Homework due Thursday, May 21, 9:00 pm, on Gradescope.
Throughout, A is a ring (commutative with 1).

(1) Let D be an integral domain, and assume $\dim D = 1$. Show that any decomposable ideal has a unique primary decomposition.

(2) Let B/A be integral and let $\iota : A \to B$ be the inclusion map. Show that $\iota^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed map. More, explicitly show that $\iota^*\big(V(\mathfrak{b})\big) = V(\mathfrak{b}^c)$ for any $b \lhd B$.

(3) Problem 5, page 67 of Atiyah-MacDonald.

(4) Problem 7, page 67 of Atiyah-MacDonald.

(Hint: Given $b \in B \setminus A$ which is integral over A, choose a monic polynomial with minimal degree so that b is a zero of it. Show that the degree may be decreased.)

(5) Let $A \subset B$ be two rings, and let C be the integral closure of A in B. Let $f, g \in B[x]$ be monic polynomials so that $fg \in C[x]$. Then $f, g \in C[x]$.

(This is problem 8, page 67 of Atiyah-MacDonald. Problem 8(i) is essentially done in the book, part (ii) requires construction of a ring where f and g split; we recall this construction, hence, giving a uniform treatment.

Let $f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$ and $g(x) = x^m + b'_{m-1}x^{m-1} + \cdots + b'_0$. We will show that $\{b_i\}$ and $\{b'_j\}$ are integral over C.

Consider the ring of polynomials $R = \mathbb{Z}[r_0, \ldots, r_{n-1}, s_0, \ldots, s_{m-1}]$. For every $0 \le k \le n+m-1$, let $t_k = t_k(\{r_i\}, \{s_j\})$ be defined by

$$x^{n+m} + t_{n+m-1}x^{n+m-1} + \cdots + t_0 = (x^n + r_{n-1}x^{n-1} + \cdots + r_0)(x^m + s_{m-1}x^{m-1} + \cdots + s_0).$$

Let F be the field of fractions of R and let E/F be the splitting field of F. Now any zero of $x^{n+m} + t_{n+m-1}x^{n+m-1} + \cdots + t_0$, hence zeros of $x^n + r_{n-1}x^{n-1} + \cdots + r_0$ and $x^m + s_{m-1}x^{m-1} + \cdots + s_0$, are integral over $R' = \mathbb{Z}[t_0, \ldots, t_{n+m-1}]$.

Since $\{r_i\}$ and $\{s_j\}$ are polynomials in these zeros, we conclude that R is a finitely generated module over R'.

We now specialize $r_i = b_i$ and $s_j = b'_j$. Then since $t_k(\{b_i\}, \{b'_j\}) \in C$, we conclude that $\{b_i\}$ and $\{b'_j\}$ are integral over C.)

(6) Let $A \subset B$ be two rings, and let C be the integral closure of A in B. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$.

(This is problem 9, page 68 of Atiyah-MacDonald.)