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A criterion for detecting m-regularity

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Let $S=k[x_1, \ldots, x_n]$ be a polynomial ring over an infinite field k, and let I be a homogeneous ideal of S.

An algorithm for computing the (first) syzygies of I is due independently to Spear [Spe 77] and Schreyer [Sch 80]: One chooses an ordering on the monomials of S, and then constructs a monomial ideal $\mathrm{in}(I)$ generated by the lead terms of all elements of I. $\mathrm{in}(I)$ can be viewed as the limit of I under the action of a 1-parameter subgroup of GL(n) on the Hilbert scheme [Bay 82], so $\mathrm{in}(I)$ occurs as the special fiber of a flat family whose general fiber is isomorphic to I. It follows from a well-known criterion for flatness [Art 76] that each syzygy of $\mathrm{in}(I)$ can be lifted to a syzygy of I; the set of syzygies thus obtained can be trimmed to give a complete set of minimal syzygies of I.

The monomial ideal $\mathrm{in}(I)$ was first studied by Macaulay [Mac 27]; an algorithm for its construction was first given by Buchberger [Buc 65], [Buc 76]. $\mathrm{in}(I)$ is studied in [Hir 64], [Bri 73], [Gal 74], [Gal 79] as part of an analogous division process for power series rings.

The following problem arises in using this syzygy algorithm in practice: $\mathrm{in}(I)$ can have minimal generators and syzygies in degrees higher than any minimal generator or syzygy of I. In this situation, computations in these higher degrees are unnecessary; one should compute the generators and syzygies of $\mathrm{in}(I)$ in only those degrees necessary to find all minimal syzygies of I.

In order to modify the syzygy algorithm to take advantage of this observation, one would like a criterion for determining when all minimal syzygies of I have been found. This problem appears to be intractable at present. However, the question of bounding the degrees of the minimal j^{th} syzygies of I, for all j, is tractable. Recall that I is defined to be m-regular if the j^{th} syzygy module of I is generated in degrees $\leqq m+j$, for all $j \geqq 0$ ([Mum 66], [EiGo 84]). The regularity of I, $\mathrm{reg}(I)$, is defined to be the least m for which I is m-regular. We have $\mathrm{reg}(\mathrm{in}(I)) \geqq \mathrm{reg}(I)$, because regularity is upper-semicontinuous in flat families. When $\mathrm{reg}(\mathrm{in}(I)) > \mathrm{reg}(I)$, computations in degrees $> \mathrm{reg}(I)+1$ are un-

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necessary. One would therefore like a criterion for determining when I is m-regular.

We give in §1 a criterion for I to be m-regular, which depends only on computations in the finite vector spaces S_m and S_{m+1} of polynomials of degrees m, $m+1$: If one can find $h_1, \ldots, h_j \in S_1$, so that the subspaces $((I, h_1, \ldots, h_{i-1}) : h_i)_m$ and $(I, h_1, \ldots, h_{i-1})_m$ are equal for $1 \leqq i \leqq j$, and $(I, h_1, \ldots, h_j)_m = S_m$, then I is m-regular. Furthermore, if I is m-regular, then a generic choice of $h_1, \ldots, h_j \in S_1$ will satisfy these conditions.

One could use this result to terminate syzygy computations early, in cases where $\operatorname{reg}(\operatorname{in}(I)) > \operatorname{reg}(I)$. However, further study reveals a close connection between this result and a particular order on the monomials of S, the reverse lexicographic order. This order is used to compute saturations in [Bay 82]. The reverse lexicographic order was then studied in characteristic zero, in generic coordinates, by several authors. It is observed in [Laz 83] that under this hypothesis in low dimensions, the generators of $\operatorname{in}(I)$ are of particularly low degree. In [Giu 84], this hypothesis is further studied, and a worst-case upper bound on the degrees of generators of $\operatorname{in}(I)$ is obtained, which improves Hermann's corresponding bound for ideal membership [Her 26]. In [Ang 84], independent of a preliminary version of our results, it is shown that under this hypothesis, the maximum of the degrees of the generators of $\operatorname{in}(I)$ is equal to a quantity which agrees with the regularity of I.

In §2, we show that for the reverse lexicographic order and generic coordinates, $\operatorname{reg}(\operatorname{in}(I)) = \operatorname{reg}(I)$ in any characteristic. This result generalizes many of the preceding results; for example, it implies that under this hypothesis, $\operatorname{in}(I)$ is generated by elements of degree $\leqq \operatorname{reg}(I)$ in any characteristic. This result also establishes the optimality of the reverse lexicographic order in generic coordinates, since for any order, $\operatorname{reg}(\operatorname{in}(I)) > \operatorname{reg}(I)$.

We also show that in characteristic zero and in generic coordinates, $\operatorname{in}(I)$ has a minimal generator of degree $\operatorname{reg}(I)$, so $\operatorname{reg}(I)$ is equal to the highest degree of a minimal generator of $\operatorname{in}(I)$. Thus the regularity of the ideal I arises naturally in studying the relationship between I and $\operatorname{in}(I)$.

We have seen that for the syzygy algorithm, the only unnecessary computations which appear to be systematically avoidable are those in degrees $> \operatorname{reg}(I)+1$. The results in §2 provide a theoretical justification of the observation that in practice, with the reverse lexicographic order this algorithm usually terminates naturally by degree $\operatorname{reg}(I)+1$. Thus the reverse lexicographic order appears to be an optimal choice for the computation of syzygies.

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§ 1. A criterion for *m*-regularity

In this section we prove a criterion for a homogeneous ideal to be m-regular (Theorem (1.10)).

For a discussion of m-regularity see [Mum 66] or [EiGo 84]. We shall be using graded local cohomology instead of sheaf cohomology: Let $\mathcal{M}$

$=(x_1, \ldots, x_n)$ be the irrelevant maximal ideal, and let M be a graded S-module. $H^i_{\mathcal{M}}(M)_d$ will denote the degree d part of the i^{th} local cohomology group of M. For properties of local cohomology, see [EiGo 84].

(1.1) *Definition.* A homogeneous ideal I is m-regular if equivalently

(a) There exists a free resolution

$$0 \to \oplus_j S(-e_{rj}) \to \ldots \to \oplus_j S(-e_{1j}) \to \oplus_j S(-e_{0j}) \to I \to 0$$

of I, with $e_{ij} - i \leqq m$ for all i, j.

(b) $H^i(\mathbf{P}^n, \mathfrak{I}(d)) = (0)$ for all $i \geqq 1$ and $d \geqq m - i$, where $\mathfrak{I}$ is the ideal sheaf on $\mathbf{P}^n$ associated to $IS[z]$, for a new variable z.

(c) $H^i_{\mathcal{M}}(I)_d = (0)$ for all i, and $d \geqq m - i + 1$.

The regularity of I is the least m for which I is m-regular.

The equivalence of these conditions follows easily using Serre duality. See for example [EiGo 84].

Recall that two ideals $I, J \subset S$ define the same subscheme of $\mathbf{P}^{n-1}$ if $I_d = J_d$ for all degrees $d \gg 0$; the saturation I^{sat} of I is the largest ideal in this equivalence class. If I is not saturated, then the vertex of the affine cone in $\mathbf{A}^n$ defined by I is an associated prime of I. To see this vertex projectively, we must add a new variable to S, and study the projective cone defined in $\mathbf{P}^n$. This is the motivation for the use of $S[z]$ instead of S in (1.1 b). By substituting the local cohomology modules $H^i_{\mathcal{M}}(I)$ for coherent sheaf cohomology, as in (1.1 c), we can avoid this difficulty.

(1.2) *Definition.* An ideal $I \subset S$ is m-saturated if $I_d = I^{\text{sat}}_d$ for all degrees $d \geqq m$.

(1.3) *Remark.* Since $H^1_{\mathcal{M}}(I) = H^0_{\mathcal{M}}(S/I) = I^{\text{sat}}/I$, I is m-saturated if and only if $H^1_{\mathcal{M}}(I)_d$ is zero for $d \geqq m$. Thus if I is m-regular, then I is m-saturated.

Since the field k is infinite, if $\mathcal{M}$ is not an associated prime of the ideal I, we can find a linear element $h \in S_1$ which is not a zero divisor on S/I.

(1.4) **Lemma.** *Let $I \subset S$ be a saturated ideal with* $\dim(S/I) \neq 0$, *and let* $h \in S$.

(a) *If h is not a zero-divisor on S/I, then $(I:h) = I$.*

(b) *If h is a zero-divisor on S/I, then $(I:h)_d \neq I_d$ for all degrees $d \gg 0$.*

Proof. (a) This is a restatement of the definition.

(b) Choose $f \in (I:h)$ so $f \notin I$; this can be done since h is a zero-divisor on S/I. Choose $g \in S_1$ not a zero-divisor on S/I. Then $gf \notin I$, but $gf \in (I:h)$. Iterating, we can find elements in $(I:h)_d$ which are not in I_d, for all $d \geqq \deg(f)$. $\square$

(1.5) *Definition.* Call $h \in S$ generic for I, if h is not a zero-divisor on S/I^{sat}. If $\dim(S/I) = 0$, interpret this to mean every $h \in S$ is generic for I.

For $j > 0$, define $U_j(I)$ to be the subset

$$\{(h_1, \ldots, h_j) \in S_1^j \mid h_i \text{ is generic for } (I, h_1, \ldots, h_{i-1}),\ 1 \leqq i \leqq j\}$$

of S_1^j.

Since k is infinite, the set of $h \in S_1$ which are generic for I form a nonempty Zariski open subset of S_1. $U_j(I)$ is likewise a nonempty Zariski open subset of S_1^j.

(1.6) **Lemma.** *Let $I \subset S$ be an ideal and let $h \in S$. The following conditions are equivalent:*

(a) $(I:h)_d = I_d$ *for all* $d \geqq m$.

(b) *I is m-saturated, and h is generic for I.*

Proof. (a) $\Rightarrow$ (b). Choose f of maximal degree so $f \in I^{\mathrm{sat}}$ but $f \notin I$. Then $hf \in I$, so $f \in (I:h)$. Thus $\deg(f) < m$, so I is m-saturated. If $\dim(S/I) = 0$, this proves (b). Otherwise

$$(I^{\mathrm{sat}}:h)_d = (I:h)_d = I_d = I^{\mathrm{sat}}_d \qquad \text{for all } d \geqq m.$$

By Lemma (1.4), h is not a zero-divisor on S/I^{sat}.

(b) $\Rightarrow$ (a). If $\dim(S/I) = 0$, (a) follows immediately. Otherwise using Lemma (1.4) we have

$$(I:h)_d = (I^{\mathrm{sat}}:h)_d = I^{\mathrm{sat}}_d = I_d \qquad \text{for all } d \geqq m. \quad \square$$

(1.7) **Lemma.** *Let $I \subset S$ be an ideal with* $\dim(S/I) = 0$. *The following conditions are equivalent:*

(a) *I is m-saturated.*

(b) *I is m-regular.*

(c) $I_m = S_m$.

Proof. (a) $\Leftrightarrow$ (c) since $I^{\mathrm{sat}} = S$; (b) $\Rightarrow$ (a) by remark (1.3).

(a) $\Rightarrow$ (b). $H^i_{\mathscr{M}}(I) = 0$ if $i \neq 1$ and $H^1_{\mathscr{M}}(I) = S/I$ since S/I is Artinian. Since $H^1_{\mathscr{M}}(I)_d = 0$ for $d \geqq m$, by hypothesis, it follows that I is m-regular. $\square$

The following lemma is implicit in [EiGo 84].

(1.8) **Lemma.** *Let $I \subset S$ be an ideal, and suppose $h \in S_1$ is generic for I. The following conditions are equivalent:*

(a) *I is m-regular.*

(b) *I is m-saturated, and (I, h) is m-regular.*

Proof. Suppose I is m-saturated. Let $Q = (I:h)/I$, so

$$0 \to I \to (I:h) \to Q \to 0.$$

By Lemma (1.6), $I_d = (I:h)_d$ for all $d \geqq m$, so $\dim(Q) = 0$. Thus $H^i_{\mathscr{M}}(Q) = 0$ for $i \neq 0$, and $H^0_{\mathscr{M}}(Q) = Q$. Thus by the long exact sequence for local cohomology we obtain

$$H^i_{\mathscr{M}}(I)_d \cong H^i_{\mathscr{M}}((I:h))_d \qquad \text{for } d \geqq m - i + 1 \text{ and all } i.$$

(a) $\Rightarrow$ (b). Assume that I is m-regular. By Remark (1.3), I is m-saturated. We need to show that (I, h) is m-regular.

Consider the exact sequence

$$0 \to I \cap (h) \to I \oplus (h) \to (I, h) \to 0.$$

Since $I \cap (h) = (I : h)h$, we have

$$(*) \qquad 0 \to (I : h)(-1) \to I \oplus (h) \to (I, h) \to 0.$$

From

$$H^i_{\mathcal{M}}(I \oplus (h))_d \to H^i_{\mathcal{M}}((I, h))_d \to H^{i+1}_{\mathcal{M}}((I : h))_{d-1}$$

and the isomorphisms

$$H^i_{\mathcal{M}}(I)_d \cong H^i_{\mathcal{M}}((I : h))_d = 0 \qquad \text{for } d \geqq m - i + 1 \text{ and all } i,$$

it follows that (I, h) is m-regular.

(b) $\Rightarrow$ (a). Suppose that (I, h) is m-regular and I is m-saturated. From the long exact sequence associated to $(*)$ it follows that

$$H^i_{\mathcal{M}}(I : h))_{d-1} \cong H^i_{\mathcal{M}}(I \oplus (h))_d,$$

for all i, and $d \geqq m - i + 2$. Using the isomorphisms

$$H^i_{\mathcal{M}}(I)_d \cong H^i_{\mathcal{M}}((I : h))_d \qquad \text{for } d \geqq m - i + 1, \text{ and all } i,$$

and the vanishing of cohomology for $d \gg 0$, it follows I is m-regular. □

(1.9) **Lemma.** *Let $I \subset S$ be an ideal generated in degrees $\leqq m$, and let $h \in S_1$. If (I, h) is m-regular, then $(I : h)$ is generated in degrees $\leqq m$.*

Proof. Choose a minimal set of generators for I of the form

$$f_1, \ldots, f_r, hf_{r+1}, \ldots, hf_s,$$

where $f_1, \ldots, f_r$, and h are minimal set of generators for (I, h). If $f \in (I : h)$, then

$$hf = g_1 f_1 + \ldots + g_r f_r + h(g_{r+1} f_{r+1} + \cdots + g_s f_s),$$

for some $g_1, \ldots, g_s$. Thus

$$(f - g_{r+1} f_{r+1} - \cdots - g_s f_s) h - g_1 f_1 - \ldots - g_r f_r = 0$$

is a syzygy of (I, h). Conversely, any syzygy of (I, h) yields in this way an element of $(I : h)$. Because (I, h) is m-regular, each syzygy of (I, h) can be expressed in terms of syzygies of (I, h) of degree $\leqq m + 1$. By expressing the above syzygy in this way,

$$f - g_{r+1} f_{r+1} - \cdots - g_s f_s$$

can be expressed in terms of elements of $(I : h)$ of degree $\leqq m$. Since $f_{r+1}, \ldots, f_s$ also belong to $(I : h)$, and have degrees $\leqq m$, $(I : h)$ can be generated by elements of degree $\leqq m$. □

(1.10) **Theorem.** (Criterion for m-regularity.) *Let $I \subset S$ be an ideal generated in degrees $\leqq m$. The following conditions are equivalent:*

(a) *I is m-regular,*

(b) *There exists $h_1, \ldots, h_j \in S_1$ for some $j \geqq 0$ so that*

$$((I, h_1, \ldots, h_{i-1}) : h_i)_m = (I, h_1, \ldots, h_{i-1})_m \quad \text{for } i = 1, \ldots, j,$$

and

$$(I, h_1, \ldots, h_j)_m = S_m.$$

(c) *Let $r = \dim(S/I)$. For all $(h_1, \ldots, h_r) \in U_r(I)$, and all $p \geqq m$,*

$$((I, h_1, \ldots, h_{i-1}) : h_i)_p = (I, h_1, \ldots, h_{i-1})_p \quad \text{for } i = 1, \ldots, r,$$

and

$$(I, h_1, \ldots, h_r)_p = S_p.$$

Furthermore, if $h_1, \ldots, h_j$ satisfy condition (b), *then $(h_1, \ldots, h_j) \in U_j(I)$.*

Proof. (c) $\Rightarrow$ (b) is immediate.

(b) $\Rightarrow$ (a). We induct on j. If $j = 0$, I is m-saturated by hypothesis. I is then m-regular by Proposition (1.7).

If $j > 0$, (I, h_1) is m-regular by induction, and $(I : h_1)_m = I_m$ by hypothesis. Also by induction, $(h_2, \ldots, h_j) \in U_j((I, h_1))$ for $j \geqq 2$. By Lemma (1.9), $(I : h_1)$ is generated in degrees $\leqq m$. Thus $(I : h_1)_d = I_d$ for $d \geqq m$. By Lemma (1.6), I is m-saturated, and h_1 is generic for I. By Lemma (1.8), I is m-regular. Furthermore, $(h_1, \ldots, h_j) \in U_j(I)$.

(a) $\Rightarrow$ (c). We prove (c) by induction on r. If $r = 0$, Remark (1.3) implies that $I_p = S_p$ for all $p \geqq m$, so (c) holds.

Let $(h_1, \ldots, h_r) \in U_r(I)$. By Lemma (1.8), I is m-saturated, and (I, h_1) is m-regular. By Lemma (1.6), $(I : h_1)_p = I_p$ for all $p \geqq m$.

By construction, $(h_2, \ldots, h_r) \in U_{r-1}((I, h_1))$. Since (I, h_1) is m-regular, it follows from the induction hypothesis for (I, h_1) that the remaining equalities hold. □

§ 2. The reverse lexicographic order and m-regularity

The division algorithm for $S = k[x_1, \ldots, x_n]$ is sensitive to the choice of order on the monomials of S; the following orders play special roles [Tri 78], [Bay 82], [Laz 83], [Giu 84]:

(2.1) *Definition.* Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be exponent vectors.

(a) The reverse lexicographic order on monomials of S of the same degree is defined by $x^A > x^B$ if the last nonzero entry of $A - B$ is negative.

(b) The lexicographic order on monomials of S of the same degree is defined by $x^A > x^B$ if the first nonzero entry of $A - B$ is positive.

Note that these orders agree on S_1: $x_1>x_2>\dots>x_n$. For $f\in S$, let $\mathrm{in}(f)\in S$ denote the greatest term of f for a given order on S; each of the above orders is characterized by a list of properties that hold for f iff they hold for $\mathrm{in}(f)$. In the case of the lexicographic order, $f\in k[x_i,\dots,x_n]$ iff $\mathrm{in}(f)\in k[x_i,\dots,x_n]$, for each i; its use in computing projections depends on this relationship. In the case of the reverse lexicographic order, $x_i|f$ iff $x_i|\mathrm{in}(f)$, for each i and each $f\in k[x_1,\dots,x_i]$. We shall study the consequences of this relationship here.

Given an order $>$ as above, and a homogeneous ideal I, define

$$\mathrm{in}(I)=\{\mathrm{in}(f)|f\in I\};$$

$\mathrm{in}(I)$ is the monomial ideal of initial forms of I.

(2.2) **Lemma.** *Let $>$ be the reverse lexicographic order, and choose i in the range $1\leqq i\leqq n$.*

(a) $\mathrm{in}(I,x_n,\dots,x_i)=(\mathrm{in}(I),x_n,\dots,x_i)$.

(b) *Let $x_n,\dots,x_{i+1}\in I$, and let $m\geqq 0$. Then*

$$(I:x_i)_m=I_m\Leftrightarrow(\mathrm{in}(I):x_i)_m=\mathrm{in}(I)_m.$$

(c) *Let $x_n,\dots,x_{i+1}\in I$, and let $m\geqq 0$. Suppose that $(I:x_i)_d=I_d$ for all $d\geqq m$, and that $\mathrm{in}(I,x_i)$ is generated by elements of degree $\leqq m$. Then $\mathrm{in}(I)$ is generated by elements of degree $\leqq m$.*

Proof. (a) $\mathrm{in}(I,x_n,\dots,x_i)\supset(\mathrm{in}(I),x_n,\dots,x_i)$ for any order $>$; we need to show that for the reverse lexicographic order, $\mathrm{in}(I,x_n,\dots,x_i)\subset(\mathrm{in}(I),x_n,\dots,x_i)$. Suppose that $f\in(I,x_n,\dots,x_i)$. If $x_j|\mathrm{in}(f)$ for some $j\geqq i$, then $\mathrm{in}(f)\in(x_j)\subset(\mathrm{in}(I),x_n,\dots,x_i)$. Otherwise write f as $g+h_nx_n+\dots+h_ix_i$ for $g\in I$ and $h_n,\dots,h_i\in S$. Since $\mathrm{in}(f)>\mathrm{in}(h_nx_n+\dots+h_ix_i)$ in the reverse lexicographic order, $\mathrm{in}(f)=\mathrm{in}(g)$, so $\mathrm{in}(f)\in\mathrm{in}(I)\subset(\mathrm{in}(I),x_n,\dots,x_i)$.

(b) Suppose that $(I:x_i)_m=I_m$, and that $x^A\in S_m$. If $x_ix^A\in\mathrm{in}(I)_{m+1}$, then $x_ix^A=\mathrm{in}(f)$ for some $f\in I_{m+1}$. If any of $x_n,\dots,x_{i+1}$ divide x^A, then $x^A\in\mathrm{in}(I)_m$. Otherwise, by subtracting multiples of $x_n,\dots,x_{i+1}$, we may assume that $f\in k[x_1,\dots,x_i]$. Then because $>$ is the reverse lexicographic order, $f=x_ig$ for some $g\in S_m$, with $\mathrm{in}(g)=x^A$. By hypothesis $g\in I_m$, so $x^A\in\mathrm{in}(I)_m$.

Suppose that $(\mathrm{in}(I):x_i)_m=\mathrm{in}(I)_m$. Let $x_if\in I_{m+1}$, and assume by induction that for all $g\in S_m$ so $\mathrm{in}(g)<\mathrm{in}(f)$ and $x_ig\in I_{m+1}$, $g\in I_m$. Since $x_i\,\mathrm{in}(f)=\mathrm{in}(x_if)\in\mathrm{in}(I)_{m+1}$, $\mathrm{in}(f)\in\mathrm{in}(I)_m$ by hypothesis. Write $\mathrm{in}(f)=\mathrm{in}(g)$ for some $g\in I_m$. Then $x_i(f-g)\in I_{m+1}$, and $\mathrm{in}(f-g)<\mathrm{in}(f)$, so by induction $f-g\in I_m$. Thus $f\in I_m$.

(c) Let $f\in I$ be homogeneous of degree $>m$. If any of $x_n,\dots,x_{i+1}$ divide $\mathrm{in}(f)$, then $\mathrm{in}(f)$ cannot be a minimal generator of $\mathrm{in}(I)$. Otherwise, by subtracting multiples of $x_n,\dots,x_{i+1}$, we may assume that $f\in k[x_1,\dots,x_i]$. If $x_i|\mathrm{in}(f)$, then $f=x_ig$ for some $g\in S_m$ because $>$ is the reverse lexicographic order. $g\in(I:x_i)_d$ for $d=\deg(f)-1\geqq m$, so $g\in I_d$. Thus $\mathrm{in}(f)=x_i\,\mathrm{in}(g)$ is not a minimal generator of $\mathrm{in}(I)$.

If none of $x_n,\dots,x_i$ divide $\mathrm{in}(f)$, write $\mathrm{in}(f)=x^A\,\mathrm{in}(g)$ for $g\in(I,x_i)$ and $x^A\neq 1$; this can be done since $f\in(I,x_i)$, but $\mathrm{in}(f)$ is of too large a degree to be a minimal generator of $\mathrm{in}(I, x_i)$. Write $g = g_1 + x_i g_2$, with $g_1 \in I$. Since $\mathrm{in}(g) > \mathrm{in}(x_i g_2)$ in the reverse lexicographic order, $\mathrm{in}(g) = \mathrm{in}(g_1)$. Thus $\mathrm{in}(f) = x^A \mathrm{in}(g_1)$ is not a minimal generator of $\mathrm{in}(I)$. □

(2.3) **Lemma.** *Let $r \geqq 0$, let $m \geqq 0$, and let $>$ be the reverse lexicographic order. The following conditions are equivalent:*

(a) $((I, x_n, \ldots, x_{i+1}) : x_i)_m = (I, x_n, \ldots, x_{i+1})_m$ *for* $i = n, \ldots, n-r+1$, *and* $(I, x_n, \ldots, x_{n-r+1})_m = S_m$.

(b) $((\mathrm{in}(I), x_n, \ldots, x_{i+1}) : x_i)_m = (\mathrm{in}(I), x_n, \ldots, x_{i+1})_m$ *for* $i = n, \ldots, n-r+1$, *and* $(\mathrm{in}(I), x_n, \ldots, x_{n-r+1})_m = S_m$.

Proof. The equivalence of (a) and (b) follows immediately from parts (a) and (b) of Lemma (2.2). □

(2.4) **Theorem.** *Let $I \subset S$ be a homogeneous ideal, let $>$ be the reverse lexicographic order, and let $r = \dim(S/I)$.*

(a) $(x_n, \ldots, x_{n-r+1}) \in U_r(I) \Leftrightarrow (x_n, \ldots, x_{n-r+1}) \in U_r(\mathrm{in}(I))$.

(b) *If* $(x_n, \ldots, x_{n-r+1}) \in U_r(I)$, *$I$ and* $\mathrm{in}(I)$ *have the same regularity.*

Proof. $r = \dim(S/\mathrm{in}(I))$, since I and $\mathrm{in}(I)$ have the same Hilbert function [Mac 27].

Suppose that $(x_n, \ldots, x_{n-r+1}) \in U_r(I)$, and let m denote the regularity of I. Then $(x_n, \ldots, x_{n-r+1})$ satisfies condition (c) of Theorem (1.10) for I. Since $(I, x_n, \ldots, x_{n-r+1})_m = S_m$, $\mathrm{in}(I, x_n, \ldots, x_{n-r+1})$ is generated by elements of degree $\leqq m$. Assume by induction that $\mathrm{in}(I, x_n, \ldots, x_i)$ is generated by elements of degree $\leqq m$; by Lemma (2.2c), $\mathrm{in}(I, x_n, \ldots, x_{i+1})$ is generated by elements of degree $\leqq m$. Thus $\mathrm{in}(I)$ is generated by elements of degree $\leqq m$.

By Lemma (2.3), $(x_n, \ldots, x_{n-r+1})$ also satisfy condition (b) of Theorem (1.10) for $\mathrm{in}(I)$. Thus $(x_n, \ldots, x_{n-r+1}) \in U_r(\mathrm{in}(I))$ and $\mathrm{in}(I)$ is m-regular, by Theorem (1.10).

Suppose that $(x_n, \ldots, x_{n-r+1}) \in U_r(\mathrm{in}(I))$, and let m denote the regularity of $\mathrm{in}(I)$. Let f be a minimal generator of I. If $\mathrm{in}(f) = x^A \mathrm{in}(g)$ for some $g \in I$ and $x^A \neq 1$, then f can be replaced by $f - x^A g$ as a minimal generator of I, where $\mathrm{in}(f - x^A g) < \mathrm{in}(f)$. By iterating this process, we can assume that $\mathrm{in}(f)$ is a minimal generator of $\mathrm{in}(I)$. Since $\mathrm{in}(I)$ is generated by elements of degree $\leqq m$, $\deg(f) \leqq m$, so I is generated by elements of degree $\leqq m$.

Again by Theorem (1.10) and Lemma (2.3), $(x_n, \ldots, x_{n-r+1}) \in U_r(I)$ and I is m-regular. □

(2.5) **Corollary.** *Let $I \subset S$ be a homogeneous ideal, let $>$ be the reverse lexicographic order, and let m be the regularity of I. If* $(x_n, \ldots, x_{n-r+1}) \in U_r(I)$, *then* $\mathrm{in}(I)$ *is generated by elements of degree $\leqq m$.*

Note that Theorem (2.4a) does not assert that $U_r(I) = U_r(\mathrm{in}(I))$, which is false.

Corollary (2.5) asserts that for the reverse lexicographic order and a generic choice of coordinates, $\mathrm{in}(I)$ is generated by monomials of degree $\leqq m$. In the remainder of this section, we show that in characteristic zero, this bound is exact: for the reverse lexicographic order and a generic choice of coordinates, $\mathrm{in}(I)$ has a minimal generator of degree m.

(2.6) *Definition.* Let $B=\{g\in \mathrm{GL}(n,k)|g_{ij}=0$ whenever $j<i\}$ denote the Borel subgroup of $\mathrm{GL}(n,k)$. An ideal I is Borel fixed if $g\cdot I=I$ whenever $g\in B$.

Any ideal which is Borel fixed is a monomial ideal, since B contains the subgroup $D(n)\subset \mathrm{GL}(n,k)$ of diagonal matrices, and the ideals fixed by $D(n)$ are precisely the monomial ideals.

For $1\leqq j<i\leqq n$, and $c\in k$, let $g_{ij}(c)\in \mathrm{GL}(n,k)$ be given by

$$\begin{aligned} g_{ij}(c)\cdot x_i &= x_i+c\,x_j,\\ g_{ij}(c)\cdot x_p &= x_p, \qquad \text{for } p\neq i. \end{aligned}$$

Recall that B is generated by $\{g_{ij}(c)|1\leqq j<i\leqq n$, and $c\in k\}$ and $D(n)$.

The following proposition describes in characteristic zero those monomial ideals which are Borel fixed.

(2.7) **Proposition.** *Suppose that k is of characteristic zero. Then a monomial ideal I is Borel fixed if and only if whenever*

$$x_1^{p_1}x_2^{p_2}\dots x_n^{p_n}\in I,$$

then for each $1\leqq j<i\leqq n$ and $0\leqq q\leqq p_i$,

$$x_1^{p_1}\dots x_j^{(p_j+q)}\dots x_i^{(p_i-q)}\dots x_n^{p_n}\in I.$$

Proof. If $x^A=x_1^{p_1}x_2^{p_2}\dots x_n^{p_n}\in I$,

$$\begin{aligned} g_{ij}(c)\cdot x^A\in I &\Leftrightarrow x_1^{p_1}\dots x_j^{p_j}\dots(x_i+c\,x_j)^{p_i}\dots x_n^{p_n}\in I\\ &\Leftrightarrow x_1^{p_1}\dots x_j^{(p_j+q)}\dots x_i^{(p_i-q)}\dots x_n^{p_n}\in I, \qquad \text{for } 0\leqq q\leqq p_i. \end{aligned}$$

The result follows since for a monomial ideal I, $g_{ij}(c)\cdot x^A\in I$ for all $x^A\in I$, $1\leqq j<i\leqq n$, and all $c\in k\Leftrightarrow I$ is Borel fixed. □

The following theorem is due to Galligo, and is proved in [Ga 74]. It is generalized to any order, and any characteristic, in [BaSt 86].

(2.8) **Theorem** (Galligo). *Let $I\subset S$ be a homogeneous ideal. Suppose that $>$ is the reverse lexicographic order, and that k is of characteristic zero. There is a Zariski open subset $U_1\subset \mathrm{GL}(n,k)$ such that for each $g\in U_1$, $\mathrm{in}(g\cdot I)$ is Borel fixed.*

(2.9) **Proposition.** *Let I be a Borel fixed monomial ideal, generated by monomials of degree $\leqq m$, and having a minimal generator of degree m. If k is of characteristic zero, then the regularity of I is precisely m.*

Proof. $x_1^m\in I$, since I contains a monomial of degree m, and I is Borel fixed. Choose $r\geqq 0$ so $x_{n-r}^q\in I$, for some q, but $x_{n-r+1}^p\notin I$, for all p. Since I is generated by monomials of degree $\leqq m$, $x_{n-r}^m\in I$.

To show that I is m-regular, it suffices by Theorem (1.10) to show that

$$((I, x_n, \ldots, x_{i+1}) : x_i)_m = (I, x_n, \ldots, x_{i+1})_m$$

for $i=n, \ldots, n-r+1$, and $(I, x_n, \ldots, x_{n-r+1})_m = S_m$.

Since $x_{n-r}^m \in I$, by the Borel condition (2.7), any monomial of degree m in the variables $x_1, \ldots, x_{n-r}$ is also in I. Thus $(I, x_n, \ldots, x_{n-r+1})_m = S_m$.

Let $J=(I, x_n, \ldots, x_{i+1})$ for some i in the range $n-r+1 \leqq i \leqq n$, and suppose that $x_i x^A \in J$ for a monomial x^A of degree m. If any of $x_n, \ldots, x_{i+1}$ divide x^A, then $x^A \in J$. Otherwise $x_i x^A \in I$. Since $\deg(x_i x^A) = m+1$, $x_i x^A$ is not a minimal generator of I. Write $x_i x^A = x_j x^B$, for some $j \leqq i$, where $x^B \in I$. If $j=i$, then $x^A = x^B \in J$. If $j<i$, write $x^B = x_i x^C$. Then $x^A = x_j x^C$. By the Borel condition (2.7), since $x^B \in I$, $x^A \in I \subset J$. Thus $(J : x_i)_m = J_m$. □

(2.10) **Lemma.** *Define* U_1 *as in Theorem* (2.8), *and define* U_2 *to be the open subset of* $\mathrm{GL}(n, k)$ *given by* $\{g \in \mathrm{GL}(n, k) | (x_n, \ldots, x_{n-r+1}) \in U_r(g \cdot I)\}$. *Then* $U_1 \subset U_2$.

Proof. For each $g \in U_1$, since $\mathrm{in}(g \cdot I)$ is Borel fixed, the associated primes of $\mathrm{in}(g \cdot I)$ are all of the form $(x_1, \ldots, x_j)$ for $1 \leqq j \leqq n$. Thus $(x_n, \ldots, x_{n-r+1}) \in U_r(\mathrm{in}(g \cdot I))$. By Theorem (2.4), $(x_n, \ldots, x_{n-r+1}) \in U_r(g \cdot I)$, so $g \in U_2$. □

The inclusion $U_1 \subset U_2$ is in general proper. For example, if $I=(x_1^5, x_2^3)$, then $1 \in \mathrm{U}_2$. I is not Borel fixed, so $1 \notin U_1$.

(2.11) **Proposition.** *Let* $I \subset S$ *be a homogeneous ideal of regularity* m. *Suppose that* k *is of characteristic zero, and define the Zariski open subset* $U_1 \subset \mathrm{GL}(n, k)$ *as in Theorem* (2.8). *Then for each* $g \in U_1$, $\mathrm{in}(g \cdot I)$ *has a minimal generator of degree* m.

Proof. For each $g \in U_1$, $\mathrm{in}(g \cdot I)$ is Borel fixed by Theorem (2.8). Since $U_1 \subset U_2$ by Lemma (2.10), $\mathrm{in}(g \cdot I)$ is of regularity m by Theorem (2.4). By Proposition (2.9), $\mathrm{in}(g \cdot I)$ has a minimal generator of degree m. □

In characteristic p, Proposition (2.11) fails: $I=(x_1^p, x_2^p)$ is Borel fixed, and of regularity $2p-1$.

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