Math 240B Winter 2020

Solution to Problems of Final Exam

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1. Suppose $\|x_k - x\| \to 0$. $\forall f \in H^*$:

$$|f(x_k) - f(x)| = |f(x_k - x)| \le \|f\| \, \|x_k - x\| \to 0.$$

Hence, $x_k \to x$ weakly.

$$\big| \|x_k\| - \|x\| \big| \le \|x_k - x\| \to 0.$$

Hence, $\|x_k\| \longrightarrow \|x\|$.

Suppose $x_k \longrightarrow x$ weakly in H and $\|x_k\| \longrightarrow \|x\|$. Note that $u \longmapsto \langle u, x\rangle$ is a bounded linear functional on H. So, $\langle x_k, x\rangle \longrightarrow \langle x, x\rangle = \|x\|^2$. Thus,

$$\|x_k - x\|^2 = \|x_k\|^2 + \|x\|^2 - 2\langle x_k, x\rangle$$
$$\longrightarrow 2\|x\|^2 - 2\|x\|^2 = 0.$$

First, $\forall x, y \in X$, the series converges, as it is dominated by $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

2. (1) Clearly $\rho(x,y) \ge 0$ $\forall x, y \in X$. $\rho(x,y) = 0$ $\iff p_n(x-y) = 0$ $(\forall n \in \mathbb{N})$ $\iff x - y = 0$, i.e., $x = y$.

Also, $\rho(x,y) = \rho(y,x)$ since $p_n(x-y) = p_n((-1)(y-x)) = |-1|\, p_n(y-x) = p_n(y-x)$. $\forall n \in \mathbb{N}$.

$\forall n \in \mathbb{N}$, $\forall x, y, z \in X$:

$$p_n(x-y) \le p_n(x-z) + p_n(z-y)$$

Since $t \longmapsto \frac{t}{1+t}$ is increasing in $t \in (0, \infty)$,

$$\frac{p_n(x-y)}{1+p_n(x-y)} \le \frac{p_n(x-z)+p_n(z-y)}{1+p_n(x-z)+p_n(z-y)}$$
$$= \frac{p_n(x-z)}{1+p_n(x-z)+p_n(z-y)} + \frac{p_n(z-y)}{1+p_n(x-z)+p_n(z-y)}$$
$$\le \frac{p_n(x-z)}{1+p_n(x-z)} + \frac{p_n(z-y)}{1+p_n(z-y)}.$$

Thus $\rho(x,y) \le \rho(x,z) + \rho(z,y)$.
Therefore ρ is a metric on X.

Let $x, y, z \in X$. Since $p_n(x+z-(y+z)) = p_n(x-y)$ for all n, $\rho(x+z, y+z) = \rho(x,y)$. Hence ρ is translationally invariant.

(2) Let $\mathcal{T}_1$ denote the topology on X defined by the seminorms p_n ($n \in \mathbb{N}$). Let $\mathcal{T}_2$ denote the topology of the metric space (X,ρ). Note that $\mathcal{T}_1$ is generated by

$$B_n(x,\varepsilon) = \{y \in X : p_n(y-x) < \varepsilon\}$$

($n \in \mathbb{N}$, $x \in X$, $\varepsilon > 0$). Fix n, x, and $\varepsilon > 0$. Let $y \in B_n(x,\varepsilon)$. Choose $\delta = \min(1, \varepsilon - p_n(x-y)) > 0$. Then, $B_n(y,\delta) \subseteq B_n(x,\varepsilon)$. Let $z \in X$ be such that $\rho(z,y) < \delta/2^{n+1}$. Then,

$$\frac{p_n(z-y)}{2^n[1+p_n(z-y)]} \le \rho(z,y) < \frac{\delta}{2^{n+1}}$$

Thus, $(1-\frac{\delta}{2})\, p_n(z-y) < \frac{\delta}{2}$. But, $0 < \delta \le 1$. Hence, $1-\frac{\delta}{2} \ge \frac{1}{2}$. Hence, $p_n(z-y) < \delta$ i.e., $z \in B_n(y,\delta)$ $\subseteq B_n(x,\varepsilon)$. Thus, $B_n(x,\varepsilon)$ is open in the metric topology, and $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Note that the metric topology $\mathcal{T}_2$ is generated by balls $B(x,\varepsilon) = \{y \in X : \rho(y,x) < \varepsilon\}$ $(x \in X, \varepsilon > 0)$. Fix $x \in X$ and $\varepsilon > 0$. Let $y \in B(x,\varepsilon)$. Then

$$\rho(y,x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(y-x)}{1+p_n(y-x)} < \varepsilon$$

Choose $N \in \mathbb{N}$ so that $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{1}{2}\left(\varepsilon - \rho(y,x)\right) =: \delta$

We claim that $y \in \bigcap_{n=1}^{N} B_n(y, \delta/N) \subseteq B(x,\varepsilon)$. In fact, clearly, $y \in \bigcap_{n=1}^{N} B_n(y,\delta/N)$. Let $z \in \bigcap_{n=1}^{N} B_n(y,\delta/N)$, i.e., $p_n(z-y) < \delta/N$ $(n=1,\cdots,N)$. Then

$$\begin{aligned}
\rho(z,x) &\le \rho(z,y) + \rho(y,x) \\
&\le \sum_{n=1}^{N} p_n(z-y) + \sum_{n=N+1}^{\infty} \frac{1}{2^n} + \rho(y,x) \\
&< \delta + \delta + \rho(y,x) \\
&= \varepsilon.
\end{aligned}$$

Thus $z \in B(x,\varepsilon)$, and hence $\bigcap_{n=1}^{N} B_n(y,\delta/N) \subseteq B(x,\varepsilon)$. Since $\bigcap_{n=1}^{N} B_n(y,\delta/N) \in \mathcal{T}_1$, we have $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Finally, $\mathcal{T}_1 = \mathcal{T}_2$.

3. Let $E = \operatorname{span}\{x_1,\cdots,x_n\}$. Then E is a finite-dimensional subspace of X, and E is thus a closed subspace of X. Any $x \in E$ can be uniquely expressed as $x = \sum_{k=1}^{n} \alpha_k x_k$, where $\alpha_k \in \mathbb{R}$ $(k=1,\cdots,n)$. Define $\hat{f}_1,\cdots,\hat{f}_n : E \to \mathbb{R}$ by $\hat{f}_j(x) = \hat{f}_j\left(\sum_{k=1}^{n} \alpha_k x_k\right) = \alpha_j$, $1 \le j \le n$. Each $\hat{f}_j$ is linear: if $x = \sum_{k=1}^{n} \alpha_k x_k$, $y = \sum_{k=1}^{n} \beta_k y_k$, then $x+y = \sum_{k=1}^{n} (\alpha_k+\beta_k) x_k$.

then $\hat{f}_j(x+y) = \alpha_j + \beta_j = \hat{f}_j(x) + \hat{f}_j(y)$. Similarly, $\hat{f}_j(\alpha x) = \alpha \hat{f}_j(x)$ if $\alpha \in \mathbb{R}$ and $x \in X$. If $x = \sum_{k=1}^{n} \alpha_k x_k \in E$ and $\alpha_j = 0$ then $\hat{f}_j(x) = 0$. If $\alpha_j \neq 0$ then by the assumption that $\inf\{\|x_j - y_j\| : y_j \in E_j\} \geq 1$,

$$|\hat{f}_j(x)| = |\alpha_j| \leq |\alpha_j| \, \| x_1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{\alpha_k}{\alpha_j} x_k \|$$

$$= \|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| = \|x\|.$$

Hence $\|\hat{f}_j\| \leq 1$.

Clearly, $\hat{f}_j(x_j) = 1$. Since $\|x_j\| = 1$, we thus have $\|\hat{f}_j\| = 1$. In particular, $\hat{f}_j \in E^*$ $(1 \leq j \leq n)$. Now it follows from the Hahn-Banach theorem that there exist $f_j \in X^*$ $(j = 1, \cdots, n)$ such that $f_j = \hat{f}_j$ on E and $\|f_j\| = \|\hat{f}_j\| = 1$. $(1 \leq j \leq n)$.

Clearly, $f_j(x_k) = \hat{f}_j(x_k) = \delta_{jk}$ $(j, k = 1, \cdots, n)$.

4. (1) Clearly all I_k $(k \in \mathbb{N})$ and I are linear functionals on $C([0,1])$. Let $f \in C([0,1])$. Let $\varepsilon > 0$. By the Weierstrass Theorem, there exists a polynomial p such that $\|p - f\|_u < \varepsilon$, where $\|\cdot\|_u$ is the uniform norm on $[0,1]$. Thus,

$$|I_k(f) - I(f)| \leq |I_k(f-p)| + |I_k(p) - I(p)| + |I(p-f)|$$

$$\leq \left(\sum_{j=0}^{k} |A_j^{(k)}|\right) \|f - p\|_u + |I_k(p) - I(p)| + \|f - p\|_u$$

$$\leq \left(1 + \sup_{m \geq 1} \sum_{j=0}^{m} |A_j^{(m)}|\right) \varepsilon + |I_k(p) - I(p)|$$

This implies that

$$\limsup_{k \to \infty} |I_k(f) - I(f)| \leq \left(1 + \sup_{m \geq 1} \sum_{j=0}^{m} |A_j^{(m)}|\right) \varepsilon,$$

as $I_k(p) \to I(p)$. Thus $I_k(f) \to I(f)$.

(2) Let $f \in C([0,1])$. Then $|I_k(f)| \le \sum_{j=0}^{k} |A_j^{(k)}| \, \|f\|_u$, and $\|I_k\| \le \sum_{j=0}^{k} |A_j^{(k)}|$. Define $f_k \in C([0,1])$ by $f_k(x_j^{(k)}) = \operatorname{sgn}(A_j^{(k)})$ $(j = 0, 1, \cdots, k)$ and f_k is affine on each $[x_{j-1}, x_j]$ $(j = 1, \cdots, n)$. Then $\|f_k\|_u = 1$ and $I_k(f_k) = \sum_{j=0}^{k} |A_j^{(k)}|$. Hence $\|I_k\| = \sum_{j=0}^{k} |A_j^{(k)}|$.

(3) Since $C([0,1])$ is a Banach space, by (2) and the Principle of Uniform Boundedness, we have
$$\sup_{k \ge 1} \sum_{j=0}^{k} |A_j^{(k)}| = \sup_{k \ge 1} \|I_k\| < \infty.$$

5. (1) We first prove the following statement:

Let (X, ρ) be a metric space and $\phi \ne E \subseteq X$. Suppose for any $y \in E$ and any $b > 0$, there exist $x \in X$ and $a > 0$ such that
$$B(x, a) \subseteq B(y, b) \cap E^c.$$
Then E is nowhere dense in X.

<u>Proof</u> If E is not nowhere dense in X, then $\operatorname{int}(\bar{E}) \ne \phi$. So, $\exists z \in X$ and $c > 0$ such that $B(z, c) \subseteq \bar{E}$. Since $z \in \bar{E}$, there exists $y \in B(z, c) \cap E$. Moreover, $\exists b > 0$ s.t. $B(y, b) \subseteq B(z, c) \subseteq \bar{E}$ with $y \in E$. By the assumption, $\exists x \in X$ and $a > 0$ such that $B(x, a) \subseteq B(y, b) \cap E^c \subseteq \bar{E} \cap E^c$. Thus, $x \in \bar{E}$, and

$\exists\, x_n \in E$ s.t. $x_n \to x$. Hence, for large n, $x_n \in B(x,a) \cap E \subseteq \bar{E} \cap E^c \cap E = \phi$. Impossible. Hence, E is nowhere dense in X.

We now fix $n \in \mathbb{N}$ and show that E_n is nowhere dense in $C([0,1])$. By the above statement, it suffices to show that $\forall f \in E_n$, $\forall \varepsilon > 0$ $\exists\, g \in C([0,1])$ $\exists\, a > 0$ such that $\|g - f\| < \varepsilon$ and $B(g,a) \cap E_n = \phi$. [Note that if we shrink a, we can have $B(g,a) \subseteq B(f,\varepsilon)$ as in the statement above.]

Let $f \in E_n$ and $\varepsilon > 0$. By the Weierstrass Theorem, f can be approximated by polynomials in $C([0,1])$. Any polynomial on $[0,1]$ has a bounded derivative. Without loss of generality, we may then assume f is a C^1-function on $[0,1]$.

Choose $M > 0$ so that $M > 2n$ and $M > \max_{0 \le x \le 1} |f'(x)|$. Choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \frac{\varepsilon}{2M}$. Denote $\delta = \frac{1}{N}$. Let $x_k = k\delta$ $(k = 0, 1, \cdots, N)$; $x_0 = 0$, $x_N = 1$. We construct $g \in C([0,1])$ to be a piecewise linear function with the slope of g on each piece $[x_{k-1}, x_k]$ $(1 \le k \le N)$ being M or $-M$, and $\|g - f\| < \varepsilon$.

Define $g(x_0) = f(x_0)$. Clearly, $|g(x_0) - f(x_0)| < \varepsilon$.

Define $g(x_1) = g(x_0) + M\delta$ if $f(x_1) \geq g(x_0)$,

$g(x_1) = g(x_0) - M\delta$ if $f(x_1) < g(x_0)$.

Suppose we have defined $g(x_0), \cdots, g(x_k)$ with $1 \leq k \leq N-1$. Then we define

$$g(x_{k+1}) = g(x_k) + M\delta \quad \text{if } f(x_{k+1}) \geq g(x_k),$$
$$g(x_{k+1}) = g(x_k) - M\delta \quad \text{if } f(x_{k+1}) < g(x_k).$$

By induction, we have defined all $g(x_k)$ $(0 \leq k \leq N)$. Since $|g(x_{k+1}) - g(x_k)| = M\delta = M(x_{k+1} - x_k)$, by connecting $(x_k, g(x_k))$ and $(x_{k+1}, g(x_{k+1}))$ for each k by a line segment, we have indeed constructed $g \in C([0,1])$ such that it is a piecewise linear function and on each piece, the slope is M or $-M$.

We now prove $\|f - g\| < \varepsilon$ in two steps. First, we show that $|g(x_k) - f(x_k)| < \varepsilon$ $(k = 0, \cdots, N)$. Second, we show $\max_{x_k \leq x \leq x_{k+1}} |g(x) - f(x)| < \varepsilon$ for all $k = 0, 1, \cdots, N-1$.

Clearly $|g(x_0) - f(x_0)| < \varepsilon$. Suppose for some k $|g(x_k) - f(x_k)| < \varepsilon$. We show $|g(x_{k+1}) - f(x_{k+1})| < \varepsilon$. Let us consider the case that $f(x_{k+1}) < g(x_k)$. (The other case that $f(x_{k+1}) \geq g(x_k)$ is similar.) In this case,

$$f(x_{k+1}) - g(x_{k+1}) = f(x_{k+1}) - g(x_k) + M\delta < M\delta < \varepsilon.$$

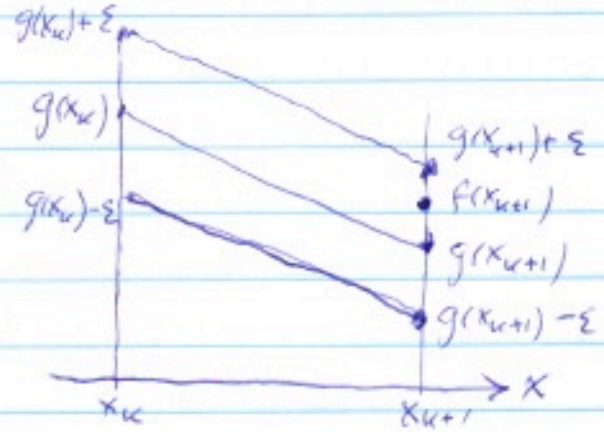
We also should have

$$f(x_{k+1}) - g(x_{k+1}) > -M\delta > -\varepsilon$$

Otherwise, $f(x_{k+1}) - g(x_{k+1}) \le -M\delta$. Combining this with $g(x_k) \le f(x_k) + M\delta$, we would get
$M\delta = g(x_k) - g(x_{k+1}) \le f(x_k) - f(x_{k+1}) = f'(\xi_k)\delta$
for some $\xi_k \in [x_k, x_{k+1}]$, leading to $|f'(\xi_k)| \ge M$, contradicting to the fact that $M > \max_{0 \le x \le 1} |f'(x)|$. Thus, $|f(x_{k+1}) - g(x_{k+1})| < M\delta \le \varepsilon$. Hence, this holds true a each of $k = 0, 1, \cdots, N-1$.

Fix $k \in \{0, \cdots, N-1\}$. Consider again $[x_k, x_{k+1}]$ and the case $f(x_{k+1}) < g(x_k)$. [The case that $f(x_{k+1}) \ge g(x_k)$ is similar.] We show

$$g(x) - \varepsilon < f(x) < g(x) + \varepsilon \quad \forall x \in (x_k, x_{k+1}).$$



Let $x \in (x_k, x_{k+1})$. If $f(x) \ge g(x) + \varepsilon$, then

$$f(x) - f(x_{k+1}) \ge g(x) + \varepsilon - f(x_{k+1})$$
$$> g(x) - g(x_{k+1})$$
$$= (x_{k+1} - x)M > 0$$

Moreover, $f(x) - f(x_{k+1}) = f'(\eta_k)(x - x_{k+1})$ for some $\eta_k \in (x_k, x_{k+1})$. Thus, $-f'(\eta_k) > M$. This is impossible by our choice of M. Thus, $f(x) < g(x) + \varepsilon$. Similarly $f(x) > g(x) - \varepsilon$. Hence, $|f(x) - g(x)| < \varepsilon \ \forall x \in [x_k, x_{k+1}]$. This is true for all k. So $\max_{0 \le x \le 1} |f(x) - g(x)| < \varepsilon$.

Let $a \in (0, \frac{n}{4N})$. Let $h \in C([0,1])$ be such that $\|h-g\| < a$. Then, for any $x, x_0 \in [0,1]$,

$$|h(x)-h(x_0)| \geq |g(x)-g(x_0)| - |h(x)-g(x)| - |h(x_0)-g(x_0)|$$
$$\geq |g(x)-g(x_0)| - 2a.$$

Suppose $x_0 \in [x_k, x_{k+1})$ for some k $(0 \leq k \leq N-1)$. Choose $x \in [x_k, x_{k+1}]$ so that $|x-x_0| \geq \frac{\delta}{2} = \frac{1}{2N}$. Then,

$$|h(x)-h(x_0)| \geq |x-x_0| M - 2a$$
$$= |x-x_0| \left(M - \frac{2a}{|x-x_0|} \right)$$
$$\geq |x-x_0| (2n - 4aN)$$
$$> n|x-x_0|.$$

Hence $h \notin E_n$, i.e., the ball $B(g,a)$ does not intersect E_n. Hence, E_n is nowhere dense in $C([0,1])$.

(2) It suffices to show that $\mathcal{N}^c \subseteq \bigcup_{k=1}^{\infty} E_k$.
Let $f \in \mathcal{N}^c$. Then, there exists $x_0 \in [0,1]$ such that $f'(x_0)$ exists. Hence, $\exists \alpha > 0$ s.t. for any $x \in [0,1]$ with $|x-x_0| < \alpha$,

$$|f(x)-f(x_0)| \leq |x-x_0| (1 + |f'(x_0)|).$$

If $x \in [0,1]$ and $|x-x_0| \geq \alpha$, then

$$\frac{|f(x)-f(x_0)|}{|x-x_0|} \leq \frac{1}{\alpha} |f(x)-f(x_0)| \leq \frac{2}{\alpha} \|f\|.$$

Let $n \in \mathbb{N}$ be s.t. $n > \max(\frac{2}{\alpha}\|f\|, 1+|f'(x_0)|)$.
Then $|f(x)-f(x_0)| \leq n|x-x_0|$ $\forall x \in [0,1]$. Hence, $f \in E_n$, and $\mathcal{N}^c \subseteq \bigcup_{k=1}^{\infty} E_k$.

6. Since $f_k \to f$ in $L^1(\mu)$, there exists a subsequence $f_{k_j} \to f$ μ-a.e. Since $\sup_{k\geq 1} \|f_k\|_p < \infty$, by Fatou's lemma, we have $\int |f|^p d\mu \leq \liminf_{j\to\infty} \int |f_{k_j}|^p d\mu < \infty$.

Hence $f \in L^p(\mu)$.

Let $\delta > 0$. Since $f_k \to f$ in $L^1(\mu)$, $f_k \to f$ in measure. Hence, for

$$E_k(\delta) = \{x \in X : |f_k(x) - f(x)| > \delta\},$$

we have $\mu(E_k(\delta)) \to 0$ as $k \to \infty$. Consequently, if $q \in (1, p)$, we have by Hölder's inequality and the boundedness of $\{\|f_k\|_p\}_{k=1}^{\infty}$ that

$$\begin{aligned}
&\limsup_{k\to\infty} \int |f_k - f|^q d\mu \\
&= \limsup_{k\to\infty} \left[\int_{E_k(\delta)} |f_k - f|^q d\mu + \int_{X \setminus E_k(\delta)} |f_k - f|^q d\mu \right] \\
&\leq \limsup_{k\geq 1} \int_{E_k(\delta)} |f_k - f|^q d\mu + \delta^q \mu(X) \\
&\leq \limsup_{k\geq 1} [\mu(E_k(\delta))]^{1-\frac{q}{p}} \left(\int_{E_k(\delta)} |f_k - f|^p d\mu \right)^{q/p} + \delta^q \mu(X) \\
&\leq \left(\|f\|_p + \sup_{k\geq 1} \|f_k\|_p \right)^q \limsup_{k\to\infty} [\mu(E_k(\delta))]^{1-q/p} + \delta^q \mu(X) \\
&= \delta^q \mu(X).
\end{aligned}$$

Since $\delta > 0$ is arbitrary, $f_k \to f$ in $L^q(\mu)$.

7. (1) Since $\mu(X)<\infty$, $\chi_E \in L^p(\mu)$ for any $E\in\mathcal{M}$. Thus $\nu(E)=F(\chi_E)\in\mathbb{R}$. Clearly, $\nu(\phi)=F(\chi_\phi)=F(0)=0$. Let $\{E_j\}_{j=1}^\infty$ be a disjoint sequence of members in $\mathcal{M}$ and $E=\bigcup_{j=1}^\infty E_j\in\mathcal{M}$. We have $\chi_E=\sum_{j=1}^\infty \chi_{E_j}$ pointwise on X. Moreover,

$$\int_X \Big|\chi_E-\sum_{j=1}^n \chi_{E_j}\Big|^p d\mu=\int_X \Big|\sum_{j=n+1}^\infty \chi_{E_j}\Big|^p d\mu$$

$$=\int_X \Big(\sum_{j=n+1}^\infty \chi_{E_j}\Big)d\mu=\int_X \chi_{\bigcup_{j=n+1}^\infty E_j}\,d\mu$$

$$=\mu\Big(\bigcup_{j=n+1}^\infty E_j\Big)=\sum_{j=n+1}^\infty \mu(E_j)\to 0,$$

since $\mu(E)=\sum_{j=1}^\infty \mu(E_j)<\infty$. Thus $\sum_{j=1}^n \chi_{E_j}\to\chi_E$ in $L^p(\mu)$. Since $F\in[L^p(\mu)]^*$,

$$\sum_{j=1}^n \nu(E_j)=\sum_{j=1}^n F(\chi_{E_j})=F\Big(\sum_{j=1}^n \chi_{E_j}\Big)\to F(\chi_E)=\nu(E).$$

Hence $\nu(E)=\sum_{j=1}^\infty \nu(E_j)$. So, ν is a signed measure on $(X,\mathcal{M})$.

If $E\in\mathcal{M}$ and $\mu(E)=0$, then $\chi_E=0$ in $L^p(\mu)$. Hence $\nu(E)=F(\chi_E)=F(0)=0$. Thus $\nu\ll\mu$. It now follows from the Radon-Nikodym theorem that $\exists\, g\in L^1(\mu)$ s.t. $\nu(E)=F(\chi_E)=\int_X \chi_E g\,d\mu=\int_E g\,d\mu$ for any $E\in\mathcal{M}$. If $f=\sum_{j=1}^m a_j\chi_{E_j}\in\Sigma$, where $E_1,\dots,E_m\in\mathcal{M}$ are disjoint and $a_1,\dots,a_m\in\mathbb{R}$, then

$$F(f)=\sum_{j=1}^m a_j F(\chi_{E_j})=\sum_{j=1}^m a_j\int_{E_j} g\,d\mu$$

$$=\sum_{j=1}^m a_j\int_X \chi_{E_j} g\,d\mu=\int_X fg\,d\mu.$$

(2) Let $f \in \Sigma$ and $\|f\|_p = 1$. Then by (1),

$$\left| \int_X f g \, d\mu \right| = |F(f)| \le \|F\| \, \|f\|_p = \|F\|.$$

Hence $M_q(g) \le \|F\|$.

Let $f : X \to \mathbb{R}$ be $\mathcal{M}$-measurable, bounded, and $\|f\|_p = 1$. (Note: $\mu(X) < \infty$. So, f is measurable and bounded $\Rightarrow f \in L^\infty(\mu) \subseteq L^p(\mu)$.) Let $\{\phi_j\}_{j=1}^\infty$ be a sequence of simple functions on X such that $0 \le |\phi_1| \le \cdots \le |\phi_j| \le \cdots \le |f|$ and $\phi_j \to f$ on X. By the dominant convergence theorem $\|\phi_j\|_p \to \|f\|_p = 1$. We may assume $\|\phi_j\|_p > 0$ for all $j \in \mathbb{N}$. Thus,

$$\left| F\left(\frac{\phi_j}{\|\phi_j\|_p}\right) \right| = \left| \int_X \frac{\phi_j}{\|\phi_j\|_p} g \, d\mu \right| \le M_q(g).$$

By the dominant convergence theorem again (as $|\phi_j g| \le |fg|$ and f is bounded and $g \in L^1(\mu)$),

$$F\left(\frac{\phi_j}{\|\phi_j\|_p}\right) = \frac{1}{\|\phi_j\|_p} \int_X \phi_j g \, d\mu$$

$$\longrightarrow \frac{1}{\|f\|_p} \int_X f g \, d\mu = \int_X f g \, d\mu.$$

Hence $\quad \left| \int_X f g \, d\mu \right| \le M_q(g).$

Let $\{\psi_j\}_{j=1}^\infty$ be a sequence of simple functions on X such that $0 \le |\psi_1| \le \cdots \le |\psi_n| \le \cdots$ and $|\psi_n| \le |g|$ $(\forall n)$, $\psi_n \to g$ on X. We may assume that $g \ne 0$ in $L^1(\mu)$. Otherwise, $\|g\|_q \le M_q(g)$ is obviously true. By the dominant convergence theorem, $\|\psi_n\|_1 \to \|g\|_1 > 0$. So we may again assume that all $\psi_n \neq 0$ in $L^1(\mu)$. Since ψ_n is a simple function, $\psi_n \in L^q(\mu)$, and $\|\psi_n\|_q > 0$. Define now

$$f_n = \frac{|\psi_n|^{q-1} \operatorname{sgn} g}{\|\psi_n\|_q^{q-1}} \quad (n \in \mathbb{N}).$$

Each f_n is measurable and bounded. Moreover, noting that $(q-1)p = q$, we have

$$\int_X |f_n|^p d\mu = \frac{1}{\|\psi_n\|_q^{(q-1)p}} \int_X |\psi_n|^{(q-1)p} d\mu = \frac{1}{\|\psi_n\|_q^q} \int_X |\psi_n|^q d\mu$$
$$= 1. \quad (\forall n \in \mathbb{N}).$$

Thus, by what has been proved that

$$\left|\int_X f_n g \, d\mu\right| \leq M_q(g) \leq \|F\| < \infty.$$

Note that for each $n \in \mathbb{N}$,

$$\int_X |f_n \psi_n| d\mu = \int_X \frac{|\psi_n|^q}{\|\psi_n\|_q^{q-1}} d\mu = \frac{\int_X |\psi_n|^q d\mu}{\|\psi_n\|_q^{q-1}}$$
$$= \frac{\|\psi_n\|_q^q}{\|\psi_n\|_q^{q-1}} = \|\psi_n\|_q.$$

By the fact that $f_n g = \frac{|\psi_n|^{q-1}|g|}{\|\psi_n\|_q^{q-1}} \geq 0$ on X and that $|\psi_n| \leq |g|$ on X, we thus have by Fatou's lemma that

$$\|g\|_q \leq \liminf_{n\to\infty} \|\psi_n\|_q = \liminf_{n\to\infty} \int_X |f_n \psi_n| d\mu$$
$$\leq \liminf_{n\to\infty} \int_X |f_n g| d\mu = \liminf_{n\to\infty} \int_X f_n g \, d\mu$$
$$\leq M_q(g) < \infty.$$

Hence $g \in L^q(\mu)$.

(3) By the part (1) and part (2), we have $g \in L^q(\mu)$ and $F(f) = \int_X fg\,d\mu$ for any simple function f on X. Since $\mu(X) < \infty$, simple functions are dense in $L^p(\mu)$ (cf. Proposition 6.7). Thus for any $f \in L^p(\mu)$, there exist simple functions $f_n \to f$ in $L^p(\mu)$. Hence $F(f_n) \to F(f)$, and by Hölder's inequality,

$$\begin{aligned} &\left|\int_X f_n g\,d\mu - \int_X fg\,d\mu\right| \\ &= \left|\int_X (f_n - f) g\,d\mu\right| \\ &\leq \|f_n - f\|_p \|g\|_q \to 0. \end{aligned}$$

Since $F(f_n) = \int_X f_n g\,d\mu$ $(n = 1, 2, \cdots)$, we have

$$F(f) = \int_X fg\,d\mu.$$

8 (1) If $x \in K \subseteq \bigcup_{j=1}^n U_j$ then $x \in U_j$ for some j. But U_j is open. So it contains a compact neighborhood N_x of x, as the space X is locally compact Hausdorff; cf. Proposition 4.30.

(2) We have $K = \bigcup_{x \in K} \{x\} \subseteq \bigcup_{x \in K} \mathring{N}_x$. But K is compact. So, the open cover $\{\mathring{N}_x\}_{x \in K}$ of K has a finite subcover $\{\mathring{N}_{x_k}\}_{k=1}^m$.

(3) Each N_{x_i} is compact, and hence the finite union F_j of N_{x_i}'s is compact.

(If A, B are compact, $A \cup B \subseteq \bigcup_{\alpha \in I} V_\alpha$, each V_α is open. then $A \subseteq \bigcup_\alpha V_\alpha$ and $B \subseteq \bigcup_\alpha V_\alpha$. Hence, each of A, B is covered by finitely many V_α's; so is $A \cup B$.) Each $N_{x_i} \subseteq U_j$ so $F_j \subseteq U_j$.

(4) If $x \in K$ then $x \in N_{x_i}$ for some i, and $N_{x_i} \subseteq F_j$ for some j. Hence $x \in F_j$ and $g_j(x) = 1$. Thus, $\sum_{j=1}^{n} g_j \geq 1$ on K.

(5) G is open, since $\sum_{j=1}^{n} g_j$ is continuous, and $G = \left(\sum_{j=1}^{n} g_j\right)^{-1}((0,\infty))$ and $(0,\infty)$ is open. (We can replace $(0,\infty)$ by $(0,1]$ which is open in $[0,1]$.) $K \subseteq G$ since $\sum_{j=1}^{n} g_j \geq 1 > 0$ on K as shown in (4).

(6) Note that $g_{n+1} = 1 - f \in [0,1]$. If $x \in G$ then $\sum_{j=1}^{n+1} g_j(x) \geq \sum_{j=1}^{n} g_j(x) > 0$ by the definition of G. If $x \in G^c$ then $f(x) = 0$ and $g_{n+1}(x) = 1$. Hence $\sum_{j=1}^{n+1} g_j(x) \geq g_{n+1}(x) = 1 > 0$. Thus $\sum_{j=1}^{n} g_{j+1} > 0$ on X.

(7) Clearly $h_j \in C(X)$, as it is continuous pointwise. $0 \leq h_j \leq 1$, since each $g_k \geq 0$ $(k = 0, 1, \cdots, n)$. Also, $\operatorname{supp}(h_j) = \operatorname{supp}(g_j) \subseteq U_j$ $(1 \leq j \leq n)$. On K, $f = 1$. So, $g_{n+1} = 0$, and $\sum_{j=1}^{n} h_j = \sum_{j=1}^{n} g_j \Big/ \sum_{k=1}^{n+1} g_k$
$= \sum_{j=1}^{n} g_j \Big/ \sum_{k=1}^{n} g_k = 1$.