Math 240B: Real Analysis, Winter 2020

Homework Assignment 4

Due Friday, February 7, 2020

1. Let $n \in \mathbb{N}$ and $-\infty < a = x_0 < x_1 < \cdots < x_n = b < \infty$. Denote by $\mathcal{P}_n$ the set of all real polynomials of degree $\le n$.
   (1) Define $l_j \in \mathcal{P}_n$ by $l_j(x) = \prod_{i=0, i\neq j}^{n} (x - x_i)/(x_j - x_i)$ $(j = 0, \ldots, n)$. Show that $l_j(x_k) = \delta_{jk}$ ($\delta_{jk} = 1$ if $j = k$ and 0 if $j \neq k$).
   (2) Let $f \in C([a, b])$ and define $L_n f \in \mathcal{P}_n$ by $(L_n f)(x) = \sum_{j=0}^{n} f(x_j) l_j(x)$ (called the Lagrange interpolation of f). Prove that $L_n f$ is the unique polynomial in $\mathcal{P}_n$ that satisfies $(L_n f)(x_j) = f(x_j)$ $(j = 0, 1, \ldots, n)$.
   (3) Prove that the operator norm of the linear operator $L_n : C([a, b]) \to C([a, b])$ (where $C([a, b])$ is equipped with the maximum norm) is given by $\|L_n\| = \max_{x\in[a,b]} \sum_{j=0}^{n} |l_j(x)|$.

2. Consider the Banach space $L^\infty([0, 1])$ and its subspace $C([0, 1])$.
   (1) Let $x_0 \in (0, 1)$ and define $F : C([0, 1]) \to \mathbb{C}$ by $F(f) = f(x_0)$. Prove that $F \in C([0, 1])^*$.
   (2) By Hahn–Banach Theorem, there exists $\tilde{F} \in L^\infty([0, 1])^*$ such that $\tilde{F} = F$ on $C([0, 1])$ and $\|\tilde{F}\|_{L^\infty([0,1])^*} = F_{C([0,1])^*}$. Prove that there exists no $g \in L^1([0, 1])$ such that $\tilde{F}(f) = \displaystyle\int_0^1 f(x)g(x)\,dx$ for all $f \in L^\infty([0, 1])$.

3. Let $\mathcal{X}$ be a normed vector space and $\mathcal{X}^*$ its dual space.
   (1) Suppose $x_n \to x$ weakly in $\mathcal{X}$ (i.e., $f(x_n) \to f(x)$ for any $f \in \mathcal{X}^*$). Prove that $\sup_{n\ge1} \|x_n\| < \infty$.
   (2) Assume in addition that $\mathcal{X}$ is a Banach space. Suppose $f_n \to f$ weak-$*$ in $\mathcal{X}^*$ (i.e., $f_n(x) \to f(x)$ for any $x \in \mathcal{X}$). Prove that $\sup_{n\ge1} \|f_n\| < \infty$.

4. Let $\mathcal{X}$ be an infinite-dimensional normed vector space. Prove that there exist $x_k \in \mathcal{X}$ $(k = 1, 2, \ldots)$ such that $\|x_k\| = 1$ for all $k \ge 1$ and $\|x_j - x_k\| \ge 1/2$ for all $j, k = 1, 2, \ldots$ with $j \neq k$. (See the hint in Exercise 19 on page 160.)

5. Let $\mathcal{X}$ be a Banach space. Assume $\mathcal{X}^*$ is separable. Prove that $\mathcal{X}$ is also separable. (See the hint in Exercise 25 on page 160.)

6. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space $\mathcal{X}$ such that $\|x\|_1 \le \|x\|_2$ for any $x \in \mathcal{X}$. Assume that $\mathcal{X}$ is complete with respect to both norms. Prove that these norms are equivalent.

7. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Let $T : \mathcal{X} \to \mathcal{Y}$ be a linear map such that $f \circ T \in \mathcal{X}^*$ for every $f \in \mathcal{Y}^*$. Prove that T is bounded.

8. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Let $T_n \in L(\mathcal{X}, \mathcal{Y})$ $(n = 1, 2, \ldots)$ be such that $\lim_{n\to\infty} T_n x$ exists for every $x \in \mathcal{X}$. Define $Tx = \lim_{n\to\infty} T_n x$ $(x \in X)$. Prove that $T \in L(\mathcal{X}, \mathcal{Y})$.