Math 240A: Real Analysis, Fall 2019

Final Exam

Name ______________________________ **ID number** ______________________________

Note: This is a close-book and close-note exam. There are 8 problems of total 200 points. To get credit, you must show your work. Partial credit will be given to partial answers.

Problem	1	2	3	4	5	6	7	8	Total
Score									

Notation: We use m to denote the Lebesgue measure.

1. (40 points) True or false? If true, then prove it. If fasls, give a counterexample. (Note: There are 4 subproblems.)

(1) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f : X \to [0, \infty)$ be μ-integrable. If $\displaystyle\int_E f \, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ μ-a.e.
Your answer: __________________

(2) Let f and g be both Lebesgue integrable on $\mathbb{R}^n$. Assume $f = g$ m-a.e. Then the Lebesgue set of f is the same as the Lebesgue set of g.
Your answer: __________________

(3) If μ is a finite Borel measure on $\mathbb{R}$, then there exists $f \in L^1_{\text{loc}}(m)$ such that $d\mu = f\,dm$.
Your answer: ____________________

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0, \infty)$ be μ-integrable. Define $f_n(x) = \min(f(x), n)$ for any $x \in X$ $(n = 1, 2, \ldots)$. Then $\displaystyle\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu$.
Your answer: ____________________

2. (25 points) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Prove the following:

(1) The measure μ is semifinite, i.e., if $A \in \mathcal{M}$ and $\mu(A) = \infty$, then there exists $B \in \mathcal{M}$ such that $B \subset A$ and $0 < \mu(B) < \infty$;

(2) If $E \in \mathcal{M}$ and $\mu(E) = \infty$, then $\sup\{\mu(F) : F \in \mathcal{M}, F \subseteq E, \text{ and } 0 < \mu(F) < \infty\} = \infty$.

3. (20 points) Let μ and ν be two finite Borel measures on $X = [0, 1]$. Suppose that

$$\int_X f \, d\mu = \int_X f \, d\nu$$

for any continuous function $f : [0, 1] \to \mathbb{R}$. Prove that $\mu = \nu$.

4. (20 points) Assume $f \in C([0,1])$, $g \in L^1(m,[0,1])$, and

$$\int_0^1 e^{-x/n} g(x)\,dx = n \int_0^1 e^{-nx} f(x)\,dx \qquad (n = 1, 2, \dots).$$

Prove that

$$\int_0^1 g(x)\,dx = f(0).$$

5. (20 points) Let $a > 0$, $f : (0, a) \to \mathbb{R}$ be Lebesgue integral on $(0, a)$, and

$$g(x) = \int_x^a t^{-1} f(t)\, dt \quad (0 < x < a).$$

Prove that g is integrable on $(0, a)$ and

$$\int_{(0,a)} g\, dm = \int_{(0,a)} f\, dm.$$

6. (30 points) Let $f(x) = x^2 - 1$ ($x \in \mathbb{R}$) and $d\nu = f\,dm$.

(1) Prove that ν is σ-finite.

(2) Find a Hahn decomposition (P, N) of $\mathbb{R}$ for ν.

(3) Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of μ. Calculate $\nu^+([0,2])$, $\nu^-([0,2])$, and $|\nu|([-1/2, 1/2])$.

7. (25 points) Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x+1 & \text{if } 0 \le x < 1, \\ 4 & \text{if } x \ge 1. \end{cases}$$

(1) Calculate the total variation of F on $\mathbb{R}$.

(2) Note that the function $F \in$ NBV and therefore there exists a unique Boreal measure, μ_F associated to F. Find explicitly the Lebesgue–Radon–Nikodym decomposition $d\mu_F = d\lambda + f dm$. (i.e., find the measure λ and also $f \in L^1(m)$.)

8. (20 points)

(1) Suppose $g : [0,1] \to \mathbb{R}$ is absolutely continuous on $[0,1]$ and $g' = 0$ m-a.e. on $[0,1]$. Prove that g is a constant function on $[0,1]$.

(2) Let $f \in L^1([0,1], m)$ be real-valued and define $F : [0,1] \to \mathbb{R}$ by $F(x) = \displaystyle\int_0^x f\,dm$. Prove that the function F is absolutely continuous on $[0,1]$ and that the total variation of F on $[0,1]$, $\mathrm{TV}\,(F;[0,1])$, satisfies $\mathrm{TV}\,(F,[0,1]) \leq \displaystyle\int_0^1 |f|\,dm$. (The inequality above is in fact an equality. But you do not need to prove that.)