u + iv)(dx + idy) = (u + iv)\, dx + (-v + iu)\, dy = P\, dx + Q\, dy$$

where

$$P = u + iv, \quad Q = -v + iu.$$

We note that

$$Q_x - P_y = -v_x + iu_x - u_y - iv_y = 0$$

using the Cauchy-Riemann equations. Using Green's theorem which applies since P, Q are of class $\mathcal{C}^1$, we find

$$\int_\gamma f\, dz = \int_\gamma P\, dx + Q\, dy = \int \int_D (Q_x - P_y)\, dx\, dy = 0.$$

Q6. Along the circle $\gamma = \{w : |w| = R\}$, we have the following identity of differential forms:

$$\begin{aligned} \frac{dw}{w} + \frac{d\overline{w}}{\overline{w}} &= \frac{(x - iy)(dx + idy) + (x + iy)(dx - idy)}{R^2} \\ &= \frac{2(xdx + ydy)}{R^2} = \frac{d(x^2 + y^2)}{R^2} = 0 \end{aligned}$$

The last equality holds since $x^2 + y^2$ is a constant function over the curve γ. Let

$$h(w) = \frac{\overline{z} f(w)}{R^2 - \overline{z} w},$$

and note that it is holomorphic over $\Delta(0, R + \epsilon)$ where ϵ is small enough to satisfy $\Delta(0, R + \epsilon) \subset U$ and $(R + \epsilon)(R - |z|) < R$. Thus

$$\int_\gamma h(w)\, dw = 0.$$

Take the conjugate to reduce the problem to showing

$$f(0) = \frac{-1}{2\pi i} \int_{|w|=R} \frac{f(w)}{(\overline{w} - \overline{z})} d\overline{w} \tag{0.1}$$

where $d\overline{w} = dx - idy$. Now we express right hand side as

$$\begin{aligned}
\frac{-1}{2\pi i}\int_\gamma \frac{f(w)}{(\overline{w}-\overline{z})}d\overline{w} &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)\overline{w}}{(\overline{w}-\overline{z})w}dw \\
&= \frac{1}{2\pi i}\int_\gamma \left(\frac{f(w)\overline{w}}{(\overline{w}-\overline{z})w} - h(w)\right)dw \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w}dw \\
&= f(0).
\end{aligned}$$

Cauchy's integral formula was used in the last step.

Q7. Let G be an open set and γ be a C^1 loop in G. Suppose $\phi : \{\gamma\} \times G \to \mathbb{C}$ is a continuous function and define $g : G \to \mathbb{C}$ by

$$g(z) = \int_\gamma \phi(w,z)dw.$$

- g is a continuous function : Let $\ell = L(\gamma)$ be the length of the loop γ. For any $\epsilon > 0$, we can find $\delta > 0$ such that for any $|h| < \delta$,

  $$|\phi(w, z+h) - \phi(w,z)| < \frac{\epsilon}{\ell},$$

  for all $w \in \gamma$. Here we are using compactness of Im γ and the continuity of ϕ.

  (Indeed, assuming otherwise. Then, there would exist $\epsilon > 0$ such that for all δ, say $\delta = \frac{1}{n}$, there exists $h_\delta = h_n$ with $|h_n| < \frac{1}{n}$ and $w_n \in \gamma$ such that

  $$|\phi(w_n, z+h_n) - \phi(w_n, z)| \geq \frac{\epsilon}{\ell}.$$

  By compactness of Im γ, we may assume $w_n \to w$ after passing to a subsequence. Making $n \to \infty$ in the above inequality we obtain $0 \geq \frac{\epsilon}{\ell}$ a contradiction.)

  Therefore for any $|h| < \delta$,

  $$\begin{aligned}
  |g(z+h) - g(z)| &= \left|\int_\gamma \Big(\phi(w,z+h) - \phi(w,z)\Big)dw\right| \\
  &< \frac{\epsilon}{\ell}\cdot L(\gamma) = \epsilon.
  \end{aligned}$$

Assume $\frac{\partial \phi}{\partial z}$ exists for each $(w,z) \in \{\gamma\} \times G$ and is continuous. We show

- g is holomorphic and $g'(z) = f(z)$ where

  $$f(z) = \int_\gamma \frac{\partial \phi}{\partial z}(w,z)dw.$$

  It is enough to show that

  $$\lim_{h\to 0}\left(\frac{g(z+h)-g(z)}{h} - f(z)\right) = 0.$$

Let us denote $\phi_2 = \frac{\partial \phi}{\partial z}(w, z)$. Given $\epsilon > 0$, there exist $\delta > 0$ such that for $|h| < \delta$ we have

$$|\phi_2(w, z + th) - \phi_2(w, z)| < \frac{\epsilon}{\ell}$$

for all $w \in \gamma$. This follows by the same reasoning as above applied to the function ϕ_2.

Note that

$$\frac{d}{dt} \frac{\phi(w, z + th)}{h} = \phi_2(w, z + th)$$

and hence by the fundamental theorem of calculus, we find

$$\begin{aligned} \left| \frac{\phi(w, z + h) - \phi(w, z)}{h} - \phi_2(w, z) \right| &= \left| \int_0^1 \phi_2(w, z + th) - \phi_2(w, z) dt \right| \\ &\leq \int_0^1 |\phi_2(w, z + th) - \phi_2(w, z)| \, dt < \frac{\epsilon}{\ell} \end{aligned}$$

Therefore, for $|h| < \delta$,

$$\left| \frac{g(z + h) - g(z)}{h} - f(z) \right| = \left| \int_\gamma \left(\frac{\phi(w, z + h) - \phi(w, z)}{h} - \phi_2(w, z) \right) dw \right| < L(\gamma) \frac{\epsilon}{\ell} = \epsilon$$

using the basic estimate and the preceding inequality at the last step. This completes the argument.