Math 200a Fall 2020 Homework 3

Due 10/23/2020 by 7pm on Gradescope

Reading: Finish reviewing the notes about free groups and presentations from class (10/14 and 10/16 lectures). Read Chapter 4 in the text and begin to read Chapter 5.

Exercises to write up and hand in:

1. Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be the functions defined by the formulas $f(x) = -x$ and $g(x) = x+1$. Let $G = \langle f, g \rangle$ be the subgroup of $\mathrm{Sym}(\mathbb{Z})$ generated by f and g. Prove carefully that

$$G \cong \langle a, b \,|\, b^2 = 1, ba = a^{-1}b \rangle$$

(G is called the *infinite dihedral group*, D_∞.)

2. Prove that the following is a presentation of the quaternion group Q_8 as defined in Exercise 1.9 of the text:

$$Q_8 \cong \langle a, b \mid a^2 = b^2, a^{-1}ba = b^{-1} \rangle.$$

3. Consider the group G given by the presentation

$$G = \langle x, y \mid xy^2 = y^3x,\, yx^2 = x^3y \rangle.$$

Show that G is trivial.
(Hint. Establish that $x^2y^8x^{-2} = y^{18}$ and $x^3y^8x^{-3} = y^{27}$ using the first relation. Using the second relation, deduce that $y^9 = e$. Conclude.)

Remark: this exercise illustrates that it can be difficult it is to predict the size of a group from a glance at the presentation. The word problem asks if given a presentation $\langle X|W \rangle$ with X and W finite, there exists an always terminating algorithm for deciding whether a reduced word in $F(X)$ is equal to 1 in the presented group. In 1955 Novikov showed that in general the word problem is undecidable, i.e. there does not always exist such an algorithm.

4. Let $G = F(a, b)$ be a free group in two variables. Let H be the subgroup of G given by $H = \langle b, a^{-1}ba, a^{-2}ba^2, \dots \rangle$. Show that H is free on the subset $\{b, a^{-1}ba, a^{-2}ba^2, \dots\}$.

(Hint: Let $G' = F(w_0, w_1, w_2 \ldots)$ be a free group on countably many variables and let $\phi : G' \to G$ be the homomorphism with $w_i \mapsto a^{-i}ba^i$. Show that ϕ is injective by proving that no nontrivial reduced word in the w_i can map to 1 under ϕ.

Remark: The Nielsen-Schreier theorem shows that in fact every subgroup of a free group is free on some set of generators.

5. Presentations are useful for finding automorphisms of groups. Let $n \geq 3$ be fixed. As shown in class, the dihedral group D_{2n} has the presentation $\langle a, b | a^n = 1, b^2 = 1, ba = a^{-1}b \rangle$. Think of D_{2n} as this presented group. Show that any automorphism $\sigma : D_{2n} \to D_{2n}$ has $\sigma(a) = a^i$, where $0 \leq i \leq n-1$ with $\gcd(i, n) = 1$ and $\sigma(b) = a^j b$ for some $0 \leq j \leq n-1$. Moreover, given any i, j satisfying $0 \leq i, j \leq n-1$ and $\gcd(i, n) = 1$, there is a unique automorphism σ with $\sigma(a) = a^i$ and $\sigma(b) = a^j b$. Conclude that $|\operatorname{Aut}(D_{2n})| = n\varphi(n)$, where φ is the Euler φ-function.

6. By using similar techniques as in Problem 5, prove that $|\operatorname{Aut}(Q_8)| = 24$.