Math 240B: Real Analysis, Winter 2020

Homework Assignment 1

Due Monday, January 13, 2020

1. Let $(X, \mathcal{M})$ be a measurable space. Let $M(X)$ be the space of complex measures on $(X, \mathcal{M})$. Prove that $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space.

2. Let $k \in \mathbb{N}$ and denote by $C^k([0,1])$ the space of functions on $[0,1]$ possessing continuous derivatives up to order k on $[0,1]$, including one-sided derivatives at the endpoints. For any $f \in C^k([0,1])$, define $\|f\|_{C^k} = \max_{0\le j\le k} \max_{x\in[0,1]} |f^{(j)}(x)|$. Prove that this is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach space.

3. Let $\alpha \in (0,1)$ and denote by $\Lambda_\alpha([0,1])$ the space of Hölder continuous functions of exponent α on $[0,1]$, defined by the following: $f \in \Lambda_\alpha$ if and only if $\|f\|_\alpha < \infty$, where
$$\|f\|_\alpha = |f(0)| + \sup_{x,y\in[0,1], x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$
Prove that $\|\cdot\|_\alpha$ is a norm that makes $\Lambda_\alpha([0,1])$ into a Banach space.

4. Let $\mathcal{X}$ be a finitely dimensional vector space. Prove the following:
   (1) Any two norms on $\mathcal{X}$ are equivalent;
   (2) The space $\mathcal{X}$ is a Banach space.

5. Let $\|\cdot\|$ be a seminorm on a vector space $\mathcal{X}$ and $\mathcal{M} = \{x \in \mathcal{X} : \|x\| = 0\}$. Prove that $\mathcal{M}$ is a vector subspace of $\mathcal{X}$ and that the map $x + \mathcal{M} \to \|x\|$ is a norm on the quotient space $\mathcal{X}/\mathcal{M}$.

6. Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper closed subspace of $\mathcal{X}$. Prove the following:
   (1) $\|x + \mathcal{M}\| = \inf\{\|x + y\| : y \in \mathcal{M}\}$ is a norm on $\mathcal{X}/\mathcal{M}$;
   (2) For any $\varepsilon \in (0,1)$ there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|x + \mathcal{M}\| > 1 - \varepsilon$;
   (3) The projection map $\pi(x) = x + \mathcal{M}$ from $\mathcal{X}$ to $\mathcal{X}/\mathcal{M}$ has norm 1;
   (4) If $\mathcal{X}$ is complete, so is $\mathcal{X}/\mathcal{M}$.

7. Prove that a linear functional f on a normed vector space $\mathcal{X}$ is continuous if and only if $f^{-1}(\{0\})$ is a closed subspace of $\mathcal{X}$.

8. Let $\mathcal{X}$ be a Banach space and $T \in L(\mathcal{X}, \mathcal{X})$.
   (1) Suppose $\|I - T\| < 1$, where I is the identity operator. Prove that T is invertible; in fact, the series $\sum_{n=0}^\infty (I-T)^n$ converges in $L(\mathcal{X}, \mathcal{X})$ to T^{-1}.
   (2) Suppose $T \in L(\mathcal{X}, \mathcal{X})$ is invertible, $S \in L(\mathcal{X}, \mathcal{X})$, and $\|S - T\| < \|T^{-1}\|^{-1}$, then S is also invertible. (Thus the set of invertibale operators in open in $L(\mathcal{X}, \mathcal{X})$.)