PIERI RULES FOR SCHUR FUNCTIONS IN SUPERSPACE

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ABSTRACT. The Schur functions in superspace s_{Λ} and \bar{s}_{Λ} are the limits q=t=0 and $q=t=\infty$ respectively of the Macdonald polynomials in superspace. We prove Pieri rules for the bases s_{Λ} and \bar{s}_{Λ} (which happen to be essentially dual). As a consequence, we derive the basic properties of these bases such as dualities, monomial expansions, and tableaux generating functions.

1. Introduction

An extension to superspace of the theory of symmetric functions was developed in [2, 6, 7]. In this extension, the polynomials $f(x,\theta)$, where $(x,\theta) = (x_1,\ldots,x_N,\theta_1,\ldots,\theta_N)$, not only depend on the usual commuting variables x_1,\ldots,x_N but also on the anticommuting variables θ_1,\ldots,θ_N ($\theta_i\theta_j=-\theta_j\theta_i$, and $\theta_i^2=0$). In this article, we are concerned with two natural generalizations to superspace of the Schur functions that arise as special limits of the Macdonald polynomials in superspace and whose combinatorics appears to be extremely rich.

The extension to superspace of the Macdonald polynomials, $\{P_{\Lambda}(x,\theta;q,t)\}_{\Lambda}$, is a basis of the ring $\mathbb{Q}(q,t)[x_1,\ldots,x_N;\theta_1,\ldots,\theta_N]^{S_N}$ of symmetric polynomials in superspace, where the superscript S_N indicates that the elements of the ring are invariant under the diagonal action of the symmetric group S_N (that is, invariant under the simultaneous interchange of $x_i \leftrightarrow x_j$ and $\theta_i \leftrightarrow \theta_j$, for any i,j). They are indexed by superpartitions Λ and defined as the unique basis such that

- (1) $P_{\Lambda}(q,t) = m_{\Lambda} + \text{smaller terms}$
- (2) $\langle P_{\Lambda}(q,t), P_{\Omega}(q,t) \rangle_{q,t} = 0$ if $\Lambda \neq \Omega$

where the scalar product $\langle \langle \cdot, \cdot \rangle \rangle_{q,t}$ is given by

$$\langle\!\langle p_{\Lambda}, p_{\Omega} \rangle\!\rangle_{q,t} = \delta_{\Lambda\Omega} \, q^{|\Lambda^a|} z_{\Lambda^s} \prod_i \frac{1 - q^{\Lambda_i^s}}{1 - t^{\Lambda_i^s}}$$

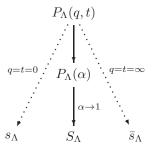
$$\tag{1.1}$$

on the power sum symmetric functions in superspace (see Section 2 for all the relevant definitions). It was shown in [2] that even though the limits q=t=0 and $q=t=\infty$ of this scalar product are degenerate and not well-defined respectively, the corresponding limits $s_{\Lambda}:=P_{\Lambda}(0,0)$ and $\bar{s}_{\Lambda}:=P_{\Lambda}(\infty,\infty)$ of the Macdonald superpolynomials exist and are related to Key polynomials [9, 11]. As we will see, the rich combinatorics of these functions makes them the genuine extensions to superspace of the Schur functions. In comparison, the a priori more relevant limit q=t=1 of the Macdonald polynomials in superspace, which corresponds to the limit $\alpha=1$ of the Jack polynomials in superspace $P_{\Lambda}(\alpha)$, does not seem to be very interesting from the combinatorial point of view (in the figure below, the limit q=t=1 of the Macdonald polynomials in superspace corresponds to S_{Λ}).

The basis s_{Λ} is especially relevant since it plays the role of the Schur functions in the generalization to superspace of the original Macdonald positivity conjectures [1]. To be more specific, let $J_{\Lambda}(q,t) = c_{\Lambda}(q,t)P_{\Lambda}(q,t)$ be the integral form of the Macdonald superpolynomials $(c_{\Lambda}(q,t))$ is a

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constant belonging to $\mathbb{Z}[q,t]$) and let $\varphi(s_{\Lambda})$ be a certain plethystically transformed version of the function s_{Λ} (see [1] for more details). Then the coefficients $K_{\Omega\Lambda}(q,t)$ appearing in

$$J_{\Lambda}(q,t) = \sum_{\Omega} K_{\Omega\Lambda}(q,t) \,\varphi(s_{\Omega}) \tag{1.2}$$

are conjectured to be polynomials in q and t with nonnegative integer coefficients (the conjecture is known to hold when the degree in the anticommuting variables is either zero [8], which corresponds to the usual Macdonald case, or sufficiently large [3]).

In this article, we will derive the basic properties of the bases s_{Λ} and \bar{s}_{Λ} (which as we will see are essentially dual) such as Pieri rules, dualities, monomial expansions, tableaux generating functions, and Cauchy identities. It is important to note that the combinatorics of the bases s_{Λ} and \bar{s}_{Λ} was first studied in [4]. Our work stems in large part from a desire to develop the right framework to prove the conjectures therein, especially those concerning Pieri rules¹.

We are confident that this work is only the tip of the iceberg and that deeper properties of the bases s_{Λ} and \bar{s}_{Λ} will be uncovered in the future, such as for instance a group-theoretical interpretation of the generalization to superspace of the Macdonald positivity conjecture. At the tableau level, we are hopeful that this work will eventually lead to a Robinson-Schensted-Knuth insertion algorithm in superspace, and ultimately to a charge statistic on tableaux that would solve the case q = 0 of (1.2).

The most technical parts of this work are the proofs of the Pieri rules. They rely on the correspondence between the Schur functions in superspace and Key polynomials [2] which allows to use the powerful machinery of divided differences [10]. Once the Pieri rules are assumed to hold, the remaining results follow somewhat easily from duality arguments or well-known techniques of symmetric function theory [12, 14]. To alleviate the presentation, the proofs of many technical results have been relegated to the appendix.

2. Symmetric function theory in superspace: basic definitions

Before introducing the basic concepts of symmetric function theory in superspace, we discuss an important identification (as vector spaces) between the ring of symmetric polynomials in superspace and the ring of bisymmetric polynomials.

A polynomial in superspace, or equivalently, a superpolynomial, is a polynomial in the usual N variables x_1, \ldots, x_N and the N anticommuting variables $\theta_1, \ldots, \theta_N$ over a certain field, which will be taken in the remainder of this article to be \mathbb{Q} . A superpolynomial $P(x, \theta)$, with $x = (x_1, \ldots, x_N)$ and $\theta = (\theta_1, \ldots, \theta_N)$, is said to be symmetric if the following is satisfied:

$$P(x_1, \dots, x_N, \theta_1, \dots, \theta_N) = P(x_{\sigma(1)}, \dots, x_{\sigma(N)}, \theta_{\sigma(1)}, \dots, \theta_{\sigma(N)}) \qquad \forall \sigma \in S_N$$
 (2.1)

¹The connection between the results of [4] and those of this article is discussed in Remarks 21 and 26.

where S_N is the symmetric group on $\{1,\ldots,N\}$. The ring of superpolynomials in N variables has a natural grading with respect to the fermionic degree m (the total degree in the anticommuting variables). We will denote by Λ_N^m the ring of symmetric superpolynomials in N variables and fermionic degree m over the field \mathbb{Q} . By symmetry, one can reconstruct a superpolynomial in Λ_N^m from its coefficient $\theta_1 \cdots \theta_m$. Furthermore, since that coefficient is necessarily antisymmetric in the variables x_1, \ldots, x_m , and symmetric in the remaining variables, there is a natural identification as vector spaces between Λ_N^m and the ring of bisymmetric polynomials $\mathbb{Q}[x_1, \ldots, x_N]^{S_m \times S_{m^c}}$, where $S_m \times S_{m^c}$ is taken as the subgroup of S_N given by the permutations that leave the sets $\{1, \ldots, m\}$ and $\{m+1, \ldots, N\}$ invariant. Given an element $P(x, \theta)$ of Λ_N^m , the identification is simply

$$P(x,\theta) \longleftrightarrow \frac{P(x,\theta)\Big|_{\theta_1\cdots\theta_m}}{\Delta_m(x)}$$
 (2.2)

where $\Delta_m(x)$ is the Vandermonde determinant $\prod_{1 \leq i < j \leq m} (x_i - x_j)$, and $P(x, \theta) \Big|_{\theta_1 \cdots \theta_m}$ is the coefficient of $\theta_1 \cdots \theta_m$ in $P(x, \theta)$.

2.1. Superpartitions. We first recall some definitions related to partitions [12]. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of degree $|\lambda|$ is a vector of non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ for $i = 1, 2, \dots$ and such that $\sum_i \lambda_i = |\lambda|$. Each partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the top to bottom. Any lattice square in the Ferrers diagram is called a cell (or simply a square), where the cell (i,j) is in the ith row and jth column of the diagram. The conjugate λ' of a partition λ is represented by the diagram obtained by reflecting λ about the main diagonal. We say that the diagram μ is contained in λ , denoted $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$ for all i. Finally, λ/μ is a horizontal (resp. vertical) n-strip if $\mu \subseteq \lambda$, $|\lambda| - |\mu| = n$, and the skew diagram λ/μ does not have two cells in the same column (resp. row).

Symmetric superpolynomials are naturally indexed by superpartitions². A superpartition Λ of degree (n|m), or $\Lambda \vdash (n|m)$ for short, is a pair $(\Lambda^{\circledast}, \Lambda^*)$ of partitions Λ^{\circledast} and Λ^* such that:

- 1. $\Lambda^* \subseteq \Lambda^{\circledast}$;
- 2. the degree of Λ^* is n;
- 3. the skew diagram $\Lambda^{\circledast}/\Lambda^*$ is both a horizontal and a vertical m-strip³

We refer to m and n respectively as the fermionic degree and total degree of Λ . Obviously, if $\Lambda^{\circledast} = \Lambda^* = \lambda$, then $\Lambda = (\lambda, \lambda)$ can be interpreted as the partition λ .

We will also need another characterization of a superpartition. A superpartition Λ is a pair of partitions $(\Lambda^a; \Lambda^s) = (\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_N)$, where Λ^a is a partition with m distinct parts (one of them possibly equal to zero), and Λ^s is an ordinary partition (with possibly a string of zeros at the end). The correspondence between $(\Lambda^\circledast, \Lambda^*)$ and $(\Lambda^a; \Lambda^s)$ is given explicitly as follows: given $(\Lambda^\circledast, \Lambda^*)$, the parts of Λ^a correspond to the parts of Λ^* such that $\Lambda^\circledast_i \neq \Lambda^*_i$, while the parts of Λ^s correspond to the parts of Λ^* such that $\Lambda^\circledast_i = \Lambda^*_i$.

The conjugate of a superpartition $\Lambda=(\Lambda^\circledast,\Lambda^*)$ is $\Lambda'=((\Lambda^\circledast)',(\Lambda^*)')$. A diagrammatic representation of Λ is given by the Ferrers diagram of Λ^* with circles added in the cells corresponding to $\Lambda^\circledast/\Lambda^*$. For instance, if $\Lambda=(\Lambda^a;\Lambda^s)=(3,1,0;2,1)$, we have $\Lambda^\circledast=(4,2,2,1,1)$ and $\Lambda^*=(3,2,1,1)$, so that

$$\Lambda^{\circledast}: \qquad \longrightarrow \qquad \Lambda: \qquad \longrightarrow \qquad \Lambda': \qquad \longrightarrow \qquad (2.3)$$

²Superpartitions correspond, using a trivial bijection, to the overpartitions studied in [5].

 $^{^{3}}$ Such diagrams are sometimes called m-rook strips.

where the last diagram illustrates the conjugation operation that corresponds, as usual, to replacing rows by columns.

From the dominance ordering on partitions

$$\mu \le \lambda$$
 iff $|\mu| = |\lambda|$ and $\mu_1 + \dots + \mu_i \le \lambda_1 + \dots + \lambda_i \quad \forall i$. (2.4)

we define the dominance ordering on superpartitions as

$$\Omega \le \Lambda \quad \text{iff} \quad \deg(\Lambda) = \deg(\Omega), \quad \Omega^* \le \Lambda^* \quad \text{and} \quad \Omega^{\circledast} \le \Lambda^{\circledast}$$
 (2.5)

where we stress that the order on partitions is the dominance ordering.

- 2.2. **Simple bases.** Four simple bases of the space of symmetric polynomials in superspace will be particularly relevant to our work [6]:
 - (1) the extension of the monomial symmetric functions, m_{Λ} , defined by

$$m_{\Lambda} = \sum_{\sigma \in S_N} \theta_{\sigma(1)} \cdots \theta_{\sigma(m)} x_{\sigma(1)}^{\Lambda_1} \cdots x_{\sigma(N)}^{\Lambda_N}, \tag{2.6}$$

where the sum is over the permutations of $\{1, ..., N\}$ that produce distinct terms, and where the entries of $(\Lambda_1, ..., \Lambda_N)$ are those of $\Lambda = (\Lambda^a; \Lambda^s) = (\Lambda_1, ..., \Lambda_m; \Lambda_{m+1}, ..., \Lambda_N)$;

(2) the generalization of the power-sum symmetric functions $p_{\Lambda} = \tilde{p}_{\Lambda_1} \cdots \tilde{p}_{\Lambda_m} p_{\Lambda_{m+1}} \cdots p_{\Lambda_\ell}$,

where
$$\tilde{p}_k = \sum_{i=1}^{N} \theta_i x_i^k$$
 and $p_r = \sum_{i=1}^{N} x_i^r$, for $k \ge 0, r \ge 1$; (2.7)

(3) the generalization of the elementary symmetric functions $e_{\Lambda} = \tilde{e}_{\Lambda_1} \cdots \tilde{e}_{\Lambda_m} e_{\Lambda_{m+1}} \cdots e_{\Lambda_{\ell}}$,

where
$$\tilde{e}_k = m_{(0;1^k)}$$
 and $e_r = m_{(\emptyset;1^r)}$, for $k \ge 0, r \ge 1$; (2.8)

(4) the generalization of the homogeneous symmetric functions $h_{\Lambda} = \tilde{h}_{\Lambda_1} \cdots \tilde{h}_{\Lambda_m} h_{\Lambda_{m+1}} \cdots h_{\Lambda_\ell}$,

where
$$\tilde{h}_k = \sum_{\Lambda \vdash (n|1)} (\Lambda_1 + 1) m_{\Lambda}$$
 and $h_r = \sum_{\Lambda \vdash (n|0)} m_{\Lambda}$, for $k \ge 0, r \ge 1$ (2.9)

Observe that when $\Lambda = (\emptyset; \lambda)$, we have that $m_{\Lambda} = m_{\lambda}$, $p_{\Lambda} = p_{\lambda}$, $e_{\Lambda} = e_{\lambda}$ and $h_{\Lambda} = h_{\lambda}$ are respectively the usual monomial, power-sum, elementary and homogeneous symmetric functions. Also note that if we define the operator $d = \theta_1 \partial/\partial_{x_1} + \cdots + \theta_N \partial/\partial_{x_N}$, we have

$$(k+1)\tilde{p}_k = d(p_{k+1}), \qquad \tilde{e}_k = d(e_{k+1}) \quad \text{and} \quad \tilde{h}_k = d(h_{k+1})$$
 (2.10)

that is, the new generators in the superspace versions of the bases can be obtained from acting with d on the generators of the usual symmetric function versions.

2.3. Scalar product and duality. The relevant scalar product in this article is the specialization q = t = 1 of the scalar product $(1.1)^4$, that is,

$$\langle\!\langle p_{\Lambda}, p_{\Omega} \rangle\!\rangle = \delta_{\Lambda\Omega} \, z_{\Lambda^s} \tag{2.11}$$

where, as usual, $z_{\lambda} = 1^{n_{\lambda}(1)} n_{\lambda}(1)! 2^{n_{\lambda}(2)} n_{\lambda}(2)! \cdots$ with $n_{\lambda}(i)$ the number of parts of λ equal to i. The homogeneous and monomial bases are dual with respect to this scalar product [6]

$$\langle\langle h_{\Lambda}, m_{\Omega} \rangle\rangle = \delta_{\Lambda\Omega} \tag{2.12}$$

We define the homomorphism ω as

$$\omega(\tilde{p}_r) = (-1)^r \tilde{p}_r \quad \text{and} \quad \omega(p_r) = (-1)^{r-1} p_r \tag{2.13}$$

⁴The scalar product (1.1) differs from that of [2] by a sign depending on the fermionic degree.

This homomorphism, which is obviously an involution and an isometry of the scalar product $\langle \langle \cdot, \cdot \rangle \rangle$, is such that [6]

$$\omega(h_{\Lambda}) = e_{\Lambda} \tag{2.14}$$

The Schur functions in superspace s_{Λ} and \bar{s}_{Λ} were defined in the introduction as the special limits q=t=0 and $q=t=\infty$ respectively of the Macdonald polynomials in superspace. Remarkably, the functions s_{Λ} and \bar{s}_{Λ} are essentially dual with respect to our scalar product.

Proposition 1 ([2]). Let s_{Λ}^* and \bar{s}_{Λ}^* be the bases dual to the bases s_{Λ} and \bar{s}_{Λ} respectively, that is, let s_{Λ}^* and \bar{s}_{Λ}^* be such that

$$\langle \langle s_{\Lambda}^*, s_{\Omega} \rangle \rangle = \langle \langle \bar{s}_{\Lambda}^*, \bar{s}_{\Omega} \rangle \rangle = \delta_{\Lambda\Omega}$$
 (2.15)

Then

$$s_{\Lambda}^* = (-1)^{\binom{m}{2}} \omega \bar{s}_{\Lambda'}$$
 and $\bar{s}_{\Lambda}^* = (-1)^{\binom{m}{2}} \omega s_{\Lambda'}$ (2.16)

where m is the fermionic degree of Λ .

When m=0, we have $s_{\Lambda}=s_{(\emptyset;\lambda)}=s_{\lambda}$ and $\bar{s}_{\Lambda}=\bar{s}_{(\emptyset;\lambda)}=s_{\lambda}$. In this case the proposition is simply stating the well known fact that the dual basis of the Schur basis $\{s_{\lambda}\}$ with respect to the Hall scalar product is the basis $\{s_{\lambda}=\omega s_{\lambda'}\}$, that is, that the Schur basis is orthonormal.

3. Relation with Key Polynomials

A connection between Key polynomials and Schur functions in superspace was established in [2]. Before stating that connection explicitly we introduce the Key polynomials [10, 11]⁵.

3.1. **Key polynomials.** Let π_i be the isobaric divided difference operator

$$\pi_i = \frac{1}{(x_i - x_{i+1})} (x_i - x_{i+1} \kappa_{i,i+1})$$
(3.1)

where $\kappa_{i,i+1}$ is the operator that interchanges the variables x_i and x_{i+1} :

$$\kappa_{i,i+1} f(x_1, \dots, x_i, x_{i+1}, \dots, x_N) = f(x_1, \dots, x_{i+1}, x_i, \dots, x_N)$$
(3.2)

The isobaric divided difference operators satisfy the relations

$$\pi_i \pi_j = \pi_j \pi_i \quad \text{if } |i - j| > 1, \qquad \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, \qquad \pi_i^2 = \pi_i$$
 (3.3)

and

$$\pi_i x_i = x_{i+1} \pi_i + x_i, \qquad \pi_i x_{i+1} = x_i \pi_i - x_i$$
(3.4)

Given $\eta \in \mathbb{Z}_{\geq 0}^N$, the Key polynomial $K_{\eta}(x_1, \ldots, x_N)$ is defined recursively as follows. If η is weakly decreasing, then

$$K_n = x^{\eta} = x_1^{\eta_1} x_2^{\eta_2} \cdots x_N^{\eta_N} \tag{3.5}$$

Otherwise, if $\eta_i < \eta_{i+1}$ then

$$K_{\eta} = \pi_i K_{s_i \eta} \tag{3.6}$$

where $s_i\eta$ is equal to η with the *i*-th and i+1-th entries interchanged. It should be noted that if η is weakly increasing then K_{η} is equal to a Schur function:

$$K_{\eta}(x_1, \dots, x_N) = s_{\lambda}(x_1, \dots, x_N) \tag{3.7}$$

where λ is the partition corresponding to the rearrangement of the entries of η .

The adjoint Key polynomial $\hat{K}_{\eta}(x_1,\ldots,x_N)$ is defined similarly. If η is weakly decreasing, then

$$\hat{K}_{\eta} = x^{\eta} = x_1^{\eta_1} x_2^{\eta_2} \cdots x_N^{\eta_N} \tag{3.8}$$

Otherwise, if $\eta_i < \eta_{i+1}$ then

$$\hat{K}_{\eta} = \hat{\pi}_i \hat{K}_{s_i \eta} \tag{3.9}$$

⁵Key polynomials are also known as Type A Demazure atoms [13].

where $\hat{\pi}_i$ is the divided difference operator

$$\hat{\pi}_i = \pi_i - 1 \tag{3.10}$$

which satisfies the relations

$$\hat{\pi}_i \hat{\pi}_j = \hat{\pi}_j \hat{\pi}_i \quad \text{if } |i - j| > 1, \qquad \hat{\pi}_i \hat{\pi}_{i+1} \hat{\pi}_i = \hat{\pi}_{i+1} \hat{\pi}_i \hat{\pi}_{i+1}, \qquad \hat{\pi}_i^2 = -\hat{\pi}_i.$$
 (3.11)

Note that $\pi_i \hat{\pi}_i = \hat{\pi}_i \pi_i = 0$ from the definition of $\hat{\pi}_i$ and the relation $\pi_i^2 = \pi_i$.

3.2. **Key polynomials and Schur functions in superspace.** To connect Key polynomials and Schur functions in superspace, we will also need the usual divided difference operator

$$\partial_i = \frac{1}{(x_i - x_{i+1})} (1 - \kappa_{i,i+1}) \tag{3.12}$$

which is such that

$$\partial_i \partial_j = \partial_j \partial_i \quad \text{if } |i - j| > 1, \qquad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}, \qquad \partial_i^2 = 0.$$
 (3.13)

Observe that $\pi_i = \partial_i x_i$, which implies that $\partial_i \pi_i = 0$.

For $s_{i_1} \dots s_{i_\ell}$ a reduced decomposition of the permutation σ , we let $\partial_{\sigma} = \partial_{i_1} \cdots \partial_{i_\ell}$ (and similarly for π_{σ} and $\hat{\pi}_{\sigma}$). We also let ω_m and ω_{m^c} be the longest permutation of $\{1, \dots, m\}$ and $\{m+1, \dots, N\}$ respectively.

We now establish the image of the Schur functions in superspace s_{Λ} and \bar{s}_{Λ} in the identification (2.2) between Λ_N^m and $\mathbb{Q}[x_1,\ldots,x_N]^{S_m\times S_{m^c}}$.

Lemma 2. Let Λ be a superpartition of fermionic degree m. In the identification (2.2) between Λ_N^m and $\mathbb{Q}[x_1,\ldots,x_N]^{S_m\times S_{m^c}}$ we have

$$s_{\Lambda} \longleftrightarrow (-1)^{\binom{m}{2}} \partial_{\omega_m} \pi_{\omega_m c} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x)$$
 (3.14)

where $(\Lambda^a)^R$, Λ^s is the composition $(\Lambda^a_m \dots, \Lambda^a_1, \Lambda^s_1, \Lambda^s_2, \dots)$ obtained by concatenating Λ^a in reverse order and Λ^s .

Proof. Let $A_m = \sum_{\sigma \in S_m} (-1)^{\ell(\sigma)} \kappa_{\sigma}$ be the antisymmetrization operator with respect to the variables x_1, \ldots, x_m . It was established in [2] that

$$s_{\Lambda}(x,\theta) = (-1)^{\binom{m}{2}} \sum_{\sigma \in S_N/(S_m \times S_{m^c})} \mathcal{K}_{\sigma}\theta_1 \cdots \theta_m A_m \left(\sum_{v \in S_{m^c}} \hat{\pi}_v \right) \hat{K}_{(\Lambda^a)^R,\Lambda^s}(x)$$
(3.15)

where \mathcal{K}_{σ} is the operator such that

$$\mathcal{K}_{\sigma}P(x_1,\ldots,x_N,\theta_1,\ldots,\theta_N) = P(x_{\sigma(1)},\ldots,x_{\sigma(N)},\theta_{\sigma(1)},\ldots,\theta_{\sigma(N)}) \qquad \forall \, \sigma \in S_N$$
 (3.16)

It is known that $\pi_{\omega_{m^c}} = \sum_{v \in S_{m^c}} \hat{\pi}_v$ and that $A_m = \Delta_m(x) \partial_{\omega_m}$ (see for instance [10]). We thus have that

$$\frac{s_{\Lambda}(x,\theta)\big|_{\theta_1\cdots\theta_m}}{\Delta_m(x)} = (-1)^{\binom{m}{2}} \partial_{\omega_m} \pi_{\omega_m c} \hat{K}_{(\Lambda^a)^R,\Lambda^s}(x)$$
(3.17)

and the lemma follows.

Lemma 3. Let Λ be a superpartition of fermionic degree m. In the identification (2.2) between Λ_N^m and $\mathbb{Q}[x_1,\ldots,x_N]^{S_m\times S_{m^c}}$ we have

$$\bar{s}_{\Lambda} \longleftrightarrow \partial'_{\omega_{(N-m)^c}} K_{(\Lambda^s)^R,\Lambda^a}(y)$$
 (3.18)

where $(\Lambda^s)^R$, Λ^a is the composition $(\Lambda_{N-m}^s, \ldots, \Lambda_1^s, \Lambda_1^a, \Lambda_2^a, \ldots)$ obtained by concatenating Λ^s in reverse order with Λ^a , where y stands for the variables $y_1 = x_N, y_2 = x_{N-1}, \ldots, y_N = x_1$ and where the ' indicates that the divided differences act on the y variables.

Proof. It was established in [2] that

$$\bar{s}_{\Lambda}(x,\theta) = (-1)^{\binom{m}{2} + \binom{N-m}{2}} \sum_{\sigma \in S_N/(S_m \times S_{m^c})} \mathcal{K}_{\sigma}\theta_1 \cdots \theta_m A_m \frac{1}{\Delta_{m^c}(x)} A_{m^c} x^{\gamma} K_{\Lambda^s,\Lambda^a}(x_N,\dots,x_1)$$
(3.19)

where $x^{\gamma} = x_N^{N-m-1} x_{N-1}^{N-m-2} \cdots x_{m+2}$, and where A_{m^c} is the antisymmetrization operator (see the proof the previous lemma) acting on the variables x_{m+1}, \ldots, x_N (and similarly for $\Delta_{m^c}(x)$). In this case, it is convenient to make the change of variables $y_i = x_{N+1-i}$ for $i = 1, \ldots, N$ to obtain

$$\bar{s}_{\Lambda}(x,\theta) = (-1)^{\binom{m}{2}} \sum_{\sigma \in S_N/(S_m \times S_{m^c})} \mathcal{K}_{\sigma}\theta_1 \cdots \theta_m A'_{(N-m)^c} \frac{1}{\Delta_{N-m}(y)} A'_{N-m} y^{\delta} K_{\Lambda^s,\Lambda^a}(y)$$
(3.20)

where $y^{\delta} = y_1^{N-m-1}y_2^{N-m-2}\cdots y_{N-m-1}$, and the ' indicate that the operator now acts on the y variables (note that \mathcal{K}_{σ} still acts on the x variables). Observe that changing $\Delta_{m^c}(x)$ to $\Delta_{N-m}(y)$ introduces a factor of $(-1)^{\binom{N-m}{2}}$. Using $A'_{(N-m)^c} = \Delta_{(N-m)^c}(y)\partial'_{\omega_{(N-m)^c}}$ and $A'_{N-m}y^{\delta} = \Delta_{N-m}(y)\pi'_{\omega_{N-m}}$, where again the ' mean that that the operators are acting on the y variables, we get

$$\frac{\bar{s}_{\Lambda}(x,\theta)\big|_{\theta_1\cdots\theta_m}}{\Delta_m(x)} = \partial'_{\omega_{(N-m)^c}} K_{(\Lambda^s)^R,\Lambda^a}(y)$$
(3.21)

since the effect of $\pi'_{\omega_{N-m}}$ on $K_{\Lambda^s,\Lambda^a}(y)$ is to reorder the entries of Λ^s in weakly increasing order. Observe that there is no sign anymore in the previous equation due to the fact that $\Delta_m(x)$ and $\Delta_{(N-m)^c}(y)$ differ by a factor $(-1)^{\binom{m}{2}}$.

To prove the Pieri rules associated to the multiplication by a fermionic quantity (a superpolynomial of fermionic degree one) we need to find the image of $\tilde{e}_{\ell} s_{\Lambda}$, $\tilde{p}_{\ell} s_{\Lambda}$ and $\tilde{e}_{\ell} \bar{s}_{\Lambda}$ in the identification between Λ_N^{m+1} and $\mathbb{Q}[x_1,\ldots,x_N]^{S_{m+1}\times S_{(m+1)^c}}$.

Lemma 4. Let Λ be a superpartition of fermionic degree m. In the identification (2.2) between Λ_N^{m+1} and $\mathbb{Q}[x_1,\ldots,x_N]^{S_{m+1}\times S_{(m+1)^c}}$ we have

$$\tilde{e}_{\ell} s_{\Lambda} \quad \longleftrightarrow \quad (-1)^{\binom{m+1}{2}} \partial_{\omega_{m+1}} e_{\ell}^{(m+1)} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) \tag{3.22}$$

where $e_{\ell}^{(m+1)}$ stands for the elementary symmetric function e_{ℓ} without the variable x_{m+1} (or equivalently, at $x_{m+1} = 0$).

Proof. Using $\tilde{e}_{\ell} = \sum_{i=1}^{N} \theta_{i} e_{\ell}^{(i)}$, we get

$$\tilde{e}_{\ell} \sum_{\sigma \in S_N/(S_m \times S_{m^c})} \mathcal{K}_{\sigma} \theta_1 \cdots \theta_m A_m \Big|_{\theta_1 \cdots \theta_{m+1}} = \left(\theta_{m+1} e_{\ell}^{(m+1)} + \sum_{i=1}^m \theta_i e_{\ell}^{(i)} \mathcal{K}_{im+1} \right) \theta_1 \cdots \theta_m A_m \Big|_{\theta_1 \cdots \theta_{m+1}} \\
= (-1)^m \left(1 + \sum_{i=1}^m \mathcal{K}_{im+1} \right) \theta_1 \cdots \theta_{m+1} A_m e_{\ell}^{(m+1)} \Big|_{\theta_1 \cdots \theta_{m+1}} \\
= (-1)^m \left(1 - \sum_{i=1}^m \kappa_{im+1} \right) \theta_1 \cdots \theta_{m+1} A_m e_{\ell}^{(m+1)} \Big|_{\theta_1 \cdots \theta_{m+1}} \\
= (-1)^m \left(1 - \sum_{i=1}^m \kappa_{im+1} \right) A_m e_{\ell}^{(m+1)} \\
= (-1)^m A_{m+1} e_{\ell}^{(m+1)} \tag{3.23}$$

Note that in the third equality we used the fact that interchanging θ_i and θ_{m+1} introduces a sign $(\mathcal{K}_{\sigma} \text{ permutes both } x$'s and θ 's while κ_{σ} only permutes x's). From (3.15), $\pi_{\omega_{m^c}} = \sum_{v \in S_{m^c}} \hat{\pi}_v$ and

 $A_{m+1} = \Delta_{m+1}(x)\partial_{\omega_{m+1}}$, we then deduce that

$$\frac{\tilde{e}_{\ell}s_{\Lambda}(x)|_{\theta_{1}\cdots\theta_{m+1}}}{\Delta_{m+1}(x)} = (-1)^{m+\binom{m}{2}}\partial_{\omega_{m+1}}e_{\ell}^{(m+1)}\pi_{\omega_{m^{c}}}\hat{K}_{(\Lambda^{a})^{R},\Lambda^{s}}(x)$$
(3.24)

Lemma 5. Let Λ be a superpartition of fermionic degree m. In the identification (2.2) between Λ_N^{m+1} and $\mathbb{Q}[x_1,\ldots,x_N]^{S_{m+1}\times S_{(m+1)^c}}$ we have

$$\tilde{e}_{\ell}\,\bar{s}_{\Lambda} \quad \longleftrightarrow \quad (-1)^{m}\partial'_{\omega_{(N-m-1)^{c}}}e^{(N-m)}_{\ell}(y)K_{(\Lambda^{s})^{R},\Lambda^{a}}(y) \tag{3.25}$$

where again y stands for the variables $y_1 = x_N, y_2 = x_{N-1}, \dots, y_N = x_1$, and the ' indicates that the divided differences act on the y variables.

Proof. As was established in the proof of the previous lemma, we have

$$\tilde{e}_{\ell} \sum_{\sigma \in S_N / (S_m \times S_{m^c})} \mathcal{K}_{\sigma} \theta_1 \cdots \theta_m A_m \Big|_{\theta_1 \cdots \theta_{m+1}} = (-1)^m A_{m+1} e_{\ell}^{(m+1)}$$
(3.26)

from which we deduce from (3.20), following the argument that led to (3.21), that

$$\frac{\tilde{e}_{\ell}\bar{s}_{\Lambda}(x)\big|_{\theta_{1}\cdots\theta_{m+1}}}{\Delta_{m+1}(x)} = (-1)^{m}\partial'_{\omega_{(N-m-1)^{c}}}e_{\ell}^{(N-m)}(y)K_{(\Lambda^{s})^{R},\Lambda^{a}}(y)$$
(3.27)

Lemma 6. Let Λ be a superpartition of fermionic degree m. In the identification (2.2) between Λ_N^{m+1} and $\mathbb{Q}[x_1,\ldots,x_N]^{S_{m+1}\times S_{(m+1)^c}}$ we have

$$\tilde{p}_{\ell} s_{\Lambda} \quad \longleftrightarrow \quad (-1)^{\binom{m+1}{2}} \partial_{\omega_{m+1}} x_{m+1}^{\ell} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) \tag{3.28}$$

Proof. Recall that $\tilde{p}_{\ell} = \sum_{i=1}^{N} \theta_{i} x_{i}^{\ell}$. As in (3.23), we have

$$\left(\sum_{i=1}^{N} \theta_i x_i^{\ell}\right) \sum_{\sigma \in S_N/(S_m \times S_{m^c})} \mathcal{K}_{\sigma} \theta_1 \cdots \theta_m A_m \Big|_{\theta_1 \cdots \theta_{m+1}} = (-1)^m A_{m+1} x_{m+1}^{\ell}$$
(3.29)

We then deduce, as in the proof of Lemma 4, that

$$\frac{\tilde{p}_{\ell} \, s_{\Lambda}(x,\theta) \big|_{\theta_{1} \cdots \theta_{m+1}}}{\Delta_{m+1}(x)} = (-1)^{m + \binom{m}{2}} \partial_{\omega_{m+1}} x_{m+1}^{\ell} \pi_{\omega_{m^{c}}} \hat{K}_{(\Lambda^{a})^{R},\Lambda^{s}}(x)$$
(3.30)

4. Pieri rules

Pieri rules for the multiplication of a Schur functions in superspace s_{Λ} or \bar{s}_{Λ} by e_{ℓ} and \tilde{e}_{ℓ} will be established in this section using Key polynomials. As we will see, these Pieri rules are essentially the transposed of those corresponding to the multiplication of s_{Λ}^* or \bar{s}_{Λ}^* by h_{ℓ} and \tilde{h}_{ℓ} .

4.1. **Pieri rules for** s_{Λ}^* **and** \bar{s}_{Λ} . In order to use Lemmas 3 and 5, it will prove convenient to express the Key polynomial appearing in these lemmas as a sequence of operators π_j 's acting on a monomial. Letting $\pi_{(b,a)} = \pi_b \pi_{b-1} \cdots \pi_a$, for $b \geq a$ and $\pi_{(b,a)} = 1$ if b < a, we get (see Lemma 31 in the Appendix for extra details)

$$K_{(\Lambda^s)^R,\Lambda^a}(x) = \pi_{\omega_{N-m}} \pi_{(N-m,\alpha_1)} \pi_{(N-m+1,\alpha_2)} \cdots \pi_{(N-1,\alpha_m)} x^{\Lambda^*}$$
(4.1)

where α_i is the row of the *i*-th circle (starting from the top) in Λ . To simplify the notation, we set, for $\alpha_1 < \alpha_2 < \cdots < \alpha_m$,

$$\mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} = \pi_{\omega_{N-m}} \pi_{(N-m,\alpha_1)} \pi_{(N-m+1,\alpha_2)} \cdots \pi_{(N-1,\alpha_m)}. \tag{4.2}$$

from which we have

$$K_{(\Lambda^s)^R,\Lambda^a}(x) = \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} x^{\Lambda^*}.$$
(4.3)

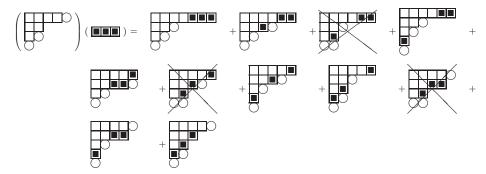
Theorem 7. Let Λ be a superpartition of fermionic degree m. Then, for $\ell \geq 1$, we have respectively

$$s_{\Lambda}^* h_{\ell} = \sum_{\Omega} s_{\Omega}^*$$
 and $\bar{s}_{\Lambda} e_{\ell} = \sum_{\Omega} \bar{s}_{\Omega}$ (4.4)

where the sum is over all superpartitions Ω of fermionic degree m such that

- (1) Ω^*/Λ^* is a horizontal (resp. vertical) ℓ -strip.
- (2) The i-th circle, starting from below, of Ω is either in the same row (resp. column) as the i-th circle of Λ if Ω^*/Λ^* does not contain a cell in that row (resp. column) or one row below that (resp. one column to the right) of the i-th circle of Λ if Ω^*/Λ^* contains a cell in the row (resp. column) of the i-th circle of Λ . In the latter case, we say that the circle was moved.

We illustrate the rules by giving the expansion of $s_{(4,1,0;2)}^* h_3$.



$$s_{(4,1,0;2)}^* h_3 = s_{(2,1,0;7)}^* + s_{(3,1,0;6)}^* + s_{(2,1,0;6,1)}^* + s_{(4,1,0;5)}^* + s_{(3,1,0;5,1)}^* + s_{(2,1,0;5,2)}^* + s_{(4,1,0;4,1)}^* + s_{(4,1,0;3,2)}^*$$

To generate all Ω 's described in Theorem 7, draw all possible horizontal strips on Λ^* . For each partition obtained this way, start from the bottom row and proceed row by row. If a new square occupies the place of a circle in Λ^{\circledast} , move the circle to the next row and slide it to the first available column. If there already is a circle occupying the row then the resulting diagram is not a superpartition and should be discarded. Note that this happens twice in our example.

Proof. Applying the involution ω on $\bar{s}_{\Lambda} e_{\ell}$, we obtain from Proposition 1 and (2.14) that

$$\bar{s}_{\Lambda} e_{\ell} = \sum_{\Omega} \bar{s}_{\Omega} \iff s_{\Lambda'}^* h_{\ell} = \sum_{\Omega} s_{\Omega'}^*$$
 (4.5)

The superpartitions Ω appearing in the expansion of $\bar{s}_{\Lambda} e_{\ell}$ are thus simply the transposes of those appearing in the expansion of $s_{\Lambda}^* h_{\ell}$. It thus suffices to prove the rule for the multiplication of \bar{s}_{Λ}

by e_{ℓ} . From Lemma 3 and the commutativity between \bar{s}_{Λ} and e_{ℓ} , this is equivalent to proving the statement

$$e_{\ell} \,\partial_{\omega_{(N-m)^c}} K_{(\Lambda^s)^R,\Lambda^a}(x) = \sum_{\Omega} \partial_{\omega_{(N-m)^c}} K_{(\Omega^s)^R,\Omega^a}(x) \tag{4.6}$$

where the sum is over all superpartitions Ω described above. Note that for simplicity we changed y by x and removed the ' on the divided differences.

Using (4.3) and the fact that e_{ℓ} commutes with all the operators, it is easily seen that we can rewrite the left hand side of (4.6) as

$$e_{\ell} \, \partial_{\omega_{(N-m)^c}} K_{(\Lambda^s)^R, \Lambda^a}(x) = \partial_{\omega_{(N-m)^c}} \mathcal{R}_{N, [\alpha_1, \dots, \alpha_m]} x^{\Lambda^*} e_{\ell} \tag{4.7}$$

Suppose that Λ^* is a partition whose entries are of exactly k distinct sizes, that is, $\Lambda^* = (a_1^{\ell_1}, \dots, a_k^{\ell_k})$ with $a_1 > a_2 > \dots > a_k$, and let I_1, I_2, \dots, I_k be intervals such that the entries a_1, a_2, \dots, a_k occupy the rows I_1, I_2, \dots, I_k respectively in Λ^* . For example, if the partition is $\Lambda^* = (5, 5, 5, 4, 3, 3, 3, 3, 1)$ then $I_1 = [1, 3], I_2 = [4], I_3 = [5, 8]$ and $I_4 = [9]$.

For I equal to the interval [a,b] and $i \leq |I|$, we let $x^{(i,I)} = x_a x_{a+1} \cdots x_{a+i-1}$ and

$$\boldsymbol{\pi}_{i,I} = \pi_{[b-i,b-1]} \cdots \pi_{[a+1,a+i]} \pi_{[a,a+i-1]} \tag{4.8}$$

where $\pi_{[c,d]} = \pi_c \pi_{c+1} \cdots \pi_d = \pi_{(d,c)}^{-1}$ for $c \leq d$ and $\pi_{[c,d]} = 1$ if c > d (note that $\pi_{i,I} = 1$ if i = |I|). As shown in Lemma 32 of the Appendix, we have the expansion

$$x^{\Lambda^*} e_{\ell} = \sum_{i_1 + i_2 + \dots + i_k = \ell} \pi_{i_1, I_1} \pi_{i_2, I_2} \cdots \pi_{i_k, I_k} x^{\Lambda^*} x^{(i_1, I_1)} x^{(i_2, I_2)} \cdots x^{(i_k, I_k)}$$

$$\tag{4.9}$$

The left hand side side of (4.6) thus becomes

$$\sum_{i_1+i_2+\dots+i_k=\ell} \partial_{\omega_{(N-m)^c}} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} \boldsymbol{\pi}_{i_1,I_1} \boldsymbol{\pi}_{i_2,I_2} \cdots \boldsymbol{\pi}_{i_k,I_k} x^{\Lambda^*} x^{(i_1,I_1)} x^{(i_2,I_2)} \cdots x^{(i_k,I_k)}$$
(4.10)

It is shown in the appendix that $\mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]}\pi_j=\mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]}$ whenever $j\neq\alpha_i-1$ for all i. By definition, the circles in Λ^\circledast can only occur at the beginning of the intervals I_1,\ldots,I_k . Therefore, a value $j=\alpha_i-1$ would correspond to the end of one such interval. Now, the π_r 's that occur in a given $\pi_{s,I}$ are such that r belongs to [a,b-1] (supposing that I=[a,b]), which implies that they cannot coincide with any π_{α_i-1} . Hence, every π_{i_j,I_j} in (4.10) acts as the identity on $\mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]}$, and we have

$$e_{\ell} \, \partial_{\omega_{(N-m)^c}} K_{(\Lambda^s)^R,\Lambda^a}(x) = \sum_{i_1+i_2+\dots+i_k=\ell} \partial_{\omega_{(N-m)^c}} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} x^{\Lambda^*} x^{(i_1,I_1)} x^{(i_2,I_2)} \dots x^{(i_k,I_k)}$$
(4.11)

Notice that this summation sums over all vertical ℓ -strips and so we get

$$e_{\ell} \, \partial_{\omega_{(N-m)^c}} K_{(\Lambda^s)^R, \Lambda^a}(x) = \sum_{\substack{\mu \\ \mu/\Lambda^* \text{ vertical } \ell\text{-strip}}} \partial_{\omega_{(N-m)^c}} \mathcal{R}_{N, [\alpha_1, \dots, \alpha_m]} x^{\mu}$$
(4.12)

We will now show that (4.6) holds (and thus also the theorem) by showing that

$$\sum_{\substack{\mu \\ \mu/\Lambda^* \text{ vertical } \ell\text{-strip}}} \partial_{\omega_{(N-m)^c}} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} x^{\mu} = \sum_{\Omega} \partial_{\omega_{(N-m)^c}} K_{(\Omega^s)^R,\Omega^a}(x)$$
(4.13)

where the sum runs over the Ω 's described in the statement of the theorem. First of all, part (1) of Theorem 7 is clearly satisfied since by construction we have that $\Omega^* = \mu$ and μ/Λ^* is a vertical ℓ -strip. Second, for a given vertical strip, there can be at most one Ω in the statement of the theorem. It thus suffices to show that for a given vertical strip μ/Λ^* , we have that $\partial_{\omega_{(N-m)^c}} \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]} x^{\mu}$ is either equal to zero (if there is no corresponding Ω) or to $\partial_{\omega_{(N-m)^c}} K_{(\Omega^s)^R,\Omega^a}(x)$ for the right value

of Ω (see part (2) of Theorem 7). We will in fact see that if $\partial_{\omega_{(N-m)^c}} \mathcal{R}_{N,[\alpha_1,...,\alpha_m]} x^{\mu}$ is not equal to zero then

$$\mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]}x^{\mu} = \mathcal{R}_{N,[\alpha'_1,\dots,\alpha'_m]}x^{\mu} = K_{(\Omega^s)^R,\Omega^a}(x). \tag{4.14}$$

where Ω is the superpartition obtained from μ by adding circles in rows $\alpha'_1, \ldots, \alpha'_m$.

We simply need to focus on what happens to α_i in $\partial_{\omega_{(N-m)^c}} \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]} x^{\mu}$. Note that circles can only occupy rows that are addable corners of Λ^* . The possible cases are:

- (1) If there is no cell in row α_i of μ/Λ^* then $\alpha'_i = \alpha_i$ and there is a circle in row α_i of Ω (hence the *i*-th circle is in the same position in Λ and in Ω).
- (2) If there is a cell in row α_i of μ/Λ^* and row α_i is still an addable corner of μ then there will be a circle in row α_i of Ω (hence the *i*-th circle of Ω is one column to the right of the *i*-th circle of Λ since μ/Λ^* is vertical).
- (3) If there is a cell in row α_i of μ/Λ^* and row α_i is not an addable corner of μ then there cannot be a circle in row α_i of Ω . But this is okay since in this case there exists a row, call it α_i' that has the same length as row α_i and is an addable corner, and such that by Corollary 35 of the Appendix

$$\mathcal{R}_{N,[\alpha_1,\ldots,\alpha_i,\ldots,\alpha_m]}x^{\mu} = \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_i',\ldots,\alpha_m]}x^{\mu}$$

Since μ/Λ^* is vertical, we thus have that again the *i*-th circle of Ω is one column to the right of the *i*-th circle of Λ .

If, however, there is a circle in row α'_i , that is $\alpha_{i-1} = \alpha'_i$, then we have

$$\mathcal{R}_{N,[\alpha_1,\ldots,\alpha_{i-1},\alpha_i,\ldots,\alpha_m]}x^{\mu} = \mathcal{R}_{N,[\alpha_1,\ldots,\alpha'_i,\alpha'_i,\ldots,\alpha_m]}x^{\mu} = 0$$

according to Lemma 37 in the Appendix.

Therefore, part (2) of Theorem 7 is satisfied, which completes the proof of the theorem.

Theorem 8. Let Λ be a superpartition of fermionic degree m. Then, for $\ell \geq 0$, we have respectively

$$s_{\Lambda}^* \tilde{h}_{\ell} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)} s_{\Omega}^*$$
 and $\bar{s}_{\Lambda} \tilde{e}_{\ell} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)} \bar{s}_{\Omega}$ (4.15)

where the sum is over all superpartitions Ω of fermionic degree m+1 such that

- (1) Ω^*/Λ^* is a horizontal (resp. vertical) ℓ -strip.
- (2) There exists a unique circle of Ω (the new circle), let's say in column c (resp. row r), such that
 - column c (resp. row r) does not contain any cell of Ω^*/Λ^* .
 - there is a cell of Ω^*/Λ^* in every column (resp. row) strictly to the left of column c (resp. strictly above row r).
- (3) If $\tilde{\Omega}$ is Ω without its new circle, then the i-th circle, starting from below, of $\tilde{\Omega}$ is either in the same row (resp. column) as the i-th circle of Λ if Ω^*/Λ^* does not contain a cell in that row (resp. column) or one row below that (resp. one column to the right) of the i-th circle of Λ if Ω^*/Λ^* contains a cell in the row (resp. column) of the i-th circle of Λ . In the latter case, we say that the circle was moved.

and where $\#(\Omega, \Lambda)$ is the number of circles in Ω below the new circle.

We illustrate this time the rules by giving the expansion of $s_{(4.1:3)}^* \tilde{h}_3$.

This rule is very similar to the previous one but in this case keep in mind that every column to the left of the new circle must have a new box. Also, multiply by (-1) for every circle below the new circle.

Proof. Applying the involution ω , we obtain from Proposition 1 and (2.14) that

$$\bar{s}_{\Lambda}\,\tilde{e}_{\ell} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)} \bar{s}_{\Omega} \quad \Longleftrightarrow \quad s_{\Lambda'}^{*}\,\tilde{h}_{\ell} = (-1)^{m} \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)} s_{\Omega'}^{*} \tag{4.16}$$

The superpartitions Ω appearing in the expansion of $\bar{s}_{\Lambda} \tilde{e}_{\ell}$ are thus simply the transposes of those appearing in the expansion of $s_{\Lambda}^* \tilde{h}_{\ell}$. Observe that $(-1)^{\#(\Omega,\Lambda)+m} = (-1)^{\#(\Omega',\Lambda')}$, which means that the sign associated to Ω is the same in both formulas of (4.15). Hence, it suffices to prove the rule for the multiplication of \bar{s}_{Λ} by \tilde{e}_{ℓ} . Using $\tilde{e}_{\ell} \bar{s}_{\Lambda} = (-1)^m \bar{s}_{\Lambda} \tilde{e}_{\ell}$, this is equivalent to proving that

$$\tilde{e}_{\ell}\,\bar{s}_{\Lambda} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)'}\bar{s}_{\Omega} \tag{4.17}$$

where $\#(\Omega, \Lambda)'$ is equal to the number of circles in Ω above the new circle. From Lemmas 3 and 5, this corresponds in the bisymmetric world $\mathbb{Q}[x_1, \dots, x_N]^{S_{m+1} \times S_{(m+1)^c}}$ to proving that

$$(-1)^{m} \partial_{\omega_{(N-m-1)^{c}}} e_{\ell}^{(N-m)} K_{(\Lambda^{s})^{R}, \Lambda^{a}}(x) = \sum_{\Omega} (-1)^{\#(\Omega, \Lambda)'} \partial_{\omega_{(N-m-1)^{c}}} K_{(\Omega^{s})^{R}, \Omega^{a}}(x)$$
(4.18)

where the sum is over all superpartitions Ω described in the statement of the theorem. From (4.3), we have

$$(-1)^m \partial_{\omega_{(N-m-1)^c}} e_\ell^{(N-m)} K_{(\Lambda^s)^R, \Lambda^a}(x) = (-1)^m \partial_{\omega_{(N-m-1)^c}} e_\ell^{(N-m)} \mathcal{R}_{N, [\alpha_1, \dots, \alpha_m]} x^{\Lambda^*}$$
(4.19)

Using Corollary 45 of the Appendix, we get

$$\partial_{\omega_{(N-m-1)^c}} e_{\ell}^{(N-m)} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} = \\ \partial_{\omega_{(N-m-1)^c}} \sum_{r \notin \{\alpha_1,\dots,\alpha_m\}} (-1)^{\operatorname{pos}(r)} \mathcal{R}_{N,[\alpha_1,\dots,r,\dots,\alpha_m]} x_1 \dots x_{r-1} e_{\ell-r+1}^{(1,2,\dots,r)}$$
(4.20)

where $e_{\ell}^{(1,2,\ldots,r)}$ is e_{ℓ} with $x_1 = x_2 = \cdots = x_r = 0$, and where pos(r) is the position of r starting from 0 in the increasing sequence $[\alpha_1,\ldots,r,\ldots,\alpha_m]$, that is, pos(r) is the number of entries of $[\alpha_1,\ldots,r,\ldots,\alpha_m]$ that are less than r. For example, if we have the increasing sequence [2,3,9,13,19,22] then pos(13) = 3 because there are 3 numbers less than 13 in the sequence. Therefore, we have

$$(-1)^{m} \partial_{\omega_{(N-m-1)^{c}}} e_{\ell}^{(N-m)} K_{(\Lambda^{s})^{R}, \Lambda^{a}}(x) =$$

$$(-1)^{m} \partial_{\omega_{(N-m-1)^{c}}} \sum_{r \notin \{\alpha_{1}, \dots, \alpha_{m}\}} (-1)^{\operatorname{pos}(r)} \mathcal{R}_{N, [\alpha_{1}, \dots, r, \dots, \alpha_{m}]} e_{\ell-r+1}^{(1,2, \dots, r)} x^{\Lambda^{*}+1^{r-1}}$$

$$(4.22)$$

where $\Lambda^* + 1^{r-1}$ is the partition obtained by adding a new cell in the first r-1 rows of Λ^* . Using an argument similar to the one that led to (4.12) in the proof of Theorem 7, we can show that

$$\mathcal{R}_{N,[\alpha_1,\dots,r,\dots,\alpha_m]} e_{\ell-r+1}^{(1,2,\dots,r)} x^{\Lambda^*+1^{r-1}} = \sum_{\mu} \mathcal{R}_{N,[\alpha_1,\dots,r,\dots,\alpha_m]} x^{\mu+1^{r-1}}$$
(4.23)

where the sum is over all μ 's such that μ/Λ^* is a vertical $(\ell-r+1)$ -strip whose cells are all in rows strictly below row r. From the previous two equations, the right hand side of equation (4.18) is seen to be equal to

$$\sum_{r \notin \{\alpha_1, \dots, \alpha_m\}} \sum_{\mu} \partial_{\omega_{(N-m-1)^c}} (-1)^{m + \operatorname{pos}(r)} \mathcal{R}_{N, [\alpha_1, \dots, r, \dots, \alpha_m]} x^{\mu + 1^{r-1}}$$
(4.24)

where the sum runs over the μ 's described in the previous equation. Now, as in the proof of Theorem 7, we have that if $\partial_{\omega_{(N-m-1)^c}} \mathcal{R}_{N,[\alpha_1,\ldots,r,\ldots,\alpha_m]} x^{\mu+1^{r-1}} \neq 0$ then

$$\mathcal{R}_{N,[\alpha_1,\dots,r,\dots,\alpha_m]} x^{\mu+1^{r-1}} = \mathcal{R}_{N,[\alpha'_1,\dots,r,\dots,\alpha'_m]} x^{\mu+1^{r-1}} = K_{(\Omega^s)^R,\Omega^a}(x)$$
(4.25)

where Ω is the superpartition obtained from $\mu+1^{r-1}$ by adding circles in certain rows $\alpha'_1,\ldots,r,\ldots,\alpha'_m$. Observe that we can always add a circle in row r since there is an addable corner in row r of $\mu+1^{r-1}$. This circle corresponds to the new circle in row r mentioned in part (2) of Theorem 8. Notice that in $(\mu+1^{r-1})/\Lambda^*$ there is, by definition, no cell in row r and a new cell in every row above row r, as required in part (2) of the theorem. The sign is also correct given that $(-1)^{m+(pos(r))}$ is the same as $(-1)^{\#(\Omega,\Lambda)'}$ (which corresponds to the number of circles above row r). Furthermore, the sum is over all vertical strips that contain cells in every row above row r and none in row r as required by part (1) and (2) of the theorem. Now, if $\alpha_i < r$ then row α_i is still an addable corner in row α_i of $\mu+1^{r-1}$. The corresponding circle is thus moved one column to the right given the new cell in its row. Finally, for the α_i below row r, we proceed exactly as in the proof of the previous theorem to show that part (3) needs to be satisfied.

Therefore, the sum over Ω in the right hand side of (4.18) is indeed as in the statement of Theorem 8, which concludes the proof.

The following corollary shows that a product of fermionic symmetric functions \tilde{e}_{ℓ} or \tilde{h}_{ℓ} corresponds to a single Schur function in superspace.

Corollary 9. We have $e_{(\Lambda^a;\emptyset)} = (-1)^{\binom{m}{2}} \bar{s}_{(\Lambda^a;\emptyset)'}$ and $h_{(\Lambda^a;\emptyset)} = s^*_{(\Lambda^a;\emptyset)}$. In particular, $\tilde{e}_{\ell} = \bar{s}_{(0;1^{\ell})}$ and $\tilde{h}_{\ell} = s^*_{(\ell;\emptyset)}$.

Proof. Let $\lambda = \Lambda^a$ and $\Lambda = (\lambda; \emptyset)$. We first show that $h_{\Lambda} = s_{\Lambda}^*$. We proceed by induction on m. If m = 1, the result holds since

$$\tilde{h}_{\lambda_1} = s_{(\emptyset,\emptyset)}^* \, \tilde{h}_{\lambda_1} = s_{(\lambda_1,\emptyset)}^* \tag{4.26}$$

from Theorem 8 (starting from the empty superpartition, the only superpartition Ω satisfying the conditions of the theorem is obviously the one consisting of a row with λ_1 squares followed by a circle).

Now, let $\hat{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{m-1})$ be the partition λ without its last part. By induction, we simply need to show that

$$s_{(\hat{\lambda}\cdot\emptyset)}^* \tilde{h}_{\lambda_m} = s_{\Lambda}^* \tag{4.27}$$

By Theorem 8, this corresponds to showing that in

$$s_{(\hat{\lambda};\emptyset)}^* \tilde{h}_{\lambda_m} = \sum_{\Omega} (-1)^{\#(\Omega,\hat{\Lambda})} s_{\Omega}^*$$

the sum only contains the term s_{Λ}^* . Observe that this is precisely the superpartition obtained by placing the new circle in column $\lambda_m + 1$ and having a new square in every column to the left of the new circle.

The new circle cannot be in a column $c > \lambda_m + 1$ because there are not enough new squares to fill every column strictly to the left of column c. So the new circle must be in row m seeing as $\lambda_m < \lambda_{m-1}$ (this is possible given that there is no circle in row m of $\hat{\Lambda} = (\hat{\lambda}; \emptyset)$).

Suppose by contradiction that the new circle is in column $c \leq \lambda_m$. Then there are new squares in every column strictly to the left of cell c. Since $c \leq \lambda_m$, there are some extra new squares that need to be placed to the right of cell c in a horizontal strip. For each possible horizontal strip to the right of column c, the leftmost new square will replace an existing circle (since every row of $\hat{\Lambda}$ ends with a circle) and move it down. But the resulting diagram cannot be a superpartition since the highest such horizontal strip will move a circle down in a row already containing a circle given that the new circle is in row m and $\hat{\Lambda}$ has circles in every row. Therefore, the only resulting superpartition is Λ , and so we have

$$s_{(\hat{\lambda};\emptyset)}^* \tilde{h}_{\lambda_m} = (-1)^{\#(\Lambda,\hat{\Lambda})} s_{\Lambda}^*$$

The result then follows since $(-1)^{\#(\Lambda,\hat{\Lambda})} = 1$ given that the new circle is in the last row.

Applying the homomorphism ω on both sides of $h_{\Lambda} = s_{\Lambda}^*$ we immediately get from Proposition 1 and (2.14) that $e_{\Lambda} = (-1)^{\binom{m}{2}} \bar{s}_{\Lambda'}$.

4.2. **Pieri rules for** \bar{s}_{Λ}^* and s_{Λ} . This time, we express the Key polynomial appearing in (3.17) as a sequence of operators $\hat{\pi}_j$'s acting on a monomial. Letting $\hat{\pi}_{[a,b]} = \hat{\pi}_a \hat{\pi}_{a+1} \cdots \hat{\pi}_b$, for $a \leq b$ and $\hat{\pi}_{[a,b]} = 1$ if a > b, we get

$$\hat{K}_{(\Lambda^a)^R,\Lambda^s}(x) = \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_m-1]} \cdots \hat{\pi}_{[1,\alpha_1-1]} x^{\Lambda^*}$$
(4.28)

where α_i is again the row of the *i*-th circle (starting from the top) in Λ . Note that it is understood that $\hat{\pi}_{[i,\alpha_i-1]}=1$ if $\alpha_i=i$. We set, for $\alpha_1<\cdots<\alpha_m$,

$$\mathcal{P}_{N,[\alpha_1,\dots,\alpha_m]} = \partial_{\omega_m} \pi_{\omega_m^c} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_m-1]} \cdots \hat{\pi}_{[1,\alpha_1-1]}$$

$$\tag{4.29}$$

which implies

$$\partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \mathcal{P}_{N, [\alpha_1, \dots, \alpha_m]} x^{\Lambda^*}$$

$$\tag{4.30}$$

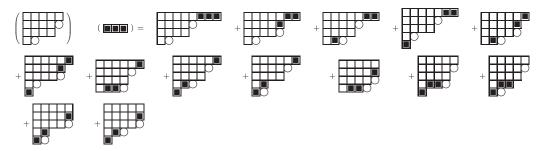
Theorem 10. Let Λ be a superpartition of fermionic degree m. Then, for $\ell \geq 1$, we have respectively

$$\bar{s}_{\Lambda}^* h_{\ell} = \sum_{\Omega} \bar{s}_{\Omega}^* \quad \text{and} \quad s_{\Lambda} e_{\ell} = \sum_{\Omega} s_{\Omega}$$
 (4.31)

where the sum is over all superpartitions Ω of fermionic degree m such that

- Ω^*/Λ^* is a horizontal (resp. vertical) ℓ -strip
- the i-th circles, starting from the first row and going down, of Ω and Λ are either in the same row or the same column. In the case where the i-th circles are in the same row r (resp. column c), the circle in row r (resp. column c) of Ω cannot be located passed row r-1 (resp. column c-1) of Λ (if r=1 or c=1 the condition does not apply).

We illustrate the rules by giving the expansion of $\bar{s}_{(4,1;5,4)}^* h_3$.



$$\bar{s}_{(4,1;5,4)}^{*}h_{3} = \bar{s}_{(4,1;8,4)}^{*} + \bar{s}_{(4,1;7,5)}^{*} + \bar{s}_{(4,2;7,4)}^{*} + \bar{s}_{(4,1;7,4,1)}^{*} + \bar{s}_{(4,2;6,5)}^{*} + \bar{s}_{(4,1;6,5,1)}^{*} + \bar{s}_{(4,3;6,4)}^{*} + \bar{s}_{(4,2;6,4,1)}^{*} + \bar{s}_{(4,1;6,4,2)}^{*} + \bar{s}_{(4,3;5,5)}^{*} + \bar{s}_{(4,3;5,4,1)}^{*} + \bar{s}_{(4,1;5,4,3)}^{*} + \bar{s}_{(4,1;5,5,2)}^{*} + \bar{s}_{(4,2;5,5,1)}^{*}$$

Notice here that the circle can be pushed down if there is room or pushed to the right. It cannot be pushed to the right farther than the original row above it.

Proof. Applying the involution ω on $s_{\Lambda} e_{\ell}$, we deduce from Proposition 1 that the superpartitions Ω appearing in the expansion of $s_{\Lambda} e_{\ell}$ are simply the transposed of those appearing in the expansion of $s_{\Lambda}^* h_{\ell}$ (we used a similar argument in the proof of Theorem 7). It thus suffices to prove the rule for the multiplication of s_{Λ} by e_{ℓ} . From Lemma 2 and the commutativity between s_{Λ} and e_{ℓ} , this is equivalent to proving the statement:

$$e_{\ell} \,\partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R,\Lambda^s}(x) = \sum_{\Omega} \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Omega^a)^R,\Omega^s}(x)$$

$$(4.32)$$

where the sum is over the superpartitions Ω described in the statement of the theorem. Using (4.30) and the commutativity of e_{ℓ} with the divided-difference operators, we can rewrite the left side of (4.32) as

$$e_{\ell} \, \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \mathcal{P}_{N, [\alpha_1, \dots, \alpha_m]} x^{\Lambda^*} e_{\ell} \tag{4.33}$$

Recall expansion (4.9)

$$x^{\Lambda^*} e_{\ell} = \sum_{i_1 + i_2 + \dots + i_k = \ell} \boldsymbol{\pi}_{i_1, I_1} \boldsymbol{\pi}_{i_2, I_2} \cdots \boldsymbol{\pi}_{i_k, I_k} x^{\Lambda^*} x^{(i_1, I_1)} x^{(i_2, I_2)} \cdots x^{(i_k, I_k)}$$

which gives

$$e_{\ell} \, \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \sum_{i_1 + i_2 + \dots + i_k = \ell} \mathcal{P}_{N, [\alpha_1, \dots, \alpha_m]} \left(\prod_{j=1}^k \pi_{i_j, I_j} x^{(i_j, I_j)} \right) x^{\Lambda^*}$$
 (4.34)

Observe that the summation corresponds to all possible ways to add a vertical ℓ -strip to Λ^* . Now, we apply the product of π_{i_j,I_j} 's to $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}$ one at a time according to the following rules (see Lemma 48 in the Appendix):

- (1) $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_i,\ldots,\alpha_m]} + \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_i+i_j,\ldots,\alpha_m]}$ if the interval I_j starts with $\alpha_i \in \{\alpha_1,\ldots,\alpha_m\}$.
- (2) $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}$ if the interval I_j starts with $\beta \notin \{\alpha_1,\ldots,\alpha_m\}$.

Since the π_{i_j,I_j} 's commute with each other, by the second rule above we can get rid of the factors π_{i_j,I_j} such that I_j start with $\beta \notin \{\alpha_1,\ldots,\alpha_m\}$. Hence (4.34) becomes

$$e_{\ell} \, \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \sum_{i_1 + i_2 + \dots + i_k = \ell} \mathcal{P}_{N, [\alpha_1, \dots, \alpha_m]} \left(\prod_{\substack{1 \le j \le k \\ I_j \text{ starts with } \alpha_i}} \boldsymbol{\pi}_{i_j, I_j} x^{(i_j, I_j)} \right) x^{\Lambda^*}$$
(4.35)

Now we interpret the right side of equation (4.34) and show that it corresponds to the left hand side of (4.32). First of all, the sum in (4.34) is over vertical strips so the first point in the characterization of the superpartitions Ω 's in Theorem 10 is satisfied. If I_j starts with α_i then x^{Λ^*} is multiplied by $x^{(i_j,I_j)}$, which means that cells are added in rows $\alpha_i, \alpha_i + 1, \ldots, \alpha_i + i_j - 1$ of Λ^* . The rule for $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j}$ given above results in this case in two terms. In the first one, the circle remains in row α_i while in the second the circle is moved from row α_i to $\alpha_i + i_j$ (thus sliding down in its column). Observe that if $i_j = |I_j|$, then $\boldsymbol{\pi}_{i_j,I_j} = 1$ and the second term does not occur, meaning that a circle cannot slide down a column past the end of its block. Furthermore, if $\Omega^*_{\alpha_i} = \Omega^*_{\alpha_i-1}$ for some α_i then by Lemma 48 of the Appendix,

$$\mathcal{P}_{N,[\alpha_1,\dots,\alpha_i,\dots,\alpha_m]} x^{\Omega^*} = \mathcal{P}_{N,[\alpha_1,\dots,\alpha_i,\dots,\alpha_m]} \pi_{\alpha_i-1} x^{\Omega^*} = 0$$

$$\tag{4.36}$$

In other words, if there is no addable corner in row Ω_{α_i} then the circle in that row cannot be pushed one cell to its right. Hence the resulting diagrams have to be superpartitions obeying the conditions of the theorem and the proof is complete.

Theorem 11. Let Λ be a superpartition of fermionic degree m. Then, for $\ell \geq 0$, we have respectively

$$\bar{s}_{\Lambda}^* \tilde{h}_{\ell} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)} \bar{s}_{\Omega}^* \quad \text{and} \quad s_{\Lambda} \tilde{e}_{\ell} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)} s_{\Omega}$$
 (4.37)

where the sum is over all superpartitions Ω of fermionic degree m+1 such that

- $\Omega^{\circledast}/\Lambda^{\circledast}$ is a horizontal (resp. vertical) $(\ell+1)$ -strip whose rightmost (resp. lowermost) cell (the new circle) belongs to $\Omega^{\circledast}/\Omega^*$.
- Let Ω be Ω without its new circle. Then the i-th circles, starting from the first row and going down, of $\tilde{\Omega}$ and Λ are either in the same row or the same column. In the case where the i-th circles are in the same row r (resp. column c), the circle in row r (column c) of Ω cannot be located passed row r-1 (resp. column c-1) of Λ (if r=1 or c=1 the condition does not apply).

and where $\#(\Omega, \Lambda)$ is again the number of circles in Ω below the new circle.

We illustrate this time the rules by giving the expansion of $\bar{s}^*_{(4.1:5,4)}\tilde{h}_2$.

$$\bar{s}_{(4,1;5,4)}^* \tilde{h}_3 = \bar{s}_{(7,4,1;4)}^* + \bar{s}_{(6,4,1;5)}^* + \bar{s}_{(6,4,2;4)}^* + \bar{s}_{(6,4,1;4,1)}^* + \bar{s}_{(5,4,2;5)}^* + \bar{s}_{(5,4,1;5,1)}^* + \bar{s}_{(5,4,3;4)}^* + \bar{s}_{(5,4,2;4,1)}^* + \bar{s}_{(5,4,1;4,2)}^* - \bar{s}_{(4,2,1;5,4)}^*$$

To generate these superpartitions, start with the expansion of $\bar{s}^*_{(4,1;5,4)} h_3$ given previously and replace the new box that is farthest to the right with the new circle. Discard every diagram that has two circles in the same row or column.

Proof. We first use an argument similar to the one given at the beginning of the proof of Theorem 8. Applying the involution ω , we obtain from Proposition 1 that

$$s_{\Lambda} \tilde{e}_{\ell} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)} s_{\Omega} \quad \Longleftrightarrow \quad \bar{s}_{\Lambda'}^{*} \tilde{h}_{\ell} = (-1)^{m} \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)} \bar{s}_{\Omega'}^{*}$$

$$(4.38)$$

The superpartitions Ω appearing in the expansion of $s_{\Lambda} \tilde{e}_{\ell}$ are thus simply the transposed of those appearing in the expansion of $\bar{s}_{\Lambda}^* \tilde{h}_{\ell}$. Observe that $(-1)^{\#(\Omega,\Lambda)+m} = (-1)^{\#(\Omega',\Lambda')}$, which means that the sign associated to Ω is the same in both formulas of (4.37). It thus suffices to prove the rule for the multiplication of s_{Λ} by \tilde{e}_{ℓ} . Using $\tilde{e}_{\ell} s_{\Lambda} = (-1)^m s_{\Lambda} \tilde{e}_{\ell}$, this is equivalent to proving that

$$\tilde{e}_{\ell} s_{\Lambda} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda)'} s_{\Omega} \tag{4.39}$$

where we recall that $\#(\Omega, \Lambda)'$ is equal to the number of circles in Ω above the new circle. From Lemmas 2 and 4, this corresponds in the bisymmetric world $\mathbb{Q}[x_1, \ldots, x_N]^{S_{m+1} \times S_{(m+1)^c}}$ to proving

$$\partial_{\omega_{m+1}} e_{\ell}^{(m+1)} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \sum_{\Omega} (-1)^{\#(\Omega, \Lambda)'} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{K}_{(\Omega^a)^R, \Omega^s}(x)$$
(4.40)

Using $\pi_{\omega_{m^c}} = \pi_{\omega_{(m+1)^c}} \pi_{[m+1,N-1]}$ and commuting $e_{\ell}^{(m+1)}$ with $\pi_{\omega_{(m+1)^c}}$, the left hand side of (4.40) becomes

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} e_{\ell}^{(m+1)} \pi_{[m+1,N-1]} \hat{K}_{(\Lambda^a)^R,\Lambda^s}(x)$$
(4.41)

Lemma 53 of the Appendix states that

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)c}} e_{\ell}^{(m+1)} \pi_{[m+1,N-1]} = \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)c}} \sum_{i=1}^{N-m} \hat{\pi}_{[m+1,m+i-1]} e_{\ell}^{\{1,\dots,m+i-1\}}$$
(4.42)

where $e_{\ell}^{\{1,\ldots,m+i-1\}}$ corresponds to $e_{\ell}(x_1,\ldots,x_{m+i-1})$. We then use (4.28) to transform (4.41) into

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \sum_{i=1}^{N-m} \hat{\pi}_{[m+1,m+i-1]} e_{\ell}^{\{1,\dots,m+i-1\}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_m-1]} \dots \hat{\pi}_{[1,\alpha_1-1]} x^{\Lambda^*}$$
(4.43)

As shown in Lemma 60 of the Appendix, this expression can be further reduced to

$$\sum_{r \notin \{\alpha_1, \dots, \alpha_m\}} (-1)^{\operatorname{pos}(r)} \mathcal{P}_{N, [\alpha_1, \dots, r, \dots, \alpha_m]} e_{\ell}^{\{1, \dots, r-1\}} x^{\Lambda^*}$$
(4.44)

where pos(r) is again the number of entries of $[\alpha_1, \ldots, r, \ldots, \alpha_m]$ that are less than r. As in (4.9), we use the identity

$$e_{\ell}^{\{1,\dots,r-1\}}x^{\Lambda^*} = \sum_{i_1+i_2+\dots+i_k=\ell} \left(\prod_{j=1}^k \pi_{i_j,I_j}x^{(i_j,I_j)}\right) x^{\Lambda^*}$$

where the I_j 's are this time the blocks of the subpartition $(\Lambda_1^*, \ldots, \Lambda_{r-1}^*)$ given that $e_\ell^{\{1,\ldots,r-1\}}$ only involves the first r-1 variables. Using a similar argument to the one that led to (4.35), we then get

$$\sum_{\substack{r \notin \{\alpha_1, \dots, \alpha_m\}}} \sum_{i_1 + i_2 + \dots + i_k = \ell} (-1)^{\operatorname{pos}(r)} \mathcal{P}_{N, [\alpha_1, \dots, r, \dots, \alpha_m]} \left(\prod_{\substack{1 \le j \le k \\ I_j \text{ starts with } \alpha_i}} \boldsymbol{\pi}_{i_j, I_j} x^{(i_j, I_j)} \right) x^{\Lambda^*}$$

$$= \sum_{\substack{r \notin \{\alpha_1, \dots, \alpha_m\} \\ I_i \text{ starts with } \alpha_i}} \sum_{\substack{1 \le j \le k \\ I_i \text{ starts with } \alpha_i}} \boldsymbol{\pi}_{i_j, I_j} x^{\mu} \qquad (4.45)$$

where the sum is over all μ 's such that μ/Λ^* is a vertical ℓ -strip whose cells are all in rows strictly less than r.

The right hand side of the previous equation is very reminiscent of (4.35), whence its interpretation in the language of Key polynomials will be very similar. The first sum goes through all possibilities of where the new circle could be. Note that if there is no addable corner in row r of μ then $\mu_{r-1} = \mu_r$, which implies that $x^{\mu} = \pi_{r-1}x^{\mu}$. Given that by Lemma 48 (4) and (5) of the Appendix there will still be a circle in row r after the operators π_{i_j,I_j} 's have been applied to $\mathcal{P}_{N,[\alpha_1,\dots,r_r,\dots,\alpha_m]}$ (recall that π_{i_j,I_j} only acts on the rows above row r), the ensuing action of π_{r-1} will annihilate it by Lemma 48 (2) of the Appendix. Thus the new circle in row r occupies an addable corner of μ . We also have that the new cells are added only in rows strictly less than r, that is, strictly above the new circle. Finally, pos(r) is exactly $\#(\Omega, \Lambda)'$ and so we have

$$\partial_{\omega_{m+1}} e_{\ell}^{(m+1)} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \sum_{\Omega} (-1)^{\#(\Omega, \Lambda)'} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{K}_{(\Omega^a)^R, \Omega^s}(x)$$
(4.46)

where the sum over Ω runs over the set of superpartitions stated in the theorem.

The following corollary shows that a product of fermionic symmetric functions \tilde{e}_{ℓ} or \tilde{h}_{ℓ} corresponds to a single Schur function in superspace (although distinct from that appearing in Corollary 9).

Corollary 12. We have $e_{(\Lambda^a;\emptyset)} = (-1)^{\binom{m}{2}} s_{(\Lambda^a;\emptyset)'}$, and $h_{(\Lambda^a;\emptyset)} = \bar{s}^*_{(\Lambda^a;\emptyset)}$. In particular, $\tilde{e}_{\ell} = s_{(0;1^{\ell})}$ and $\tilde{h}_{\ell} = \bar{s}^*_{(\ell;\emptyset)}$.

Proof. Let $\lambda = \Lambda^a$ and $\Lambda = (\lambda; \emptyset)$. We first show that $h_{\Lambda} = \bar{s}_{\Lambda}^*$. We proceed by induction on m. If m = 1, the result holds since

$$\tilde{h}_{\lambda_1} = \bar{s}^*_{(\emptyset;\emptyset)} \, \tilde{h}_{\lambda_1} = \bar{s}^*_{(\lambda_1;\emptyset)} \tag{4.47}$$

from Theorem 11 (starting from the empty superpartition, the only superpartition Ω satisfying the conditions of the theorem is obviously the one consisting of a row with λ_1 squares followed by a circle).

Now, let $\hat{\lambda} = (\lambda_2, \dots, \lambda_m)$ be the partition λ without its first part. By induction, we simply need to show that

$$\tilde{h}_{\lambda_1} \, \bar{s}^*_{(\hat{\lambda};\emptyset)} = \bar{s}^*_{\Lambda} \tag{4.48}$$

By Theorem 11 and the fact that $\bar{s}^*_{(\hat{\lambda};\emptyset)} = (-1)^{\binom{m}{2}} \bar{s}^*_{(\hat{\lambda};\emptyset)} \tilde{h}_{\lambda_1}$, this corresponds to showing that in

$$\tilde{h}_{\lambda_1} \, \bar{s}^*_{(\hat{\lambda};\emptyset)} = \sum_{\Omega} (-1)^{\#(\Omega,\hat{\Lambda})'} \bar{s}^*_{\Omega}$$

the sum only contains the term s_{Λ}^* (with positive sign), where we recall that $\#(\Omega, \hat{\Lambda})'$ is the number of circles above the new circle. Observe that Λ is precisely the superpartition obtained by placing the new circle in column $\lambda_1 + 1$ and having a new square in every column to the left of the new circle.

The new circle cannot be in a column $c < \lambda_1 + 1$ because by the first conditions of Theorem 11, all of the new squares must be to the left of the new circle. So the new circle must be in the first row. This means that the circle in the first row must be moved to the second row. By the conditions of Theorem 11, this implies that the second row must be completed. Now the circle in the second row must be moved to the third row and the third row must thus also be completed. Since there is a circle in every row, repeating the argument again and again we see that all rows must be completed (including row m). Therefore, the new circle must be in column $\lambda_1 + 1$ (otherwise it would be impossible to complete all the rows) and there is a new square in every column to the left of it. As previously commented, this corresponds to the superpartition Λ . Hence,

$$\tilde{h}_{\lambda_1} \bar{s}^*_{(\hat{\lambda},\emptyset)} = (-1)^{\#(\Lambda,\hat{\Lambda})} \bar{s}^*_{\Lambda} \tag{4.49}$$

and the result holds since $\#(\Lambda, \hat{\Lambda})' = 0$ (the new circle is in the first row).

Applying the homomorphism ω on both sides of $h_{\Lambda} = \bar{s}_{\Lambda}^*$ we immediately get, from Proposition 1 and (2.14), that $e_{\Lambda} = (-1)^{\binom{m}{2}} s_{\Lambda'}$.

5. Kostka Coefficients in Superspace

Define the Kostka coefficients in superspace $K_{\Omega\Lambda}$ and $\bar{K}_{\Omega\Lambda}$ to be respectively such that

$$h_{\Lambda} = \sum_{\Omega} K_{\Omega\Lambda} \, \bar{s}_{\Omega}^* \quad \text{and} \quad h_{\Lambda} = \sum_{\Omega} \bar{K}_{\Omega\Lambda} \, s_{\Omega}^*$$
 (5.1)

As expected, the Kostka coefficients in superspace give the monomial expansion of the Schur functions in superspace.

Proposition 13. We have

 $s_{\Lambda} = \sum_{\Omega \le \Lambda} \bar{K}_{\Lambda\Omega} m_{\Omega}$ and $\bar{s}_{\Lambda} = \sum_{\Omega \le \Lambda} K_{\Lambda\Omega} m_{\Omega}$ (5.2)

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Proof. The triangularity in both equations follows from the fact that the Schur functions in superspace are special cases of Macdonald polynomials in superspace, which are triangular by construction.

Suppose that $s_{\Lambda} = \sum_{\Omega} A_{\Lambda\Omega} m_{\Omega}$. On the one hand, by duality of the bases s_{Λ} and s_{Λ}^* , we have $\langle\langle h_{\Lambda}, s_{\Gamma} \rangle\rangle = \bar{K}_{\Gamma\Lambda}$ by the second formula of (5.1). On the other hand, by the duality (2.12) between the bases m_{Λ} and h_{Λ} , we have

$$\langle \langle h_{\Lambda}, s_{\Gamma} \rangle \rangle = \langle \langle h_{\Lambda}, \sum_{\Omega} A_{\Gamma\Omega} m_{\Omega} \rangle \rangle = A_{\Gamma\Lambda}$$
 (5.3)

Therefore $A_{\Lambda\Omega} = \bar{K}_{\Lambda\Omega}$ and the first equality holds.

Similarly, suppose that $\bar{s}_{\Lambda} = \sum_{\Omega} B_{\Lambda\Omega} m_{\Omega}$. As before, by duality of the bases \bar{s}_{Λ} and \bar{s}_{Λ}^* , we get that $\langle\langle h_{\Lambda}, \bar{s}_{\Gamma} \rangle\rangle = K_{\Gamma\Lambda}$ by the first formula of (5.1), while by duality of the bases m_{Λ} and h_{Λ} we get and $\langle\langle h_{\Lambda}, \bar{s}_{\Gamma} \rangle\rangle = B_{\Gamma\Lambda}$. Hence $B_{\Lambda\Omega} = K_{\Lambda\Omega}$ and the second equality also holds.

The Kostka coefficients in superspace turn out to be nonnegative integers, which is relatively surprising given that signs show up in the Pieri rules.

Proposition 14. The Kostka coefficients in superspace $K_{\Omega\Lambda}$ and $\bar{K}_{\Omega\Lambda}$ are nonnegative integers.

Proof. We first prove the nonnegativity of $K_{\Omega\Lambda}$. From Corollary 12, we have that if $\Lambda = (\Lambda^a; \emptyset)$, then $h_{\Lambda} = \bar{s}_{\Lambda}^*$ and the proposition holds in that case. For Λ generic, we thus have $h_{\Lambda} = h_{(\Lambda^a; \emptyset)} h_{(\emptyset; \Lambda^s)} = \bar{s}_{(\Lambda^a; \emptyset)}^* h_{(\emptyset; \Lambda^s)}$. The expansion of h_{Λ} in terms of Schur functions in superspace s_{Ω} of (5.1) is thus obtained by using repeatedly, starting from $\bar{s}_{(\Lambda^a; \emptyset)}^*$, the Pieri rule for the multiplication by h_{ℓ} of Theorem 10. But since that Pieri rule does not involve signs, the positivity of $K_{\Omega\Lambda}$ is immediate.

The positivity of $\bar{K}_{\Omega\Lambda}$ follows similarly using Corollary 9 and the Pieri rule for the multiplication by h_{ℓ} of Theorem 7.

A combinatorial interpretation in terms of tableaux for the coefficients $\bar{K}_{\Lambda\Omega}$ and $K_{\Lambda\Omega}$ will be given in Corollary 23 and Corollary 25 respectively. As will be discussed in Remark 26, a different combinatorial interpretation for $\bar{K}_{\Lambda\Omega}$ and $K_{\Lambda\Omega}$ is conjectured in [4].

6. Extra Pieri rules

In order to establish the duality that will appear in Section 7, we need to derive another Pieri rule whose terms are exactly those that appear in the multiplication of h_{ℓ} by \bar{s}_{Λ}^* obtained in Theorem 10.

Theorem 15. Let Λ be a superpartition of fermionic degree m. For $\ell \geq 1$, we have

$$s_{\Lambda} h_{\ell} = \sum_{\Omega} s_{\Omega} \tag{6.1}$$

where the sum is over all superpartitions Ω of fermionic degree m such that

- Ω^*/Λ^* is a horizontal ℓ -strip
- the i-th circles, starting from the first row and going down, of Ω and Λ are either in the same row or the same column. In the case where the i-th circles are in the same row r, the circle in row r of Ω cannot be located passed row r-1 of Λ (if r=1 the condition does not apply).

Proof. Using Lemma 2 and the commutativity of h_{ℓ} and s_{Λ} , it is equivalent to prove that

$$h_{\ell} \, \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \sum_{\Omega} \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Omega^a)^R, \Omega^s}(x)$$
(6.2)

where the sum is over all superpartitions Ω described in the statement of the theorem. With the help of (4.30) and the fact that h_{ℓ} commutes with all the operators, we can rewrite the left hand side of (6.2) as

$$h_{\ell} \, \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \mathcal{P}_{N, [\alpha_1, \dots, \alpha_m]} x^{\Lambda^*} h_{\ell}. \tag{6.3}$$

As shown in Lemma 46 of the Appendix, we have

$$x^{\Lambda^*} h_{\ell} = \sum_{\substack{\mu/\Lambda^* \\ \text{horizontal } \ell\text{-strip}}} \left(\prod_{\substack{i \\ \mu_{i+1} = \Lambda^*_i}}^{\text{(down)}} \pi_i \right) x^{\mu}$$

where the superscript "down" means that the π_i 's in the product are ordered from the largest to the smallest index i. Hence,

$$h_{\ell} \, \partial_{\omega_m} \pi_{\omega_m c} \, \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x) = \sum_{\substack{\mu/\Lambda^* \\ \text{horizontal } \ell\text{-strip}}} \mathcal{P}_{N, [\alpha_1, \dots, \alpha_m]} \left(\prod_{\substack{i \\ \mu_{i+1} = \Lambda_i^*}}^{\text{(down)}} \pi_i \right) x^{\mu}$$
(6.4)

We say that row i+1 of Λ^* has been completed whenever, as in the product above, $\mu_{i+1} = \Lambda_i^*$. Note that this also include the case where $\mu_{i+1} = \Lambda_{i+1}^* = \Lambda_i^*$ where no real completion occurred.

We need to interpret the right hand side of equation (6.4) and show that it sums over superpartitions satisfying the two conditions given in the statement of Theorem 15. The fist condition is satisfied because the sum in (6.4) is over horizontal ℓ -strips. To show that the second condition is satisfied is more complex. In order to do so, we need to understand the behavior or the π_i 's associated to completed rows. This relies essentially on the following rules (see Lemma 48 of the Appendix):

- (1) $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\pi_{\alpha_i} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]} + \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_m]}$ (2) $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\pi_{\alpha_i-1} = 0$ (3) $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\pi_{\beta} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}$ if $\beta \notin \{\alpha_i,\alpha_i-1\}$ for $i=1,\ldots,m$.

Since the product $\prod_{i:\mu_{i+1}=\Lambda_i^*}^{\text{(down)}} \pi_i$ is decreasing, we start from row $\rho=N$ with the greatest value and follow the steps.

- (1) If row ρ has not been completed then $\pi_{\rho-1}$ does not appear in the product $\prod_{i:u_{i+1}=\Lambda^*}^{\text{(down)}} \pi_i$ and so $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}$ is unaffected and we move on to row $\rho-1$.
- (2) If row ρ has been completed but there is no circle in both rows ρ and $\rho-1$ then $\pi_{\rho-1}$ belongs to the product $\prod_{i:\mu_{i+1}=\Lambda_i^*}^{\text{(down)}} \pi_i$ but $\rho-1 \notin \{\alpha_i, \alpha_i-1\}$ for $i=1,\ldots,m$. Hence $\pi_{\rho-1}$ acts as the identity on $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}$ and we move on to row $\rho-1$.
- (3) If row ρ has been completed and there is a circle in row ρ then $\rho = \alpha_i$ for some i and π_{α_i-1} is in the product $\prod_{i:\mu_{i+1}=\Lambda_i^*}^{\text{(down)}} \pi_i$. Therefore in this case $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}\pi_{\alpha_i-1}=0$ and we stop
- (4) If row ρ has been completed and there is no circle in row ρ while there is a circle in row $\rho-1$ then $\rho-1=\alpha_i$ for some i and π_{α_i} is in the product $\prod_{i:\mu_{i+1}=\Lambda_i^*}^{\text{(down)}}\pi_i$. There are two
 - If $\mu_{\alpha_i} = \mu_{\alpha_i+1}$, then $\mu_{\alpha_i} = \Lambda_{\alpha_i}^* < \Lambda_{\alpha_i-1}^* \le \mu_{\alpha_i-1}$ since there is a circle in row α_i . Thus row α_i is not a completed row, which means that π_{α_i-1} is not in the product $\prod_{i:\mu_{i+1}=\Lambda_{i}^{*}}^{(\text{down})}\pi_{i}$. Therefore, $\pi_{\alpha_{i}}$ commutes with all the π_{i} 's to its right and then acts as the identity on x^{μ} (recall that $\mu_{\alpha_i} = \mu_{\alpha_i+1}$). We can thus consider that $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\pi_{\alpha_i} =$ $P_{N,[\alpha_1,\ldots,\alpha_m]}$ and we move on to row $\rho-1$.

• Otherwise, $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}\pi_{\alpha_i} = \mathcal{P}_{N,[\alpha_1,...,\alpha_m]} + \mathcal{P}_{N,[\alpha_1,...,\alpha_i+1,...,\alpha_m]}$ and we split the process into the one for $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}$ and that for which $\mathcal{P}_{N,[\alpha_1,...,\alpha_i,...,\alpha_m]}$ is replaced by $\mathcal{P}_{N,[\alpha_1,...,\alpha_i+1,...,\alpha_m]}$ (the two cases correspond to leaving the circle in its row α_i and sliding it down to row $\alpha_i + 1$ respectively). We then move on to row $\rho - 1$.

We thus have, by construction, that

$$\mathcal{P}_{N,[\alpha_1,\dots,\alpha_m]} \left(\prod_{\substack{i \\ \mu_{i+1} = \Lambda_i^*}}^{\text{(down)}} \pi_i \right) x^{\mu} = \sum_{\alpha'_1,\dots,\alpha'_m} \mathcal{P}_{N,[\alpha'_1,\dots,\alpha'_m]} x^{\mu} = \sum_{\Omega} \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Omega^a)^R,\Omega^s}(x)$$
(6.5)

where the sum is over all superpartitions Ω (obtained from the partition $\mu = \Omega^*$ by adding circles in rows $\alpha'_1, \ldots, \alpha'_m$) such that

- 1. If a square was added to a row with a circle then that circle may move to the right if there is room or down if there is room (that is, if the row below has been completed)
- 2. A square cannot push a circle to the right past the length of the original length of the row above.

For a given $\Omega^* = \mu$, these are precisely the superpartitions Ω described in the statement of the theorem. Since the sum in (6.4) is over all μ 's such that μ/Λ^* is a horizontal ℓ -strip, the other condition in the statement of the theorem is satisfied and the proof is complete.

The Pieri rule corresponding to the multiplication of s_{Λ} by \tilde{p}_{ℓ} will be established in full generality in Corollary 20. For our purposes, it is sufficient at this point to prove the very special case when the superpartition $\Lambda = (\Lambda^a; \Lambda^s)$ is such that Λ^s is the empty partition.

Lemma 16. Let $\lambda = (\lambda_1, \dots, \lambda_m, \lambda_{m+1})$ be a partition and let $\hat{\lambda} = (\lambda_1, \dots, \lambda_m)$ be λ without its last part. We have

$$s_{(\hat{\lambda};\emptyset)}\,\tilde{p}_{\lambda_{m+1}} = s_{(\lambda;\emptyset)} \tag{6.6}$$

Proof. We will instead show that $\tilde{p}_{\lambda_{m+1}} s_{(\hat{\lambda};\emptyset)} = (-1)^m s_{(\lambda;\emptyset)}$. Let $\ell = \lambda_{m+1}$ and note that $\hat{\lambda}$ is a partition of length m. Using Lemma 2 and Lemma 6, we need to show that

$$\partial_{\omega_{m+1}} x_{m+1}^{\ell} \pi_{\omega_{m^c}} \hat{K}_{(\hat{\lambda})^R, 0^{N-m}}(x) = (-1)^m \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{K}_{\lambda^R, 0^{N-m-1}}(x)$$
(6.7)

We start with the left hand side and use $\pi_{\omega_{m^c}} = \pi_{\omega_{(m+1)^c}} \pi_{[m+1,N-1]}$ to get

$$\partial_{\omega_{m+1}} x_{m+1}^{\ell} \pi_{\omega_{m^c}} \hat{K}_{(\hat{\lambda}^R \cdot 0^{N-m})}(x) = \partial_{\omega_{m+1}} x_{m+1}^{\ell} \pi_{\omega_{(m+1)^c}} \pi_{[m+1,N-1]} \hat{K}_{(\hat{\lambda}^R \cdot 0^{N-m})}(x) \tag{6.8}$$

By definition, $\hat{K}_{\hat{\lambda}^R,0^{N-m}}(x) = \hat{\pi}_{\omega_m} x^{\hat{\lambda}}$, which means that $\pi_{m+1},\ldots,\pi_{N-1}$ act on the right as the identity (since rows $m+1,\ldots,N$ of $\hat{\lambda}$ are all zero). After commuting x_{m+1}^{ℓ} with $\pi_{\omega_{(m+1)^c}}\hat{\pi}_{\omega_m}$ and combining $x_{m+1}^{\ell} x^{\hat{\lambda}} = x^{\lambda}$, we then get

$$\partial_{\omega_{m+1}} x_{m+1}^{\ell} \pi_{\omega_{m^c}} \hat{K}_{(\hat{\lambda})^R, 0^{N-m}}(x) = \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} x^{\lambda}$$

$$\tag{6.9}$$

Owing to the fact that $\partial_{\omega_{m+1}} = (-1)^m \partial_{\omega_{m+1}} \hat{\pi}_{[1,m]}$, we have

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} x^{\lambda} = (-1)^m \partial_{\omega_{m+1}} \hat{\pi}_{[1,m]} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} x^{\lambda}
= (-1)^m \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{[1,m]} \hat{\pi}_{\omega_m} x^{\lambda}
= (-1)^m \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_{m+1}} x^{\lambda}
= (-1)^m \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{K}_{\lambda^R, 0^{N-m-1}}(x)$$
(6.10)

where we used the fact that $\hat{\pi}_{\omega_{m+1}}x^{\lambda} = \hat{K}_{\lambda^{R},0^{N-m-1}}(x)$ since λ is a partition of length m+1. Therefore (6.7) holds from the previous two equations and the claim follows.

The following corollary is immediate.

Corollary 17. If $\Lambda = (\Lambda^a; \emptyset)$ then $p_{(\Lambda^a; \emptyset)} = \tilde{p}_{\Lambda^a_1} \cdots \tilde{p}_{\Lambda^a_m} = s_{(\Lambda^a; \emptyset)} = s_{\Lambda}$. In particular, $\tilde{p}_{\ell} = s_{(\ell; \emptyset)}$.

7. Duality

We will now see that there is a natural duality that relates s_{Λ} and $s_{\Lambda'}$. Unexpectedly, for reasons we will see later in this section, no such simple duality exists in the case of \bar{s}_{Λ} .

Applying ω on the first formula of (5.1), we obtain from (2.14) and Proposition 1 that

$$e_{\Lambda} = (-1)^{\binom{m}{2}} \sum_{\Omega} K_{\Omega \Lambda} \, s_{\Omega'} \tag{7.1}$$

Now, let $H_{\Lambda} = p_{(\Lambda^a;\emptyset)} h_{(\emptyset;\Lambda^s)}$, that is, H_{Λ} is h_{Λ} with the superspace generators \tilde{h}_r replaced by \tilde{p}_r .

Proposition 18. We have

$$H_{\Lambda} = \sum_{\Omega} K_{\Omega \Lambda} \, s_{\Omega} \tag{7.2}$$

where we stress that the coefficients $K_{\Omega\Lambda}$ are exactly those that appear in (7.1).

Proof. From Corollary 12 and 17, we have that if $\Lambda = (\Lambda^a; \emptyset)$, then $e_{\Lambda} = (-1)^{\binom{m}{2}} s_{\Lambda'}$ and $H_{\Lambda} = p_{(\Lambda^a; \emptyset)} = s_{\Lambda}$. Which means that the proposition holds in that case with $K_{\Omega\Lambda} = \delta_{\Omega\Lambda}$. For Λ generic, we thus have $e_{\Lambda} = e_{(\Lambda^a; \emptyset)} e_{(\emptyset; \Lambda^s)} = (-1)^{\binom{m}{2}} s_{(\Lambda^a; \emptyset)'} e_{(\emptyset; \Lambda^s)}$, and similarly, $H_{\Lambda} = p_{(\Lambda^a; \emptyset)} h_{(\emptyset; \Lambda^s)} = s_{(\Lambda^a; \emptyset)} h_{(\emptyset; \Lambda^s)}$. The expansion of e_{Λ} in terms of Schur functions in superspace s_{Ω} is thus obtained by using repeatedly, starting from $(-1)^{\binom{m}{2}} s_{(\Lambda^a; \emptyset)'}$, the Pieri rule for the multiplication by e_{ℓ} of Theorem 10. Similarly, the expansion of H_{Λ} in terms of Schur functions in superspace s_{Ω} is obtained by using repeatedly, starting from $s_{(\Lambda^a; \emptyset)}$, the Pieri rule for the multiplication by h_{ℓ} of Theorem 15. But since those Pieri rules are transposed of each other and do not involve any sign, the proposition follows immediately given that we start on the transposed superpartitions $(\Lambda^a; \emptyset)'$ and $(\Lambda^a; \emptyset)$ and with signs that differ by a factor of $(-1)^{\binom{m}{2}}$.

The previous proposition suggests a natural duality between the H_{Λ} and e_{Λ} bases. In effect, let φ be the homomorphism defined by

$$\varphi(\tilde{p}_r) = \tilde{e}_r \quad \text{and} \quad \varphi(h_r) = e_r$$
 (7.3)

that is, such that $\varphi(H_{\Lambda}) = e_{\Lambda}$. By (7.1) and (7.2), the following result is essentially immediate:

Corollary 19. We have

$$\varphi(s_{\Lambda}) = (-1)^{\binom{m}{2}} s_{\Lambda'} \tag{7.4}$$

As a consequence, the homomorphism φ is an involution, that is, $\varphi \circ \varphi$ is the identity.

Proof. Applying the homomorphism φ on both sides of (7.2), we have from (7.1)

$$e_{\Lambda} = \sum_{\Omega} K_{\Omega \Lambda} \, \varphi(s_{\Omega}) = (-1)^{\binom{m}{2}} \sum_{\Omega} K_{\Omega \Lambda} \, s_{\Omega'} \tag{7.5}$$

Therefore $\varphi(s_{\Lambda}) = (-1)^{\binom{m}{2}} s_{\Lambda'}$ since the matrix $\{K_{\Omega\Lambda}\}_{\Omega,\Lambda}$ is invertible (it is, up to a sign, the change of basis matrix between the bases e_{Λ} and $s_{\Lambda'}$). The involutivity of φ is then immediate.

Using Corollary 19, the Pieri rules for the multiplication of s_{Λ} by \tilde{p}_{ℓ} and h_{ℓ} are seen, from Theorems 10 and 11, to be identical to those for the multiplication of \bar{s}_{Λ}^* by \tilde{h}_{ℓ} and h_{ℓ} respectively. The first formula in the corollary is the content of Theorem 15. But the point here is to understand how it follows from the duality (which unfortunately we were not able to prove without first proving Theorem 15). We should note that these Pieri rules were conjectured to hold in [4].

Corollary 20. We have, for $k \geq 1$ and $\ell \geq 0$,

$$s_{\Lambda} h_k = \sum_{\Omega} s_{\Omega}$$
 and $s_{\Lambda} \tilde{p}_{\ell} = \sum_{\Omega} (-1)^{\#(\Omega, \Lambda)} s_{\Omega}$ (7.6)

where $\#(\Omega, \Lambda)$ is as usual the number of circles of Ω below the new circle and where the sums run over the superpartitions Ω obeying the conditions of Theorem 10 (in the $\bar{s}_{\Lambda}^*h_{\ell}$ case) and Theorem 11 (in the $\bar{s}_{\Lambda}^*\tilde{h}_{\ell}$ case) respectively.

Proof. We prove the second formula, as it involves a non-trivial sign. The first formula follows from the same argument (without the sign issue) using Corollary 19 and Theorem 10.

Applying φ on both sides of the second equation in (4.37), we obtain from Corollary 19

$$s_{\Lambda'} \, \tilde{p}_{\ell} = \sum_{\Omega} (-1)^{\#(\Omega,\Lambda) + m} s_{\Omega'} \tag{7.7}$$

where the extra m in the sign comes from the difference between $\binom{m}{2}$ and $\binom{m+1}{2}$, and where the sum is over the Ω 's described in Theorem 11 (in the $s_{\Lambda} \tilde{e}_{\ell}$ case). Hence

$$s_{\Lambda} \tilde{p}_{\ell} = \sum_{\Omega'} (-1)^{\#(\Omega', \Lambda') + m} s_{\Omega} = \sum_{\Omega'} (-1)^{\#(\Omega, \Lambda)} s_{\Omega}$$
 (7.8)

since $\#(\Omega', \Lambda') + \#(\Omega, \Lambda) = m$. The result then follows by transposing the conditions on Ω (which means considering $\bar{s}_{\Lambda}^* \tilde{h}_{\ell}$ instead of $s_{\Lambda} \tilde{e}_{\ell}$).

It is immediate from Proposition 1 that the linear map $\omega \circ \varphi^+ \circ \omega$ sends \bar{s}_{Λ} to $\bar{s}_{\Lambda'}$ (up to a sign), where φ^+ is the adjoint of φ with respect to the scalar product (2.11). But φ^+ turns out not to be a homomorphism since otherwise the map $\omega \circ \varphi^+ \circ \omega$ would also be a homomorphism, contradicting the fact that the Littlewood-Richardson coefficients associated to the products of Schur functions in superspace \bar{s}_{Λ} do not have a symmetry under conjugation (see Remark 29). As such the duality between \bar{s}_{Λ} and $\bar{s}_{\Lambda'}$ is less natural (for instance, it does not lead to any analog of Corollary 20).

Remark 21. In Conjecture 4.1 of [4], Pieri rules for the multiplication of s_{Λ} by $s_{(\emptyset;r)}$, $s_{(r;\emptyset)}$, $s_{(\emptyset;1^r)}$ and $s_{(0;1^r)}$ are stated. Given that $s_{(\emptyset;r)} = s_r = h_r$, $s_{(r;\emptyset)} = \tilde{p}_r$ (see Corollary 17), $s_{(\emptyset;1^r)} = e_r$ and $s_{(0;1^r)} = m_{(0;1^r)} = \tilde{e}_r$ (by the triangularity in (5.2)), these Pieri rules are proven in Corollary 20 $(h_r \text{ and } \tilde{p}_r)$, Theorem 10 (e_r) and Theorem 11 (\tilde{e}_r) .

8. Tableaux

In this section, we show that the Schur functions in superspace are generating series of certain types of tableaux.

We will refer to $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \dots\}$ as the set of *fermionic* nonnegative integers. In this spirit, we will also refer to the set of nonnegative integers $\{0, 1, 2, 3, \dots\}$ as the set of *bosonic* nonnegative integers. For $\alpha \in \{0, \bar{0}, 1, \bar{1}, 2, \bar{2}, \dots\}$, we will say that $\operatorname{type}(\alpha)$ is bosonic or fermionic depending on whether the corresponding integer is fermionic or bosonic. Finally, define

$$|\alpha| = \begin{cases} a & \text{if } \alpha = \bar{a} \text{ is fermionic} \\ a & \text{if } \alpha = a \text{ is bosonic} \end{cases}$$
 (8.1)

8.1. s-tableaux. By (5.1), the tableaux needed to represent the Schur function in superspace s_{Λ} are those stemming from the Pieri rules associated to the multiplication of s_{Λ}^* by h_r or \tilde{h}_r given in Theorem 7 and 8. We say that the sequence $\Omega = \Lambda_{(0)}, \Lambda_{(1)}, \ldots, \Lambda_{(n)} = \Lambda$ is an s-tableau of shape Λ/Ω and weight $(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \{0, \bar{0}, 1, \bar{1}, 2, \bar{2}, \ldots\}$, if $\Omega = \Lambda_{(i)}$ and $\Lambda = \Lambda_{(i-1)}$ obey the conditions of Theorem 7 with $\ell = \alpha_i$ whenever α_i is bosonic or the conditions of Theorem 8 with $\ell = |\alpha_i|$ whenever α_i is fermionic. An s-tableau can be represented by a diagram constructed recursively in the following way:

- (1) the cells of $\Lambda_{(i)}^*/\Lambda_{(i-1)}^*$, which form a horizontal strip, are filled with the letter i. In the fermionic case, the new circle is also filled with a letter i.
- (2) the circles of $\Lambda_{(i-1)}$ that are moved a row below keep their fillings.

The sign of an s-tableau T, which corresponds to the product of the signs appearing in the fermionic horizontal strips, can be extracted quite efficiently from an s-tableau. Read the fillings of the circles from top to bottom to obtain a word (without repetition): the sign of the tableau T is then equal to $(-1)^{\text{inv}(T)}$, where inv(T) is the number of inversions of the word.

Given a diagram of an s-tableau, we define the path of a given circle (filled let's say with letter i) in the following way. Let c be the leftmost column that does not contain a square (a cell of Ω^*) filled with an i. The path starts in the position of the smallest entry larger than i (let's say j) in column c. The path then moves to the smallest entry (let's say k) larger than j in the row below (if there are many such k's the path goes through the leftmost such k). We continue this way until we reach the row above that of the circle filled with an i.

It is important to realize that a tableau can be identified with its diagram given that the sequence $\Omega = \Lambda_{(0)}, \Lambda_{(1)}, \dots, \Lambda_{(n)} = \Lambda$ can be recovered from the diagram. We obtain the diagram corresponding to $\Omega = \Lambda_{(0)}, \Lambda_{(1)}, \dots, \Lambda_{(n-1)}$ by removing the letters n from the diagram (including, possibly, the circled one), and by moving the remaining circle one row above according to the following rule. A circle (filled let's say with letter i) in a given row r is moved to row r-1 if there is an n in row r-1 that belongs to its path. Otherwise the circle in row r stays in its position.

Consider the tableau $\begin{array}{c} 1 & 2 & 2 & 3 & 4 \\ 2 & 4 & 6 & 6 \\ 4 & 5 & 4 \\ 6 & 6 & 2 \\ \end{array}$ of weight $(1,\bar{3},1,\bar{3},1,3)$ and shape (2,0;5,3,2) $(\Omega=\emptyset)$ in the example). The path for the $\begin{array}{c} 2 & 3 & 4 \\ 2 & 4 & 6 \\ \hline 2 & 4 & 6 \\ \hline 6 & 6 & 6 \\ \end{array}$ which is seen as follows: the leftmost column without a

non-circled 2 is column 4. Since there is only one entry, a 3, in that column, the path starts there. The smallest entry larger than 3 in the row below, the second one, is 4. Then the smallest entry larger than 4 in the third row is 5. Finally, the smallest entry larger than 4 in row 4 is 6, and the path goes through the leftmost 6. The path then stops since the circled 2 is in row 5. The path for

4 can similarly be seen to be $\frac{12234}{666}$. It is obvious from the example that the non-circled letters

of an s-tableau form an ordinary tableau. However it is not obvious where the circled letters can be added, and the paths above could only be constructed because the tableau was valid.

The sequence of superpartitions associated to that s-tableau can then be recovered by stripping successively the tableau of its largest letter and possibly moving circled letters one row above according to their paths:

Now, define the skew Schur function in superspace $s_{\Lambda/\Omega}$ as

$$s_{\Lambda/\Omega} = \sum_{T} (-1)^{\text{inv}(T)} (x\theta)^T$$
(8.2)

where the sum is over all s-tableaux of shape Λ/Ω , and where

$$(x\theta)^{T} = \prod_{i} x_{i}^{|\alpha_{i}|} \prod_{j: \text{type}(\alpha_{j}) = \text{fermionic}} \theta_{j}$$
(8.3)

if T is of weight $(\alpha_1, \ldots, \alpha_n)$. We stress that the product over anticommuting variables is ordered from left to right over increasing indices.

Proposition 22. $s_{\Lambda/\Omega}$ is a symmetric function in superspace. Moreover,

$$s_{\Lambda/\Omega} = \sum_{\Gamma} \bar{K}_{\Lambda/\Omega,\Gamma} m_{\Gamma} \tag{8.4}$$

where $\bar{K}_{\Lambda/\Omega,\Gamma} = \sum_{T} (-1)^{\mathrm{inv}(T)}$, with the sum over all s-tableaux T of weight

$$(\bar{\Gamma}_1, \dots, \bar{\Gamma}_m, \Gamma_{m+1}, \dots, \Gamma_N)$$
 for $\Gamma = (\Gamma_1, \dots, \Gamma_m; \Gamma_{m+1}, \dots, \Gamma_N)$

Proof. By definition, the coefficient of $(x\theta)^{\gamma}$ in $s_{\Lambda/\Omega}$ is equal to $\sum_{T} (-1)^{\text{inv}(T)}$, where the sum is over all s-tableaux T of weight γ . From the construction of the s-tableaux from the Pieri rules in Theorem 7 and 8, we have

$$s_{\Omega}^* h_{\gamma} = \sum_{\Lambda} \bar{K}_{\Lambda/\Omega,\gamma} s_{\Lambda}^* \tag{8.5}$$

where $\bar{K}_{\Lambda/\Omega,\gamma}$ is equal to the signed sum over all s-tableaux of weight γ . By the commutation/anticommutation of the h_i and \tilde{h}_i , we have that if β corresponds to γ with two fermionic integers interchanged then $\bar{K}_{\Lambda/\Omega,\beta} = -\bar{K}_{\Lambda/\Omega,\gamma}$, while if β corresponds to γ with two bosonic integers interchanged or with a bosonic integer and a fermionic integer interchanged then $\bar{K}_{\Lambda/\Omega,\beta} = \bar{K}_{\Lambda/\Omega,\gamma}$. Thus the coefficients of $(x\theta)^{\gamma}$ and $(x\theta)^{\beta}$ in $s_{\Lambda/\Omega}$ are equal up to the right sign and the first statement follows. The other statement is then immediate since the coefficient of $(x\theta)^{(\bar{\Gamma}_1,\dots,\bar{\Gamma}_m,\Gamma_{m+1},\dots,\Gamma_N)}$ in m_{Γ} is equal to 1.

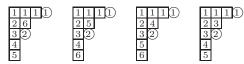
Corollary 23. We have that $s_{\Lambda/\emptyset} = s_{\Lambda}$. Hence

$$s_{\Lambda} = \sum_{T} (-1)^{\text{inv}(T)} (x\theta)^{T}$$
(8.6)

where the sum is over all s-tableaux T of shape Λ .

Proof. We have by (8.5) and (5.1) that $\bar{K}_{\Lambda/\emptyset,\Gamma} = \bar{K}_{\Lambda\Gamma}$. The corollary then follows from (5.2).

We thus obtain the monomial expansion of $s_{(3,1;2,1,1)}$ by listing every filling of the shape (3,1;2,1,1) whose weight corresponds to a superpartition:



Therefore, $s_{(3,1;2,1,1)} = 3 \, m_{(3,1;1,1,1,1)} + m_{(3,1;2,1,1)}$ since there are 3 tableaux of weight $(\bar{3}, \bar{1}, 1, 1, 1, 1, 1)$ and one tableau of weight $(\bar{3}, \bar{1}, 2, 1, 1)$. We stress that we don't have an easy criteria in general to determine whether a given filling is a valid tableau. However, in the example above, the rules for constructing tableaux immediately imply that we need to have three non-circled 1's and one non-circled 2 (otherwise the circled 2 could never be in the second column). Then there are very few possibilities to fill the rest of the tableau with a weight corresponding to a superpartition. The case of weight $(\bar{3},\bar{1},1,1,1,1)$ where a 3 is above the circled 2 is not allowed since again this would prevent the circled 2 from being in the second column.

Note that when $\Omega \neq \emptyset$, the coefficient $\bar{K}_{\Lambda/\Omega,\Gamma}$ can be a negative integer.

- 8.2. \bar{s} -tableaux. By (5.1), the tableaux needed to represent the Schur function in superspace \bar{s}_{Λ} are this time those stemming from the Pieri rules associated to the multiplication of \bar{s}_{Λ}^* by h_r or \tilde{h}_r given in Theorem 10 and 11. We say that the sequence $\Omega = \Lambda_{(0)}, \Lambda_{(1)}, \ldots, \Lambda_{(n)} = \Lambda$ is an \bar{s} -tableau of shape Λ/Ω and weight $(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \{0, \bar{0}, 1, \bar{1}, 2, \bar{2}, \ldots\}$, if $\Omega = \Lambda_{(i)}$ and $\Lambda = \Lambda_{(i-1)}$ obey the conditions of Theorem 10 with $\ell = \alpha_i$ whenever α_i is bosonic or the conditions of Theorem 11 with $\ell = |\alpha_i|$ whenever α_i is fermionic. An \bar{s} -tableau can be represented by a diagram constructed recursively in the following way:
 - (1) the cells of $\Lambda_{(i)}^*/\Lambda_{(i-1)}^*$ are filled with the letter i. In the fermionic case, the new circle is also filled with a letter i
 - (2) the circles of $\Lambda_{(i-1)}$ that moved along a column or a row keep their fillings.

As is the case for s-tableaux, the sign of an \bar{s} -tableau T is equal to $(-1)^{\mathrm{inv}(T)}$, where $\mathrm{inv}(T)$ is the number of inversions of the word obtained by reading the filling of the circles from top to bottom.

It is important to realize that the sequence $\Omega = \Lambda_{(0)}, \Lambda_{(1)}, \dots, \Lambda_{(n)} = \Lambda$ can be recovered from the diagram. We obtain the diagram corresponding to $\Omega = \Lambda_{(0)}, \Lambda_{(1)}, \dots, \Lambda_{(n-1)}$ by removing the letters n from the diagram (including, possibly, the circled one), and by moving the circled letters one cell above if there is a letter n above them or to their left if there are letters n to their left and none above them. For instance, if one considers the \bar{s} -tableau T below of weight $(3, \bar{1}, \bar{2}, 1, \bar{3}, 5, \bar{0})$, the sequence of superpartitions associated to it can then be recovered by stripping successively the tableaux of their largest letter:

We should stress that there is no immediate criteria to determine whether a given filling of a shape is a valid tableau. It is only after checking that every removal of a letter corresponds to an application of a Pieri rule that we know that the filling is valid. In the example above, removing the 6 corresponds to an application of the Pieri rule h_5 since the the 6's form a horizontal strip and when moving the circled 2 and 3 above and the circled 5 to its left there are no collisions (two circles in the same row or column). Similarly, removing the 5's corresponds to the Pieri rule \tilde{h}_3 since the 5's form a horizontal strip with the circled one being the rightmost and when moving the circled 2 to its left and the circled 3 above there are no collisions.

As was done in the previous subsection, define the Schur function in superspace $\bar{s}_{\Lambda/\Omega}$ as

$$\bar{s}_{\Lambda/\Omega} = \sum_{T} (-1)^{\operatorname{sign}(T)} (x\theta)^{T}$$
(8.7)

where the sum is over all \bar{s} -tableaux of shape Λ/Ω .

The proof of the next proposition and its corollary are as in the previous subsection.

Proposition 24. $\bar{s}_{\Lambda/\Omega}$ is a symmetric function in superspace. Moreover,

$$\bar{s}_{\Lambda/\Omega} = \sum_{\Gamma} K_{\Lambda/\Omega,\Gamma} m_{\Gamma} \tag{8.8}$$

where $K_{\Lambda/\Omega,\Gamma} = \sum_{T} (-1)^{\operatorname{sign}(T)}$, the sum over all \bar{s} -tableaux T of weight

$$(\bar{\Gamma}_1, \dots, \bar{\Gamma}_m, \Gamma_{m+1}, \dots, \Gamma_N)$$
 for $\Gamma = (\Gamma_1, \dots, \Gamma_m; \Gamma_{m+1}, \dots, \Gamma_N)$

Corollary 25. We have that $\bar{s}_{\Lambda/\emptyset} = \bar{s}_{\Lambda}$. Hence

$$\bar{s}_{\Lambda} = \sum_{T} (-1)^{\operatorname{sign}(T)} (x\theta)^{T}$$
(8.9)

where the sum is over all \bar{s} -tableaux in superspace of shape Λ .

The monomial expansion of $\bar{s}_{(2,0;3)}$ is thus obtained by listing every filling of the shape (2,0;3) whose weight is that of a superpartition

Hence,

$$\bar{s}_{(2,0;3)} = 3\,m_{(1,0;1,1,1,1)} + 2\,m_{(1,0;2,1,1)} + m_{(1,0;2,2)} + m_{(1,0;3,1)} + m_{(2,0;1,1,1)} + m_{(2,0;2,1)} + m_{(2,0;3)}\,.$$

As was mentioned before, we don't have an easy criteria in general to determine whether a given filling is a valid tableau. For instance, the tableau $\frac{135}{461}$ is not valid because the circled 1 and the circled 2 collide at the moment of removing letter 4:

We note that, as is the case for $\bar{K}_{\Lambda/\Omega,\Gamma}$, the coefficient $K_{\Lambda/\Omega,\Gamma}$ can be a negative integer when $\Omega \neq \emptyset$.

Remark 26. Combinatorial formulas, as tableau generating series, for the monomial expansions of s_{Λ} and \bar{s}_{Λ} are conjectured in [4]. Their formulas essentially only concern the cases when the fermionic Pieri rules are applied last, that is, when the weights are of the type $(a_1, ..., a_{\ell}, \bar{b}_1, ..., \bar{b}_m)$. In those cases, their sums are cancellation free while ours are not (we have cancellation free sums when the fermionic Pieri rules are applied first). We have not tried to prove their conjectures in this article, even though it would be interesting to understand their combinatorial formula for \bar{s}_{Λ} since it involves a subset of the \bar{s} -tableaux.

9. Further properties of skew Schur functions in superspace

In this section, we first show that skew Schur functions in superspace can be obtained from Schur functions in superspace by taking their adjoint with respect to the scalar product $\langle \langle \cdot, \cdot \rangle \rangle$. We then connect the generalization to superspace of the Littlewood-Richardson coefficients to skew Schur functions in superspace. These basic properties of skew Schur functions in superspace generalize well known properties in the classical case (m=0).

Corollary 27. We have

$$\langle\!\langle s_{\Omega}^* f, s_{\Lambda} \rangle\!\rangle = \langle\!\langle f, s_{\Lambda/\Omega} \rangle\!\rangle \quad \text{and} \quad \langle\!\langle \bar{s}_{\Omega}^* f, \bar{s}_{\Lambda} \rangle\!\rangle = \langle\!\langle f, \bar{s}_{\Lambda/\Omega} \rangle\!\rangle$$
 (9.1)

for all symmetric functions in superspace f.

Proof. Since the two cases are similar, we will only prove the first formula. It suffices to consider that $f = h_{\Gamma}$ since the h_{Λ} 's form a basis of the space. From Proposition 22 and the duality (2.12) between the h_{Λ} and m_{Λ} bases, we have

$$\langle \langle h_{\Gamma}, s_{\Lambda/\Omega} \rangle \rangle = \bar{K}_{\Lambda/\Omega, \Gamma} \tag{9.2}$$

On the other hand, using (8.5), we get from Proposition 1 that

$$\langle \langle s_{\Omega}^* h_{\Gamma}, s_{\Lambda} \rangle \rangle = \langle \langle \sum_{\Lambda} \bar{K}_{\Delta/\Omega, \Gamma} s_{\Delta}^*, s_{\Lambda} \rangle \rangle = \bar{K}_{\Lambda/\Omega, \Gamma}$$

$$(9.3)$$

and the result follows. \Box

Define $\bar{c}^{\Lambda}_{\Gamma\Omega}$ and $c^{\Lambda}_{\Gamma\Omega}$ to be respectively such that

$$\bar{s}_{\Gamma} \, \bar{s}_{\Omega} = \sum_{\Lambda} \bar{c}_{\Gamma\Omega}^{\Lambda} \, \bar{s}_{\Lambda} \quad \text{and} \quad s_{\Gamma} \, s_{\Omega} = \sum_{\Lambda} c_{\Gamma\Omega}^{\Lambda} \, s_{\Lambda}$$
 (9.4)

It is immediate from the (anti-)commutation relations between the Schur functions in superspace that if Γ and Ω are respectively of fermionic degrees a and b, then $\bar{c}_{\Gamma\Omega}^{\Lambda}=(-1)^{ab}\bar{c}_{\Omega\Gamma}^{\Lambda}$ and $c_{\Gamma\Omega}^{\Lambda}=(-1)^{ab}c_{\Omega\Gamma}^{\Lambda}$. Even though $\bar{c}_{\Gamma\Omega}^{\Lambda}$ and $c_{\Gamma\Omega}^{\Lambda}$ are not always nonnegative from these relations, we can consider them as generalizations to superspace of the Littlewood-Richardson coefficients.

We now extend to superspace the well-known connection between Littlewood-Richardson coefficients and skew Schur functions.

Proposition 28. We have

$$s_{\Lambda/\Omega} = \sum_{\Gamma} \bar{c}_{\Gamma'\Omega'}^{\Lambda'} s_{\Gamma} \quad \text{and} \quad \bar{s}_{\Lambda/\Omega} = \sum_{\Gamma} c_{\Omega\Gamma}^{\Lambda} \bar{s}_{\Gamma}$$
 (9.5)

Furthermore, $c_{\Omega\Gamma}^{\Lambda} = c_{\Gamma'\Omega'}^{\Lambda'}$.

Proof. Suppose that Γ and Ω are respectively of fermionic degrees a and b. The symmetry $c_{\Omega\Gamma}^{\Lambda} = (-1)^{ab} c_{\Omega'\Gamma'}^{\Lambda'}$ is from Corollary 19 an immediate consequence of applying the homomorphism ϕ on $s_{\Omega} s_{\Gamma} = \sum_{\Lambda} c_{\Omega\Gamma}^{\Lambda} s_{\Lambda}$. In effect, applying the homomorphism gives

$$(-1)^{\binom{a}{2} + \binom{b}{2}} s_{\Omega'} s_{\Gamma'} = (-1)^{\binom{a+b}{2}} \sum_{\Lambda} c_{\Omega\Gamma}^{\Lambda} s_{\Lambda'}$$
(9.6)

since Λ is necessarily of fermionic degree a+b. Hence $s_{\Omega'} s_{\Gamma'} = (-1)^{ab} \sum_{\Lambda} c_{\Omega\Gamma}^{\Lambda} s_{\Lambda'}$, which gives the symmetry $c_{\Omega\Gamma}^{\Lambda} = (-1)^{ab} c_{\Omega'\Gamma'}^{\Lambda'} = c_{\Gamma'\Omega'}^{\Lambda'}$.

We now prove the first formula. Using the previous corollary, we have

$$\langle \langle s_{\Omega}^* s_{\Gamma}^*, s_{\Lambda} \rangle \rangle = \langle \langle s_{\Gamma}^*, s_{\Lambda/\Omega} \rangle \rangle \tag{9.7}$$

Therefore, if $d_{\Omega\Gamma}^{\Lambda}$ is such that

$$s_{\Omega}^* s_{\Gamma}^* = \sum_{\Lambda} d_{\Omega\Gamma}^{\Lambda} s_{\Lambda}^* \tag{9.8}$$

then we have by Proposition 1

$$s_{\Lambda/\Omega} = \sum_{\Gamma} d_{\Omega\Gamma}^{\Lambda} s_{\Gamma} \tag{9.9}$$

Applying ω on both sides of (9.8), we get as in equation (9.6),

$$(-1)^{\binom{a}{2} + \binom{b}{2}} \bar{s}_{\Omega'} \, \bar{s}_{\Gamma'} = \sum_{\Lambda} (-1)^{\binom{a+b}{2}} d^{\Lambda}_{\Omega\Gamma} \, \bar{s}_{\Lambda'} \tag{9.10}$$

Therefore

$$d_{\Omega\Gamma}^{\Lambda} = (-1)^{ab} c_{\Omega'\Gamma'}^{\Lambda'} = \bar{c}_{\Gamma'\Omega'}^{\Lambda'}$$

$$(9.11)$$

and the first formula follows.

We can prove in a similar way that $\bar{s}_{\Lambda/\Omega} = \sum_{\Gamma} c_{\Gamma'\Omega'}^{\Lambda'} \bar{s}_{\Gamma}$. Using the symmetry $c_{\Gamma'\Omega'}^{\Lambda'} = c_{\Omega\Gamma}^{\Lambda}$, the second formula is then seen to hold.

Remark 29. Somewhat surprisingly, the coefficient $\bar{c}_{\Omega\Gamma}^{\Lambda}$ does not have in general any symmetry under conjugation. For instance, it can be checked that if $\Gamma = (1;)$, $\Omega = (0;)$ and $\Lambda = (1,0;)$ then $\bar{c}_{\Omega\Gamma}^{\Lambda} = 1$ while $\bar{c}_{\Gamma'\Omega'}^{\Lambda'} = 0$.

10. Large m and N limit

Let Λ be a superpartition of fermionic degree m. Lemmas 2 and 3 state that in the identification (2.2) between Λ_N^m and $\mathbb{Q}[x_1,\ldots,x_N]^{S_m\times S_{m^c}}$ we have

$$s_{\Lambda} \longleftrightarrow (-1)^{\binom{m}{2}} \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{K}_{(\Lambda^a)^R, \Lambda^s}(x)$$
 (10.1)

and

$$\bar{s}_{\Lambda} \longleftrightarrow \partial'_{\omega_{(N-m)^c}} K_{(\Lambda^s)^R,\Lambda^a}(y)$$
 (10.2)

where y stands for the variables $y_1 = x_N, y_2 = x_{N-1}, \dots, y_N = x_1$ and where the ' indicates that the divided differences act on the y variables.

Let $\mu = \Lambda^s$ and $\lambda = \Lambda^a - \delta_m$, where $\delta_m = (m-1, m-2, ..., 0)$. It was shown in [3] that when $m \ge |\lambda| + |\mu|$ and $N - m \ge |\lambda| + |\mu|$ (the large m and N limit), the identification becomes

$$s_{\Lambda} \longleftrightarrow s_{\lambda}(x_1, \dots, x_m) s_{\mu}(x_{m+1}, \dots, x_N)$$
 (10.3)

and

$$\bar{s}_{\Lambda} \longleftrightarrow s_{\lambda}(x_1, \dots, x_N) s_{\mu}(x_{m+1}, \dots, x_N)$$
 (10.4)

We thus have the following proposition which does not seem to have an easy proof in the Key world.

Proposition 30. Let $\mu = \Lambda^s$ and $\lambda = \Lambda^a - \delta_m$, where $\delta_m = (m-1, m-2, ..., 0)$. If $m \ge |\lambda| + |\mu|$ and $N - m \ge |\lambda| + |\mu|$ then

$$(-1)^{\binom{m}{2}} \partial_{\omega_m} \pi_{\omega_m^c} \hat{K}_{(\Lambda^a)R,\Lambda^s}(x) = s_{\lambda}(x_1, \dots, x_m) s_{\mu}(x_{m+1}, \dots, x_N)$$

$$(10.5)$$

and

$$\partial_{\omega_{m^c}} K_{(\Lambda^s)^R,\Lambda^a}(x_N, x_{N-1}, \dots, x_1) = s_{\lambda}(x_1, \dots, x_N) s_{\mu}(x_{m+1}, \dots, x_N)$$
(10.6)

11. Cauchy formulas

As is the case in symmetric function theory, the dualities of Proposition 1 translate into Cauchy type formulas. The one most relevant to this work is the following. Given two bases $\{f_{\Lambda}\}_{\Lambda}$ and $\{g_{\Lambda}\}_{\Lambda}$ dual to each other with respect to the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$, we have [6]

$$\prod_{i,j} (1 + x_i y_j + \theta_i \phi_j) = \sum_{\Lambda} (-1)^{\binom{m}{2}} f_{\Lambda}(x,\theta) \,\omega(g_{\Lambda})(y,\phi) \tag{11.1}$$

where m is the fermionic degree of Λ , and where the variables y_1, y_2, \ldots are ordinary variables while the variables ϕ_1, ϕ_2, \ldots are anticommuting (they also anticommute with the θ_i 's). Using Proposition 1, we then get

$$\prod_{i,j} (1 + x_i y_j + \theta_i \phi_j) = \sum_{\Lambda} s_{\Lambda}(x, \theta) \,\bar{s}_{\Lambda'}(y, \phi) \tag{11.2}$$

The tableaux generating series of Propositions 23 and 25 suggest that there should exist a bijective proof of that formula using an extension to superspace of the dual Robinson-Schensted-Knuth algorithm [14].

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APPENDIX A.

In the appendix, we prove various technical results that were needed to prove the Pieri rules. They range from the very elementary to the quite intricate (see Lemma 60 for instance). Although the proofs of the elementary ones may probably be found in the literature, we include them for completeness.

Lemma 31. Using the notation of Section 4, we have

$$K_{(\Lambda^s)^R,\Lambda^a}(x) = \pi_{\omega_{N-m}} \pi_{(N-m,\alpha_1)} \pi_{(N-m+1,\alpha_2)} \cdots \pi_{(N-1,\alpha_m)} x^{\Lambda^*}$$
(A.1)

where α_i is the row of the i-th circle (starting from the top) in Λ .

Proof. By definition, we have $K_{\Lambda^*}(x) = x^{\Lambda^*}$. It is then easy to see, again from the definition of Key polynomials, that $\pi_{(N-1,\alpha_m)}x^{\Lambda^*} = K_{\gamma}(x)$, where γ is the composition obtained by acting with the cycle $(\alpha_m, N, N-1, \ldots, \alpha_m+1)$ on the rows of Λ^* , that is, by sending row α_m to row N, and shifting rows α_m+1 to N one row up. The action of $\pi_{(N-m,\alpha_1)}\pi_{(N-m+1,\alpha_2)}\cdots\pi_{(N-1,\alpha_m)}$ on x^{Λ^*} thus produces $K_{\Lambda^s,\Lambda^a}(x)$, since (Λ^s,Λ^a) corresponds to the composition obtained by sending row α_i to row N-m+i, for $i=1,\ldots,m$, and shifting the remaining rows up if necessary. Finally, acting with $\pi_{\omega_{N-m}}$ on $K_{\Lambda^s,\Lambda^a}(x)$ simply reorders the entries of Λ^s in a non-decreasing way to produce $K_{(\Lambda^s)^R,\Lambda^a}(x)$.

Lemma 32. Using the notation of Section 4, we have

$$x^{\Lambda^*} e_{\ell} = \sum_{i_1 + i_2 + \dots + i_k = \ell} \pi_{i_1, I_1} \pi_{i_2, I_2} \cdots \pi_{i_k, I_k} x^{\Lambda^*} x^{(i_1, I_1)} x^{(i_2, I_2)} \cdots x^{(i_k, I_k)}$$
(A.2)

Proof. By construction, x^{Λ^*} commutes with $\pi_{i_1,I_1}\pi_{i_2,I_2}\cdots\pi_{i_k,I_k}$. We thus only need to prove that

$$e_{\ell} = \sum_{i_1 + i_2 + \dots + i_k = \ell} \pi_{i_1, I_1} \pi_{i_2, I_2} \cdots \pi_{i_k, I_k} x^{(i_1, I_1)} x^{(i_2, I_2)} \cdots x^{(i_k, I_k)}$$
(A.3)

Let X_I be the alphabet made out of the variables x_j for $j \in I$. It is well known (see for instance [12]) that

$$e_{\ell}(x_1, \dots, x_N) = \sum_{i_1 + i_2 + \dots + i_k = \ell} e_{i_1}(X_{I_1}) e_{i_2}(X_{I_2}) \cdots e_{i_k}(X_{I_k})$$
(A.4)

Since the π_{i_j,I_j} 's commute among themselves, the result will then follow if we can show that $e(X_{I_j}) = \pi_{i_j,I_j} x^{(i_j,I_j)}$ for all j. We only consider the case j=1, the remaining ones being similar. For simplicity we let $i_1=i$ and $I_1=\{1,\ldots,r\}$. By definition of Key polynomials, $K_{1^i,0^{r-i}}=x^{1^i,0^{r-i}}$. Hence, from (3.7), we have

$$s_{1^{i}}(x_{1},\ldots,x_{r})=K_{0^{r-i},1^{i}}=\pi_{[r-i,r-1]}\ldots\pi_{[2,i+1]}\pi_{[1,i]}K_{1^{i},0^{r-i}}=\pi_{[r-i,r-1]}\ldots\pi_{[2,i+1]}\pi_{[1,i]}x^{1^{i},0^{r-i}}$$
(A.5) and the result follows since it is well known that $e_{i}(x_{1},\ldots,x_{r})=s_{1^{i}}(x_{1},\ldots,x_{r}).$

Lemma 33. $\mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]}$ obeys the following properties:

$$\begin{array}{ll} (1) \ \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]}\pi_{\alpha_i-1} = \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},\ldots,\alpha_m]}. \\ (2) \ \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]}\pi_{\beta} = \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_m]} \ \text{if} \ \beta \not\in \{\alpha_1-1,\ldots,\alpha_m-1\}. \end{array}$$

where we use the notation of Section 4.

Proof. Let v = N - m. Consider first Case (1). We have

$$\mathcal{R}_{N,[\alpha_{1},...,\alpha_{m}]}\pi_{\alpha_{i}-1} = \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(N-1,\alpha_{m})}\pi_{\alpha_{i}-1}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-1,\alpha_{i})}\pi_{\alpha_{i}-1}\cdots\pi_{(N-1,\alpha_{m})}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-1,\alpha_{i}-1)}\cdots\pi_{(N-1,\alpha_{m})}
= \mathcal{R}_{N,[\alpha_{1},...,\alpha_{i}-1...,\alpha_{m}]}$$

Case (2) has three cases. If $\beta < \alpha_1 - 1$ then

$$\mathcal{R}_{N,[\alpha_{1},...,\alpha_{m}]}\pi_{\beta} = \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(N-1,\alpha_{m})}\pi_{\beta}
= \pi_{\omega_{v}}\pi_{\beta}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(N-1,\alpha_{m})}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(N-1,\alpha_{m})}
= \mathcal{R}_{N,[\alpha_{1},...,\alpha_{m}]}$$

If $\beta = \alpha_i$ for some i then

$$\mathcal{R}_{N,[\alpha_{1},...,\alpha_{m}]}\pi_{\alpha_{i}} = \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(N-1,\alpha_{m})}\pi_{\alpha_{i}}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-1,\alpha_{i})}\pi_{\alpha_{i}}\cdots\pi_{(N-1,\alpha_{m})}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-1,\alpha_{i})}\cdots\pi_{(N-1,\alpha_{m})}
= \mathcal{R}_{N,[\alpha_{1},...,\alpha_{m}]}$$

Finally, if $\alpha_i < \beta < \alpha_{i+1} - 1$ (the argument is the same for $\alpha_m < \beta$) then

$$\mathcal{R}_{N,[\alpha_{1},...,\alpha_{m}]}\pi_{\beta} = \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(N-1,\alpha_{m})}\pi_{\beta}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-1,\alpha_{i})}\pi_{\beta}\cdots\pi_{(N-1,\alpha_{m})}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-1,\beta-1)}\pi_{\beta}\pi_{(\beta-2,\alpha_{i})}\cdots\pi_{(N-1,\alpha_{m})}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-1,\beta+1)}\pi_{\beta}\pi_{\beta-1}\pi_{\beta}\pi_{(\beta-2,\alpha_{i})}\cdots\pi_{(N-1,\alpha_{m})}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-1,\beta+1)}\pi_{\beta-1}\pi_{\beta}\pi_{\beta-1}\pi_{(\beta-2,\alpha_{i})}\cdots\pi_{(N-1,\alpha_{m})}
= \pi_{\omega_{v}}\pi_{(v,\alpha_{1})}\pi_{(v+1,\alpha_{2})}\cdots\pi_{(v+i-2,\alpha_{i-1})}\pi_{\beta-1}\pi_{(v+i-1,\alpha_{i})}\cdots\pi_{(N-1,\alpha_{m})}$$

At this point, we repeat as in the second line until we get to the beginning of the product.

$$\pi_{\omega_v} \pi_{\beta - i} \pi_{(v,\alpha_1)} \pi_{(v+1,\alpha_2)} \cdots \pi_{(N-1,\alpha_m)} = \pi_{\omega_v} \pi_{(v,\alpha_1)} \pi_{(v+1,\alpha_2)} \cdots \pi_{(N-1,\alpha_m)}$$
$$= \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]}$$

Lemma 34. If $\mu_{\alpha_i} = \mu_{\alpha_i-1}$ then

$$\mathcal{R}_{N,[\alpha_1,\dots,\alpha_i,\dots,\alpha_m]}x^{\mu} = \mathcal{R}_{N,[\alpha_1,\dots,\alpha_i-1,\dots,\alpha_m]}x^{\mu}.$$
(A.6)

Proof. Since $\mu_{\alpha_i} = \mu_{\alpha_i-1}$, we have $x^{\mu} = \pi_{\alpha_i-1}x^{\mu}$. By Lemma 33, we then have that

$$\mathcal{R}_{N,[\alpha_1,\ldots,\alpha_i,\ldots,\alpha_m]}x^{\mu} = \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_i,\ldots,\alpha_m]}\pi_{\alpha_i-1}x^{\mu} = \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_i-1,\ldots,\alpha_m]}x^{\mu}.$$

Corollary 35. If there is no addable corner in row α_i of μ then there exists a row, call it α'_i such that $\mu_{\alpha_i} = \mu_{\alpha'_i}$ and there is an addable corner in row α'_i of μ . Furthermore,

$$\mathcal{R}_{N,[\alpha_1,\dots,\alpha_i,\dots,\alpha_m]} x^{\mu} = \mathcal{R}_{N,[\alpha_1,\dots,\alpha_i',\dots,\alpha_m]} x^{\mu}. \tag{A.7}$$

Proof. Follows by repeated applications of Lemma 34.

Lemma 36. We have

$$\pi_{(n-1,i)}\pi_{(n,i)} = \pi_n(\pi_{n-1}\pi_n)(\pi_{n-2}\pi_{n-1})\dots(\pi_i\pi_{i+1})$$
(A.8)

Proof. Easy by induction.

Lemma 37. If $\alpha_i = \alpha_{i+1}$ for some i then

$$\partial_{\omega_{(N-m)^c}} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} = 0 \tag{A.9}$$

Proof. Let v = N - m. We have

$$\partial_{\omega_{v^c}} \mathcal{R}_{N,[\alpha_1,\ldots,\alpha_i,\alpha_i,\ldots,\alpha_m]} = \partial_{\omega_{v^c}} \pi_{\omega_v} \pi_{(v,\alpha_1)} \cdots \pi_{(v+i-1,\alpha_i)} \pi_{(v+i,\alpha_i)} \cdots \pi_{(N-1,\alpha_m)}$$

Use Lemma 36 to rewrite $\pi_{(v+i-1,\alpha_i)}\pi_{(v+i,\alpha_i)}$ as π_{v+i} to the left times a string of isobaric divided differences. Then π_{v+i} commutes with every isobaric divided difference to its left, and the result follows given that $\partial_{\omega_{v}c}\pi_{v+i}=0$.

Lemma 38. For i < v,

$$x_v \pi_{\omega_{v-1}} \pi_{(v-1,i)} = \pi_{\omega_{v-1}} \pi_{(v-1,i+1)} (\pi_i - 1) x_i. \tag{A.10}$$

Proof. We first commute x_v and $\pi_{\omega_{v-1}}$. Then we use the relation $x_{r+1}\pi_r = (\pi_r - 1)x_r$ repeatedly to get $x_v\pi_{(v-1,i)} = (\pi_{v-1} - 1)(\pi_{v-2} - 1)\dots(\pi_i - 1)x_i$. Since $\pi_{\omega_{v-1}}(\pi_r - 1) = 0$ for all r < v - 1, we have

$$x_{v}\pi_{\omega_{v-1}}\pi_{(v-1,i)} = (\pi_{v-1} - 1)(\pi_{v-2} - 1)(\pi_{v-3} - 1)\dots(\pi_{i} - 1)x_{i}$$

$$= \pi_{v-1}(\pi_{v-2} - 1)(\pi_{v-3} - 1)\dots(\pi_{i} - 1)x_{i}$$

$$= \pi_{v-1}\pi_{v-2}(\pi_{v-3} - 1)\dots(\pi_{i} - 1)x_{i}$$
(A.11)

where we used the fact that π_{v-1} and $(\pi_{v-3}-1)$ commute. Continuing in this way, we get that $x_v\pi_{\omega_{v-1}}\pi_{(v-1,i)}=\pi_{\omega_{v-1}}\pi_{v-1}\pi_{v-2}\pi_{v-3}\cdots\pi_{i+1}(\pi_i-1)x_i$, and the lemma follows.

Recall that $e_{\ell}^{(v)}$ and $e_{\ell}^{(1,\dots,v)}$ were defined in Section 4. The following two lemmas are immediate.

Lemma 39.

$$e_{\ell}^{(v)} = e_{\ell} - x_v e_{\ell-1}^{(v)} \tag{A.12}$$

Lemma 40.

$$e_{\ell}^{(1,\dots,v)} = e_{\ell}^{(1,\dots,v-1)} - x_v e_{\ell-1}^{(1,\dots,v)}$$
 (A.13)

Lemma 41.

$$e_{\ell}^{(1,\dots,r)}\pi_r = \pi_r \left(e_{\ell}^{(1,\dots,r+1)} + x_r e_{\ell-1}^{(1,\dots,r+1)} \right) - x_r e_{\ell-1}^{(1,\dots,r+1)}$$
(A.14)

Proof. From Lemma 40, we have that

$$e_{\ell}^{(1,\dots,r)} = e_{\ell}^{(1,\dots,r+1)} + x_{r+1}e_{\ell-1}^{(1,\dots,r+1)}$$

and by the fact that $x_{r+1}\pi_r = (\pi_r - 1)x_r$, we have

$$\begin{array}{lcl} e_{\ell}^{(1,\ldots,r)}\pi_{r} & = & e_{\ell}^{(1,\ldots,r+1)}\pi_{r} + x_{r+1}e_{\ell-1}^{(1,\ldots,r+1)}\pi_{r} \\ & = & \pi_{r}e_{\ell}^{(1,\ldots,r+1)} + x_{r+1}\pi_{r}e_{\ell-1}^{(1,\ldots,r+1)} \\ & = & \pi_{r}e_{\ell}^{(1,\ldots,r+1)} + ((\pi_{r}-1)x_{r})e_{\ell-1}^{(1,\ldots,r+1)} \\ & = & \pi_{r}\left(e_{\ell}^{(1,\ldots,r+1)} + x_{r}e_{\ell-1}^{(1,\ldots,r+1)}\right) - x_{r}e_{\ell-1}^{(1,\ldots,r+1)} \end{array}$$

Lemma 42. We have

$$e_{\ell}^{(v)} \pi_{\omega_v} = \sum_{j=1}^{v} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)}$$
(A.15)

where we note that $e_0^{(1,2,\ldots,j)}=1$ and $e_n^{(1,2,\ldots,j)}=0$ for n<0 .

Proof. We proceed by induction on ℓ . If $\ell = 0$ then there is only the term j = 1 in the sum:

$$e_0^{(v)} \pi_{\omega_v} = \pi_{\omega_v} = \pi_{\omega_{v-1}} \pi_{(v-1,1)}$$
(A.16)

and the result is seen to hold. Now, suppose that the result is true for $k \leq \ell$. By Lemma 39, we have

$$e_{\ell+1}^{(v)} \pi_{\omega_v} = (e_{\ell+1} - x_v e_{\ell}^{(v)}) \pi_{\omega_v} = e_{\ell+1} \pi_{\omega_v} - x_v e_{\ell}^{(v)} \pi_{\omega_v}$$
(A.17)

Hence, by induction,

$$\begin{aligned} e_{\ell+1}^{(v)} \pi_{\omega_v} &= \pi_{\omega_v} e_{\ell+1} - x_v \sum_{j=1}^v \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)} \\ &= \pi_{\omega_v} e_{\ell+1} - x_v \pi_{\omega_{v-1}} x_1 \dots x_{v-1} e_{\ell-v+1}^{(1,2,\dots,v)} - x_v \sum_{j=1}^{v-1} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)} \\ &= \pi_{\omega_v} e_{\ell+1} - \pi_{\omega_{v-1}} x_1 \dots x_{v-1} x_v e_{\ell-v+1}^{(1,2,\dots,v)} - \sum_{j=1}^{v-1} x_v \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)} \end{aligned}$$

Then by Lemma 38, we have

$$\begin{split} e_{\ell+1}^{(v)} \pi_{\omega_v} &= \pi_{\omega_v} e_{\ell+1} - \pi_{\omega_{v-1}} x_1 \dots x_v e_{\ell-v+1}^{(1,2,\dots,v)} \\ &- \sum_{j=1}^{v-1} \pi_{\omega_{v-1}} \pi_{(v-1,j+1)} (\pi_j - 1) x_j x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)} \\ &= \pi_{\omega_v} e_{\ell+1} - \pi_{\omega_{v-1}} x_1 \dots x_v e_{\ell-v+1}^{(1,2,\dots,v)} - \sum_{j=1}^{v-1} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_j e_{\ell-j+1}^{(1,2,\dots,j)} \\ &+ \sum_{j=1}^{v-1} \pi_{\omega_{v-1}} \pi_{(v-1,j+1)} x_1 x_2 \dots x_j e_{\ell-j+1}^{(1,2,\dots,j)} \end{split}$$

We combine the second term with the sum to get

$$e_{\ell+1}^{(v)} \pi_{\omega_v} = \pi_{\omega_v} e_{\ell+1} - \sum_{j=1}^v \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_j e_{\ell-j+1}^{(1,2,\dots,j)} + \sum_{j=1}^{v-1} \pi_{\omega_{v-1}} \pi_{(v-1,j+1)} x_1 x_2 \dots x_j e_{\ell-j+1}^{(1,2,\dots,j)}$$

We then take out the first term of the first sum and renumber the second sum to obtain

$$e_{\ell+1}^{(v)} \pi_{\omega_v} = \pi_{\omega_v} e_{\ell+1} - \pi_{\omega_v} x_1 e_{\ell}^{(1)} - \sum_{j=2}^{v} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_j e_{\ell-j+1}^{(1,2,\dots,j)}$$

$$+ \sum_{j=2}^{v} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+2}^{(1,2,\dots,j-1)}$$

We now use Lemma 39 to combine the first two terms and combine the sums

$$e_{\ell+1}^{(v)}\pi_{\omega_v} = \pi_{\omega_v}e_{\ell+1}^{(1)} + \sum_{i=2}^v \pi_{\omega_{v-1}}\pi_{(v-1,j)}x_1x_2\dots x_{j-1}\left(e_{\ell-j+2}^{(1,2,\dots,j-1)} - x_je_{\ell-j+1}^{(1,2,\dots,j)}\right)$$

We use Lemma 40 to combine the terms within the sum to get

$$e_{\ell+1}^{(v)}\pi_{\omega_v} = \pi_{\omega_v}e_{\ell+1}^{(1)} + \sum_{j=2}^v \pi_{\omega_{v-1}}\pi_{(v-1,j)}x_1x_2\dots x_{j-1}e_{\ell-j+2}^{(1,2,\dots,j)}$$

We finally combine to make one sum

$$e_{\ell+1}^{(v)}\pi_{\omega_v} = \sum_{j=1}^v \pi_{\omega_{v-1}}\pi_{(v-1,j)}x_1x_2\dots x_{j-1}e_{\ell-j+2}^{(1,2,\dots,j)}$$

Lemma 43. Suppose $\alpha_d < j + d$ and let r = j + d. Then we have that

$$\partial_{\omega_{(v-1)^c}} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)} \prod_{i=1}^d \pi_{(v+i-1,\alpha_i)} =$$

$$(-1)^d \partial_{\omega_{(v-1)^c}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^d \pi_{(v+i-2,\alpha_i)} \right) \pi_{(v+d-1,r)} x_1 x_2 \dots x_{r-1} e_{\ell-r+1}^{(1,2,\dots,r)}$$
(A.18)

Proof. We prove the lemma by induction on d for a fixed j. The base case d=1 is easily seen to hold by applying the same logic as the inductive step. Now assume the lemma is true for $k \leq d$, and suppose $\alpha_{d+1} < j+d+1$. Then

$$\partial_{\omega_{(v-1)c}} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)} \prod_{i=1}^{d+1} \pi_{(v+i-1,\alpha_i)} =$$

$$(-1)^d \partial_{\omega_{(v-1)c}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^d \pi_{(v+i-2,\alpha_i)} \right) \pi_{(v+d-1,r)} x_1 x_2 \dots x_{r-1} e_{\ell-r+1}^{(1,2,\dots,r)} \pi_{(v+d,\alpha_{d+1})}$$
(A.19)

Since $\alpha_{d+1} < r+1$, then $\pi_{(v+d,\alpha_{d+1})}$ cannot commute with $x_1x_2 \dots x_{r-1}e_{\ell-r+1}^{(1,2,\dots,r)}$.

We use $\pi_{(v+d,\alpha_{d+1})} = \pi_{(v+d,r+1)}\pi_r\pi_{(r-1,\alpha_{d+1})}$ and then commute $\pi_{(v+d,r+1)}$ with $x_1x_2\dots x_{r-1}e_{\ell-r+1}^{(1,2,\dots,r)}$ to transform the right hand side of (A.19) into

$$(-1)^{d} \partial_{\omega_{(v-1)^{c}}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^{d} \pi_{(v+i-2,\alpha_{i})} \right) \pi_{(v+d-1,r)} \pi_{(v+d,r+1)} x_{1} x_{2} \dots x_{r-1} e_{\ell-r+1}^{(1,2,\dots,r)} \pi_{r} \pi_{(r-1,\alpha_{d+1})}$$

Now, applying Lemma 41, the previous expression becomes

$$(-1)^{d} \partial_{\omega_{(v-1)c}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^{d} \pi_{(v+i-2,\alpha_{i})} \right) \pi_{(v+d-1,r)} \pi_{(v+d,r+1)} \pi_{r}$$

$$\times x_{1} x_{2} \dots x_{r-1} \left(e_{\ell}^{(1,\dots,r+1)} + x_{r} e_{\ell-1}^{(1,\dots,r+1)} \right) \pi_{(r-1,\alpha_{d+1})}$$

$$+ (-1)^{d+1} \partial_{\omega_{(v-1)c}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^{d} \pi_{(v+i-2,\alpha_{i})} \right) \pi_{(v+d-1,r)} \pi_{(v+d,r+1)}$$

$$\times x_{1} x_{2} \dots x_{r-1} x_{r} e_{\ell-1}^{(1,\dots,r+1)} \pi_{(r-1,\alpha_{d+1})}$$

The first term of the expression is 0 because by Lemma 36 one can extract π_{v+d} from the left in $\pi_{(v+d-1,r)}\pi_{(v+d,r+1)}\pi_r$.

$$\partial_{\omega_{(v-1)^c}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^d \pi_{(v+i-2,\alpha_i)} \right) \pi_{(v+d-1,r)} \pi_{(v+d,r+1)} \pi_r = 0.$$

The second term is

$$(-1)^{d+1}\partial_{\omega_{(v-1)^c}}\pi_{\omega_{v-1}}\left(\prod_{i=1}^d(-1)\pi_{(v+i-2,\alpha_i)}\right)\pi_{(v+d-1,r)}\pi_{(v+d,r+1)}\pi_{(r-1,\alpha_{d+1})}x_1x_2\dots x_{r-1}x_re_{\ell-1}^{(1,\dots,r+1)}$$

Using $\pi_{(v+d-1,r)}\pi_{(v+d,r+1)}\pi_{(r-1,\alpha_{d+1})} = \pi_{(v+d-1,\alpha_{d+1})}\pi_{(v+d,r+1)}$, the left hand side of (A.19) finally becomes

$$(-1)^{d+1} \partial_{\omega_{(v-1)^c}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^{d+1} \pi_{(v+i-2,\alpha_i)} \right) \pi_{(v+d,r+1)} x_1 x_2 \dots x_r e_{\ell-1}^{(1,\dots,r+1)}$$

and the lemma holds.

Corollary 44. For a given j, let d be the maximum value such that $\alpha_d < j + d$. We then have

$$\partial_{\omega_{(v-1)^c}} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)} \prod_{i=1}^m \pi_{(v+i-1,\alpha_i)} =$$

$$(-1)^d \partial_{\omega_{(v-1)^c}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^d \pi_{(v+i-2,\alpha_i)} \right) \pi_{(v+d-1,r)} \left(\prod_{i=d+1}^m \pi_{(v+i-1,\alpha_i)} \right) x_1 x_2 \dots x_{r-1} e_{\ell-r+1}^{(1,2,\dots,r)}$$

$$(A.20)$$

where r = j + d.

Proof. We first break up the product
$$\prod_{i=1}^m \pi_{(v+i-1,\alpha_i)} = \prod_{i=1}^d \pi_{(v+i-1,\alpha_i)} \prod_{i=d+1}^m \pi_{(v+i-1,\alpha_i)}.$$

Using Lemma 43 to multiply by the first factor, the left hand side of (A.20) becomes

$$(-1)^{d} \partial_{\omega_{(v-1)^{c}}} \pi_{\omega_{v-1}} \left(\prod_{i=1}^{d} \pi_{(v+i-2,\alpha_{i})} \right) \pi_{(v+d-1,r)} x_{1} x_{2} \dots x_{r-1} e_{\ell-r+1}^{(1,2,\dots,r)} \prod_{i=d+1}^{m} \pi_{(v+i-1,\alpha_{i})}.$$

By hypothesis, $\alpha_{d+1} \geq j + d + 1$ which implies that the rightmost product in the previous expression commutes with $x_1 x_2 \dots x_{r-1} e_{\ell-r+1}^{(1,2,\dots,r)}$. The left hand side of (A.20) is thus equal to its right hand side and the result follows.

Corollary 45. Let v = N - m. We have

$$\partial_{\omega_{(v-1)^c}} e_{\ell}^{(v)} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} = \partial_{\omega_{(v-1)^c}} \sum_{r \notin \{\alpha_1,\dots,\alpha_m\}} (-1)^{\operatorname{pos}(r)} \mathcal{R}_{N,[\alpha_1,\dots,r,\dots,\alpha_m]} x_1 \dots x_{r-1} e_{\ell-r+1}^{(1,2,\dots,r)}$$
(A.21)

where we recall that pos(r) is the amount of elements of $\{\alpha_1, \ldots, \alpha_m\}$ that are less than r.

Proof.

$$\partial_{\omega_{(v-1)^c}} e_{\ell}^{(v)} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} = \partial_{\omega_{(v-1)^c}} e_{\ell}^{(v)} \pi_{\omega_v} \pi_{(v,\alpha_1)} \pi_{(v+1,\alpha_2)} \cdots \pi_{(N-1,\alpha_m)}$$

$$= \sum_{j=1}^v \partial_{\omega_{(v-1)^c}} \pi_{\omega_{v-1}} \pi_{(v-1,j)} x_1 x_2 \dots x_{j-1} e_{\ell-j+1}^{(1,2,\dots,j)} \pi_{(v,\alpha_1)} \pi_{(v+1,\alpha_2)} \cdots \pi_{(N-1,\alpha_m)}$$

by Lemma 42

For each j in the sum define d(j) to be the maximum value such that $\alpha_{d(j)} < j + d(j)$. If no such d(j) exists, then set d(j) = 0. Then by Corollary 44 we have

$$\partial_{\omega_{(v-1)^c}} e_{\ell}^{(v)} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} = \sum_{j=1}^v (-1)^{d(j)} \partial_{\omega_{(v-1)^c}} \pi_{\omega_{v-1}} \times \left(\prod_{i=1}^{d(j)} \pi_{(v+i-2,\alpha_i)} \right) \pi_{(v+d(j)-1,r(j))} \left(\prod_{i=d(j)+1}^m \pi_{(v+i-1,\alpha_i)} \right) x_1 x_2 \dots x_{r(j)-1} e_{\ell-r(j)+1}^{(1,2,\dots,r(j))}$$

where r(j) = j + d(j). We can simplify this by using

$$\mathcal{R}_{N,[\alpha_{1},...,r(j),...,\alpha_{m}]} = \pi_{\omega_{v-1}} \left(\prod_{i=1}^{d(j)} \pi_{(v+i-2,\alpha_{i})} \right) \pi_{(v+d(j)-1,r(j))} \left(\prod_{i=d(j)+1}^{m} \pi_{(v+i-1,\alpha_{i})} \right)$$

And so now we have

$$\partial_{\omega_{(v-1)^c}} e_{\ell}^{(v)} \mathcal{R}_{N,[\alpha_1,\dots,\alpha_m]} = \partial_{\omega_{(v-1)^c}} \sum_{j=1}^{v} (-1)^{d(j)} \mathcal{R}_{N,[\alpha_1,\dots,r(j),\dots,\alpha_m]} x_1 x_2 \dots x_{r(j)-1} e_{\ell-r(j)+1}^{(1,2,\dots,r(j))}$$

Note here that by construction, $\{r(1), \ldots, r(v)\}$ is precisely the complement of $\{\alpha_1, \ldots, \alpha_m\}$. Therefore, we can change the summation to sum over the complement of $\{\alpha_1, \ldots, \alpha_m\}$ and furthermore, we can set pos(r(j)) = d(j), which happens to agree with the definition in Section 4 to be the amount of numbers in $\{\alpha_1, \ldots, \alpha_m\}$ that are less than r(j).

So we conclude that

$$\partial_{\omega_{(v-1)^c}} e_{\ell}^{(v)} \mathcal{R}_{N,[\alpha_1,...,\alpha_m]} = \partial_{\omega_{(v-1)^c}} \sum_{r \not \in \{\alpha_1,...,\alpha_m\}} (-1)^{\text{pos}(r)} \mathcal{R}_{N,[\alpha_1,...,r,...,\alpha_m]} x_1 \dots x_{r-1} e_{\ell-r+1}^{(1,2,...,r)}.$$

Lemma 46. Let λ be a partition. Using the notation of Section 4.2, we have

$$x^{\lambda} h_k = \sum_{\mu} \left(\prod_{\substack{i \\ \mu_{i+1} = \lambda_i}}^{\text{(down)}} \pi_i \right) x^{\mu}$$
(A.22)

where the sum is over all partitions μ such that μ/λ is a horizontal k-strip.

Proof. We proceed by induction on the number of variables N. The result is easily checked when N=1. Now suppose (A.22) is true when the number of variables is less than N. We let $\lambda=(\lambda_1,\ldots,\lambda_N),\ \hat{\lambda}=(\lambda_1,\ldots,\lambda_{N-1}),\ d=\lambda_{N-1}-\lambda_N,$ and denote by $h_k^{[\ell]}$ the complete symmetric function $h_k(x_1,\ldots,x_\ell)$ in ℓ variables. Using the simple identity $h_k^{[N]}=x_N^d h_{k-d}^{[N]}+\sum_{j=0}^{d-1}x_N^j h_{k-j}^{[N-1]},$ we have

$$x^{\lambda}h_{k}^{[N]} = x^{\lambda} \left[x_{N}^{d}h_{k-d}^{[N]} + \sum_{j=0}^{d-1} x_{N}^{j}h_{k-j}^{[N-1]} \right]$$
$$= \left(x^{\hat{\lambda}}x_{N}^{\lambda_{N}} \right) x_{N}^{d} \left(\pi_{N-1}h_{k-d}^{(N-1)} \right) + \sum_{j=0}^{d-1} \left(x^{\hat{\lambda}}x_{N}^{\lambda_{N}} \right) x_{N}^{j}h_{k-j}^{[N-1]}$$
(A.23)

$$= \pi_{N-1} x_N^{\lambda_N + d} x^{\hat{\lambda}} h_{k-d}^{[N-1]} + \sum_{j=0}^{d-1} x_N^{\lambda_N + j} x^{\hat{\lambda}} h_{k-j}^{[N-1]}$$
(A.24)

where we used the fact that $x^{\hat{\lambda}}x_N^{\lambda_N+d}$, being symmetric in x_{N-1} and x_N , commutes with π_{N-1} . Hence, by induction,

$$\begin{split} x^{\lambda}h_k^{[N]} &= \pi_{N-1}x_N^{\lambda_N+d} \sum_{\substack{\nu/\hat{\lambda} \text{ is a} \\ \text{horiz. } k-d\text{-strip}}} \left(\prod_{\substack{i< N-1 \\ \nu_{i+1}=\lambda_i}}^{\text{(down)}} \pi_i\right) x^{\nu} + \sum_{j=0}^{d-1} x_N^{\lambda_N+j} \sum_{\substack{\nu/\hat{\lambda} \text{ is a} \\ \text{horiz. } k-j\text{-strip}}} \left(\prod_{\substack{i< N-1 \\ \nu_{i+1}=\lambda_i}}^{\text{(down)}} \pi_i\right) x^{\nu} \\ &= \sum_{\substack{\nu/\hat{\lambda} \text{ is a} \\ \text{horiz. } k-d\text{-strip}}} \left(\prod_{\substack{i=\lambda_i \\ \nu_{i+1}=\lambda_i}}^{\text{(down)}} \pi_i\right) x^{\nu,\lambda_N+d} + \sum_{j=0}^{d-1} \sum_{\substack{\nu/\hat{\lambda} \text{ is a} \\ \text{horiz. } k-j\text{-strip}}} \left(\prod_{\substack{i=\lambda_i \\ \nu_{i+1}=\lambda_i}}^{\text{(down)}} \pi_i\right) x^{\nu,\lambda_N+j} \\ &= \sum_{\substack{\mu/\lambda \text{ is a} \\ \text{horiz. } k\text{-strip}}} \left(\prod_{\substack{i=\lambda_i \\ \mu_{i+1}=\lambda_i}}^{\text{(down)}} \pi_i\right) x^{\mu} \end{split}$$

and the lemma follows.

Lemma 47. If $\beta > i$, then $\partial_{\omega_m} \pi_{\omega_m c} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_m-1]} \cdots \hat{\pi}_{[i+1,\alpha_{i+1}-1]} \hat{\pi}_{[i,\beta+1]} \hat{\pi}_{\beta} = 0$, where we use the notation of Section 4.2

Proof.

$$\begin{split} \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_m-1]} &\cdots \hat{\pi}_{[i+1,\alpha_{i+1}-1]} \hat{\pi}_{[i,\beta+1]} \hat{\pi}_{\beta} \\ &= \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_m-1]} &\cdots \hat{\pi}_{[i+1,\alpha_{i+1}-1]} \hat{\pi}_{[i,\beta-1]} \hat{\pi}_{\beta} \hat{\pi}_{\beta+1} \hat{\pi}_{\beta} \\ &= \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_m-1]} &\cdots \hat{\pi}_{[i+1,\alpha_{i+1}-1]} \hat{\pi}_{[i,\beta-1]} \hat{\pi}_{\beta+1} \hat{\pi}_{\beta} \hat{\pi}_{\beta+1} \\ &= \partial_{\omega_m} \pi_{\omega_{m^c}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_m-1]} &\cdots \hat{\pi}_{[i+1,\alpha_{i+1}-1]} \hat{\pi}_{\beta+1} \hat{\pi}_{[i,\beta+1]} \end{split}$$

We repeat the above process through each $\hat{\pi}_{[j,\alpha_i-1]}$ to get

$$\partial_{\omega_m} \pi_{\omega_m} \hat{\pi}_{\omega_m} \hat{\pi}_{\beta+m-i} \hat{\pi}_{[m,\alpha_m-1]} \cdots \hat{\pi}_{[i+1,\alpha_{i+1}-1]} \hat{\pi}_{[i,\beta+1]}$$

The result then holds since, for $\beta > i$, $\hat{\pi}_{\beta+m-i}$ commutes with $\hat{\pi}_{\omega_m}$ and is such that $\pi_{\omega_m c} \hat{\pi}_{\beta+m-i} = 0$ (given that $\pi_j \hat{\pi}_j = 0$).

Lemma 48. $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}$ obeys the following properties (we use the notation of Section 4.2)

- (1) $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}\pi_{\alpha_i} = \mathcal{P}_{N,[\alpha_1,...,\alpha_m]} + \mathcal{P}_{N,[\alpha_1,...,\alpha_i+1,...,\alpha_m]}$
- (2) $\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}\pi_{\alpha_i-1}=0$
- (3) $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\pi_\beta = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}$ for $\beta \notin \{\alpha_i,\alpha_i-1\}$ for $i=1,\ldots,m$.
- $(4) \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_i,\ldots,\alpha_m]} + \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_i+i_j,\ldots,\alpha_m]} \text{ if } I_j \text{ starts with } \alpha_i.$
- (5) $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}$ if I_j starts with β such that $\beta \notin \{\alpha_1,\ldots,\alpha_m\}$.

Proof. To prove (1), consider

$$\mathcal{P}_{N,[\alpha_{1},...,\alpha_{m}]}\pi_{\alpha_{i}} = \partial_{\omega_{m}}\pi_{\omega_{m}c}\hat{\pi}_{\omega_{m}}\hat{\pi}_{[m,\alpha_{m}-1]}\cdots\hat{\pi}_{[1,\alpha_{1}-1]}\pi_{\alpha_{i}}$$

$$= \partial_{\omega_{m}}\pi_{\omega_{m}c}\hat{\pi}_{\omega_{m}}\hat{\pi}_{[m,\alpha_{m}-1]}\cdots\hat{\pi}_{[i,\alpha_{i}-1]}\pi_{\alpha_{i}}\hat{\pi}_{[i-1,\alpha_{i-1}-1]}\cdots\hat{\pi}_{[1,\alpha_{1}-1]}$$

$$= \partial_{\omega_{m}}\pi_{\omega_{m}c}\hat{\pi}_{\omega_{m}}\hat{\pi}_{[m,\alpha_{m}-1]}\cdots\hat{\pi}_{[i,\alpha_{i}-1]}(1+\hat{\pi}_{\alpha_{i}})\hat{\pi}_{[i-1,\alpha_{i-1}-1]}\cdots\hat{\pi}_{[1,\alpha_{1}-1]}$$

$$= \mathcal{P}_{N,[\alpha_{1},...,\alpha_{m}]} + \mathcal{P}_{N,[\alpha_{1},...,\alpha_{i}+1,...,\alpha_{m}]}$$

For (2), we have

$$\mathcal{P}_{N,[\alpha_{1},...,\alpha_{m}]}\pi_{\alpha_{i}-1} = \partial_{\omega_{m}}\pi_{\omega_{m}c}\hat{\pi}_{\omega_{m}}\hat{\pi}_{[m,\alpha_{m}-1]}\cdots\hat{\pi}_{[i,\alpha_{i}-1]}\pi_{\alpha_{i}-1}\hat{\pi}_{[i-1,\alpha_{i-1}-1]}\cdots\hat{\pi}_{[1,\alpha_{1}-1]} = 0$$
 since $\hat{\pi}_{\alpha_{i}-1}\pi_{\alpha_{i}-1} = 0$.

For (3), suppose that $\alpha_i < \beta < \alpha_{i+1} - 1$ (the cases $\beta < \alpha_1 - 1$ and $\beta > \alpha_m$ are similar). We have

$$\mathcal{P}_{N,[\alpha_{1},...,\alpha_{m}]}\pi_{\beta} = \partial_{\omega_{m}}\pi_{\omega_{m}c}\hat{\pi}_{\omega_{m}}\hat{\pi}_{[m,\alpha_{m}-1]}\cdots\hat{\pi}_{[i,\beta+1]}\pi_{\beta}\hat{\pi}_{[\beta+2,\alpha_{i}-1]}\hat{\pi}_{[i-1,\alpha_{i-1}-1]}\cdots\hat{\pi}_{[1,\alpha_{1}-1]}$$

$$= \partial_{\omega_{m}}\pi_{\omega_{m}c}\hat{\pi}_{\omega_{m}}\hat{\pi}_{[m,\alpha_{m}-1]}\cdots\hat{\pi}_{[i,\beta+1]}(\hat{\pi}_{\beta}+1)\hat{\pi}_{[\beta+2,\alpha_{i}-1]}\hat{\pi}_{[i-1,\alpha_{i-1}-1]}\cdots\hat{\pi}_{[1,\alpha_{1}-1]}$$

$$= \mathcal{P}_{N,[\alpha_{1},...,\alpha_{m}]}$$

since $\partial_{\omega_m} \pi_{\omega_m c} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_{m-1}]} \cdots \hat{\pi}_{[i,\beta+1]} \hat{\pi}_{\beta} = 0$ by Lemma 47 (observe that the lemma applies since $i \leq \alpha_i < \beta$).

To prove (4), let $I_j = [a, b]$ with $a = \alpha_i$. By definition, we have

$$\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{[b-i_j,b-1]}\cdots\boldsymbol{\pi}_{[\alpha_i+1,\alpha_i+i_j]}\boldsymbol{\pi}_{[\alpha_i,\alpha_i+i_j-1]}$$

Now, by construction of I_j , we have $b < \alpha_{i+1}$ since rows α_i and α_{i+1} need to be distinct. Hence, $\pi_{[b-i_j,b-1]} \cdots \pi_{[\alpha_i+1,\alpha_i+i_j]}$ acts as the identity on $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}$ from (3). We thus have

$$\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{[\alpha_i,\alpha_i+i_j-1]}.$$

By successive applications of (1) and (2) we finally get

$$\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]} + \mathcal{P}_{N,[\alpha_1,\ldots,\alpha_i+i_j,\ldots,\alpha_m]}$$

since $\alpha_i + i_j \leq b < \alpha_{i+1}$ by construction.

We finally prove (5). Let $I_j = [a, b]$. By construction, we have that $\alpha_i < a \le b < \alpha_{i+1}$ for some i (the cases $a \le b < \alpha_1$ and $\alpha_m < a \le b$ are similar). By definition, we have

$$\mathcal{P}_{N,[\alpha_1,...,\alpha_m]} \boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,...,\alpha_m]} \boldsymbol{\pi}_{[b-i_j,b-1]} \cdots \boldsymbol{\pi}_{[a+1,a+i_j]} \boldsymbol{\pi}_{[a,a+i_j-1]}$$
(A.25)

As can be seen in the previous equation, of the π_j 's acting from the right on $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}$, the highest j is such that $j=b-1<\alpha_{i+1}-1$ while the lowest j is such that $\alpha_i< a=j$. We thus get from (3) that all π_j 's in (A.25) act as the identity on $\mathcal{P}_{N,[\alpha_1,\ldots,\alpha_m]}$, which gives

$$\mathcal{P}_{N,[\alpha_1,...,\alpha_m]}\boldsymbol{\pi}_{i_j,I_j} = \mathcal{P}_{N,[\alpha_1,...,\alpha_m]}$$

Recall that for $S \subseteq \{1, 2, ..., N\}$, e_{ℓ}^{S} and h_{ℓ}^{S} respectively stand for $e_{\ell}(x_1, ..., x_N)$ and $h_{\ell}(x_1, ..., x_N)$ with $x_i = 0$ for $i \notin S$, that is, to e_{ℓ} and h_{ℓ} in the variables x_i for $i \in S$. The next two lemmas, being elementary, are stated without proof.

Lemma 49.

$$e_{\ell}^{(m+1)} = \sum_{k=0}^{\ell} (-1)^{\ell-k} x_{m+1}^{\ell-k} e_k$$

Lemma 50.

$$e_{\ell}^{\{1,...,i\}} = \sum_{k=0}^{\ell} (-1)^{\ell-k} h_{\ell-k}^{\{i+1,...,N\}} e_k$$

Lemma 51.

$$h_k^{\{i+1,...,N\}}\pi_N=h_k^{\{i+1,...,N+1\}}+\hat{\pi}_N h_k^{\{i+1,...,N-1,N+1\}}$$

Proof.

$$\begin{split} h_k^{\{i+1,\dots,N\}} \pi_N &= \sum_{i=0}^k x_N^i h_{k-i}^{\{i+1,\dots,N-1\}} \pi_N \\ &= \sum_{i=0}^k x_N^i \pi_N h_{k-i}^{\{i+1,\dots,N-1\}} \\ &= \sum_{i=0}^k \left(h_i^{\{N,N+1\}} + \hat{\pi}_N x_{N+1}^i \right) h_{k-i}^{\{i+1,\dots,N-1\}} \\ &= \sum_{i=0}^k h_i^{\{N,N+1\}} h_{k-i}^{\{i+1,\dots,N-1\}} + \hat{\pi}_N \sum_{i=0}^k x_{N+1}^i h_{k-i}^{\{i+1,\dots,N-1\}} \\ &= h_k^{\{i+1,\dots,N+1\}} + \hat{\pi}_N h_k^{\{i+1,\dots,N-1,N+1\}} \end{split}$$

Lemma 52.

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} x_{m+1}^k \pi_{[m+1,N-1]} = \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \sum_{i=m}^{N-1} \hat{\pi}_{[m+1,i]} h_k^{\{i+1,\dots,N\}}$$

Proof. Proof by induction on N. For the base case, let N = m + 1 then we have

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} x_{m+1}^k \pi_{[m+1,m]} = \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} x_{m+1}^k$$

and
$$x_{m+1}^k = h_k^{\{m+1\}}$$

Now suppose it is true for N and lets show that it is true for N+1. (Note here that $\pi_{\omega_{(m+1)^c}}$ depends on N so lets say $\pi^N_{\omega_{(m+1)^c}}$ is for N variables. Also note that $\pi^{N+1}_{\omega_{(m+1)^c}} = \pi_{[m+2,N]} \pi^N_{\omega_{(m+1)^c}}$.) So we have

$$\begin{array}{ll} \partial_{\omega_{m+1}} \pi^{N+1}_{\omega_{(m+1)^c}} x^k_{m+1} \pi_{[m+1,N]} &= \partial_{\omega_{m+1}} \pi_{[m+2,N]} \pi^N_{\omega_{(m+1)^c}} x^k_{m+1} \pi_{[m+1,N-1]} \pi_N \\ &= \pi_{[m+2,N]} \partial_{\omega_{m+1}} \pi^N_{\omega_{(m+1)^c}} x^k_{m+1} \pi_{[m+1,N-1]} \pi_N \end{array}$$

Now use the induction hypothesis to get

$$=\pi_{[m+2,N]}\partial_{\omega_{m+1}}\pi^{N}_{\omega_{(m+1)^{c}}}\sum_{i=m}^{N-1}\hat{\pi}_{[m+1,i]}h^{\{i+1,...,N\}}_{k}\pi_{N}.$$

And now use Lemma 51 to get

$$= \partial_{\omega_{m+1}} \pi_{[m+2,N]} \pi_{\omega_{(m+1)^c}}^N \sum_{i=m}^{N-1} \hat{\pi}_{[m+1,i]} \left(h_k^{\{i+1,\dots,N+1\}} + \hat{\pi}_N h_k^{\{i+1,\dots,N-1,N+1\}} \right)$$

$$= \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}}^{N+1} \left(\sum_{i=m}^{N-1} \hat{\pi}_{[m+1,i]} h_k^{\{i+1,\dots,N+1\}} + \sum_{i=m}^{N-1} \hat{\pi}_{[m+1,i]} \hat{\pi}_N h_k^{\{i+1,\dots,N-1,N+1\}} \right).$$

Notice that in the second sum, the π_N commutes with $\hat{\pi}_{[m+1,i]}$ when $i \neq N-1$ and $\pi_{\omega_{(m+1)^c}}^{N+1} \hat{\pi}_N = 0$ so only the last summand survives and we have

$$= \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}}^{N+1} \left(\sum_{i=m}^{N-1} \hat{\pi}_{[m+1,i]} h_k^{\{i+1,\dots,N+1\}} + \hat{\pi}_{[m+1,N-1]} \hat{\pi}_N h_k^{\{N+1\}} \right)$$

$$= \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}}^{N+1} \sum_{i=m}^{N} \hat{\pi}_{[m+1,i]} h_k^{\{i+1,\dots,N+1\}}.$$

Lemma 53.

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} e_{\ell}^{(m+1)} \pi_{[m+1,N-1]} = \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \sum_{i=1}^{N-m} \hat{\pi}_{[m+1,m+i-1]} e_{\ell}^{\{1,\dots,m+i-1\}}$$
(A.26)

Proof. For simplicity, assume that each expression begins with $\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}}$. By Lemma 49, we have

$$e_{\ell}^{(m+1)}\pi_{[m+1,N-1]} = \left(\sum_{k=0}^{\ell} (-1)^{\ell-k} x_{m+1}^{\ell-k} e_k\right) \pi_{[m+1,N-1]}$$

Commuting e_k and $\pi_{[m+1,N-1]}$ and using Lemma 52, we obtain

$$e_{\ell}^{(m+1)}\pi_{[m+1,N-1]} = \sum_{k=0}^{\ell} (-1)^{\ell-k} \left(\sum_{i=m}^{N-1} \hat{\pi}_{[m+1,i]} h_{\ell-k}^{\{i+1,\dots,N\}} \right) e_k$$

Switching the order of the sums, we get by Lemma 50 that

$$e_{\ell}^{(m+1)}\pi_{[m+1,N-1]} = \sum_{i=m}^{N-1} \hat{\pi}_{[m+1,i]} e_{\ell}^{\{1,\dots,i\}}$$

The lemma is then seen to hold after reindexing the sum and putting back $\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}}$ on both sides of the equation.

The following lemma is immediate.

Lemma 54.

$$e_{\ell}^{\{1,\dots,k\}} = e_{\ell}^{\{1,\dots,k-1\}} + x_k e_{\ell-1}^{\{1,\dots,k-1\}}$$
(A.27)

Lemma 55.

$$e_{\ell}^{\{1,\dots,k\}}\hat{\pi}_{k} = \hat{\pi}_{k}e_{\ell}^{\{1,\dots,k-1\}} + \pi_{k} x_{k+1}e_{\ell-1}^{\{1,\dots,k-1\}} \tag{A.28}$$

Proof. The lemma is immediate from the previous lemma and the identity $x_k \hat{\pi}_k = \pi_k x_{k+1}$.

Lemma 56. We have

$$\hat{\pi}_{[a,b]}\hat{\pi}_{[a-1,b-1]}\hat{\pi}_b = \hat{\pi}_{a-1}\hat{\pi}_{[a,b]}\hat{\pi}_{[a-1,b-1]} \tag{A.29}$$

Proof. Easy by induction.

Lemma 57. If $h + 1 \le \alpha \le \beta$ and $h \le m$, then

$$\partial_{\omega_{m+1}}\hat{\pi}_{[h+1,\alpha]}\hat{\pi}_{[h,\beta]} = -\partial_{\omega_{m+1}}\hat{\pi}_{[h+1,\beta]}\hat{\pi}_{[h,\alpha-1]}$$
(A.30)

Proof. We have from the previous lemma

$$\hat{\pi}_{[h+1,\alpha]}\hat{\pi}_{[h,\beta]} = \hat{\pi}_{[h+1,\alpha]}\hat{\pi}_{[h,\alpha-1]}\hat{\pi}_{\alpha}\hat{\pi}_{\alpha+1}\cdots\hat{\pi}_{\beta} = \hat{\pi}_{h}\hat{\pi}_{[h+1,\alpha]}\hat{\pi}_{[h,\alpha-1]}\hat{\pi}_{\alpha+1}\cdots\hat{\pi}_{\beta}$$

Hence, $\hat{\pi}_{[h+1,\alpha]}\hat{\pi}_{[h,\beta]} = \hat{\pi}_h\hat{\pi}_{[h+1,\beta]}\hat{\pi}_{[h,\alpha-1]}$ and the lemma follows since $\partial_{\omega_{m+1}}\hat{\pi}_h = -\partial_{\omega_{m+1}}$.

Lemma 58. Suppose that $\alpha_h \geq h + j$. Then, for $1 \leq h \leq m$, we have

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)c}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m+1,m+j-1]} e_{\ell}^{\{1,\dots,m+j-1\}} \left(\prod_{i=m}^{h} \hat{\pi}_{[i,\alpha_i-1]} \right)$$

$$= (-1)^{m+1-h} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)c}} \hat{\pi}_{\omega_m} \left(\prod_{i=m}^{h} \hat{\pi}_{[i+1,\alpha_i-1]} \right) \hat{\pi}_{[h,h+j-2]} e_{\ell}^{\{1,\dots,h+j-2\}}$$
(A.31)

where the products are decreasing.

Proof. We prove the lemma by induction on h for a fixed j in the decreasing direction. We will not show the base case h=m since it can be proven using the same ideas as in the general case. Therefore, suppose the lemma holds for h and we will show that it holds for h-1. Assume $\alpha_{h-1} \geq h-1+j$. We then have

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m+1,m+j-1]} e_{\ell}^{\{1,\dots,m+j-1\}} \left(\prod_{i=m}^{h-1} \hat{\pi}_{[i,\alpha_i-1]} \right)$$

$$= \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m+1,m+j-1]} e_{\ell}^{\{1,\dots,m+j-1\}} \left(\prod_{i=m}^{h} \hat{\pi}_{[i,\alpha_i-1]} \right) \hat{\pi}_{[h-1,\alpha_{h-1}-1]}$$

$$= (-1)^{m+1-h} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \left(\prod_{i=m}^{h} \hat{\pi}_{[i+1,\alpha_i-1]} \right) \hat{\pi}_{[h,h+j-2]} e_{\ell}^{\{1,\dots,h+j-2\}} \hat{\pi}_{[h-1,\alpha_{h-1}-1]}$$

Since $\alpha_{h-1} \ge h-1+j$, $\hat{\pi}_{[h-1,\alpha_{h-1}-1]}$ does not commute with $e_{\ell}^{\{1,\dots,h+j-2\}}$. Using $\hat{\pi}_{[h-1,\alpha_{h-1}-1]} = \hat{\pi}_{[h-1,h+j-3]}\hat{\pi}_{h+j-2}\hat{\pi}_{[h+j-1,\alpha_{h-1}-1]}$ and Lemma 55, the right hand side of the previous equation becomes

$$(-1)^{m+1-h}\partial_{\omega_{m+1}}\pi_{\omega_{(m+1)^c}}\hat{\pi}_{\omega_m}\left(\prod_{i=m}^h\hat{\pi}_{[i+1,\alpha_i-1]}\right)\hat{\pi}_{[h,h+j-2]}\hat{\pi}_{[h-1,h+j-3]}\times$$

$$\left(\hat{\pi}_{h+j-2}e_{\ell}^{\{1,\dots,h+j-3\}}+\pi_{h+j-2}x_{h+j-1}e_{\ell-1}^{\{1,\dots,h+j-3\}}\right)\hat{\pi}_{[h+j-1,\alpha_{h-1}-1]}$$
(A.33)

The second term of the sum is zero from Lemma 56 (note that $\pi_a = \hat{\pi}_a + 1$ satisfies the usual braid relation with $\hat{\pi}_b$) and the fact that $\partial_{\omega_{m+1}} \pi_m = 0$. The previous expression is thus equal to

$$(-1)^{m+1-h} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \left(\prod_{i=m}^h \hat{\pi}_{[i+1,\alpha_i-1]} \right) \hat{\pi}_{[h,h+j-2]} \hat{\pi}_{[h-1,h+j-3]} \times$$

$$\hat{\pi}_{h+j-2} e_{\ell}^{\{1,\dots,h+j-3\}} \hat{\pi}_{[h+j-1,\alpha_{h-1}-1]}$$
(A.34)

Now, $e_{\ell}^{\{1,\dots,h+j-3\}}$ commutes and the expression becomes

$$(-1)^{m+1-h}\partial_{\omega_{m+1}}\pi_{\omega_{(m+1)^c}}\hat{\pi}_{\omega_m}\left(\prod_{i=m}^h\hat{\pi}_{[i+1,\alpha_i-1]}\right)\left(\hat{\pi}_{[h,h+j-2]}\hat{\pi}_{[h-1,\alpha_{h-1}-1]}\right)e_{\ell}^{\{1,\dots,(h-1)+j-2\}}$$

Using Lemma 57, we finally get for the right hand side of (A.32)

$$(-1)^{m+1-h}\partial_{\omega_{m+1}}\pi_{\omega_{(m+1)c}}\hat{\pi}_{\omega_{m}}\left(\prod_{i=m}^{h}\hat{\pi}_{[i+1,\alpha_{i}-1]}\right) \times$$

$$\left((-1)\hat{\pi}_{[h,\alpha_{h-1}-1]}\hat{\pi}_{[h-1,(h-1)+j-2]}\right)e_{\ell}^{\{1,\dots,(h-1)+j-2\}}$$

$$(-1)^{m+1-(h-1)}\partial_{\omega_{m+1}}\pi_{\omega_{(m+1)c}}\hat{\pi}_{\omega_{m}}\left(\prod_{i=m}^{h-1}\hat{\pi}_{[i+1,\alpha_{i}-1]}\right)\hat{\pi}_{[h-1,(h-1)+j-2]}e_{\ell}^{\{1,\dots,(h-1)+j-2\}}$$

and the lemma holds. $\hfill\Box$

Corollary 59. For a given j, let h be the minimum value such that $\alpha_h \geq h + j$. Then

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m+1,m+j-1]} e_{\ell}^{\{1,\dots,m+j-1\}} \left(\prod_{i=m}^{1} \hat{\pi}_{[i,\alpha_i-1]} \right)$$

$$= (-1)^{m+1-h} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \left(\prod_{i=m}^{h} \hat{\pi}_{[i+1,\alpha_i-1]} \right) \hat{\pi}_{[h,h+j-2]} \left(\prod_{i=h-1}^{1} \hat{\pi}_{[i,\alpha_i-1]} \right) e_{\ell}^{\{1,\dots,h+j-2\}}$$

Proof. We first break up the product as $\prod_{i=m}^{1} \hat{\pi}_{[i,\alpha_i-1]} = \left(\prod_{i=m}^{h} \hat{\pi}_{[i,\alpha_i-1]}\right) \left(\prod_{i=h-1}^{1} \hat{\pi}_{[i,\alpha_i-1]}\right)$. Using Lemma 58 to multiply by the first factor, we get

$$(-1)^{m+1-h} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \left(\prod_{i=m}^h \hat{\pi}_{[i+1,\alpha_i-1]} \right) \hat{\pi}_{[h,h+j-2]} e_{\ell}^{\{1,\dots,h+j-2\}} \left(\prod_{i=h-1}^1 \hat{\pi}_{[i,\alpha_i-1]} \right).$$

By hypothesis $\alpha_{h-1} < h-1+j$, which implies that the rightmost product in the previous expression commutes with $e_\ell^{\{1,\dots,h+j-2\}}$. Therefore we have

$$(-1)^{m+1-h}\partial_{\omega_{m+1}}\pi_{\omega_{(m+1)c}}\hat{\pi}_{\omega_m}\left(\prod_{i=m}^h\hat{\pi}_{[i+1,\alpha_i-1]}\right)\hat{\pi}_{[h,h+j-2]}\left(\prod_{i=h-1}^1\hat{\pi}_{[i,\alpha_i-1]}\right)e_{\ell}^{\{1,\dots,h+j-2\}}.$$

Lemma 60. We have

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \sum_{j=1}^{N-m} \hat{\pi}_{[m+1,m+j-1]} e_{\ell}^{\{1,\dots,m+j-1\}} \hat{\pi}_{\omega_m} \hat{\pi}_{[m,\alpha_{m-1}]} \dots \hat{\pi}_{[1,\alpha_{1}-1]} x^{\Lambda^*}$$

$$= \sum_{r \notin \{\alpha_1,\dots,\alpha_m\}} (-1)^{\operatorname{pos}(r)} \mathcal{P}_{N,[\alpha_1,\dots,r,\dots,\alpha_m]} e_{\ell}^{\{1,\dots,r-1\}} x^{\Lambda^*}$$
(A.37)

Proof. For each j in the sum on the left hand side of (A.37), define $h(j) \ge 1$ to be the minimum value such that $\alpha_{h(j)} \ge h(j) + j$. If no such h(j) exists, set h(j) = m + 1. Then, using Corollary 59 and straightforward manipulations, the left hand side of (A.37) can be reexpressed as

$$\begin{split} \sum_{j=1}^{N-m} (-1)^{m+1-h(j)} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \times \\ \left(\prod_{i=m}^{h(j)} \hat{\pi}_{[i+1,\alpha_i-1]} \right) \hat{\pi}_{[h(j),h(j)+j-2]} \left(\prod_{i=h(j)-1}^{1} \hat{\pi}_{[i,\alpha_i-1]} \right) e_{\ell}^{\{1,...,h(j)+j-2\}} x^{\Lambda^*} \\ = \sum_{j=1}^{N-m} (-1)^{m+1-h(j)} \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} \times \\ \left(\prod_{i=m}^{h(j)} \hat{\pi}_{[i+1,\alpha_i-1]} \right) \hat{\pi}_{[h(j),r(j)-1]} \left(\prod_{i=h(j)-1}^{1} \hat{\pi}_{[i,\alpha_i-1]} \right) e_{\ell}^{\{1,...,r(j)-1\}} x^{\Lambda^*} \end{split}$$

where we used the substitution r(j) = h(j) + j - 1 in the last sum. Observe that we almost have the form for

$$\mathcal{P}_{N,[\alpha_{1},...,r(j),...,\alpha_{m}]} = \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^{c}}} \hat{\pi}_{\omega_{m+1}} \left(\prod_{i=m}^{h(j)} \hat{\pi}_{[i+1,\alpha_{i}-1]} \right) \hat{\pi}_{[h(j),r(j)-1]} \left(\prod_{i=h(j)-1}^{1} \hat{\pi}_{[i,\alpha_{i}-1]} \right)$$

in the last expression except that we need $\hat{\pi}_{\omega_{m+1}}$ instead of $\hat{\pi}_{\omega_m}$. Using $\hat{\pi}_{\omega_{m+1}} = \hat{\pi}_1 \cdots \hat{\pi}_m \hat{\pi}_{\omega_m}$, we have

$$\partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_m} = (-1)^m \partial_{\omega_{m+1}} \pi_{\omega_{(m+1)^c}} \hat{\pi}_{\omega_{m+1}}$$

The left hand side of (A.37) can thus be rewritten as

$$\sum_{j=1}^{N-m} (-1)^{m+1-h(j)} (-1)^m \mathcal{P}_{N,[\alpha_1,\dots,r(j),\dots,\alpha_m]} e_{\ell}^{\{1,\dots,r(j)-1\}} x^{\Lambda^*}$$

$$= \sum_{j=1}^{N-m} (-1)^{h(j)-1} \mathcal{P}_{N,[\alpha_1,\dots,r(j),\dots,\alpha_m]} e_{\ell}^{\{1,\dots,r(j)-1\}} x^{\Lambda^*}$$

Note here that by construction, $\{r(1), \ldots, r(N-m)\}$ is precisely the complement of $\{\alpha_1, \ldots, \alpha_m\}$. Therefore, we can change the summation to sum over the complement of $\{\alpha_1, \ldots, \alpha_m\}$ and furthermore, we can set pos(r(j)) = h(j) - 1, which happens to agree with the definition of pos(r) given in Section 4 (which corresponds to the number of elements of $\{\alpha_1, \ldots, \alpha_m\}$ smaller than r). Hence the left hand side of (A.37) becomes

$$\sum_{r \notin \{\alpha_1, \dots, \alpha_m\}} (-1)^{\operatorname{pos}(r)} \mathcal{P}_{N, [\alpha_1, \dots, r, \dots, \alpha_m]} e_{\ell}^{\{1, \dots, r-1\}} x^{\Lambda^*}$$

and the lemma holds.

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