Pieri Rules for Schur functions in superspace

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Overview

Symmetric function theory

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Symmetric function theory
Symmetric function theory IN SUPER SPACE!!!!!

Symmetric Function Theory

$$\mathbb{K}[z_1, \ldots, z_N]^{S_N}$$

1. partitions

$(5, 4, 2, 2, 1)$

2. simple bases $(m_\lambda, p_\lambda, e_\lambda, h_\lambda, s_\lambda, \ldots)$

3. other bases: Macdonald polynomials, Jack polynomials

Symmetric Function Theory

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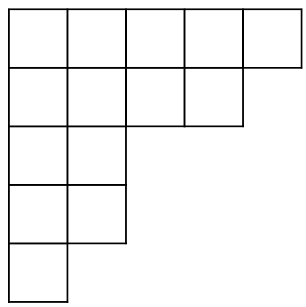

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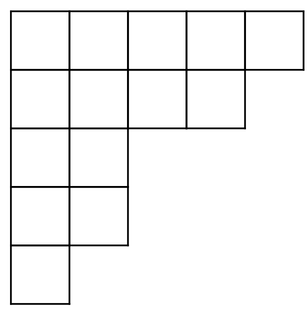

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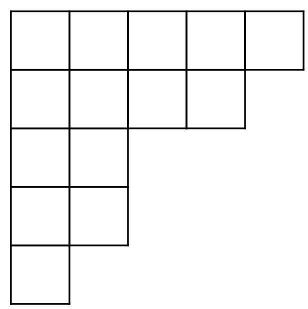

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Symmetric function theory

$$M_\lambda^{(q,t)}$$

Symmetric function theory

$$\begin{tikzcd} M_\lambda^{(q,t)} \arrow[d, "\substack{q=t^\alpha \\ t\to 1}"] \\ {} \end{tikzcd}$$

Symmetric function theory

$$\begin{array}{c} M_\lambda^{(q,t)} \\ \Big\downarrow {\scriptstyle q=t^\alpha,\ t\to 1} \\ J_\lambda^{(\alpha)} \end{array}$$

Symmetric function theory

$$\begin{tikzcd} M_\lambda^{(q,t)} \arrow[d, "{q=t^\alpha \atop t\to 1}"] \\ J_\lambda^{(\alpha)} \arrow[d, "\alpha\to 1"] \\ {} \end{tikzcd}$$

Symmetric function theory

$$
\begin{tikzcd}
M_\lambda^{(q,t)} \arrow[d, "q=t^\alpha" , "t\to 1"'] \\
J_\lambda^{(\alpha)} \arrow[d, "\alpha\to 1"] \\
s_\lambda
\end{tikzcd}
$$

Symmetric function theory

$$
\begin{array}{ccc}
M_\lambda^{(q,t)} & \qquad & \text{COMBINATORICS} \\
\Big\downarrow {\scriptstyle q=t^\alpha,\ t\to 1} & & \\
J_\lambda^{(\alpha)} & & \text{PHYSICS} \\
\Big\downarrow {\scriptstyle \alpha\to 1} & & \\
s_\lambda & & \text{combinatorics}
\end{array}
$$

Symmetric function theory

Common characterization:

- $M_\lambda^{(q,t)} = m_\lambda + \text{smaller terms}$ (triangularity)
- $\langle M_\lambda^{(q,t)}, M_\mu^{(q,t)} \rangle = 0 \quad \text{if } \lambda \neq \mu$ (orthogonality)

Scalar product:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

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Macdonald positivity

$$\tilde{M}_{\lambda}^{(q,t)} = \sum_{\mu} K_{\mu\lambda}(q,t)\, s_{\mu} \qquad \text{with} \qquad K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]$$

$K_{\mu\lambda}(1,1) =$ number of standard tableaux of shape μ

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Macdonald positivity

$$\tilde{M}^{(q,t)}_{\ydiagram{2,1}} = t\, s_{\ydiagram{3}} + (1 + qt)\, s_{\ydiagram{2,1}} + q\, s_{\ydiagram{1,1,1}}$$

$$\begin{ytableau} 1 & 2 & 3 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$$

Macdonald positivity

$$\tilde{M}^{(q,t)}_{\yng(2,1)} = t\, s_{\yng(3)} + (1+qt)\, s_{\yng(2,1)} + q\, s_{\yng(1,1,1)}$$

$$\young(123) \qquad \young(12,3) \quad \young(13,2) \qquad \young(1,2,3)$$

Supersymmetry

2 types of particles in nature

bosons (integer spin: $0, 1, 2, \ldots$)

fermions (half integer spin: $1/2, 3/2, \ldots$)

$\Psi \longrightarrow \Psi$
exchange of two bosons

$\Psi \longrightarrow -\Psi$
exchange of two fermions
(*Pauli's exclusion principle*)

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A symmetric function theory in superspace

$$\mathbb{K}[z_1,\dots,z_N,\theta_1,\dots,\theta_N]^{S_N} \qquad \text{with} \qquad (\theta_i\theta_j = -\theta_i\theta_j \quad \text{and} \quad \theta_i^2 = 0)$$

$\underline{N=2:}$ $\qquad (z_1 - z_2)\,\theta_1\theta_2$

Power sums: $p_1, p_2, \dots, \tilde{p}_0, \tilde{p}_1, \dots$

$$p_r = z_1^r + z_2^r + \cdots \qquad \text{and} \qquad \tilde{p}_k = \theta_1 z_1^k + \theta_2 z_2^k + \cdots$$

$$\tilde{p}_i\tilde{p}_j = -\tilde{p}_j\tilde{p}_i \qquad \text{and} \qquad \tilde{p}_i^2 = 0$$

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A symmetric function theory in superspace

$\mathbb{K}[z_1, \ldots, z_N, \theta_1, \ldots, \theta_N]^{S_N}$ with $(\theta_i\theta_j = -\theta_i\theta_j$ and $\theta_i^2 = 0)$

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$$\tilde{p}_i\tilde{p}_j = -\tilde{p}_j\tilde{p}_i \quad \text{and} \quad \tilde{p}_i^2 = 0$$

Superpartitions

$$\Lambda = (\Lambda^a; \Lambda^s) \qquad \begin{cases} \Lambda^s \text{ is usual partition} \\ \Lambda^a \text{ has no repeated parts} \end{cases}$$

$$(4, 2, 0; 3, 2, 1, 1) \quad \longleftrightarrow \quad (4, 3, 2, 2, 1, 1, 0) \quad \longleftrightarrow$$

simple bases $(m_\Lambda, p_\Lambda, e_\Lambda, h_\Lambda, s_\Lambda, \ldots)$

other bases: Macdonald polynomials and Jack polynomials in superspace.

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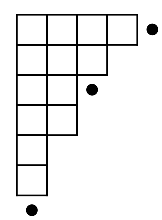

simple bases $(m_\Lambda, p_\Lambda, e_\Lambda, h_\Lambda, s_\Lambda, \dots)$

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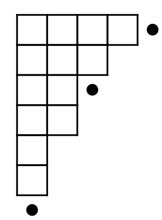
Superpartitions

$$\Lambda = ({\color{red}\Lambda^a}; {\color{blue}\Lambda^s}) \qquad \begin{cases} \Lambda^s \text{ is usual partition} \\ \Lambda^a \text{ has no repeated parts} \end{cases}$$

$$({\color{red}4}, {\color{red}2}, {\color{red}0}; {\color{blue}3}, {\color{blue}2}, {\color{blue}1}, {\color{blue}1}) \quad \longleftrightarrow \quad ({\color{red}4}, {\color{blue}3}, {\color{red}2}, {\color{blue}2}, {\color{blue}1}, {\color{blue}1}, {\color{red}0}) \quad \longleftrightarrow$$



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Symmetric function theory in superspace

Common characterization:

- $M_\Lambda^{(q,t)} = m_\Lambda + \text{smaller terms}$ (triangularity)
- $\langle M_\Lambda^{(q,t)}, M_\Omega^{(q,t)} \rangle = 0 \quad \text{if } \Lambda \neq \Omega$ (orthogonality)

Scalar product:

$$\langle p_\Lambda, p_\Omega \rangle_{qt} = \delta_{\Lambda\Omega} q^{|\Lambda^a|} z_{\Lambda^s} \prod_i \frac{1 - q^{\Lambda_i^s}}{1 - t^{\Lambda_i^s}}$$

$$\Lambda = (\Lambda^a; \Lambda^s) \qquad \text{and} \qquad p_\Lambda = \tilde{p}_{\Lambda^a} \, p_{\Lambda^s}$$

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Symmetric function theory in superspace

$M_{\Lambda}^{(q,t)}$

Symmetric function theory in superspace

$$\begin{tikzcd} M_\Lambda^{(q,t)} \arrow[d, "q=t^\alpha" near start, "t\to 1" near end] \\ J_\Lambda^{(\alpha)} \end{tikzcd}$$

Symmetric function theory in superspace

$$\begin{CD} M_\Lambda^{(q,t)} \\ @V{q=t^\alpha}V{t\to 1}V \\ J_\Lambda^{(\alpha)} \\ @V{\alpha\to 1}VV \\ s_\Lambda^1 \end{CD}$$

Symmetric function theory in superspace

$$
\begin{array}{ccc}
M_\Lambda^{(q,t)} & & \text{COMBINATORICS} \\
\Big\downarrow{\scriptstyle q=t^\alpha,\ t\to 1} & & \\
J_\Lambda^{(\alpha)} & & \text{PHYSICS} \\
\Big\downarrow{\scriptstyle \alpha\to 1} & & \\
s_\Lambda^{1} & & \text{yucky combinatorics}
\end{array}
$$

Symmetric function theory in superspace

$$
\begin{tikzcd}
 & M_\Lambda^{(q,t)} \arrow[d] \arrow[dl, dotted, "q=t=0"'] \arrow[ddl, phantom] \\
 & J_\Lambda^{(\alpha)} \arrow[d, "\alpha\to 1"] \\
s_\Lambda^0 & s_\Lambda^1
\end{tikzcd}
$$

Symmetric function theory in superspace

$$
\begin{tikzcd}
 & M_\Lambda^{(q,t)} \arrow[d] \arrow[ddl, dotted, "q=t=0"'] \\
 & J_\Lambda^{(\alpha)} \arrow[d, "\alpha\to 1"] \\
s_\Lambda^0 \arrow[r, leftrightarrow, "\textit{different}"'] & s_\Lambda^1
\end{tikzcd}
$$

$$\langle p_\Lambda, p_\Omega \rangle_{qt} = \delta_{\Lambda\Omega} q^{|\Lambda^a|} z_{\Lambda^s}(q,t)$$

Symmetric function theory in superspace

$$\begin{tikzcd} M_\Lambda^{(q,t)} \arrow[d, dotted, "q=t=0"'] \\ s_\Lambda^0 \end{tikzcd}$$

Macdonald positivity conjecture in superspace!!

$$\tilde{M}_{\Lambda}^{(q,t)} = \sum_{\Omega} K_{\Omega\Lambda}(q,t)\, s_{\Omega}^{0} \qquad \text{with} \qquad K_{\Omega\Lambda}(q,t) \in \mathbb{N}[q,t]???$$

Macdonald positivity conjecture in superspace

$$\tilde{M}^{(q,t)}_{\begin{ytableau} \, & \none[\bullet] \\ \, \end{ytableau}} = t\, s^0_{\begin{ytableau} \, & \, & \none[\bullet] \end{ytableau}} + s^0_{\begin{ytableau} \, & \none[\bullet] \\ \, \end{ytableau}} + qt\, s^0_{\begin{ytableau} \, & \, \\ \none[\bullet] \end{ytableau}} + q\, s^0_{\begin{ytableau} \, \\ \, \\ \none[\bullet] \end{ytableau}}$$

$$\tilde{M}^{(q,t)}_{\begin{ytableau} \, & \, \\ \, \end{ytableau}} = t\, s_{\begin{ytableau} \, & \, & \, \end{ytableau}} + (1+qt)\, s_{\begin{ytableau} \, & \, \\ \, \end{ytableau}} + q\, s_{\begin{ytableau} \, \\ \, \\ \, \end{ytableau}}$$

Refinement of the original problem!!

Macdonald positivity conjecture in superspace

$$\tilde{M}^{(q,t)}_{\begin{smallmatrix}\square & \bullet\\ \square & \end{smallmatrix}} = t\, s^{0}_{\begin{smallmatrix}\square\square & \bullet\end{smallmatrix}} + s^{0}_{\begin{smallmatrix}\square & \bullet\\ \square & \end{smallmatrix}} + qt\, s^{0}_{\begin{smallmatrix}\square\square\\ \bullet\end{smallmatrix}} + q\, s^{0}_{\begin{smallmatrix}\square\\ \square\\ \bullet\end{smallmatrix}}$$

$$\tilde{M}^{(q,t)}_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} = t\, s_{\begin{smallmatrix}\square\square\square\end{smallmatrix}} + (1+qt)\, s_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} + q\, s_{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}$$

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$$\tilde{M}^{(q,t)}_{\ydiagram{2,1}} = t\, s_{\ydiagram{3}} + s_{\ydiagram{2,1}} + qt\, s_{\ydiagram{2,1}} + q\, s_{\ydiagram{1,1,1}}$$

Refinement of the original problem!!

Symmetric function theory in superspace

$$\begin{tikzcd} M_\Lambda^{(q,t)} \arrow[d, dotted, "q=t=0"'] \\ s_\Lambda^0 \end{tikzcd}$$

$$\langle p_\Lambda, p_\Omega \rangle_{qt} = \delta_{\Lambda\Omega} q^{|\Lambda^a|} z_{\Lambda^s}(q,t)$$

Symmetric function theory in superspace

$$\begin{tikzcd}
 & M_\Lambda^{(q,t)} \arrow[dl, dotted, "q=t=0"'] \arrow[dr, dotted, "q=t=\infty"] & \\
s_\Lambda^0 \arrow[rr, leftrightarrow, "\textit{essentially dual}"'] & & s_\Lambda^\infty
\end{tikzcd}$$

Symmetric function theory in superspace

Pieri rules, tableaux generating functions (monomial expansions), Cauchy identities (RSK).

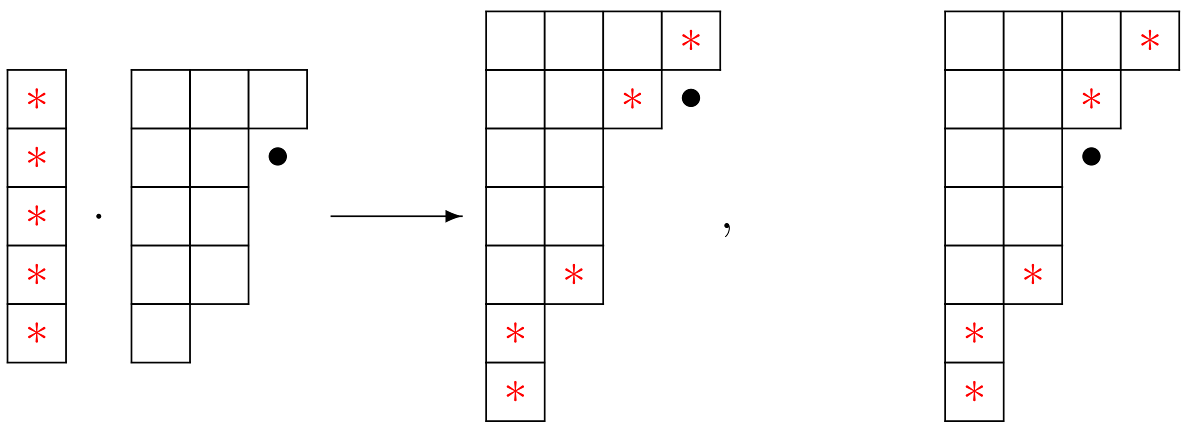
$$
\begin{tikzcd}
 & M_\Lambda^{(q,t)} \arrow[dl, dotted, "q=t=0"'] \arrow[dr, dotted, "q=t=\infty"] & \\
s_\Lambda^0 & & s_\Lambda^\infty
\end{tikzcd}
$$

Symmetric function theory in superspace

- Mathieu
- Blondeau-Fournier
- Pieri Rule

$$
\begin{tikzcd}
 & M_\Lambda^{(q,t)} \arrow[dl, dotted, "q=t=0"'] \arrow[dr, dotted, "q=t=\infty"] & \\
s_\Lambda^0 & & s_\Lambda^\infty
\end{tikzcd}
$$

Pieri Rule for $e_r \cdot s^0_\Lambda$



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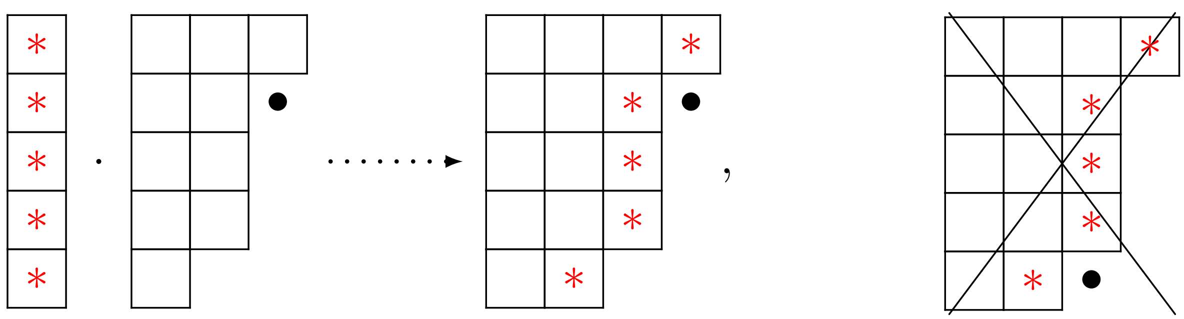
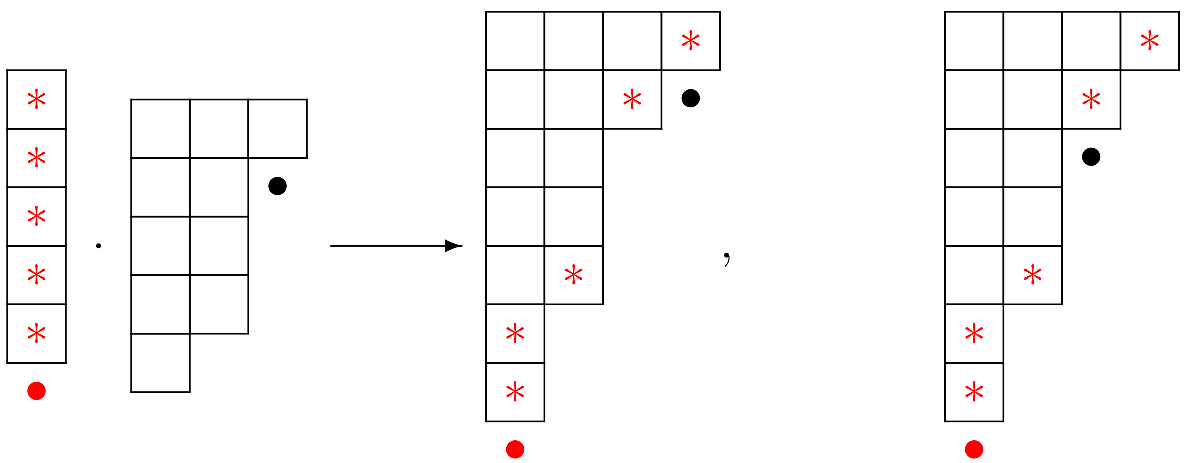
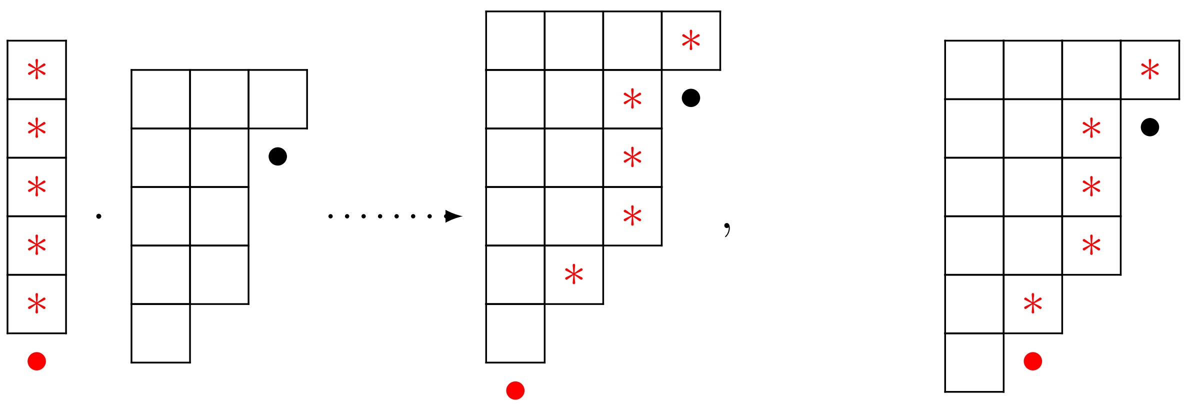


Pieri Rule for $\tilde{e}_r \cdot s^0_\Lambda$



Pieri Rule for $\tilde{e}_r \cdot s_\Lambda^0$



Monomial expansions

$$s_{\Lambda}^{0} \xleftarrow{\textit{essentially}} \xrightarrow{\textit{dual}} s_{\Lambda}^{\infty}$$

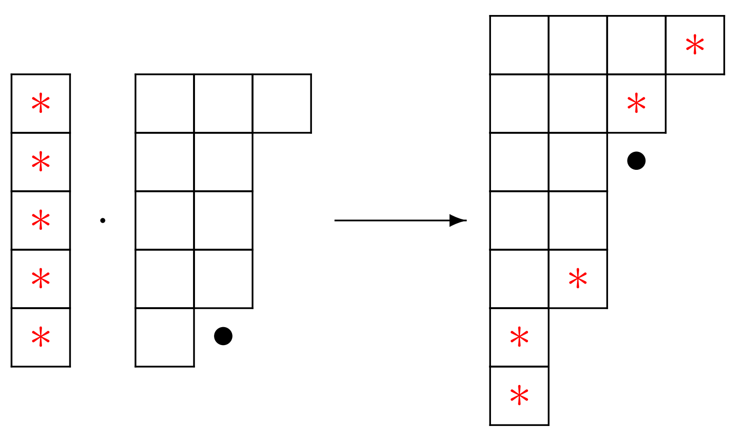
$$s_{\Lambda}^{\infty} = \sum_{T \in Tab_{\Lambda}^{0}} (x\theta)^{T}$$

Symmetric function theory in superspace

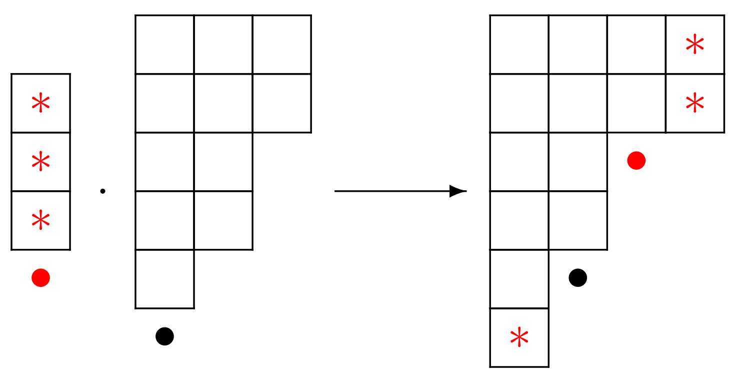
- Lapointe
- Preville-Ratelle
- MJ
- Pieri Rule for s_Λ^∞.

$$
\begin{tikzcd}
 & M_\Lambda^{(q,t)} \arrow[dl, dotted, "q=t=0"'] \arrow[dr, dotted, "q=t=\infty"] & \\
s_\Lambda^0 & & s_\Lambda^\infty
\end{tikzcd}
$$

Pieri Rule for $e_r \cdot s_\Lambda^\infty$



Pieri Rule for $\tilde{e}_r \cdot s_\Lambda^\infty$



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Cauchy Identity

$$\sum_\Lambda s_\Lambda^0(x,\theta) s_{\Lambda'}^\infty(y,\phi) = \prod_{i,j}(1 + x_i y_j + \theta_i \phi_j)$$

Pieri Rules proofs

$$M_\Lambda^{qt} = \mathcal{O}^{qt} E_{\eta(\Lambda)}^{qt} \theta_1 \dots \theta_m + \text{other terms}_{(\text{symmetrization})}$$

$\eta(\Lambda)$ is a composition based on Λ
$E_{\eta(\Lambda)}^{qt}$ is a non-symmetric Macdonald polynomial.
$\mathcal{O}^{qt}$ is some operator.

$$\begin{tikzcd}
 & \hat{K}_{\eta(\Lambda)} \\
E_{\eta(\Lambda)}^{qt} \arrow[ur, "q=t=0"'] \arrow[dr, "q=t=\infty"'] & \\
 & K_{\eta(\Lambda)}
\end{tikzcd}$$

Key polynomials

Pieri Rules proofs

$$M_\Lambda^{qt} = \mathcal{O}^{qt} E^{qt}_{\eta(\Lambda)} \theta_1 \dots \theta_m + \text{other terms}_{\text{(symmetrization)}}$$

$\eta(\Lambda)$ is a composition based on Λ
$E^{qt}_{\eta(\Lambda)}$ is a non-symmetric Macdonald polynomial.
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$$\begin{tikzcd} & \mathcal{O}^{00} \hat{K}_{\eta(\Lambda)} \\ \mathcal{O}^{qt} E^{qt}_{\eta(\Lambda)} \arrow[ur, "q=t=0"'] \arrow[dr, "q=t=\infty"'] & \\ & \mathcal{O}^{\infty\infty} K_{\eta(\Lambda)} \end{tikzcd}$$

Pieri Rules proofs

$$M_\Lambda^{qt} = \mathcal{O}^{qt} E_{\eta(\Lambda)}^{qt} \theta_1 \dots \theta_m + \text{other terms}_{(\text{symmetrization})}$$

$\eta(\Lambda)$ is a composition based on Λ
$E_{\eta(\Lambda)}^{qt}$ is a non-symmetric Macdonald polynomial.
$\mathcal{O}^{qt}$ is some operator.

$$\begin{tikzcd} & s_\Lambda^0 \\ M_\Lambda^{qt} \arrow[ur, "q=t=0"'] \arrow[dr, "q=t=\infty"'] & \\ & s_\Lambda^\infty \end{tikzcd}$$

Symmetric function theory in superspace

$$
\begin{tikzcd}
& M_\Lambda^{(q,t)} \arrow[d, "\substack{q=t^\alpha \\ t\to 1}"] \arrow[ddl, dotted, "q=t=0"'] \arrow[ddr, dotted, "q=t=\infty"] & \\
& J_\Lambda^{(\alpha)} \arrow[d, "\alpha\to 1"] & \\
s_\Lambda^0 & s_\Lambda^1 & s_\Lambda^\infty
\end{tikzcd}
$$

COMBINATORICS

PHYSICS

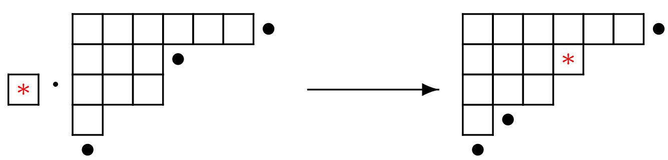
$J_\Lambda^{(\alpha)}$

The Pieri Rules for $e_1 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.

$$\frac{3\alpha(5\alpha+2)}{(3\alpha+2)^2(5\alpha+3)}$$

$J_\Lambda^{(\alpha)}$

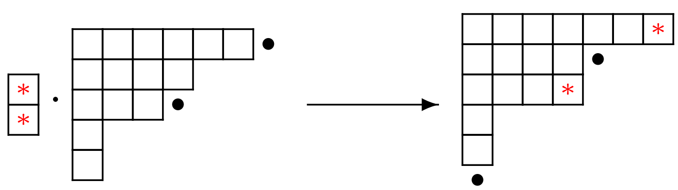
The Pieri Rules for $e_1 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.



$$\frac{3\alpha(5\alpha+2)}{(3\alpha+2)^2(5\alpha+3)}$$

$J_\Lambda^{(\alpha)}$

The Pieri Rules for $e_2 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.
Sometimes there are quadratic factors

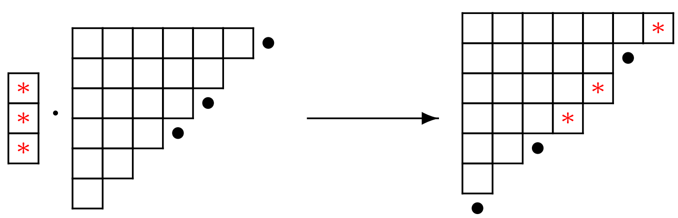


$$-\frac{2\alpha^3(3\alpha^2+\alpha-1)}{(6\alpha+5)(7\alpha+5)(\alpha+1)(\alpha+2)(3\alpha+1)(2\alpha+1)}$$

Sum of 2 terms

$J_\Lambda^{(\alpha)}$

The Pieri Rules for $e_3 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.
Sometimes there are degree 6 factors!!!!!!!!

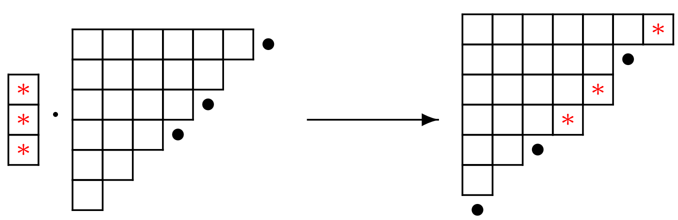


$$\frac{1}{1152}\,\frac{\alpha^4(2\alpha+3)(3\alpha+4)(416\alpha^6+2000\alpha^5+3484\alpha^4+2608\alpha^3+559\alpha^2-256\alpha-108)}{(4\alpha+3)(5\alpha+4)(7\alpha+6)(2\alpha+1)(\alpha+1)^{10}}$$

Sum of 6 terms???????

$J_\Lambda^{(\alpha)}$

The Pieri Rules for $e_3 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.
Sometimes there are degree 6 factors!!!!!!!!



$$\frac{1}{1152}\,\frac{\alpha^4(2\alpha+3)(3\alpha+4)(416\alpha^6+2000\alpha^5+3484\alpha^4+2608\alpha^3+559\alpha^2-256\alpha-108)}{(4\alpha+3)(5\alpha+4)(7\alpha+6)(2\alpha+1)(\alpha+1)^{10}}$$

Sum of 7 terms!!!!!!!!!!!!!

Alternating Sign Matrices!!!!!

$$1, 1, 2, 7, 42, 429, \ldots$$

Square Ice!!!!

Sum corresponds to partition function of square ice!!!!!!!

$$
\begin{array}{ccccccccccccccccc}
H & - & O & & H & - & O & - & H & & O & - & H & & O & - & H \\
 & & | & & & & & & & & | & & & & | & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & & & & & | & & & & & & & & & & \\
H & - & O & - & H & & O & & H & - & O & - & H & & O & - & H \\
 & & & & & & | & & & & & & & & | & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & | & & & & & & & & | & & & & & & \\
H & - & O & & H & - & O & & H & - & O & & H & - & O & - & H \\
 & & & & & & | & & & & & & & & & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & | & & & & & & & & | & & & & | & & \\
H & - & O & & H & - & O & - & H & & O & - & H & & O & - & H
\end{array}
$$

Thank you