Math 240C: Real Analysis, Spring 2020

Homework Assignment 1

Due 12:00 noon, Monday, April 6, 2020

1. Let X be a locally compact Hausdorff (LCH) space, Y a closed subset of X (which is an LCH space in the relative topology), and μ a Radon measure on Y. Define $I : C_c(X) \to \mathbb{C}$ by
$$I(f) = \int_Y (f|_Y)\, d\mu \qquad \forall f \in C_c(X),$$
where $f|_Y$ is the restriction of f onto Y. Prove that I is a positive linear functional on $C_c(X)$ and that the induced Radon measure ν on X is given by $\nu(E) = \mu(E \cap Y)$ for any Borel measurable subset E of X.

2. Let X be a locally compact Hausdorff space and I a positive linear functional on $C_c(X)$. Prove that for any compact subset K of X there exists $C_K \in \mathbb{R}$, depending on K, such that $|I(f)| \le C_K \|f\|$ for any $f \in C_c(X)$ such that $\operatorname{supp}(f) \subseteq K$.

3. Let μ be a Radon measure on a locally compact Hausdorff space X.
   (1) Let N be the union of all open subsets $U \subseteq X$ such that $\mu(U) = 0$. Prove that N is the largest open subset of X such that $\mu(N) = 0$. The complement of N is called the **support** of μ and is denoted by $\operatorname{supp}(\mu)$.
   (2) Prove that $x \in \operatorname{supp}(\mu)$ if and only if
$$\int_X f\, d\mu > 0 \quad \text{for all } f \in C_c(X, [0,1]) \quad \text{such that } f(x) > 0.$$

4. Let μ be a Radon measure on a locally compact Hausdorff space X and $\phi \in L^1(\mu)$ with $\phi \ge 0$ on X. Define
$$\nu(E) = \int_E \phi\, d\mu \qquad \forall E \in \mathcal{B}_X.$$
Prove that ν is a Radon measure on X and that $\operatorname{supp}(\nu) \subseteq \operatorname{supp}(\phi) \cap \operatorname{supp}(\mu)$.

5. Let X be a locally compact Hausdorff space and $x_0 \in X$. Define $I(f) = f(x_0)$ for any $f \in C_c(X)$. Prove that $I : C_c(X) \to \mathbb{C}$ is a positive linear functional on $C_c(X)$ and that the Radon measure associated with the functional I is the Dirac mass δ_{x_0} at x_0.

6. Let μ be a Radon measure on a compact Hausdorff space X with $\mu(X) = 1$. Prove that there is a compact set $K \subseteq X$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K.

7. Prove that a Borel measure on $\mathbb{R}^n$ is a Radon measure if and only if it is finite on each compact subset of $\mathbb{R}^n$.