Homework due Thursday, April 9, 9:00 pm, on Gradescope.

(1) Let A be a ring and x and indeterminate. Prove that
   (a) $\mathrm{Nil}\big(A[x]\big) = \mathrm{Nil}(A)[x]$.
   (b) $A[x]^\times = \{\sum a_i x^i \in A[x] : a_0 \in A^\times, a_i \in \mathrm{Nil}(A), i > 0\}$
   (c) $\mathrm{J}\big(A[x]\big) = \mathrm{Nil}\big(A[x]\big) = \mathrm{Nil}(A)[x]$.
   (This is parts of Exercise 2 of chapter 1 in the book.)

   Hint: part (1): one inclusion is clear, for the other inclusion use the fact that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$ for any prime ideal $\mathfrak{p} \lhd A$.
   Part (2): For one inclusion show and use the fact that if B is a ring, $u \in B^\times$ and $n \in \mathrm{Nil}(B)$, then $n + u \in B^\times$. For the other inclusion prove and use the fact that if B is an integral domain $B[x]^\times = B^\times$ —apply this to $A/\mathfrak{p}$ for prime ideals $\mathfrak{p}$.
   Part (3): One inclusion is clear, for the other use part (2).

(2) Complete the proof of McCoy's Theorem: Let $\mathfrak{a}, \mathfrak{b}_1, \dots, \mathfrak{b}_n \lhd A$. Suppose

$$\mathfrak{a} \subset \bigcup_{i=1}^n \mathfrak{b}_i \quad \text{and} \quad \mathfrak{a} \not\subset \bigcup_{i, i\neq j} \mathfrak{b}_i \quad \text{for every } j.$$

   Then there exists some $k \in \mathbb{N}$ so that $\mathfrak{a}^k \subset \bigcap_i \mathfrak{b}_i$.

(3) Prove the following statements:
   (a) $\big((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}\big) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = \big((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b}\big)$.
   (b) $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.
   (c) If $\mathfrak{p}$ is a prime ideal, $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$.
   (d) $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}, \text{prime}} \mathfrak{p}$.

(4) Prove the following statements:
   (a) Any non-empty closed subset of $\mathrm{Spec}(A)$ intersects $\mathrm{Max}(A)$ non-trovially.
   (b) $\{\mathfrak{p} \in \mathrm{Spec}(A) : \{\mathfrak{p}\} \text{ is closed}\} = \mathrm{Max}(A)$.
   (c) If A is an integral domain, $\{0\}$ is dense in $\mathrm{Spec}(A)$.

(5) Let $\mathfrak{a} \lhd A$ and let $\pi : A \to A/\mathfrak{a}$ be the natural map. Then π^* induces a bijection from $\mathrm{Spec}(A/\mathfrak{a})$ to $V(\mathfrak{a})$.

(6) Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then either $M = 0$ or $N = 0$. (Exercise 3, chapter 2 in the book.)