Math 240A, Fall 2019
Solution to Problems of HW#6
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1. Note that f is Lebesgue measurable and the function $V:\mathbb{R}\to\mathbb{R}$, defined by $V(x)=\frac{1}{x}$ if $x\neq 0$ and $V(0)=0$, is also measurable. Hence $g(x)=f(x)V(x)$ is also measurable.

Since $f(0)=0$ and $f'(0)$ exists,

$$|g(x)| = \left|\frac{f(x)}{x}\right| = \left|\frac{f(x)-f(0)}{x}\right| \longrightarrow |f'(0)| \text{ as } x\to 0.$$

Thus, $\exists\, \delta>0$ such that

$$|g(x)| \le 1+|f'(0)| \quad \forall\, x\in[-\delta,\delta].$$

(Note that $g(0)=0$). Therefore,

$$\begin{aligned}
\int_{\mathbb{R}}|g|\,dm &= \int_{[-\delta,\delta]}|g|\,dm + \int_{\mathbb{R}\setminus[-\delta,\delta]}|g|\,dm \\
&\le \int_{[-\delta,\delta]}\left[1+|f'(0)|\right]dm + \int_{\{|x|>\delta\}}\frac{|f(x)|}{|x|}\,dm(x) \\
&\le 2\delta\left[1+|f'(0)|\right] + \int_{\{|x|>\delta\}}\frac{1}{\delta}|f(x)|\,dm(x) \\
&\le 2\delta\left[1+|f'(0)|\right] + \frac{1}{\delta}\int_{\mathbb{R}}|f|\,dm \\
&< \infty.
\end{aligned}$$

Hence, $g\in L^1(m)$. $\square$

2 (1) $\log(1+e^t) < c+t \quad (\forall t>0)$

$\Longleftrightarrow \quad 1+e^t < e^{c+t} = e^c e^t \quad (\forall t>0)$.

Hence $c>0$. Write $c = \log a$ for some $a>1$.

$\log(1+e^t) < c+t \Longleftrightarrow \log(1+e^t) < \log a + t$

$\Longleftrightarrow \log \frac{1+e^t}{a} < t \Longleftrightarrow \frac{1+e^t}{a} < e^t \Longleftrightarrow 1+e^t < ae^t$

$\Longleftrightarrow 1 < (a-1)e^t \quad \forall t>0.$

Let $t \to 0^+$. We get $1 \le a-1$, i.e., $a \geq 2$. So, the smallest $c = \log 2$.

(2) Let $g_n(x) = \frac{1}{n}\log\left[1+e^{nf(x)}\right] \quad (x \in [0,1])$

$g(x) = \ln 2 + f(x) \quad (x \in [0,1])$.

By L'Hospital's rule

$$\lim_{t\to+\infty} \frac{\ln(1+e^t)}{t} = \lim_{t\to+\infty} \frac{\frac{e^t}{1+e^t}}{1} = 1.$$

Thus, if $f(x) > 0$ for some $x \in [0,1]$ then

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \log\left\{\left[1+e^{nf(x)}\right]^{\frac{1}{nf(x)}}\right\}^{f(x)}$$

$$= \log e^{f(x)} = f(x).$$

If $f(x) = 0$ at $x \in [0,1]$ then

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{1}{n}\log 2 = 0 = f(x).$$

Hence $\lim_{n\to\infty} g_n(x) = f(x) \quad \forall x \in [0,1]$.

By Part (1), $|g_n(x)| \le \frac{1}{n}\left(\log 2 + nf(x)\right)$

$\le \log 2 + f(x) = g(x). \quad \forall x \in [0,1]$

Since $f \in L^1([0,1])$, we have $g \in L^1([0,1])$. Thus it follows from the Lebesgue Dominanted Convergence Theorem that $\lim_{n\to\infty}\int_0^1 g_n(x)\,dx = \int_0^1 f(x)\,dx$.

i.e., $\lim_{n\to\infty} \frac{1}{n}\int_0^1 \log\left[1+e^{nf(x)}\right]dx = \int_0^1 f(x)\,dx.$ $\square$

3. For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $1 \le k \le 2^n$, define

$$g_{n,k}(x) = \begin{cases} (-1)^n & \text{if } x = 0, \\ \chi_{(0,1]\setminus I_{n,k}}(x) & \text{if } 0 < x \le 1, \end{cases}$$

where

$$I_{n,k} = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right].$$

Note that $(0,1] = \bigcup_{k=1}^{2^n} I_{n,k}$ is a disjoint union.

Also, $$g_{n,k}(x) = \begin{cases} 1 & \text{if } x \neq 0,\ x \notin I_{n,k} \\ 0 & \text{if } x \in I_{n,k}. \end{cases}$$

Clearly, $$\int_0^1 g_{n,k}\, dm = 1 - m(I_{n,k}) = 1 - \frac{1}{2^n} \to 0$$

as $n \to \infty$.

Clearly $g_{n,k}(0) = (-1)^n$ diverges. If $x \in (0,1]$, then since $(0,1] = \bigcup_{k=1}^{2^n} I_{n,k}$ for each $n \in \mathbb{N}$, there exists $k = k(n,x)$ such that $x \in I_{n,k}$. Thus, for each $n \in \mathbb{N}$, $\exists k$ such that $g_{n,k}(x) = 0$. But, by the same disjoint union, there are other k's $\in$ (in fact all other k's) such that $g_{n,k}(x) = 1$. Thus $\{g_{n,k}(x)\}$ diverges.

Now, relabel $g_{n,k}$ $(n = 1, 2, \cdots,\ k = 1, 2, \cdots, 2^n)$ into $\{f_m\}_{m=1}^{\infty}$ in a natural order ($n = 1, 2, \cdots$ for each n, $k = 1, \cdots, 2^n$) with $m = 2^n - 1 + k$ $(n = 1, 2, \cdots$ $k = 1, \cdots, 2^n - 1)$. Then, $\int_0^1 f_m\, dm \to 1$ as $m \to \infty$ and $\{f_m(x)\}$ diverges at any $x \in [0,1]$. $\square$

4. Since $|f_n| \le g$ on X and $f_n \to f$ a.e. we have $|f| \le g$ a.e. Hence $|f_n - f| \le 2g$ a.e. If $k \in \mathbb{N}$ then $\exists A_n(k) \in \mathcal{M}_\mu$ such that $\mu(A_n(k)) = 0$ and

$$\{|f_n - f| \ge \tfrac{1}{k}\} \subseteq \{2g \ge \tfrac{1}{k}\} \cup A_n(k), \quad n = 1, 2, \cdots;$$

Fix $k \ge 1$. Let $E_n(k) = \bigcup_{m=n}^{\infty} \{|f_m - f| \ge \frac{1}{k}\}$ $(n = 1, 2, \cdots)$.
Then $\bigcap_{n=1}^{\infty} E_n(k) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{|f_m - f| \ge \frac{1}{k}\}$

$= \{x \in X :$ there are infinitely many $n \in \mathbb{N}$ such that $|f_n(x) - f(x)| \ge \frac{1}{k}\}$.

Since $f_n \to f$ a.e., $\mu(\bigcap_{n=1}^{\infty} E_n(k)) = 0$.

Note that $E_n(k)$ decreases as n increases. Moreover,

$$E_1(k) = \bigcup_{m=1}^{\infty} \{|f_m - f| \ge \tfrac{1}{k}\} \subseteq \{2g \ge \tfrac{1}{k}\} \cup \left(\bigcup_{n=1}^{\infty} A_n(k)\right)$$

But $g \in L^1(\mu)$. So, $\mu(\{|g| < \infty\}) < \infty$. Also, $\mu(\bigcup_{n=1}^{\infty} A_n(k)) \le \sum_{n=1}^{\infty} \mu(A_n(k)) = 0$. Therefore, $\mu(E_1(k)) < \infty$.
Consequently, by the continuity of measure from above

$$\lim_{n\to\infty} \mu(E_n(k)) = \mu\left(\bigcap_{n=1}^{\infty} E_n(k)\right) = 0.$$

Let $\varepsilon > 0$. We can choose $n_k \uparrow$ and $\mu(E_{n_k}(k)) < \varepsilon/2^k$ $(k = 1, 2, \cdots)$. Let $E = \bigcup_{k=1}^{\infty} E_{n_k}(k)$. Then $\mu(E) \le \sum_{k=1}^{\infty} \mu(E_{n_k}(k)) < \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon$. Moreover, $E^c = \bigcap_{k=1}^{\infty} \bigcap_{m \ge n_k} \{|f_m - f| < \frac{1}{k}\}$.
Thus, for any $k \in \mathbb{N}$, $\exists n_k$ such that if $m \ge n_k$ then $E^c \subseteq \{|f_m - f| < \frac{1}{k}\}$, i.e., $|f_m(x) - f(x)| < \frac{1}{k}$ $\forall x \in E^c$. Hence, $f_n \to f$ uniformly on E^c. $\square$

5. Since $f = \mathrm{Re}f + i\,\mathrm{Im}f$, we can just consider the case that f is real-valued. Further, if f is real-valued, $f = f^+ - f^-$. So, we can assume that f is nonnegative.

Suppose f is a simple function: $f = \sum_{j=1}^{n} a_j \chi_{E_j}$, where $a_j \in \mathbb{R}$, distinct, $E_j \in \mathcal{L}$ (class of Lebesgue measurable subsets of $[a,b]$), pairwise disjoint, and $\bigcup_{j=1}^{n} E_j = [a,b]$. Let $\varepsilon > 0$. By the inner regularity of the Lebesgue measure m, for each j there exists a compact set $K_j \subseteq E_j$ such that $m(K_j) > m(E_j) - \frac{\varepsilon}{n}$. Let $E = \bigcup_{1}^{n} K_j$. Then E is compact. Moreover $f|_E$ is continuous, since f is constant on each E_j and hence each K_j, and $\mathrm{dist}(K_i, K_j) > 0$ if $i \neq j$ (since $K_i \subseteq E_i$, $K_j \subseteq E_j$, $E_i \cap E_j = \phi$, $i \neq j$, and K_i, K_j are compact.) Finally,

$$m(E^c) = m([a,b] \cap E^c) = m\left(\left(\bigcup_{j=1}^{n} E_j\right) \cap E^c\right)$$

$$= m\left(\bigcup_{j=1}^{n} (E_j \cap E^c)\right) = m\left(\bigcup_{j=1}^{n} (E_j \setminus E)\right)$$

$$= m\left(\bigcup_{j=1}^{n} (E_j \setminus K_j)\right) \leq \sum_{j=1}^{n} m(E_j \setminus K_j) < \varepsilon.$$

Now, suppose $f: [a,b] \to [0,\infty)$ is measurable. There exist simple functions ϕ_n $(n = 1, 2, \cdots)$ such that $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_n \leq \cdots$ and $\phi_n \to f$ pointwise on $[a,b]$. $\forall \varepsilon > 0$. By the previously proved result, for each ϕ_n, $\exists$ compact $K_n \subseteq [a,b]$ such that

$m([a,b]\setminus K_n) < \frac{\varepsilon}{2^{n+1}}$ and $\Phi_n|_{K_n}$ is continuous.

Let $E_0 = \bigcap_{n=1}^{\infty} K_n$. E_0 is compact, and

$$m([a,b]\setminus E_0) = m\left([a,b]\cap\left(\bigcup_{n=1}^{\infty} K_n^c\right)\right)$$
$$= m\left(\bigcup_{n=1}^{\infty}[a,b]\cap K_n^c\right) = m\left(\bigcup_{n=1}^{\infty}([a,b]\setminus K_n)\right)$$
$$\le \sum_{n=1}^{\infty} m([a,b]\setminus K_n) < \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}.$$

Now, $\Phi_n \to f$ on E_0 pointwise. By Egoroff's Thm, there exists a measurable subset $E \subseteq E_0$ with $m(E_0\setminus E) < \frac{\varepsilon}{2}$ and $\Phi_n \to f$ uniformly on E. By the inner regularity, we may assume ~~the~~ E is compact. Then, $m(E^c) = m([a,b]\setminus E) \le m([a,b]\setminus E_0) + m(E_0\setminus E) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. Moreover $f|_E$ is continuous since $f|_E$ is the uniform limit of a sequence of continuous function Φ_n. $\square$

6. (1) Since $f: X\to\mathbb{C}$ and $g: Y\to\mathbb{C}$ are measurable, there exist simple functions $\phi_n: X\to\mathbb{C}$ and $\psi_n: Y\to\mathbb{C}$ $(n=1,2,\cdots)$ such that $0\le|\phi_1|\le|\phi_2|\le\cdots\le|f|$, $\phi_n\to f$ pointwise on X, and $0\le|\psi_1|\le|\psi_2|\le\cdots\le|g|$ and $\psi_n\to g$ pointwise on Y. Therefore $\phi_n(x)\psi_n(y)\to h(x,y) = f(x)g(y)$ $\forall x\in X$ $\forall y\in Y$. Fix $n\in\mathbb{N}$, write $\phi=\phi_n$ and $\psi=\psi_n$. We have

$$\phi(x) = \sum_{j=1}^{m} a_j \chi_{E_j}(x),\quad a_j\in\mathbb{C},\ E_j\in\mathcal{M},\ \bigcup_j E_j = X$$
(disjoint).
$$\psi(y) = \sum_{k=1}^{\ell} b_k \chi_{F_k}(y),\quad b_k\in\mathbb{C},\ F_k\in\mathcal{N},\ \bigcup_k F_k = Y$$
(disjoint).

Thus, $\phi(x)\psi(y) = \sum_{j=1}^{m} \sum_{k=1}^{\ell} a_j b_k \chi_{E_j}(x) \chi_{F_k}(y)$

$$= \sum_{j=1}^{m} \sum_{k=1}^{\ell} a_j b_k \chi_{E_j \times F_k}(x,y).$$

This is a simple function on $X \times Y$ with respect to $\mu \times \nu$, and is therefore $\mathcal{M} \otimes \mathcal{N}$-measurable. Since all $\phi_n(x)\psi_n(y)$ are $\mathcal{M} \otimes \mathcal{N}$-measurable, their pointwise limit $h(x,y) = f(x)g(y) = \lim_{n\to\infty} \phi_n(x)\psi_n(y)$ is also $\mathcal{M} \otimes \mathcal{N}$-measurable.

(2) Since $f \in L^1(\mu)$ and $g \in L^1(\nu)$, $\mu(\{|f|=\infty\})=0$, $\nu(\{|g|=\infty\})=0$, and μ, ν are σ-finite on $\{0<|f|<\infty\}$, $\{0<|g|<\infty\}$, respectively. Hence, $\mu \times \nu$ is σ-finite on $\{0<|h|<\infty\} = \{0<|f|<\infty\} \times \{0<|g|<\infty\}$. Note that $\{|h|=\infty\} \subseteq (\{|f|=\infty\} \times Y) \cup (X \cup \{|h|=\infty\})$. Thus $\mu \times \nu(\{|h|=\infty\}) = 0$ $(0 \cdot \infty = 0!)$ Hence, applying Tonelli's Theorem on $\{0<|h|<\infty\} = \{0<|f|<\infty\} \times \{0<|g|<\infty\}$, we have

$$\int_{X \times Y} |h| \, d\mu \, d\nu = \int_{\{0<|h|<\infty\}} |h| \, d(\mu \times \nu) = \int_{\{0<|f|<\infty\}} \left[\int_{\{0<|g|<\infty\}} |g(y)| \, d\nu(y) \right] |f(x)| \, d\mu(x)$$

$$= \int_X |f| \, d\mu \cdot \int_Y |g| \, d\nu < \infty.$$ Hence, $h \in L^1(\mu \times \nu)$.

Now, applying Fubini's Theorem, we get

$$\int_{X \times Y} h(x,y) \, d\mu(x) \, d\nu(y)$$

$$= \int_X \left[\int_Y h(x,y) \, d\nu(y) \right] d\mu(x)$$

$$= \int_X \left[\int_Y f(x) g(y) \, d\nu(y) \right] d\mu(x)$$

$$= \int_X f(x) \left(\int_Y g \, d\nu \right) d\mu(x)$$

$$= \left(\int_X f \, d\mu \right) \left(\int_Y g \, d\nu \right). \quad \square$$

7. Define $\alpha: X \times [0,\infty) \to \mathbb{R}$, $\beta: X \times [0,\infty) \to [0,\infty)\times[0,\infty)$, and $\gamma: [0,\infty)\times[0,\infty) \to \mathbb{R}$ by $\alpha(x,y) = f(x) - y$, $\beta(x,y) = (f(x), y)$, and $\gamma(u,v) = u - v$, respectively. Clearly

$$(\gamma\circ\beta)(x,y) = \gamma(\beta(x,y)) = \gamma(f(x), y) = f(x) - y = \alpha(x,y).$$ So, $\alpha = \gamma\circ\beta$.

Clearly γ is a continuous function, so it is measurable. Let $A, B \subseteq [0,\infty)$ be Borel sets. We have $\beta^{-1}(A\times B) = f^{-1}(A)\times B \in \mathcal{M}_\xi \otimes \mathcal{B}_{\mathbb{R}}$.
In fact, $(a,b) \in \beta^{-1}(A\times B) \Leftrightarrow \beta(a,b) = (f(a), b) \in A\times B \Leftrightarrow f(a)\in A$ and $b \in B \Leftrightarrow a \in f^{-1}(A)$ and $b\in B \Leftrightarrow (a,b) \in f^{-1}(A)\times B$.
Since $\mathcal{B}_{\mathbb{R}}\otimes\mathcal{B}_{\mathbb{R}}$ on $[0,\infty)\times[0,\infty)$ is generated by $\{A\times B : A\subseteq[0,\infty), B\subseteq[0,\infty), A, B \in \mathcal{B}_{\mathbb{R}}\}$, β is $\mathcal{M}_\xi\otimes\mathcal{B}_{\mathbb{R}}$-measurable. Thus, $\alpha = \gamma\circ\beta$ is $\mathcal{M}_\xi\otimes\mathcal{B}_{\mathbb{R}}$-measurable. Since $G_f = \{\alpha \geq 0\}$, G_f is $\mathcal{M}_\xi\otimes\mathcal{B}_{\mathbb{R}}$-measurable.

To prove the identity $(\mu\times m)(G_f) = \int_X f\,d\mu$, we proceed in several steps. Step 1: we assume f is bounded. Step 1.1. $\mu(X) < \infty$. Step 1.2: μ is σ-finite. Step 2. consider a general $f: X \to [0,\infty)$.

Assume f is bounded, i.e., $\exists M > 0$ such that $0 \leq f < M$ for all $x\in X$. Assume $\mu(X) < \infty$. Let $\{\phi_n\}$ be a sequence of simple functions on X such that $0 \leq \phi_n \uparrow f$ on X.

Then $X \times [0,M] \setminus G_f = \{(x,y) \in X \times [0,M] : y > f(x)\}$
$$= \bigcap_{n=1}^{\infty} \{(x,y) \in X \times [0,M] : y > \phi_n(x)\}.$$
Since $G_f \subseteq X \times [0,M]$ we have
$$M\mu(X) - (\mu \times m)(G_f)$$
(✱) $$= (\mu \times m)(X \times [0,M] \setminus G_f)$$
$$= \lim_{n\to\infty} (\mu \times m)(\{(x,y) \in X \times [0,M] : y > \phi_n(x)\})$$
Fix n and write $\phi(x) = \phi_n(x) = \sum_{j=1}^{m} a_j \chi_{E_j}(x)$ with $E_j \in \mathcal{M}$ $(1 \le j \le m)$, $X = \bigcup_1^m E_j$ (disjoint), and $a_j \ge 0$. Note $\phi_n \le f < M$ so, $a_j < M$ $(1 \le j \le m)$. Moreover,
$$\{(x,y) \in X \times [0,M] : y > \phi(x)\}$$
$$= \bigcup_{j=1}^{m} \{(x,y) \in E_j \times [0,M] : y > a_j\}$$
$$= \bigcup_{j=1}^{m} E_j \times (a_j, M].$$
This is a disjoint union. Thus,
$$(\mu \times m)(\{(x,y) \in X \times [0,M] : y > \phi(x)\})$$
$$= \sum_{j=1}^{m} \mu(E_j)(M - a_j)$$
$$= M\mu(X) - \sum_1^m a_j \mu(E_j)$$
$$= M\mu(X) - \int_X \phi \, d\mu$$
Consequently, from (✱), by the Monotone Convergence Thm.
$$M\mu(X) - (\mu \times m)(G_f) = M\mu(X) - \lim_{n\to\infty} \int_X \phi_n \, d\mu$$
$$= M\mu(X) - \int_X f \, d\mu$$
Hence $(\mu \times m)(G_f) = \int_X f \, d\mu$.

If μ is σ-finite, then $\exists X_j \in \mathcal{M}$, $X_j \uparrow \bigcup_{j=1}^{\infty} X_j = X$, and $\mu(X_j) < \infty$ $(\forall j \ge 1)$. Denote
$$G_f(X_j) = \{(x,y) \in X_j \times [0,\infty) : y \le f(x)\}.$$
Then, as shown before, each $G_f(X_j)$ is measurable.

Since X_j increases, $G_f(X_j)$ increases. Since $\bigcup_1^\infty X_j = X$, $\bigcup_1^\infty G_f(X_j) = G_f$. Then

$$
\begin{aligned}
(\mu\times m)(G_f) &= \lim_{j\to\infty} (\mu\times m)(G_f(X_j)) \quad \left[\text{continuity of measure}\right] \\
&= \lim_{j\to\infty} \int_{X_j} f\,d\mu \quad \left[\text{since } \mu(X_j)<\infty \text{ and the previously proven result}\right]. \\
&= \lim_{j\to\infty} \int_X \chi_{X_j} f\,d\mu \\
&= \int_X f\,d\mu \quad [\text{MCT}].
\end{aligned}
$$

Now, assume a general case: $f: X \to [0,\infty)$. Let $f_n = \min(f, n)$. Then $f_n : X \to [0,\infty)$ is measurable. $0 \le f_1 \le f_2 \le \cdots$ $f_n(x) \to f(x)$ $\forall x \in X$. Define

$$G_{f_n} = \{(x,y) \in X \times [0,\infty) : y \le f_n(x)\}. \quad (n=1,2,\cdots)$$

Then $G_{f_n} \subseteq G_{f_{n+1}}$ $(n=1,2\cdots)$. Clearly $\bigcup_{n=1}^\infty G_{f_n} \subseteq G_f$. If $(x,y) \in G_f$, then $x \in X$, $0 \le y \le f(x)$. Let $n \in \mathbb{N}$ be such that $n > f(x)$ [possible since $f(x)<\infty$]. then $f(x) = f_n(x) = \min(f(x), n)$. Hence, $(x,y) \in G_{f_n}$. Thus $G_f = \bigcup_{n=1}^\infty G_{f_n}$. Consequently, by the continuity of measure, what's proved above, and the MCT,

$$
\begin{aligned}
(\mu\times m)(G_f) &= \lim_{n\to\infty} (\mu\times m)(G_{f_n}) \\
&= \lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu. \quad \square
\end{aligned}
$$

8. The left-hand side is

$$\int_0^1 \int_0^\infty (e^{-xy} - 2e^{-2xy})\,dy\,dx$$

$$= \int_0^1 \left(-\frac{1}{x} e^{-xy} + \frac{1}{x} e^{-2xy}\right)\Big|_{y=0}^{y=\infty} dx$$

$$= \int_0^1 0\,dx = 0.$$

The right-hand side is

$$\int_0^\infty \int_0^1 (e^{-xy} - 2e^{-2xy})\,dx\,dy$$

$$= \int_0^\infty \left(-\frac{1}{y} e^{-xy} + \frac{1}{y} e^{-2xy}\right)\Big|_{x=0}^{x=1} dy$$

$$= \int_0^\infty \frac{1}{y}(e^{-2y} - e^{-y})\,dy$$

< 0 since $e^{-2y} - e^{-y} < 0$ for $y > 0$. $\square$

9. Let $f(x,t) = \sin x\, e^{-xt}$ $(x > 0,\ t > 0)$. Let $A > 0$.
We have

$$\int_0^A \int_0^\infty |f(x,t)|\,dt\,dx = \int_0^A |\sin x| \left[\int_0^\infty e^{-xt}\,dt\right] dx$$

$$= \int_0^A |\sin x| \left(-\frac{1}{x} e^{-xt}\right)\Big|_{t=0}^{t=\infty} dx$$

$$= \int_0^A \frac{|\sin x|}{x}\,dx < \infty$$

Since $\frac{\sin x}{x} \to 1$ as $x \to 0$. Thus by Tonelli's Thm, $f \in L^1((0,A)\times(0,\infty))$. Hence, using the fact that $1/x = \int_0^\infty e^{-xt}\,dt$ $(\forall x > 0)$, we have by Fubini's thm that

$$\int_0^A \frac{\sin x}{x}\,dx = \int_0^A \int_0^\infty \sin x\, e^{-xt}\,dt\,dx$$

$$= \int_0^\infty \int_0^A \sin A\, e^{-xt}\,dx\,dt.$$

By the integration by parts, we get

$$\int_0^A \sin x\, e^{-xt}\,dx = \int_0^A (-\cos x)' e^{-xt}\,dx$$
$$= -\cos x\, e^{-xt}\Big|_{x=0}^{x=A} - \int_0^A (-\cos x)(-t)e^{-xt}\,dx$$
$$= -(\cos A)e^{-At} + 1 - t\int_0^A \cos x\, e^{-xt}\,dx$$
$$= -(\cos A)e^{-At} + 1 - t\int_0^A (\sin x)' e^{-xt}\,dx$$
$$= -(\cos A)e^{-At} + 1 - t\left[\sin x\, e^{-xt}\Big|_{x=0}^{x=A} - \int_0^A \sin x(-t)e^{-xt}\,dx\right]$$
$$= -(\cos A)e^{-At} + 1 - t\sin A\, e^{-At} - t^2\int_0^A \sin x\, e^{-xt}\,dx.$$

Thus. $\int_0^A \sin x\, e^{-xt}\,dx = \frac{1}{1+t^2}\left[-e^{-At}\cos A + 1 - t e^{-At}\sin A\right]$

This is controlled by $\frac{1}{1+t^2}$ for large A. So, by the dominant convergence theorem, we have

$$\int_0^A \frac{\sin x}{x}\,dx = \int_0^\infty \frac{1}{1+t^2}\,dt - \int_0^\infty \frac{e^{-At}(\cos A + t\sin A)}{1+t^2}\,dt$$
$$\to \arctan t\Big|_0^\infty - \int_0^\infty 0\,dt = \frac{\pi}{2}. \qquad \square$$

10. Let $H(x,t) = \chi_{(x,a)}(t)\,\frac{f(t)}{t}$ $(0<x,\ t<a)$.

$$\chi_{(x,a)}(t) = \chi_{(0,t)}(x) = \begin{cases} 1 & \text{if } x<t \\ 0 & \text{if } x\geq t \end{cases}$$

Thus. $\int_0^a \int_0^a |H(x,t)|\,dx\,dt$

$$= \int_0^a \frac{|f(t)|}{t}\left[\int_0^a \chi_{(0,t)}(x)\,dx\right]dt$$
$$= \int_0^a \frac{|f(t)|}{t}\, t\,dt$$
$$= \int_0^a |f(t)|\,dt < \infty$$

Since $f \in L^1((0,a))$. Thus. $H \in L^1((0,a)\times(0,a))$.

Now, by Fubini's Theorem,

$$\int_{(0,a)} g\,dm = \int_0^a g(x)\,dx = \int_0^a \int_x^a \frac{f(t)}{t}\,dt\,dx$$

$$= \int_0^a \int_0^a \chi_{(x,a)}(t)\,\frac{f(t)}{t}\,dt\,dx$$

$$= \int_0^a \int_0^a H(x,t)\,dt\,dx$$

$$= \int_0^a \frac{f(t)}{t}\left[\int_0^a \chi_{(x,a)}(t)\,dx\right]dt$$

$$= \int_0^a \frac{f(t)}{t}\left[\int_0^a \chi_{(0,t)}(x)\,dx\right]dt$$

$$= \int_0^a f(t)\,dt = \int_{(0,a)} f\,dm. \qquad \square$$