Homework due Thursday, April 23, 9:00 pm, on Gradescope.
Throughout, A is a ring (commutative with 1).

(1) Assume A is non-trivial.
  (a) Show that the set of prime ideals in A contain a minimal element.
  (b) Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \lhd A$ and $\mathfrak{p} \in \operatorname{Spec}(A)$. Assume $\cap_i \mathfrak{a}_i \subset \mathfrak{p}$. Then $\mathfrak{a}_i \subset \mathfrak{p}$ for some i. If $\cap_i \mathfrak{a}_i = \mathfrak{p}$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i.

(2) Suppose for every $a \in A$ we have $a^{n(a)} = a$ for some integer $n(a) > 1$. Show that $\max(A) = \operatorname{Spec}(A)$.

(3) Let $\varphi : M \to N$ be a homomorphism of A-modules. The following are equivalent
  (a) φ is onto.
  (b) $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is onto for every prime ideal $\mathfrak{p} \lhd A$.
  (c) $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is onto for every maximal ideal $\mathfrak{m} \lhd A$.

(4) Let $\mathfrak{p} \in \operatorname{Spec}(A)$ and let $S \subset A$ be a multiplicatively closed subset so that $\mathfrak{p} \cap S = \emptyset$. Then

$$\theta : A_{\mathfrak{p}} \to \left(S^{-1}A\right)_{S^{-1}\mathfrak{p}}$$

defined as $\theta(\frac{a}{t}) = \frac{a/1}{t/1}$ is an isomorphism of rings.

(5) Suppose for any $\mathfrak{m} \in \operatorname{Spec}(A)$ we have $\operatorname{Nil}(A_{\mathfrak{m}}) = 0$. Prove or disprove $\operatorname{Nil}(A) = 0$.
   (Hint: Let $a \in \operatorname{Nil}(A)$ and consider $\operatorname{Ann}(a)$.)

(6) Let X be a set and let $A = P(X)$ be the power set of X together with $X_1 + X_2 = X_1 \Delta X_2$ and $X_1 \cdot X_2 = X_1 \cap X_2$. Then $a^2 = a$ for every $a \in A$.
  (a) Prove that for any $\mathfrak{p} \in \operatorname{Spec}(A)$ we have $A/\mathfrak{p} \simeq \mathbb{Z}/2\mathbb{Z}$.
  (b) Prove that for any $\mathfrak{p} \in \operatorname{Spec}(A)$ we have $A_{\mathfrak{p}} \simeq \mathbb{Z}/2\mathbb{Z}$. (Hint: If $a \in \mathfrak{p}$, $\frac{a}{1} = \frac{a(1-a)}{1-a} = 0$.)
  (c) Show that A is Noetherian if and only if X is finite. Conclude that being Noetherian is not a local property.
  (d) Assume X is infinite. Does the ideal $\{0\} \lhd A$ have a primary decomposition?