Math 240B. Winter 2020

Solution to Problems of HW #4

B. Li, Feb. 2020

1. (1) $l_j(x_j) = \prod_{\substack{i=0 \\ i\neq j}}^{n} \frac{x_j - x_i}{x_j - x_i} = 1$.

If $k \neq j$ then
$$l_j(x_k) = \prod_{\substack{i=0 \\ i\neq j}}^{n} \frac{x_k - x_i}{x_j - x_i} = 0$$
as one of the factors $x_k - x_i$ is $x_k - x_k = 0$.

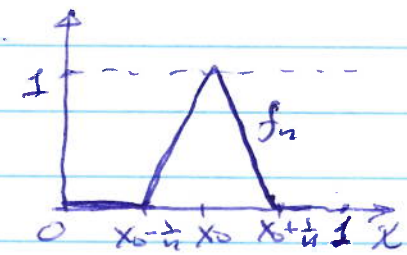
(2) Since each $l_j \in P_n$, $L_n f \in P_n$. By (1),
$$(L_n f)(x_k) = \sum_{j=0}^{n} f(x_j) l_j(x_k) = \sum_{j=0}^{n} f(x_j)\delta_{jk} = f(x_k)$$
$(k = 0, 1, \dots, n)$.

(3) Clearly $L_n : C([a,b]) \to C([a,b])$ is linear. $\forall f \in C([a,b])$, $\forall x \in [a,b]$
$$|(L_n f)(x)| \le \sum_{j=0}^{n} |f(x_j)|\,|l_j(x)| \le \|f\| \max_{x\in[a,b]} \sum_{j=0}^{n} |l_j(x)|$$
Hence, $\|L_n f\| \le \alpha \|f\|$, $\alpha := \max_{x\in[a,b]} \sum_{j=0}^{n} |l_j(x)|$

In particular, L_n is a bounded operator. Let $c \in [a,b]$ be such that $\alpha = \sum_{j=0}^{n} |l_j(c)|$. Define $g : [a,b] \to \mathbb{R}$ by $g(x_j) := \operatorname{sgn} l_j(c)$ $(j = 0, 1, \dots, n)$ and g is piecewise linear and g is continuous on $[a,b]$. On each $[x_{j-1}, x_j]$, g is linear (or more precisely, affine), connecting $(x_{j-1}, g(x_{j-1}))$ and $(x_j, g(x_j))$. Thus, $g \in C([a,b])$ and $\|g\| \le 1$. Moreover $(L_n g)(c) = \sum_{j=0}^{n} g(x_j) l_j(c) = \alpha$.
Hence $\|L_n\| \ge \|L_n g\| \ge |(L_n g)(c)| = \alpha$. So $\|L_n\| = \alpha$.

2. (1) If α, β are scalars and $f, g \in C([0,1])$, then
$$F(\alpha f+\beta g) = (\alpha f+\beta g)(x_0) = (\alpha f)(x_0)+(\beta g)(x_0)$$
$$= \alpha f(x_0)+\beta g(x_0) = \alpha F(f)+\beta F(g).$$
Hence F is linear. $|F(f)| = |f(x_0)| \le \|f\|$. Hence F is bounded with $\|F\| \le 1$. So, $F \in C([0,1])^*$.

(2) If there existed $g \in L^1([0,1])$ such that $\hat{F}(f) = \int_0^1 f(x)g(x)dx$ for all $f \in L^\infty([0,1])$, then for any $f \in C([0,1]) \subseteq L^\infty([0,1])$
$$f(x_0) = F(f) = \hat{F}(f) = \int_0^1 f(x)\,g(x)dx.$$
Let $n \in \mathbb{N}$ with $x_0 - \frac{1}{n} > 0$, $x_0 + \frac{1}{n} < 1$. Define $f_n \in C([0,1])$ as a piecewise linear function such that $f_n(x_0) = 1$, $f_n(x_0 - \frac{1}{n}) = 0$, and $f_n(x_0+\frac{1}{n}) = 0$. Also, $f_n = 0$ on $[0, x_0 - \frac{1}{n})$ and $[x_0+\frac{1}{n}, 1)$.



Then
$$1 = f_n(x_0) = \int_0^1 f_n(x)\,g(x)\,dx$$
$$\le \int_{x_0-\frac{1}{n}}^{x_0+\frac{1}{n}} |g(x)|dx \to 0$$
as $n \to \infty$, a contradiction. Hence, there exists no $g \in L^1([0,1])$ with the desired property.

3. (1) Let $\hat{x}_n, \hat{x} \in X^{**}$ be given by $\hat{x}_n(f) = f(x_n)$, $\hat{x}(f) = f(x)$ $\forall f \in X^*$. Hence, $\hat{x}_n(f) = f(x_n) \longrightarrow f(x) = \hat{x}(f)$ as $x_n \to x$ weakly. Since X^* is complete, the Principle of Uniform Boundedness implies that $\sup_{n\ge1} \|\hat{x}_n\| = \sup_{n\ge1} \|x_n\| < \infty$ (since $\sup_{n\ge1} \|\hat{x}_n(f)\| < \infty$ $\forall f \in X^*$ and since $\|\hat{x}_n\| = \|x_n\|$.)

(2) Since $f_n(x) \to f(x)$ $\forall x \in \mathcal{X}$, we have $\sup_{n\geq 1} |f_n(x)| < \infty$ for any $x \in \mathcal{X}$. Since $\mathcal{X}$ is complete, by the Principle of Uniform Boundedness, $\sup_{n \geq 1} \|f_n\| < \infty$.

4. Choose any $\hat{x}_1 \in \mathcal{X}$, $\hat{x}_1 \neq 0$, and set $x_1 = \hat{x}_1 / \|\hat{x}_1\|$. Then $\|x_1\| = 1$. Let $M_1 = \text{span}\{x_1\}$. Then M_1 is a one-dimensional subspace of $\mathcal{X}$. Hence M_1 is complete, and closed. By Riesz's Lemma, and the fact that $\dim \mathcal{X} = \infty$, $\exists x_2 \in \mathcal{X} \setminus M_1$ s.t. $\|x_2\| = 1$, and $\text{dist}(x_2, M_1) \geq \frac{1}{2}$. In particular, $\|x_1 - x_2\| \geq \frac{1}{2}$. Suppose we have $x_1, \cdots, x_n \in \mathcal{X}$ s.t. $\|x_j\| = 1$, $j = 1, \cdots, n$, and $\|x_j - x_k\| \geq \frac{1}{2}$, $j \neq k$. Then, $M_n = \text{span}\{x_1, \cdots x_n\}$ is a subspace of $\mathcal{X}$ of dimension n.

$\text{dist}(x_{j+1}, M_j) \geq \frac{1}{2}$ $(j = 1, \cdots, n-1)$
$M_j = \text{span}\{x_1, \cdots, x_j\}$

Hence it is closed. Again, by Riesz's Lemma (cf. Prob. 12(b) on page 156 of the text or Prob. 6 of HW#1), $\exists x_{n+1} \in \mathcal{X}$ with $\|x_{n+1}\| = 1$ such that $\text{dist}(x_{n+1}, M_n) \geq \frac{1}{2}$. Hence, $\|x_{n+1} - x_j\| \geq \frac{1}{2}$ $(j = 1, 2, \cdots, n)$. By induction, we have constructed $\{x_n\}_{n=1}^{\infty}$ with the desired properties.

5. Let $\{f_n\}_{n=1}^{\infty}$ be a countable dense subset of $\mathcal{X}^*$. By the definition of $\|f_n\|$, there exists $x_n \in \mathcal{X}$ such that $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. $(n = 1, 2, \cdots)$.
Let M denote the closure of $\operatorname{span}\{x_1, x_2, \cdots\}$. M is a closed subspace of $\mathcal{X}$. We show that $M = \mathcal{X}$

If not, $\exists\, x \in \mathcal{X} \setminus M$. By a consequence of the Hahn-Banach theorem, $\exists\, f \in \mathcal{X}^*$ such that $\|f\| = 1$, $f = 0$ on M, and $f(x) = \operatorname{dist}(x, M) > 0$. Since $\{f_1, f_2, \cdots, f_n, \cdots\}$ is dense in $\mathcal{X}^*$, there exists some n such that $\|f_n - f\| < \frac{1}{4}$. Thus,

$$
\begin{aligned}
1 = \|f\| &\leq \|f - f_n\| + \|f_n\| < \tfrac{1}{4} + 2|f_n(x_n)| \\
&= \tfrac{1}{4} + 2|f_n(x_n) - f(x_n)| \longleftarrow \left(\begin{array}{l}\text{since } f(x_n) = 0 \text{ as} \\ x_n \in M \text{ and } f = 0 \\ \text{on } M.\end{array}\right) \\
&\leq \tfrac{1}{4} + 2\|f_n - f\|\,\|x_n\| \\
&< \tfrac{1}{4} + 2 \cdot \tfrac{1}{4} = \tfrac{3}{4}.
\end{aligned}
$$

This is impossible. Hence $M = \mathcal{X}$.

Since K ($= \mathbb{R}$ or $\mathbb{C}$) is separable, there exists $A \subseteq K$, A is countable, and A is dense in K.
Denote $S = \{\sum_{j=1}^{n} a_j x_j : a_j \in A,\ j = 1, 2, \cdots, n;\ n = 1, 2, \cdots\}$.
Then S is countable. $\forall\, y \in \mathcal{X}$. $\forall\, \varepsilon > 0$. Since $M = \mathcal{X}$, $\exists\, n \in \mathbb{N}$ and $b_j \in K$ $(j = 1, \cdots, n)$ such that $\|\sum_{j=1}^{n} b_j x_j - y\| < \frac{\varepsilon}{2}$. Since A is dense in K, $\exists\, a_j \in A$ $(j = 1, \cdots, n)$ such that $|b_j - a_j| < \frac{\varepsilon}{2n}$ $(j = 1, \cdots, n)$. Thus, $\sum_{j=1}^{n} a_j x_j \in S$ and

$$
\begin{aligned}
\Big\|\sum_{j=1}^{n} a_j x_j - y\Big\| &\leq \Big\|\sum_{j=1}^{n} a_j x_j - \sum_{j=1}^{n} b_j x_j\Big\| + \Big\|\sum_{j=1}^{n} b_j x_j - y\Big\| \\
&\leq \sum_{j=1}^{n} |a_j - b_j|\,\|x_j\| + \frac{\varepsilon}{2} \leq n \cdot \frac{\varepsilon}{2n} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Hence, S is dense in $\mathcal{X}$, and $\mathcal{X}$ is separable.

6. The identity map $I:(\mathcal{X}, \|\cdot\|_2) \to (\mathcal{X}, \|\cdot\|_1)$ is linear, and bounded, since
$$\|Ix\|_1 = \|x\|_1 \le \|x\|_2 \quad \forall x \in \mathcal{X}.$$
Since I is bijective, the Open Mapping Theorem implies that the inverse $I:(\mathcal{X}, \|\cdot\|_1) \to (\mathcal{X}, \|\cdot\|_2)$ is continuous, i.e., bounded. Hence, $\exists M > 0$ such that
$$\|x\|_2 = \|Tx\|_2 \le M\|x\|_1 \quad \forall x \in \mathcal{X}.$$
Thus, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.

7. We need to show that there exists $M>0$ such that $\|Tx\| \le M$ for any $x \in \mathcal{X}$ with $\|x\|=1$.

Let $x \in \mathcal{X}$ with $\|x\|=1$. By a corollary of the Hahn-Banach theorem, $\exists g \in \mathcal{Y}^*$ s.t. $\|g\|=1$ and $g(Tx) = \|Tx\|$. Hence,

(*) $$\|Tx\| = g(Tx) \le \sup_{f \in \mathcal{Y}^*, \|f\|=1} |f(Tx)|.$$

(This inequality holds true for the case $Tx=0$)

For any $u \in \mathcal{X}$ define $F_u(f) = f(Tu) = (f\circ T)(u)$ $\forall f \in \mathcal{Y}^*$. Clearly $F_u : \mathcal{Y}^* \to K$ is linear. Moreover, $|F_u(f)| \le \|f\|\,\|Tu\|$. $\forall f \in \mathcal{Y}^*$. Hence F_u is bounded, i.e., $F_u \in \mathcal{Y}^{**}$. Consider F_u for all $u \in \mathcal{X}$ with $\|u\|=1$. We have for any $f \in \mathcal{Y}^*$ that $\sup_{u \in \mathcal{X}, \|u\|=1} |F_u(f)| = \sup_{u \in \mathcal{X}, \|u\|=1} |(f\circ T)u| \le \|f \circ T\| < \infty$.

Hence, by the Principle of Uniform Boundedness,

$$M := \sup_{u \in \mathcal{X}, \|u\|=1} \|F_u\| < \infty.$$

Back to (*), we get therefore

$$\|Tx\| = \sup_{f \in \mathcal{Y}^*, \|f\|=1} |F_x(f)| \le \|F_x\| \le M.$$

8. Let $x, y \in \mathcal{X}$ and $\alpha, \beta \in K$. Then

$$T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$$

$$= \lim_{n\to\infty}(\alpha T_n x + \beta T_n y) = \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y$$

$$= \alpha T x + \beta T y.$$

Hence, $T: \mathcal{X} \to \mathcal{Y}$ is linear.

Since for any $x \in \mathcal{X}$, $\lim_{n\to\infty} T_n x$ exists, $\{T_n x\}$ is bounded: $\sup_{n\ge 1} \|T_n x\| < \infty$. The Principle of Uniform Boundedness then implies that $\sup_{n\ge 1} \|T_n\| < \infty$. Denote $M = \sup_{n\ge 1} \|T_n x\|$. Then for any $x \in \mathcal{X}$, $\|T_n x\| \le M \|x\|$. Hence,

$$\|Tx\| = \|\lim_{n\to\infty} T_n x\| = \lim_{n\to\infty} \|T_n x\| \le M\|x\|.$$

Thus, T is also bounded. Hence, $T \in L(\mathcal{X}, \mathcal{Y})$.