Math 240B Winter 2020
Solution to Problems of HW #5
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1. We first show that $(M^\perp)^\perp = M$, provided that $M$ is a closed subspace of a Hilbert space $H$.

If $x \in M$, $y \in M^\perp$, then $\langle x, y \rangle = 0$. Hence, $M \subseteq (M^\perp)^\perp$. To show $M \supseteq (M^\perp)^\perp$, first note that $(M^\perp)^\perp$ is a closed subspace of $H$, and hence $(M^\perp)^\perp$ is complete. Moreover, $M$ is a closed subspace of $(M^\perp)^\perp$. Let $x \in (M^\perp)^\perp$. By the "Projection" theorem, $\exists !\, y \in M$, $\exists !\, z \in M^\perp$ such that $x = y + z$. Since $M \subseteq (M^\perp)^\perp$, $x - y = z \in (M^\perp)^\perp \cap M^\perp = \{0\}$. Hence $x = y \in M$. Thus, $(M^\perp)^\perp \subseteq M$. Finally, $(M^\perp)^\perp = M$.

We now show that for $E \subseteq H$, $(E^\perp)^\perp$ is the smallest closed subspace of that contains $E$, i.e., $(E^\perp)^\perp = \overline{\mathrm{Span}(E)}$.

By the linearity of the inner product, $E^\perp = \mathrm{Span}(E)$. By the continuity of the inner product, $\mathrm{Span}(E)^\perp = \overline{\mathrm{Span}(E)}^\perp$. This is a closed subspace of $H$. Hence, by what has been proved above,

$$(E^\perp)^\perp = \overline{\mathrm{Span}(E)}^{\perp\perp} = \overline{\mathrm{Span}(E)}$$

2. Let $\delta = \inf_{y' \in M} \|x - y'\|$. $\delta > 0$ since $x \notin M$ and $M$ is closed. Let $y_n \in M$ $(n = 1, 2, \cdots)$ be such that $\|x - y_n\| \to \delta$. By the Paralellogram Law,

$$2(\|y_n - x\|^2 + \|y_m - x\|^2) = \|y_n - y_m\|^2 + \|y_n + y_m - 2x\|^2$$

Since $M$ is convex, $\frac{1}{2}(y_n + y_m) \in M$. So

$$\begin{aligned}\|y_n - y_m\|^2 &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 \\ &\quad - 4\|\tfrac{1}{2}(y_n + y_m) - x\|^2 \\ &\le 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\delta^2. \\ &\to 0 \text{ as } n, m \to \infty.\end{aligned}$$

Hence $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence and $\exists y \in H$ s.t. $y_n \to y$. Since $M$ is closed and all $y_n \in M$, $y \in M$, and $\delta = \lim_{n\to\infty} \|x - y_n\| = \|x - y\|$.

Before we show that there is only one $y \in M$ such that $\delta = \|x - y\|$, we prove

$$\mathrm{Re}\langle x - y, z - y\rangle \le 0 \quad \forall z \in M.$$

(where $y \in M$, $\delta = \|x - y\|$.)

$\forall z \in M$. $\|x - y\|^2 \le \|x - z\|^2$

$$\begin{aligned}&\Leftrightarrow \langle x - y, x - y\rangle - \langle x - z, x - z\rangle \le 0 \\ &\Leftrightarrow \langle x - z, x - y\rangle + \langle z - y, x - y\rangle - \langle x - z, x - z\rangle \le 0 \\ &\Leftrightarrow \langle x - z, z - y\rangle + \langle z - y, x - y\rangle \le 0 \\ &\Leftrightarrow \langle x - y + y - z, z - y\rangle + \langle z - y, x - y\rangle \le 0 \\ &\Leftrightarrow \langle x - y, z - y\rangle + \langle z - y, x - y\rangle \le -\|z - y\|^2 \le 0 \\ &\Leftrightarrow \quad 2\,\mathrm{Re}\langle x - y, z - y\rangle \le -\|z - y\|^2 \le 0.\end{aligned}$$

What we have shown is that, for $y \in M$ the necessary and sufficient conditions that $\delta = \|x-y\|$ are

$$Re\langle x-y, z-y\rangle \le -\|z-y\|^2 \quad \forall z \in M.$$

Thus, if $y \in M$ satisfies $\delta = \|x-y\|$, then

(*) $$Re\langle x-y, z-y\rangle \le 0 \quad \forall z \in M.$$

If $\exists\, \hat{y} \in M$ s.t. $\delta = \|x-\hat{y}\|$. Then we also have

(**) $$Re\langle x-\hat{y}, z-\hat{y}\rangle \le 0 \quad \forall z \in M.$$

From (*) and (**), we have

$$Re\langle x-y, \hat{y}-y\rangle \le 0$$
$$Re\langle x-\hat{y}, y-\hat{y}\rangle \le 0$$

Hence $$Re\langle x-y, \hat{y}-y\rangle \le 0$$
$$Re\langle \hat{y}-x, \hat{y}-y\rangle \le 0$$

Thus. $$Re\langle \hat{y}-y, \hat{y}-y\rangle = \|\hat{y}-y\|^2 \le 0$$

and $\hat{y} = y$.

3. $\forall x \in M$, $\forall y \in M^\perp$, $\|y\|=1$:

$$|\langle x_0, y\rangle| = |\langle x_0 - x, y\rangle| \le \|x_0 - x\|\,\|y\| = \|x - x_0\|$$

Hence $$\inf\{\|x-x_0\| : x \in M\} \ge \sup\{|\langle x_0, y\rangle| : y \in M^\perp, \|y\|=1\}.$$

But, by the "projection" theorem the inf. can be replaced by min. By a consequence of the Hahn-Banach theorem, $\exists f \in H$ such that $\|f\|=1$, $f=0$ on $M$, and $f(x_0) = \min\{\|x-x_0\| : x \in M\}$. But, the Riesz representation theorem implies that $\exists\, y \in H$ such that $\langle z, y\rangle = f(z)$ $\forall z \in H$ and $\|y\| = \|f\| = 1$. If $z \in M$, then $\langle z, y\rangle = f(z) = 0$. Hence,

$y \in M^\perp$. Therefore

$$|\langle x_0, y\rangle| = |f(x_0)| = \min\{\|x - x_0\| : x \in M\}.$$

Consequently,

$$\min\{\|x - x_0\| : x \in M\} = \max\{|\langle x_0, y\rangle : y \in M^\perp, \|y\|=1\}.$$

4. Since $y_n \to y$ weakly, $f(y_n) \to f(y)$ $\forall f \in H^*$. i.e., $\hat{y}_n(f) \to \hat{y}(f)$ $\forall f \in H^*$. Hence for any $z \in H$, $\hat{z} \in H^{**}$ is defined by $\hat{z}(f) = f(z)$. and it is a consequence of the Hahn-Banach theorem that $\|\hat{z}\| = \|z\|$. We have thus

$$\sup_{n\geq 1} |\hat{y}_n(f)| = \sup_{n\geq 1} |f(y_n)| < \infty$$

which implies, by the Principle of Uniform Boundedness that

$$\sup_{n\geq 1} \|y_n\| = \sup_{n\geq 1} \|\hat{y}_n\| < \infty.$$

Now

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle|$$

$$\leq |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle|$$

$$|\langle x_n - x, y_n\rangle| \leq \|x_n - x\|\,\|y_n\| \leq \|x_n - x\|\left(\sup_{k\geq 1} \|y_k\|\right) \to 0.$$

Note that $z \mapsto \langle z, x\rangle$ defines a bounded linear functional on $H$. Since $y_n \to y$ weakly, $\langle x, y_n - y\rangle \to 0$. Hence, $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

5. Let $a = \sup\{|\langle Tx, x\rangle| : x \in H, \|x\|=1\}$.
If $x \in H$, $\|x\|=1$, then
$$|\langle Tx, x\rangle| \le \|Tx\|\,\|x\| = \|Tx\| \le \|T\|\,\|x\| = \|T\|.$$
Hence, $a \le \|T\|$.
Let $x, y \in H$. Direct verifications lead to
$$\langle Tx, y\rangle = \tfrac{1}{4}\left[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right] + \tfrac{1}{4} i\left[\langle T(x+iy), x+iy\rangle - \langle T(x-iy), x-iy\rangle\right].$$
Since $T$ is self-adjoint, $\langle Tz, z\rangle = \langle z, Tz\rangle = \overline{\langle Tz, z\rangle}$ $(z \in H)$, which is real. Hence,
$$(*)\qquad \operatorname{Re}\langle Tx, y\rangle = \tfrac{1}{4}\left[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right]$$
By the definition of $a$, $|\langle Tz, z\rangle| \le a\|z\|^2$ $\forall z \in H$.
Hence, by $(*)$ and the parallelogram law,
$$\begin{aligned}\operatorname{Re}\langle Tx, y\rangle &\le \tfrac{1}{4} a\left(\|x+y\|^2 + \|x-y\|^2\right)\\ &= \tfrac{1}{4} a\left(2\|x\|^2 + 2\|y\|^2\right)\\ &= \tfrac{1}{2} a\left(\|x\|^2 + \|y\|^2\right)\end{aligned}$$
Now, $\forall x \in H$. Suppose $\|x\|=1$ and $Tx \ne 0$. Let $y = \frac{Tx}{\|Tx\|}$.
Then $\|Tx\| = \operatorname{Re}\langle Tx, y\rangle \le \frac{1}{2} a\left(\|x\|^2 + \|y\|^2\right) = a$.

6. (1) The operator $P : H \to M$ is defined by $Px \in M$ and $\langle x - Px, y\rangle = 0$ $\forall y \in M$, for any $x \in H$. By the "Projection" theorem, $Px \in M$ is the unique element in $M$ s.t. $\|Px - x\| = \inf_{y' \in M} \|y' - x\|$.

Let $x_1, x_2 \in H$. $\langle x_1 - Px_1, y\rangle = 0$, $\langle x_2 - Px_2, y\rangle = 0$ $\forall y \in M$.
Hence, $\langle x_1 + x_2 - (Px_1 + Px_2), y\rangle = 0$ $\forall y \in M$.
But $\langle x_1 + x_2 - P(x_1 + x_2), y\rangle = 0$ $\forall y \in M$.

Hence $\langle Px_1 + Px_2 - P(x_1+x_2), y\rangle = 0 \ \forall y \in M$.
Let $y = Px_1 + Px_2 - P(x_1+x_2) \in M$. Then

$$\| Px_1 + Px_2 - P(x_1+x_2)\|^2 = 0.$$

Hence $P(x_1+x_2) = Px_1 + Px_2$.
Let $x \in H$ and $\alpha$ be a scalar. Then

$$\langle \alpha x - P(\alpha x), y\rangle = 0 \quad \forall y \in M$$

$$\alpha\langle x - Px, y\rangle = \langle \alpha x - \alpha Px, y\rangle = 0 \quad \forall y \in M.$$

A similar argument then leads to $P(\alpha x) = \alpha Px$.
Hence $P$ is linear. For any $x \in H$,

$$\|x\|^2 = \|x - Px\|^2 + \|Px\|^2 \geq \|Px\|^2$$

as $\langle x - Px, Px\rangle = 0$ (since $Px \in M$). Thus $\|Px\| \leq \|x\|$. i.e., $P \in L(H,H)$ and $\|P\| \leq 1$.

Let $x, y \in H$. Since $Px \in M$ and $Py \in M$, we have
$\langle x - Px, Py\rangle = 0$, i.e., $\langle x, Py\rangle = \langle Px, Py\rangle$
$\langle y - Py, Px\rangle = 0$, i.e., $\langle y, Px\rangle = \langle Py, Px\rangle$. Hence
$\langle Px, y\rangle = \langle Px, Py\rangle = \langle x, Py\rangle$. Hence $P^* = P$.

If $x \in H$, then $Px \in M$, and $P^2x = P(Px) = Px$, i.e., $P^2 = P$. [We use the fact that $y \in M \Longrightarrow Py = y$. This follows from $\langle y - Py, z\rangle = 0 \ \forall z \in M$. Choose $z = y - Py \in M$. Thus $\|y - Py\|^2 = 0$. $Py = y$.]

Clearly, $x \in M \Longrightarrow Px = x \in M$. Hence $Range(P) = M$.
Let $x \in Kernel(P)$, i.e., $x \in H$ and $Px = 0$. Then, for any $y \in M$, $\langle x - Px, y\rangle = 0 \Longrightarrow \langle x, y\rangle = 0 \Longrightarrow x \in M^\perp$. So, $Kernel(P) \subseteq M^\perp$. Conversely,

Let $x \in M^\perp$. Then, since $\langle x - Px, y \rangle = 0$ $\forall y \in M$, $\langle Px, y \rangle = \langle x, y \rangle = 0$ $\forall y \in M$. Let $y = Px \in M$. We get $\langle Px, Px \rangle = 0$ $Px = 0$. $x \in \text{Kernel}(P)$. Hence, $M^\perp \subseteq \text{Kernel}(P)$. Thus $\text{Kernel}(P) = M^\perp$.

(2) Let $M = \text{Range}(P) = \{Px : x \in H\}$. Then $M$ is a subspace of $H$ as $P$ is a linear operator. Suppose $x_n \in H$ $(n = 1, 2, \cdots)$ $Px_n \to y$ in $H$ for some $y \in H$. Then, since $P^2 = P$, $P^2 x_n \to Py$ as $P \in L(H, H)$, $P^2 x_n = Px_n$. So $Py = y$. i.e., $y \in \text{Range}(P)$. Hence, $M = \text{Range}(P)$ is a closed subspace. $P : H \to M$ is linear and continuous.

Let $x, y \in H$. Since $P^* = P$ (which implies that $\langle u, Pv \rangle = \langle Pu, v \rangle$ $\forall u, v \in H$) and $P^2 = P$, we have

$$
\begin{aligned}
\langle x - Px, Py \rangle &= \langle x, Py \rangle - \langle Px, Py \rangle \\
&= \langle x, Py \rangle - \langle x, P^2 y \rangle \\
&= \langle x, Py \rangle - \langle x, Py \rangle \\
&= 0.
\end{aligned}
$$

Thus, $\langle x - Px, z \rangle = 0$ for any $z \in M = \text{Range}(P)$. Hence, $P$ is the orthogonal projection onto the closed subspace $M = \text{Range}(P)$.

8. Assume $\sum_{n=1}^{\infty} \delta_n^2 = \infty$. Define $x_n = \sum_{j=1}^{n} \delta_j u_j \in S$ $(n=1,2,\cdots)$. Then $\|x_n\|^2 = \sum_{j=1}^{n} |\delta_j|^2 \to \infty$. Thus, $S$ is unbounded, and hence, not compact.

Assum $\sum_{n=1}^{\infty} \delta_n^2 < \infty$. Note that the compactness of $S$ is equivalent to that $S$ is complete and $S$ is totally bounded (cf. Theorem 0.25 on page 15 of the textbook).

Let $y^{(k)} = \sum_{n=1}^{\infty} a_n^{(k)} u_n \in S$ $(k=1,2,\cdots)$ and assume $y^{(k)} \to y$ in $H$ for some $y \in H$. $\forall n \in \mathbb{N}$. $a_n^{(k)} = \langle y^{(k)}, u_n \rangle \to \langle y, u_n \rangle$ as $k \to \infty$. Since $y^{(k)} \in S$, $|a_n^{(k)}| \le \delta_n$. So, $|\langle y, u_n \rangle| \le \delta_n$. Since $\sum_{n=1}^{\infty} \delta_n^2 < \infty$, $\sum_{j=n+1}^{m} |\langle y, u_n \rangle|^2 \to 0$ as $n, m \to \infty$. Hence

$$\left\| \sum_{j=1}^{n} \langle y, u_j \rangle u_j - \sum_{j=1}^{m} \langle y, u_j \rangle u_j \right\| \to 0 \quad \text{as } n, m \to \infty.$$

Since $H$ is complete, $\sum_{n=1}^{\infty} \langle y, u_n \rangle u_n$ converges in $H$. Let $z = y - \sum_{n=1}^{\infty} \langle y, u_n \rangle u_n$. We show $z=0$. $\forall \varepsilon > 0$. Since $\sum_{n=1}^{\infty} \delta_n^2 < \infty$, there exists $N \in \mathbb{N}$ such that $\sum_{n \ge N+1}^{\infty} \delta_n^2 < \varepsilon/8$. Now

$$\begin{aligned}
\|z\|^2 &= \left\| (y - y^{(k)}) + y^{(k)} - \sum_{n=1}^{\infty} \langle y, u_n \rangle u_n \right\|^2 \\
&\le 2\|y - y^{(k)}\|^2 + 2\left\| y^{(k)} - \sum_{n=1}^{\infty} \langle y, u_n \rangle u_n \right\|^2 \\
&= 2\|y - y^{(k)}\|^2 + 2\left\| \sum_{n=1}^{\infty} [a_n^{(k)} - \langle y, u_n \rangle] u_n \right\|^2 \\
&\overset{\text{Bessel's}}{\le} 2\|y - y^{(k)}\|^2 + 2\sum_{n=1}^{N} |a_n^{(k)} - \langle y, u_n \rangle|^2 \\
&\quad + 2\sum_{n \ge N+1}^{\infty} |a_n^{(k)} - \langle y, u_n \rangle|^2
\end{aligned}$$

$$\leq 2\|y-y^{(k)}\|^2 + 2\sum_{n=1}^{N} |a_n^{(k)} - \langle y, u_n\rangle|^2 + 8\sum_{n=N+1}^{\infty} \delta_n^2$$
$$\leq 2\|y-y^{(k)}\|^2 + 2\sum_{n=1}^{N} |a_n^{(k)} - \langle y, u_n\rangle|^2 + \varepsilon.$$

Sending $k \to \infty$, we have $\|z\|^2 \leq \varepsilon$. Thus $z=0$. Hence $y = \sum_{n=1}^{\infty} \langle y, u_n\rangle u_n \in S$. Hence $S$ is closed. But, $H$ is complete. Hence, $S$ is complete.

We now show that $S$ is totally bounded. Let $\varepsilon > 0$ again. Since $\sum_{n=1}^{\infty} \delta_n^2 < \infty$, there exists $N \in \mathbb{N}$ s.t. $\sum_{n=N+1}^{\infty} \delta_n^2 < (\varepsilon/\sqrt{2})^2$. Choose $J \in \mathbb{N}$ large enough so that $2|\delta_n|/J < \varepsilon/\sqrt{2N}$ $(n=1,2,\ldots,N)$. Consider the finitely many points in $S$: $\sum_{n=1}^{N} a_n^{(j_n)} u_n$, where

$$a_n^{(j_n)} \in \left\{ -\delta_n + \frac{2j\delta_n}{J} : j=0,1,\ldots,J \right\}.$$

If $|a_n| \leq \delta_n$ for all $n \in \mathbb{N}$, then for each $n: 1 \leq n \leq N$, $\exists\, j_n \in \{0,1,\ldots,J\}$ s.t. $|a_n^{(j_n)} - a_n| \leq \frac{\varepsilon}{\sqrt{2N}}$. Thus,

$$\left\| \sum_{n=1}^{N} a_n^{(j_n)} u_n - \sum_{n=1}^{\infty} a_n u_n \right\|^2$$
$$\leq \sum_{n=1}^{N} |a_n^{(j_n)} - a_n|^2 + \sum_{n=N+1}^{\infty} \delta_n^2$$
$$\leq (\varepsilon/\sqrt{2})^2 + (\varepsilon/\sqrt{2})^2 = \varepsilon^2.$$

This means any point in $S$ is within $\varepsilon$-distance to one of those finitely many points $\sum_{n=1}^{N} a_n^{(j_n)} u_n$. Thus $S$ is totally bounded.