HW2 - SOLUTIONS

Q1.

(i) Recall

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

Substituting

$$G(-a_1) \cdots G(-a_d) = \prod_{n=1}^{\infty} \left(1 - \frac{a_1}{n}\right) e^{\frac{a_1}{n}} \ldots \prod_{n=1}^{\infty} \left(1 - \frac{a_d}{n}\right) e^{\frac{a_d}{n}}$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{a_1}{n}\right) \cdots \left(1 - \frac{a_d}{n}\right) e^{\frac{a_1}{n} + \ldots + \frac{a_d}{n}} = \prod_{n=1}^{\infty} \frac{P(n)}{n^d} e^{\frac{a_1 + \ldots + a_d}{n}}.$$

Similarly

$$G(-b_1) \cdots G(-b_d) = \prod_{n=1}^{\infty} \frac{Q(n)}{n^d} e^{\frac{a_1 + \ldots + a_d}{n}}.$$

Thus

$$\frac{G(-a_1) \cdots G(-a_d)}{G(-b_1) \cdots G(-b_d)} = \frac{\prod_{n=1}^{\infty} P(n) n^{-d} e^{\frac{a_1 + \ldots + a_d}{n}}}{\prod_{n=1}^{\infty} Q(n) n^{-d} e^{\frac{a_1 + \ldots + a_d}{n}}} = \prod_{n=1}^{\infty} \frac{P(n)}{Q(n)} = \prod_{n=1}^{\infty} R(n).$$

In the above we used $\sum_{k=1}^{d} a_k = \sum_{k=1}^{d} b_k$ to cancel the exponentials. Using that

$$\Gamma(z) = \frac{e^{-\gamma z}}{z G(z)} \implies G(z) = \frac{e^{-\gamma z}}{z \Gamma(z)}$$

the above expression becomes

$$\prod_{n=1}^{\infty} R(n) = \frac{(-b_1) \cdots (-b_d)}{(-a_1) \cdots (-a_d)} \cdot \frac{\Gamma(-b_1) \cdots \Gamma(-b_d)}{\Gamma(-a_1) \cdots \Gamma(-a_d)}.$$

The constant γ does not appear in the final answer since $\sum_{k=1}^{d} a_k = \sum_{k=1}^{d} b_k$ forces the cancellation.

Remark: Using $\Gamma(z+1) = z\Gamma(z)$, the answer can also be re-written as

$$\frac{\Gamma(1 - b_1) \cdots \Gamma(1 - b_d)}{\Gamma(1 - a_1) \cdots \Gamma(1 - a_d)}.$$

(ii) We have

$$P(n) = n^2 + n - 4/9 = \left(n - \frac{1}{3}\right)\left(n + \frac{4}{3}\right), \quad Q(n) = n^2 + n - 5/16 = \left(n - \frac{1}{4}\right)\left(n + \frac{5}{4}\right).$$

By (i) we have

$$\prod_{n=1}^{\infty} \frac{n^2+n-4/9}{n^2+n-5/16} = \frac{(-1/3)(4/3)}{(-1/4)(5/4)} \cdot \frac{\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}.$$

We showed in class that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Thus

$$\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right) = -\frac{\pi}{\sin \pi/3}$$

$$\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right) = -\frac{\pi}{\sin \pi/4}.$$

By direct substitution, we obtain

$$\prod_{n=1}^{\infty} \frac{n^2+n-4/9}{n^2+n-5/16} = \frac{45}{64} \cdot \frac{\sin \pi/3}{\sin \pi/4} = \frac{45}{128}\sqrt{6}.$$

Q2. Clearly,

$$\prod_{n=-\infty}^{\infty} E_1\left(\frac{z}{n-\alpha}\right)$$

converges absolutely and locally uniformly to an entire function with zeroes only at $n-\alpha$ for $n \in \mathbb{Z}$. This is ensured by the Weierstraß theorem, because the sum

$$\sum_n \frac{1}{|n-\alpha|^2}$$

converges (use the limit comparison test with the series $\sum_n \frac{1}{n^2}$ for instance). The function $\frac{\sin \pi(z+\alpha)}{\sin \pi\alpha}$ also has zeros at $n-\alpha$ for $n \in \mathbb{Z}$. Therefore, by the Weierstraß theorem, we must have

$$\frac{\sin \pi(z+\alpha)}{\sin \pi\alpha} = e^{g(z)} \prod_{n=-\infty}^{\infty} E_1\left(\frac{z}{n-\alpha}\right).$$

Setting $z=0$, we see that $e^{g(0)}=1$ hence $g(0)=0$. To conclude, it suffices to show

$$g'(z) = \pi \cot \pi\alpha.$$

In the above expression, take the logarithmic derivative. This yields

$$\pi \cot \pi(z+\alpha) = g'(z) + \sum_{n=-\infty}^{\infty} \left(\frac{1}{z-(n-\alpha)} + \frac{1}{n-\alpha}\right).$$

Problem 5 on HWK6, Math 220A reads

$$\pi \cot \pi w = \frac{1}{w} + 2w \sum_{n=1}^{\infty} \frac{1}{w^2-n^2} = \frac{1}{w} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{1}{w-n} - \frac{1}{n}\right)$$

where in the last line we group the terms for n and $-n$ together. With the aid of this identity for $w = z + \alpha$, we obtain

$$g'(z) = \pi \cot \pi(z+\alpha) - \sum_{n=-\infty}^{\infty} \left(\frac{1}{z-(n-\alpha)} + \frac{1}{n-\alpha} \right)$$

$$= \frac{1}{z+\alpha} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{1}{z+\alpha-n} - \frac{1}{n} \right) - \sum_{n=-\infty}^{\infty} \left(\frac{1}{z+\alpha-n} + \frac{1}{n-\alpha} \right)$$

$$= \frac{1}{\alpha} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{1}{\alpha-n} - \frac{1}{n} \right) = \pi \cot \pi\alpha,$$

after applying the homework problem from Math 220A again.

Let $\alpha = -\frac{1}{4}$ and change z to $\frac{z}{4}$. By trigonometry

$$\sin\left(\frac{\pi}{4}z - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \left(\cos\left(\frac{\pi}{4}z\right) - \sin\left(\frac{\pi}{4}z\right) \right).$$

(The usual trigonometric identities hold true in complex analysis: they hold in $\mathbb{R}$ and they extend everywhere via the identity theorem.) Substituting into the above formula and carrying out the arithmetic, we obtain

$$\cos\left(\frac{\pi}{4}z\right) - \sin\left(\frac{\pi}{4}z\right) = e^{-\frac{\pi z}{4}} \prod_{n=-\infty}^{\infty} E_1\left(\frac{z}{4n+1}\right).$$

Q3. Assume the zeroes of f occur at a set A and assume the zeroes of g occur at a set B. Let C be the set of common zeroes for f and g. Each $c \in C$ will be enumerated with multiplicity $\min(m, m')$ where m, m' are the orders of c for f and g. Since $C \subset A, B$, the set C has no limit point in $\mathbb{C}$. Thus, the Weierstraß problem can be solved for C. Let h denote the solution. Let $F = f/h$ and $G = g/h$. Since

$$\operatorname{ord}(F, c) = \operatorname{ord}(f, c) - \operatorname{ord}(h, c) \geq 0$$

it follows F is holomorphic at c for all $c \in C$, hence F is entire. Similarly G is entire. Finally,

$$f = hF, \quad g = hG$$

and by construction F, G have no common zeroes.

Q4. If $f = g^n$, then for any zero a of g we have

$$\operatorname{ord}(f, a) = \operatorname{ord}(g^n, a) = n \cdot \operatorname{ord}(g, a)$$

is divisible by n. Conversely, if ord (f, a) is divisible by n for any zero a of f, by Weierstraß, we can construct an entire function h with zeros exactly at the zeros of f and of order $\frac{1}{n} \cdot \operatorname{ord}(f, a)$. Clearly f and h^n have exactly the same zeros with the same multiplicity, hence f/h^n is entire and zero free. In particular,

$$f/h^n = e^F$$

for some entire function F. Then $f = g^n$ where $g = h \cdot e^{\frac{1}{n}F}$.

Q5.

(i) We show that there exist constants $c > 0$ such that

$$|t\omega_1 + \omega_2| \geq c(|t| + 1),$$

for all t real. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \frac{|t|+1}{|t\omega_1+\omega_2|}$. Note that the denominator never vanishes since $\omega_1/\omega_2 \notin \mathbb{R}$. Thus f is continuous. We have

$$\lim_{t \to \pm\infty} f(t) = \frac{1}{|\omega_1|}$$

so we can find δ such that $|f(t)| < \frac{2}{|\omega_1|}$ for $|t| \geq \delta$. Over the interval $[-\delta, \delta]$, f is continuous so it achieves a maximum M. Therefore letting $1/c = \max(M, \frac{2}{|\omega_1|})$ we conclude

$$f(t) \leq c^{-1} \implies |t\omega_1 + \omega_2| \geq c(|t| + 1)$$

for all $t \in \mathbb{R}$.

If $n \neq 0$, letting $t = m/n$, we have therefore established that

$$|m\omega_1 + n\omega_2| \geq c(|m| + |n|).$$

(When $n = 0$, we can arrange that the same inequality hold as well.) By the comparison test, $\sum_{\lambda \in \Lambda, \lambda \neq 0} \frac{1}{|\lambda|^3}$ provided we show

$$\sum_{(m,n) \neq (0,0)} \frac{1}{(|m| + |n|)^3}$$

converges. For each $k > 0$, there are $4k$ integer solutions to the equation $|m| + |n| = k$. Thus

$$\sum_{(m,n) \neq (0,0)} \frac{1}{(|m| + |n|)^3} = \sum_{k=1}^{\infty} \frac{1}{k^3} \cdot 4k = \sum_{k=1}^{\infty} \frac{4}{k^2} < \infty.$$

This completes the proof.

(ii) The fact that the product

$$\prod_{\lambda \in \Lambda, \lambda \neq 0} E_2\left(\frac{z}{\lambda}\right)$$

converges absolutely and locally uniformly was established in class during the proof of the Weierstraß factorization theorem. The convergence of $\sum_{\lambda \in \Lambda, \lambda \neq 0} \frac{1}{|\lambda|^3}$ is used to ensure convergence of the product. The statement about the zeros of the σ-function also follows from the Weierstraß factorization theorem.