Math 240A: Real Analysis, Fall 2019

Homework Assignment 4

Due Friday, October 25, 2019

1. Let $(X, \mathcal{M})$ be a measurable space. Prove the following:
   (1) A function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}((r, \infty)) \in \mathcal{M}$ for any $r \in \mathbb{Q}$;
   (2) If $f_n : X \to \mathbb{R}$ $(n = 1, 2, \dots)$ are measurable functions, then $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

2. If $f : \mathbb{R} \to \mathbb{R}$ is a monotonic function then it is Borel measurable.

3. Let $(X, \mathcal{M})$ be a measurable space. If $X = A \cup B$ for some $A, B \in \mathcal{M}$, then a function $f : X \to \mathbb{R}$ is measurable if and only if f is measurable on A and on B.

4. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is Lebesgue measurable but nowhere continuous.

5. Let μ be a finite Borel measure on $\mathbb{R}$. Let V be a nonempty, bounded, open subset of $\mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \mu(V + x)$. Is f necessary continuous?

6. Let $f : [0, 1] \to [0, 1]$ be the Cantor function (§1.5), and let $g(x) = f(x) + x$.
   (1) Show that g is a bijection from $[0, 1]$ to $[0, 2]$ and that $h = g^{-1}$ is continuous from $[0, 2]$ to $[0, 1]$.
   (2) Let C be the Cantor set. Show that $m(g(C)) = 1$.
   (3) By Problem 10 of Homework Assignment 3, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Prove that B is Lebesgue measurable but not Borel measurable.
   (4) There exist a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

7. Let $f, g : (0, 1) \to \mathbb{R}$ be monotonically decreasing and left-continuous. Assume $m(\{x \in (0, 1) : f(x) \geq \alpha\}) = m(\{x \in (0, 1) : g(x) \geq \alpha\})$ for any $\alpha \in \mathbb{R}$. Prove that $f = g$ on $(0, 1)$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space and denote by L^+ the set of all measurable functions from X to $[0, \infty]$. Let $f \in L^+$. Define $\lambda(E) = \int_E f\, d\mu$ for any $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$. Moreover, $\int_X g\, d\lambda = \int_X gf\, d\mu$ for any $g \in L^+$.

9. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be a measurable function such that $\int_X f\, d\mu < \infty$. Prove the following:
   (1) The set $\{x \in X : f(x) = \infty\}$ is a μ-null set and the set $\{x \in X : f(x) > 0\}$ is σ-finite;
   (2) For any $\varepsilon > 0$, there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f\, d\mu > \int_X f\, d\mu - \varepsilon$.

10. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ a measurable function. Prove that the following are equivalent:
    (1) $f = 0$ μ-a.e. on X;
    (2) $\int_X |f|\, d\mu = 0$;
    (3) $\int_E f\, d\mu = 0$ for any $E \in \mathcal{M}$;
    (4) $\int_X fg\, d\mu = 0$ for any measurable function $g : X \to \mathbb{R}$.