Math 240A: Real Analysis, Fall 2019

Homework Assignment 1

Due Friday, October 4, 2019

1. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets. Define

$$\limsup_{n\to\infty} E_n = \{x : x \in E_n \text{ for infinitely many } n\},$$
$$\liminf_{n\to\infty} E_n = \{x : x \in E_n \text{ for all but finitely many } n\}.$$

Prove that

$$\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \quad \text{and} \quad \liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

2. Let X and Y be two sets and $f : X \longrightarrow Y$ a mapping. Let $\{Y_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of subsets of Y. Prove

$$f^{-1}\left(\bigcup_{\alpha\in\mathcal{A}} Y_\alpha\right) = \bigcup_{\alpha\in\mathcal{A}} f^{-1}(Y_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha\in\mathcal{A}} Y_\alpha\right) = \bigcap_{\alpha\in\mathcal{A}} f^{-1}(Y_\alpha).$$

3. Find a bijection from $\mathbb{N}$ to $\mathbb{N}^2$.

4. Construct a sequence of open sets U_n $(n = 1, 2, \dots)$ of $\mathbb{R}$ such that $\cap_{n=1}^{\infty} U_n$ is not open.

5. Let X be a complete metric space and E a non-empty subset of X. Prove that E is closed if and only if E is complete.

6. Let $(Y, \mathcal{B})$ be a measurable space and X a nonempty set. For any $f : X \to Y$, define $\mathcal{A} = \{f^{-1}(B) : B \in \mathcal{B}\}$. Prove that $\mathcal{A}$ is a σ-algebra of subsets of X.

7. An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e., if $E_n \in \mathcal{A}$ for all $n = 1, 2, \dots$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\cup_{n=1}^{\infty} E_n \in \mathcal{A}$.)

8. Does there exist an infinite σ-algebra which has only countably many members? If yes, provide an example. If no, prove it.

9. Let X be a nonempty set and $\mathcal{E}$ a class of subsets of X. Let $\mathcal{M}$ be the σ-algebra of subsets of X generated by $\mathcal{E}$. Prove that $\mathcal{M}$ is the union of the σ-algebra generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

10. Prove Part c and Part d of Proposition 1.2 of the textbook.