Math 240B: Real Analysis, Winter 2020

Homework Assignment 5

Due Wednesday, February 19, 2020

1. Let H be a Hilbert space. Prove the following:
   (1) If M is a closed subspace of H then $(M^\perp)^\perp = M$;
   (2) If E is a (nonempty) subset of H then $(E^\perp)^\perp = \overline{\text{Span}\,(E)}$.

2. Let H be a Hilbert space, M a closed and convex subset of H, and $x \in H \setminus M$. Prove that there exists a unique $y \in M$ such that $\|x - y\| = \inf_{z\in M} \|x - z\|$. Moreover, y is the unique element in M such that $\text{Re}\,\langle x - y, z - y\rangle \le 0$ for any $z \in M$.

3. Let M be a closed subspace of a Hilbert space H and $x_0 \in H \setminus M$. Prove that

$$\min\{\|x - x_0\| : x \in M\} = \max\{|\langle x_0, y\rangle| : y \in M^\perp, \|y\| = 1\}.$$

4. Suppose $x_n \to x$ strongly and $y_n \to y$ weakly in a Hilbert space. Prove that $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

5. Let H be a Hilbert space. Let $T : H \to H$ be a bounded linear and self-adjoint operator. Prove that $\|T\| = \sup\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\}$.

6. Let M be a closed subspace of a Hilbert space H.
   (1) For any $x \in H$, let $Px \in M$ be the unique element in M such that $x - Px \in M^\perp$. Prove that $P \in L(H,H)$, $P^* = P$ and $P^2 = P$, Range $(P) = M$, and Kernel $(P) = M^\perp$. (P is called the orthogonal projection onto M.)
   (2) Suppose $P \in L(H,H)$ satisfies $P^* = P$ and $P^2 = P$. Prove that Range (P) is a closed subspace of H and P is the orthogonal projection onto Range (P).

7. Let H be an infinitely-dimensional Hilbert space. Prove the following:
   (1) Any infinite sequence of orthonormal vectors in H is bounded, converges to 0 weakly, and is not pre-compact;
   (2) The unit sphere $S = \{x \in H : \|x\| = 1\}$ is weakly dense in the closed unit ball $B = \{x \in H : \|x\| \le 1\}$. (In fact, every $x \in B$ is the weak limit of a sequence of points in S.)

8. Let H be an infinitely-dimensional Hilbert space, $\{u_n\}_{n=1}^\infty$ an orthonormal basis for H, $\delta_n \in (0,\infty)$ $(n = 1, 2, \dots)$, and $S = \{\sum_{n=1}^\infty c_n u_n : |c_n| \le \delta_n\,(n = 1, 2, \dots)\}$. Prove that S is compact in H if and only if $\sum_{n=1}^\infty \delta_n^2 < \infty$.