Homework 1

Due 11:59 PM January 17 on Gradescope

All the problems are nontrivial (some are challenging). Try to solve as many problems as you can. You can discuss with your classmates or the instructor. But you must write the solution in your own words and acknowledge all resources (people you discussed with, book you used) at the beginning of your homework. **At least one** out of the four problems will be carefully graded. **This problem is worth 10 points.** The rest problems worth 2 points each. For these problems, you will get 2 points as long as you write down some reasonable arguments.

Problem 1. MIT 18.905: Problem Set II Problem 8 (a). See `https://ocw.mit.edu/courses/mathematics/18-905-algebraic-topology-i-fall-2016/assignments/` This is about split short exact sequences.

Problem 2. Use the Mayer-Vietoris sequence to compute the homology (in all dimensions) of the torus and the Klein bottle. (Hint: Find a cover of the torus by two annuli $S^1 \times (0,1)$ and find a cover of the Klein bottle by two mobius trips.)

Problem 3. (1) Treat S^1 as the unit circle in the complex plane $\mathbb{C}$. Consider the map $f : S^1 \to S^1$ defined by $f(z) = z^n$. Show that $\deg(f) = n$. (Hint: You may use the natural identification between the first homology group and the fundamental group.)

(2) For $m \geqslant 1$, let treat S^{m+1} as the unit sphere in $\mathbb{C} \oplus \mathbb{R}^m$. Namely, we set

$$S^{m+1} := \{(z, \vec{x}) \in \mathbb{C} \oplus \mathbb{R}^m : \ |z|^2 + |\vec{x}|^2 = 1\}$$

consider the map $f : S^{m+1} \to S^{m+1}$ defined by $f(z, \vec{x}) = (z^n, \vec{x})$. Show that $\deg(f) = n$. (Hint: Prove by induction. Decompose S^{m+1} into $D^{m+1} \cup_{S^m} D^{m+1}$ and use naturality of the Mayer-Vietoris sequence.)

Problem 4. (1) Let f, g be two continuous maps from some space X to $S^n = \{\vec{x} \in \mathbb{R}^{n+1} : |\vec{x}| = 1\}$. Suppose we have $f(p) \neq -g(p)$ for any $p \in X$. Show that f is homotopic to g.

(2) Consider the antipodal map $l : S^n \to S^n$ defined by $l(\vec{x}) = -\vec{x}$. Compute the mapping degree of l and show that l is not homotopic to 1_{S^n} if n is even.

(3) Let $\vec{x}$ be a point on S^n. A **tangent vector** of S^n at $\vec{x}$ is a vector $\vec{v} \in \mathbb{R}^{n+1}$ that satisfies $\vec{x} \cdot \vec{v} = 0$. A **vector field** on S^n is a continuous map $f : S^n \to \mathbb{R}^{n+1}$ such that $f(\vec{x}) \cdot \vec{x} = 0$ for any $\vec{x} \in S^n$. We say the vector field f is **nowhere vanishing** if $f(\vec{x}) \neq \vec{0}$ for any $\vec{x} \in S^n$.

Prove that there doesn't exist a nowhere vanishing vector field on S^n if n is a positive even integer. (Hint: Prove by contradiction. Suppose such f exists. Use it to construct a map $S^n \to S^n$ which is homotopic to both l and 1_{S^n}.)