Math 240A: Real Analysis, Fall 2019

Homework Assignment 7

Due Friday, November 22, 2019

Unless otherwise stated, we assume that $(X, \mathcal{M})$ is a measurable space.

1. Prove the following:
   (1) If ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$, then E is ν-null if and only if $|\nu|(E) = 0$.
   (2) If μ and ν be two signed measures on $(X, \mathcal{M})$, then $\nu \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

2. Let μ be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$. Prove the following:
   (1) $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M} \text{ and } F \subseteq E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M} \text{ and } F \subseteq E\}$;
   (2) $|\nu|(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \dots, E_n \in \mathcal{M} \text{ are disjoint, and } \cup_{j=1}^n E_j = E\right\}$.

3. Let ν be a signed measure on $(X, \mathcal{M})$. Prove the following:
   (1) $L^1(\nu) = L^1(|\nu|)$;
   (2) If $f \in L^1(\nu)$ then $\left|\int_X f\, d\nu\right| \le \int_X |f|\, d|\nu|$;
   (3) If $E \in \mathcal{M}$ then $|\nu|(E) = \sup\left\{\left|\int_E f\, d\nu\right| : |f| \le 1\right\}$.

4. Let μ be a positive measure on $(X, \mathcal{M})$ and $f \in L^1(\mu)$ a real-valued function on X. Define $\nu(E) = \int_E f\, d\mu$ for each $E \in \mathcal{M}$.
   (1) Prove that ν is a signed measure on $(X, \mathcal{M})$.
   (2) Describe the Hahn decomposition of ν and the positive, negative, and the total variation of ν in terms of μ and f.

5. Let μ be a positive measure and ν a signed measure on $(X, \mathcal{M})$. Prove that the following are equivalent: (1) $\nu \ll \mu$; (2) $|\nu| \ll \mu$; (3) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n \to f$ in $L^1(\mu)$. Prove that $\{f_n\}_{n=1}^\infty$ is uniformly integrable.

7. Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, m = Lebesgue measure, and μ = counting measure. Prove the following:
   (1) $m \ll \mu$ but $dm \ne f d\mu$ for any $f \in L^1(\mu)$;
   (2) μ has no Lebesgue decomposition with respect to m.

8. Let μ and ν be two σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$. Let $\lambda = \mu + \nu$. Assume that $f = d\nu/d\lambda$. Prove that $0 \le f < 1$ μ-a.e. and $d\nu/d\mu = f/(1 - f)$.

9. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, and $\nu = \mu|_{\mathcal{N}}$. Let $f \in L^1(\mu)$. Prove that there exists $g \in L^1(\nu)$ such that $\int_E f\, d\mu = \int_E g\, d\nu$ for all $E \in \mathcal{N}$. If h is another such function, then $g = h$ ν-a.e.

10. Let ν be a complex measure on $(X, \mathcal{M})$ such that $\nu(X) = |\nu|(X)$. Prove that $\nu = |\nu|$.