Math 240B, Winter 2020
Solution to Problems of HW#1
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1. We first prove some auxiliary results regarding complex measures (Cf. Exercise 21 on page 94 of the textbook). Let $E \in \mathcal{M}$. Denote

$$\xi(E) = \sup\left\{ \left|\int_E f\,d\mu\right| : f \in L^1(\mu),\ |f| \le 1 \text{ on } X \right\}$$

$$\eta(E) = \sup\left\{ \sum_{j=1}^{\infty} |\mu(E_j)| : E_j \in \mathcal{M}\ (j=1,2,\dots) \text{ disjoint},\ E = \bigcup_{j=1}^{\infty} E_j \right\}.$$

We claim that

$$(*) \qquad |\mu|(E) = \xi(E) = \eta(E).$$

If $f \in L^1(\mu)\ (= L^1(|\mu|))$ and $|f| \le 1$ on X then

$$\left|\int_E f\,d\mu\right| \le \int_E |f|\,d|\mu| \le \int_E d|\mu| = |\mu|(E).$$

Hence $\xi(E) \le |\mu|(E)$. Let $f = \overline{d\mu/d|\mu|} \in L^1(|\mu|)$. Note that $|f| \le 1$ μ-a.e. on X. We can modify f on a set of μ-measure zero so that $|f| \le 1$ on X. We have $d\mu = \bar{f}\,d|\mu|$. So,

$$\left|\int_E f\,d\mu\right| = \left|\int_E f\bar{f}\,d|\mu|\right| = \left|\int_E d|\mu|\right| = |\mu|(E).$$

Thus, $\xi(E) = |\mu|(E)$.

If $E = \bigcup_{j=1}^{\infty} E_j$, $E_j \in \mathcal{M}\ (j=1,2,\dots)$, disjoint, then

$$\sum_{j=1}^{\infty} |\mu(E_j)| \le \sum_{j=1}^{\infty} |\mu|(E_j) = |\mu|(E)$$

Hence $\eta(E) \le |\mu|(E)$. Let $f \in L^1(\mu)$ and $|f| \le 1$ on X. There exist simple functions ϕ_n $(n=1,2,\dots)$ such that $\varphi_n \to f$ on X and $|\varphi_n| \le |\varphi_{n+1}| \le |f| = 1$. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_E \varphi_n \, d\mu = \int_E f \, d\mu.$$

Each of the simple functions has the form

$$\varphi = \sum_{k=1}^{m} a_k \chi_{E_k}, \quad E_1, \dots, E_m \in \mathcal{M}_E, \text{ disjoint}$$

$E = \bigcup_{k=1}^{m} E_k$, all $|a_k| \le 1$ since $|\varphi| \le |f| \le 1$. Thus

$$\left|\int_E \varphi \, d\mu\right| = \left|\sum_{k=1}^{m} a_k \mu(E_k)\right|$$

$$\le \sum_{k=1}^{m} |a_k| \, |\mu(E_k)| \le \sum_{k=1}^{m} |\mu(E_k)|$$

$$= \sum_{k=1}^{\infty} |\mu(E_k)| \le \eta(E)$$

where all $E_{k+1} = E_{k+2} = \cdots = \phi$. Thus $\left|\int_E f \, d\mu\right| \le \eta(E)$

Hence $\xi(E) \le \eta(E)$. Hence $\xi(E) = \eta(E) = |\mu|(E)$.

We now verify that $\|\mu\| = |\mu|(X)$ is a norm of $M(X)$.

(1) Clearly $\|\mu\| \ge 0$. If $\mu = 0$ then the corresponding $\eta(E) = 0$ $\forall E \in \mathcal{M}_E$. By $(*)$, $|\mu|(E) = 0$ for any $E \in \mathcal{M}$. Hence $\mu(E) = 0$ $\forall E \in \mathcal{M}_E$. So, $\mu = 0$. If $\mu \in M(X)$ and $\|\mu\| = |\mu|(X) = 0$ then $\forall E \in \mathcal{M}_E$, $|\mu(E)| \le |\mu|(E) \le |\mu|(X) = 0$. So $\mu = 0$.

(2) If α is a scalar (i.e., $\alpha \in \mathbb{R}$ or $\mathbb{C}$) and $\mu \in M(X)$, then $(\alpha\mu)(E) = \alpha\mu(E)$ $\forall E \in \mathcal{M}$. By $(*)$,

$$|\alpha\mu|(E) = \sup\left\{\sum_{j=1}^{\infty} |(\alpha\mu)E_j| : E_j \in \mathcal{M}_E \ (j=1,2,\dots) \text{ disjoint}, E = \bigcup_{j=1}^{\infty} E_j\right\}$$

$$= |\alpha| \sup\left\{\sum_{j=1}^{\infty} |\mu(E_j)| : E_j \in \mathcal{M}_E, (j=1,2,\dots), \text{ disjoint}, E = \bigcup_{j=1}^{\infty} E_j\right\}$$

$$= |\alpha| \, |\mu|(E). \qquad \forall E \in \mathcal{M}$$

(3) Let $\mu, \nu \in M(X)$. By Proposition 3.14

$$\|\mu+\nu\| = |\mu+\nu|(X) \le |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|.$$

Hence, by (1)–(3), $\|\cdot\|$ is a norm on $M(X)$.

We show finally $M(X)$ is a Banach space. Let $\mu_n \in M(X)$ be such that $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$. For any $E \in \mathcal{M}$, we have $\sum_{n=1}^{\infty} |\mu_n(E)| \le \sum_{n=1}^{\infty} |\mu_n|(E)$
$\le \sum_{n=1}^{\infty} |\mu_n|(X) = \sum_{n=1}^{\infty} \|\mu_n\| < \infty$. Hence $\sum_{n=1}^{\infty} \mu_n(E)$ converges absolutely. Let $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E) \in \mathbb{C}$.
Clearly $\mu(\phi) = \phi$. Suppose $E = \bigcup_{j=1}^{\infty} E_j$ is a disjoint union of $E_j \in \mathcal{M}$ $(j = 1, 2, \cdots)$. Since $|\mu_n(E_j)| \le |\mu_n|(E_j)$ $(\forall j, n \in \mathbb{N})$ and

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |\mu_n|(E_j) = \sum_{n=1}^{\infty} |\mu_n|(E) < \infty,$$

we have

$$\sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \mu_n(E_j)$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \mu_n(E_j) = \sum_{n=1}^{\infty} \mu_n(E) = \mu(E)$$

Thus $\mu \in M(X)$. We show that $\left\| \sum_{n=1}^{N} \mu_n - \mu \right\| \to 0$ as $N \to \infty$. Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$ there exists $N_0 \in \mathbb{N}$ s.t. $N \ge N_0 \Rightarrow \sum_{n=N+1}^{\infty} \|\mu_n\| \le \varepsilon$.
Now, let $E_j \in \mathcal{M}$ $(j = 1, 2, \cdots)$ be disjoint and $X = \bigcup_{j=1}^{\infty} E_j$. Then

$$\sum_{j=1}^{\infty} \left| \left(\sum_{n=1}^{N} \mu_n - \mu \right)(E_j) \right| = \sum_{j=1}^{\infty} \left| \sum_{n=N+1}^{\infty} \mu_n(E_j) \right|$$

$$\le \sum_{j=1}^{\infty} \sum_{n=N+1}^{\infty} |\mu_n|(E_j) = \sum_{n=N+1}^{\infty} \sum_{j=1}^{\infty} |\mu_n|(E_j)$$

$$= \sum_{n=N+1}^{\infty} |\mu_n|(X) = \sum_{n=N+1}^{\infty} \|\mu_n\| \le \varepsilon \quad \text{if } N \ge N_0$$

Thus, by replacing E and μ in the definition of $\eta(E)$ by X and $\sum_{n=1}^{N} \mu_n - \mu$, respectively, we have $\|\sum_{n=1}^{N} \mu_n - \mu\| \le \varepsilon$ if $N \ge N_0$. Hence, $\sum_{n=1}^{N} \mu_n \to \mu$ in $M(X)$.

2. Let $f \in C^k([0,1])$. Clearly $\|f\|_{C^k} \ge 0$. If $f=0$ then $\|f\|_{C^k} = 0$. Suppose $f \in C^k([0,1])$ and $\|f\|_{C^k} = 0$ then $\max_{0\le x\le 1} |f(x)| = 0$ Hence $f=0$. If $f \in C^k([0,1])$ and α is a scalar ($\alpha \in K = \mathbb{R}$ or $\mathbb{C}$) then

$$\begin{aligned}\|\alpha f\|_{C^k} &= \max_{0\le j\le k} \max_{0\le x\le 1} |(\alpha f)^{(j)}(x)| \\ &= \max_{0\le j\le k} \max_{0\le x\le 1} |\alpha f^{(j)}(x)| \\ &= |\alpha| \max_{0\le j\le k} \max_{0\le x\le 1} |f^{(j)}(x)| \\ &= |\alpha| \, \|f\|_{C^k}.\end{aligned}$$

Let $f, g \in C^k([0,1])$ and $0 \le j \le k$. Let $x \in [0,1]$. Then

$$\begin{aligned}|(f+g)^{(j)}(x)| &= |f^{(j)}(x) + g^{(j)}(x)| \\ &\le |f^{(j)}(x)| + |g^{(j)}(x)| \le \|f\|_{C^k} + \|g\|_{C^k}.\end{aligned}$$

Hence, $\|f+g\|_{C^k} \le \|f\|_{C^k} + \|g\|_{C^k}$. We have thus verified that $\|\cdot\|_{C^k}$ is a norm on $C^k([0,1])$.

We now show that $C^k([0,1])$ is a Banach space. Let $f_n \in C^k([0,1])$ $(n=1,2,\cdots)$ and assume that $\|f_m - f_n\|_{C^k} \to 0$ as $m,n \to \infty$. Let $0 \le j \le k$ and $x \in [0,1]$. Then $|f_m^{(j)}(x) - f_n^{(j)}(x)| \to 0$ as $m,n \to \infty$. Hence, $\exists\, g_j(x)$ s.t. $f_n^{(j)}(x) \to g_j(x)$ as $n \to \infty$.
$\forall \varepsilon > 0$ $\exists N$, $n,m \ge N \Longrightarrow \|f_m - f_n\|_{C^k} \le \varepsilon$. Thus

$|f_m^{(j)}(x) - f_n^{(j)}(x)| \le \varepsilon \quad \forall x \in [0,1],\ 0 \le j \le k,\ m, n \ge N.$

Hence, $|f_m^{(j)}(x) - g_j(x)| \le \varepsilon \quad \forall x \in [0,1]\ 0 \le j \le k,\ m \ge N.$

This means $f_m^{(j)} \to g_j$ uniformly on $[0,1]$ as $m \to \infty$. Fix $m = N$, we get for any $x, y \in [0,1]$ that

$$|g_j(x) - g_j(y)| \le |g_j(x) - f_N^{(j)}(x)| + |f_N^{(j)}(x) - f_N^{(j)}(y)| + |f_N^{(j)}(y) - g_j(y)|$$
$$\le 2\varepsilon + |f_N^{(j)}(x) - f_N^{(j)}(y)|.$$

Since $f_N^{(j)}$ is continuous, $|f_N^{(j)}(x) - f_N^{(j)}(y)| \to 0$ as $y \to x$.

So, $\limsup_{y \to x} |g_j(x) - g_j(y)| \le 2\varepsilon$. Hence $\lim_{y \to x} |g_j(x) - g_j(y)| = 0$ i.e., g_j is continuous on $[0,1]$.

Let $f = g_0$. We finally show that $g_j = f^{(j)}$ on $[0,1]$ for $j = 1, \cdots, k$. Since $f_n' \to g_1$ uniformly on $[0,1]$,

$$\int_0^x f_n'(t)\,dt \to \int_0^x g_1(t)\,dt \quad \forall x \in [0,1].$$

But $\int_0^x f_n'(t)\,dt = f_n(x) - f_n(0) \to f(x) - f(0)$.

Hence $f(x) - f(0) = \int_0^x g_1(t)\,dt \quad \forall x \in [0,1]$.

Consequently $f'(x) = g_1(x) \quad \forall x \in [0,1]$.

Suppose $1 \le m \le k-1$ and $g_m = f^{(m)}$. Then

$$\int_0^x f_n^{(m+1)}(t)\,dt \to \int_0^x g_{m+1}(t)\,dt. \quad \text{and}$$

$$\int_0^x f_n^{(m+1)}(t)\,dt = f_n^{(m)}(x) - f_n^{(m)}(0) \to f^{(m)}(x) - f^{(m)}(0)$$

Hence, $f^{(m+1)}(x) = \frac{d}{dx}\int_0^x g_{m+1}(t)\,dt = g_{m+1}(x)$.

Therefore, $f^{(j)} = g_j \quad (1 \le j \le k)$. The proof is complete, as $f_n^{(j)} \to f^{(j)}$ uniformly on $[0,1]$ for all $j = 0, 1, \cdots, k$. which means $\|f_n - f\|_{C^k} \to 0$ as $n \to \infty$.

3. We first show that $\Lambda_\alpha([0,1])$ is a vector subspace of $C^0([0,1]) = C([0,1])$. If $f \in \Lambda_\alpha([0,1])$ then

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \le \|f\|_\alpha < \infty \quad \forall x, y \in [0,1].$$

Thus $|f(x)-f(y)| \le \|f\|_\alpha |x-y|^\alpha \quad \forall x, y \in [0,1]$. Hence, $f \in C([0,1])$. If both $f, g \in \Lambda_\alpha([0,1])$ then

$$\|f+g\|_\alpha = |f(0)+g(0)| + \sup_{\substack{x,y\in[0,1] \\ x\ne y}} \frac{|f(x)-f(y)+g(x)-g(y)|}{|x-y|^\alpha}$$

$$\le |f(0)| + |g(0)| + \sup_{\substack{x,y\in[0,1] \\ x\ne y}} \left(\frac{|f(x)-f(y)|}{|x-y|^\alpha} + \frac{|g(x)-g(y)|}{|x-y|^\alpha} \right)$$

$$\le |f(0)| + |g(0)| + \sup_{\substack{x,y\in[0,1] \\ x\ne y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} + \sup_{\substack{x,y\in[0,1] \\ x\ne y}} \frac{|g(x)-g(y)|}{|x-y|^\alpha}$$

$$= \|f\|_\alpha + \|g\|_\alpha < \infty$$

Hence, $f+g \in \Lambda_\alpha([0,1])$. If a is a scalar, then clearly $\|af\|_\alpha = |a|\,\|f\|_\alpha < \infty$. Hence $af \in \Lambda_\alpha([0,1])$. Thus, $\Lambda_\alpha([0,1])$ is a vector subspace of $C([0,1])$.

If $f=0$ then clearly $\|f\|_\alpha = 0$. If $f \in \Lambda_\alpha([0,1])$ and $\|f\|_\alpha = 0$ then $f(0)=0$ and $f(x)=f(y)$ $\forall x, y \in [0,1]$, $x \ne y$. Thus $f(x) = f(0) = 0$ $\forall x \in [0,1]$. $f=0$. We already showed $\|f+g\|_\alpha \le \|f\|_\alpha + \|g\|_\alpha$ $\forall f, g \in \Lambda_\alpha([0,1])$ and $\|af\|_\alpha = |a|\,\|f\|_\alpha$ if a is a scalar. Thus, $\Lambda_\alpha([0,1])$ is a normed vector space.

Let $\{f_n\}$ be a Cauchy sequence in $\Lambda_\alpha([0,1])$. Then $|f_m(0) - f_n(0)| \le \|f_m - f_n\|_\alpha \to 0$ as $m, n \to \infty$. So $f_n(0) \to f(0)$ as $n \to \infty$ for some $f(0)$. Also, for any $x \in [0,1]$.

$$|f_m(x) - f_n(x) - [f_m(0) - f_n(0)]| \le \|f_m - f_n\|_\alpha |x|^\alpha \to 0$$

as $m, n \to \infty$. Hence $\{f_m(x) - f_m(0)\}$ converges. Since $f_m(0) \to f(0)$, we have $f_m(x) \to f(x)$ for some $f(x)$ ($\forall x \in (0,1]$). Thus $f_n \to f$ pointwise on $[0,1]$. We show that $f \in \Lambda_\alpha([0,1])$ and $f_n \to f$ in $\Lambda_\alpha([0,1])$. Since $\|f_n - f_m\|_\alpha \to 0$ as $m, n \to \infty$ we have for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\|_\alpha \le \varepsilon$ if $n, m \ge N$. Thus, for any $x, y \in [0,1]$

$$\begin{aligned} |[f_m(x) - f_n(x)] - [f_m(y) - f_n(y)]| &\le \|f_m - f_n\|_\alpha |x-y|^\alpha \\ &\le \varepsilon |x-y|^\alpha \end{aligned}$$

and also $|f_m(0) - f_n(0)| \le \varepsilon$. By taking $n \to \infty$ we get

$$|[f_m(x) - f(x)] - [f_m(y) - f(y)]| \le \varepsilon |x-y|^\alpha \quad \forall m \ge N$$

$$|f_m(0) - f(0)| \le \varepsilon \quad \forall m \ge N.$$

These imply that

$$\|f_m - f\|_\alpha \le \varepsilon \quad \text{if } m \ge N$$

In particular, $\|f_N - f\|_\alpha \le \varepsilon$. So, $f_N - f \in \Lambda_\alpha([0,1])$. Hence $f = f_N - (f_N - f) \in \Lambda_\alpha([0,1])$ as $\Lambda_\alpha([0,1])$ is a vector space. Moreover, $f_m \to f$ in $\Lambda_\alpha([0,1])$.

4. Assume $\dim \mathcal{X} = n \geq 1$. Let $\{e_1, \cdots, e_n\}$ be a basis for $\mathcal{X}$. Let $K = \mathbb{R}$ or $\mathbb{C}$.

(1) Any $x \in \mathcal{X}$ can be uniquely expressed as $x = \sum_{j=1}^n \xi_j e_j$, i.e., $x \longmapsto (\xi_1, \cdots, \xi_n) \in K^n$ is a bijection between $\mathcal{X}$ and K^n. In fact, this is an isomorphism (preserving the linearity). Define

$$\|x\|_0 = \sum_{j=1}^n |\xi_j| \quad \forall x = \sum_{j=1}^n \xi_j e_j \in \mathcal{X}.$$

Then $\|x\|_0 \geq 0$. $\|x\|_0 = 0 \iff$ all $\xi_j = 0$ $(1 \leq j \leq n)$, i.e., $x = 0$. If $\alpha \in K$, $x = \sum_{j=1}^n \xi_j e_j$, then $\alpha x = \sum_{j=1}^n (\alpha \xi_j) e_j$. Hence, $\|\alpha x\|_0 = \sum_{j=1}^n |\alpha \xi_j| = |\alpha| \sum_{j=1}^n |\xi_j| = |\alpha| \|x\|_0$.
If $y = \sum_{j=1}^n \eta_j e_j \in \mathcal{X}$ then $x + y = \sum_{j=1}^n (\xi_j + \eta_j) e_j$. Hence $\|x+y\|_0 = \sum_{j=1}^n |\xi_j + \eta_j| \leq \sum_{j=1}^n |\xi_j| + \sum_{j=1}^n |\eta_j| = \|x\|_0 + \|y\|_0$. There for $\|\cdot\|_0$ is a norm on $\mathcal{X}$.

To show that any two norms on $\mathcal{X}$ are equivalent, we need only to show that any norm $\|\cdot\|$ on $\mathcal{X}$ is equivalent to the norm $\|\cdot\|_0$. Let $x = \sum_{j=1}^n \xi_j e_j \in \mathcal{X}$. Then

$$\|x\| \leq \sum_{j=1}^n |\xi_j| \, \|e_j\| \leq A \sum_{j=1}^n |\xi_j| = A \|x\|_0,$$

where $A = \max_{1 \leq j \leq n} \|e_j\| > 0$. Define $f(\xi) = \left\| \sum_{j=1}^n \xi_j e_j \right\|$ $\forall \xi = (\xi_1, \cdots, \xi_n) \in K^n$. Then

$$|f(\xi) - f(\eta)| \leq \left\| \sum_{j=1}^n (\xi_j - \eta_j) e_j \right\| \leq A \sum_{j=1}^n |\xi_j - \eta_j|$$

Hence $f : K^n \to \mathbb{R}$ is continuous. Let

$$S = \left\{ \xi = (\xi_1, \cdots, \xi_n) \in K^n : \sum_{j=1}^n |\xi_j| = 1 \right\}.$$

Then S is bounded and closed in K^n (hence compact in K^n). Thus, $\exists \, \bar{\xi} \in S$ such that

$$|f(\hat{\xi})| = \min_{\xi \in S} |f(\xi)| =: a > 0$$

(If $a=0$ then $f(\hat{\xi})=0$ which implies all $\hat{\xi}_j = 0$ where $\hat{\xi} = (\hat{\xi}_1, \cdots, \hat{\xi}_n)$. This is impossible as $\hat{\xi} \in S$.) Now $\forall x \in \mathcal{X}$, $x \neq 0$. Let $x = \sum_{j=1}^n \xi_j e_j$. Then not all $\xi_j = 0$. i.e., $\sum_{j=1}^n |\xi_j| > 0$. Note that for $\xi = (\xi_1, \cdots, \xi_n)$ $\xi / \sum_{j=1}^n |\xi_j| \in S$. Hence

$$f\left(\frac{\xi}{\sum_{j=1}^n |\xi_j|}\right) = \frac{1}{\sum_{j=1}^n |\xi_j|} \|x\| \geq a. \quad \text{Thus } \|x\| \geq a\|x\|_0.$$

Summarize: $a\|x\|_0 \leq \|x\| \leq A\|x\| \quad \forall x \in \mathcal{X}$.

(2) It suffices to show that $(\mathcal{X}, \|\cdot\|_0)$ is a Banach space. Let $x^{(k)} = \sum_{j=1}^n \xi_j^{(k)} e_j \in \mathcal{X}$ $(k=1,2,\cdots)$ be such that

$$\|x^{(k)} - x^{(m)}\|_0 = \sum_{j=1}^n |\xi_j^{(k)} - \xi_j^{(m)}| \to 0 \text{ as } m, k \to \infty.$$

For each j, $1 \leq j \leq k$, $\{\xi_j^{(k)}\}_{k=1}^\infty$ is a Cauchy sequence in K. Hence $\exists\, \xi_j \in K$ such that $\xi_j^{(k)} \to \xi_j$ as $k \to \infty$. Let $x = \sum_{j=1}^n \xi_j e_j \in \mathcal{X}$. Then $\|x^{(k)} - x\|_0 = \sum_{j=1}^n |\xi_j^{(k)} - \xi_j| \to 0$ as $k \to \infty$. Hence $(\mathcal{X}, \|\cdot\|_0)$ is a Banach space.

5. If $x, y \in \mathcal{M}$, then $\|x+y\| \leq \|x\| + \|y\| = 0$. Hence $x+y \in \mathcal{M}$. If $x \in \mathcal{M}$ and $\alpha \in K$ ($=\mathbb{R}$ or $\mathbb{C}$) then $\|\alpha x\| = |\alpha| \|x\| = 0$. Hence $\alpha x \in \mathcal{M}$. Thus $\mathcal{M}$ is a vector subspace of $\mathcal{X}$.

Note that $\mathcal{X}/\mathcal{M} = \{x + \mathcal{M} : x \in \mathcal{X}\}$ is a vector space with $(x+\mathcal{M}_\xi)+(y+\mathcal{M}_\xi)=x+y+\mathcal{M}_\xi$ and $\alpha(x+\mathcal{M}_\xi)=\alpha x+\mathcal{M}_\xi$ $(x,y\in\mathcal{X}, \alpha\in K)$

Define $|||x+\mathcal{M}_\xi|||=\|x\|$ $\forall x\in\mathcal{X}$. Note that $x+\mathcal{M}_\xi=y+\mathcal{M}_\xi \Leftrightarrow x-y\in\mathcal{M}_\xi \Leftrightarrow \|x-y\|=0$.

In this case, $\|x\|=\|x-y+y\|\le\|x-y\|+\|y\|=\|y\|$ and $\|y\|\le\|x\|$. Hence $\|x\|=\|y\|$. Thus $|||x+\mathcal{M}_\xi|||=\|x\|$ is well-defined (i.e., it is independent of the representative x).

Clearly $|||x+\mathcal{M}_\xi|||=\|x\|\ge0$ $\forall x\in\mathcal{X}$. If $|||x+\mathcal{M}_\xi|||=\|x\|=0$ then $x\in\mathcal{M}_\xi$ and $x+\mathcal{M}_\xi=\mathcal{M}_\xi$ which is the zero vector in $\mathcal{X}/\mathcal{M}_\xi$. Also, $|||\mathcal{M}_\xi|||=|||0+\mathcal{M}_\xi|||=\|0\|=0$. If $x\in\mathcal{X}$ and $\alpha\in K$ then $|||\alpha(x+\mathcal{M}_\xi)|||=|||\alpha x+\mathcal{M}_\xi|||=\|\alpha x\|=|\alpha|\,\|x\|=|\alpha|\,|||x+\mathcal{M}_\xi|||$. If $x,y\in\mathcal{X}$ then $|||(x+\mathcal{M}_\xi)+(y+\mathcal{M}_\xi)|||=|||x+y+\mathcal{M}_\xi|||=\|x+y\|\le\|x\|+\|y\|=|||x+\mathcal{M}_\xi|||+|||y+\mathcal{M}_\xi|||$. Hence, $|||x+\mathcal{M}_\xi|||=\|x\|$ is a norm on $\mathcal{X}/\mathcal{M}_\xi$.

6. (1) Denote $[x]=x+\mathcal{M}_\xi\in\mathcal{X}/\mathcal{M}_\xi$ $(x\in\mathcal{X})$. Note that $[0]=\mathcal{M}_\xi$ is the zero vector in $\mathcal{X}/\mathcal{M}_\xi$. Clearly, $\|[x]\|\ge0$ $\forall x\in\mathcal{X}$. Also, $\|[0]\|=\inf_{y\in\mathcal{M}_\xi}\|y\|=\|0\|=0$. If $x\in\mathcal{X}$ and $\|[x]\|=0$, i.e., $\inf_{y\in\mathcal{M}_\xi}\|x+y\|=0$, then $\exists\, y_n\in\mathcal{M}_\xi$ $(n\in\mathbb{N})$ s.t. $\|x+y_n\|\to0$ as $n\to\infty$. i.e., $-y_n\to x$. But all $-y_n\in\mathcal{M}_\xi$ and $\mathcal{M}_\xi$ is closed. Hence $x\in\mathcal{M}_\xi$ and $[x]=[0]=\mathcal{M}_\xi$. (the zero vector of $\mathcal{X}/\mathcal{M}_\xi$).

If α is a scalar and $\alpha \neq 0$, and $x \in \mathcal{X}$, then

$$\begin{aligned}\|[\alpha x]\| &= \inf_{y \in \mathcal{M}_\xi} \|\alpha x + y\| \\ &= |\alpha| \inf_{y \in \mathcal{M}_\xi} \|x + \tfrac{y}{\alpha}\| \\ &= |\alpha| \inf_{z \in \mathcal{M}_\xi} \|x + z\| \\ &= |\alpha| \, \|[x]\|\end{aligned}$$

Let $x, y \in \mathcal{X}$. $\forall \varepsilon > 0$ $\exists u, v \in \mathcal{M}_\xi$ such that $\|[x]\| \geq \|x+u\| - \varepsilon$ and $\|[y]\| \geq \|y+v\| - \varepsilon$.

Hence
$$\begin{aligned}\|[x]\| + \|[y]\| &\geq \|x+u\| - \varepsilon + \|y+v\| - \varepsilon \\ &\geq \|x+y+u+v\| - 2\varepsilon \\ &\geq \|[x+y]\| - 2\varepsilon \\ &= \|[x]+[y]\| - 2\varepsilon.\end{aligned}$$

where we used the fact that $u+v \in \mathcal{M}_\xi$. Thus, $\|[x]\| + \|[y]\| \geq \|[x]+[y]\|$. Hence $\mathcal{X}/\mathcal{M}_\xi$ is a normed vector space.

(2) Let $x_1 \in \mathcal{X} \setminus \mathcal{M}_\xi$. Since $\mathcal{M}_\xi$ is closed,

$$d := \operatorname{dist}(x_1, \mathcal{M}_\xi) = \inf_{y \in \mathcal{M}_\xi} \|x_1 - y\| > 0.$$

Let $\varepsilon \in (0,1)$ and $\varepsilon_1 = d\varepsilon/(1-\varepsilon) > 0$. There exists $m_1 \in \mathcal{M}_\xi$ such that $\|x_1 - m_1\| < d + \varepsilon_1$. Let $x = \dfrac{x_1 - m_1}{\|x_1 - m_1\|} \in \mathcal{X}$. Then $\|x\| = 1$. Moreover,

$$\begin{aligned}\|x + \mathcal{M}_\xi\| &= \inf_{y \in \mathcal{M}_\xi} \|x + y\| = \inf_{m \in \mathcal{M}_\xi} \left\| \frac{x_1 - m_1}{\|x_1 - m_1\|} - m \right\| \\ &= \inf_{m \in \mathcal{M}_\xi} \frac{1}{\|x_1 - m_1\|} \|x_1 - m_1 - \|x_1 - m_1\| m\| \\ &= \inf_{m' \in \mathcal{M}_\xi} \frac{1}{\|x_1 - m_1\|} \|x_1 - m_1'\|\end{aligned}$$

$$> \frac{d}{d+\varepsilon_1} = 1-\varepsilon,$$

where we used the fact that $m \in \mathcal{M}$ if and only $m_1 + \|x_1 - m_1\| m \in \mathcal{M}$, since $m_1 \in \mathcal{M}$ and $\|x_1 - m_1\| > 0$.

(3) The projection $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$, defined by $\pi(x) = [x] = x + \mathcal{M}$ $(x \in \mathcal{X})$, is clearly a linear map. Moreover, if $x \in \mathcal{X}$, then

$$\|\pi(x)\| = \|[x]\| = \inf_{y \in \mathcal{M}} \|x+y\| \le \|x+0\| = \|x\|.$$

Hence π is bounded, and $\|\pi\| \le 1$.

Let $\varepsilon \in (0,1)$. By part (2), $\exists\, x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|\pi(x)\| = \|[x]\| > 1-\varepsilon$. Hence $\|\pi\| > 1-\varepsilon$. Thus $\|\pi\| = 1$.

(4) Assume $\mathcal{X}$ is complete. Let $x_n \in \mathcal{X}$ $(n = 1, 2, \cdots)$ be such that $\sum_{n=1}^{\infty} \|[x_n]\| < \infty$. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, $\exists\, y_n \in \mathcal{M}$ such that

$$\|[x_n]\| = \inf_{y \in \mathcal{M}} \|x_n + y\| \ge \|x_n + y_n\| - \frac{1}{2^n}.$$

Hence

$$\sum_{n=1}^{\infty} \|x_n + y_n\| \le \sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \|[x_n]\| \right) = 1 + \sum_{n=1}^{\infty} \|[x_n]\| < \infty.$$

Since $\mathcal{X}$ is complete, $\sum_{n=1}^{\infty} (x_n + y_n)$ converges in $\mathcal{X}$. Let $z = \sum_{n=1}^{\infty} (x_n + y_n) \in \mathcal{X}$, i.e.,

$$\left\| z - \sum_{k=1}^{n} (x_k + y_k) \right\| \to 0 \text{ as } n \to \infty.$$

Since $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ is linear and bounded,

$$[z] = \pi(z) = \lim_{n\to\infty} \sum_{k=1}^{n} \pi(x_k + y_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \pi(x_n)$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} [x_n], \quad \text{in } \mathcal{X}/\mathcal{M}.$$

i.e., $\left\| \sum_{k=1}^{n} [x_n] - [z] \right\| \to 0$ as $n \to \infty$. So, $[z] = \sum_{k=1}^{\infty} [x_k]$ in $\mathcal{X}/\mathcal{M}$.

Hence $\mathcal{X}/\mathcal{M}$ is complete.

7. Denote $K = \mathbb{R}$ or $\mathbb{C}$. Let $f : \mathcal{X} \to K$ be linear.

Assume f is continuous. Since $\{0\}$ is closed in K, $f^{-1}(\{0\})$ is closed in $\mathcal{X}$. Since f is linear, $f^{-1}(\{0\})$ is a vector subspace of $\mathcal{X}$. Hence, $f^{-1}(\{0\})$ is a closed subspace of $\mathcal{X}$.

Assume now $f^{-1}(\{0\})$ is a closed subspace of $\mathcal{X}$. If $f^{-1}(\{0\}) = \mathcal{X}$ then $f = 0$ and it is continuous. Assume $f^{-1}(\{0\}) \neq \mathcal{X}$. Then $\exists \hat{x}_0 \in \mathcal{X} \setminus f^{-1}(\{0\})$, i.e., $f(\hat{x}_0) \neq 0$. Let $x_0 = \frac{\hat{x}_0}{f(\hat{x}_0)} \in \mathcal{X}$. Then $f(x_0) = 1$. and $x_0 \in \mathcal{X} \setminus f^{-1}(\{0\})$. If f were not continuous then $\exists x_n \in \mathcal{X}$ $(n \in \mathbb{N})$ such that $x_n \to 0$ but $|f(x_n)| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. Let $y_n = x_n - f(x_n) x_0$ $(n \in \mathbb{N})$. Then $f(y_n) = f(x_n) - f(x_n) f(x_0) = 0$. Hence $y_n \in f^{-1}(\{0\})$. Thus

$$\|x_n\| = \|f(x_n) x_0 + y_n\|$$
$$= |f(x_n)| \, \|x_0 + \frac{y}{f(x_n)}\| \geq \varepsilon_0 \inf_{z \in f^{-1}(\{0\})} \|x_0 + z\|.$$

$(n \in \mathbb{N})$. Since $f^{-1}(\{0\})$ is a closed subspace and $x_0 \notin f^{-1}(\{0\})$, $\delta := \inf_{z \in f^{-1}(\{0\})} \|x_0 + z\| > 0$. Hence $\|x_n\| \geq \varepsilon_0 \delta$ $(n = 1, 2, \cdots)$. This contradicts to $x_n \to 0$. Hence f is continuous.

8. (1) Note that if $A, B \in L(\mathcal{X}, \mathcal{X})$ then $AB \in L(\mathcal{X}, \mathcal{X})$ and

$$\|(AB)x\| = \|A(Bx)\| \leq \|A\| \, \|Bx\|$$
$$\leq \|A\| \, \|B\| \, \|x\| \qquad \forall x \in \mathcal{X}.$$

Hence $\|AB\| \leq \|A\| \, \|B\|$. In particular, $\|A^n\| \leq \|A\|^n$ $(n \in \mathbb{N})$.

Since $\|I-T\|<1$, we have
$$\sum_{n=0}^{\infty}\|(I-T)^n\| \le \sum_{n=0}^{\infty}\|I-T\|^n < \infty.$$
Since X is a Banach space, $L(X,X)$ is also a Banach space. Thus, $\sum_{n=0}^{\infty}(I-T)^n$ converges in $L(X,X)$. Let $S=\sum_{n=0}^{\infty}(I-T)^n$, i.e., $S\in L(X,X)$ and $\|S-\sum_{n=0}^{N}(I-T)^n\|\to 0$ as $N\to\infty$. Now, we have
$$ST-I = \left[S-\sum_{j=0}^{n}(I-T)^j\right]T + \left[\sum_{j=0}^{n}(I-T)^j\right](T-I) + \sum_{j=0}^{n}(I-T)^j - I$$
$$= \left[S-\sum_{j=0}^{n}(I-T)^j\right]T - \sum_{j=0}^{n}(I-T)^{j+1} + \sum_{j=0}^{n}(I-T)^j - I$$
$$= \left[S-\sum_{j=0}^{n}(I-T)^j\right]T - (I-T)^{n+1}$$
Hence $\|ST-I\| \le \|S-\sum_{j=0}^{n}(I-T)^j\|\,\|T\| + \|I-T\|^{n+1} \to 0$

So, $ST=I$. Similarly, $TS=I$. Hence $S=T^{-1}$.

(2) We have
$$\|I-ST^{-1}\| = \|TT^{-1}-ST^{-1}\|$$
$$= \|(T-S)T^{-1}\| \le \|T-S\|\,\|T^{-1}\| < 1.$$
Hence, by (1), $ST^{-1}\in L(X,X)$ is invertible. Let $P=ST^{-1}\in L(X,X)$. Then P is invertible. Since $S=PT\in L(X,X)$, and both P and T are invertible, S is invertible. In fact, $S^{-1}=T^{-1}P^{-1}$.