Solution and grading guidelines for the midterm exam.

Math 240C, Spring 2020. B. Li, 5/1/20

Problem #1

If $x \in K$ then $f(x) = \chi_K(x) = 1$. If $x \in K^c$ then $f(x) \geq 0$ and $\chi_K(x) = 0$. Hence $f \geq \chi_K$.

$$\text{Now, } \mu(K) = \int_K 1\, d\mu = \int_K f\, d\mu \overset{\text{since } K \subseteq U,\ f \geq 0.}{\leq} \int_U f\, d\mu$$

$$\overset{\text{since } 0 \leq f \leq 1}{\leq} \int_U d\mu = \mu(U).$$

Problem #2

Assume μ is a Radon measure. Since μ is finite, it is σ-finite. Thus by Proposition 7.5, $\forall E \in \mathcal{B}_X$, $\forall \varepsilon > 0$, $\exists K$: compact, U open. s.t. $K \subseteq E \subseteq U$ and $\mu(U \setminus K) < \varepsilon$.

Suppose for any $E \in \mathcal{B}_X$ any $\varepsilon > 0$, there exist K: compact and U: open, s.t. $K \subseteq E \subseteq U$ and $\mu(U \setminus K) < \varepsilon$. Then, $\mu(K) > \mu(E) - \varepsilon$ (as $\mu(E \setminus K) \leq \mu(U \setminus K) < \varepsilon$) and $\mu(U) < \mu(E) + \varepsilon$ (as $\mu(U \setminus E) \leq \mu(U \setminus E) < \varepsilon$). Thus μ is regular. Since μ is finite, it is finite on compact sets. Thus μ is a Radon measure.

Problem #3

(1) True. If μ is a finite Borel measure on $\mathbb{R}^n$ then it is finite on compact sets. If U is open in $\mathbb{R}^n$, then it is the countable union of compact sets $\overline{B(x,r)}$ where $x \in U \cap \mathbb{Q}^n$ and $r \in (0,\infty) \cap \mathbb{Q}$. Hence, by Thm 7.8, μ is a Radon measure.

(2) False. Example. $X = \mathbb{R}^n$. μ = Lebesgue measure. $f(x) = \frac{1}{1+|x|}$, $x \in \mathbb{R}$. $f \in C_0(\mathbb{R})$ but $f \in L^1(\mathbb{R})$.

Problem 4

Let $U \subseteq X$ be open. Let $\varepsilon > 0$. By the Riesz Representation Thm for positive and linear functionals on $C_c(X)$:

$\mu(U) = \sup\{ \int f \, d\mu : f \in C_c(X), f \prec U \}$.

Since μ is finite, $\exists f \in C_c(X), f \prec U$ s.t. $\mu(U) < \int f \, d\mu + \varepsilon$. Since $\mu_n \to \mu$ vaguely. $\int f \, d\mu_n \to \int f \, d\mu$. Hence, $\mu(U) < \lim_n \int f \, d\mu_n + \varepsilon$.

But each $\int f \, d\mu_n \leq \mu_n(U)$ $(n = 1, 2, \cdots)$

Thus, $\lim_n \int f \, d\mu_n = \liminf_{n\to\infty} \int f \, d\mu_n \leq \liminf_{n\to\infty} \mu_n(U)$

i.e., $\mu(U) < \liminf_n \mu_n(U) + \varepsilon$

Thus $\mu(U) \leq \liminf_{n\to\infty} \mu_n(U) \leq \limsup_{n\to\infty} \mu_n(U)$.

Let $F \subseteq X$ be compact. Then by the Riesz Representation Thm for positive linear functionals,

$$\mu(F) = \inf\left\{\int f\,d\mu : f \in C_c(X),\ f \geq \chi_F\right\}.$$

$\forall \varepsilon > 0$. Since $\mu(F) < \infty$ (as μ is a Radon measure), $\exists f \in C_c(X)$, $f \geq \chi_F$ s.t.

$\mu(F) > \int f\,d\mu - \varepsilon$. Thus, since $\int f\,d\mu_n \to \int f\,d\mu$,

$$\mu(F) > \lim_{n\to\infty} \int f\,d\mu_n - \varepsilon = \limsup_{n\to\infty} \int f\,d\mu_n - \varepsilon$$

$$\geq \limsup_{n\to\infty} \mu_n(F) - \varepsilon.$$

Thus, $\liminf_{n\to\infty} \mu_n(F) \leq \limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$.