Problem 0

I agree

Problem 1

So the sequence is $\{n \in \mathbb{Z}_{\geq 0}\}$ and at each n, we require the principal part of Laurent series to be $g_n = \frac{\sqrt{n}}{z-n} = \frac{-1}{\sqrt{n}} \frac{1}{1-z/n}$.

Expand it at $z = 0$, we have $\sum_{i=0} \frac{-1}{\sqrt{n}} \frac{z^i}{n^i} = \frac{-1}{\sqrt{n}} + \sum_{i=1} \frac{-1}{\sqrt{n}} \frac{z^i}{n^i}$.

Take $h_n = \frac{-1}{\sqrt{n}} = \frac{-\sqrt{n}}{n}$. Hence

$$\begin{aligned}
|g_n - h_n| &= |\frac{\sqrt{n}}{z-n} + \frac{\sqrt{n}}{n}| \\
&= |\frac{n\sqrt{n} + \sqrt{n}(z-n)}{n(z-n)}| \\
&= |\frac{\sqrt{n}z}{n(z-n)}| \\
&\leq \frac{\sqrt{n}r_n}{n(n-r_n)} \\
&= c_n
\end{aligned}$$

Now simply let $r_n = n^{1/3}$ and c_n will be roughly at the order of $\frac{n^{5/6}}{n^2} = \frac{1}{n^{7/6}}$. Hence $\sum_n c_n < \infty$.

Hence the solution to Mittag-Leffler problem is simply $\sum_{n\geq 1} \frac{\sqrt{n}z}{n(z-n)} + \frac{1}{z}$

Problem 2

If we consider $\{a_n\}$ as zeros, then we can construct an entire function $f(z)$ such that it has simple zeros at $\{a_n\}$. Similarly, we can construct a meromorphic function $g(z)$ such that $\{a_n\}$ are the poles of the function and have principal part of Laurent series $\frac{\beta_n}{z-a_n}$. Now expand the Taylor series of $f(z)$ at a_n, we obtain $\alpha_n(z-a_n) + ...$, where the omission represents terms with higher power of $(z-a_n)$ and $\alpha_n \neq 0$.

To obtain the value $f(a_n)g(a_n)$, we can check the corresponding value in the multiplication of their Laurent series, which start with $\alpha_n\beta_n+$ polynomials of

$(z-a_n)$. Hence we simply let $\beta_n = A_n/\alpha_n$, so that we can define $fg = A_n$ at the removable pole a_n. Do this for all n, we see that fg is entire and satisfy $fg(a_n) = A_n$ for arbitrary A_n.

Problem 3

(a) We take the logarithmic derivative

$$\begin{aligned}\zeta(z) &= (log(\sigma(z))' \\ &= (log(z) + \sum_\lambda log(1-\frac{z}{\lambda}) + (\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}))' \\ &= \frac{1}{z} + \sum_\lambda \frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\end{aligned}$$

By the previous homework, $\sigma(z)$ converges absolutely. Hence its logarithmic derivative $\zeta(z)$ also converges. Hence it is a solution to some Mittag Lefler problem. By observation, we see that it have Λ as its set of poles, $\frac{1}{z-\lambda}$ as principal part of Laurent series centered at any λ

(b) Given a solution

$$\begin{aligned}f(z) &= z^{k_0}e^g \prod_n (E_{p_n}(z/a_n))^{k_n} \\ &= z^{k_0}e^g \prod_n (1-z/a_n)^{k_n} e^{k_n h_n}\end{aligned}$$

, where g and h_n are entire functions, k_0 and k_n are zero multiplicities. We take the logarithmic derivative

$log(f)' = k_0/z + g + \sum_n \frac{k_n}{z-a_n} + k_n h_n$

Again, the convergence of the logarithmic derivative is ensured by the convergence of the original function and therefore it is a solution to some ML problem. By observation, we see that it has 0 and $\{a_n\}$ as its set of poles and have $\frac{k_n}{z-a_n}$ as its principal part of Laurent series at a_n and $\frac{k_0}{z}$ as its principal part of Laurent series at 0.

Problem 4

(a) $\zeta(z) = \frac{1}{z} + \sum_\lambda \frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}$. Hence we have

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} - (\sum_\lambda -\frac{1}{(z-\lambda)^2} + \frac{1}{\lambda^2}) \\ &= \frac{1}{z^2} + \sum_\lambda \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\end{aligned}$$

(b) $\wp$ is obviously meromorphic as all of its poles are in the discrete set Λ. We have

$$\begin{aligned}\wp(z+w_1) &= \frac{1}{(z+w_1)^2} + (\sum_{\lambda\neq 0} \frac{1}{(z+w_1-\lambda)^2} - \sum_{\lambda\neq 0} \frac{1}{\lambda^2} \\ &= \frac{1}{(z+w_1)^2} + (\sum_{\lambda\neq -w_1} \frac{1}{(z-\lambda)^2} - \sum_{\lambda\neq 0} \frac{1}{\lambda^2}\end{aligned}$$

which is obviously equal to $\wp(z)$ after we pull $\frac{1}{z^2}$ out of the sum and push $\frac{1}{(z+w_1)^2}$ into the sum. Same logic holds for w_2 and hence $\wp$ is doubly periodic.

(c) $\wp(z) = \wp(z+w_1) = \wp(z+w_2)$, hence taking derivative, we obtain $\wp'(z) = \wp'(z+w_1) = \wp'(z+w_2)$

(d/e) the only part that contains the pole $z=0$ is $\frac{1}{z^2}$, hence that is the principal part of the Laurent series at 0. When plugging in $z=0$, $\wp(z) - \frac{1}{z^2} = 0$ and hence the constant term is 0. Now $\wp(-z) = \frac{1}{(-z)^2} + \sum_\lambda \frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} = \wp(z)$ as the mapping $\Lambda \to \Lambda$ that sends λ to $-\lambda$ is a bijection. Hence expanding the Laurent series of $\wp(-z)$ at $z=0$ and subtract it from that from $\wp(z)$ we must have all odd power terms vanish. Hence the Laurent series of $\wp(z) = \frac{1}{z^2} + az^2 + bz^4 + ..$ for some $a, b \in \mathbb{C}$. Taking derivative on both side, we obtain $\wp'(z) = \frac{-2}{z^3} + 2az + 4bz^3 + ...$

(f) We consider the principal part of Laurent series of $\frac{1}{4}\wp'(z)^2 - (\wp(z)^3 + A\wp(z) + B)$, which can be written as $\frac{1}{4}(\frac{4}{z^6} + \frac{-8a}{z^2}) - (\frac{1}{z^6} + \frac{3a}{z^2} + \frac{A}{z^2})$. Since $\frac{1}{z^6}$ terms cancel each other, we can obtain A, in terms of a, such that $\frac{1}{z^2}$ terms vanish. Then B can be used to offset whatever constants generated by the previous three terms to ensure the entire expression vanish at $z=0$.

(g)

We see that $l(z) = \frac{1}{4}\wp'(z)^2 - (\wp(z)^3 + A\wp(z) + B)$ is entire as every pole in Λ is no longer a pole for this expression (by (f) and the doubly periodic property). Now using doubly periodic property again and the fact that w_1 and w_2 represents two linearly independent vectors on $\mathbb{C}$, given any $z \in \mathbb{C}$, there exists $a, b \in [0, 1)$, such that $\wp(z) = \wp(aw_1 + bw_2)$. But since the set $\{aw_1 + bw_2 : a, b \in [0, 1)\}$ is compact, the image of l is compact and therefore bounded. Hence by Louville theorem, l is a constant function, which means $l \equiv 0$ as $l(0) = 0$.