Math 240 A, Fall 2019
Solution to Problems of HW #8
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1. Since $f \in L^1(\mathbb{R}^n)$ and $f \neq 0$, $0 < \int_{\mathbb{R}^n} |f|\, dm < \infty$. By the Monotone Convergence Theorem,
$$\lim_{r\to\infty} \int_{B(0,r)} |f|\, dm = \int_{\mathbb{R}^n} |f|\, dm,$$
there exists $R > 0$, such that $0 < \int_{B(0,R)} |f|\, dm < \infty$.

Let $x \in \mathbb{R}^n$ with $|x| > R$. Then, $B(0,R) \subseteq B(x, 2|x|)$. Hence
$$
\begin{aligned}
(Hf)(x) &\geq A_{2|x|}|f|(x) \\
&= \frac{1}{m(B(x,2|x|))} \int_{B(x,2|x|)} |f|\, dm \\
&\geq \frac{1}{m(B(0,2|x|))} \int_{B(0,R)} |f|\, dm \\
&= \frac{1}{|x|^n m(B(0,2))} \int_{B(0,R)} |f|\, dm \\
&= \frac{C}{|x|^n},
\end{aligned}
$$
where $C = \frac{1}{m(B(0,2))} \int_{B(0,R)} |f|\, dm > 0$.

Now, let $0 < \alpha < \frac{C}{2R^n}$. Then, for $R < |x| < \left(\frac{C}{\alpha}\right)^{\frac{1}{n}}$, we have
$$(Hf)(x) \geq \frac{C}{|x|^n} > \frac{C}{\left(\frac{C}{\alpha}\right)} = \alpha.$$
Thus, $\{Hf > \alpha\} \supseteq \{R < |x| < \left(\frac{C}{\alpha}\right)^{1/n}\}$ and
$$
\begin{aligned}
m(\{Hf > \alpha\}) &\geq m\left(\{R < |x| < \left(\tfrac{C}{\alpha}\right)^{1/n}\}\right) \\
&= \left(\frac{C}{\alpha} - R^n\right) m(B(0,1))
\end{aligned}
$$

$$\geq \frac{c\, m(B(0,1))}{2\alpha}$$

Since $\frac{c}{\alpha} - R^n \geq \frac{c}{2\alpha} \Longleftrightarrow \frac{c}{2\alpha} > R^n$ which is true with our choice of $\alpha$ : $0<\alpha<\frac{c}{2R^n}$.

Now, set $C' = \frac{c}{2} m(B(0,1)) > 0$; then

$$m(\{Hf > \alpha\}) \geq \frac{C'}{\alpha} \quad \text{if } 0 < \alpha < \frac{c}{2R^n}.$$

2. $\forall \varepsilon > 0$. Since $F$ is continuous at $x$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ provided that $|x-y| < \delta$. Thus, if $0 < r < \delta$ then $|f(y)-f(x)| < \varepsilon$ if $y \in B(x,r)$. Hence,

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy$$

$$\leq \frac{1}{m(B(x,r))} \int_{B(x,r)} \varepsilon\, dy$$

$$= \varepsilon.$$

Thus, $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)|\, dy = 0$,

and hence $x \in L_f$ (the Lebesgue set of $f$), i.e., $x$ is a Lebesgue point of $f$.

3. Let $x \in L_f$. Note that $|f| \in L^1_{loc}(\mathbb{R}^n)$. Moreover,

$$\left| \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy - |f(x)| \right|$$

$$= \left| \frac{1}{m(B(x,r))} \int_{B(x,r)} \left[ |f(y)| - |f(x)| \right] dy \right|$$

$$\leq \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\,dy$$

$$\longrightarrow 0 \quad \text{as } r\to 0.$$

Thus, $x$ is also a Lebesgue point of $|f|$.

Now, we have

$$(Hf)(x) = \sup_{r>0} A_r|f|(x)$$

$$= \sup_{r>0} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)|\,dy$$

$$\geq \lim_{r\to 0} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)|\,dy$$

$$= |f(x)|.$$

4. (1) Let $f=\chi_E \in L^1_{loc}(\mathbb{R}^n)$. Then $\forall x\in L_f$

$$D_E(x) = \lim_{r\to 0} \frac{m(E\cap B(x,r))}{m(B(x,r))}$$

$$= \lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)} f\,dm$$

$$= f(x)$$

$$= \chi_E(x).$$

Then $D_E(x)=1 \quad \forall x\in E\cap L_f$

$D_E(x)=0 \quad \forall x\in E^c\cap L_f$.

But, $m(L_f^c)=0$. Hence $m(E\cap L_f^c)=0$ and $m(E^c\cap L_f^c)=0$. Thus for m-a.e. $x\in E$, $D_E(x)=1$ and for m-a.e. $x\in E^c$, $D_E(x)=0$.

(2) Consider $\mathbb{R}^2$. Let $x=(0,0)$ Define

$$E=\{(r,\theta): r>0,\ 0<\theta<2\pi\alpha\}$$

(using the polar coordinates). Then $\forall \tilde{r}>0$

$$\frac{m(E\cap B(0,\tilde{r}))}{m(B(0,\tilde{r}))}=\frac{\frac{1}{2}2\pi\alpha\tilde{r}^2}{\pi\tilde{r}^2}=\alpha$$

Hence $D_E(x)=\alpha$.

Now consider $\mathbb{R}^1$. Let $E=\bigcup_{n=1}^{\infty}(2^{-n},2^{-n}+2^{-n-1})$.

Then, for $N\in\mathbb{N}$, we have

$$E\cap B(0,2^{-N})=\bigcup_{n=N+1}^{\infty}(2^{-n},2^{-n}+2^{-n-1}) \text{ disjoint}$$

So
$$\frac{m(E\cap B(0,2^{-N}))}{m(B(0,2^{-N}))}=\frac{\sum_{n=N+1}^{\infty}2^{-n-1}}{2\cdot 2^{-N}}=\frac{2^{-N-1}}{2^{-N+1}}=\frac{1}{4}.$$

$$E\cap B(0,2^{-N}+2^{-N-1})=\left(\bigcup_{n=1}^{\infty}(2^{-n},2^{-n}+2^{-n-1})\right)\cup(2^{-N},2^{-N}+2^{-N-1})$$

disjoint. Hence,

$$\frac{m(E\cap B(0,2^{-N}+2^{-N-1}))}{m(B(0,2^{-N}+2^{-N-1}))}=\frac{\left(\sum_{n=N+1}^{\infty}2^{-n-1}\right)+2^{-N-1}}{2(2^{-N}+2^{-N-1})}=\frac{2^{-N}}{3\cdot 2^{-N}}=\frac{1}{3}.$$

Thus $D_E(0)$ does not exist.

5. Clearly $F$ and $G$ are differentiable at any $x\neq 0$.

$$F'(0)=\lim_{x\to 0}\frac{F(x)-F(0)}{x}=\lim_{x\to 0}\frac{F(x)}{x}$$
$$=\lim_{x\to 0}x\sin\frac{1}{x}=0, \text{ exists.}$$

Similarly $G'(0)=\lim_{x\to 0}x\sin\frac{1}{x^2}=0$, exists.

If $x\neq 0$, then

$$F'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

$$|F'(x)|\le 2\left|\frac{\sin\frac{1}{x}}{\frac{1}{x}}\right|+\left|\cos\frac{1}{x}\right|\le 2+1=3$$

Thus $F\in BV([-1,1])$.

We show that $G \notin BV([-1,1])$. In fact, we have $G \notin BV([0,1])$. Define

$$x_n = \sqrt{\frac{1}{2n\pi + \frac{\pi}{4} - (-1)^n \frac{\pi}{4}}} \quad (n=1,2,\cdots).$$

Then

$$G(x_n) = \begin{cases} \frac{1}{2n\pi + \frac{\pi}{2}} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

Thus, $1 > x_1 > \cdots > x_{2n} > 0 \quad \forall n \in \mathbb{N}$, and

$$\sum_{k=1}^{2n-1} |G(x_k) - G(x_{k+1})|$$
$$\geq \sum_{k=1}^{n} |G(x_{2k-1}) - G(x_{2k})|$$
$$= \sum_{k=1}^{n} \frac{1}{2(2k-1)\pi + \frac{\pi}{2}}$$
$$\to \infty \quad \text{as } n \to \infty.$$

Hence $G \notin BV([-1,1])$, In fact, $G \notin BV([0,1])$.

6. Disaprove. Let $f(x) = G(x)$ $(x \in [0,1])$ as in Problem #5. $f = G \notin BV([0,1])$ as shown above. Let

$$f_n(x) = \begin{cases} f(x) & \text{if } \sqrt{\frac{1}{2n\pi}} < x \leq 1, \\ 0 & \text{if } 0 \leq x \leq \sqrt{\frac{1}{2n\pi}}, \end{cases}$$

$n = 1, 2, \cdots$,

$$\sup_{0 \leq x \leq 1} |f_n(x) - f(x)| = \sup_{0 \leq x \leq \sqrt{\frac{1}{2n\pi}}} |f(x)|$$
$$= \sup_{0 \leq x \leq \sqrt{\frac{1}{2n\pi}}} \left| x^2 \sin \frac{1}{x^2} \right| \leq \frac{1}{2\pi n} \to 0$$

as $n \to \infty$. Hence $f_n \to f$ uniformly on $[0,1]$. But $f = G \notin BV([0,1])$.

7. Define $G(x) = \begin{cases} F(x^+) & \text{if } a \le x < b, \\ F(b) & \text{if } x = b. \end{cases}$

Then, $G$ is increasing on $[a,b]$, $G' = F'$ m-a.e. on $[a,b]$, and $G$ is right continuous. Moreover $G(a) \ge F(a)$.

We may extend $G$ by $G(x) = G(a)$ if $x < a$ and $G(x) = G(b)$, if $x > b$. Then, $G : \mathbb{R} \to \mathbb{R}$ is increasing, right continuous, and $G \in BV$. Let $\widehat{G}(x) = G(x) - G(a)$. $(x \in \mathbb{R})$. Then $\widehat{G} \in NBV$. Let $\mu_{\widehat{G}}$ be the complex Borel measure on $\mathbb{R}$ associated with $\widehat{G}$. Then $\mu_{\widehat{G}}$ is a finite positive measure, $d\mu_{\widehat{G}} = d\lambda + \widehat{G}'dm = d\lambda + G'dm$. Here $\lambda \ge 0$ is finite, $\widehat{G}' \ge 0$ and $\widehat{G}' \overset{a.e.}{=} G' \in L^1(m)$. Thus

$$F(b) - F(a) \ge G(b) - G(a) = \widehat{G}(b) - \widehat{G}(a)$$

$$= \mu_{\widehat{G}}((a,b]) = \lambda((a,b]) + \int_a^b G'dm$$

$$\ge \int_a^b G'dm = \int_a^b F'dm.$$

8. Define $\widehat{F}(x) = \begin{cases} 0 & \text{if } x < a, \\ F(x) - F(a) & \text{if } a \le x \le b, \\ F(b) - F(a) & \text{if } x > b, \end{cases}$

and $\widehat{G}(x) = \begin{cases} 0 & \text{if } x < a, \\ G(x) - G(a) & \text{if } a \le x \le b, \\ G(b) - G(a) & \text{if } x > b. \end{cases}$

Since $F, G \in BV([a,b])$, $\widehat{F}, \widehat{G} \in BV(\mathbb{R}^1)$. In fact, $TV(\widetilde{F}; \mathbb{R}^1) = TV(F; [a,b]) < \infty$, $TV(\widehat{G}; \mathbb{R}^1) = TV(G; [a,b]) < \infty$. Since $\widehat{F}(-\infty) = 0$, $\widehat{G}(-\infty) = 0$, and both $\widehat{F}, \widehat{G}$ are continuous, $\widehat{F}, \widehat{G} \in NBV$. Let $\mu_{\widehat{F}}, \mu_{\widehat{G}}$ be the complex Borel measures associated with $\tilde{F}$, $\tilde{G}$, respectively. Since $\tilde{F}, \tilde{G} \in NBV$ and they are absolutely continuous, $d\mu_{\tilde{F}} = \tilde{F}' dm$, and $d\mu_{\tilde{G}} = \tilde{G}' dm$. $\tilde{F}', \tilde{G}' \in L^1(m)$. Hence

$$\int_{(a,b]} \tilde{F}\, d\tilde{G} = \int_{(a,b]} F\, d\mu_{\tilde{G}} = \int_{(a,b]} F G' dm = \int_a^b F G' dm.$$

Similarly $\int_{(a,b]} \tilde{G}\, d\tilde{F} = \int_a^b G F' dm$.

Thus, by Thm 3.36,

$$\begin{aligned} &F(b)G(b) - F(a)G(a) \\ &= \tilde{F}(b)\tilde{G}(b) - \tilde{F}(a)\tilde{G}(a) \\ &= \int_{(a,b]} \tilde{F}\, d\tilde{G} + \int_{(a,b]} \tilde{G}\, d\tilde{F} \\ &= \int_a^b (FG' + GF')\, dm. \end{aligned}$$

9. Assume $\exists L \geq 0$ s.t. $|F(x) - F(y)| \leq L|x-y|$ $\forall x, y \in \mathbb{R}$. $\forall \varepsilon > 0$. Let $\delta = \frac{\varepsilon}{L} > 0$. Then, if $(a_j, b_j)$ are disjoint, finite intervals $(j = 1, \cdots, J)$, then

$$\sum_1^J (b_j - a_j) < \delta \Longrightarrow \sum_1^J |F(b_j) - F(a_j)| \leq L \sum_1^J |b_j - a_j| < L\delta = \varepsilon.$$

Thus $F$ is absolutely continuous. Hence, $F'$ exists $m$-a.e. on any finite interval, and hence $F'$ exists $m$-a.e. on $\mathbb{R}$. If $x \in \mathbb{R}$ and $F'(x)$ exists, then $|F'(x)| = \lim_{y \to x} \left| \frac{F(y) - F(x)}{y - x} \right| \leq L$.

Conversely, assume $F$ is absolutely continuous and $|F'| \le L$ $m$-a.e. for some $L \ge 0$. Let $-\infty < a < b < \infty$. We show that $|F(b)-F(a)| \le L(b-a)$.

Clearly $F \in BV[a,b] \cap AC([a,b])$. Define

$$\tilde{F}(x) = \begin{cases} 0 & \text{if } x < a, \\ F(x)-F(a) & \text{if } a \le x \le b, \\ F(b)-F(a) & \text{if } x > b. \end{cases}$$

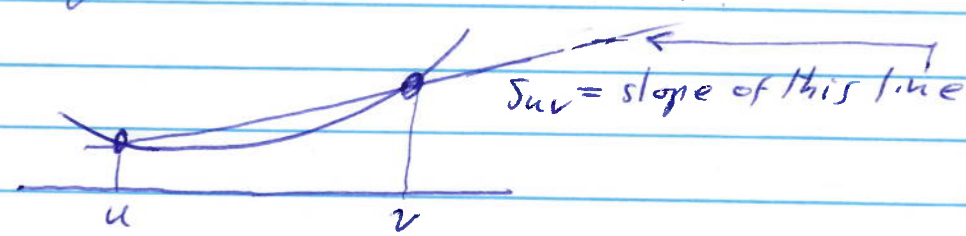
Then $\tilde{F} \in NBV$ and $\tilde{F}$ is AC on $\mathbb{R}$. If $\mu_{\tilde{F}}$ is the complex Borel measure on $\mathbb{R}$ associated to $\tilde{F}$, then $d\mu_{\tilde{F}} = \tilde{F}'\,dm$. Note that $\tilde{F}' = F'$ a.e. on $[a,b]$. So,

$$|F(b)-F(a)| = |\tilde{F}(b)-\tilde{F}(a)| = |\mu_{\tilde{F}}((a,b])|$$
$$= \left|\int_{(a,b]} \tilde{F}'\,dm\right| = \left|\int_a^b F'\,dm\right| \le \int_a^b |F'|\,dm$$
$$\le \int_a^b L\,dm = L|b-a|.$$

10. (1) Assume $F$ is convex on $(a,b)$ and $s, t, s', t' \in (a,b)$ with $s < s' < t$ and $s < t < t'$. We show that

$$\frac{F(t)-F(s)}{t-s} \le \frac{F(t')-F(s')}{t'-s'}.$$

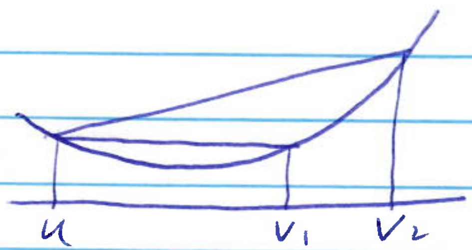
Denote for $a < u < v < b$ $S_{uv}$ = slope of line connecting $(u, F(u))$ and $(v, F(v))$



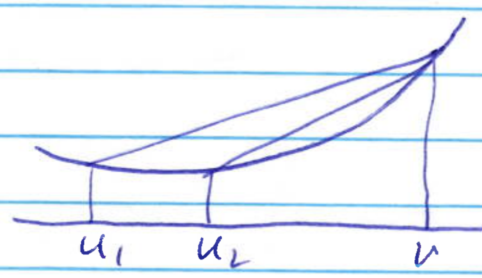
We show that the slope increases as $u$ increases or $v$ increases. i.e.,

(A) $\dfrac{F(u)-F(v_1)}{u-v_1} \le \dfrac{F(u)-F(v_2)}{u-v_2}$ if $a<u<v_1<v_2<b$,

(B). $\dfrac{F(u_1)-F(v)}{u_1-v} \le \dfrac{F(u_2)-F(v)}{u_2-v}$ if $a<u_1<u_2<v<b$.




(A)



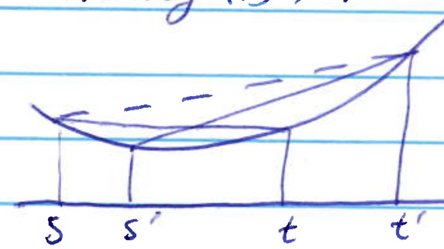

(B)

Note that (A) is equivalent to

$$F(v_1) \le \frac{v_2-v_1}{v_2-u} F(u) + \frac{v_1-u}{v_2-u} F(v_2),$$

which is true, since $F$ is convex, $v_1 = \lambda u + (1-\lambda) v_2$ with $\lambda = \frac{v_2-v_1}{v_2-u} \in (0,1)$. Similarly (B) is true.

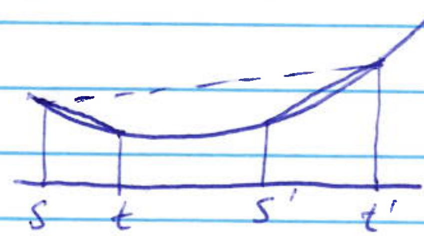
Now, if $a<s<s'<t<t'<b$, then




$$S_{st} \le S_{st'} \le S_{s't'}$$

as desired.

If $a<s<t<s'<t'<b$, then




$$S_{st} \le S_{st'} \le S_{s't'}$$

as desired.

Conversely, for any $u, v \in (a,b)$, $u<v$, and any $\lambda \in (0,1)$. Let $w = \lambda u + (1-\lambda) v \in (u,v) \subset (a,b)$. Setting $u=s$, $v=t'$, $w=s'=t$, we get

$$\frac{F(w)-F(u)}{w-u} \le \frac{F(v)-F(w)}{v-w},$$

which is the same as

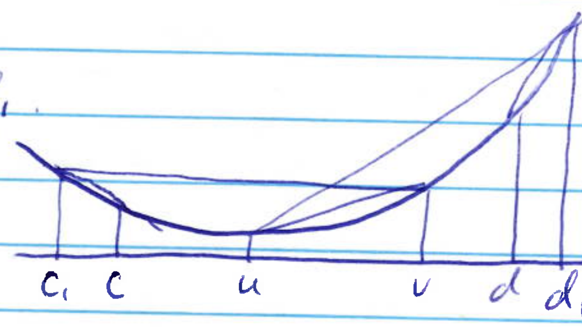
$$F(w) \le \frac{v-w}{v-u} F(u) + \frac{w-u}{v-u} F(v)$$
$$= \lambda F(u) + (1-\lambda) F(v).$$

Hence $F$ is convex on $(a,b)$.

(2) Assume $F$ is convex on $(a,b)$ and $[c,d] \subset (a,b)$. Choose $c_1, d_1$ so that $a < c_1 < c < d < d_1 < b$. Then for any $u, v$ $c \le u < v \le d$, we have the increasing property of the slope as proved in part (1) that

$$S_{c_1 c} \le S_{c_1 v} \le S_{uv} \le S_{u d_1} \le S_{d d_1}.$$

Let $M = \max(|S_{c_1 c}|, |S_{d d_1}|)$. Then




$$\left| \frac{F(v)-F(u)}{v-u} \right| \le M.$$

i.e., $|F(v)-F(u)| \le M|v-u|$ $\forall u,v:$ $c \le u < v \le d$.

Now it follows from the definition that $F$ is absolutely continuous on $[c,d]$. In particular $F$ is continuous on $(a,b)$.

Suppose now $a < x < y < b$ and $F'(x)$, $F'(y)$ exist. Then, the slope $S_{xz} \le S_{xy} \le S_{zy}$ for any $z \in (x,y)$. Thus

$$F'(x) = \lim_{z \to x} S_{xz} \le S_{xy}.$$

$$F'(y) = \lim_{z \to y} S_{zy} \ge S_{xy}.$$

Hence $F'(x) \le F'(y)$.

Conversely, assume that $F$ is absolutely continuous on any compact subinterval of $(a,b)$ and $F'$ increasing on the set where it is defined. Let $a < x < y < b$ and $\lambda \in (0,1)$. Let $z = \lambda x + (1-\lambda) y$. Denote

$$m = \sup\{F'(t) : t \in [x, z],\ F'(t) \text{ exists}\}$$

$$M = \inf\{F'(t) : t \in [z, y],\ F'(t) \text{ exists}\}.$$

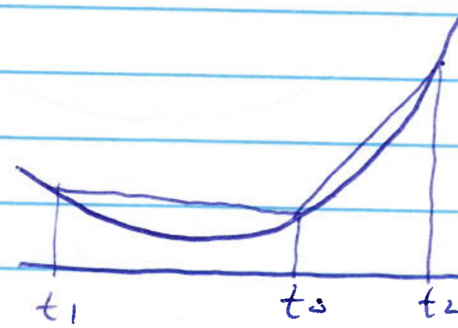
Then

$$\frac{F(z) - F(x)}{z - x} = \frac{1}{z-x}\int_x^z F'(t)\,dt \le \frac{1}{z-x}\int_x^z m\,dt = m$$

$$\le M = \frac{1}{y-z}\int_z^y M\,dt \le \frac{1}{y-z}\int_z^y F'(t)\,dt = \frac{F(y)-F(z)}{y-z}.$$

Thus. $$F(z) \le \frac{y-z}{y-x}F(x) + \frac{z-x}{y-x}F(y) = \lambda F(x) + (1-\lambda)F(y).$$

Hence $F$ is convex.




(3) If $a < t_1 < t_0 < t_2 < b$, then the slope $S_{t_0 t_1} \le S_{t_0 t_2}$. Moreover, as shown in Par (1),

$S_{t_0 t_1}$ increases as $t_1 \in (a, t_0)$ increases,

$S_{t_0 t_2}$ decreases as $t_2 \in (t_0, b)$ decreases.

Thus, $$\beta_1 := \lim_{t_1 \to t_0^-} S_{t_0 t_1} \text{ exisits},$$

$$\beta_2 = \lim_{t_2 \to t_0^+} S_{t_0 t_2} \text{ exists},$$

and $\beta_1 \le \beta_2$.

Set $\beta = \frac{1}{2}(\beta_1 + \beta_2)$ so $\beta_1 \le \beta_2 \le \beta_2$. Let $t \in (a,b)$.
If $t = t_0$ then $F(t) - F(t_0) = \beta(t - t_0) = 0$.
If $t > t_0$ then $S_{tt_0} \ge \beta_2 \ge \beta$, i.e. $F(t) - F(t_0) \ge \beta(t - t_0)$.
If $t < t_0$ then $S_{tt_0} \le \beta_1 \le \beta$, i.e. $F(t) - F(t_0) \ge \beta(t - t_0)$
(since $t - t_0 < 0$).

(4) Let $t_0 = \int_X g\,d\mu$. Since $a < g(x) < b$ $\forall x \in X$ and $\mu(X) = 1$,

$$\int_X (b - g(x))\,d\mu(x) > 0$$

Hence $b = \int_X b\,d\mu > \int_X g\,d\mu = t_0$. Similarly $t_0 > a$.
i.e., $a < t_0 < b$. Let $x \in X$ and $t = g(x) \in (a,b)$.
By Part (b), $F(t) - F(t_0) \ge \beta(t - t_0)$ $\forall t \in (a,b)$.
where $\beta$ is a constant depending on $t_0$ but not on $t$. Thus

$$F(t_0) \le F(t) - \beta(t - t_0)$$
$$= F(g(x)) - \beta(g(x) - t_0) \quad \forall x \in X.$$

Since $\mu(X) = 1$, we get

$$F\left(\int_X g\,d\mu\right) = F(t_0) = \int_X F(t_0)\,d\mu \le \int_X F(g(x))\,d\mu(x) - \beta\left(\int_X g\,d\mu - \int_X t_0\,d\mu\right)$$
$$= \int_X F\circ g\,d\mu - \beta(t_0 - t_0) = \int_X F\circ g\,d\mu.$$