Math 200a Fall 2020 Homework 8

Due 12/11/2020 by 7pm on Gradescope

Reading: Continue to read the course notes. All of the material covered is also in Dummit and Foote Chapters 7-9 (but we are not covering all of that).

Exercises to write up and hand in:

1. Let R be a commutative ring, and let $I = (r_1, \ldots, r_n)$ be a nonzero finitely generated ideal of R. Prove that there is an ideal J of R which is maximal among ideals which do not contain I.

2. A *minimal prime* in a commutative ring R is a prime ideal I of R such that there does not exist any prime ideal J with $J \subsetneq I$. In other words, I is a minimal prime if it is a minimal element of the poset of prime ideals of R under inclusion.

Prove that any commutative ring R has a minimal prime. (Hint: you may assume the following variation of Zorn's Lemma: If every chain in a poset has a lower bound, then the poset has a minimal element.)

3. Let R be a commutative ring.

(a). Show that an ideal I is equal to an intersection of finitely many maximal ideals of R if and only if R/I is isomorphic to a direct product of finitely many fields.

(b). Show that if I is an intersection of finitely many distinct maximal ideals of R, say $I = M_1 \cap \cdots \cap M_n$, then the ideals M_i are uniquely determined (up to rearrangement).

(c). Give an example showing that the same property as in (b) does not hold in groups. In other words, find a group G and a subgroup H such that H can be written as an intersection of maximal subgroups of G in multiple ways.

4. Let R be a commutative ring. The ring of formal Laurent series over R is the ring $R((x))$ given by

$$R((x)) = \{\sum_{n \geq N}^{\infty} a_n x^n | a_n \in R, N \in \mathbb{Z}\}.$$

Note that this is similar to the power series ring $R[[x]]$, except that Laurent series are allowed to include finitely many negative powers of x. The product and sum in this ring are defined similarly as for power series.

(a). Prove that if F is a field, then $F((x))$ is a field. (Hint: you may want to derive this from the result you proved on the previous homework that an element in the power series ring $F[[x]]$ is a unit in $F[[x]]$ if and only if it has nonzero constant term).

(b). Prove that if F is a field, then $F((x))$ is isomorphic to the field of fractions of $F[[x]]$. (Hint: use the universal property of the localization to show there is a map from the field of fractions to $F((x))$, then show it is surjective).

(c). Show that $\mathbb{Q}((x))$ is *not* the field of fractions of its subring $\mathbb{Z}[[x]]$. (Hint: consider the power series representation of e^x.)

5. Let R be an integral domain. Let X be a multiplicative system in R not containing 0, and let $S = RX^{-1}$. Show that if R is a Euclidean domain, so is S.

6. Recall that when D is a squarefree integer, then the *ring of integers* in the field $\mathbb{Q}(\sqrt{D}) = \{x + y\sqrt{D} | x, y \in \mathbb{Q}\}$ is the subring $\mathcal{O} = \{a + b\omega | a, b \in \mathbb{Z}\}$ of $\mathbb{Q}(\sqrt{D})$, where $\omega = \sqrt{D}$ if D is congruent to 2 or 3 modulo 4, while $\omega = (1 + \sqrt{D})/2$ if D is congruent to 1 modulo 4. The field $\mathbb{Q}(\sqrt{D})$ has the norm function $N(a + b\sqrt{D}) = a^2 - Db^2$, which is multiplicative, i.e. $N(z_1 z_2) = N(z_1)N(z_2)$ for $z_1, z_2 \in \mathbb{Q}(\sqrt{D})$.

(a). Consider the ring of integers $\mathcal{O}$ in $\mathbb{Q}(\sqrt{D})$. Suppose that for every $z \in \mathbb{Q}(\sqrt{D})$, there exists an element $y \in \mathcal{O}$ such that $|N(z - y)| < 1$. Prove that $\mathcal{O}$ is a Euclidean domain with respect to the function d with $d(z) = |N(z)|$. (Hint: follow the method of proof we used to show that $\mathbb{Z}[i]$ is a Euclidean domain).

(b). Show that the ring of integers $\mathcal{O}$ is a Euclidean domain when $D = -2, 2, -3, -7$, or -11. (In each case show that part (a) applies).