13th Jan

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If $M \cong N$ (free mods).

X, Y be basis of M & N.

Then $|X| = |Y|$.

$M \xrightarrow{\varphi} N$.

$\underbrace{M \xrightarrow{\varphi} N \xrightarrow{\text{quotient}} N/IN}$

$\ker(\text{quo} \circ \varphi) = IM$.

$IM \subseteq \ker$

$\ker \subseteq I \cdot M$ follows from the fact that φ is an iso.

Hence, $M/I \cdot M \underset{\overline{\varphi}}{\xrightarrow{\simeq}} N/I \cdot N$

$\{m_i\}_i$ is a set of basis of M.

then : $\{\overline{m_i}\}_i$ is a set of basis $M/I \cdot M$.

① They generate all elements in M/IM

② Independent.

t

(2). Independent.

$$\sum_{k=1}^{t} \overline{a_{i_k}}\, \overline{m_{i_k}} = 0 \quad \text{in } M/IM.$$

$$\Downarrow$$

$$\sum_{k=1}^{t} a_{i_k} m_{i_k} \in \underset{\sim}{IM} \quad \text{in } M.$$

But every elem. in $I \cdot M$ can be written as

$$\sum_{j=1}^{s} r_{i_j} \cdot m_{i_j} \quad \text{for } r_{i_j} \in I.$$

Then this forces $a_{i_k} \in I \Rightarrow \overline{a_{i_k}} = 0$

F-reps of G $\longleftrightarrow$ FG-modules.

V : F-rep of G. (V an F-v.s.).

$\phi : G \to GL(V)$.

Define a FG-mod structure on V.

$x \in F$ $\quad xv$

$$g \cdot v = \phi(g) v.$$

This makes V a (left) FG-module.

FG-mod. V. $\quad$! V is an F-v.s

for any g the action of g on V. is a linear transformation

We have. $G \to \mathrm{End}_F(V)$. $\quad (F \subseteq Z(FG))$.

In fact. $\quad G \to \mathrm{Aut}_F(V) = GL_F(V) \quad$ (g is invertible).

This is a group homomorphism

Induced modules.

$$\underbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathbb{Z})}_{\phi^g \qquad g\cdot\phi.} \quad \text{G-module?}$$

$g \cdot \phi(x)$

$= \phi(x \cdot g)$.

This def coincides with Problem 3.

V is a k-v.s.

$R = \mathrm{End}_k(V)$. $\{v_i\}$ basis

$R \cong R \oplus R$.

$\phi \in R$ $\quad \phi(v_i) = \begin{cases} v_{\frac{1}{2}i} & \text{if } 2 \mid i \\ 0 & 2 \nmid i \end{cases}$.

$V = V_1 \oplus V_2$

$V_1 = \mathrm{span}\{ v_2, v_4, v_6 \cdots v \cdot \}$.

$V_2 = \mathrm{span}\{ v_1, v_3 \cdots \}$

$\phi : V_1 \xrightarrow{\sim} V$

$\psi : V_2 \xrightarrow{\sim} V$.

isomorphisms.

$R = \mathrm{Hom}_k(V, V) . = \mathrm{End}_k(V)$

$= \mathrm{Hom}_k(V_1 \oplus V_2, V)$.

$= \mathrm{Hom}_k(V_1, V) \oplus \mathrm{Hom}_k(V_2, V)$. (property of Hom).

$\underset{\text{via } \phi \text{ and } \psi}{=}$

$\mathrm{Hom}_k(V, V) \oplus \mathrm{Hom}_k(V, V)$

$= R \oplus R.$

$R\phi \cong R$ means. $\mathrm{Hom}(V_1, V) \cong \mathrm{Hom}(V, V)$ via ϕ.

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/5, \mathbb{Z}/5)$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}.$

$\mathrm{End}(\mathbb{Z}/5)$

"Pontryagin dual"

Let $T = \{$ set of roots of unity $\} \subseteq \mathbb{C}$.

T is a torsion $\mathbb{Z}$-module.

Consider $\mathrm{Hom}_{\mathbb{Z}}(T, T)$.

Since T is dense in S^1,

a gp homo $T \to T$ gives a continuous homormorphism $S^1 \to S^1$

Thus $\mathrm{Hom}_{\mathbb{Z}}(T, T) \cong \mathrm{Hom}_{cts}(S^1, S^1)$

The latter is classified by widing # (homotopy).

You will have $\mathrm{Hom}_{cts}(S^1, S^1) \cong \mathbb{Z}$.

F is a field. $\quad K \supseteq F$

$$0 \to g(x) \to F[x] \to F[x]/g(x) \to 0.$$

$$g(x) \otimes K \to F[x] \otimes K \to \left(F[x]/g(x)\right) \otimes K \to 0$$

$$\left(F[x]/g(x)\right) \otimes K \cong \left(F[x] \otimes K\right) / \operatorname{im}(g(x) \otimes K)$$

$$\cong K[x]/\langle g(x) \rangle$$

$$\underbrace{\operatorname{im}\left(g(x) \otimes K \longrightarrow F[x] \otimes K \right)}_{} \quad \cong K[x]$$

$$\in F[x] \otimes K$$

$$\sum g(x)\alpha_i(x) \otimes k_i \longrightarrow \sum g(x)\alpha_i(x) \otimes k_i$$

$$(k_i \sum \alpha_i(x)) g(x) \in \langle g(x) \rangle.$$

$1 \otimes m = 0.$ $\quad X \cong R^X$

$X^{-1}R$ is flat

" ... "

" F flat over R. "

$X^{-1}R$ is flat

$$\begin{array}{ccc} r\otimes m & R\otimes M & 1\otimes m = 0 \\ \downarrow & \cong & \downarrow \\ r\cdot m & M & m = 0 \end{array}$$

$$\begin{array}{ccc} 1\otimes m & & 1\otimes m \\ R\otimes_R M & \hookrightarrow & F\otimes_R M \\ \cong & & \cong \\ M & \hookrightarrow & X^{-1}M \\ m & & \frac{m}{1} \end{array}$$

$X = R\setminus\{0\}$

$X \subseteq R\setminus\{0\}$

$X^{-1}M$

2. (a)

$$0 \to I \to R \to R/I \to 0$$

$\otimes_R R/J$

$$I\otimes R/J \to R\otimes R/J \to (R/I)\otimes(R/J) \to 0.$$

$$(R/I)\otimes_R(R/J) \cong (R/J)\,/\,\mathrm{im}(I\otimes R/J)$$

$$I\cdot(R/J) \overset{\text{corres}}{=} I+J/J$$

$$(R/J) \big/ (I+J)/J \cong R/I+J.$$

$$K \otimes F[x]$$

$$\begin{tikzcd}
K \times F[x] \arrow[r, "\text{bilinear}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & M \\
K[x] \arrow[ur] &
\end{tikzcd}$$

uniqueness $K[x] \cong K \otimes F[x]$.

(b). $M/\mathrm{tors}(M)$ is divisible.

any p prime, $\hat{y} \in M/\mathrm{tors}(M)$

can find $\hat{x}$ s.t. $\underbrace{p\hat{x} = \hat{y}}$.

$p \cdot x = y + z$ for some $z \in \mathrm{tors}(M)$.

for any p, y find z.

$y = (y_q)_q \quad y_q \in \mathbb{Z}/q$

Note that p is invertible in $\mathbb{Z}/q$ if $p \neq q$.

$x = (x_q)_q, \quad x_q = y_q \cdot \frac{1}{p}$ for $q \neq p$.

$x_p = 0$. $\underbrace{y_p + z_p = 0}$

$z = (z_q) \quad z_q = 0 \quad z_p = p - y_p$

$z = (z_q)$ $\quad z_q = 0$ $\quad z_p = P - y_p.$

$q \neq p.$

(c). no nonzero divisible submod of M.

$N \lhd M$ $\quad N \neq 0$ $\quad N$ divisible.

$x \in$ is some p.

$x_p \neq 0.$ $\quad pZ = x_p.$

~~no~~ element in $\mathbb{Z}/p$,

s.t. $p \cdot z_p = x_p.$

$I \lhd R.$

I is f.g. (R is Noe.)

$I = (r_1, r_2, \ldots r_t)$ free.

nonzero $x \in R$.

consider (as an R-mod)

$\mathrm{Frac}(R) = M \cong R^{\oplus I}$.

$\{t_i\}_{i \in I}$ is a set of basis.

$t_i = \frac{a_i}{b_i}$

if $|I| \geq 2$: $t_1 \cdot b_1 a_2 = t_2 \cdot b_2 \cdot a_1$

$|I| = 1$.

~~$\mathrm{Frac}(R) \cong R$.~~ (as an R-mod).

$\mathrm{Frac}(R) = R \cdot t$ $\quad t \in R$.

$\mathrm{Frac}(R) = K$.

$t = \frac{p}{q}$, q not trivial.

Then $\frac{p^2}{q^2} \notin \frac{p}{q} \cdot R$.

CRT

$R/\langle a \rangle \qquad a = p_1^{t_1} \cdots p_s^{t_s}$.

$R/\langle p_1^{t_1} \cdots p_s^{t_s} \rangle$

$\Delta \cong R/\langle p_1^{t_1} \rangle \oplus \cdots \oplus R/\langle p_s^{t_s} \rangle$

$\cong C_1 \oplus \cdots \oplus C_{s+1} \oplus \cdots$

maximal an

C_1 is killed by $p_1^{t_1} \cdots p_s^{t_s}$.

$\Downarrow$

C_1 is killed $p_1^{t_1}$

C_1 is killed $P_1^{t_1}$

C_i is killed by $P_{i_j}^{t_{i_j}}$

$$C_1 = R/\langle P_1^{\lambda_1} P_2^{\lambda_2} \rangle$$

$$\cong R/\langle P_1^{\lambda_1} \rangle \oplus R/\langle P_2^{\lambda_2} \rangle$$

C_i $R/\langle P_{i_j}^{t_{i_j}} \rangle$

$N \subseteq M$

$\downarrow$

largest free submod. $\{n_1, \ldots n_r\} = S$

Consider the quotient. M/N.

$t\hat{m} = 0$

$tm \in N$

$tm = \sum c_i n_i$

$m \to \hat{m}$

$c_i n_i - tm = 0$

Claim. M/N torsion.

(Suppose for contra) $\hat{m} \in M/N$ is not torsion,

Consider $T = S \cup \{m\}$ Claim: T is an independant set.

(for contrad.) $\sum a_i n_i + tm = 0$. $(\in M)$. $t \neq 0$,

LHS in M/N $t \cdot \hat{m} = 0$ Contradiction.

$\Rightarrow$ $(M/N) \otimes_R F = 0$ (part (a))

$0 \to N \to M \to M/N \to 0$. of R-mods.

F is flat:

$0 \to N \otimes_R F \to M \otimes_R F \to \underbrace{\cancel{(M/N)} \otimes_R F}_{0} \to 0$.

$N \otimes_R F \cong M \otimes_R F$

$$N \otimes_R F \cong M \otimes_R F$$

$$F \otimes_R M.$$

$$\sum \frac{r_i}{t_i} \otimes m = \sum \frac{r_i \prod_{j \neq i} t_j}{\prod t_i} \otimes m$$

$$= \sum \frac{1}{\prod t_i} \otimes r_i \prod t_j m$$

$$= \frac{1}{\prod t_i} \otimes \left(\sum r_i \prod_j t_j m \right).$$

R pid. M - R.

$S = \{ n \in \mathbb{N} \mid M \cong \oplus\ n$ cyclic mods $\}$.

t invariant factors

$$M \cong \bigoplus_{i=1}^{S} M_i \qquad j < t.$$

$M \cong \oplus R/a_i$

$a_1 \mid a_2 \mid \cdots \mid a_t.$

$M_i \cong R/(\lambda_i).$

Choose p with $p \mid a_1$

P-power. elementary divs

p-power. elementary divs

$$\lambda_i = \prod_{j=1}^{k(i)} p_j^{(i)\, c_j^{(i)}} \qquad (R/\lambda_i) \cong \prod_{j=1}^{k(i)} \left(R/p_j^{(i)\, c_j^{(i)}}\right)$$

Each λ_i contribute at most $\underline{1}$ p-power elem. div.

$\oplus R/\lambda_i$ give you at most s

p-power eleme divs.

$$X^4 - 1 = 0 . = (X^2+1)(X^2-1).$$

char poly = min poly. deg = (2).

$\parallel$

X^2+1 or X^2-1.

$$M \times N \xrightarrow{\varphi} S \qquad M \times N \longrightarrow S$$

$$\varphi\left((sr, m) - (s, rm)\right) = 0.$$

$M \otimes N \cong S$?

$\varphi : M \otimes N \to S.$

$$V \underset{F}{\otimes} V \qquad \mathrm{Hom}(V, V)$$

$$\underbrace{V \underset{F}{\otimes} \check{V} \cong \underset{F}{\mathrm{Hom}}(V, V).}$$

(finite dim)

$$\begin{pmatrix} a_1 \\ \\ a_n \end{pmatrix} \quad \begin{pmatrix} b_1 & b_n \end{pmatrix}$$

$$\begin{pmatrix} \sum a_i b_j \\ \\ \end{pmatrix}.$$

$$K \otimes F[x]/g \xrightarrow{\varphi} K[x]/g$$

$$k \otimes \widehat{f(x)}. \to k\widehat{f(x)}$$

$$K[x]/g \to K \otimes (F[x])/g.$$

$$f : \underbrace{k[x] \to k \otimes F[x]} \to k \otimes (F[x] / g)$$

$$H(x) \to \sum h_i \otimes x^i .$$

$$\sum h_i x^i \qquad \langle g(x) \rangle \subseteq \ker f .$$

$$\left(\hat{f} : k[x] / g(x)) \to k \otimes_F (F[x] / g) \right)$$

$$\varphi \circ \hat{f} = id \qquad \hat{f} \circ \varphi = id .$$

$$J^2 = \begin{pmatrix} J_1 & \\ & J_2 \end{pmatrix} .$$

$$M \sim \begin{pmatrix} J_0^{(1)} & & & \\ & J_0^{(2)} & & \\ & & \ddots & \\ & & & J_0^{(k)} \\ & & & \end{pmatrix}$$

$$\begin{pmatrix} J_0^{(1)} & \\ & J_0^{(2)} \end{pmatrix} = J^2$$

$J_0^{(2)}$ has size $= J_0^{(1)}$ or 1 larger.

Q

$A^4 = I$.

$X^4 - 1 = 0$,

$(X^2+1)(X^2-1) = 0$,

$\min_A(x)$ $\quad \mathrm{char}_A(x)$

$\deg(\min_A(x)) \neq 1$ $\quad X+1=0 \quad X-1=0$ $\quad \cancel{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}} \cancel{\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}}$

$\deg(\min(x)) = 2$. $\quad \min \mid \mathrm{char} \quad \deg(\mathrm{char}) = 2$

$X^2+1=0$ $\quad \cancel{X^2-1=0}$ $\quad \cancel{X^2 = I}$

min poly $=$ char poly. $= X^2+1$

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

$X^4+1 = (X+1)^4 = 0$,

char $= (X+1)^2 =$ min $= X^2-1$ $\quad \cancel{A^2 - I = 0}$,

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}^2 = I$$

$$\begin{pmatrix} \pm i & \\ & \pm 1 \end{pmatrix} \quad \begin{pmatrix} i & \\ & i \end{pmatrix} (X-i)$$

$$\begin{pmatrix} & \\ & \pm 1 \end{pmatrix} \qquad \begin{pmatrix} & \\ & i \end{pmatrix}^{\sim} (X-i)$$

$$\begin{pmatrix} i & \\ & -i \end{pmatrix}$$

$$\begin{pmatrix} -i & \\ & -i \end{pmatrix}^{\sim} (X+i)$$

$$X^4 + 1 = 0$$

$$m(X) \mid (X+1)(X-1)(X-i)(X+i) = 0$$

$A \in M_n(F)$

$A^k = 0$ for some k

$\Downarrow$

$A^n = 0$

$\mathrm{char}_A(X) = \prod_i^n (X - \alpha_i)$ $\qquad \alpha_i = 0$

$F \subseteq \overline{F}$

$M_n(F) \subseteq M_n(\overline{F})$

$A^n = 0 \qquad A^n = 0$

$$\min_{C_f}(x) = f(x).$$

$\min(x) = h(x)$ $\qquad h(x) \mid \text{char} = f(x)$

$$h(C_f) = 0. \qquad \deg h < n.$$

$$h(C_f)e_1 = \sum_{i=0}^{n-1} a_i C_f^{i} e_1 = \sum_{i=0}^{n-1} a_i e_{i+1} = 0$$

$$\Downarrow$$

$$h(x) = \sum_{i=0}^{n-1} a_i x^i \qquad a_i = 0 \ \forall i.$$

$$C_f^{i} e_1 = C_f^{i-1} e_2 = \cdots e_{i+1}$$

$$C_f e_i = e_{i+1} \qquad i < n.$$

$$C_f e_n = -\sum_i c_i(f) e_i$$

$$\begin{pmatrix} & & & -a_n \\ 1 & & & -a_{n-1} \\ & \ddots & & \\ & & 1 & \end{pmatrix}.$$

$(C_h)^t$ has $\begin{cases} \text{char poly} \\ \text{min poly} \end{cases} = h$

r.c.f. $= C_h$.

$A = C_h^t$ apply (b).

$$A \sim \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix},$$

$$A^t \sim \begin{pmatrix} A_1^t & & \\ & \ddots & \\ & & A_k^t \end{pmatrix}.$$

$A_1 \sim A_1^t$

$T_1 A_1 T_1^{-1} = A_1^t$

$$\begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_k \end{pmatrix} \qquad \begin{pmatrix} T_1^{-1} & & \\ & \ddots & \\ & & T_k^{-1} \end{pmatrix}$$

M is free $\Rightarrow$ M is torsion-free.

$$M \cong \oplus R^r \oplus R/a_1 \oplus \cdots R/a_n.$$

☆ M is flat, then any direct summand is flat.

$M \cong K \oplus L$.

WTS: K is flat. $\qquad (M \otimes A) \cong K \otimes A \oplus L \otimes A$

$$0 \to A \to B \to C \to 0.$$

α. $\xrightarrow{\text{inj}}$ 0

$$\begin{tikzcd}
0 \arrow[r] & A \otimes K \arrow[r] \arrow[d, hook] & B \otimes K \arrow[r] \arrow[d, hook] & C \otimes K \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & A \otimes M \arrow[r] & B \otimes M \arrow[r] & C \otimes M \arrow[r] & 0
\end{tikzcd}$$

exact.

0

$\alpha = 0$

$$\begin{tikzcd}
\alpha \arrow[r] \arrow[d] & 0 \arrow[d] \\
? \arrow[r] \arrow[d, "\text{inj}"] & 0 \\
{} &
\end{tikzcd}$$

$? = 0$

△. R/a is not flat. (for nontrivial a.)

Δ. $R/\mathfrak{a}$ is not flat. (for nontrivial $\mathfrak{a}$.)

$\dim_F(R) < \infty$ $\Rightarrow$ $K_1 K_2 \subseteq K$, $\dim_F(K_1 K_2) < \infty$.

$\phi_a(x) = a \cdot x$.

$\forall a \quad \phi_a : R \to R$ injective.

Regard ϕ_a as an F-vector space morphism.

$\ker(\phi_a) = 0$.

$$V \xrightarrow{\phi_a} V$$

$$\dim(V) = \cancel{\dim(\ker)} + \dim(\mathrm{Im}).$$

$\dim_F R < \infty$

$\{1, a, a^2, \dots a^n \dots\}$.

$C_n a^n + C_{n-1} a^{n-1} \dots + \dots + C_0 = 0$. $\quad C_i \in F$.

$C_n \neq 0$

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0. \qquad c_0 \neq 0.$$

$$\underbrace{a\left(a^{n-1} + \cdots + c_1\right)}_{-c_0} = 1.$$

$$K_1 = K(\alpha_1, \ldots \alpha_m).$$

$$K_2 = F(\beta_1, \ldots \beta_n),$$

$K[\alpha_2]$

$K[\alpha_1, \alpha_2]$ finite.

$K[\alpha_1]$.

$$M = F[\alpha_1, \ldots \alpha_m, \beta_1, \ldots \beta_n]$$

$\dim_K M < \infty.$ — K is an F-algebra

M is integral domain b/c. $M \subseteq K$.

HW5. P5 $\Rightarrow$ M is a field.

$$S = \{ \sum a_i b_i \mid a_i \in K \ \& \ b_i \in K \}.$$

$$\text{expand } a_i = \left(\sum c_s^{(i)} \cdot \alpha_s\right)$$

expand $a_i = \left(\sum c_s \cdot \alpha_s\right)$

$b_i = \left(\sum d_t^{(i)} \beta_t\right).$

$\sum a_i \cdot b_i \in M.$

$S \subseteq M.$ $S = M.$

$M \subseteq S$

$$K_1 \otimes_F K_2 \xrightarrow{\theta} K_1 K_2$$

iff $[K_1 K_2 : F] = \underbrace{[K_1 : F] \cdot [K_2 : F]}$

θ is a F-vector space morphism.

θ surjective. morphism $V_1 \to V_2$ with $\dim V_1 = \dim V_2$

then θ is injective.

$[K_1 K_2 : F] = \dim_F (K_1 K_2).$

$$\begin{tikzcd}[column sep=small] & K_1K_2 \arrow[dl, dash] \arrow[dr, dash] & \\ K_1 \arrow[dr, dash] & & K_2 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

$$[K_1K_2 : F]$$

$$= [K_2K_1 : K_1][K_1 : F]$$

$$[K_2K_1 : K_1] \leq [K_2 : F],$$

"$=$" iff $K_1 \cap K_2 = F$

$$\begin{tikzcd}[column sep=small] K_1K_2 \arrow[d, dash] \arrow[dr, dash] & \\ K_1 \arrow[dr, dash] & K_2 \arrow[d, dash] \\ & K_1 \cap K_2 \arrow[d, dash] \\ & F \end{tikzcd}$$

$$[K_2K_2 : K_1]$$

$$= [K_2 : K_1 \cap K_2]$$

$$\leq [K_2 : F]$$

$$\begin{tikzcd}[column sep=small] & K_1K_2 \arrow[dl, dash] \arrow[dr, dash] & \\ K_1 \arrow[dr, dash] & & K_2. \arrow[dl, dash] \\ & M = K_1 \cap K_2 & \end{tikzcd}$$

$\{\alpha_1, \ldots \alpha_t\}$ is M-basis for K_2

then $\{\alpha \ldots \alpha_t\}$ is K_1-basis for K_1K_2.

$$f = \sum b_i x^{ip}$$

$$= g(x^p). \quad \text{for some } g.$$

$= \underbrace{g(x^p)}$. for some g

g is separable

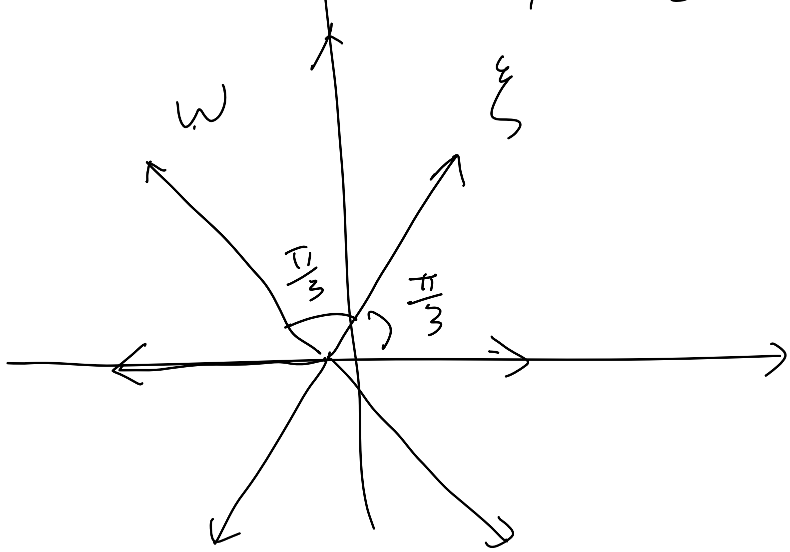
g is NOT separable,

$\underbrace{g} = \underbrace{h}(x^p) \Rightarrow f = h(x^{p^2})$

$f = h_2(x^{p^3})$

check h

$\sqrt{-3} = \sqrt{3} \cdot i$.



$\boxed{\mathbb{Q}(\xi, \sqrt[3]{2}).} / \mathbb{Q}$.

$$[F[\sqrt{a},\sqrt{b}]:F]=4$$

$$\Updownarrow$$

ab has no sqrt in F

$\Downarrow$ $[F[\sqrt{a}][\sqrt{b}]:F[\sqrt{a}]]=2.$

$ab=f^2$ then $\sqrt{b}=\pm f\frac{1}{\sqrt{a}}$

$\sqrt{b}\in F[\sqrt{a}].$

$\Uparrow$ ab has no sqrt in F

$[F[\sqrt{a}][\sqrt{b}]:F[\sqrt{a}]]=1$

$\sqrt{b}\in F[\sqrt{a}].$

$\Theta: K_1\otimes_F K_2\to K_1K_2.$

if $[\quad]=[\quad]\cdot[\quad]$ then Θ is iso

①. Θ is a homomorphism of rings.

② Regard Θ as a map between F-vector spaces

①. θ is a homomorphism of rings.

②. Regard θ as a map between F-vector spaces

θ is surjective

$V_1 \xrightarrow{\theta} V_2$ $\dim(V_1) = \dim(V_2)$ θ surj $\Rightarrow \theta$ inj

$K_1 \otimes K_2$ $F^{d_1} \otimes F^{d_2}$

$K_1 = F^{[K_1:F] = d_1}$ $\cong F^{d_1 \cdot d_2}$ (as an F-vector space)

$K_2 = F^{[K_2:F] = d_2}$

$\alpha \in K \setminus K^p$ $F = K^p$

$\beta = \alpha^p \in F.$

α is a root of $x^p - \beta \in F[x]$

$f(x) = x^p - \beta$ is the minpoly of α over F.

$f(x) = (x - \alpha)^p$ in $K[x]$.

$\deg(f_i) \cdot t$ factor into irreducibles

$f(x) = (x-\alpha)^{-}$ in $K[x]$.

$p = \deg(f_1) \cdot t$

factor into irreducibles

$f(x) = f_1(x) \cdots f_t(x)$ then $f_i(x) = (x-\alpha)^{n_i}$ (in $K[x]$).

Point is $\deg(f_1) = \deg(f_2) = \cdots \deg(f_t)$

Why? Assume f_1 has smallest deg.

$\deg(f_2) \geq \deg(f_1)$

$f_1 = (x-\alpha)^{n_1}$ $\qquad$ $f_2 = (x-\alpha)^{n_2}$.

$f_2 = f_1 \cdot (\quad)$