Math 240B Winter 2020

Solution to Problems of HW#6

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1. Since $A \subseteq A\cup B$, $\bar{A} \subseteq \overline{A\cup B}$. Similary $\bar{B} \subseteq \overline{A\cup B}$. Thus, $\bar{A} \cup \bar{B} \subseteq \overline{A\cup B}$. On the otherhand, $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$, hence $A \cup B \subseteq \bar{A}\cup\bar{B}$. Note that $\bar{A}\cup\bar{B}$ is closed. Hence, $\overline{A\cup B} \subseteq \overline{\bar{A}\cup\bar{B}} = \bar{A}\cup\bar{B}$. Therefore, $\overline{A\cup B} = \bar{A}\cup\bar{B}$.

2. Clearly, $U \supseteq U \cap A$. So, $\bar{U} \supseteq \overline{U\cap A}$. On the other hand,

$$A = A \cap X = A \cap (U \cup U^c)$$
$$= (A\cap U) \cup (A \cup U^c)$$
$$\subseteq \overline{A\cap U} \cup U^c$$

Hence,

$$U \subseteq X = \bar{A} = \overline{(A\cap U) \cup U^c}$$
$$= \overline{A\cap U} \cup U^c$$

(See Prob. 1). But $U \cap U^c = \phi$. So, $U \subseteq \overline{A\cap U}$. Hence, $\bar{U} \subseteq \overline{A\cap U}$. Finally, $\bar{U} = \overline{U\cap A}$.

3. Let X be a separable metric space with metric ρ. Let $S = \{x_n : n \geq 1\}$ be a countable dense subset of X. We show that

$$\mathcal{B} = \{ B(x_n, \tfrac{1}{k}) : n, k \in \mathbb{N} \}$$

is a countable base. Clearly, $\mathcal{B}$ is countable. Let $x \in X$. For any $m \in \mathbb{N}$, $\exists\, n_m \in \mathbb{N}$ s.t. $\rho(x_{n_m}, x) < \frac{1}{2m}$ i.e., $x \in B(x_{n_m}, \frac{1}{2m})$. If $y \in B(x_{n_m}, \frac{1}{2m})$ then $\rho(y, x_{n_m}) < \frac{1}{2m}$. Hence, $\rho(y,x) \leq \rho(y, x_{n_m}) + \rho(x_{n_m}, x) < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}$. Thus,

$B(x_{n_m}, \frac{1}{2m}) \subseteq B(x, \frac{1}{m})$. Now $\{B(x_{n_m}, \frac{1}{2m}) : m \in \mathbb{N}\}$ is a neighborhood base at x: each $B(x_{n_m}, \frac{1}{2m})$ contains x; if U is a neighborhood of x, then $\exists\, m \in \mathbb{N}$ s.t. $x \in B(x, \frac{1}{m}) \subseteq U$. Hence, $B(x_{n_m}, \frac{1}{2m}) \subseteq B(x, \frac{1}{m}) \subseteq U$. Since $x \in X$ is arbitrary, $\mathcal{B} = \{B(x_n, \frac{1}{k}) : n, k \in \mathbb{N}\}$ is a base.

4. Let A, B be two disjoint, nonempty, closed subsets of X. Define for $x \in X$, $E \subseteq X$, $\rho(x, E) = \inf_{y \in E} \rho(x, y)$. Set $U = \{x \in X : \rho(x, A) < \rho(x, B)\} \subseteq X$,

$V = \{x \in X : \rho(x, B) < \rho(x, A)\} \subseteq X$.

Clearly, $U \cap V = \phi$. If $x \in A$, then $\rho(x, A) = 0$. Moreover, $\rho(x, B) = 0$; for otherwise, $\rho(x, B) = 0$, then $\exists\, b_n \in B$ $(n = 1, 2, \cdots)$ such that $\rho(x, b_n) \to 0$. Since B is closed, this would imply that $x \in B$, and further that $A \cap B \neq \phi$, a contradiction. Hence $\rho(x, B) > 0$, and $x \in U$. So, $A \subseteq U$. Similarly, $B \subseteq V$.

We finally show that U and V are open. Define $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ by $f(x) = \rho(x, A) - \rho(x, B)$ and $g(x) = -f(x)$, $\forall x \in X$. It suffices to show that f and g are continuous, as if so, $U = f^{-1}((-\infty, 0))$ and $V = g^{-1}((-\infty, 0))$ are open. The continuity of f and g is a consequence of the following:

If $\phi \neq E \subseteq X$ then

$$|\rho(x,E)-\rho(y,E)| \leq \rho(x,y) \quad \forall x,y \in E.$$

Pf. Let $x,y \in X$. Let $\varepsilon>0$. By the definition of $\rho(x,E)$, $\exists\, a \in E$ such that

$$\rho(x,E) > \rho(x,a)-\varepsilon.$$

Thus,
$$\begin{aligned}&\rho(y,E)-\rho(x,E)\\ &\leq \rho(y,a)-\rho(x,a)+\varepsilon\\ &\leq \rho(y,x)+\varepsilon.\end{aligned}$$

Similarly, $\rho(x,E)-\rho(y,E) \leq \rho(x,y)+\varepsilon$.

Thus, $|\rho(x,E)-\rho(y,E)| \leq \rho(x,y)+\varepsilon$.

But, ε is arbitrary. So $|\rho(x,E)-\rho(y,E)| \leq \rho(x,y)$.

5. (1) $\Longrightarrow$ (2). We have $f(A) \subseteq \overline{f(A)}$. Hence, $A \subseteq f^{-1}(\overline{f(A)})$. Since f is continuous and $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is also closed, and thus $\overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$. Finally, $\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$.

(2) $\Longrightarrow$ (3) Since $\overline{B}$ is closed and f is continuous, $f^{-1}(\overline{B})$ is closed: $\overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$. Therefore, since $f^{-1}(B) \subseteq f^{-1}(\overline{B})$, we have $\overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$.

(3) $\Longrightarrow$ (1) Let $B \subseteq Y$ be closed. Since

$$f^{-1}(B) \subseteq \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B) \subseteq \overline{f^{-1}(B)},$$

$f^{-1}(B) = \overline{f^{-1}(B)}$, which is closed. Hence, f is continuous.

6. Let F be a nonempty and closed subset of $\mathbb{C}$. We show that $g^{-1}(F)$ is closed in X, which then implies that $g: X \to \mathbb{C}$ is continuous.

Note first that $g^{-1}(F) \cap A = \{x \in A : g(x) \in F\}$ is a closed subset of A, since g is continuous on A. Further, since A is closed in X, the subset $g^{-1}(F) \cap A$ is also closed in X. [The relative topology of A consists of open sets $A \cap U$, U: open in X. The closed subsets of A consist of $A \setminus (A \cap U) = A \cap (X \setminus U) = A \cap U^c$, U: open in X. Since A is closed in X, $A \cap U^c$ is closed in X.]

If $0 \notin F$ then $g^{-1}(F) \cap A^c = \phi$, since $g = 0$ on A^c. In this case,

$$\begin{aligned} g^{-1}(F) &= g^{-1}(F) \cap (A \cup A^c) \\ &= (g^{-1}(F) \cap A) \cup (g^{-1}(F) \cap A^c) \\ &= (g^{-1}(F) \cap A) \cup \phi \\ &= g^{-1}(F) \cap A, \end{aligned}$$

which is closed in X.

Consider now the case $0 \in F$. Note for any $E \subseteq X$ that $\partial E = \overline{E} \setminus E^\circ$, and hence $\overline{E} = \partial E \cup E^\circ = \partial E \cup E$. Also, $\partial E = \overline{E} \cap \overline{E^c} = \partial E^c$. Therefore, since on $\overline{A^c} = \partial A^c \cup A^c = \partial A \cup A$, $g = 0$, we have

$$\begin{aligned} g^{-1}(F) &= g^{-1}(F) \cap (A \cup \overline{A^c}) \\ &= (g^{-1}(F) \cap A) \cup (g^{-1}(F) \cap \overline{A^c}) \\ &= (g^{-1}(F) \cap A) \cup \overline{A^c}. \end{aligned}$$

where $g^{-1}(F)\cap\bar{A}^c = \{x\in\bar{A}^c : g(x)\in F\}$
$$= \{x\in\bar{A}^c : g(x)=0\}$$
$$= \bar{A}^c.$$

Since both $g^{-1}(F)\cap A$ and $\bar{A}^c$ are closed in X, the union is also closed in X.

In any case $g^{-1}(F)$ is closed in X if F is closed in $\mathbb{C}$. Hence, $g: X\to\mathbb{C}$ is continuous.

7. (1) Let $A=\{x\in X : f(x)\neq g(x)\}$. Since
$$\{x\in X : f(x)=g(x)\}=A^c$$
it suffices to show that A is open in X.
Let $x\in A$. Since $f(x)\neq g(x)$ in Y, and Y is Hausdorff, there exist disjoint open subsets U and V of Y such that $f(x)\in U$ and $g(x)\in V$.
Since f and g are continuous, there exist open sets P and Q in X such that $x\in P\subseteq f^{-1}(U)$ and $x\in Q\subseteq g^{-1}(V)$. Now $P\cap Q$ is an open subset of X and $x\in P\cap Q$.
Moreover, $f(P\cap Q)\subseteq f(P)\subseteq U$,
$$g(P\cap Q)\subseteq g(Q)\subseteq V.$$
Since $U\cap V=\phi$, $f\neq g$ on $P\cap Q$. Hence, $x\in P\cap Q\subseteq A$, $P\cap Q$: open. Thus, A is open.

(2) Let A be defined as above. We need to show that $A=\phi$. If not, $\exists\, x\in A$. But A is open as shown in (1), and A^c is dense in X by our assumption. Then, $\exists\, y\in A^c\cap A=\phi$, impossible!
Therefore, $A=\phi$, and $f=g$ on X.

8. (1) Let $x \in \prod_{n=1}^{\infty} X_n$. Then, for each $n \in \mathbb{N}$, $x_n = \pi_n(x) \in X_n$. Since X_n is first countable, there exists a countable neighborhood base, $\mathcal{N}_n$, at x_n for the topology of X_n. Let $\mathcal{N}$ be the class of subsets of $\prod_{n=1}^{\infty} X_n$ of the form $\bigcap_{j=1}^{k} \pi_{n_j}^{-1}(U_{n_j})$, where $n_1 < n_2 < \cdots < n_k$, each $n_j \in \mathbb{N}$, $U_{n_j} \in \mathcal{N}_{n_j}$, $1 \leq j \leq k$, and $k \in \mathbb{N}$. We show that $\mathcal{N}$ is a countable neighborhood base at $x \in \prod_{n=1}^{\infty} X_n$.

Since $U_{n_j} \in \mathcal{N}_{n_j}$, U_{n_j} is open in X_{n_j}. Hence, $\bigcap_{j=1}^{k} \pi_{n_j}^{-1}(U_{n_j})$ is open in $\prod_{n=1}^{\infty} X_n$. Moreover, $x_{n_j} \in U_{n_j}$ for each j, and $x_{n_j} = \pi_{n_j}(x)$. Thus, $x \in \pi_{n_j}^{-1}(U_{n_j})$ for each j. Hence, $x \in \bigcap_{j=1}^{k} \pi_{n_j}^{-1}(U_{n_j})$. Hence, each member (which is an open set in $\prod_{n=1}^{\infty} X_n$) contains x. If U is an open set in the product space $\prod_{n=1}^{\infty} X_n$ and $x \in U$, then $\exists\, n_j \in \mathbb{N}$, $n_1 < \cdots < n_k$, $k \in \mathbb{N}$, V_{n_j}: open set in X_{n_j}, $x_{n_j} \in V_{n_j}$ such that $x \in \bigcap_{j=1}^{k} \pi_{n_j}^{-1}(V_{n_j}) \subseteq U$. But, $\mathcal{N}_{n_j}$ is a neighborhood base at $x_{n_j} = \pi_{n_j}(x)$ for each j. Thus, $\exists\, U_{n_j} \in \mathcal{N}_{n_j}$ such that $x_{n_j} \in U_{n_j} \subseteq V_{n_j}$. Hence, the member in $\mathcal{N}$, $\bigcap_{j=1}^{k} \pi_{n_j}^{-1}(U_{n_j}) \subseteq \bigcap_{j=1}^{k} \pi_{n_j}^{-1}(V_{n_j}) \subseteq U$. Thus, $\mathcal{N}$ is a neighborhood base at x in the product topology.

There are countably many $(n_1, n_2, \ldots, n_k)$ with $n_j \in \mathbb{N}$, $j = 1, 2, \ldots, k$, $k \in \mathbb{N}$ and $n_1 < n_2 < \cdots < n_k$. Since $\mathcal{N}_{n_j}$ is countable, $\{\bigcap_{j=1}^{k} \pi_{n_j}^{-1}(U_{n_j}) : U_{n_j} \in \mathcal{N}_{n_j},$

$j=1,\cdots,k\}$ is countable when all $k, n_1<\cdots<n_k$, are fixed. Therefore $\mathcal{N}$ is countable. It is a countable neighborhood base at x.

(2) Let $\mathcal{N}_n$ be now a countable base for X_n $(n\in\mathbb{N})$. Let $\mathcal{N}$ denote again the sets of form $\bigcap_{j=1}^{k}\pi_{n_j}^{-1}(U_{n_j})$, each $U_{n_j}\in\mathcal{N}_{n_j}$, $n_1<\cdots<n_k$, $j=1,2,\cdots,k$, $k\in\mathbb{N}$. As in the proof of (1), $\mathcal{N}$ is a countable collection of open sets of the product space $\prod_{n=1}^{\infty}X_n$. Let $x\in\prod_{n=1}^{\infty}X_n$. For each $n\in\mathbb{N}$, $\mathcal{N}_n$ contains a neighborhood base at $x_n=\pi_n(x)\in X_n$. The subclass of $\mathcal{N}$, containing $\bigcap_{j=1}^{k}\pi_{n_j}^{-1}(U_{n_j})$ with U_{n_j} in that neighborhood base at x_{n_j} for each j, $(1\le j\le k$, $n_j\in\mathbb{N}$, $k\in\mathbb{N})$, is therefore a neighborhood base at x in $\prod_{n=1}^{\infty}X_n$ as shown in Part (1). Therefore, $\mathcal{N}$ is a base for $\prod_{n=1}^{\infty}X_n$. Since $\mathcal{N}$ is countable, $\prod_{n=1}^{\infty}X_n$ is second countable.

9. Let $\varepsilon>0$. Since $\sup_{x\in X}\rho(f_n(x),f_m(x))\to 0$ as $m,n\to\infty$, there exists $N\in\mathbb{N}$ such that

$$\rho(f_n(x),f_m(x))\le\varepsilon \quad \text{if } n,m\ge N,\ x\in X.$$

Thus, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y for each $x\in X$. Since Y is complete, there exists $f(x)\in Y$ such that $\lim_{n\to\infty}\rho(f_n(x),f(x))=0$. for each $x\in X$. In the above inequality, fixing n and sending $m \to \infty$, we get $\rho(f_n(x), f(x)) \le \varepsilon$ if $n \ge N$ and $x \in X$. Thus $\sup_{x \in X} \rho(f_n(x), f(x)) \le \varepsilon$ if $n \ge N$. Consequently, $\sup_{x \in X} \rho(f_n(x), f(x)) \to 0$ as $n \to \infty$.

Assume now each f_n is continuous. Let $x \in X$. Consider the open ball $B(f(x), \varepsilon) = \{ y \in Y : \rho(f(x), y) < \varepsilon \}$ for a given and arbitrary $\varepsilon > 0$. Since $\sup_{x' \in X} \rho(f_n(x'), f(x')) \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $\rho(f_N(x'), f(x')) < \frac{\varepsilon}{3}$ for any $x' \in X$. Since f_N is continuous at x, there exists an open set U in X with $x \in U$, such that $f_N(U) \subseteq B(f_N(x), \varepsilon/3)$, i.e.,

$$\rho(f_N(x'), f_N(x)) < \varepsilon/3 \quad \text{if } x' \in U.$$

Consequently, for any $x' \in U$,

$$\begin{aligned} &\rho(f(x'), f(x)) \\ &\le \rho(f(x'), f_N(x')) + \rho(f_N(x'), f_N(x)) + \rho(f_N(x), f(x)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Thus $f(U) \subseteq B(f(x), \varepsilon)$. Hence f is continuous at x.