Math 240 B. Winter 2020
Solution to Problems of HW #8
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1. Let X be a second countable and locally compact Hausdorff space. Let $\mathcal{U}$ be a countable base for X. Let V be an open subset of X. Since X is locally compact Hausdorff, for each $x \in V$ there exists a compact neighborhood N_x of x such that $x \in \mathring{N}_x \subseteq N_x \subseteq V$. Since $\mathcal{U}$ is a base of X, there exists $U = U(x) \in \mathcal{U}$ such that $x \in U(x) \subseteq \mathring{N}_x \subseteq N_x \subseteq V$. Note $\overline{U(x)} \subseteq N_x$ and N_x is compact. Since X is Hausdorff, $\overline{U(x)}$ is compact. Since

$$V = \bigcup_{x \in V} U(x) \subseteq \bigcup_{x \in V} \overline{U(x)} \subseteq \bigcup_{x \in V} N_x \subseteq V,$$

we have $V = \bigcup_{x \in V} \overline{U(x)}$ and each $\overline{U(x)}$ is compact. Since there are only countably many members in $\mathcal{U}$, the union $V = \bigcup_{x \in V} \overline{U(x)}$ is a countable union. Hence V is σ-compact.

2 (1) Let $X = \prod_{j=1}^{n} X_j$ with each X_j a locally compact topological space. Let $x = (x_1, \cdots, x_n) \in X$ with each $x_j \in X_j$. Since X_j is locally compact, there exists a compact neighborhood N_j of x_j in X_j: $x_j \in \mathring{N}_j \subseteq N_j$. Then $\prod_{j=1}^{n} \mathring{N}_j$ is an open subset of X and $x = (x_1, \cdots, x_n) \in \prod_{j=1}^{n} \mathring{N}_j$. Moreover, $\prod_{j=1}^{n} N_j$ is compact

and $\prod_{j=1}^{n} \mathring{N}_j \subseteq \prod_{j=1}^{n} N_j$. Thus, $\prod_{j=1}^{n} N_j$ is a compact neighborhood of x in $X = \prod_{j=1}^{n} X_j$. Hence, X is also locally compact.

(2) Let $X = \prod_{n=1}^{\infty} X_n$ with each X_n sequentially compact. Let $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)}, \dots) \in X$ $(k = 1, 2 \cdots)$ be a sequence of points of X with $x_n^{(k)} \in X_n$ for each $k \geq 1$ and $n \geq 1$. Since X_1 is sequentially compact, the sequence $\{x_1^{(k)}\}_{k=1}^{\infty}$ of points in X_1 has a subsequence $\{x_1^{1(k)}\}_{k=1}^{\infty}$ of $\{x_1^{(k)}\}_{k=1}^{\infty}$ (here $1(1), 1(2), 1(3), \dots$ is a subsequence of $1, 2, 3 \dots$) such that $x_1^{1(k)} \to x_1 \in X_1$ for some $x_1 \in X_1$. Now, the sequence $\{x_2^{1(k)}\}_{k=1}^{\infty}$ has a subsequence $\{x_2^{2(k)}\}_{k=1}^{\infty}$ in X_2 such that $x_2^{2(k)} \to x_2$ in X_2 for some $x_2 \in X_2$. By the induction, we have for each n, a subsequence $\{x_n^{n(k)}\}_{k=1}^{\infty}$ of $\{x_n^{(k)}\}_{k=1}^{\infty}$ such that $x_n^{n(k)} \to x_n$ as $k \to \infty$ for some $x_n \in X_n$. Moreover, each $\{n(k)\}_{k=1}^{\infty}$ is a subsequence of $\{(n-1)(k)\}_{k=1}^{\infty}$. Now, $\{x^{n(n)}\}_{n=1}^{\infty}$ is a subsequence of $\{x^{(k)}\}_{k=1}^{\infty}$ in $X = \prod_{j=1}^{\infty} X_j$. For each $j \in \mathbb{N}$ $x_j^{n(n)} \to x_j$ in X_j as $n \to \infty$.

Let $x = (x_1, x_2, \cdots) \in X = \prod_{j=1}^{\infty} X_j$. We show that $x^{n(n)} \to x$ in the product topology. Let U be an open set in X with $x \in U$. Then, there are $n_j \in \mathbb{N}$: $n_1 < n_2 < \cdots < n_m$ such that $x \in \bigcap_{k=1}^{m} \pi_{n_k}^{-1}(U_{n_k}) \subseteq U$, where U_{n_k} is an open neighborhood of x_{n_k} in X_{n_k} $(1 \leq k \leq m)$. But $\{x_{n_k}^{n(n)}\}_{n=1}^{\infty}$ is eventually in U_{n_k}.

as $x_{n_k}^{n(n)} \to x_{n_k}$ as $n \to \infty$ for each k $(1 \le k \le m)$. Hence, $\{x^{n(n)}\}_{n=1}^{\infty}$ is eventually in $\mathcal{U}$, and $x^{n(n)} \to x$ in $X = \prod_{j=1}^{\infty} X_j$. Hence, $X = \prod_{j=1}^{\infty} X_j$ is sequentially compact.

3. If $f \in C([0,1])$ and $x_1, x_2 \in [0,1]$, then

$$|Tf(x_1) - Tf(x_2)| = \left| \int_0^1 [K(x_1,y) - K(x_2,y)] f(y)\,dy \right|$$
$$\le \|f\|_u \int_0^1 |K(x_1,y) - K(x_2,y)|\,dy.$$

Since K is uniformly continuous on $[0,1] \times [0,1]$,

$$\lim_{x_2 \to x_1} \int_0^1 |K(x_1,y) - K(x_2,y)|\,dy$$
$$= \int_0^1 \lim_{x_2 \to x_1} |K(x_1,y) - K(x_2,y)|\,dy = 0.$$

Hence $\lim_{x_2 \to x_1} |Tf(x_1) - Tf(x_2)| = 0$, and $Tf \in C([0,1])$.

From the above inequality, we have for any $f \in C([0,1])$ with $\|f\|_u \le 1$ that

$$\sup_{\|f\|_u \le 1} |Tf(x_1) - Tf(x_2)| \le \int_0^1 |K(x_1,y) - K(x_2,y)|\,dy \to 0$$

as $x_2 \to x_1$. Hence $\{Tf : f \in C([0,1]) \text{ and } \|f\|_u \le 1\}$ is equicontinuous on $[0,1]$.

Since $K \in C([0,1] \times [0,1])$, $C := \sup_{(x,y) \in [0,1] \times [0,1]} |K(x,y)| < \infty$.

Thus, if $f \in C([0,1])$ and $\|f\|_u \le 1$, we have for any $x \in [0,1]$ that

$$|Tf(x)| \le \|f\|_u \int_0^1 |K(x,y)|\,dy \le C.$$

Hence $\{Tf : f \in C([0,1]) \text{ and } \|f\|_u\}$ is pointwise bounded. By Arzelà-Ascoli Theorem, the set $\{Tf : f \in C([0,1]) \text{ and } \|f\|_u \le 1\}$ is precompact in $C([0,1])$.

4. Let X be a separable normed vector space over $\mathbb{C}$ (the case for $\mathbb{R}$ is similar). Let $B^* = \{f \in X^* : \|f\| \le 1\}$. Denote for any $x \in X$, $a \in \mathbb{C}$ and $\varepsilon \in (0, \infty)$

$$U_{x,a,\varepsilon} = \hat{x}^{-1}(B(a,\varepsilon)) = \{f \in X^* : f(x) \in B(a,\varepsilon)\}$$
$$= \{f \in X^* : |f(x) - a| < \varepsilon\}.$$

Let $\mathcal{E} = \{$ finite intersections of $U_{x,a,\varepsilon}$ with $x \in X$, $a \in \mathbb{C}$ and $\varepsilon \in (0,\infty)\}$. Then $\mathcal{E}$ is a base for the weak-$*$ topology on X^*. Since X is separable, there exists a countable dense subset S of X. Let $A \subseteq \mathbb{C}$ be a countable dense subset of $\mathbb{C}$. Define $\mathcal{E}_0 = \{$ finite intersections of $U_{x,a,\varepsilon}$ with $x \in S$, $a \in A$, and $\varepsilon \in \mathbb{Q} \cap (0,\infty)\}$. Then $\mathcal{E}_0$ is countable. Hence $\mathcal{E}_0 \cap B^* = \{U_0 \cap B^* : U_0 \in \mathcal{E}_0\}$ is also countable. We show that $\mathcal{E}_0 \cap B^*$ is a base for the weak-$*$ topology on B^*.

It suffices to show that for any $f \in B^*$ and any open neighborhood $\mathcal{U}$ of f in X^* there exists $\mathcal{U}_0 \in \mathcal{E}_0$ such that $f \in \mathcal{U}_0 \subseteq \mathcal{U}$. Since $\mathcal{U}$ is open in X^* in the weak-$*$ topology and $f \in \mathcal{U}$, there exists $\bigcap_{j=1}^{m} U_{x_j,a_j,\varepsilon_j} \in \mathcal{E}$ such that $f \in \bigcap_{j=1}^{m} U_{x_j,a_j,\varepsilon_j} \subseteq \mathcal{U}$. So, it suffices to show that there exists $U_{y,b,\delta} \in \mathcal{E}_0$ such that $f \in U_{y,b,\delta} \cap B^* \subseteq U_{x,a,\varepsilon} \cap B^*$ if $f \in U_{x,a,\varepsilon} \cap B^*$. The key here is that $y \in S$, $b \in A$, and $\delta \in \mathbb{Q} \cap (0,\infty)$

Since $f \in U_{x,a,\varepsilon} \cap B^*$, $\|f\| \le 1$ and $f(x) \in B(a,\varepsilon)$, there exists $\delta \in (0,\infty) \cap \mathbb{Q}$ such that $B(f(x), 2\delta) \subseteq B(a,\varepsilon)$. Choose $y \in S$ such that $\|y - x\| < \delta/2$, and choose $b \in A$ such that $|f(x) - b| < \delta/2$. Then

$$\begin{aligned} |f(y)-b| &\le |f(y)-f(x)| + |f(x)-b| \\ &\le \|f\|\,\|y-x\| + |f(x)-b| \\ &< 1 \cdot \delta/2 + \delta/2 = \delta \end{aligned}$$

Hence $f \in U_{y,b,\delta} \cap B^*$. Now, if $g \in U_{y,b,\delta} \cap B^*$ then $\|g\| \le 1$ and $|g(y) - b| < \delta$. Hence

$$\begin{aligned} |g(x)-f(x)| &\le |g(x)-g(y)| + |g(y)-b| + |b-f(x)| \\ &< \|g\|\,\|x-y\| + \delta + \delta/2 \le \delta/2 + \delta + \tfrac{\delta}{2} = 2\delta. \end{aligned}$$

This means $g(x) \in B(f(x), 2\delta) \subseteq B(a,\varepsilon)$, i.e., $g \in U_{x,a,\varepsilon}$. Hence $U_{y,b,\delta} \cap B^* \subseteq U_{x,a,\varepsilon} \cap B^*$.

We show now that B^* with respect to the weak-$*$ topology is metrizable. First, the weak-$*$ topology on B^* is Hausdorff. In fact, if $f \in B^*$ and $\hat{x}(f) = f(x) = 0$ $\forall x \in X$, then $f = 0$. Hence B^* is Hausdorff in the weak-$*$ topology. B^* is also compact in the weak-$*$ topology, by Alaoglu's Theorem. Thus B^* is normal in the weak-$*$ topology. By part (1) and Urysohn's metrization theorem, B^* is metrizable

5. (1) Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in B and $x_\alpha \to x$ weakly for some $x \in X$. Then $f(x_\alpha) \to f(x)$ for any $f \in X^*$. If $x \neq 0$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$ (cf. Theorem 5.8). For this f, $|f(x_\alpha)| \leq \|f\| \|x_\alpha\| = \|x_\alpha\| \leq 1$. Hence $|f(x)| = \|x\| \leq 1$, i.e., $x \in B$. Hence B is weakly closed.

(2) Let E be norm-bounded subset of X. Let $C = \sup_{x \in E} \|x\| < \infty$. Denote by $\overline{E}^w$ the weak closure of E (i.e., the closure of E with respect to the weak topology). Let $\in \overline{E}^w$. There exists a net $\langle x_\alpha \rangle_{\alpha \in A}$ in E such that $x_\alpha \to x$ weakly. Let $f \in X^*$ be such that $\|f\| = 1$ and $f(x) = \|x\|$. Then, $f(x_\alpha) \to f(x) = \|x\|$. But $|f(x_\alpha)| \leq \|f\| \|x_\alpha\| \leq C < \infty$. Hence $\|x\| \leq C$. Hence $\sup_{x \in \overline{E}^w} \|x\| \leq C$. i.e., $\overline{E}^w$ is also norm-bounded.

(3) Let $S \subseteq X^*$ be such that $a := \sup_{f \in S} \|f\| < \infty$. If $g \in X^*$ is in the weak-* closure of S, then there exists a net $\langle f_\alpha \rangle_{\alpha \in A}$ in S such that $f_\alpha \to g$ in the weak-* topology. This means that $f_\alpha(x) \to g(x)$ $\forall x \in X$. Since $f_\alpha \in S$, $\|f_\alpha\| = \sup_{\|x\|=1} |f_\alpha(x)| \leq a$ $\forall \alpha \in A$. Hence, $\|g\| = \sup_{\|x\|=1} |g(x)| \leq a$. Thus, the weak-* closure of S is also norm-bounded in X^*.

(4) Let $\{f_n\}_{n=1}^{\infty}$ be a weak-* Cauchy sequence in X^*. (This means $f_n - f_m \to 0$ as $n, m \to \infty$ in the weak-* topology.) Then, for any $x \in X$, $f_n(x) - f_m(x) \to 0$ as $n, m \to \infty$. Hence $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $K \in \mathbb{R}$ or $\mathbb{C}$. Thus $\exists\, f(x) \in K$ such that $f_n(x) \to f(x)$ as $n \to \infty$. Clearly f is linear as each f_n is. Note that $\sup_{n \geq 1} |f_n(x)| < \infty$ $\forall x \in X$. The Principle of Uniform Boundedness implies that $b := \sup_{n \geq 1} \|f_n\| < \infty$. Hence $|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq b\|x\|$ for any $x \in X$. Thus $\|f\| \leq b$ and $f \in X^*$, and $f_n \to f$ weak-* since $f_n(x) \to f(x)$ $\forall x \in X$.

6. (1) Let $f \in C^\infty(\mathbb{R})$. If all $p_{j,k}(f) = 0$ $\forall j, \forall k$, then $p_{j,0}(f) = \max_{-j \leq x \leq j} |f(x)| = 0$ $\forall j \in \mathbb{N}$. Hence $f = 0$ on $[-j, j]$ for any $j \in \mathbb{N}$. Thus $f \equiv 0$ on $\mathbb{R}$. Hence, the topology defined by $\{p_{j,k}\}$ is Hausdorff. Since $p_{j,k}$ ($j = 1, 2, \cdots$; $k = 0, 1, \cdots$) are countable, the topology is metrizable.

(2) $f_n \to f \iff \forall j, k$ $p_{j,k}(f_n - f) \to 0$ $\iff \forall k$: $f_n^{(k)} \to f^{(k)}$ on any $[-j, j]$ uniformly. Each $[-j, j]$ is compact in $\mathbb{R}^1$ and any compact $E \subseteq \mathbb{R}^1$ is a subset of $[-j, j]$ for some j. Hence, $f_n \to f$ in this topology $\iff f_n^{(k)} \to f^{(k)}$ uniformly on any compact subset of $\mathbb{R}$, for any $k = 0, 1, \cdots$.

7. Suppose M is weakly closed. Let $x_n \in M$ $(n \in \mathbb{N})$, $x \in X$, and $x_n \to x$ in norm. Then for any $f \in X^*$ $|f(x_n) - f(x)| = |f(x_n - x)| \le \|f\| \|x_n - x\|$ $\to 0$ as $n \to \infty$. Hence, $x_n \to x$ weakly. Thus, $x \in M$, and M is norm-closed.

Let M be norm-closed. Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in M, $x \in X$, and $x_\alpha \to x$ weakly. Hence, $f(x_\alpha) \to f(x)$ $\forall f \in X^*$. We claim $x \in M$, and hence M is weakly closed. Otherwise, $x \notin M$. Then, there exists $f \in X^*$ such that $f = 0$ on M and $f(x) = \inf_{y \in M} \|x - y\| > 0$. For this f, $f(x_\alpha) = 0$. But $f(x_\alpha) \to f(x) \neq 0$. That is a contradiction.

8. Note first that $\sup_{n \ge 1} \|T_n\| < \infty$ as $T_n \to T$ in $L(X, X)$. Thus,

$$\|T_n x_n - Tx\| \le \|T_n(x_n - x)\| + \|(T_n - T)x\|$$
$$\le \|T_n\| \|x_n - x\| + \|T_n - T\| \|x\|$$
$$\to 0 \text{ as } n \to \infty,$$

hence $T_n x_n \to Tx$ in X. Let $y \in X$. Then, since $S_n \to S$ in $L(X, X)$, $\|S_n y - Sy\| \le \|S_n - S\| \|y\| \to 0$. Thus,

$$\|T_n S_n y - TSy\| \le \|(T_n - T) S_n y\| + \|T(S_n y - Sy)\|$$
$$\le \|T_n - T\| \|S_n y\| + \|T\| \|S_n y - Sy\| \to 0 \text{ as } n \to \infty$$

since $\|S_n y\|$ is bounded, $T_n \to T$ in $L(X, X)$ and $S_n y \to Sy$ in X. Hence $(T_n S_n) y \to (TS) y$ $\forall y \in X$. Thus $T_n S_n \to TS$ strongly.