Math 240C: Real Analysis, Spring 2020

Homework Assignment 2

Due 12:00 noon, Monday, April 13, 2020

1. Let μ be a Radon measure on a locally compact Hausdorff space X. Let $\phi \in C(X, (0, \infty))$. Define
$$\nu(E) = \int_E \phi \, d\mu \qquad \forall E \in \mathcal{B}_X.$$
Define also
$$I(f) = \int f\phi \, d\mu \qquad \forall f \in C_c(X).$$
Clearly $I : C_c(X) \to \mathbb{C}$ is linear and positive. Let ν' be the Radon measure associated with I. Prove the following:
   (1) If U is an open subset of X, then $\nu(U) = \nu'(U)$;
   (2) The Borel measure ν is outer regular on all Borel sets;
   (3) $\nu = \nu'$ and hence ν is a Radon measure.

   (See Exercise 9 on page 220 for some hints.)

2. Let μ be a Radon measure on a locally compact Hausdorff space X such that $\mu(\{x\}) = 0$ for any $x \in X$. Assume $A \in \mathcal{B}_X$ such that $0 < \mu(A) < \infty$. Prove that for any $\alpha \in \mathbb{R}$ with $0 < \alpha < \mu(A)$, there exists $B \in \mathcal{B}_X$ such that $B \subseteq A$ and $\mu(B) = \alpha$.

3. Let μ be a Radon measure on a locally compact Hausdorff space X. Assume $f \in L^1(\mu)$ is real-valued. Let $\varepsilon > 0$. Prove that there exist a lower semi-continuous function g on X and an upper semi-continuous function h on X such that
$$h \le f \le g \quad \text{on } X \qquad \text{and} \qquad \int_X (g - h) \, d\mu < \varepsilon.$$

4. Prove that the Banach space $C([0, 1])$ is not reflexive.

5. Let X be a locally compact Hausdorff space and $f, f_n \in C_0(X)$ $(n = 1, 2, \dots)$. Prove that $f_n \to f$ weakly in $C_0(X)$ if and only $\sup_{n\ge1} \|f_n\| < \infty$ and $f_n(x) \to f(x)$ for all $x \in X$.

6. Let $\mu_n = (1/n) \sum_{k=1}^n \delta_{k/n}$ $(n = 1, 2, \dots)$, where δ_a denotes the Dirac measure concentrated at $a \in \mathbb{R}$. Let m denote the Lebesgue measure. Denote
$$\langle f, \mu \rangle = \int_X f \, d\mu$$
for any Randon measure μ on $[0, 1]$ and any $f \in C([0, 1]$.
   (1) Prove that $\langle f, \mu_n \rangle \to \langle f, m \rangle$ for any $f \in C([0, 1])$.
   (2) Is it true that $\mu_n(E) \to m(E)$ for any Borel set $E \subseteq [0, 1]$?