Math 240C: Real Analysis, Spring 2020

Homework Assignment 3

Due 12:00 noon, Monday, April 20, 2020

1. Prove Lusin's Theorem: Let μ be a Radon measure on a locally compact Hausdorff space X. Let $f : X \to \mathbb{C}$ be a Borel-measurable function such that $\mu(\{f \neq 0\}) < \infty$. Then, for any $\varepsilon > 0$, there exists $\phi \in C_{\mathrm{c}}(X)$ such that $\mu(\{\phi \neq f\}) < \varepsilon$. If f is bounded, then ϕ can be taken to satisfies $\|\phi\|_{\mathrm{u}} \leq \|f\|_{\mathrm{u}}$.

2. Let X be a locally compact Hausdorff space, $I \in C_0(X, \mathbb{R})^*$, and let I^+, I^- be the functionals constructed in the proof of Lemma 7.15. Let μ be the signed Radon measure associated to I. Prove that the positive and negative variations of μ are the Radon measures associated to I^+ and I^-, respectively.

3. Let X be a locally compact Hausdorff space. Prove that every positive linear functional on $C_0(X)$ is bounded.

4. Let μ be a σ-finite Radon measure on a locally compact Hausdorff space X. Suppose $\nu \in M(X)$ and $\nu = \nu_1 + \nu_2$ is the Lebesgue decomposition of ν with respect to μ. Prove that both ν_1 and ν_2 are Radon measures on X.

5. Let X be a locally compact Hausdorff space. Let $\mu, \mu_n \in M(X)$ $(n = 1, 2, \dots)$ be such that $\mu_n \to \mu$ vaguely and $\|\mu_n\| \to \|\mu\|$. Let $f : X \to \mathbb{C}$ be bounded and continuous. Prove that
$$\lim_{n\to\infty} \int_X f \, d\mu_n = \int_X f \, d\mu.$$

6. Let $k \in \mathbb{N}$ and $I \in C^k([0,1])^*$. Prove that there exist $\mu \in M([0,1])$ and constants $c_0, \dots, c_{k-1}$, all unique, such that
$$I(f) = \int_0^1 f^{(k)} d\mu + \sum_{j=0}^{k-1} c_j f^{(j)}(0) \qquad \forall f \in C^k([0,1]).$$