Math 240A: Real Analysis, Fall 2019

Homework Assignment 2

Due Friday, October 11, 2019

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $E, F \in \mathcal{M}$ then $\mu(E)+\mu(F) = \mu(E\cup F)+\mu(E\cap F)$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Define $\mu_E(A) = \mu(A\cap E)$ for any $A \in \mathcal{M}$. Show that μ_E is a measure.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $E_n \in \mathcal{M}$ $(n = 1, 2, \dots)$. Prove the following:
   (1) $\mu\left(\liminf_{n\to\infty} E_n\right) \le \liminf_{n\to\infty} \mu(E_n)$;
   (2) $\mu\left(\limsup_{n\to\infty} E_n\right) \ge \limsup_{n\to\infty} \mu(E_n)$, provided that $\mu(\cup_{n=1}^{\infty} E_n) < \infty$.

4. Let $(X, \mathcal{M})$ be a measurable space and $\mu : \mathcal{M} \to [0, \infty]$ be such that $\mu(\phi) = 0$ and μ is finitely additive.
   (1) Prove that μ is a measure if and only if it is continuous from below as in Theorem 1.8 (c) of the textbook.
   (2) Assume in addition that $\mu(X) < \infty$. Prove that μ is a measure if and only if is continuous from above as in Theorem 1.8 (d) of the textbook.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E_n \in \mathcal{M}$ $(n = 1, 2, \dots)$ satisfy $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Prove that $\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ if and only if $\mu(E_i \cap E_j) = 0$ for any i, j with $i \ne j$.

6. If μ is a σ-finite measure on a measure space $(X, \mathcal{M})$, then it is semifinite, i.e., for any $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ such that $F \subset E$ and $0 < \mu(F) < \infty$.

7. Let μ be a semifinite measure on a measurable space $(X, \mathcal{M})$. Suppose $E \in \mathcal{M}$ and $\mu(E) = \infty$. Show that for any $C > 0$ there exists $F \in \mathcal{M}$ such that $F \subset E$ and $C < \mu(F) < \infty$.

8. Let μ^* be an outer measure on X. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of disjoint μ^*-measurable sets. Prove that $\mu^*(E \cap (\cup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} \mu^*(E \cap A_n)$ for any $E \subseteq X$.

9. Let $\mathcal{A}$ be an algebra of a set X. Denote by $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$ and by $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure on X. Prove the following:
   (1) For any $E \subseteq X$ and $\epsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ such that $E \subseteq A$ and $\mu^*(A) \le \mu^*(E) + \epsilon$;
   (2) If $E \subseteq X$ and $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exsits a $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.

10. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let μ^* be the outer measure induced by μ. Suppose $E \subseteq X$ satisfies $\mu^*(E) = \mu^*(X)$ (but not that $E \in \mathcal{M}$).
    (1) If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.
    (2) Let $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$. Define $\nu : \mathcal{M}_E \to [0, \infty]$ by $\nu(A \cap E) = \mu(A)$ (which makes sense by Part (1)). Then $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.