Math 240A: Real Analysis, Fall 2019

Homework Assignment 3

Due Friday, October 18, 2019

1. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Let $n \in \mathbb{N}$ and $A_j \in \mathcal{M}$ $(j = 1, \dots, n)$ be such that $\sum_{j=1}^{n} \mu(A_j) > n - 1$. Prove that $\mu(\cap_{j=1}^{n} A_j) > 0$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume $E_j \in \mathcal{M}$ $(j = 1, 2, \dots)$ and $\sum_{j=1}^{\infty} \mu(E_j) < \infty$. Let $E = \{x \in X : x \in E_j \text{ for infinitely many } j\}$. Prove that $E \in \mathcal{M}$ and $\mu(E) = 0$.

3. Is it true that $m(G) = m(\overline{G})$ for any open subset $G \subseteq \mathbb{R}$? If yes, prove it. If no, provide a counter example.

4. Let A and B be two Lebesgue measurable subsets of $\mathbb{R}$ such that $m(A) = m(B) < \infty$. Suppose $E \subseteq \mathbb{R}$ satisfies $A \subseteq E \subseteq B$. Prove that E is also Lebesgue measurable and $m(E) = m(A)$. ‘

5. Let μ be the Borel measure on $\mathbb{R}$ defined by the nondecreasing and right-continuous function

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 < x < 1, \\ x^2 + 3 & \text{if } 1 \leq x \leq 2, \\ 7 & \text{if } 2 < x. \end{cases}$$

   Calculate: (1) $\mu((-\infty, 1))$; (2) $\mu((-\infty, 1])$; (3) $\mu(\mathbb{R})$; and (4) $\mu(\{2\})$.

6. The Dirac measure δ concentrated on $\{0\}$ is a Borel measure on $\mathbb{R}$. Find all the increasing and right-continuous functions $F : \mathbb{R} \to \mathbb{R}$ such that $\mu_F = \delta$.

7. Prove Proposition 1.20 in the textbook.

8. Let $0 < \alpha < 1$. Construct a closed set $F \subseteq [0, 1]$ such that it is nowhere dense and $m(F) = \alpha$.

9. Let $E \subseteq \mathbb{R}$ be Lebesgue measurable with $0 < m(E) < \infty$. Then for any $\alpha \in (0, 1)$ there exists an open interval I such that $m(E \cap I) > \alpha m(I)$.

10. Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set and N the nonmeasurable set described in §1.1.
    (1) If $E \subseteq N$, then $m(E) = 0$.
    (2) If $m(E) > 0$, then E contains a nonmeasurable set. (It suffices to assume $E \subseteq [0, 1]$. In the notation of §1.1, $E = \cup_{r \in \mathbb{Q} \cap [0,1)} E \cap N_r$.)