Math 240A, Fall 2019

Solution to Problems of HW#3

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1. We have

$$\mu\left(\bigcap_{j=1}^{n} A_j\right) = \mu\left(\left(\bigcup_{j=1}^{n} A_j^c\right)^c\right) = 1-\mu\left(\bigcup_{j=1}^{n} A_j^c\right)$$

$$= 1-\sum_{j=1}^{n}\mu(A_j^c) = 1-\sum_{j=1}^{n}\left(1-\mu(A_j)\right)$$

$$= 1-n+\sum_{j=1}^{n}\mu(A_j) > 1-n+n-1=0. \quad \square$$

2. We have $E=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}E_k \in \mathcal{M}$. The sequence $\bigcup_{k=n}^{\infty}E_k$ decreases with n, and $\mu\left(\bigcup_{k=1}^{\infty}E_k\right) \le \sum_{k=1}^{\infty}\mu(E_k) < \infty$. By the continuity from above of a measure, we have

$$\mu(E) = \lim_{n\to\infty}\mu\left(\bigcup_{k=n}^{\infty}E_k\right).$$

But, $\mu\left(\bigcup_{k=n}^{\infty}E_k\right) \le \sum_{k=n}^{\infty}\mu(E_k) \to 0$ as $\sum_{k=1}^{\infty}\mu(E_k)<\infty$. Thus, $\mu(E)=0$. $\square$

3. No. Let $\mathbb{Q}=\{r_1, r_2, \cdots\}$ be the set of all rational numbers. Then $G=\bigcup_{k=1}^{\infty}\left(r_k-\frac{1}{2^k}, r_k+\frac{1}{2^k}\right)$ is open and $m(G)\le \sum_{k=1}^{\infty} 2\cdot\frac{1}{2^k}=2$. But $\overline{G}=\mathbb{R}$, and $m(\overline{G})=\infty$. $\square$

4. We have $m(B\setminus A)=m(B)-m(A)=0$, and $E\setminus A\subseteq B\setminus A$. Since m is complete, we have $E\setminus A\in\mathcal{L}$ (i.e., $E\setminus A$ is Lebesgue measurable) and $m(E\setminus A)=0$. Now, $E=(E\setminus A)\cup A$ is also Lebesgue measurable. $m(E)=m(E\setminus A)+m(A)=m(A)$. $\square$

5 (1) We have $(-\infty, 1) = \bigcup_{n=1}^{\infty} (-n, 1-\frac{1}{2^n}]$, a union of increasing sets. (h-intervals). Hence,

$$\mu((-\infty,1)) = \mu\left(\bigcup_{n=1}^{\infty} (-n, 1-\tfrac{1}{2^n}]\right)$$
$$= \lim_{n\to\infty} \mu((-n, 1-\tfrac{1}{2^n}])$$
$$= \lim_{n\to\infty} [F(1-\tfrac{1}{2^n}) - F(-n)]$$
$$= 1 - 0 = 1.$$

(2) Similarly,

$$\mu((-\infty,1]) = \lim_{n\to\infty} \mu((-n,1])$$
$$= \lim_{n\to\infty} [F(1) - F(-n)]$$
$$= 4 - 0$$
$$= 4.$$

(3) $\mu(\mathbb{R}) = \mu((-\infty,1]) + \mu((1,2]) + \mu((2,\infty))$

$$= 4 + F(2) - F(1) + \lim_{n\to\infty} \mu((2, n+2])$$
$$= 4 + 7 - 4 + \lim_{n\to\infty} [F(n+2) - F(2)]$$
$$= 7 + 7 - 7 = 7.$$

(4) $\mu(\{2\}) = \mu\left(\bigcap_{n=1}^{\infty} (2-\frac{1}{n}, 2+\frac{1}{n}]\right)$

$$= \lim_{n\to\infty} \mu((2-\tfrac{1}{n}, 2+\tfrac{1}{n}])$$
$$= \lim_{n\to\infty} [F(2+\tfrac{1}{n}) - F(2-\tfrac{1}{n})]$$
$$= 7 - 7 = 0.$$

6. Let $H(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x \geq 0 \end{cases}$, for any $x \in \mathbb{R}$. (This is called the Heaviside function.) H is increasing and righ continuous. We show that $\mu_H = \delta$. Hence, all increasing and right continuous functions F with $\mu_F = \delta$ are just $H(x)+C$ for some constants C.

Recall the Dirac measure δ is defined by

$$\delta(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \end{cases}$$

for any $E \in \mathcal{B}_{\mathbb{R}}$. We have

$$\delta(\{0\}) = 1, \quad \delta(\mathbb{R}') = 1.$$

$$\mu_H(\{0\}) = \lim_{n\to\infty} \mu_H\left(\left(-\tfrac{1}{n}, \tfrac{1}{n}\right]\right) = \lim_{n\to\infty}\left[H\left(\tfrac{1}{n}\right) - H\left(-\tfrac{1}{n}\right)\right] = 1,$$

and $\mu_H(\mathbb{R}) = \mu_H\left(\bigcup_{n=1}^{\infty} (-n, n]\right) = \lim_{n\to\infty} \mu_H(n) - \mu_H(-n) = 1$.

Hence, both δ and μ_H are concentrated on $\{0\}$.

Finally, we show that

$$\delta((a,b]) = H(b) - H(a). \quad \forall a, b \in \mathbb{R}, a<b.$$

$\delta((a,b]) = 1$ or 0 if $0 \in (a,b]$ or $0 \notin (a,b]$.

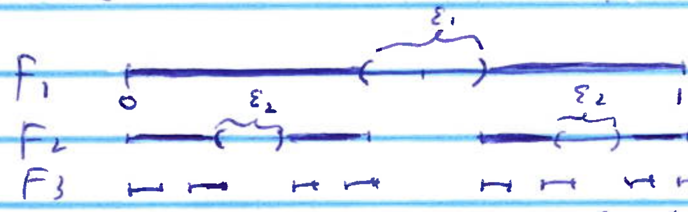
Same for the right-hand side: $H(b) - H(a) = \mu_H((a,b])$ $= 1$ or 0 if $0 \in (a,b]$ or $0 \notin (a,b]$ as $\mu_H(\{0\}) = \mu_H(\mathbb{R}) = 1$. $\square$

7. Proposition 1.20. If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for any $\varepsilon > 0$ there exists a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \varepsilon$.

Let $\varepsilon > 0$. By Theorem 1.18, $\exists$ open set $U \supseteq E$ such that $\mu(U) < \mu(E) + \frac{\varepsilon}{2}$. If U is already a finite union of open intervals, then let $A = U$.
We have
$$\mu(E \triangle A) = \mu((E \setminus A) \cup (A \setminus E))$$
$$\leq \mu(E \setminus A) + \mu(A \setminus E)$$
$$= \mu(A) - \mu(E) < \varepsilon/2 < \varepsilon$$
Otherwise, $U = \bigcup_{j=1}^{\infty} (a_j, b_j)$ a disjoint union for some $a_j, b_j \in \overline{\mathbb{R}}$, $a_j < b_j$ $(j = 1, 2 \ldots)$. [Note that any open set in $\mathbb{R}$ is a countable disjoint union of open intervals]
Hence,
$$\mu(U) = \sum_{j=1}^{\infty} \mu((a_j, b_j)) < \mu(E) + \frac{\varepsilon}{2} < \infty.$$
There exists $N \in \mathbb{N}$ such that
$$\sum_{n=N+1}^{\infty} \mu((a_j, b_j)) < \varepsilon/2.$$
Let $A = \bigcup_{n=1}^{N} (a_j, b_j) \subseteq U$. A is a finite union of open intervals. $\mu(U \setminus A) = \sum_{n=N+1}^{\infty} \mu((a_j, b_j)) < \varepsilon/2$. Hence
$$\mu(E \triangle A) \leq \mu(E \setminus A) + \mu(A \setminus E)$$
$$\leq \mu(U \setminus A) + \mu(U \setminus E)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \square$$

8. Let $\varepsilon = \frac{1-\alpha}{2-\alpha} \in (0,1)$ and $\varepsilon_k = \varepsilon^k / 2^{k-1}$ $(k=1,2,\cdots)$

Divide $[0,1]$ into 3 intervals with the middle open



one having length ε_1 and the other two closed intervals having same length. Denote by F_1 the union of the two closed intervals.

For each of the two closed intervals of F_1, divide it into 3 intervals with the middle open interval of length ε_2 and the other two closed intervals having same length. Denote by F_2 the union of the 4 ~~closed~~ remaining disjoint closed intervals.

Continuing, by induction, we have a sequence of closed sets F_k $(k=1,2,\cdots)$. Each F_k ~~cons~~ is the union of 2^k disjoint closed intervals of same length.

Let $F = \bigcap_{k=1}^{\infty} F_k$. Then $F \subseteq [0,1]$. F is closed and hence compact.

If $0 \le a < b \le 1$ and $(a,b) \subseteq F$. Then $(a,b) \subseteq F_k$ $(k=1,2,\cdots)$ But F_k is the union of 2^k disjoint closed intervals. So, (a,b) is contained in one such interval which has the length $\le \frac{1}{2^k}$ as $m(F_k) \le 1$. But as k is large enough $b-a > \frac{1}{2^k}$. This is impossible. Thus F contains no open interval. Hence F is nowhere dense.

Finally $m(F) = 1 - \varepsilon_1 - 2\varepsilon_2 - 3\varepsilon_3 - \cdots = 1 - \varepsilon - \varepsilon^2 - \cdots$

$= 1 - \frac{\varepsilon}{1-\varepsilon} = 1 - \frac{1-\alpha}{2-\alpha} \Big/ 1 - \frac{1-\alpha}{2-\alpha} = \alpha.$ □

9. If the statement were not true, then there exists $\alpha \in (0,1)$ such that $m(E \cap I) \le \alpha\, m(I)$ for any open interval I.

$\forall \varepsilon > 0$. There exists open set $U \supseteq E$ such that $m(U) < m(E) + \varepsilon$. Let $U = \bigcup_{i=1}^{\infty} I_i$ with each I_i an open interval and $I_i \cap I_j = \phi$ if $i \ne j$. (The case that U is a finite union of disjoint open intervals can be treated similarly.) Since $m(E \cap I_i) \le \alpha m(I_i)$, we have for each i

$$m(I_i) = m(I_i \setminus E) + m(I_i \cap E) \le m(I_i \sim E) + \alpha\, m(I_i),$$

and hence $\quad m(I_i) \le \frac{1}{1-\alpha} m(I_i \setminus E)$.

Consequently,

$$0 < m(E) \le m(U) = \sum_{i=1}^{\infty} m(I_i) \le \frac{1}{1-\alpha} \sum_{i=1}^{\infty} m(I_i \setminus E)$$

$$= \frac{1}{1-\alpha} m\Big(\bigcup_{i=1}^{\infty} (I_i \setminus E)\Big) = \frac{1}{1-\alpha} m(U \setminus E) < \frac{\varepsilon}{1-\alpha}.$$

Since $\varepsilon > 0$ is arbitrary, this implies $m(E) = 0$, a contradiction. $\square$

10. (1) Recall that $N_r = (N \cap [0, 1-r) + r) \cup (N \cap [1-r, 1) - (1-r))$ for each $r \in \mathbb{Q} \cap [0,1)$. Define similarly

$$E_r = (E \cap [0, 1-r) + r) \cup (E \cap [1-r, 1) - (1-r)) \subseteq N_r \subseteq [0,1]$$

Since $N_r \cap N_s = \phi$ if $r \ne s$, $r, s \in \mathbb{Q} \cap [0,1)$, we have $E_r \cap E_s = \phi$ if $r \ne s$, $r, s \in \mathbb{Q} \cap [0,1)$. It is clear by the translation invariance of the Lebesgue measure that $m(E_r) = m(E)$. Thus, if $m(E) > 0$ then

$$\infty = \sum_{r \in \mathbb{Q} \cap [0,1)} m(E) = \sum_{r \in \mathbb{Q} \cap [0,1)} m(E_r) = m\Big(\bigcup_{r \in \mathbb{Q} \cap [0,1)} E_r\Big) \le m([0,1]) = 1.$$

a contradiction. Hence $m(E) = 0$.

(2) Note that $E = \bigcup_{n=-\infty}^{\infty} (E \cap [n, n+1))$, a disjoint union.
Since $m(E) > 0$, there exists $n \in \mathbb{Z}$ such that $m(E \cap [n, n+1)) > 0$. Let $F = E \cap [n, n+1) - n \subseteq [0,1)$. $m(F) > 0$. If we can show that F contains a Lebesgue nonmeasurable subset $\hat{N}$ then $E \cap [n, n+1)$ and hence E, contains a Lebesgue nonmeasurable set $\hat{N} + n$.
So, it suffices to assume $E \subseteq [0,1)$.

Suppose $m(E) > 0$ but any subset of E is Lebesgue measurable.

Observe that for Part (1) holds true with N replaced by N_r for any $r \in \mathbb{Q} \cap [0,1)$, since N_r consists points exactly one in $[0,1)$ from ~~that~~ each equivalent class defined by $x \sim y \iff x - y \in \mathbb{Q}$. Thus, $m(E \cap N_r) = 0$
Consequently, since $[0,1) = \bigcup_{r \in \mathbb{Q} \cap [0,1)} N_r$ disjoint, we get

$$0 < m(E) = m(E \cap [0,1)) = m\Big(\bigcup_{r \in \mathbb{Q} \cap [0,1)} E \cap N_r\Big)$$
$$= \sum_{r \in \mathbb{Q} \cap [0,1)} m(E \cap N_r) = 0.$$

This is a contradiction. Hence, E contains a nonmeasurable set. $\square$