MATH 287A Winter 2020 Final exam —due Monday March 16 by 2pm. Leave it in a sealed envelope in Prof. Politis' mailbox but *KEEP A COPY FOR YOUR RECORDS!*

You may use your textbooks, notes and calculator but **do not collaborate with anybody on this exam.** *All ten problems have equal weight.*

1. Give an example of a dependent, white noise sequence. Be sure to prove that your example is white (uncorrelated) but *not* independent.

2. Let X_t be a mean zero, (weakly) stationary time series, with an autocovariance $\gamma(k)$ that is absolutely summable, i.e., $\sum_k |\gamma(k)| < \infty$. Define the *innovations* $W_t = X_t - \hat{X}_t$ where $\hat{X}_t$ is the projection of X_t on $\bar{sp}\{X_s, s < t\}$. Show that W_t is a white noise and give an expression for its variance.

3. Let $X_t = Y_t + W_t$ where the Y series is independent of the W series. Assume Y_t satisfies an AR(1) model (with respect to some white noise), and W_t satisfies a different AR(1) model (with respect to some other white noise). Show that X_t is not AR(1) but it is ARMA(p,q) and identify p and q. [Hint: show that the spectral density of X_t is of the form of an ARMA(p,q).]

4. Do problem 5.20 of Brockwell and Davis.

5. Let $|c| < 1$ and consider the matrix identity:

$$\begin{bmatrix} 1 & c & c^2 & \ldots & c^{n-2} & c^{n-1} \\ c & 1 & c & \ldots & c^{n-3} & c^{n-2} \\ c^2 & c & 1 & \ldots & c^{n-4} & c^{n-3} \\ & \vdots & & \ddots & & \vdots \\ c^{n-2} & c^{n-3} & c^{n-4} & \ldots & 1 & c \\ c^{n-1} & c^{n-2} & c^{n-3} & \ldots & c & 1 \end{bmatrix}^{-1} = \frac{1}{1-c^2} \begin{bmatrix} 1 & -c & 0 & \ldots & 0 & 0 & 0 \\ -c & 1+c^2 & -c & \ldots & 0 & 0 & 0 \\ 0 & -c & 1+c^2 & \ldots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \ldots & 1+c^2 & -c & 0 \\ 0 & 0 & 0 & \ldots & -c & 1+c^2 & -c \\ 0 & 0 & 0 & \ldots & 0 & -c & 1 \end{bmatrix}$$

   How can you explain this matrix identity in light of what we know about the autocovariance and inverse autocovariance of a stationary sequence? Can you tell what the stationary sequence in question should be by looking at this matrix formula? (Hint: you may treat the 2nd matrix as approximately Toeplitz).

6. In view of the above matrix identity, compute the BLUE (best linear unbiased estimator) of $\mu = EX_t$ based on data $X_1, \ldots, X_n$ satisfying the AR(1) model: $X_t - \mu = \rho(X_{t-1} - \mu) + Z_t$ where $Z_t \sim$iid (0,1) and $|\rho| < 1$. (Hint: the form of BLUE is given in problem 7.2 of the book).

7. In the setting of the above problem, show that the sample mean $\bar{X}$ is asymptotically efficient by showing that $Var(\bar{X})/Var(BLUE) \to 1$ as $n \to \infty$.

8. Do problem 7.1 of Brockwell and Davis.

9. Do problem 7.4 of Brockwell and Davis. [Note: eq. (7.2.5) in the book (and the equation for W appearing in Theorem 7.2.1) are often called 'Bartlett's formula'; note that these expressions only hold true for *linear* time series, i.e., MA(∞) with iid innovations.]

10. Do problem 7.11 of Brockwell and Davis. [Hint: just verify the identity $\sum_{k=-\infty}^{\infty} \hat{\gamma}(k) e^{ikw} = n^{-1} |\sum_{t=1}^{n} (X_t - \bar{X}) e^{itw}|^2$ which can also be used to prove problem 7.3 in the book.]