HW4 - SOLUTIONS

Q1. Using Montel's theorem, it is enough to show that : $\mathcal{F}$ is a locally bounded family iff f is constant. When f is constant, $\mathcal{F}$ is the constant family, hence $\mathcal{F}$ is normal.

Suppose that $\mathcal{F} = \{f(kz) : k \in \mathbb{N}\}$ is a locally bounded family on the annulus $\Delta(0; r_1, r_2)$. Pick positive numbers a_1, a_2 such that

$$r_1 < a_1 < a_2 < r_2.$$

Define $D_k = \overline{\Delta}(0; ka_1, ka_2)$ be the closed annulus. Since D_1 is a compact subset of $\Delta(0; r_1, r_2)$, there exists a number M such that $|f(kz)| < M$ for all $k \in \mathbb{N}$ and $z \in D_1$. Equivalently,

$$|f(z)| < M \qquad \forall z \in \bigcup_{k=1}^{\infty} D_k.$$

We claim that the union $\cup_k D_k$ contains a neighborhood of infinity, that is, there exists $R > 0$ such that

$$\{z : |z| \geq R\} \subset \cup_k D_k.$$

In particular, this implies that

$$|z| \geq R \implies |f(z)| < M.$$

Since f is continuous, f is also bounded on the compact set $|z| \leq R$ by a constant M' and thus f is everywhere bounded by $\max(M, M')$. By Liouville's theorem, it follows that f is constant.

The claim of the previous paragraph can be easily justified. Pick an integer ℓ sufficiently large so that

$$\ell a_2 > (\ell + 1)a_1 \iff \ell > \frac{a_1}{a_2 - a_1}.$$

Then for each $k \geq \ell$ the outer radius of D_k is larger than inner radius of D_{k+1}. Therefore, letting $R = \ell a_1$ be the inner radius of D_ℓ we see that

$$\bigcup_{k=\ell}^{\infty} D_k = \{z \in \mathbb{C} : |z| \geq R\}.$$

Q2. Let $\{f_n\}$ be a sequence in $\mathcal{F}$. To prove $\mathcal{F}$ is normal, we wish to find a subsequence of $\{f_n\}$ that converges locally uniformly.

Step 1: Let ϕ be the Cayley (like) transform mapping the half plane $\mathrm{Re}\ z > 0$ to the unit disc $\Delta = \Delta(0, 1)$. It is given explicitly by the formula

$$\phi(z) = C(iz) = \frac{z-1}{z+1},$$

where C is the usual Cayley transform. The multiplication of the argument by i achieves the rotation needed to take us to the correct half plane. Let

$$\psi : \Delta(0, 1) \to \{z : \mathrm{Re}\ z > 0\}$$

be the inverse of ϕ.

Step 2: Define $g_n = \phi \circ f_n$. Note that each

$$g_n : \Delta(0, 1) \to \Delta(0, 1), \quad g_n(0) = \phi(f(0)) = \phi(1) = 0.$$

In particular, $\{g_n\}$ is bounded, so by Montel's theorem, after passing to a subsequence, we may assume g_n converges locally uniformly to a holomorphic function

$$g : \Delta(0, 1) \to \overline{\Delta}(0, 1), \quad g(0) = 0.$$

Note that a-priori the target of g is the closed unit disc – this is necessary since we are taking a limit.

Step 3: We claim that in fact g takes values in $\Delta(0, 1)$. Indeed, if $|g(z_0)| = 1$ for some $z_0 \in \Delta(0, 1)$ then z_0 is a local maximum for g, hence g is constant and thus $g(z_0) = g(0) = 0$. This contradicts $|g(z_0)| = 1$.

Let $f = \psi \circ g$. The composition is well-defined since g takes values in $\Delta(0, 1)$, the domain of ψ.

Step 4: It remains to show that if

$$g_n \overset{c}{\rightrightarrows} g \quad \text{then} \quad f_n = \psi \circ g_n \overset{c}{\rightrightarrows} f = \psi \circ g$$

in $\Delta(0, 1)$. This part is real analysis.

Let $K \subset \Delta(0, 1)$ be compact. We first show that the images

$$g_n(K) \subset L, \quad g(K) \subset L$$

are contained in a common compact subset $L \subset \Delta(0, 1)$. From here, note that the function ψ is continuous in L, hence uniformly continuous. That is, for all $\epsilon > 0$, we can find $\delta > 0$ with

$$|w - w'| < \delta, \quad w, w' \in L \implies |\psi(w) - \psi(w')| < \epsilon.$$

Since $g_n \rightrightarrows g$ on K, we can find N such that for all $n \geq N$,

$$z \in K \implies |g_n(z) - g(z)| < \delta.$$

Using the above inequalities for $w = g_n(z) \in L$, $w' = g(z) \in L$ for arbitrary $z \in K$ and $n \geq N$, we obtain that

$$z \in K \implies |w - w'| < \delta \implies |\psi(w) - \psi(w')| < \epsilon \implies |f_n(z) - f(z)| < \epsilon,$$

proving uniform convergence of the f_n's on K as claimed.

To see that we can pick a compact L bounding the images of g_n and g, define first

$$L_0 = g(K) \subset \Delta(0,1)$$

which is compact. We will enlarge L_0 slightly so that we can contain the images of the g_n as well. Let $d = d(L_0, \partial\Delta)$. In particular

$$|g(z)| \le 1 - d$$

for all $z \in K$ and $g(z) \in L_0$. Let

$$L = \overline{\Delta}\left(0, 1 - \frac{d}{2}\right).$$

This is clearly compact and furthermore $g(K) \subset L$ as noted just above. We claim that $g_n(K) \subset L$ for n sufficiently large. Since $g_n \rightrightarrows g$ in K, we can find M such that for all $n \ge M$ we have

$$z \in K \implies |g_n(z) - g(z)| < \frac{d}{2}.$$

But then by the triangle inequality

$$|g_n(z)| \le |g(z)| + |g_n(z) - g(z)| \le 1 - d + \frac{d}{2} = 1 - \frac{d}{2} \implies g_n(z) \in L,$$

as needed.

Q3. Suppose $\{f_n\}$ does not converge locally uniformly to f. Then there exist a compact set K, a number $\epsilon > 0$ and a sub-sequence $\{g_k\}$ such that

$$\sup_K |g_k - f| \ge \epsilon \tag{0.1}$$

for all k.

Since $\{g_k\}$ is also a locally bounded sequence of holomorphic functions over U, Montel's theorem implies that there is sub-sequence $\{g_{k_j}\}$ converging (locally uniformly) to a holomorphic function g in U.

Since locally uniform convergence implies point-wise converge, for all $a \in A$,

$$\lim_{j \to \infty} g_{k_j}(a) = g(a).$$

But we also know that

$$\lim_{j \to \infty} g_{k_j}(a) = f(a),$$

hence $g(a) = f(a)$ for all $a \in A$. Since A has a limit point in U, the identity principle (Lecture 10, Math 220A) implies $g = f$ on U. This is a contradiction since

$$\sup_K |g - f| \ge \epsilon.$$

Q4. Using Montel's theorem, it is enough to show that $\{g \circ f : f \in \mathcal{F}\}$ is a locally bounded family of holomorphic functions on U. Let $K \subset U$ be a compact set, then there is $M > 0$ such that $|f(z)| < M$ for all $z \in K$ and $f \in \mathcal{F}$.

Since g is bounded on bounded sets, there exist an $R > 0$ such that $|g(w)| < R$ for $w \in \Delta(0, M) \cap \Omega$. Therefore, for all $z \in K$ and $f \in \mathcal{F}$,

$$|g \circ f(z)| < R.$$

Hence $\{g \circ f : f \in \mathcal{F}\}$ is locally bounded as well, hence a normal family.

Q5. Note that by the root test the series

$$C(r) := \sum_{n=0}^{\infty} M_n r^n$$

converges for any $0 < r < 1$. For any $f \in \mathcal{F}$, let

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

be the Taylor expansion at 0 valid in $\Delta(0, 1)$. By the triangle inequality

$$|f(z)| \leq \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| |z|^n \leq \sum_{n=0}^{\infty} M_n r^n = C(r)$$

for all $|z| \in \overline{\Delta}(0, r)$ and $f \in \mathcal{F}$. It follows $\mathcal{F}$ is a bounded family on all compact sets, hence $\mathcal{F}$ is locally bounded. By Montel's theorem, $\mathcal{F}$ must be a normal family.

Conversely, let $\mathcal{F}$ be a normal family of holomorphic functions over $\Delta(0, 1)$. Let $\{r_k\}$ be a strictly increasing sequence of positive numbers converging to 1. Define

$$B_k := \sup_{f \in \mathcal{F}} \big(\max_{|z|=r_k} |f(z)| \big).$$

Since $\mathcal{F}$ is bounded on the compact set $|z| = r_k$, we conclude B_k is finite for each k.

By the maximum modulus theorem,

$$\max_{|z|=r_k} |f(z)| \leq \max_{|z|=r_{k+1}} |f(z)| \implies B_k \leq B_{k+1}.$$

Let n be a positive integer, define k_n to be the largest integer such that

$$B_{k_n} \leq n.$$

When $\{B_k\}$ is bounded, pick $\{k_n\}$ to be any unbounded increasing sequence satisfying the above inequality. We define

$$M_n = \frac{B_{k_n}}{r_{k_n}^n}.$$

We see below that this choice of M_n satisfies both required properties :

- By Cauchy's estimate, for any $f \in \mathcal{F}$, we have

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq \frac{\max_{|z|=r_{k_n}} |f(z)|}{r_{k_n}^n} \leq \frac{B_{k_n}}{r_{k_n}^n} = M_n.$$

- Note that

$$M_n^{\frac{1}{n}} = \frac{B_{k_n}^{\frac{1}{n}}}{r_{k_n}} \leq \frac{n^{\frac{1}{n}}}{r_{k_n}}.$$

Since k_n, by construction, is an increasing unbounded sequence of integers,

$$\lim_{n\to\infty} r_{k_n} = 1.$$

Thus, we get

$$\limsup M_n^{\frac{1}{n}} \leq \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{r_{k_n}} = 1.$$