Math 240B: Real Analysis, Winter 2020

Homework Assignment 2

Due Wednesday, January 22, 2020

Unless otherwise stated, $(X, \mathcal{M}, \mu)$ denotes a measure space.

1. Prove the following:
   (1) The space $L^\infty(\mu)$ of all essentially bounded functions (with two functions being equal if and only if they are equal μ-a.e.) equipped with $\|\cdot\|_\infty$ is a Banach space;
   (2) If $f,\ f_n \in L^\infty(\mu)$ $(n = 1, 2, \dots)$, then $f_n \to f$ in $L^\infty(\mu)$ if and only if there exists $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E;
   (3) Simple functions are dense in $L^\infty(\mu)$.

2. Assume $f \in L^p(\mu) \cap L^\infty(\mu)$ for some $p \in [1, \infty)$. Prove that $f \in L^q(\mu)$ for any $q \in (p, \infty)$ and that $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$.

3. Prove that the space $L^p(\mathbb{R}^n, m)$ (where m denotes the Lebesgue measure) is separable if $1 \le p < \infty$ but not separable if $p = \infty$. (This is Exercise 13 on page 187 of the textbook. See some hints there.)

4. Suppose $\mu(X) = 1$ and $f \in L^p(\mu)$ for some $p > 0$. Prove the following:

   (1) $\quad f \in L^q(\mu) \quad \forall q \in (0, p)$;

   (2) $\quad \log \|f\|_q \ge \int \log |f| \, d\mu$;

   (3) $\quad \frac{1}{q}\left(\int |f|^q d\mu - 1\right) \ge \log \|f\|_q \quad$ and $\quad \lim_{q\to 0^+} \frac{1}{q}\left(\int |f|^q d\mu - 1\right) = \int \log |f| \, d\mu$;

   (4) $\quad \lim_{q\to 0^+} \|f\|_q = \exp\left(\int \log |f| \, d\mu\right)$.

5. Let $1 \le p < \infty$. Prove the following:
   (1) If $f_n \to f$ in $L^p(\mu)$, then $f_n \to f$ in measure, and hence there exists a subsequence of $\{f_n\}_{n=1}^\infty$ converging to f μ-a.e.
   (2) If $f_n \to f$ in measure and $|f_n| \le g$ on X for some $g \in L^p(\mu)$ $(n = 1, 2, \dots)$, then $\|f_n - f\|_p \to 0$.

6. Assume $1 \le p < \infty$, all $f_n, f \in L^p(\mu)$ $(n = 1, 2, \dots)$, and $f_n \to f$ μ-a.e. Prove that $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$.

7. Let $g \in L^\infty(\mu)$ and $1 < p < \infty$. Prove that the operator $T : L^p(\mu) \to L^p(\mu)$ defined by $Tf = fg$ is bounded on $L^p(\mu)$, and its operator norm is at most $\|g\|_\infty$ with equality if μ is semifinite.

8. Assume $1 < p < \infty$, all $f, f_n \in L^p(\mu)$, $\sup_{n\ge1} \|f_n\|_p < \infty$, and $f_n \to f$ a.e. Prove that $f_n \to f$ weakly in $L^p(\mu)$. (See hints in Exercise 20 on page 192 of the textbook.)