Math 240C: Real Analysis, Spring 2020

Midterm Exam

Please read carefully:

- Please write your name on your exam and email your exam (after it is done) to your grader.
- You can look at the textbook, your own notes and homework solutions, and the instructor's lecture notes and homework solutions. But you are not allowed to discuss with any other people, and you are not allowed to search internet for solutions and material.
- If you you need to cite some result from exercise or homework problems, then you need to prove that.
- To get credit, you must show your work. Partial credit will be given to partial answers.
- This exam will be 50 minutes. But additional 10 minutes will be given for submitting your exams.

Problem 1. (25 points) Let X be a LCH space and μ a Radon measure on X. Let K be a compact subset and U an open subset of X such that $K \subseteq U$. Let $f \in C_c(X, [0,1])$ be such that $K \prec f \prec U$ (i.e., $f = 1$ on K and $\operatorname{supp}(f) \subseteq U$). Prove that $f \geq \chi_K$ and that $\mu(K) \leq \int_X f \, d\mu \leq \mu(U)$.

Problem 2. (25 points) Let X be a LCH space and μ a finite Borel measure on X. Prove that μ is a Radon measure if and only if the following holds true: For any Borel set E and any $\varepsilon > 0$, there exist a compact set K and an open set U such that $K \subseteq E \subseteq U$ and $\mu(U \setminus K) < \varepsilon$.

Problem 3. (25 points) True or false: (If your answer ture, please give a brief proof. If you answer no, please give a counter example.)

(1) A finite Borel meausre on $\mathbb{R}^n$ is a Radon measure.

(2) If X is a LCH space, μ is a Radon measure on X, and $f \in C_0(X)$, then $f \in L^1(\mu)$.

Problem 4. (25 points) Let X be a LCH space. Let μ_n $(n = 1, 2, \dots)$ and μ be all finite Radon measures on X. Assume $\mu_n \to \mu$ vaguely, i.e.,

$$\lim_{n\to\infty} \int_X f \, d\mu_n = \int_X f \, d\mu \qquad \forall f \in C_c(X).$$

Prove the following:

$$\liminf_{n\to\infty} \mu_n(U) \geq \mu(U) \qquad \text{for any open set } U \subseteq X;$$

$$\limsup_{n\to\infty} \mu_n(F) \leq \mu(K) \qquad \text{for any compact set } K \subseteq X.$$