Math 240B: Real Analysis, Winter 2020

Homework Assignment 8

Due Friday, March 13, 2020

1. Prove that every open set in a second countable locally compact Hausdorff space is σ-compact.

2. Prove the following:
   (1) The product of finitely many locally compact spaces is locally compact;
   (2) The product of countably many sequentially compact spaces is sequentially compact.

3. Let $K \in C([0,1] \times [0,1])$. For $f \in C([0,1])$, define
$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy \qquad \forall x \in [0,1].$$
Prove that $Tf \in C([0,1])$ and that the set $\{Tf : \|f\|_u \le 1\}$ is precompact in $C([0,1])$.

4. Let $\mathcal{X}$ be a separable normed vector space. Prove that the weak-$*$ topology on the closed unit ball in $\mathcal{X}^*$ is second countable and hence metrizable.

5. Let $\mathcal{X}$ be a Banach space. Prove the following:
   (1) The norm-closed unit ball $B = \{x \in \mathcal{X} : \|x\| \le 1\}$ is also weakly closed;
   (2) The weak closure of any bounded subset of $\mathcal{X}$ is also bounded;
   (3) The weak-$*$ closure of any bounded subset of $\mathcal{X}^*$ is also bounded;
   (4) Every weak-$*$ Cauchy sequence in $\mathcal{X}^*$ converges.

6. Let $C^\infty(\mathbb{R})$ denotes the set of all infinitely differentiable functions on $\mathbb{R}$. For each $j \in \mathbb{N}$ and $f \in C^\infty(\mathbb{R})$, define
$$p_{j,k}(f) = \max_{-j \le x \le j} |f^{(k)}(x)| \qquad (k = 0, 1, \dots).$$
It is clear that each $p_{j,k}$ is a seminorm on $C^\infty(\mathbb{R})$. Prove the following:
   (1) The topological vector space defined by the family of seminorms $p_{j,k}$ ($j = 1, 2, \dots$ and $k = 0, 1, \dots$) is Hausdorff and metrizable;
   (2) The convergence $f_n \to f$ in this space is equivalent to $f_n^{(k)} \to f^{(k)}$ uniformly on any compact subsets of $\mathbb{R}$ for all $k \ge 0$.

7. Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a vector subspace of $\mathcal{X}$. Prove that $\mathcal{M}$ is norm-closed if and only if it is weakly closed.

8. Let $\mathcal{X}$ be a Banach space, $T_n \to T$ and $S_n \to S$ in $L(\mathcal{X}, \mathcal{X})$, and $x_n \to x$ in $\mathcal{X}$. Prove that $T_n x_n \to Tx$ in $\mathcal{X}$ and $T_n S_n \to TS$ strongly in $L(\mathcal{X}, \mathcal{X})$.