Math 240A, Fall 2019

Solution to Problems of HW#5.

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1. Since $f_n \to f$ uniformly, $f_n \to f$ pointwise. Hence f is measurable. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, $\exists N \in \mathbb{N}$ such that $n \geq N \Longrightarrow |f_n(x) - f(x)| \leq \varepsilon$ $\forall x \in X$. Thus,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)|$$
$$\leq \varepsilon + |f_N(x)| \quad \forall x \in X.$$

Since $\mu(X) < \infty$ and $f_N \in L^1(\mu)$, we have

$$\int |f| \, d\mu \leq \varepsilon \mu(X) + \int |f_N| \, d\mu < \infty.$$

So $f \in L^1(\mu)$. Moreover, if $n \geq N$, then

$$\int_X |f_n - f| \, d\mu \leq \int_X \varepsilon \, d\mu = \varepsilon \mu(X).$$

Since $\mu(X) < \infty$ and ε is arbitrarily, $f_n \to f$ in $L^1(\mu)$.

If $\mu(X) = \infty$ then f may not be in $L^1(\mu)$.
Example: $f(x) = \frac{1}{1+|x|}$ $(x \in \mathbb{R})$. $f_n = \chi_{(-n,n)} f$.
$X = \mathbb{R}$, $\mathcal{M} = \mathcal{L}$, $\mu = m$. $\square$

2. We have $g_n + f_n \geq 0$ and $g_n - f_n \geq 0$ on X.
Fatou's lemma implies that

$$\int \liminf_n (g_n + f_n) \leq \liminf_n \int (g_n + f_n)$$
$$\int \lim_n (g_n - f_n) \leq \liminf_n \int (g_n - f_n).$$

Hence
$$\int g + \int f \le \int g + \liminf_n \int f_n,$$
$$\int g - \int f \le \int g - \limsup_n \int f_n.$$
Thus $\limsup_n \int f_n \le \int f \le \liminf_n \int f_n$.

Hence $\lim_{n\to\infty} \int f_n = \int f$. $\square$

3. If $\int |f_n - f|\,d\mu \to 0$ then
$$\left| \int |f_n|\,d\mu - \int |f|\,d\mu \right| = \left| \int (|f_n| - |f|)\,d\mu \right|$$
$$\le \int |f_n - f_n|\,d\mu \to 0.$$
Suppose $\int |f_n| \to \int |f|$. Let $F_n = |f_n - f|$, $F = 0$, $G_n = |f_n| + |f|$, $G = 2|f|$. Then $F_n \to F$ a.e. $G_n \to G$ a.e. $F_n \le G_n$ $(n = 1, 2, \cdots)$ and all $F_n, F, G_n, G \in L^1(\mu)$ and $\int G_n \to \int G$. Thus by Prob. 2 replacing f_n, f, g_n, g there by F_n, F, G_n, G respectively, we have $\int F_n \to \int F$ i.e. $\int |f_n - f|\,d\mu \to 0$. $\square$

4. Let $x \in \mathbb{R}$ and $x_n \downarrow x$ (i.e., x_n decreases and converges to x) Then
$$|F(x_n) - F(x)| = \left| \int_{(x, x_n]} f\,dm \right|$$
$$\le \int_{(x, x_n]} |f|\,dm = \int_{\mathbb{R}} \chi_{(x, x_n]} |f|\,dm$$
Since $\chi_{(x,x_n]} |f| \to 0$ a.e. $\mathbb{R}$.
$$|\chi_{(x,x_n]} |f|| \le |f| \quad \text{and } f \in L^1(m).$$
The Dominated Convergence Theorem implies that $\int_{\mathbb{R}} \chi_{(x,x_n]} |f|\,dm \to 0$. Hence F is right continuous at x. Similarly F is left continuous at x. Hence it is continuous at x. $\square$

5. (1) Let $f_n(x) = \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)$. Then $f_n(x) \to e^{-x} \cdot 0 = 0$ for any $x > 0$. Moreover,

$$|f_n(x)| \le \left(1+\frac{x}{n}\right)^{-n} = \frac{1}{\left(1+\frac{x}{n}\right)^n} \le \frac{1}{1+n\cdot\frac{x}{n}+\frac{n(n-1)}{2}\left(\frac{x}{n}\right)^2}$$

$$\le \frac{1}{1+\left(\frac{n-1}{2n}\right)x^2} \le \frac{1}{1+\frac{1}{4}x^2} \quad \text{if } n \ge 2.$$

(The last step: $1+\left(\frac{n-1}{2n}\right)x^2 \ge 1+\frac{1}{4}x^2 \iff \frac{n-1}{2n} \ge \frac{1}{4} \iff 4n-4 \ge 2n \iff n \ge 2$.)

Since $g(x) = \frac{1}{1+\frac{1}{4}x^2}$ $(x \in [0,\infty))$ is integrable we have by the Dominated Convergence Theorem that $\int f_n \to \int 0 = 0$. i.e.

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx = 0.$$

(2) Let $f_n(x) = (1+nx^2)(1+x^2)^{-n}$, $x \in (0,1)$.

Then $f_n(x) = \frac{1+nx^2}{(1+x^2)^n} \to 0 \quad \forall x \in (0,1)$.

$$|f_n(x)| \le \frac{1+nx^2}{1+nx^2} = 1.$$

So, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0. \qquad \square$$

6. Assume $f_n \to f$ in measure. Then $\forall \varepsilon > 0$,

$$\mu(\{|f_n - f| \ge \varepsilon\}) \to 0 \quad \text{as } n \to \infty.$$

Thus for the same $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$n \ge N \Rightarrow \mu(\{|f_n - f| \ge \varepsilon\}) < \varepsilon$.

Conversely, for any $\varepsilon > 0$, we show

$$\lim_{n\to\infty} \mu(\{|f_n - f| \ge \varepsilon\}) = 0.$$

i.e., $\forall \delta > 0$ $\exists N \in \mathbb{N}$ such that

$$\mu(\{|f_n - f| \ge \varepsilon\}) < \delta \quad \text{if } n \ge N.$$

Let $\eta = \min(\varepsilon, \delta) > 0$. Then, by the assumption, $\exists N \in \mathbb{N}$ such that $\mu(\{|f_n - f| \geq \eta\}) < \eta$ if $n \geq N$.
But $\eta \leq \varepsilon$. So $\{|f_n - f| \geq \varepsilon\} \subseteq \{|f_n - f| \geq \eta\}$.
Since $\delta \geq \eta$, we have

$$\mu(\{|f_n - f| \geq \varepsilon) \leq \mu(\{|f_n - f| \geq \eta) < \eta \leq \delta \quad \forall n \geq N.$$

Hence, $\mu(\{|f_n - f \geq \varepsilon\}) \to 0$ as $n \to \infty$, and $f_n \to f$ in measure. $\square$

7. (1) Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that

$$\liminf_{n \to \infty} \int f_n \, d\mu = \lim_{k \to \infty} \int f_{n_k} \, d\mu.$$

Since $f_n \to f$ in measure, $f_{n_k} \to f$ in measure. Thus, $\{f_{n_k}\}$ has a further subsequence $\{f_{n_{k_j}}\}$ that converges to f a.e. Fatou's lemma now implies that

$$\int f \, d\mu \leq \liminf_j \int f_{n_{k_j}} d\mu = \lim_{k \to \infty} \int f_{n_k} d\mu = \liminf_n \int f_n \, d\mu.$$

(2) If $f_n \not\to f$ in $L^1(\mu)$, then $\exists \delta > 0$ and $\exists \{f_{n_k}\}$ a subsequence of $\{f_n\}$, such that

$$\int |f_{n_k} - f| \, d\mu \geq \delta \quad (k = 1, 2, \cdots).$$

Since $f_{n_k} \to f$ in measure, $\{f_{n_k}\}$ has a subsequence $\{f_{n_{k_j}}\}$ such that $f_{n_{k_j}} \to f$ a.e. Since $|f_{n_{k_j}} - f| \leq 2g$ and $2g \in L^1(\mu)$, the Dominated Convergence Theorem implies that

$$\lim_{j \to \infty} \int |f_{n_{k_j}} - f| \, d\mu = \int \lim_{j \to \infty} |f_{n_{k_j}} - f| \, d\mu = 0.$$

This contradicts $\int |f_{n_{k_j}} - f| \, d\mu \geq \delta$ $(j = 1, 2, \cdots)$ since $\{f_{n_{k_j}}\}$ is a subsequence of $\{f_{n_k}\}$. Thus, $\int |f_n - f| \, d\mu \to 0$ as $n \to \infty$. $\square$

8. Since $\chi_{E_n} \to f$ in $L^1(\mu)$, there exists a subsequence $\{E_{n_k}\}$ of $\{E_n\}$ such that $\chi_{E_{n_k}} \to f$ a.e. Since $\chi_{n_k}(x) = 0$ or 1 for all $k \geq 1$ and for all $x \in X$, we have $f(x) = 0$ or 1 a.e. Let $E = \{f = 1\}$. Then $E = \bigcap_{n=1}^{\infty} \{1 - \frac{1}{2^n} < f < 1 + \frac{1}{2^n}\}$ is measurable and $f = \chi_E$ a.e. $\square$

9. $\forall \varepsilon > 0$. Since $f_n \to f$ and $g_n \to g$ in measure,

$$\mu(\{|f_n + g_n - (f+g)| \leq \varepsilon\})$$
$$\leq \mu(\{|f_n - f| \geq \tfrac{\varepsilon}{2}\}) + \mu(\{|g_n - g| \geq \tfrac{\varepsilon}{2}\}) \to 0.$$

Thus $f_n + g_n \to f + g$ in measure.

Suppose $f_n g_n \not\to fg$ in measure. Then $\exists\, \varepsilon_0 > 0$ such that $\mu(\{|f_n g_n - fg| \geq \varepsilon_0\}) \not\to 0$. Hence $\exists\, \delta_0 > 0$ and a subsequence $\{n_k\}$ such that

(*) $$\mu(\{|f_{n_k} g_{n_k} - fg| \geq \varepsilon_0\}) \geq \delta_0 > 0 \quad (k = 1, 2, \cdots).$$

Since f, g are finitely valued (~~The~~ convergence in measure is defined for complex-valued functions. If each f_n is finite-valued and $f_n \to f$ in measure. Then $\mu(\{|f| = \infty\}) = 0$.), there exists $A > 0$ such that $\mu(\{|f| \geq A\} \cup \{|g| \geq A\}) < \delta_0/4$. This follows from the fact that $\mu(X) < \infty$ and

$$\infty > \mu(X) = \mu(\{|f| \geq 0\}) = \mu\left(\bigcup_{n=0}^{\infty} \{|f| \geq n\}\right)$$
$$= \lim_{n\to\infty} \mu(\{|f| \geq n\})$$

Similarly, $\lim_{n \to \infty} \mu(\{|g| \geq n\}) = \mu(X) < \infty$.

Since $g_{n_k} \to g$ in measure, there exists a further subsequence $\{g_{n_{k_j}}\}$ such that $g_{n_{k_j}} \to g$ uniformly on some $E^c = X \setminus E$ with $E \in \mathcal{M}$ and $\mu(E) < \delta_0/4$.

Thus, $\exists j_0 \in \mathbb{N}$ such that

$$j \geq j_0 \Longrightarrow |g_{n_{k_j}}(x) - g(x)| \leq 1 \quad \forall x \in E^c$$

In particular,

$$j \geq j_0 \Longrightarrow |g_{n_{k_j}}(x)| \leq 1 + |g(x)| \leq 1 + A \quad \forall x \in E^c \cap \{|g| < A\}$$

Let $F = E \cup \{|f| \geq A\} \cup \{|g| \geq A\}$. Then $F \in \mathcal{M}$ and $\mu(F) \leq \mu(E) + \mu(\{|f| \geq A\} \cup \{|g| \geq A\}) < \frac{\delta_0}{2}$.

Thus, for $j \geq j_0$,

$$\mu(\{|f_{n_{k_j}} g_{n_{k_j}} - fg| \geq \varepsilon_0\})$$

$$\leq \mu(F) + \mu(\{x \in F^c : |f_{n_{k_j}}(x) g_{n_{k_j}}(x) - f(x) g(x)| \geq \varepsilon_0\})$$

$$< \frac{\delta_0}{2} + \mu(\{x \in F^c : |f_{n_{k_j}}(x) - f(x)||g_{n_{k_j}}(x)| \geq \frac{\varepsilon_0}{2}\})$$

$$+ \mu(\{x \in F^c : |g_{n_{k_j}}(x) - g(x)||f(x)| \geq \frac{\varepsilon_0}{2}\})$$

$$< \frac{\delta_0}{2} + \mu(\{x \in F^c : |f_{n_{k_j}}(x) - f(x)| \geq \frac{\varepsilon_0}{2(1+A)}\})$$

$$+ \mu(\{x \in F^c : |g_{n_{k_j}}(x) - g(x)| \geq \frac{\varepsilon_0}{2A}\})$$

Thus, $\limsup_{j \to \infty} \mu(\{|f_{n_{k_j}} g_{n_{k_j}} - fg| \geq \varepsilon_0\}) \leq \frac{\delta_0}{2}$.

This contradicts $(*)$. Hence $f_n g_n \to fg$ in measure.

If $\mu(X) = \infty$, then the result is not true in general.

Example. $f_n(x) = \cancel{f(x)=} \frac{1}{n}$ $(x \in \mathbb{R})$, $f(x) = 0$. $g_n(x) = g(x) = x$ $(x \in \mathbb{R})$

$f_n(x) g_n(x) = \frac{x}{n}$. $f(x) g(x) = 0$. $\forall x \in \mathbb{R}$.

$f_n \to f$, $g_n \to g$ in measure. But $f_n g_n \not\to fg$ in measure. $\square$

10. Since μ is σ-finite, $\exists\, \hat{F}_n \in \mathcal{M}_\xi$ with $\mu(\hat{F}_n) < \infty$ $(n=1,2,\dots)$ such that $X = \bigcup_{n=1}^{\infty} \hat{F}_n$. Let $F_n = \bigcup_{k=1}^{n} \hat{F}_k$ $(n=1,2,\dots)$. Then $F_n \in \mathcal{M}_\xi$, $\mu(F_n) \le \sum_{k=1}^{n} \mu(\hat{F}_k) < \infty$, $F_1 \subseteq F_2 \subseteq \cdots$, and $X = \bigcup_{n=1}^{\infty} F_n$.

For each $n \ge 1$, Fatou's lemma implies that $\exists\, E_n \subseteq F_n$ such that $f_n \to f$ uniformly on E_n, $E_n \in \mathcal{M}_\xi$, and $\mu(F_n \setminus E_n) < \frac{1}{n}$. We have

$$\mu\left(\left(\bigcup_{k=1}^{\infty} E_k\right)^c\right) = \mu\left(X \cap \left(\bigcap_{k=1}^{\infty} E_k^c\right)\right)$$
$$= \mu\left(\left(\bigcup_{n=1}^{\infty} F_n\right) \cap \bigcap_{k=1}^{\infty} E_k^c\right) = \mu\left(\bigcup_{n=1}^{\infty}\left(F_n \cap \bigcap_{k=1}^{\infty} E_k^c\right)\right)$$

Since $F_n \uparrow$, $F_n \cap \left(\bigcap_{k=1}^{\infty} E_k^c\right)$ also $\uparrow$. Thus

$$\mu\left(\left(\bigcup_{k=1}^{\infty} E_k\right)^c\right) = \lim_{n\to\infty} \mu\left(F_n \cap \bigcap_{k=1}^{\infty} E_k^c\right)$$

But for each n,

$$\mu\left(F_n \cap \bigcap_{k=1}^{\infty} E_k^c\right) \le \mu(F_n \cap E_n^c) = \mu(F_n \setminus E_n) \le \frac{1}{n}.$$

Thus $\mu\left(\left(\bigcup_{1}^{\infty} E_k\right)^c\right) = 0$. $\square$