Student full name:

MATH 281B – Midterm 1 – Winter 2019

Please tear off the last page (reference sheet) and do not turn it in. Use the notation defined there (and the notation used in lecture); otherwise, define any symbol and name any result you use from the reference sheet. If you are using a result from the sheet, or another known result, name it (e.g., "Using Fact 1 [...]" or "By the Rao-Blackwell Theorem [...]"). Be concise and clear.

Problem 1. Consider a setting where T is complete and sufficient. Let $\delta(X)$ be any statistic with finite variance and define $g(\theta) = \mathbb{E}_\theta[\delta(X)]$.

1. Show that $\eta(T) \stackrel{\text{def}}{=} \mathbb{E}[\delta(X) \mid T]$ is also unbiased for $g(\theta)$.

2. Under which distribution is the last expectation taken? Explain.

3. Argue (rigorously) that $\eta(T)$ is in fact the UMVU estimator for $g(\theta)$.

4. Use all this to obtain the UMVU estimator for θ from a normal sample $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \mathcal{N}(\theta, 1)$.

Problem 2. Consider minimizing $\phi(a) = \mathbb{E}[\rho(Z - a)]$ over $a \in \mathbb{R}$, where ρ is a function and Z is a random variable such that $\phi(a)$ is well defined and finite for all $a \in \mathbb{R}$.

1. Assume that ρ is convex. Show that ϕ is convex.

2. Assume that ρ is even and that Z has a symmetric distribution about 0. Show that ϕ is even.

3. Show that any convex and even function on $\mathbb{R}$ attains its minimum at zero.

4. Use all this (in conjunction with one of the facts in the summary sheet) to obtain the MRE estimator for θ from a normal sample $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\theta, 1)$ when ρ is an arbitrary convex and even function.

5. Provide an alternate way to obtain the MRE estimator using Problem 1(4).