Math 240A, Fall 2019.
Solution to Problems of HW #1
B. Li, Oct. 2019

1. If $x \in \limsup_{n\to\infty} E_n$, then $x \in E_n$ for infinitely many n. Hence, for each $k \geq 1$, $\exists\, n_k \geq k$ such that $x \in E_{n_k}$. thus $x \in \bigcup_{n=k}^{\infty} E_n$. and hence $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.
If $x \notin \limsup_{n\to\infty} E_n$, then $x \in E_n$ for at most finitely many n. Thus, $\exists N$ s.t. $x_n \notin E_n$ for all $n \geq N$. Hence, $x \notin \bigcup_{n=N}^{\infty} E_N$, and $x \notin \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.
Thus, $\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Now, $x \in \liminf_{n\to\infty} E_n \iff \exists\, N = N(x)$ s.t. $x \in E_n$ for all $n \geq N \iff x \in \bigcap_{n=N}^{\infty} E_n \iff x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$.
Hence $\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. $\square$

2. We have $x \in f^{-1}\left(\bigcup_{\alpha\in\mathcal{A}} Y_\alpha\right) \iff f(x) \in \bigcup_{\alpha\in\mathcal{A}} Y_\alpha \iff$ $\exists\, \alpha_0 \in \mathcal{A}$ s.t. $f(x) \in Y_{\alpha_0} \iff x \in f^{-1}(Y_{\alpha_0})$ for some $\alpha_0 \in \mathcal{A}$ $\iff x \in \bigcup_{\alpha\in\mathcal{A}} f^{-1}(Y_\alpha)$. This proves the first identity.

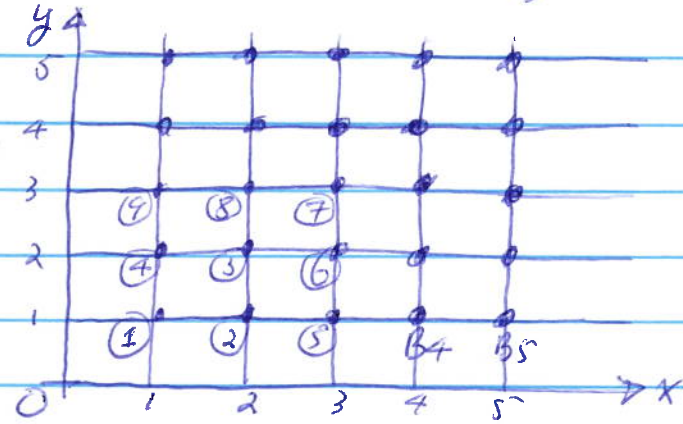
Similarly, $x \in f^{-1}\left(\bigcap_{\alpha\in\mathcal{A}} Y_\alpha\right) \iff f(x) \in \bigcap_{\alpha\in\mathcal{A}} Y_\alpha \iff$ $f(x) \in Y_\alpha$ $(\forall \alpha \in \mathcal{A}) \iff x \in f^{-1}(Y_\alpha)$ $(\forall \alpha\in\mathcal{A}) \iff$ $x \in \bigcap_{\alpha\in\mathcal{A}} f^{-1}(Y_\alpha)$. This proves the second identity. $\square$

3. There are many different ways to map $\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$ bijectively. Here is a very intuitive way:

$1 \longrightarrow (1,1)$

$2 \to (2,1),\ 3 \longrightarrow (2,2),\ 4 \to (1,2)$

$5, 6, 7, 8, 9 \longrightarrow (3,1), (3,2), (3,3), (2,3), (1,3)$, respectively, etc.



Let $k \in \mathbb{N}$ and define

$$A_k = \{(k-1)^2+1, (k-1)^2+2, \cdots, k^2\} \subset \mathbb{N},$$

$$B_k = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \max\{m,n\} = k\}.$$

Then $\mathbb{N} = \bigcup_{k=1}^{\infty} A_k$ and $\mathbb{N}\times\mathbb{N} = \bigcup_{k=1}^{\infty} B_k$ are both disjoint unions.

Define $\phi_k : A_k \longrightarrow B_k$ by

$$\phi_k((k-1)^2+j) = \begin{cases} (k,j) & \text{if } 1 \le j \le k, \\ (2k-j, k) & \text{if } k+1 \le j \le 2k-1. \end{cases}$$

Here we use the fact that

$$A_k = \{(k-1)^2+j : j = 1, 2, \cdots, 2k-1\}.$$

If $1 \le j_1 < j_2 \le 2k-1$, then $\phi_k((k-1)^2+j_1) \ne \phi_k((k-1)^2+j_2)$

So, ϕ_k is injective. Since

$$B_k = \underbrace{\{(k,j) : 1 \le j \le k\}}_{\text{(The vertical side)}} \cup \underbrace{\{(2k-j, k) : k+1 \le j \le 2k-1\}}_{\text{(The horizontal side)}}$$

ϕ_k is surjective.

Now, define $\phi : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ by

$$\phi(n) = \phi_k((k-1)^2+j) \in B_k,$$

where $k = \lceil\sqrt{n}\,\rceil$ (the smallest integer $\ge \sqrt{n}$)

and $j = n-(k-1)^2 \in \{1, 2, \dots, 2k-1\}$.

Show that ϕ is injective. Let $n_1, n_2 \in \mathbb{N}$, and assume $n_1 \neq n_2$. If $\lceil \sqrt{n_1} \rceil = \lceil \sqrt{n_2} \rceil =: k$ then $\phi(n_1) = \phi_k(n_1) \neq \phi_k(n_2) = \phi(n_2)$ as ϕ_k is injective. If $\lceil \sqrt{n_1} \rceil = \lceil \sqrt{n_2} \rceil$, then $\phi(n_1) \in B_{\lceil \sqrt{n_1} \rceil}$ and $\phi(n_2) \in B_{\lceil \sqrt{n_2} \rceil}$ are different as $B_{k_1} \cap B_{k_2} = \emptyset$ for $k_1 \neq k_2$. Thus, ϕ is injective.

Show that ϕ is surjective. If $(m,n) \in \mathbb{N} \times \mathbb{N}$ and $k = \max\{m, n\}$, then $\exists\, l \in \cancel{\mathbb{N}}\ A_k$ s.t. $\phi(l) = \phi_k(l) = (m,n) \in B_k$, since ϕ_k is surjective. Thus ϕ is surjective.

The map $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is thus bijective. $\square$

4. $U_n = (-\frac{1}{n}, \frac{1}{n})$, $n \in \mathbb{N}$. $\bigcap_{n=1}^{\infty} U_n = \{0\}$ is not open.

5. Suppose E is closed. Let $x_n \in E$ $(n = 1, 2, \dots)$ be a Cauchy sequence. Since X is complete, $\exists\, x \in X$ such that $x_n \to x$. But E is closed, so $x \in E$. Thus, E is complete.

Suppose E is complete. Let $x_n \in E$ $(n = 1, 2, \dots)$ and $x_n \to x \in X$. Then $\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in E. But E is complete. So $x_n \to x'$ for some $x' \in E$. The uniqueness of limit implies that $x' = x \in E$. Thus, E is closed. $\square$

6. $\mathcal{A}$ is nonempty since $f^{-1}(Y) = X \in \mathcal{A}$.
If $A_j = f^{-1}(B_j) \in \mathcal{A}$ with $B_j \in \mathcal{B}$ ($j = 1, 2, \cdots$), then $\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} f^{-1}(B_j) = f^{-1}(\bigcup_{j=1}^{\infty} B_j) \in \mathcal{A}$, since $\bigcup_{j=1}^{\infty} B_j \in \mathcal{B}$. Similarly, if $A = f^{-1}(B) \in \mathcal{A}$ with $B \in \mathcal{B}$ then $A^c = X \setminus A = f^{-1}(Y) \setminus f^{-1}(B)$ $= f^{-1}(\cancel{X}\, Y \setminus B) = f^{-1}(B^c) \in \mathcal{A}$ since $B^c \in \mathcal{B}$.
Thus $\mathcal{A}$ is a σ-algebra. $\square$

7. Let $\mathcal{A}$ be an algebra of subsets of a set $X \neq \phi$.
If $\mathcal{A}$ is a σ-algebra then it is closed under countable unions, regardless increasing or not.

Assume $\mathcal{A}$ is closed under countable increasing unions. Let $A_j \in \mathcal{A}$ ($j = 1, 2, \cdots$). Define $E_n = \bigcup_{j=1}^{n} A_j$. Since $\mathcal{A}$ is an algebra, $E_n \in \mathcal{A}$ ($n = 1, 2, \cdots$). Moreover $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup_{j=1}^{\infty} A_j = \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$ as $\mathcal{A}$ is closed under countable increasing unions. Hence $\mathcal{A}$ is closed under countable unions, and $\mathcal{A}$ is a σ-algebra.

8. No. Suppose $\mathcal{A}$ ~~were~~ is an ~~countably~~ infinite σ-algebra of subsets of a set $X \neq \phi$. We show there exists $A_j \in \mathcal{A}$ ($j = 1, 2, \cdots$) such that each $A_j \neq \phi$, $A_j \cap A_k = \phi$ ($j \neq k$), and $X = \bigcup_{j=1}^{\infty} A_j$.

Since $\mathcal{A}$ has infinitely many members, there exists $A_1 \in \mathcal{A}$ such that $A_1 \neq \phi$, $A_1 \neq X$. Let $A_2 = A_1^c \in \mathcal{M}$.
Then, $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 = \phi$, $A_1 \neq \phi$, $A_2 \neq \phi$, and $X = A_1 \cup A_2$.

Suppose for $n \geq 2$ there exist $A_1, \cdots, A_n \in \cancel{\mathcal{M}}\ \mathcal{A}$ pairwise disjoint, each $A_j \neq \phi$, and $\bigcup_{j=1}^{n} A_j = X$. Since $\mathcal{A}$ is infinite, and since there are only finitely many ~~finite~~ unions of different members of $A_1, \cdots, A_n$, there exist $A \in \mathcal{A}$, $A \neq \phi$, such that A is not a union of some or all of $A_1, \cdots, A_n$. Then, there must exist j $(1 \leq j \leq n)$ such that $A_j \cap A \neq \phi$ and $A_j \cap A^c \neq \phi$, for otherwise each A_i $(1 \leq i \leq n)$ will be either a subset of A or a subset of A^c, and A will be a union of some or all of $A_1, \cdots, A_n$, a contradiction. Now,

replace A_j by $A_j \cap A \neq \phi$ and

set $A_{n+1} = A_j \cap A^c \neq \phi$, $A_{n+1} \in \cancel{\mathcal{M}}\ \mathcal{A}$. Then, all $A_1, \cdots, A_{n+1} \in \cancel{\mathcal{M}}\ \mathcal{A}$ pairwise disjoint, each $A_i \neq \phi$, and $\bigcup_{i=1}^{n+1} A_i = X$.

By induction, we have constructed $A_j \in \mathcal{A}$ $(j = 1, 2, \cdots)$, nonempty, pairwise disjoint, and $X = \bigcup_{j=1}^{\infty} A_j$.

Define $f: \mathcal{P}(\mathbb{N}) \to \mathcal{A}$ by $f(\Lambda) = \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathcal{A}$ for any $\Lambda \subseteq \mathbb{N}$, $\Lambda \neq \phi$. $f(\phi) = \phi \in \mathcal{A}$.
Then, since all A_j $(j = 1, 2, \cdots)$ are pairwise disjoint, f is injective. Hence, $\mathrm{card}(\mathcal{A}) = \mathrm{card}\,\mathcal{P}(\mathbb{N}) > \mathrm{card}(\mathbb{N})$. So, $\mathcal{A}$ is uncountable. $\square$

9. Let $\mathcal{M}(\mathcal{H})$ denote the σ-algebra generated by a nonempty class $\mathcal{H}$ of subsets of X.
We show $\mathcal{M}(\mathcal{E}) = \bigcup_{\substack{\mathcal{F} \subseteq \mathcal{E} \\ \mathcal{F}:\text{ countable}}} \mathcal{M}(\mathcal{F})$ (*)

Since $\mathcal{E} \supseteq \mathcal{F}$ implies $\mathcal{M}(\mathcal{E}) \supseteq \mathcal{M}(\mathcal{F})$, we have
$$\mathcal{M}(\mathcal{E}) \supseteq \bigcup_{\substack{\mathcal{F} \subseteq \mathcal{E} \\ \mathcal{F}:\text{countable}}} \mathcal{M}(\mathcal{F}).$$
Let us denote the right-hand side of (*) by $\mathcal{A}$. Clearly $\mathcal{A}$ is nonempty. If $E \in \mathcal{A}$, then $E \in \mathcal{M}(\mathcal{F})$ for some $\mathcal{F} \subseteq \mathcal{E}$ countable. Since $\mathcal{M}(\mathcal{F})$ is a σ-algebra, $E^c \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{A}$. Now, let $E_1, E_2, \ldots$ be members of $\mathcal{A}$. Then $\exists\, \mathcal{F}_j \subseteq \mathcal{E}$ $\mathcal{F}_j$ is countable $(j=1,2,\ldots)$ such that $E_j \in \mathcal{M}(\mathcal{F}_j)$ for each $j \geq 1$. Let $\mathcal{F} = \bigcup_{j=1}^{\infty} \mathcal{F}_j$. $\mathcal{F} \subseteq \mathcal{E}$ and $\mathcal{F}$ is still countable. Moreover, $\mathcal{M}(\mathcal{F}) \supseteq \mathcal{M}(\mathcal{F}_j)$ and hence $E_j \in \mathcal{M}(\mathcal{F}_j) \subseteq \mathcal{M}(\mathcal{F})$. Hence $\bigcup_{j=1}^{\infty} E_j \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{A}$. Thus $\mathcal{A}$ is a σ-algebra. Clearly $\mathcal{A} \supseteq \mathcal{E}$. Hence $\mathcal{A} \supseteq \mathcal{M}(\mathcal{E})$. $\square$

10. Part c. We have $(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$
So $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1)$. Similarly $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_1)$.
But $(a,b) = \bigcup_{\substack{n=1 \\ b-a>\frac{1}{n}}}^{\infty} (a, b-\frac{1}{n}]$. So, $\mathcal{M}(\mathcal{E}_3) \supseteq \mathcal{M}(\mathcal{E}_1)$.
Similarly $\mathcal{M}(\mathcal{E}_4) \supseteq \mathcal{M}(\mathcal{E}_1)$. Thus
$$\mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_1).$$
But, it is proved that $\mathcal{M}(\mathcal{E}_1) = \mathcal{B}_{\mathbb{R}}$.

Part d. For $a<b$, $(a,b] = (a,\infty) \setminus (b,\infty)$, and $(a,\infty) = \bigcup_{n=1}^{\infty} (a, a+n]$. So $\mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}}$. Similarly $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_4) = \mathcal{B}_{\mathbb{R}}$. $\square$