**Math 220, Problem Set 3.**

**1.** Let $\gamma(t) = e^{it}$ for $t \in [-\pi/2, \pi/2]$. Compute

$$\int_\gamma z e^{iz}\, dz.$$

**2.** Let $C$ be the half unit circle joining $1+i$ to $1-i$ clockwise. By direct parametrization, calculate the integral

$$\int_C \sqrt{z-1}\, dz,$$

where the principal branch of the square root is used for the integrand.

**3.** Assume that

$$f(z) = c\prod_{\ell=1}^{k}(z - a_\ell)^{m_\ell}$$

is a polynomial with roots at $a_1, \ldots, a_k$ with multiplicities $m_1, m_2, \ldots, m_k$. Show that for any closed loop $\gamma$ avoiding $a_1, \ldots, a_k$ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{\ell=1}^{k} m_\ell \cdot n(\gamma, a_\ell).$$

In particular, if $R$ is sufficiently large, and $\gamma(t) = Re^{it}$ for $0 \le t \le 2\pi$, show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \deg f.$$

*Remark:* This is a version of the argument principle to be proved later.

**4.** Calculate the following integrals using the local form of Cauchy's integral formula:

(i)

$$\int_{|z|=2} \frac{e^z}{(z-1)(z-3)^2} dz$$

(ii)

$$\int_{|z|=2} \frac{\sin z}{z+i}\, dz$$

(iii)

$$\int_{|z|=1} \frac{e^z}{(z-2)^3} dz$$

(iv)

$$\int_{|z|=1} \frac{dz}{z^5 + iz - 4}.$$

**5.** Let $\gamma$ be a piecewise $\mathcal{C}^1$-closed curve in a disc $\Delta$ with no self intersections so that $\gamma$ bounds a domain $D$ with $\overline{D} \subset \Delta$. Let $f : \Delta \to \mathbb{C}$ be a holomorphic function with continuous derivative. (We will see shortly that this assumption holds for all holomorphic functions.) Use Green's theorem from multivariable calculus to prove that

$$\int_\gamma f dz = 0$$

thus giving an alternate proof for this (weaker) version of a result proved in class.

*Hint:* Write $f dz = (u + iv)(dx + idy) = P dx + Q dy$ and apply Green's theorem to the $\mathcal{C}^1$-form $P dx + Q dy$.

**6.** Let $f : U \to \mathbb{C}$ be holomorphic and let $\overline{\Delta}(0, R) \subset U$. Show that

$$\overline{f}(0) = \frac{1}{2\pi i} \int_{|w|=R} \frac{\overline{f}(w)}{w - z} \, dw$$

for any $z \in \Delta(0, R)$.

*Hint:* It is easier to prove the equivalent statement obtained by taking conjugates, also noting that

$$\frac{dw}{w} + \frac{d\bar{w}}{\bar{w}} = 0$$

along $|w| = R$ where $d\bar{w} = dx - idy$. For the required manipulations, you can give an elegant argument considering the function $h(w) = \frac{\bar{z} f(w)}{R^2 - \bar{z} w}$.

**7.** Solve Conway IV.2.2. You may assume of course that $\gamma$ is a $\mathcal{C}^1$ loop (as opposed to rectifiable). The book uses the word ”analytic” to mean “holomorphic,” using the definitions given in class.