Math 200a Fall 2020 Homework 7

Due 12/04/2020 by 7pm on Gradescope

Reading: Finish reading Chapter 8 in the text. Section 8F is optional reading, it won't be covered in the lectures or homework. Also, we did not cover all of the results in the other sections of Chapter 8 in class, and you are primarily responsible for the results covered in lecture.

The remainder of the course will cover basic ring theory. We will use Dummit and Foote instead of Isaacs as our reference text for this material; we will cover roughly Chapters 7 and 8 of D+F. However, we will also begin to produce some course notes as a running experiment.

Exercises to write up and hand in:

1. Suppose that G is finite and has the property that every maximal subgroup of G has prime index. Prove that G is solvable, in the following steps.

(a). Prove that if P is a Sylow p-subgroup of G and $N_G(P) \leq H \leq G$ for some subgroup H, then $|G : H| \equiv 1 (\mathrm{mod}\, p)$. (This part is true in any finite group.) (Hint: P is a Sylow p-subgroup of H and $N_G(P) = N_H(P)$.)

(b). Taking p to be the largest prime dividing the order of the group G, show that G has a normal Sylow p-subgroup.

(c). Conclude the proof by induction on the order.

2. Let G be finite and let P be a Sylow p-subgroup of G. Suppose that $N_G(P) \subseteq H \subseteq G$ for some subgroup H of G. Show that $N_G(H) = H$. (This generalizes Problem 5.8. Hint: use the Frattini Argument).

3. Let G be a finite group.

(a). Prove that G is nilpotent if and only if whenever $a, b \in G$ are elements with relatively prime orders, then a and b commute.

(b). Prove that the dihedral group D_{2n} is nilpotent if and only if n is a power of 2.

4. Let P be a finite p-group for a prime p. Recall from the text that the *Frattini subgroup* of a group G, denoted $\Phi(G)$, is the intersection of all of the maximal subgroups of G. Recall also that an *elementary abelian p-group* is a group which is isomorphic to a finite direct product of copies of $\mathbb{Z}_p$.

(a). Show that $P/\Phi(P)$ is an elementary abelian p-group. (Hint: Consider the natural map from P to the product of all of the P/M as M ranges over maximal subgroups of P.)

(b). Show that if N is any normal subgroup of P such that P/N is elementary abelian, then $\Phi(P) \subseteq N$. Thus $\Phi(P)$ is the uniquely smallest normal subgroup with the property that factoring it out gives an elementary abelian p-group.

5. Let R be a commutative ring, and consider the ring $R[[x]]$ of formal power series in one variable.

(a). Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in the ring $R[[x]]$ if and only if a_0 is a unit in R.

(b). Suppose that R is a field. Show that the set of power series in $R[[x]]$ whose constant term is 0 is a maximal ideal I of $R[[x]]$. Prove that I is the unique maximal ideal of $R[[x]]$. (remark: a ring with a unique maximal ideal is called *local*.)

6. Recall that a division ring is a ring such that every nonzero element of the ring is a unit. Show that D is a division ring if and only if the only *left* ideals of D are 0 and D.

7. Let R be a ring, and consider the matrix ring $M_n(R)$ for some $n \geq 1$. Given an ideal I of R, let $M_n(I)$ be the set of matrices (a_{ij}) such that $a_{ij} \in I$ for all i, j.

Show that every ideal of $M_n(R)$ is of the form $M_n(I)$ for some ideal I of R. Conclude that if R is a division ring, then $M_n(R)$ is a *simple ring*, that is, that $\{0\}$ and $M_n(R)$ are the only ideals of $M_n(R)$. Show however that $M_n(R)$ is not itself a division ring when $n \geq 2$.