Math 240A: Real Analysis, Fall 2019

Homework Assignment 8

Due Friday, December 6, 2019

1. Let $f \in L^1(\mathbb{R}^n)$ and $f \neq 0$. Prove that there exist $C, R > 0$ such that $(Hf)(x) \geq C|x|^{-n}$ for $|x| > R$ and that there exists $C' > 0$ such that $m(\{Hf > \alpha\}) \geq C'/\alpha$ if $\alpha > 0$ is small enough.

2. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be continuous at $x \in \mathbb{R}^n$. Prove that x in the Lebesgue set of f.

3. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Prove that $|f(x)| \leq (Hf)(x)$ at every Lebesgue point x of f.

4. Let E be a Borel set in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, define $D_E(x) = \lim_{r \to 0} \dfrac{m(E \cap B(x,r))}{m(B(x,r)}$ if it exists.
   (1) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.
   (2) Find examples of E and x such that $D_E(x)$ is a given number $\alpha \in (0,1)$, or such that $D_E(x)$ does not exist.

5. Let $F(x) = x^2 \sin(x^{-1})$ and $G(x) = x^2 \sin(x^{-2})$ for $x \neq 0$ and $F(0) = G(0) = 0$. Prove that F and G are differentiable everywhere (including $x = 0$), $F \in BV([-1,1])$, and $G \notin BV([-1,1])$.

6. Prove or disprove: If $f_n \in BV([0,1])$ $(n = 1, 2, \dots)$ and $f_n \to f$ uniformly on $[0,1]$, then $f \in BV([0,1])$.

7. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and $-\infty < a < b < \infty$. Prove that $F(b) - F(a) \geq \displaystyle\int_a^b F'(t)\,dt$.

8. Let $F, G : [a,b] \to \mathbb{R}$ be absolutely continuous. Prove that $FG : [a,b] \to \mathbb{R}$ is also absolutely continuous and that $\displaystyle\int_a^b (FG' + GF')(x)\,dx = F(b)G(b) - F(a)G(a)$.

9. Prove that a function $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous with a Lipschitz constant $L \geq 0$ (i.e., $|F(x) - F(y)| \leq L|x - y|$ for any $x, y \in \mathbb{R}$) if and only if F is absolutely continuous and $|F'| \leq L$ a.e. $\mathbb{R}$.

10. Let $a, b \in \mathbb{R}$ with $a < b$. A function $F : (a,b) \to \mathbb{R}$ is convex if $F(\lambda s + (1 - \lambda t) \leq \lambda F(s) + (1 - \lambda)F(t)$ for any $s, t \in (a,b)$ and any $\lambda \in [0,1]$. Prove the following:
    (1) A function $F : (a,b) \to \mathbb{R}$ is convex if and only if $\dfrac{F(t) - F(s)}{t - s} \leq \dfrac{F(t') - F(s')}{t' - s'}$ for all $s, t, s', t' \in (a,b)$ such that $s < s' < t'$ and $s < t < t'$;
    (2) A function $F : (a,b) \to \mathbb{R}$ is convex if and only if F is absolutely continuous on every compact subinterval of (a,b) and F' is increasing on the set where it is defined;
    (3) If $F : (a,b) \to \mathbb{R}$ is convex and $t_0 \in (a,b)$, then there exists $\beta \in \mathbb{R}$ such that $F(t) - F(t_0) \geq \beta(t - t_0)$ for all $t \in (a,b)$;
    (4) (Jensen's Inequality) If $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) = 1$, $g : X \to (a,b)$ is in $L^1(\mu)$, and F is a convex function on (a,b), then $F\left(\displaystyle\int_X g\,d\mu\right) \leq \displaystyle\int_X F \circ g\,d\mu$.