Math 240C: Real Analysis, Spring 2020

Homework Assignment 4

Due 12:00 noon, Monday, April 27, 2020

1. Let X and Y be locally compact Hausdorff spaces. Let μ and ν be Radon measures on X and Y, respectively. Assume $f \in C_{\rm c}(X \times Y)$. Prove that the functions

$$x \mapsto \int_Y f_x(y)\, d\nu(y) \qquad \text{and} \qquad y \mapsto \int_X f^y(x)\, d\mu(x)$$

are continuous functions on X and Y, respectively.

2. Let X and Y be locally compact Hausdorff spaces. Let μ and ν be Radon measures on X and Y, respectively. (They are not necessary σ-finite.) Assume $f : X \times Y \to \mathbb{R}$ is nonnegative lower semi-continuous. Prove that the functions

$$x \mapsto \int_Y f_x(y)\, d\nu(y) \qquad \text{and} \qquad y \mapsto \int_X f^y(x)\, d\mu(x)$$

are Borel-measurable functions on X and Y, respectively, and

$$\iint f\, d(\mu \hat{\times} \nu) = \iint f\, d\mu\, d\nu = \iint f\, d\nu\, d\mu.$$

3. Let $p = p(x)$ $(x \in \mathbb{R}^n)$ be a polynomial and define $f(x) = p(x)e^{-|x|^2}$ $(x \in \mathbb{R}^n)$. Prove that $f \in \mathcal{S}$.

4. Let $\eta_t = e^{-1/t}$ for $t > 0$ and $\eta(t) = 0$ for $t \le 0$. Prove the following:
   (1) If $k \in \mathbb{N}$ and $t > 0$ then $\eta^{(k)}(t) = P_k(1/t)e^{-1/t}$, where P_k is a polynomial of degree $2k$;
   (2) $\eta^{(k)}(0)$ exists and equals zero for all $k \in \mathbb{N}$.

5. If $f \in L^\infty(\mathbb{R}^n)$ be such that $\|\tau_y f - f\|_\infty \to 0$ as $y \to 0$. Prove that f agrees a.e. with a uniformly continuous function. (See the hint for Exercise 4 on page 239.)

6. Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $g \in C^k_{\rm c}(\mathbb{R}^n)$. Prove that $f * g \in C^k(\mathbb{R}^n)$ and $\partial^\alpha(f * g) = f * \partial^\alpha g$ for all α with $|\alpha| \le k$.