Math 240A: Real Analysis, Fall 2019

Homework Assignment 6
Due Friday, November 15, 2019

1. Let $f \in L^1(m)$. Assume $f(0) = 0$ and $f'(0)$ exists. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(0) = 0$ and $g(x) = f(x)/x$ if $x \neq 0$. Prove that $g \in L^1(m)$.

2. (1) Find the smallest $c \in \mathbb{R}$ such that $\log\left(1 + e^t\right) < c + t$ for all $t \in (0, \infty)$.

   (2) Let $f : [0, 1] \to [0, \infty)$ be Lebesgue integrable. Show that the following limit exists and calculate its value:
$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\left[1 + e^{nf(x)}\right] dx.$$

3. Construct Lebesgue integrable functions $f_n : [0, 1] \to [0, 1]$ $(n = 1, 2, \dots)$ such that $\lim_{n\to\infty} \int_0^1 f_n \, dm = 1$ and $\{f_n(x)\}$ diverges for any $x \in [0, 1]$.

4. Prove the following variant of Egoroff's theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume: (1) $f, f_n : X \to \mathbb{C}$ are all measurable and $f_n \to f$ a.e.; (2) there exists $g \in L^1(\mu)$ such that $|f_n| \le g$ on X for all n. Then, for any $\varepsilon > 0$, there exists $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.

5. Prove Lusin's Theorem: Let $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{C}$ be Lebesgue measurable. For any $\varepsilon > 0$, there exists a compact set $E \subseteq [a, b]$ such that $m(E^c) < \varepsilon$ and $f|_E$ is continuous.

6. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. Let $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be two functions and define $h : X \times Y \to \mathbb{C}$ by $h(x, y) = f(x)g(y)$ for any $x \in X$ and $y \in Y$. Prove the following:

   (1) If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable and $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, then $h : X \times Y \to \mathbb{C}$ is $\mathcal{M} \otimes \mathcal{N}$-measurable;

   (2) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int_{X\times Y} h \, d(\mu \times \nu) = \left(\int_X f \, d\mu\right)\left(\int_Y g \, d\nu\right)$.

7. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $f : X \to [0, \infty)$ a measurable function, and
$$G_f = \{(x, y) \in X \times [0, \infty) : y \le f(x)\}.$$
   Prove that G_f is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$-measurable and that $(\mu \times m)(G_f) = \int_X f \, d\mu$.

8. Prove
$$\int_0^1 \int_0^\infty \left(e^{-xy} - 2e^{-2xy}\right) dydx \neq \int_0^\infty \int_0^1 \left(e^{-xy} - 2e^{-2xy}\right) dxdy.$$

9. Use Fubini's Theorem and the formula $\frac{1}{x} = \int_0^\infty e^{-xt} \, dt$ $(x > 0)$ to prove $\lim_{A\to\infty} \int_0^A \frac{\sin x}{x} \, dx = \frac{\pi}{2}$.

10. Let $a > 0$, $f : (0, a) \to \mathbb{R}$ be Lebesgue integral on $(0, a)$, and $g(x) = \int_x^a t^{-1} f(t) \, dt$ $(0 < x < a)$. Prove that g is integrable on $(0, a)$ and $\int_{(0,a)} g \, dm = \int_{(0,a)} f \, dm$.