Problem 1

$i : A \to B$ $p : B \to C$ Suppose we are given $\pi : B \to A$, then we can naturally define $\alpha : B \to A \oplus C$ by $\alpha(b) = (\pi(b), 0)$. Suppose we are given $\sigma : C \to B$, we can naturally define $\alpha' : A \oplus C \to B$ by $\alpha'(a, c) = \sigma(c)$.

Now suppose we are given $\alpha : B \to A \oplus C$, then we can define $\pi : B \to A$ by $\pi(b) = \alpha(b)[1]$ (i.e. take the first coordinate)

Similarly if we are given $\alpha' : A \oplus C \to B$, then we can naturally define $\sigma(c) = \alpha'(0, c)$.

If (iii) is isomorphic, then the case of α and α' become consistent and we obtain all the bijections.

Problem 2

For torus T we use two annuli A_1, A_2 that cover the upper part and lower part of the torus respectively and their intersection are two disjoint circles. Hence we have the following reduce MV sequence (as A_1, A_2 are homotopic equivalent to S^1 and their intersection is homeomorphic to $S^1 \sqcup S^1$) $0 \to H_2(T) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j} H_1(T) \xrightarrow{\partial} \mathbb{Z} \to 0$.

where i is induced by the inclusion map from $A_1 \cap A_2$ into A_1 and A_2, which can be represented by matrix $[1, 1; 1, 1]$, j is induced by the inclusion map from A_1 and A_2 into T.

Hence $H_2(T) \cong ker(i)$, but that is simply $[1, -1]\mathbb{Z} \cong \mathbb{Z}$. Hence $H_2(T) \cong \mathbb{Z}$.

$$
\begin{aligned}
H_1(T)/ker\partial &\cong \mathbb{Z} \\
&\cong H_1(T)/imgj \\
&\cong H_1(T)/(\mathbb{Z} \oplus \mathbb{Z}/kerj) \\
&\cong H_1(T)/(\mathbb{Z} \oplus \mathbb{Z}/imgi) \\
&\cong H_1(T)/\mathbb{Z}
\end{aligned}
$$

Hence $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Obviously we have (unreduced) $H_0(T) = \mathbb{Z}$. And $H_n(T) = 0$ for any $n > 2$ (as $H_n(S^1) = 0$ when $n \geq 2$.)

We decompose the Klein bottle K into two Mobius strip M_1, M_2. Since M_1, M_2 can deformation retract to S^1 and $M_1 \cap M_2$ is homeomorphic to S^1, $H_2(M_1) \oplus H_2(M_2) = 0$ and we have the following (reduced) MV sequence $0 \to H_2(K) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j} H_1(K) \xrightarrow{\partial} 0$

where i is induced by the inclusion map from $M_1 \cap M_2$ to M_1 and M_2, and j is induced by the inclusion map from M_1 and M_2 to K.

Hence $H_2(K) = 0$ as the inclusion map from $M_1 \cap M_2$ to M_1 and M_2 maps the generator to 2 times of the circular generator by properties of mobius strip.

We also have $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/ker(j) = \mathbb{Z} \oplus \mathbb{Z}/img(i) = \mathbb{Z} \oplus \mathbb{Z}_2$.

And obviously unreduced $H_0(K) = \mathbb{Z}$. And $H_n(K) = 0$ for any $n > 2$ (as $H_n(S^1) = 0$ when $n \geq 2$.)

Problem 3

(a) The induced map $f^* : \pi_1(S^1) \to \pi_1(S^1)$ maps the generator (call it 1 in this case) into n. As $\mathbb{Z}$ is already abelian, the abelianization (from π_1 to H_1) has trivial effect. Hence, the degree of f by definition is n.

(b) The base case $m = 0$ is proven in (a). Now suppose the conclusion (of degree) holds for S^{m+1}. Then consider S^{m+2} as $D^{m+2} \cup_{S^{m+1}} D^{m+2}$ (i.e. two hemisphere with ≥ 0 and ≤ 0 on the last coordinate, respectively). Hence by naturality of MV sequence we have the following commutative diagram:

$$\begin{array}{ccccc} 0 \to & H_k(S^{m+2}) & \xrightarrow{\partial} & H_{k-1}(S^{m+1}) & \to 0 \\ & \downarrow & & \downarrow & \\ & H_k(S^{m+2}) & \xrightarrow{\partial} & H_{k-1}(S^{m+1}) & \end{array}$$

where the down arrow represent f^* (and we know that f^* preserve coordinates in $\vec{x}$.). Since MV sequence tells us that ∂ is an isomorphism, the degree of f^* remains n for all m.

Problem 4

(a) we construct the homotopy $H(x,t) = \frac{tf(x)+(1-t)g(x)}{||tf(x)+(1-t)g(x)||}$. If the denominator never equals to 0 because that only can happen when $f(x)$ and $g(x)$ are on the same line and point towards the opposite direction, which in this means they are antipodal (i.e. $f(x) = -g(x)$). Hence f and g are homotopic.

(b) The antipodal map $S^n \to S^n$ is simply a composition of $n+1$ reflection map, each having degree -1. Hence the degree of antipodal map is $(-1)^{n+1}$. By Hopf's theorem, when n is even, the degree of antipodal map is -1 while the degree of 1_{S^n} is 1. Hence they are not homotopic to each other.

(c) Suppose there exists such non vanishing tangent field f. Then normalize it by $v = f/||f||$. We can now construct a homotopy between the identity map and the antipodal map $H(x,t) = cos(t)x + sin(t)v(x)$, where $t \in [0, \pi]$. This means the degree of the antipodal map has to equal to that of the identity map, which by (b) means n must be odd.