Math 240B, Winter 2020
Solution to Problems of HW #3
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1. (1) Clearly $L^p(\mu) \cap L^r(\mu)$ is a vector space, since it is a vector subspace of the space of all measurable functions. Since $\|\cdot\|_p$ and $\|\cdot\|_r$ are norms on $L^p(\mu)$ and $L^r(\mu)$, respectively, $\|\cdot\| = \|\cdot\|_p + \|\cdot\|_r$ is a norm on $L^p(\mu) \cap L^r(\mu)$. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence, i.e., $f_n \in L^p(\mu) \cap L^r(\mu)$ $(n = 1, 2, \cdots)$ and $\|f_n - f_m\| = \|f_n - f_m\|_p + \|f_n - f_m\|_r \to 0$. Then, $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(\mu)$, and in $L^r(\mu)$. Thus, $\exists f \in L^p(\mu)$ such that $f_n \to f$ in $L^p(\mu)$ and $\exists g \in L^r(\mu)$ such that $f_n \to g$ in $L^r(\mu)$.

Since $f_n \to f$ in $L^p(\mu)$, there exists a subsequence $f_{n_k} \to f$ μ-a.e. Since $f_{n_k} \to g$ in $L^r(\mu)$, there exists a subsequence of f_{n_k} that converges μ-a.e. to g. Hence $f = g$ μ-a.e. and $f \in L^p(\mu) \cap L^r(\mu)$, $\|f_n - f\| = \|f_n - f\|_p + \|f_n - f\|_r \to 0$. Hence, $L^p(\mu) \cap L^r(\mu)$ is a Banach space.

It follows from Proposition 6.10 that $\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ for some $\lambda \in (0,1)$. Thus, if

$$\|f_n - f\| = \|f_n - f\|_p + \|f_n - f\|_r \to 0$$

then $\|f_n - f\|_p \to 0$ and $\|f_n - f\|_r \to 0$, and hence

$$\|f_n - f\|_q \le \|f_n - f\|_p^\lambda \, \|f_n - f\|_r^{1-\lambda} \to 0.$$

Thus, the inclusion map $L^p(\mu) \cap L^r(\mu) \to L^q(\mu)$ is continuous.

(2) Clearly $L^p(\mu)+L^r(\mu)$ is a vector space as both $L^p(\mu)$ and $L^r(\mu)$ are vector subspaces of the space of measurable functions.

Let $f \in L^p(\mu)+L^r(\mu)$. Then, clearly $\|f\| \geq 0$.
If $f=0$ then $f=g+h$ with $g=0$ $h=0$. So $\|f\|=\|0\|=0$.
If $\|f\|=0$. then $\exists\, g_n \in L^p(\mu)$, $h_n \in L^r(\mu)$ such that $f=g_n+h_n$ and $\|g_n\|_p+\|h_n\|_r \to 0$. Since $g_n \to 0$ in $L^p(\mu)$, $\exists\, g_{n_k} \to 0$ a.e. Since $h_{n_k} \to 0$ in $L^r(\mu)$, $\exists\, h_{n_{k_j}} \to 0$ a.e. Thus $f = g_{n_{k_j}} + h_{n_{k_j}} \to 0$ a.e.
Hence $f=0$ a.e. i.e., $f=0$ in $L^p(\mu)+L^r(\mu)$.
Clearly $\|\lambda f\| = |\lambda|\,\|f\|$ $\forall \lambda \in K (= \mathbb{R}$ or $\mathbb{C})$ and $\forall f \in L^p(\mu)+L^r(\mu)$. To show the triangle inequality let $f_1, f_2 \in L^p(\mu)+L^r(\mu)$ and let $\varepsilon>0$. Then, $\exists\, g_j \in L^p(\mu)$ and $h_j \in L^r(\mu)$ $(j=1,2)$ such that $\|f_j\| \geq \|g_j\|_p + \|h_j\|_r - \varepsilon$, $f_j = g_j + h_j$, $j=1,2$. Thus,
$f_1+f_2 = g_1+h_1+g_2+h_2 = (g_1+g_2)+(h_1+h_2)$ with $g_1+g_2 \in L^p(\mu)$ and $h_1+h_2 \in L^r(\mu)$. Thus

$$\|f_1+f_2\| \leq \|g_1+g_2\|_p + \|h_1+h_2\|_r \leq \|f_1\| + \|f_2\| + 2\varepsilon$$

Hence $\|f_1+f_2\| \leq \|f_1\|+\|f_2\|$. Therefore $L^p(\mu)+L^r(\mu)$ is a normed vector space.

Let $f_n \in L^p(\mu)+L^r(\mu)$ $(n=1,2,\cdots)$ and assume $\sum_{n=1}^{\infty}\|f_n\| < \infty$. $\forall n \geq 1$, $\exists\, g_n \in L^p(\mu)$ and $\exists\, h_n \in L^r(\mu)$ such that $\|f_n\| \geq \|g_n\| + \|h_n\| - \frac{1}{2^n}$ and $f_n = g_n + h_n$.
Thus, $\sum_{n=1}^{\infty}\|g_n\|_p < \infty$ and $\sum_{n=1}^{\infty}\|h_n\|_r < \infty$. But both $L^p(\mu)$ and $L^r(\mu)$ are complete. Hence $g = \sum_{n=1}^{\infty} g_n \in L^p(\mu)$ and $h = \sum_{n=1}^{\infty} h_n \in L^r(\mu)$. These mean that $\|\sum_{n=1}^{N} g_n - g\|_p \to 0$ and $\|\sum_{n=1}^{N} h_n - h\|_r \to 0$ as $N \to \infty$.

Let $f = g+h \in L^p(\mu)+L^r(\mu)$. Then

$$\left\| \sum_{n=1}^{N} f_n - f \right\| \le \left\| \sum_{n=1}^{N} (g_n+h_n) - (g+h) \right\|$$
$$= \left\| \left(\sum_{n=1}^{N} g_n - g\right) + \left(\sum_{n=1}^{N} h_n - h\right) \right\|$$
$$\le \left\| \sum_{n=1}^{N} g_n - g \right\|_p + \left\| \sum_{n=1}^{N} h_n - h \right\|_r$$
$$\to 0 \text{ as } N \to \infty.$$

Hence $\sum_{n=1}^{\infty} f_n = f$ with respect to the norm $\|\cdot\|$.
Thus $L^p(\mu)+L^r(\mu)$ is a Banach space.

2. Since $[L^p(\mu)]^* = L^q(\mu)$, where $\frac{1}{p}+\frac{1}{q}=1$, we need only to show that for any $g \in L^q(\mu)$,

$$\lim_{n\to\infty} \int f_n g \, d\mu = \int f g \, d\mu. \qquad (*)$$

Fix $g \in L^q(\mu)$, $\forall \varepsilon > 0$ there exists a simple function $\phi \in L^q(\mu)$ such that $\|\phi - g\|_q < \varepsilon$, $|\phi| \le |g|$.
Let $C = \sup_{n \ge 1} \|f_n\|_p < \infty$. Then

$$\left| \int (f_n - f) g \, d\mu \right|$$
$$\le \left| \int (f_n - f)(g-\phi) \, d\mu \right| + \left| \int (f_n - f)\phi \, d\mu \right|$$
$$\le \|f_n - f\|_p \|g-\phi\|_q + \left| \int (f_n - f)\phi \, d\mu \right|$$
$$\le (C + \|f\|_p)\varepsilon + \left| \int (f_n - f)\phi \, d\mu \right|$$

So, $\limsup_{n\to\infty} \left| \int (f_n - f) g \, d\mu \right| \le (C+\|f\|_p)\varepsilon$
as $\int (f_n - f)\phi \, d\mu \to 0$ by the assumption.
Hence $\int (f_n - f) g \, d\mu \to 0$. $(*)$ is true.

3. (1) Let $f(x)=0$ $\forall x\in[0,1]$. For each $n\in\mathbb{N}$, let $I_{n,k}=\left[\frac{k-1}{2^n},\frac{k}{2^n}\right)$ $(k=1,\cdots,2^n)$ and define $f_n(x)=(-1)^k$ if $x\in I_{n,k}$ $(1\le k\le 2^n)$ and $f_n(1)=1$. We show that $f_n\to f$ weakly in $L^2([0,1])$. We use the criteria for such convergence stated in the above problem. Clearly $\|f_n\|_{L^2([0,1])}=1$ for all $n\ge 1$. So, we need only to show that if $E\subseteq[0,1]$ is measurable, then $\int_E f_n\,dm\to\int_E f\,dm$. ($m$ denotes the Lebesgue measure.)

First, consider $E=(a,b)\subseteq[0,1]$. Let $N\in\mathbb{N}$ be such that $\frac{1}{2^N}<\frac{b-a}{2}$. Let $n\ge N$. Let $1\le j<k\le 2^n$ be such that $0\le\frac{j}{2^n}-a\le\frac{1}{2^n}$ and $0\le b-\frac{k}{2^n}\le\frac{1}{n}$. Then

$$\left|\int_{(a,b)}f_n\,dm-\int_{(a,b)}f\,dm\right|=\left|\int_{(a,b)}f_n\,dm\right|$$

$$\le\left|\int_{(a,\frac{j}{2^n})}f_n\,dm\right|+\left|\int_{(\frac{k}{2^n},b)}f_n\,dm\right|+\left|\int_{(\frac{j}{2^n},\frac{k}{2^n})}f_n\,dm\right|$$

$$\le\frac{1}{2^n}+\frac{1}{2^n}+\frac{1}{2^n}\to 0 \text{ as } n\to\infty$$

[For the estmate $\int_{(j/2^n,k/2^n)}f_n\,dm$, use the fact that f_n takes 1 and -1 alternatively on intervals of length $1/2^n$.] Hence $\int_{(a,b)}f_n\,dm\to\int_{(a,b)}f\,dm$.

Now, if $E\subseteq[0,1]$ is Lebesgue-measurable. Then, $\forall\varepsilon>0$, there exist disjoint open intervals $(a_1,b_1),\ldots,(a_k,b_k)$ such that $m\left(E\,\Delta\,\bigcup_{j=1}^{k}(a_j,b_j)\right)<\varepsilon$.

Hence, $\left|\int_E f_n\,dm-\int_E f\,dm\right|=\left|\int_E f_n\,dm\right|$

$$\le\left|\int_E f_n\,dm-\int_{\bigcup_{j=1}^{k}(a_j,b_j)}f_n\,dm\right|+\sum_{j=1}^{k}\left|\int_{(a_j,b_j)}f_n\,dm\right|$$

$$\leq m\left(E \setminus \left(\bigcup_{j=1}^{k}(a_j, b_j)\right)\right) + \sum_{j=1}^{k}\left|\int_{(a_j,b_j)} f_n\,dm\right|.$$

$$\leq m\left(E \triangle \left(\bigcup_{j=1}^{k}(a_j, b_j)\right)\right) + \sum_{j=1}^{k}\left|\int_{(a_j,b_j)} f_n\,dm\right|$$

$$\leq \varepsilon + \sum_{j=1}^{k}\left|\int_{(a_j,b_j)} f_n\,dm\right|$$

Since each $\int_{(a_j,b_j)} f_n\,dm \to \int_{(a_j,b_j)} f\,dm = 0$ as $n \to \infty$, $\limsup_{n\to\infty}\left|\int_E f_n\,dm - \int_E f\,dm\right| \leq \varepsilon$.

Thus, $\int_E f_n\,dm \to \int_E f\,dm$. Hence $f_n \to f$ weakly in $L^2([0,1])$.

Clearly, $m(\{|f_n - f| \geq \frac{1}{2}\}) = m([0,1]) = 1$ for all $n \geq 1$. Hence, $f_n \not\to f$ in measure.

Let $A = \{\frac{k}{2^n} \in [0,1] : k = 0, 1, \cdots, 2^n;\ n = 1, 2, \cdots\}$.

Then $A \subseteq [0,1]$ and $m(A) = 0$. Let $x \in [0,1] \setminus A$.

Suppose $f_n(x) = 1$ for some $n \in \mathbb{N}$. Then, $x \in \left(\frac{2k-1}{2^n}, \frac{2k}{2^n}\right)$ for some $n \in \mathbb{N}$ and $k \in \mathbb{N}$ $(1 \leq k \leq 2^{n-1})$. Let $J_p = \left(\frac{2k-1}{2^n}, \frac{2k-1}{2^n} + \frac{1}{2^p}\right)$ $(p = 1, 2, \cdots)$.

Since the length of $J_p \to 0$, there exists the smallest positive integer p such that $x \notin J_p$. Then, $f_{n+p}(x) = -1$. Similarly, if $f_s(x) = -1$ $(s \in \mathbb{N})$ then $\exists\, q \in \mathbb{N}$ s.t. $f_{s+q}(x) = 1$. Hence, there exist infinitely many n such that $f_n(x) = 1$ and also there exist infinitely many n such that $f_n(x) = -1$. Hence, $\forall x \in [0,1] \setminus A$ $\{f_n(x)\}_{n=1}^{\infty}$ diverges.

(2) Let $f \equiv 0$ on $[0,1]$. $f_n(x) = n\chi_{[0,\frac{1}{n}]}$ $(n=1,2,\cdots)$. Clearly each $f_n \in L^2([0,1])$. For any $\varepsilon > 0$, $m(\{|f_n - f| \geq \varepsilon\}) \leq \frac{1}{n} \to 0$. Hence $f_n \to f$ in measure. If $0 < x \leq 1$ then $f_n(x) = 0$ if $\frac{1}{n} < x$ that is $n > \frac{1}{x}$. Hence $f_n(x) \to f(x) = 0$. Hence, $f_n \to f$ a.e. Finally, let $g(x) \equiv 1$ on $[0,1]$.

$$\int_0^1 f_n(x) g(x) dx = 1 \quad (n=1,2\cdots)$$

$$\int_0^1 f(x) g(x) dx = 0 \qquad \text{Note: } g \in L^2([0,1])$$

So, $f_n \not\to f$ weakly in $L^2([0,1])$.

4. Define $K(x,y) = \frac{1}{x+y}$, $x, y \in (0,\infty)$. K is measurable. $K(\lambda x, \lambda y) = \lambda^{-1} K(x,y)$ $\forall \lambda > 0$, $\forall x, y > 0$. Moreover,

$$C_p = \int_0^\infty |K(x,1)| x^{-\frac{1}{p}} dx = \int_0^\infty \frac{1}{x^{\frac{1}{p}}(x+1)} dx$$

$$\leq \int_0^1 \frac{1}{x^{1/p}} dx + \int_1^\infty \frac{1}{x^{1+1/p}} dx < \infty$$

By Theorem 6.20, we have $\|Tf\|_p \leq C_p \|f\|_p$ $\forall f \in L^p((0,\infty))$.

5. (1) By Hölder's inequality,

$$\int_X \left(\int_Y |K(x,y) f(y)| d\nu(y) \right)^2 d\mu(x)$$

$$\leq \int_X \left(\int_Y |K(x,y)|^2 d\nu(y) \cdot \int_Y |f(y)|^2 d\nu(y) \right) d\mu(x)$$

$$= \|f\|^2_{L^2(\nu)} \, \|K\|^2_{L^2(\mu \times \nu)} < \infty$$

Hence, $\int_Y |K(x,y) f(y)| d\nu(y) < \infty$, μ-a.e.

(2) By Minkowski's inequality for integrals, we have

$$\|Tf\|^2_{L^2(\mu)} = \left\| \int_Y K(x,y) f(y)\, d\nu(y) \right\|^2_{L^2(\mu)}$$

$$\le \left(\int_Y \|K(\cdot, y) f(y)\|_{L^2(\mu)}\, d\nu(y) \right)^2$$

$$= \left(\int_Y \left(\int_X |K(x,y) f(y)|^2 d\mu(x) \right)^{\frac{1}{2}} d\nu(y) \right)^2$$

$$\le \left(\int_Y \left(\int_X |K(x,y)|^2 d\mu(x) \right)^{\frac{1}{2}} |f(y)|\, d\nu(y) \right)^2$$

$$\le \int_Y \int_X |K(x,y)|^2 d\mu(x) d\nu(y) \cdot \int_Y |f(y)|^2 d\nu(y)$$

$$= \|K\|^2_{L^2(\mu\times\nu)} \|f\|^2_{L^2(\nu)}.$$

Hence, $Tf \in L^2(\mu)$ and $\|Tf\|_{L^2(\mu)} \le \|K\|_{L^2(\mu\times\nu)} \|f\|_{L^2(\nu)}$.

6. Let $f \in L^q((0,\infty))$. Then for any $x > 0$, $f \in L^q((0,x))$ which implies that $f \in L^1((0,x))$. Hence, $Tf(x) = x^{-1/p} \int_0^x f(t)dt$ is well-defined. Clearly, T is linear. Since $x^{\frac{1}{p}}$ is a continuous function for $x \in (0,\infty)$ and since $\int_0^x f(t)dt$ is absolutely continuous for $x \in (0,\infty)$, Tf is a continuous function on $(0,\infty)$.

For any $x > 0$, we have by Hölder's inequality that

$$|Tf(x)| \le x^{-\frac{1}{p}} \int_0^x |f(t)|dt$$

$$\le x^{-\frac{1}{p}} \left(\int_0^x |f(t)|^q dt \right)^{1/q} x^{\frac{1}{p}}$$

$\leq \|f\|_q \quad \forall x \in (0,\infty).$

Hence, $Tf \in C((0,\infty))$. $\|Tf\|_C \leq \|f\|_q$. We have shown $T: L^q((0,\infty)) \to C((0,\infty))$ is linear and $\|Tf\|_C \leq \|f\|_q \quad \forall f \in L^q((0,\infty))$. We need to show that $Tf(x) \to 0$ as $x \to 0^+$ and $Tf(x) \to 0$ as $x \to \infty$.

By the above inequality,

$$|Tf(x)| \leq \left(\int_0^x |f(t)|^q dt\right)^{\frac{1}{q}} \to 0 \quad \text{as } x \to 0^+$$

since $f \in L^q((0,\infty))$ and we can use the absolute continuity of the integral of $|f|^q$.

$\forall \varepsilon > 0$, since $f \in L^q((0,\infty))$, $\exists A > 0$ such that $\int_A^\infty |f|^q dt < \varepsilon^q$. This follows from the monotone convergence theorem:

$$\lim_{x \to +\infty} \int_0^x |f(t)|^q dt = \lim_{x \to +\infty} \int_0^\infty \chi_{(0,x)}(t) |f(t)|^q dt$$
$$= \int_0^\infty |f(t)|^q dt.$$

If $x > A$ then

$$|Tf(x)| \leq x^{-\frac{1}{p}} \int_0^A |f(t)| dt + x^{-\frac{1}{p}} \int_A^x |f(t)| dt$$
$$\leq x^{-\frac{1}{p}} \int_0^A |f(t)| dt + x^{-\frac{1}{p}} \left(\int_A^x |f(t)|^q dt\right)^{\frac{1}{q}} (x-A)^{\frac{1}{p}}$$
$$\leq x^{-\frac{1}{p}} \int_0^A |f(t)| dt + \left(\int_A^\infty |f(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq x^{-\frac{1}{p}} \int_0^A |f(t)| dt + \varepsilon$$

Hence, $\limsup_{x \to \infty} |Tf(x)| \leq \varepsilon$.

Thus $|Tf(x)| \to 0$ as $x \to +\infty$.