Math 200a Fall 2020 Homework 6

Due 11/20/2020 by 7pm on Gradescope

Reading: Read Chapter 8 in the text. Section 8F is optional reading, it won't be covered in the lectures or homework. The problems on this homework are about direct and semidirect products. Problems about Chapter 8 will appear on a later homework.

Exercises not from Isaacs to write up and hand in:

1. Let H and K be groups and let $G = H \times K$. Identify H and K with subgroups of G as usual.

(a) Suppose that D is a subgroup of G such that $D \cap H = D \cap K = 1$. Prove that there are subgroups $H' \subseteq H$ and $K' \subseteq K$ and an isomorphism of groups $\phi : H' \to K'$ such that $D = \{(h, \phi(h)) | h \in H'\}$. In other words, D is the graph of a partial isomorphism from H to K.

(b) If D is as in part (a), show that if D is normal in G if and only if $H' \subseteq Z(H)$ and $K' \subseteq Z(K)$.

(c). Suppose that H and K are nonabelian simple groups. Show that the only normal subgroups of G are 1, H, K, and G.

2. Recall that given a homomorphism $\psi : H \to \mathrm{Aut}(K)$, we define a semidirect product $G = H \ltimes_\psi K$, with product $(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1^{h_2} k_2)$, where k^h means the right action of h on k using ψ, that is $k^h = (k)\psi(h)$.

While a semidirect product depends in general on the choice of homomorphism ψ, sometimes different choices of ψ lead to isomorphic semidirect products. This problem explores some cases where this happens. All compositions are left to right in this problem as is our usual convention following the text.

(a). Suppose that $\theta \in \mathrm{Aut}(K)$ and let $\phi_\theta : \mathrm{Aut}(K) \to \mathrm{Aut}(K)$ be the inner automorphism of $\mathrm{Aut}(K)$ given by $\rho \mapsto \theta^{-1} \circ \rho \circ \theta$. Let $\psi_2 = \psi \circ \phi_\theta : H \to \mathrm{Aut}(K)$. Prove that $H \ltimes_\psi K$ and $H \ltimes_{\psi_2} K$ are isomorphic groups. (Hint: Try the map $H \ltimes_\psi K \to H \ltimes_{\psi_2} K$ given by $(h, k) \mapsto (h, (k)\theta)$.)

(b) Suppose that $\rho : H \to H$ is an automorphism of H and define $\psi_2 = \rho \circ \psi : H \to \mathrm{Aut}(K)$. Prove that $H \ltimes_\psi K$ and $H \ltimes_{\psi_2} K$ are isomorphic groups.

3. Suppose that p and q are primes with $p < q$ where p divides $q - 1$. Show that there are precisely two groups of order pq up to isomorphism. (Use problem 2).

4. Classify groups G of order 20 up to isomorphism (there are 5 such groups). Find a presentation for each of the three non-Abelian groups you find (one for each isomorphism type).

5. Let p be a prime and consider the group $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ under addition, where there are n factors. Write $G = (\mathbb{Z}_p)^n$.

Show that G is naturally a vector space V over the field $\mathbb{Z}_p$, and explain why any group homomorphism from G to itself is the same as a linear transformation of this vector space V to itself. Conclude that the automorphism group $\mathrm{Aut}(G)$ is isomorphic to the matrix group $\mathrm{GL}_n(\mathbb{Z}_p)$.

6. Classify groups of order 75 up to isomorphism. (Hint: Find the order of the group $\mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5)$ and show that all subgroups of order 3 in this group are conjugate. You don't need to find any of the elements of order 3 explicitly.)