SYMMETRIC POLYNOMIALS IN SUPERSPACE

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Overview

- Symmetric Polynomials
  - Useful bases
  - Schur Functions
  - Pieri Rules
  - Macdonald polynomials
- Superspace
  - Motivation
  - Analogs to the classical bases
  - Analogs to the Pieri Rules
- Future Research
  - Murnaghan-Nakayama Rules
  - Jacobi-Trudi Identities

Symmetric Polynomials

Symmetric polynomials

- Let $\mathbb{Q}[x_1, \dots, x_N]^{S_N}$ be the polynomial ring in the variables $x_1, \dots, x_N$ over $\mathbb{Q}$ such that for any polynomial:

$$f(x_1, \dots, x_N) = f\left(x_{\sigma(1)}, \dots, x_{\sigma(N)}\right)$$

  For all permutations $\sigma \in S_N$.
- Homogeneous symmetric polynomials of degree n are symmetric functions such that each term is a degree n monomial.
- The set of these is called Λ_N^n. This is a vector space over $\mathbb{Q}$. (we will generally just refer to it as Λ^n.)

Symmetric polynomials

- Notice that if $f \in \Lambda^n$ and $g \in \Lambda^m$ then $fg \in \Lambda^{n+m}$
- So $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \ldots$ (vector space direct sum) is a graded $\mathbb{Q}$-algebra.

Symmetric polynomials (examples)

- $x_1 + x_2 + x_3 + x_1x_2x_3$
- $x_1x_2x_3$
- $x_1x_2 + x_1x_3 + x_2x_3$
- $x_1^5x_2 + x_1^5x_3 + x_2^5x_3 + x_1x_2^5 + x_1x_3^5 + x_2x_3^5$

Monomial base

- $x_1 + x_2 + x_3 + x_1x_2x_3 = m_{(1)} + m_{\cancel{(3)}}$ (111)
- $x_1x_2x_3 = \cancel{m_{(3)}} \in \Lambda^3$ $m_{(111)}$
- $x_1x_2 + x_1x_3 + x_2x_3 = m_{\cancel{(2)}} \in \Lambda^2$ (11)
- $x_1^5x_2 + x_1^5x_3 + x_2^5x_3 + x_1x_2^5 + x_1x_3^5 + x_2x_3^5 = m_{(5,1)} \in \Lambda^6$

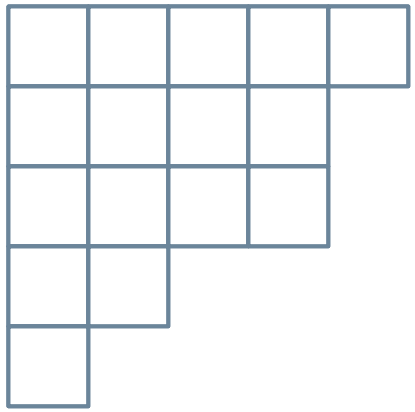
- In general,

$$m_\lambda = \sum_\alpha x^\alpha$$

- $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \dots x_N^{\alpha_N}$
- The sum is over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \dots)$ of the partition $\lambda = (\lambda_1, \lambda_2, \dots)$.

Partitions

- m_λ for all $\lambda \vdash n$ form a basis of Λ^n.
- $\lambda \vdash n$ means that λ *partitions* n.
- In other words $\lambda = (\lambda_1, \dots, \lambda_k)$ is a non-increasing sequence of integers such that $\sum_i \lambda_i = n$
- Young Diagram Example: $(5,4,4,2,1)$



Partitions

- $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ^n
- $\dim(\Lambda^n) = p(n)$ the number of partitions of n.

- There are some other bases of particular interest…

Multiplicative bases

- Elementary: $e_\lambda = e_{\lambda_1} \dots e_{\lambda_k}$

$$e_n = m_{(1^n)} = \sum_{i_1 < \cdots < i_n} x_{i_1} \dots x_{i_n}$$

Multiplicative bases

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- Power: $p_\lambda = p_{\lambda_1} \ldots p_{\lambda_k}$

$$p_n = m_{(n)} = \sum_i x_i^n$$

Multiplicative bases

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$$p_n = m_{(n)} = \sum_i x_i^n$$

- Complete Homogeneous: $h_\lambda = h_{\lambda_1} \dots h_{\lambda_k}$

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \dots x_{i_n}$$

Elementary

- Elementary: $e_\lambda = e_{\lambda_1} \dots e_{\lambda_k}$

$$e_\lambda = m_{(1^{\lambda_1})} m_{(1^{\lambda_2})} \dots m_{(1^{\lambda_k})}$$

Let $e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$.

Then the $M_{\lambda\mu}$'s are positive integers.

e_λ is m-positive and e_λ is m-integral. This suggests there may be a combinatorial interpretation of $M_{\lambda\mu}$.

$$e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$$

- The $M_{\lambda\mu}$ can be described as the number of (0,1) matrices A such that
- row sums sum to λ and
- column sums sum to μ.
- Reasoning: let $X = \begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ x_1 & x_2 & x_3 & \dots \\ \dots & \dots & & \dots \end{pmatrix}$ $\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}$

$\mu_1 \quad \mu_2$

- You want to pick λ_i factors from the i^{th} row.
- Make sure that you picked x_j, μ_j times.

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- You want to pick λ_i factors from the i^{th} row.
- Make sure that you picked x_j, μ_j times.

The matrix $M_{\lambda\mu}$.

- Notice that $M_{\lambda\lambda'} = 1$ because there is only one way to fill a (0,1)-matrix such that the row sums are λ and column sums are λ'.

$\lambda = (5, 2, 1)$

$\lambda' = (3, 2, 1, 1, 1)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & \end{pmatrix}$$

The matrix $M_{\lambda\mu}$.

- Notice that $M_{\lambda\lambda'} = 1$ because there is only one way to fill a (0,1)-matrix such that the row sums are λ and column sums are λ'.
- Also, $M_{\lambda\mu'} = 0$ if $\lambda < \mu$ in dominance order.

$\{e_\lambda \vdash n\}$ is a basis for Λ^n

- Let $\overrightarrow{e_n} = \left(e_{\lambda^1}, \dots e_{\lambda^{p(n)}}\right)^T$ with the dominance order on all partitions of n.
- Let $(\overrightarrow{m_n})' = \left(m_{\lambda^{1\prime}}, \dots m_{\lambda^{p(n)\prime}}\right)^T$ with the dominance order on the conjugate partitions.
- Then consider the matrix M such that: $\overrightarrow{e_n} = M(\overrightarrow{m_n})'$
- Then M is upper triangular and has ones down the diagonal so it must be invertible. Furthermore, the entries of M^{-1} must be integers. ($\det(M) = \pm 1$)
- (This is called the fundamental theorem of symmetric functions.)

$\{h_\lambda \vdash n\}$ is a basis for Λ^n

- There is a really slick generating function proof to show that $\{h_\lambda \vdash n\}$ is a basis.
- $H(t) = \sum_i h_i t^i = \prod_i \frac{1}{1 - x_i t}$
- $E(t) = \sum_i e_i t^i = \prod_i (1 + x_i t)$
- Then $H(t)E(-t) = 1$.

The isomorphism $\omega: \Lambda \to \Lambda$

- Consider the endomorphism ω defined to be $\omega(e_n) = h_n$.
- Then since $H(t)E(-t) = 1$,
- $\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = 0$, for all $n \geq 1$.
- Then apply ω to the sum to get:

$$\sum_{i=0}^{n}(-1)^i h_i \omega(h_{n-i}) = (-1)^n \sum_{i=0}^{n}(-1)^i \omega(h_i) h_{n-i} = 0$$

- And so $\omega(h_n) = e_n$ is an involution.
- Therefore $\{h_\lambda \vdash n\}$ is a basis for Λ^n.

$\{p_\lambda \vdash n\}$ is a basis for Λ^n

- I'm not going to go into the details of why the power symmetric functions are a basis.
- Let's consider a few facts about p_λ:
- $\omega(p_n) = (-1)^{n-1} p_n$.
- $h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda$
- $e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda$

- Where for $\lambda = \langle 1^{m_1} 2^{m_2} \ldots \rangle$, we have

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \ldots$$

$$\varepsilon_\lambda = (-1)^{m_2 + m_4 + \cdots} = (-1)^{n - \ell(\lambda)}$$

Scalar product.

- We can define a scalar product by letting the h's and the m's be dual bases:
- $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$
- Then the p's form an orthogonal basis:
- $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$
- So $\left\{ \frac{p_\lambda}{\sqrt{z_\lambda}} : \lambda \vdash n \right\}$ is an orthonormal basis of $\Lambda^n_{\mathbb{R}}$.

Is there an orthonormal basis of $\Lambda^n_{\mathbb{Z}}$?

Is there an orthonormal basis of $\Lambda^n_{\mathbb{Z}}$?

- The Schur functions $\{s_\lambda : \lambda \vdash n\}$ are a very special basis of Λ^n.

- $s_\lambda = m_\lambda + \text{smaller terms}$
- $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

- $s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$

Schur function facts!!!!

- $\omega(s_\lambda) = s_{\lambda'}$
- Cauchy Identities:

$$\prod_{i,j}\left(\frac{1}{1 - x_i y_j}\right) = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$$

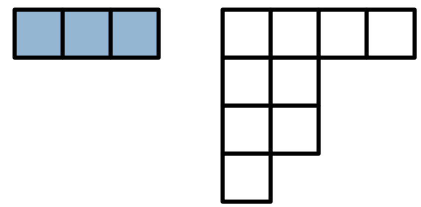
$$\prod_{i,j}(1 + x_i y_j) = \sum_{\lambda} s_\lambda(x) s_{\lambda'}(y)$$

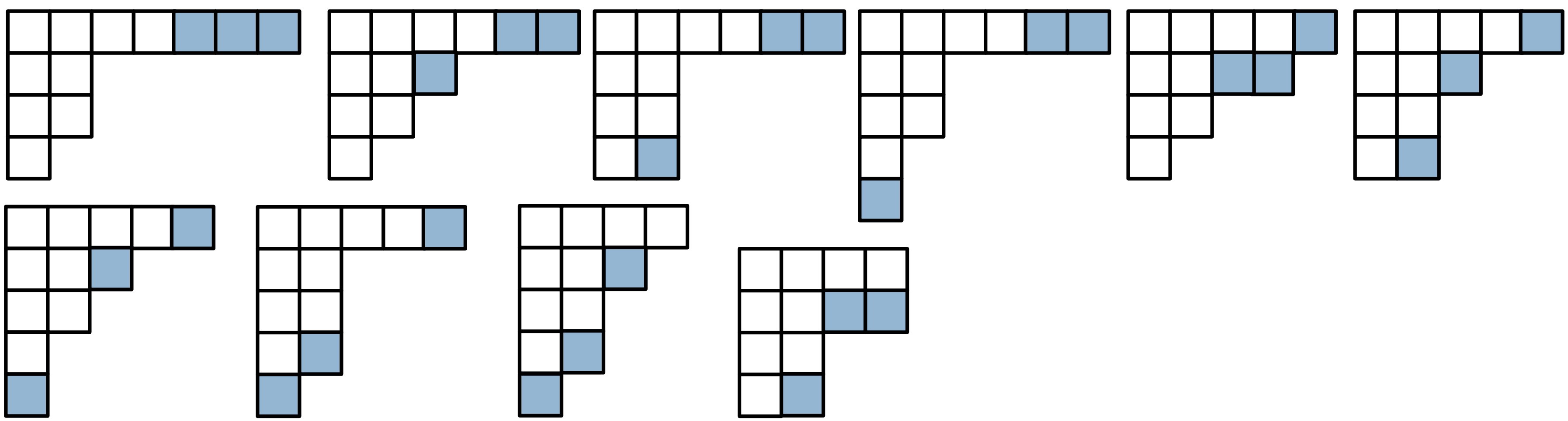
Pieri Rules:

- What is $h_n s_\lambda$?
- It is $\sum_\mu s_\mu$ where it sums over all partitions μ such that μ/λ is a horizontal n-strip.

Pieri Rules:

- Example: $h_3 s_{(4,2,2,1)}$

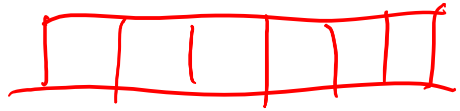




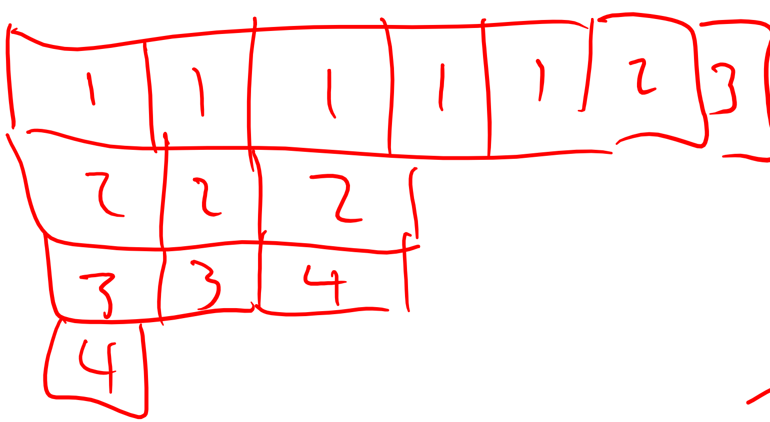
- Example: $h_3 s_{(4,2,2,1)} = s_{(7,2,2,1)} + s_{(6,3,2,1)} + s_{(6,2,2,2)} + s_{(6,2,2,1,1)} + s_{(5,4,2,1)} + s_{(5,3,2,2)} + s_{(5,3,2,1,1)} + s_{(5,2,2,2,1)} + s_{(4,3,2,2,1)} + s_{(4,4,2,2)}$

Pieri Rules:

- $h_n = s_{(n)}$



- So $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_{\ell(\lambda)}} = \sum_{\mu \vdash n} A_{\lambda\mu}\, s_\mu$.
- How to interpret $A_{\lambda\mu}$?



$h_5 \cdot h_4 h_3 h_2$

h

Pieri Rules:

- $h_n = s_{(n)}$

- So $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_{\ell(\lambda)}} = \sum_{\mu \vdash n} A_{\lambda\mu}\, s_\mu$.
- How to interpret $A_{\lambda\mu}$?

- Build a Young tableau with λ_1 1's, λ_2 2's and so on so that each new number is a horizontal strip.

- These are called semi-standard young tableau.

Semi-standard Young tableaux:

- $A_{\lambda\mu}$ is the number of SSYT's of shape λ, content μ.
- An SSYT of shape λ, content μ is a Young diagram of shape λ with μ_1 1's, μ_2 2's, and so on such that the rows are weakly increasing and the columns are strictly increasing.
- Example: an SSYT of
- shape $(5,3,2,1,1)$ and
- content $(3,3,2,2,1,1)$

$$\begin{ytableau} 1 & 1 & 1 & 2 & 4 \\ 2 & 2 & 3 \\ 3 & 4 \\ 5 \\ 6 \end{ytableau}$$

Kostka numbers.

- The Kostka numbers $K_{\lambda\mu}$ are the coefficients of the expansion of the Schur functions in terms of the monomial functions.

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$$

Kostka numbers.

$$h_\lambda = \sum_{\mu \vdash n} A_{\lambda\mu} s_\mu \qquad\qquad s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$$

- Apply the scalar product of h_λ and s_γ

$$\langle h_\lambda, s_\gamma \rangle$$

$$\left\langle h_\lambda, \sum_{\mu \vdash n} K_{\gamma\mu} m_\mu \right\rangle = \left\langle \sum_{\mu \vdash n} A_{\lambda\mu} s_\mu \, , s_\gamma \right\rangle$$

$$K_{\gamma\lambda} = A_{\lambda\gamma}$$

Kostka numbers.

- Recall that $A_{\lambda\mu}$ is the number of SSYT's of shape λ content μ.
- Then $K_{\lambda\mu}$ is the number of SSYT's of shape μ, content λ.

- Consequences:
- Schur functions are m-positive and m-integral.

Macdonald Polynomials

- Macdonald Polynomials are a generalization of Schur functions.
- We introduce two more variables q, t and a new scalar product:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

- The Macdonald Polynomials are defined to be the unique basis
  - $M_\lambda^{(q,t)} = m_\lambda +$ smaller terms
  - $\left\langle M_\lambda^{(q,t)}, M_\mu^{(q,t)} \right\rangle_{q,t} = 0$ if $\lambda \neq \mu$

Macdonald polynomials

- The Macdonald polynomials are a basis of the algebra of symmetric functions in variables $x_1, x_2, \ldots$ with coefficients in the field $\mathbb{Q}(q,t)$ of rational functions in two parameters q and t.
- They were introduced in 1988 by Macdonald to unify the two well-known one parameter bases of the algebra of symmetric functions, namely, the Hall-Littlewood polynomials and the Jack polynomials
- It promptly became clear that the discovery of Macdonald polynomials was fundamental and sure to have many ramifications.
- Developments in the years since have borne this out, notably, Cherednik's proof of the Macdonald constant-term identities and other discoveries relating Macdonald polynomials to representation theory of quantum groups and affine Hecke algebras the Calogero–Sutherland model in particle physics and combinatorial conjectures on diagonal harmonics

Taken from HILBERT SCHEMES, POLYGRAPHS, AND THE MACDONALD POSITIVITY CONJECTURE by Mark Haiman

Macdonald polynomials

$$\begin{tikzcd} M_\lambda^{(q,t)} \arrow[d, "\begin{array}{l}\bullet\ q = t^\alpha \\ \bullet\ t \to 1\end{array}"] \\ J_\lambda^{(\alpha)} \arrow[d, "\bullet\ \alpha \to 1"] \\ s_\lambda \end{tikzcd}$$

- COMBINATORICS
- PHYSICS/combinatorics
- combinatorics

Macdonald positivity

- $\widetilde{M}_{\lambda}^{(q,t)} = \sum_{\mu \vdash n} K_{\lambda\mu}(q,t) s_{\mu}$
- Recall $K_{\lambda\mu} = K_{\lambda\mu}(0,1)$ is the number of SSYT of shape μ and content λ.
- $K_{\lambda\mu}(1,1)$ is the number of standard tableaux of shape μ.

Macdonald positivity

- It has been proven that $K_{\lambda\mu}(q,t) \in \mathbb{N}(q,t)$ or in other words that the q,t-Kostka numbers are non-negative integers.

- (It is still an open problem to find a combinatorial interpretation of $K_{\lambda\mu}(q,t)$)

Macdonald positivity

- Example:
- $\widetilde{M}_{(2,1)}^{(q,t)} = t s_{(3)} + (1 + qt) s_{(2,1)} + q s_{(1,1,1)}$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Motivation for Superspace

- Example:
- $\widetilde{M}^{(q,t)}_{(2,1)} = {\color{blue}t}s_{(3)} + ({\color{red}1 + qt})s_{(2,1)} + {\color{green}q}s_{(1,1,1)}$

$$\color{blue}\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \qquad \color{red}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \color{green}\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

- $\widetilde{M}^{(q,t)}_{(1;1)} = {\color{blue}t}s_{(2;)} + 1s_{(1;1)} + {\color{red}qt}s_{(0;2)} + {\color{green}q}s_{(0;1,1)}$

Superspace

Supersymmetry

- Motivated by the Calogero-Sutherland model.
- There are 2 types of particles in nature:
  - Bosons
  - Fermions

$$\Psi \to \Psi$$

Exchange of two bosons

$$\Psi \to -\Psi$$

Exchange of two fermions

(Pauli's exclusion principle.)

Symmetric polynomials in superspace

- Let $\mathbb{Q}[x_1, \ldots, x_N, \theta_1, \ldots, \theta_N]^{S_N}$ be the polynomial ring in the commuting variables $x_1, \ldots, x_N$ and the anti-commuting variables $\theta_1, \ldots, \theta_N$ over $\mathbb{Q}$ such that for any polynomial:

$$f(x_1, \ldots, x_N, \theta_1, \ldots, \theta_N) = f\left(x_{\sigma(1)}, \ldots, x_{\sigma(N)}, \theta_{\sigma(1)}, \ldots, \theta_{\sigma(N)}\right)$$

For all permutations $\sigma \in S_N$.

With $\theta_i \theta_j = -\theta_j \theta_i$ and $\theta_i^2 = 0$.

Symmetric polynomials in superspace

- Examples:
- $x_1\theta_1\theta_2 - x_2\theta_1\theta_2$
- $x_1^2x_2\theta_1 + x_1x_2^2\theta_2$
- $x_1^2x_2x_3^2\theta_1\theta_2 + x_1^2x_2^2x_3\theta_1\theta_3 - x_1x_2^2x_3^2\theta_1\theta_2 - x_1^2x_2x_3^2\theta_2\theta_3 + x_1^2x_2^2x_3\theta_2\theta_3 - x_1x_2^2x_3^2\theta_1\theta_3$

- Just like before, we can define a monomial basis by describing a monomial and taking all rearrangements.

Superpartitions

- As you can imagine, these polynomials have a bigger dimension.
- The bases are indexed by Superpartitions.
- You need information about the θ's and the x's.

- Let's say the typical monomial looks like:

$$x_1^2 x_2^4 x_3 x_4^3 x_5^8 \theta_3 \theta_4$$

Supermonomials

- Let's say the typical monomial looks like:

$$x_1^2 x_2^4 x_3 x_4^3 x_5^8 \theta_3 \theta_4$$

- Then if a symmetric polynomial in superspace has this as a term, it must have all rearrangements.
- Let's pick a representative of these arrangements:
  - The representative should have $\theta_1 \theta_2 \ldots \theta_m$ as a factor.

Supermonomials

- Let's say the typical monomial looks like:

$$x_1^2 x_2^4 x_3 x_4^3 x_5^8 \theta_3 \theta_4$$

- Then if a symmetric polynomial in superspace has this as a term, it must have all rearrangements.
- Let's pick a representative of these arrangements:
  - The representative should have $\theta_1 \theta_2 \ldots \theta_m$ as a factor.

$$x_3^2 x_4^4 x_1 x_2^3 x_5^8 \theta_1 \theta_2$$

  - Then arrange the variables $x_1, \ldots, x_m$ in decreasing order of powers. And the remaining x's in decreasing order of powers.

$$-x_1^3 x_2 x_3^8 x_4^3 x_5^2 \theta_1 \theta_2$$

Supermonomials

- So we need a partition with m parts for $x_1, \dots, x_m$: Λ_a
- And a partition for the remaining $x_{m+1}, \dots, x_N$: Λ_s

- We say that $\Lambda = (\Lambda_a; \Lambda_s)$

$$x_1^3 x_2 x_3^8 x_4^3 x_5^2 \theta_1 \theta_2$$

- In this example, it would be $(3,1; 8,3,2)$

- So $m_{(3,1;8,3,2)}$ is the term above plus all other rearrangements.

Supermonomials

- Examples: $m_{(\Lambda_a;\Lambda_s)}$
- $x_1\theta_1\theta_2 - x_2\theta_1\theta_2 \in m_{(1,0;\emptyset)}$
- $x_1^2x_2\theta_1 + x_1x_2^2\theta_2 \in m_{(2;1)}$
- $x_1^2x_2x_3^2\theta_1\theta_2 + x_1^2x_2^2x_3\theta_1\theta_3 - x_1x_2^2x_3^2\theta_1\theta_2 - x_1^2x_2x_3^2\theta_2\theta_3 + x_1^2x_2^2x_3\theta_2\theta_3 - x_1x_2^2x_3^2\theta_1\theta_3 \in m_{(2,1;2)}$

$$m_{(\Lambda_a;\Lambda_s)} = \sum_{\substack{\text{all distinct} \\ \text{rearrangements}}} x^{\Lambda}\,\theta_1 \ldots \theta_m$$

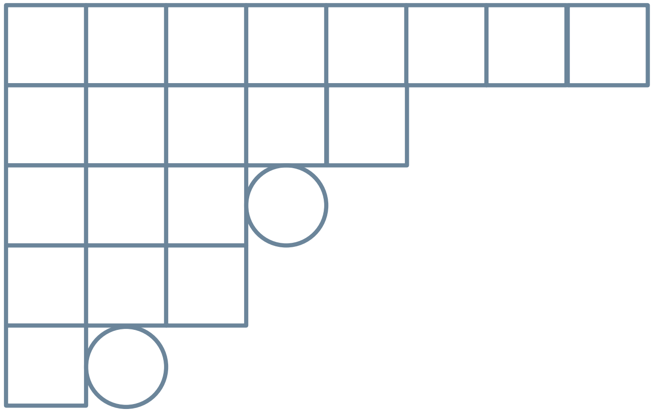
Where $(\Lambda_a;\Lambda_s) = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots \Lambda_N)$

Superpartitions

- $\Lambda = (\Lambda_a; \Lambda_s)$ is a superpartition of fermionic degree m if Λ_a is a partition with m distinct parts (possibly including 0) and Λ_s is a partition.

- It's Young diagram is the Young diagram of
$$\Lambda^* = \Lambda_a \cup \Lambda_s$$
with circles in the rows of Λ_a

$\Lambda = (3,1; 8,3,5)$



Supermonomials

- $\{m_\Lambda : \mathrm{fd}(\Lambda) = m \text{ and } \Lambda^* \vdash n\}$ is a basis for $\mathbb{L}^{m,n}$, the ring of homogeneous symmetric functions in superspace of degree n and fermionic degree m

Superelementary

- Superelementary:

$$e_{(\Lambda_1,\ldots,\Lambda_m;\Lambda_{m+1},\ldots\Lambda_N)} = \tilde{e}_{\Lambda_1}, \ldots \tilde{e}_{\Lambda_m} e_{\Lambda_{m+1}} \ldots e_{\Lambda_N}$$

$$e_n = m_{(\emptyset;1^n)}$$

$$\tilde{e}_n = m_{(0;1^n)}$$

Supercomplete

- Supercomplete:

$$h_{(\Lambda_1,\ldots,\Lambda_m;\Lambda_{m+1},\ldots\Lambda_N)} = \tilde{h}_{\Lambda_1}, \ldots \tilde{h}_{\Lambda_m} h_{\Lambda_{m+1}} \ldots h_{\Lambda_N}$$

$$h_n = \sum_{\Lambda \vdash (0,n)} m_\Lambda$$

$$\tilde{h}_n = \sum_{\Lambda \vdash (1,n)} (\Lambda_1 + 1) m_\Lambda$$

Superpower

- Superpower:

$$p_{(\Lambda_1,\ldots,\Lambda_m;\Lambda_{m+1,\ldots}\Lambda_N)} = \tilde{p}_{\Lambda_1}, \ldots \tilde{p}_{\Lambda_m} p_{\Lambda_{m+1}} \ldots p_{\Lambda_N}$$

$$p_n = m_{(\emptyset;n)}$$

$$\tilde{p}_n = m_{(n;\emptyset)}$$

The isomorphism $\omega: \Lambda \rightarrow \Lambda$

- It still works!!!

- $\omega(\tilde{e}_n) = \tilde{h}_n$ implies $\omega(e_\Lambda) = h_\Lambda$
- $\omega(\tilde{p}_n) = (-1)^n \tilde{p}_n$

Scalar product.

- We can define a scalar product by letting the h's and the m's be dual bases:
- $\langle m_\Lambda, h_\Gamma \rangle = \delta_{\Lambda\Gamma}$

- Then the p's form an orthogonal basis:
- $\langle p_\Lambda, p_\Gamma \rangle = z_{\Lambda_s} \delta_{\Lambda\Gamma}$

Is there an orthonormal basis of $\mathbb{L}_{\mathbb{Z}}^{m,n}$?

- In fact, yes there is one. But it is ugly……

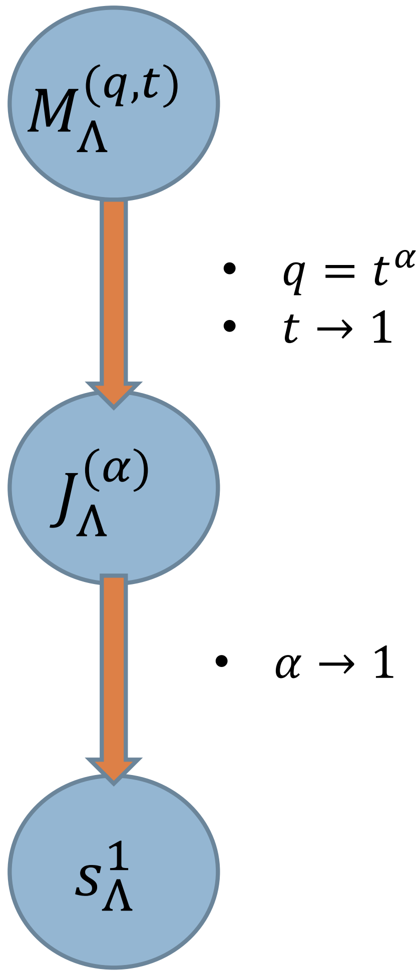
SuperMacdonald polynomials

- We introduce two more variables q, t and a new scalar product:

$$\langle p_\Lambda, p_\Gamma \rangle_{q,t} = \delta_{\Lambda\Gamma} q^{|\Lambda^a|} z_{\Lambda^s} \prod_{i}^{\ell(\lambda)} \frac{1 - q^{\Lambda_i^s}}{1 - t^{\Lambda_i^s}}$$

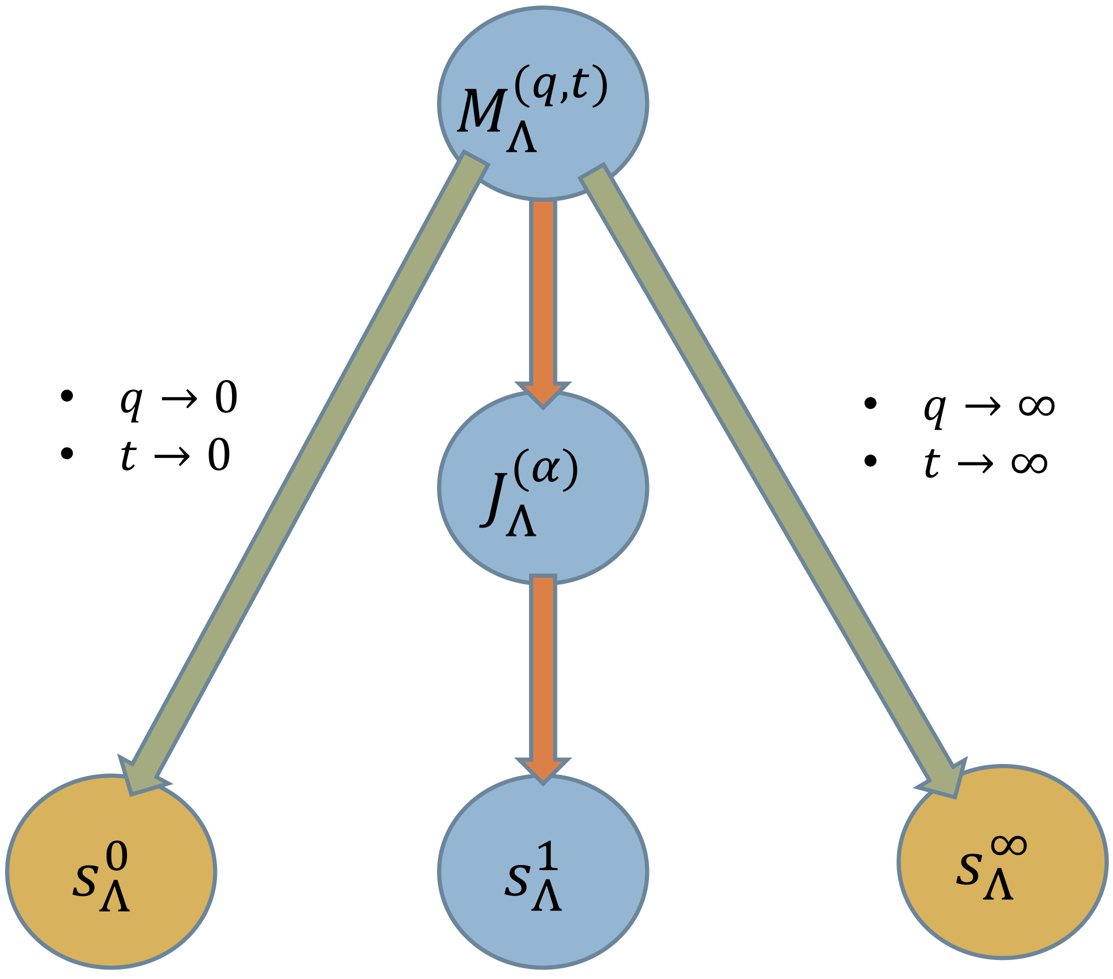
- The Macdonald Polynomials are defined to be the unique basis
  - $M_\Lambda^{(q,t)} = m_\Lambda +$ smaller terms
  - $\left\langle M_\Lambda^{(q,t)}, M_\Gamma^{(q,t)} \right\rangle_{q,t} = 0$ if $\lambda \neq \mu$

Macdonald polynomials



- COMBINATORICS
- PHYSICS/combinatorics
- Yucky combinatorics

Macdonald polynomials



Is there an orthonormal basis of $\mathbb{L}_{\mathbb{Z}}^{m,n}$?

- s_Λ^1 is not m_Λ-integral.

- s_Λ^0 and s_Λ^∞ are m_Λ-integral!!!!

Properties of s_Λ^0 and s_Λ^∞

- Pieri Rules
  - s_Λ^0 (Mathieu, Blondeau-Fournier)
  - s_Λ^∞ (Lapointe, Preville-Ratelle, MJ)
- Tableaux generating function
  - Supermonomial expansion
- Cauchy Identities
  - RSK

Properties of s_Λ^0 and s_Λ^∞

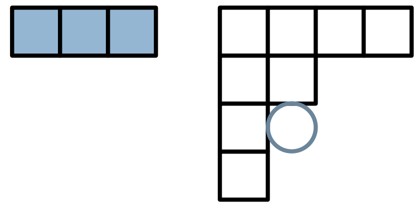
- Define $\bar{s}_\Lambda^0$ and $\bar{s}_\Lambda^\infty$ to be the duals of s_Λ^0 and s_Λ^∞.

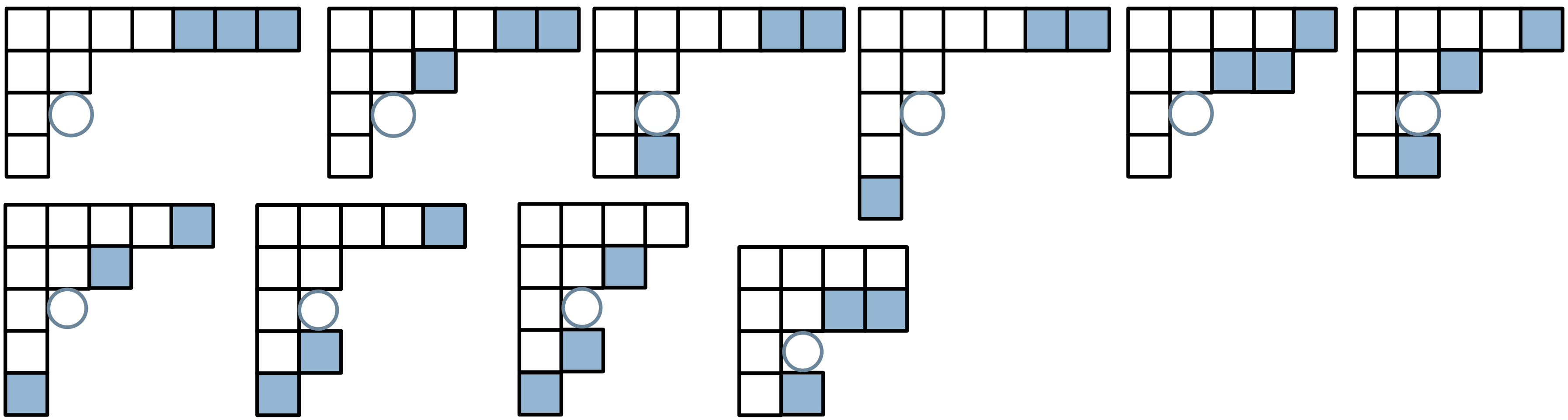
- $\left\langle s_\Lambda^0, \bar{s}_\Gamma^0 \right\rangle = \langle s_\Lambda^\infty, \bar{s}_\Gamma^\infty \rangle = \delta_{\Lambda\Gamma}$.

- Then:
- $\bar{s}_\Lambda^\infty = (-1)^{\binom{m}{2}} \omega\left(s_{\Lambda'}^0\right)$
- $\bar{s}_\Lambda^0 = (-1)^{\binom{m}{2}} \omega\left(s_{\Lambda'}^\infty\right)$

Pieri Rules:

- Example: $h_3 \bar{s}^{\infty}_{(1;\,4,2,1)}$





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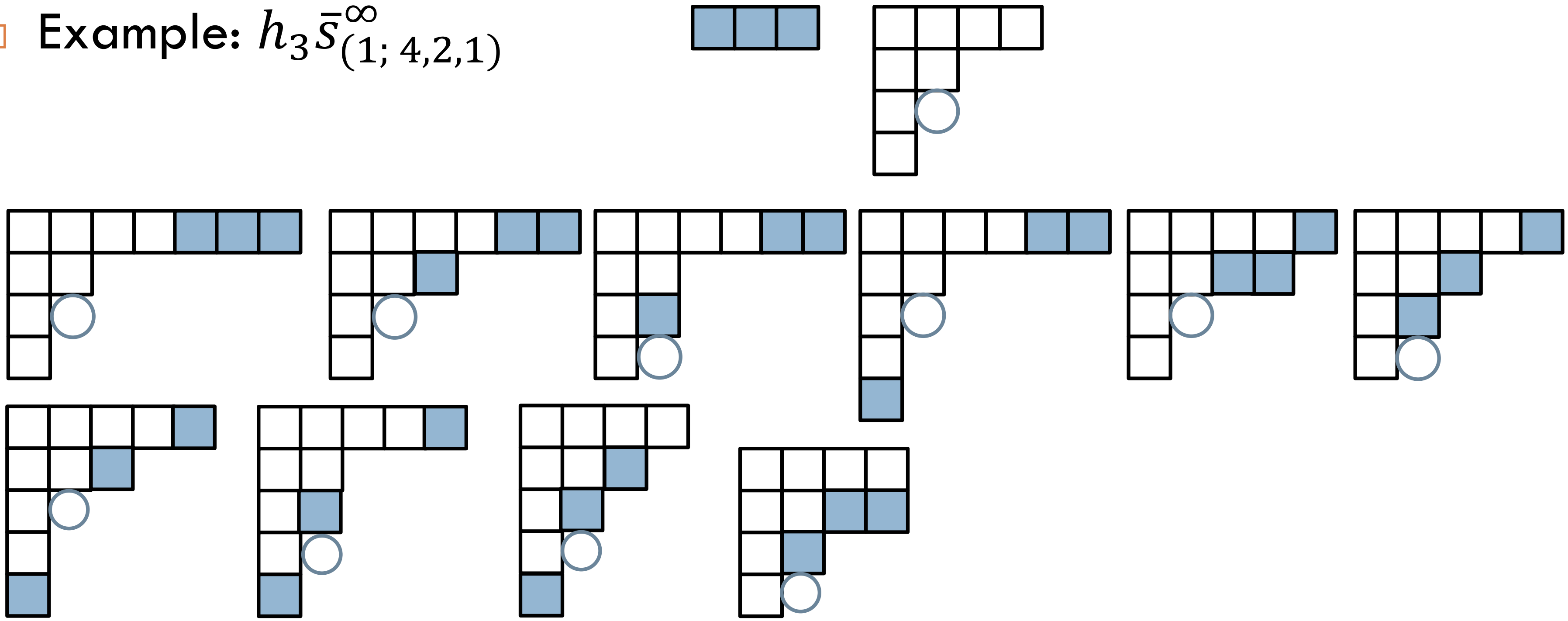


- Example: $h_3\bar{s}^{\infty}_{(1;\,4,2,1)} = \bar{s}^{\infty}_{(1;\,7,2,1)} + \bar{s}^{\infty}_{(1;\,6,3,1)} + \bar{s}^{\infty}_{(1;\,6,2,2)} + \bar{s}^{\infty}_{(1;\,6,2,1,1)} + \bar{s}^{\infty}_{(1;\,5,4,1)} + \bar{s}^{\infty}_{(1;\,5,3,2)} + \bar{s}^{\infty}_{(1;\,5,3,1,1)} + \bar{s}^{\infty}_{(1;\,5,2,2,1)} + \bar{s}^{\infty}_{(1;\,4,3,2,1)} + \bar{s}^{\infty}_{(1;\,4,4,2)}$

Pieri Rules consequences:

- So $h_\Lambda = \sum_{\Gamma \vdash (m|n)} A_{\Lambda\Gamma}\, \bar{s}_\Gamma^\infty$.

$A_{\Lambda\Gamma}$ is some sort of super SSYT of shape Λ content Γ given by the Pieri rules.

Super SSYT

$$h_\Lambda = \sum_{\Gamma \vdash (m|n)} A^{\infty}_{\Lambda\Gamma} \bar{s}^{\infty}_{\Gamma} \qquad s^{\infty}_{\lambda} = \sum_{\mu \vdash n} K^{\infty}_{\lambda\mu} m_\mu$$

- Apply the scalar product of h_λ and s_γ

$$\langle h_\Lambda, s^{\infty}_{\Omega} \rangle$$

$$\left\langle h_\Lambda, \sum_{\Gamma} K^{\infty}_{\Omega\Gamma} m_\Gamma \right\rangle = \left\langle \sum_{\mu \vdash n} A^{\infty}_{\Lambda\Gamma} \bar{s}^{\infty}_{\Gamma}, s^{\infty}_{\Omega} \right\rangle$$

$$K^{\infty}_{\Omega\Lambda} = A^{\infty}_{\Lambda\Omega}$$

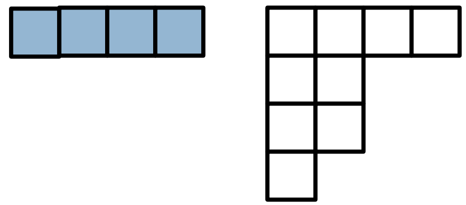
Murnaghan-Nakayama Rule

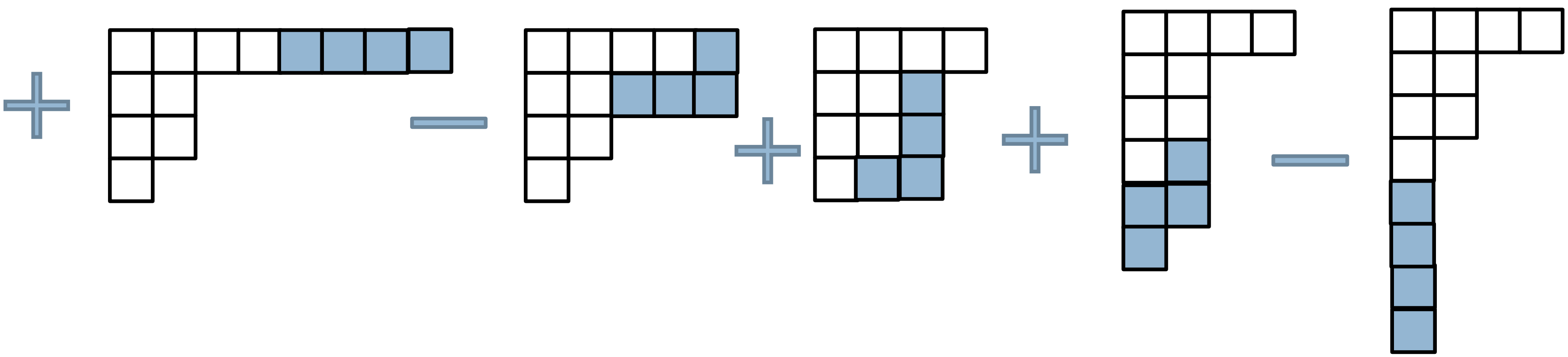
M-N Rule (classical)

- $p_n s_\lambda = \sum_\mu (-1)^{1+height\ of\ ribbon} s_\mu$ where μ is any result of adding an n-ribbon to λ.

M-N Rules:

- Example: $p_4 s_{(4,2,2,1)}$



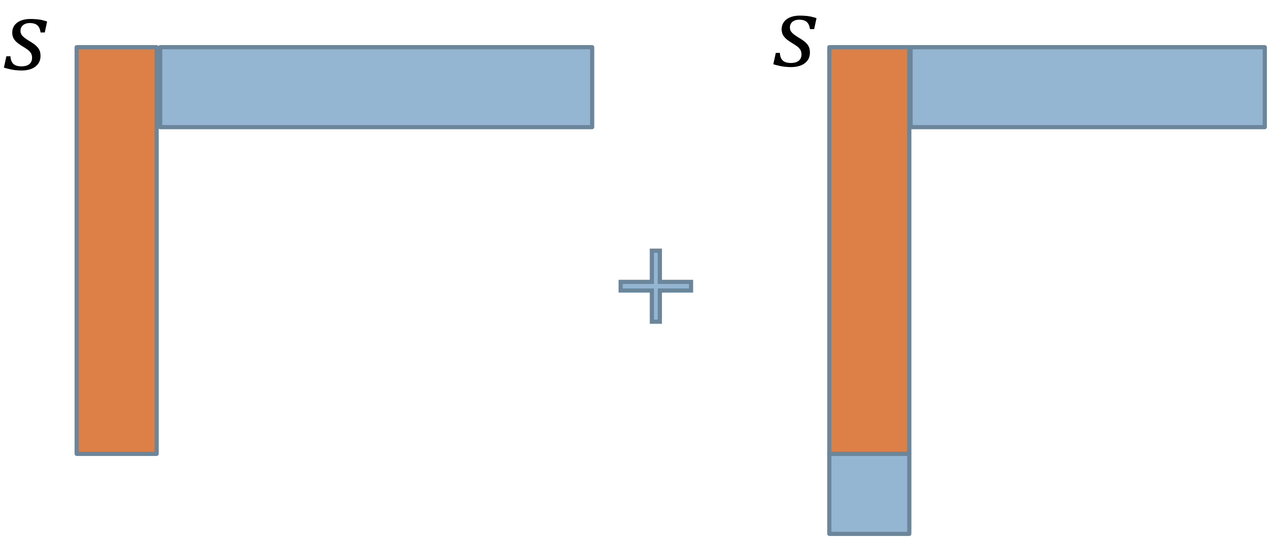


M-N Rule: Combinatorial Proof

- $s_{(n-k,1^k)} = \sum_{j=0}^{k} (-1)^{k-j} h_{n-j} e_j$
- $p_n = \sum_{k=0}^{n-1} (-1)^k s_{(n-k,1^k)}$

M-N Rule: Combinatorial Proof

- $s_{(n-k,1^k)} = \sum_{j=0}^{k}(-1)^{k-j} h_{n-j} e_j$
- $h_k e_\ell = s_{(n)} s_{(1^\ell)} = s_{(n+1,1^{k-1})} + s_{(n,1^k)}$



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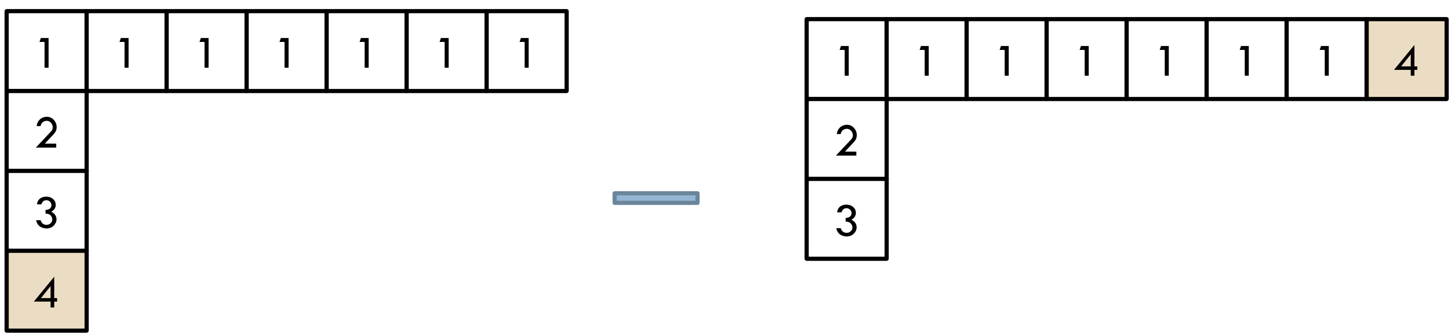
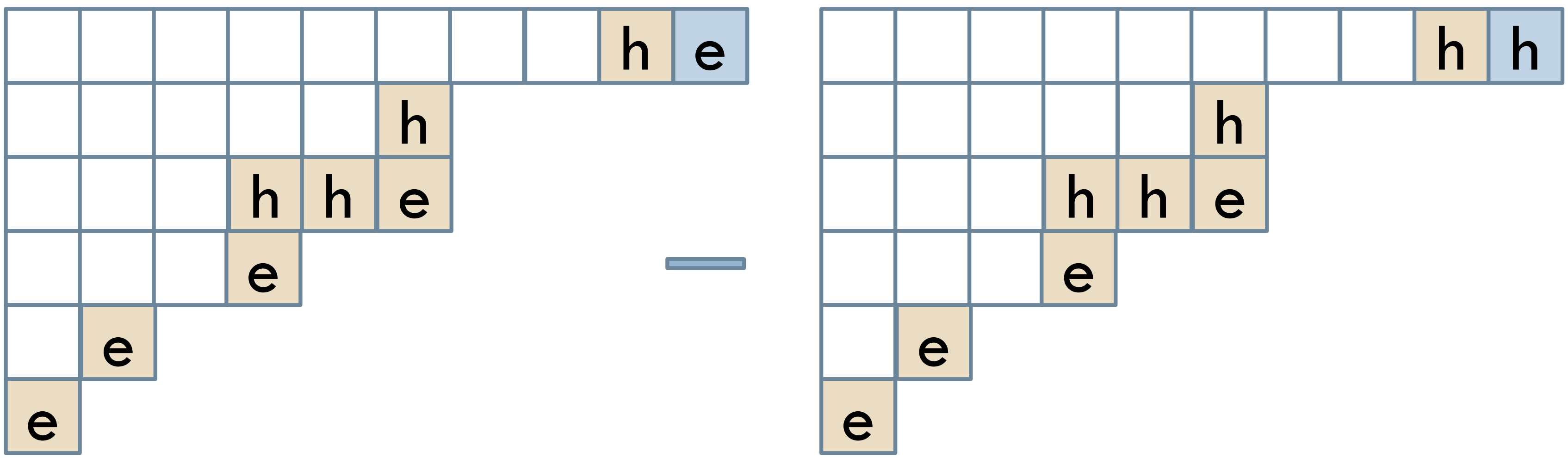


- Fixed points are horizontal strips

M-N Rule: Combinatorial Proof

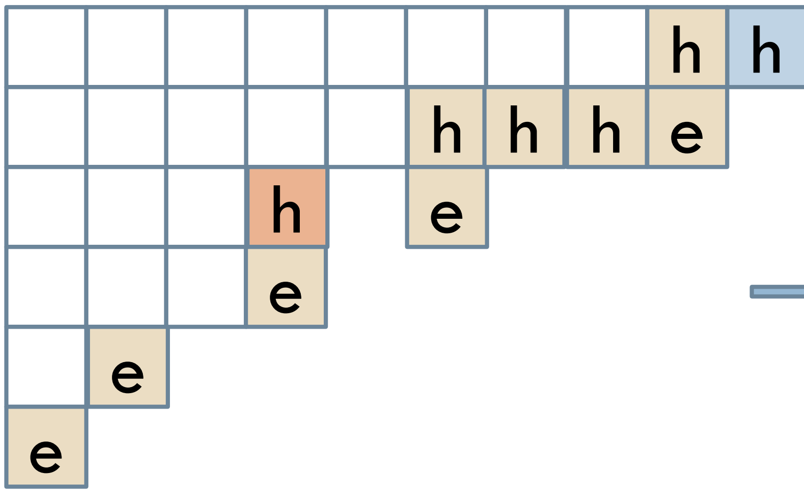
- $s_{(n-k,1^k)} = \sum_{j=0}^{k} (-1)^{k-j} h_{n-j} e_j$



- Fixed points are all broken rim hooks with exactly k $\boxed{e}$'s and $n-k$ $\boxed{h}$'s that has an h in the rightmost position.

M-N Rule: Combinatorial Proof

- $p_n = \sum_{k=0}^{n-1}(-1)^k s_{(n-k,1^k)}$



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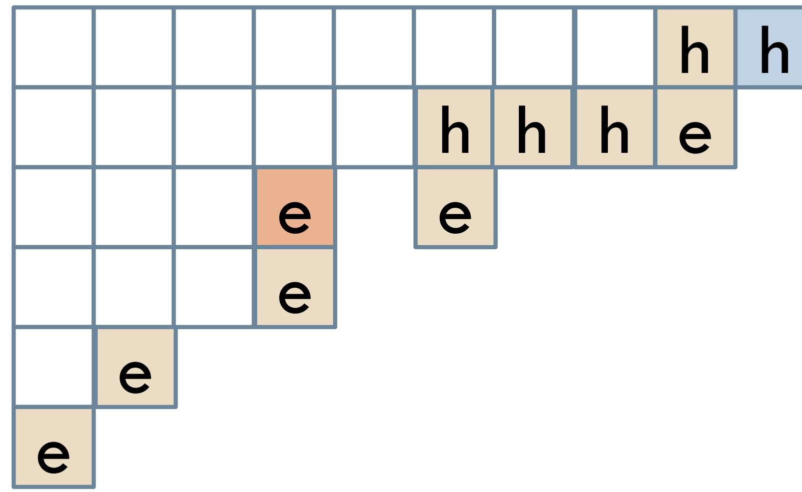
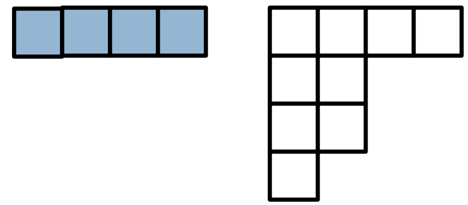
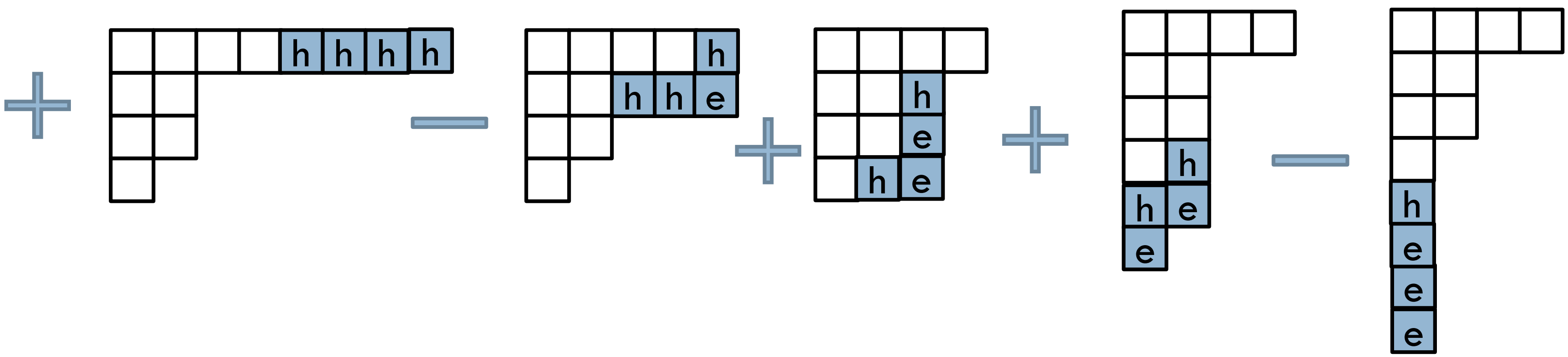


- Fixed points are all rim hooks of length n with $(-1)^{\#\ of\ e's}$

M-N Rules:

- Example: $p_4 s_{(4,2,2,1)}$





Jacobi-Trudi

Jacobi-Trudi

- The Jacobi-Trudi Identity is another way of defining a Schur function:

$$s_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} = det\begin{pmatrix} h_{\lambda_1} & h_{\lambda_2-1} & h_{\lambda_3-2} & h_{\lambda_4-3} \\ h_{\lambda_1+1} & h_{\lambda_2} & h_{\lambda_3-1} & h_{\lambda_4-2} \\ h_{\lambda_1+2} & h_{\lambda_2+1} & h_{\lambda_3} & h_{\lambda_4-1} \\ h_{\lambda_1+3} & h_{\lambda_2+2} & h_{\lambda_3+1} & h_{\lambda_4} \end{pmatrix}$$

Jacobi-Trudi

- The Jacobi-Trudi Identity is another way of defining a Schur function:

$$s_{(3,3,2)} = det\begin{pmatrix} h_3 & h_2 & h_0 \\ h_4 & h_3 & h_1 \\ h_5 & h_4 & h_2 \end{pmatrix}$$

Jacobi-Trudi

- In superspace, we are trying to define a new type of super Schur function using this as a guideline.

$$s_{(5,3;\ 7,3)} = det \begin{pmatrix} h_7 & \tilde{h}_4 & h_1 & \tilde{h}_0 \\ h_8 & \tilde{h}_5 & h_2 & \tilde{h}_1 \\ h_9 & \tilde{h}_6 & h_3 & \tilde{h}_2 \\ h_{10} & \tilde{h}_7 & h_4 & \tilde{h}_3 \end{pmatrix}$$