Math 240A: Real Analysis, Fall 2019

Midterm Exam

Solution

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Name ______________________ **ID number** ______________________

Note: This is a close-book and close-note exam. There are 5 problems of total 100 points. To get credit, you must show your work. Partial credit will be given to partial answers.

Problem	1	2	3	4	5	Total
Score						

1. (20 points) Determine if each of the following statements is true or false. If it is true, prove it. If it is false, give a counter example.

(1) Let $(X, \mathcal{M})$ be a measurable space. A function $f : X \to \mathbb{R}$ is measurable if $\{x \in X : f(x) > r\}$ is measurable for any rational number r.

(2) If E is a Lebesgue measurable, infinite subset of $\mathbb{R}$, then $m(E) = \infty$.

Solution.

(1) Ture. Let $a \in \mathbb{R}$. Then there exist rational numbers r_k $(k = 1, 2, \dots)$ such that they decrease and converge to a. Then we have $\{f > a\} = \cup_{k=1}^{\infty} \{f > r_k\}$. But each $\{f > r_k\} \in \mathcal{M}$. Hence $\{f > a\} \in \mathcal{M}$. Thus, f is measurable.

(2) False. Example. $E = \mathbb{N}$. First, for any $x \in \mathbb{R}$, we have

$$m(\{x\}) = m\left(\bigcap_{n=1}^{\infty}(x - 1/n, x + 1/n)\right) = \lim_{n\to\infty} m((x - 1/n, x + 1/n)) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

Hence

$$m(\mathbb{N}) = \sum_{n=1}^{\infty} m(\{n\}) = 0.$$

2. (20 points) Let μ be the Lebesgue–Stieltjes measure associated to the following increasing and right-continuous function $F : \mathbb{R} \to \mathbb{R}$:

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x+1 & \text{if } 0 \leq x < 1, \\ x^2+2 & \text{if } 1 \leq x < \infty. \end{cases}$$

Calculate $\mu(\{0\})$, $\mu((1,2])$, and $\mu([3,\infty))$.

Solution.

(1) $\mu(\{0\}) = \lim_{n\to\infty} \mu((-1/n, 1/n]) = \lim_{n\to\infty} [F(1/n) - F(-1/n)] = \lim_{n\to\infty}(1 + 1/n - 0) = 1$.

(2) $\mu((1,2]) = F(2) - F(1) = (2^2+2) - (1^2+2) = 3$.

(3) $\mu([3,\infty)) \geq \mu((3,\infty)) \geq \mu((3,n]) = F(n) - F(3) \to \infty \quad$ as $n \to \infty$. Hence $\mu([3,\infty)) = \infty$.

3. (20 points) Calculate the following limit with justification: (Note that $|\sin u| \le |u|$ for any $u \in \mathbb{R}$.)

$$\lim_{n\to\infty} \int_0^n \frac{n}{x(1+x^2)} \sin\left(\frac{x}{n}\right) dx.$$

Solution. Let $f_n(x) = \chi_{(0,n)}(x) \frac{n}{x(1+x^2)} \sin\left(\frac{x}{n}\right)$ $(0 < x < \infty)$. Each f_n is Lebesgue measurable as it is the product of two Lebesgue measurable functoions, one a simple function and the other continuous function.

Note that $|f_n(x)| \le 1/(1+x^2)$ for all $x > 0$ since $|n \sin(x/n)/x| = |\sin(x/n)/(x/n)| \le 1$ for all n and all $x > 0$. Moreover, $1/(1+x^2)$ is integrable in $(0, \infty)$. In addition, for any $x > 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \chi_{(0,n)}(x) \lim_{n\to\infty} \frac{\sin(x/n)}{x/n} \frac{1}{1+x^2} = \frac{1}{1+x^2}.$$

Hence by the Lebesgue Dominant Convergence Theorem, we have

$$\lim_{n\to\infty} \int_0^n \frac{n}{x(1+x^2)} \sin\left(\frac{x}{n}\right) dx = \int_0^\infty \lim_{n\to\infty} f_n(x)\, dx = \int_0^\infty \frac{1}{1+x^2}\, dx = \arctan x|_0^\infty = \frac{\pi}{4}.$$

4. (20 points) Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume all $f, f_n \in L^1(\mu)$ $(n = 1, 2, \dots)$ and $f_n \to f$ in $L^1(\mu)$. Prove that $f_n \to f$ in measure.

Proof. Let $\varepsilon > 0$ and $E_n = \{|f_n - f| \ge \varepsilon\}$ $(n = 1, 2, \dots)$. We have

$$\int_X |f_n - f|\, d\mu \ge \int_{E_n} |f_n - f|\, d\mu \ge \int_{E_n} \varepsilon\, d\mu = \varepsilon \mu(E_n).$$

Thus,

$$\mu(E_n) \le \frac{1}{\varepsilon} \int_X |f_n - f|\, d\mu \to 0,$$

as $n \to \infty$, since $f_n \to f$ in $L^1(\mu)$. Hence $f_n \to f$ in measure. **Q.E.D.**

5. (20 points) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to [0, \infty)$ be a measurable function.

(1) Prove that $f \in L^1(\mu)$ if and only if $\displaystyle\sum_{n=1}^{\infty} n\,\mu(\{n-1 < f \le n\}) < \infty$.

(2) Prove that if $f \in L^1(\mu)$ then $\displaystyle\lim_{N\to\infty} N\mu(\{f \ge N\}) = 0$.

Note. If $\mu(X) < \infty$ the proof below is also correct.

Proof. (1) Suppose $f \in L^1(\mu)$. Let $E_0 = \{f = 0\}$ and $E_n = \{n-1 < f \le n\}$ $(n \in \mathbb{N})$. Then X is the disjoint union of $E_0, E_1, \ldots$ and moreover $1 = \chi_X = \chi_{\cup_{n=0}^{\infty} E_n} = \sum_{n=0}^{\infty} \chi_{E_n}$ on X. Consequently, since $f = 0$ on E_0 and $f > n-1$ on E_n, we have

$$\begin{aligned}\infty > \int_X f\,d\mu &= \int_X \left(\sum_{n=0}^{\infty} \chi_{E_n}\right) f\,d\mu = \sum_{n=0}^{\infty} \int_X \chi_{E_n} f\,d\mu \\ &= \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu \ge \sum_{n=1}^{\infty} \int_{E_n} (n-1)\,d\mu = \sum_{n=1}^{\infty} (n-1)\mu(E_n).\end{aligned}$$

But $(n-1)/n \to 1$ as $n \to \infty$. Thus the convergence of $\sum_{n=1}^{\infty}(n-1)\mu(E_n)$ is equivalent to the convergence of $\sum_{n=1}^{\infty} n\mu(E_n)$. Hence, $\sum_{n=1}^{\infty} n\,\mu(\{n-1 < f \le n\}) < \infty$.

Conversely, assume $\sum_{n=1}^{\infty} n\,\mu(\{n-1 < f \le n\}) < \infty$. Then, as before, and by the fact that $f \le n$ on E_n for each $n \ge 1$, we have

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu \le \sum_{n=1}^{\infty} \int_{E_n} n\,d\mu = \sum_{n=1}^{\infty} n\mu(E_n) < \infty.$$

Hence $f \in L^1(\mu)$.

(2) By Part (1), $\sum_{n=1}^{\infty} n\,\mu(\{n-1 < f \le n\}) < \infty$. Hence, $\sum_{n=N+1}^{\infty} n\,\mu(\{n-1 < f \le n\}) \to 0$ as $N \to \infty$. But

$$\begin{aligned}\sum_{n=N+1}^{\infty} n\,\mu(\{n-1 < f \le n\}) &\ge \sum_{n=N+1}^{\infty} N\,\mu(\{n-1 < f \le n\}) \\ &= N\mu\left(\cup_{n=N+1}^{\infty}\{n-1 < f \le n\}\right) = N\mu(\{f \ge N\}).\end{aligned}$$

Hence $\lim_{N\to\infty} N\mu(\{f \ge N\}) = 0$. **Q.E.D.**