Math 240C: Real Analysis, Spring 2020

Homework Assignment 5

Due 12:00 noon, Friday, May 8, 2020

1. Given $f:\mathbb{R}^n\to\mathbb{C}$. Define $K:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ by $K(x,y)=f(x-y)$. Prove the following:
   (1) If f is Borel-measurable on $\mathbb{R}^n$, then K is Borel-measurable on $\mathbb{R}^n\times\mathbb{R}^n$;
   (2) If f is Lebesgue-measurable on $\mathbb{R}^n$, then K is Lebesgue-measurable on $\mathbb{R}^n\times\mathbb{R}^n$.

2. Let $1\le p,q,r\le\infty$ and $p^{-1}+q^{-1}=r^{-1}+1$. Let $f\in L^p$ and $g\in L^q$.
   (1) Use the generalized Hölder's inequality (cf. Exercise 31 in Section 6.3) to prove

$$|f*g(x)|^r\le \|f\|_p^{r-p}\|g\|_q^{r-q}\int |f(y)|^p|g(x-y)|^q dy \qquad \text{a.e. } x\in\mathbb{R}^n.$$

   (2) Prove that $f*g\in L^r$ and the Young's inequality: $\|f*g\|_r\le\|f\|_p\|g\|_q$.

3. Let $f(x)=(1/2)-x$ on $[0,1]$ and extend f to be periodic on $\mathbb{R}$. Prove the following:
   (1) $\hat f(0)=0$ and $\hat f(k)=(2\pi ik)^{-1}$ if $k\ne 0$;
   (2) $\sum_{k=1}^\infty k^{-2}=\pi^2/6$. (Hint: Use the Parseval identity.)

4. **(Wirtinger's Inequality)** Let $a<b$. If $f\in C^1([a,b])$ and $f(a)=f(b)=0$, then

$$\int_a^b |f(x)|^2dx\le\left(\frac{b-a}{\pi}\right)^2\int_a^b|f'(x)|^2dx.$$

   (See some hint for Problem 14 on page 254.)

5. Let $f_k=\chi_{[-1,1]}*\chi_{[-k,k]}$ $(k\in\mathbb{N})$.
   (1) Compute $f_k(x)$ explicitly and show that $\|f_k\|_u=2$.
   (2) Show that $f_k^\vee(x)=(\pi x)^{-2}\sin 2\pi kx\sin 2\pi x$ and $\|f_k^\vee\|_1\to\infty$ as $k\to\infty$.
   (3) Show that $\mathcal{F}(L^1)$ is a proper subset of C_0.

   (See some hints for problem 16 on page 255.)

6. Let $f\in L^1(\mathbb{R}^{n+m})$. Define

$$Pf(x)=\int_{\mathbb{R}^m} f(x,y)\,dy \qquad \forall x\in\mathbb{R}^n.$$

   Prove that $Pf\in L^1(\mathbb{R}^n)$, $\|Pf\|_1\le\|f\|_1$, and $(Pf)\hat{}(\xi)=\hat f(\xi,0)$.

7. Define $\operatorname{sinc} x=(\sin\pi x)/\pi x$ if $x\ne 0$ and 1 if $x=0$. Let $a>0$. Prove that

$$\hat\chi_{[-a,a]}(x)=\chi^\vee_{[-a,a]}(x)=2a\operatorname{sinc}2ax.$$