Math 240A, Fall 2019
Solution to Problems of HW#4
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1. (1) Assume f is measurable. Then for any $r \in \mathbb{Q} \subseteq \mathbb{R}$, $(r,\infty) \in \mathcal{B}_{\mathbb{R}}$. Hence $f^{-1}((r,\infty)) \in \mathcal{M}_E$.

Conversely assume $f^{-1}((r,\infty)) \in \mathcal{M}_E$ for all $r \in \mathbb{Q}$. Let $a \in \mathbb{R}$. Choose $r_n \in \mathbb{Q}$ such that r_n decreases and $r_n \to a$. Then, $(a,\infty) = \bigcup_{n=1}^{\infty} (r_n,\infty)$ and $f^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} f^{-1}((r_n,\infty)) \in \mathcal{M}_E$ since each $f^{-1}((r_n,\infty)) \in \mathcal{M}_E$. But $\{(a,\infty) : a \in \mathbb{R}\}$ generates $\mathcal{B}_{\mathbb{R}}$. Hence, f is measurable.

(2) Let $\underline{f}(x) = \liminf_{n\to\infty} f_n(x)$ and $\bar{f}(x) = \limsup_{n\to\infty} f_n(x)$ for any $x \in X$. Then $\bar{f}(x) \geq \underline{f}(x)$ $(x \in X)$. Both $\underline{f}$ and $\bar{f}$ are measurable. Thus

$$\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\} = \{x \in X : \underline{f}(x) = \bar{f}(x) \notin \{-\infty,\infty\}\}$$
$$= \{x \in X : (\bar{f} - \underline{f})(x) = 0\} = (\bar{f} - \underline{f})^{-1}(\{0\}) \in \mathcal{M}_E$$

as $\{0\} \in \mathcal{B}_{\mathbb{R}}$ and $\bar{f} - \underline{f}$ is measurable.

2. Assume f is increasing. (The case that f is decreasing can be treated similarly.)
Let $\alpha \in \mathbb{R}$ and define $A_\alpha = \{x \in \mathbb{R} : f(x) > \alpha\}$. If $A_\alpha = \phi$ then $A_\alpha \in \mathcal{B}_{\mathbb{R}}$. So, assume $A_\alpha \neq \phi$. Set $x_0 = \inf A_\alpha$. If $x_0 = -\infty$ then $A_\alpha = \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$. Indeed, let $x \in \mathbb{R}$, then since $x_0 = -\infty$ there exists $\hat{x} \in A_\alpha$ such that $\hat{x} < x$. Then

$f(x) \geq f(\hat{x}) > \alpha$. So, $x \in A_\alpha$ and $\mathbb{R} \subseteq A_\alpha$. But $A_\alpha \subseteq \mathbb{R}$ So $A_\alpha = \mathbb{R} \in \mathcal{M}_\mathcal{L}$. Finally, assume $x_0 = \inf A_\alpha > -\infty$. Then, clearly, $x < x_0 \Rightarrow x \notin A_\alpha$. Moreover, if $x > x_0$ then $\exists\, x_1 \in A_\alpha$ such that $x_0 \leq x_1 < x$. Hence $f(x) \geq f(x_1) > \alpha$ and $x \in A_\alpha$. Therefore $(-\infty, x_0) \cap A_\alpha = \phi$ and $A_\alpha \supseteq (x_0, \infty)$. Consequently either $A_\alpha = (x_0, \infty)$ or $A_\alpha = [x_0, \infty)$. In both cases $A_\alpha \in \mathcal{B}_\mathbb{R}$. Thus f is Borel measurable.

4. Let $f(x) = \chi_\mathbb{Q}(x)$ $(x \in \mathbb{R})$. Then f is Lebesgue measurable since $f(x) = 0$ a.e. But f is nowhere continuous as $f = 1$ at $x \in \mathbb{Q}$ and 0 at $x \in \mathbb{Q}^c$, and $\mathbb{Q}$ is dense in $\mathbb{R}$, and $\mathbb{Q}^c$ is also dense in $\mathbb{R}$, as $m(\mathbb{Q}^c \cap (a,b)) = b-a$ for any $a < b$.

3. Recall that $f : X \to \mathbb{R}$ is measurable on $A \in \mathcal{M}_\mathcal{L}$ means that for any $E \in \mathcal{B}_\mathbb{R}$, $f^{-1}(E) \cap A \in \mathcal{M}_\mathcal{L}$. If f is measurable then $f^{-1}(E) \in \mathcal{M}_\mathcal{L}$ for any $E \in \mathcal{B}_\mathbb{R}$. If $A, B \in \mathcal{M}_\mathcal{L}$, then $f^{-1}(E) \cap A \in \mathcal{M}_\mathcal{L}$ and $f^{-1}(E) \cap B \in \mathcal{M}_\mathcal{L}$. Thus f is measurable on A and on B.

Conversely, suppose $X = A \cup B$, $A, B \in \mathcal{M}_\mathcal{L}$, and $f : X \to \mathbb{R}$ is measurable on A and on B. Let

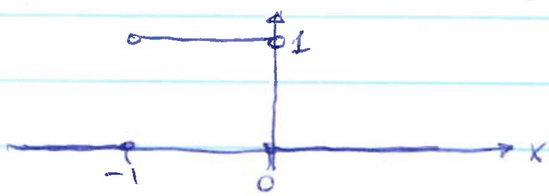
$E \in \mathcal{B}_{\mathbb{R}}$. We have

$$f^{-1}(E) = f^{-1}(E) \cap X = f^{-1}(E) \cap (A \cup B)$$
$$= (f^{-1}(E) \cap A) \cup (f^{-1}(E) \cap B) \in \mathcal{M}.$$

Hence f is measurable. $\square$

5. No. Example. $\mu = \delta$: the Dirac mass concentrated on $\{0\}$. i.e., $\delta(E) = 1$ if $0 \in E$, $\delta(E) = 0$ if $0 \notin E$ where $E \in \mathcal{B}_{\mathbb{R}}$. Let $V = (0,1)$. Then

$$f(x) = \delta((0,1) + x) = \delta((x, x+1))$$
$$= \begin{cases} 1 & \text{if } x < 0 < x+1, \text{ i.e. } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$



Clearly, $f(x) = \chi_{(-1,0)}(x)$ it is discontinuous at $x = 0, -1$. $\square$

6. (1) Note that $f : [0,1] \to [0,1]$ is continuous, nondecreasing, and $f([0,1]) = [0,1]$. Clearly $g(x) = f(x) + x$ is strictly increasing on $[0,1]$. Hence g is injective on $[0,1]$. Moreover, $g(0) = 0$ $g(1) = f(1) + 1 = 2$. By the Intermediate-Value Theorem, for any $\xi \in [0,2]$, $\exists\, x \in [0,1]$ such that $g(x) = \xi$, since g is continuous on $[0,1]$. Thus, $g : [0,1] \to [0,2]$ is surjective. Hence, $g : [0,1] \to [0,2]$ is a bijection. Consequently, $h = g^{-1} : [0,2] \to [0,1]$ is continuous as g is continuous. [In general, if $g : [0,1] \to [a,b]$ is strictly increasing, continuous, and bijective, then $g^{-1} : [a,b] \to [0,1]$ is continuous.

Otherwise, $\exists x_0 \in [a,b]$, $\exists x_n \in [a,b]$ such that $x_n \to x_0$. But $g^{-1}(x_n) \not\to g^{-1}(x_0)$. Without loss of generality, we may assume that $\exists \delta > 0$ such that

$$g^{-1}(x_n) \geq \delta + g^{-1}(x_0) \quad (n = 1, 2 \cdots).$$

Thus, $x_n \geq g(\delta + g^{-1}(x_0)) > g(g^{-1}(x_0)) = x_0$. Hence, $x_n \not\to x_0$, a contradiction.]

(2) Note that the Cantor function $f : [0,1] \to [0,1]$ is constant on any interval of $[0,1] \setminus C$. If I is such an interval, then g translates I by the constant and $m(g(I)) = m(I)$. But the closed set $[0,1] \setminus C$ is a countable union of disjoint such intervals. Thus $m(g([0,1] \setminus C)) = m([0,1] \setminus C) = 1$. By Part (1), $m(g(C)) + m(g[0,1] \setminus C)) = m([0,2]) = 2$. Hence, $m(g(C)) = 1$.

(3) We have $g(B) = A \subseteq g(C)$. Hence $B \subseteq C$. But $m(C) = 0$, and m is complete. Hence B is Lebesgue measurable, and $m(B) = 0$. If B were Borel measurable, $A = g^{-1}(B)$ would be also Boreal measurable, since g^{-1} is continuous by Part (1). Hence, A would be Lebesgue measurable, a contradiction.

(4) Let $F = \chi_B$, B as in Part (3). This is Lebesgue measurable, since B is. Let $G = g^{-1}$ as above. Then $(F \circ G)^{-1}(\{1\}) = G^{-1}(F^{-1}(\{1\})) = G^{-1}(B) = g(B) = A$. Since A is non Lebesgue measurable, $F \circ G$ is not Lebesgue measurable.

$\square$

7. Suppose there existed $x_0 \in (0,1)$ such that $f(x_0) \neq g(x_0)$, say, $f(x_0) > g(x_0)$. Let $\alpha = f(x_0)$. Since f decreases, $\{f \geq \alpha\} = \{f \geq f(x_0)\} \supseteq (0, x_0]$. Since g is decreasing and left continuous, $\exists\, x_1 : 0 < x_1 < x_0$, such that $f(x_0) > g(x_1) \geq g(x_0)$. Thus. $\{g \geq \alpha\} = \{g \geq f(x_0)\}$ $\subseteq \{g > g(x_1)\} \subseteq (0, x_1]$. Therefore,

$m(\{f \geq \alpha\}) \geq m((0, x_0]) = x_0$

$m(\{g \geq \alpha\}) \leq m((0, x_1]) = x_1 < x_0 = m(\{f \geq \alpha\})$, this is a contradiction. Hence $f = g$ on $(0,1)$. $\square$

8. $\lambda(\phi) = \int \chi_\phi f \, d\mu = 0$.

If $E_j \in \mathcal{M}$ $(j = 1, 2, \cdots)$ are disjoint, then $\lambda(\bigcup_{j=1}^{\infty} E_j)$

$$= \int_{\bigcup_{j=1} E_j} f \, d\mu = \int \chi_{\bigcup_{j=1} E_j} f \, d\mu = \int \left(\sum_{j=1}^{\infty} \chi_{E_j}\right) f \, d\mu$$

$$= \int \sum_{j=1}^{\infty} (\chi_{E_j} f) \, d\mu = \sum_{j=1}^{\infty} \int \chi_{E_j} f = \sum_{j=1}^{\infty} \int_{E_j} f = \sum_{j=1}^{\infty} \lambda(E_j).$$

Hence λ is a measure.

If $\phi \in L^+$ is a simple function with $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$, $a_j \geq 0$, $E_j \in \mathcal{M}$, disjoint $\bigcup_{j=1}^{n} E_j = X$. Then.

$$\int_X \phi \, d\lambda = \sum_{j=1}^{n} a_j \lambda(E_j) = \sum_{j=1}^{n} a_j \int_{E_j} f \, d\mu.$$

$$= \sum_{j=1}^{n} a_j \int_X \chi_{E_j} f \, d\mu = \int_X \left(\sum_{j=1}^{n} a_j \chi_{E_j}\right) f \, d\mu$$

$$= \int_X \phi f \, d\mu.$$

Let $g \in L^+$. Let $\{\phi_n\}$ be a sequence of increasing simple functions in L^+ such that $\phi_n \to g$. Then, $0 \leq \phi_1 f \leq \phi_2 f \leq \cdots$ and $\phi_n f \to gf$. By the Monotone convergence Theorem,

$$\int_X g \, d\lambda = \lim_{n\to\infty} \int_X \phi_n \, d\lambda = \lim_{n\to\infty} \int_X \phi_n f \, d\mu = \int_X g f \, d\mu. \quad \square$$

9. (1) Denote $E_\infty = \{f = \infty\}$ and $F = \{f > 0\}$.

For any $n \in \mathbb{N}$,

$$\infty > \int_X f\,d\mu \geq \int_{E_\infty} f\,d\mu \geq \int_{E_\infty} n\,d\mu = n\mu(E_\infty).$$

So $0 \leq \mu(E_\infty) \leq \frac{1}{n}\int_X f\,d\mu \to 0$ as $n \to \infty$.

Hence $\mu(E_\infty) = 0$.

Let $\hat{F} = \{f \geq 1\}$ and $F_n = \{\frac{1}{n+1} \leq f < \frac{1}{n}\}$ $(n = 1, 2, \cdots)$.

Then $X = E_\infty \cup \hat{F} \cup \left(\bigcup_{n=1}^\infty F_n\right)$. This is a disjoint union of measurable sets. By Part (1), we have

$$\infty > \int_X f\,d\mu = \int_{X \setminus E_\infty} f\,d\mu + \int_{E_\infty} f\,d\mu = \int_{X \setminus E} f\,d\mu$$

$$= \int_{\hat{F}} f\,d\mu + \int_{\bigcup_{n=1}^\infty F_n} f\,d\mu = \int_{\hat{F}} f\,d\mu + \sum_{n=1}^\infty \int_{F_n} f\,d\mu$$

$$\geq \int_{\hat{F}} 1\,d\mu + \sum_{n=1}^\infty \int_{F_n} \frac{1}{n+1}\,d\mu$$

$$= \mu(\hat{F}) + \sum_{n=1}^\infty \frac{1}{n+1}\mu(F_n).$$

Thus, $\mu(\hat{F}) < \infty$, $\mu(F_n) < \infty$ $(n = 1, 2, \cdots)$. Hence μ is σ-finite.

(2) $\forall \varepsilon > 0$. Continuing from the above, we have

$$\int_X f\,d\mu = \int_{\hat{F}} f\,d\mu + \sum_{n=1}^\infty \int_{F_n} f\,d\mu < \infty$$

Thus, $\exists N \in \mathbb{N}$ such that

$$\int_X f\,d\mu - \varepsilon < \int_{\hat{F}} f\,d\mu + \sum_{n=1}^N \int_{F_n} f\,d\mu = \int_E f\,d\mu.$$

where $E = \hat{F} \cup \left(\bigcup_{n=1}^{\infty} F_n\right) \in \mathcal{M}$, and $\mu(E) \leq \mu(\hat{F}) + \sum_{n=1}^{N} \mu(F_n) < \infty$ as shown above. $\square$

10. We show $(1) \Rightarrow (2)$, $(2) \Rightarrow (1)$,
$(2) \Rightarrow (3)$, $(3) \Rightarrow (2)$,
$(4) \Rightarrow (3)$, $(1) \Rightarrow (4)$.

$(1) \Rightarrow (2)$ We have $|f| \in L^+$ and $|f| = 0$ a.e.
If $\phi \in L^+$ is a simple function and $\phi = 0$ a.e. Then by definition we have $\int \phi \, d\mu = 0$.
If ϕ is a simple function, $0 \le \phi \le |f|$, then $\phi = 0$, a.e.
Hence $\int \phi \, d\mu = 0$. Thus.

$$\int |f| \, d\mu = \sup \left\{ \int \phi \, d\mu : 0 \le \phi \le |f|, \phi : \text{simple} \right\} = 0.$$

$(2) \Rightarrow (1)$ By ~~the~~ disjoint union of measurable sets

$$X = \{|f| = 0\} \cup \{|f| \ge 1\} \cup \bigcup_{n=1}^{\infty} \left\{ \frac{1}{n+1} \le |f| < \frac{1}{n} \right\},$$

we have

$$0 = \int_X |f| \, d\mu = \int_{\{|f|=0\}} |f| \, d\mu + \int_{\{|f| \ge 1\}} |f| \, d\mu + \sum_{n=1}^{\infty} \int_{\{\frac{1}{n+1} \le |f| < \frac{1}{n}\}} |f| \, d\mu$$

$$\ge \int_{\{|f| \ge 1\}} 1 \, d\mu + \sum_{n=1}^{\infty} \int_{\{\frac{1}{n+1} \le |f| < \frac{1}{n}\}} \frac{1}{n+1} \, d\mu$$

$$= \mu(\{|f| \ge 1\}) + \sum_{n=1}^{\infty} \mu\left(\left\{\frac{1}{n+1} \le |f| < \frac{1}{n}\right\}\right)$$

$$= \mu\left(\{|f| \ge 1\} \cup \left(\bigcup_{n=1}^{\infty} \left\{\frac{1}{n+1} \le |f| < \frac{1}{n}\right\}\right)\right)$$

$$= \mu(\{|f| > 0\}) \ge 0$$

Hence $\mu(\{|f| > 0\}) = 0$ and $f = 0$. a.e.

(2) $\Rightarrow$ (3)

$\forall E \in \mathcal{M}$. $\left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu \le \int_X |f|\,d\mu = 0$.

(3) $\Rightarrow$ (2) Let $E = \{f \ge 0\} \in \mathcal{M}$. Then,

$$0 = \int_{\{f\ge 0\}} f\,d\mu = \int_{\{f \ge 0\}} f^+\,d\mu = \int_X f^+\,d\mu = 0.$$

Similarly, $\int_X f^-\,d\mu = 0$. Hence,

$$\int_X |f|\,d\mu = \int_X (f^+ + f^-)\,d\mu = \int_X f^+\,d\mu + \int_X f^-\,d\mu = 0.$$

(4) $\Rightarrow$ (3) Let $g = \chi_E$ for $E \in \mathcal{M}$. Then

$$\int_X fg\,d\mu = \int_E f\,d\mu = 0.$$

(1) $\Rightarrow$ (4) $\forall g : X \to \mathbb{R}$, measurable. Since $f = 0$ a.e. $fg = 0$ a.e. Hence, as shown that (1) $\Rightarrow$ (2), we have $\int_X |fg|\,d\mu = 0$ ← This implies $fg \in L^1(\mu)$. But then

$$\left|\int_X fg\,d\mu\right| \le \int_X |fg|\,d\mu = 0.$$

$\square$