Math 240B, Winter 2020
Solution to Problems of HW#2
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1. We prove Part (2) first, then Part (1), and finally Part (3).

(2) Assume $f_n \to f$ in $L^\infty(\mu)$, i.e., $\|f_n - f\|_\infty \to 0$. For each n, let $E_n = \{|f_n - f| \le \|f_n - f\|_\infty\} \in \mathcal{M}$. Then $\mu(E_n^c) = 0$. Let $E = \bigcap_{n=1}^\infty E_n \in \mathcal{M}$. Then $\mu(E^c) = \mu(\bigcup_{n=1}^\infty E_n^c) \le \sum_{n=1}^\infty \mu(E_n^c) = 0$. Moreover, since $E_n \supseteq E$ for each n,

$$\sup_{x \in E} |f_n(x) - f(x)| \le \sup_{x \in E_n} |f_n(x) - f(x)| \le \|f_n - f\|_\infty \to 0.$$

Hence $f_n \to f$ uniformly on E.

Assume $\exists\, E \in \mathcal{M}$ with $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E. We show $\|f_n - f\|_\infty \to 0$. Let $g \in L^\infty(\mu)$. We claim $\|g\|_\infty = \|g\chi_E\|_\infty$.

In fact, for any $a \geq 0$ let us denote

$$A_a = \{x \in X : |g(x)| > a\},$$
$$B_a = \{x \in X : |g(x)\chi_E(x)| > a\}.$$

Then, $B_a = A_a \cap E$. Since $\mu(E^c) = 0$, we have

$$\begin{aligned}\mu(B_a) &= \mu(A_a \cap E) + \mu(A_a \cap E^c) \\ &= \mu((A_a \cap E) \cup (A_a \cap E^c)) \\ &= \mu(A_a \cap (E \cup E^c)) \\ &= \mu(A_a).\end{aligned}$$

Thus, $\|g\|_\infty = \inf\{a \geq 0 : \mu(A_a) = 0\}$
$= \inf\{a \geq 0 : \mu(B_a) = 0\}$
$= \|g\chi_E\|_\infty$.

Since $f_n \to f$ uniformly on E, $\sup_{x \in E} |f_n(x) - f(x)| \to 0$.

Hence $\|f_n - f\|_\infty = \|(f_n - f)\chi_E\|_\infty \leq \sup_{x \in E} |f_n(x) - f(x)| \to 0$.

(1) Let $f \in L^\infty(\mu)$. Clearly $\|f\|_\infty \geq 0$. If $f = 0$ then $\|f\|_\infty = 0$ (by definition of $\|\cdot\|_\infty$ norm). Suppose $f \in L^\infty$ and $\|f\|_\infty = 0$. Then $\mu(\{|f| > 0\}) = \mu(\{|f| > \|f\|_\infty\}) = 0$.
Hence $f = 0$ a.e., i.e., $f = 0$ in $L^\infty(\mu)$.

Let $\alpha \in \mathbb{C}$ and $f \in L^\infty(\mu)$. If $\alpha = 0$ then $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty = 0$.
If $\alpha \neq 0$ then

$$\|\alpha f\|_\infty = \inf\{a \geq 0 : \mu(\{|\alpha f| > a\}) = 0\}$$
$$= \inf\{a \geq 0 : \mu(\{|f| > \tfrac{a}{|\alpha|}\}) = 0\}$$
$$= |\alpha| \inf\{\tfrac{a}{|\alpha|} \geq 0 : \mu(\{|f| > \tfrac{a}{|\alpha|}\}) = 0\}$$
$$= |\alpha| \inf\{b \geq 0 : \mu(\{|f| > b\}) = 0\}$$
$$= |\alpha| \, \|f\|_\infty .$$

Now, let $f, g \in L^\infty(\mu)$. Let $\varepsilon > 0$. Then $\exists a_\varepsilon \geq 0, b_\varepsilon \geq 0$ s.t. $\|f\|_\infty \geq a_\varepsilon - \varepsilon$ and $\|g\|_\infty \geq b_\varepsilon - \varepsilon$, with $\mu(\{|f| > a_\varepsilon\}) = 0$ and $\mu(\{|g| > b_\varepsilon\}) = 0$.
Now, $\{|f+g| > a_\varepsilon + b_\varepsilon\} \subseteq \{|f| > a_\varepsilon\} \cup \{|g| > b_\varepsilon\}$
So, $\mu(\{|f+g| > a_\varepsilon + b_\varepsilon\}) \leq \mu(\{|f| > a_\varepsilon\}) + \mu(\{|g| > b_\varepsilon\}) = 0$
Thus, $\|f+g\|_\infty \leq a_\varepsilon + b_\varepsilon \leq \|f\|_\infty + \|g\|_\infty + 2\varepsilon$.
Consequently $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Hence, $L^\infty(\mu)$ is a normed vector space.

Let now $\{f_n\}$ be a Cauchy sequence in $L^\infty(\mu)$: $\|f_n - f_m\|_\infty \to 0$ as $n,m \to \infty$. Let

$$E_{m,n} = \{ |f_n - f_m| \le \|f_n - f_m\|_\infty \} \in \mathcal{M}.$$

Then $\mu(E_{m,n}^c) = 0$. Set $E = \bigcap_{m,n=1}^{\infty} E_{m,n} \in \mathcal{M}$. Then $\mu(E^c)=0$. Moreover,

$$\sup_{x\in E} |f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty \to 0 \text{ as } n,m\to\infty.$$

Thus, $\{f_n(x)\}$ $(x \in E)$ is a Cauchy sequence of complex numbers. So it converges to some $f(x)$.

Define $f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } x\in E \\ 0 & \text{if } x \in E^c. \end{cases}$

Then f is measurable.

$\forall \varepsilon > 0$, $\exists N$ s.t. $n, m \ge N \Longrightarrow \|f_n - f_m\| \le \varepsilon$.

Hence $\forall x \in E$, $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty \le \varepsilon$ if $n, m \ge N$. Let $n \ge N$, $m \to \infty$. then $|f_n(x) - f(x)| \le \varepsilon$ $\forall n \ge N$. Hence on E $|f| \le \|f_N\| + \varepsilon$. Hence $f \in L^\infty(\mu)$. Since $\sup_{x\in E} |f_n(x) - f(x)| \le \varepsilon$ $\forall n \ge N$. $f_n \to f$ uniformly on E. Since $\mu(E^c)=0$ by Part (2), $f_n \to f$ in $L^\infty(\mu)$. Thus, $L^\infty(\mu)$ is a Banach space.

(3) Let $f \in L^\infty(\mu)$ and $E = \{ |f| \le \|f\|_\infty \} \in \mathcal{M}$. Then $\mu(E^c)=0$. Let $\{f_n\}$ be a sequence of simple functions such that $f_n \to f\chi_E$ uniformly on X. Note that, since $f\chi_E$ is bounded, we have the uniform convergence. Now, $\sup_{x\in E} |f_n(x) - f(x)| \to 0$ i.e., $f_n \to f$ uniformly on E and $\mu(E^c) = 0$. Hence, by Part (2), $f_n \to f$ in $L^\infty(\mu)$. Hence, simple functions are dense in $L^\infty(\mu)$.

2. Let us assume $\|f\|_\infty > 0$ for otherwise the result is clearly true. Let $0 < \alpha < \min(1, \|f\|_\infty)$, and set $A = \{|f| \geq \alpha\}$ and $B = \{|f| < \alpha\}$. We have $\mu(A) > 0$ by the definition of $\|f\|_\infty$. We have also

$$\mu(A) = \int_A d\mu \leq \int_A \left(\frac{|f|}{\alpha}\right)^p d\mu \leq \frac{1}{\alpha^p}\|f\|_p^p < \infty.$$

Now, for any $q \in (p, \infty)$,

$$\int |f|^q d\mu = \int_A |f|^q d\mu + \int_B |f|^q d\mu \qquad (*)$$

$$\leq \|f\|_\infty^q \mu(A) + \int_B |f|^p d\mu$$

$$\leq \|f\|_\infty^q \mu(A) + \|f\|_p^p$$

$$< \infty.$$

Hence $f \in L^q(\mu)$ $(p < q < \infty)$.

Since $|f(x)|^q \to 0$ as $q \to \infty$ for any $x \in B$, and $|f(x)|^q \leq |f(x)|^p$ $(\forall x \in B)$, $f \in L^p(\mu)$, the DCT implies that $\int_B |f|^q d\mu \to 0$. Since $\|f\|_\infty > 0$ and $\mu(A) > 0$, we have for $q >> 1$ that

$$\int_B |f|^q d\mu \leq \|f\|_\infty^q \mu(A).$$

[This statement means that $\exists\, q_0 \in (p, \infty)$ such that if $q \geq q_0$ then this inequality holds true.]

Thus, for $q >> 1$, we have by $(*)$ that

$$\int |f|^q d\mu \leq 2\|f\|_\infty^q \mu(A)$$

i.e., $\quad \|f\|_q \leq [2\mu(A)]^{1/q}\|f\|_\infty$

and $\quad \limsup_{q \to \infty} \|f\|_q \leq \|f\|_\infty.$

Let $\varepsilon \in (0, \|f\|_\infty)$ and set $E_\varepsilon = \{|f| > \|f\|_\infty - \varepsilon\}$. As above, we have $0 < \mu(E_\varepsilon) < \infty$. Thus, if $q \in (p, \infty)$ then

$$\|f\|_q^q \geq \int_{E_\varepsilon} |f|^q \, d\mu \geq (\|f\|_\infty - \varepsilon)^q \mu(E_\varepsilon)$$

i.e., $\|f\|_q \geq (\|f\|_\infty - \varepsilon)\, [\mu(E_\varepsilon)]^{1/q}$.

Hence, $\liminf\limits_{q\to\infty} \|f\|_q \geq \|f\|_\infty - \varepsilon$.

Consequently, $\liminf\limits_{q\to\infty} \|f\|_q \geq \|f\|_\infty$.

Combining the two lim sup, liminf inequalities, we obtain $\lim\limits_{q\to\infty} \|f\|_q = \|f\|_\infty$.

3. Let $1 \leq p < \infty$. Simple functions of the form $\phi = \sum_{j=1}^{m} a_j \chi_{E_j}$, with $a_j \in \mathbb{C}$, $m(E_j) < \infty$, are dense in $L^p(\mathbb{R}^n, m)$ [See Proposition 6.7.] Each $a_j \in \mathbb{C}$ can be approximated by $\alpha_j + i\beta_j$ where $i^2 = -1$, $\alpha_j, \beta_j \in \mathbb{Q}$ (the set of all rational numbers). Moreover, if $m(E) < \infty$, and $\varepsilon > 0$, then there exist disjoint rectangles $R_1, \ldots, R_K$ whose sides are intervals of finite length, with end points being rational numbers, such that $m(E \triangle \bigcup_{k=1}^{K} R_k) < \varepsilon$. Let S denote the collection of simple functions of the form $\sum_{j=1}^{N} (\alpha_j + i\beta_j) \chi_{R_j}$, where $\alpha_j, \beta_j \in \mathbb{Q}$, R_j is a rectangle whose sides are finite intervals with endpoints rational numbers. Then S is countable and S is dense in $L^p(\mathbb{R}^n, m)$. Hence $L^p(\mathbb{R}^n, m)$ is separable.

Let $p=\infty$. For each $t\in(0,1)$, we define $E_t=(0,t)^n$.
Then $\{\chi_{E_t}\}_{t\in(0,1)}$ is an uncountable family of functions in $L^\infty(\mathbb{R}^n, m)$, as

$$\|\chi_{E_t}-\chi_{E_s}\|_\infty = 1 \quad \text{if } s,t\in(0,1),\ s\neq t.$$

If $\mathcal{F}\subseteq L(\mathbb{R}^n, m)$ is dense in $L^\infty(\mathbb{R}^n, m)$, then for each $t\in(0,1)$, there exists $f_t\in\mathcal{F}$ such that

$$\|\chi_{E_t}-f_t\|_\infty < \frac{1}{4}.$$

Thus $\|f_t-f_s\|_\infty \geq \|\chi_{E_t}-\chi_{E_s}\|_\infty - \|\chi_{E_t}-f_t\|_\infty - \|\chi_{E_s}-f_s\|_\infty$

$$> 1-\frac{1}{4}-\frac{1}{4}=\frac{1}{2}.$$

Hence $\{f_t\}_{t\in(0,1)}\subseteq\mathcal{F}$ is an uncountable subfamily of functions in $\mathcal{F}$. Thus $\mathcal{F}$ is uncountable. Hence, there exists no countable dense subset of $L^\infty(\mathbb{R}^n, m)$. So $L^\infty(\mathbb{R}^n, m)$ is not separable.

4. (1) Let $q\in(0,p)$. Let $r=p/q$ and $s=\frac{r}{r-1}$.
$r,s\in(1,\infty)$. $\frac{1}{r}+\frac{1}{s}=1$. We have by Hölder's inequality that

$$\|f\|_q^q = \int |f|^q d\mu = \int |f|^q\cdot 1\, d\mu$$
$$\leq \left(\int |f|^{qr} d\mu\right)^{1/r}\left(\int 1^s d\mu\right)^{1/s}$$
$$= \left(\int |f|^p d\mu\right)^{q/p} \qquad \left[\int 1\, d\mu = \mu(X)=1\right]$$

Hence $\|f\|_q \leq \|f\|_p<\infty$. i.e., $f\in L^q(\mu)$.

In Part (2) – Part (4), we assume $f\neq 0$.

(2) Note that $\ln|f| \geq 0 \iff |f| \geq 1$. Hence,

$$\int (\ln|f|)^+ d\mu = \int_{\{|f|\geq 1\}} \ln|f|\,d\mu = \frac{1}{p}\int_{\{|f|\leq 1\}} \ln|f|^p d\mu$$

$$\leq \frac{1}{p}\int_{\{|f|\leq 1\}} |f|^p d\mu \leq \frac{1}{p}\int |f|^p d\mu < \infty,$$

where we used the fact that $\ln s \leq s$ if $s \geq 1$.

If $\int (\ln|f|)^- d\mu = \int_{\{|f|<1\}} \ln|f|\,d\mu = -\infty$, then

$$\int \ln|f|\,d\mu = -\infty$$

Hence $\log\|f\|_p \geq \int \ln|f|\,d\mu$.

Suppose $\int \ln|f|\,d\mu$ is integrable, which is equivalent to $\int \ln|f|\,d\mu \in (-\infty, \infty)$.

Notation: $\log = \ln$

Let $\phi(t) = -\log t$ $(t > 0)$. Then $\phi''(t) = \frac{1}{t^2} > 0$ and hence ϕ is convex on $(0,\infty)$. By Jensen's inequality and the fact that $\mu(X) = 1$, we get

$$\phi\left(\int |f|^p d\mu\right) \leq \int \phi \circ |f|^p d\mu.$$

Thus,
$$\log\|f\|_p = -\frac{1}{p}\log\left(\int |f|^p d\mu\right)$$
$$= -\frac{1}{p}\phi\left(\int |f|^p d\mu\right)$$
$$\geq -\frac{1}{p}\int \phi \circ |f|^p d\mu$$
$$= -\frac{1}{p}\int -\log|f|^p d\mu$$
$$= \frac{1}{p}\int \log|f|^p d\mu$$
$$= \int \log|f|\,d\mu.$$

(3) For $q \in (0, p)$, we have

$$\frac{1}{q}\left(\int |f|^q d\mu - 1\right) \geq \log \|f\|_q$$

$$\Longleftrightarrow \int |f|^q d\mu - 1 \geq \log\left(\int |f|^q d\mu\right).$$

Let $s = \int |f|^q d\mu \geq 0$. Then, we need to show $s - 1 \geq \log s$. But this is true for any $s > 0$.
In fact, let $g(s) = s - \ln s - 1$. Then $g'(s) = 1 - \frac{1}{s}$. $g''(s) = \frac{1}{s^2} > 0$. So, g attains its minimum at $s = 1$ on $(0, \infty)$. Hence, $g(s) \geq g(1) = 0$, i.e., $s - 1 \geq \log s$ on $(0, \infty)$.

We now prove $\lim_{q \to 0^+} \frac{1}{q}\left(\int |f|^q d\mu - 1\right) = \int \log |f| d\mu$.
Since $\mu(X) = 1$, we need only to show that

$$\lim_{q \to 0^+} \int \frac{|f|^q - 1}{q} d\mu = \int \log |f| d\mu.$$

We should consider the integrals over $E = \{|f| > 0\}$ and $F = \{|f| = 0\}$, respectively.

Fix $s > 0$ and let $\alpha(q) = \frac{s^q - 1}{q}$ $(q \in (0, p])$.

Then $\alpha'(q) = \frac{q s^q \ln s - s^q + 1}{q^2} = \frac{s^q \ln s^q - s^q + 1}{q^2}$.

Let $t = s^q \in (0, s^p]$ and $\beta(t) = t \ln t - t + 1$. Then $\beta'(t) = \ln t$. $\beta'(t) = 0 \Leftrightarrow t = 1$, $\beta''(t) = \frac{1}{t} > 0$.
Hence $\beta(t) \geq \beta(1) = 0$ $\forall t \in (0, s^p]$. Thus, $\alpha'(q) \geq 0$ $\forall q \in (0, p]$. and $\alpha(q) \leq \alpha(p) = \frac{s^p - 1}{p}$
$\alpha(q) \geq \lim_{q' \to 0} \alpha(q') = \ln s$. i.e., $\ln s \leq \alpha(q) \leq \frac{s^p - 1}{p}$ for $q \in (0, p]$.

If $|f| \geq 1$ then $0 \leq \frac{|f|^q - 1}{q} \leq \frac{|f|^p - 1}{p}$ $\forall q \in (0, p)$

then, since $f \in L^p(\mu)$, the Dominated Convergence Theorem implies that

$$\lim_{q \to 0^+} \int_{\{|f| \geq 1\}} \frac{|f|^q - 1}{q} d\mu = \int_{\{|f| \geq 1\}} \lim_{q \to 0^+} \frac{|f|^q - 1}{q} d\mu = \int_{\{|f| \geq 1\}} \ln|f| d\mu .$$

[Use: $\frac{d}{dq} s^q = s^q \ln s$ if $s > 0$ and

$$\lim_{q \to 0^+} \frac{s^q - 1}{q} = \frac{d}{dq} s^q \Big|_{q=0} . \text{]}$$

If $0 < |f| < 1$ then $0 \leq \frac{1 - |f|^q}{q}$ and $\frac{1 - |f|^q}{q}$ increases as q decreases. The Monotone Convergence Theorem implies that

$$\lim_{q \to 0^+} \int_{\{0 < |f| < 1\}} \frac{|f|^q - 1}{q} d\mu = - \lim_{q \to 0^+} \int_{\{0 < |f| < 1\}} \frac{1 - |f|^q}{q} d\mu$$

$$= - \int_{\{0 < |f| < 1\}} \lim_{q \to 0^+} \frac{1 - |f|^q}{q} d\mu = - \int_{\{0 < |f| < 1\}} (-\ln|f|) d\mu$$

$$= \int_{\{0 < |f| < 1\}} \ln|f| d\mu \leq 0. \text{ [This can be } -\infty \text{]}$$

If $\mu(\{|f| = 0\}) > 0$ then

$$\lim_{q \to 0^+} \int_{\{|f| = 0\}} \frac{|f|^q - 1}{q} d\mu = -\infty$$

and $\int_{\{|f| = 0\}} \ln|f| d\mu = -\infty$.

Since $\int_{\{|f| \geq 1\}} \ln|f| d\mu$ is finite (cf. Part (2)) then adding the three parts we obtain

$$\lim_{q \to 0^+} \int \frac{|f|^q - 1}{q} d\mu = \int \ln|f| d\mu .$$

(4) By Part (2) and Part (3),

$$\int \ln|f|\,d\mu \le \ln\|f\|_q \le \frac{1}{q}\left(\int |f|^q d\mu - 1\right). \quad (0<q\le p)$$

By Part (3), and the Squeez Theorem,

$$\lim_{q\to 0^+} \ln\|f\|_q = \int \ln|f|\,d\mu.$$

Hence, $\displaystyle\lim_{q\to 0^+} \|f\|_q = \lim_{q\to 0^+} e^{\ln\|f\|_q} = e^{\int \ln|f|\,d\mu}.$

5. (1) Let $\varepsilon>0$. Since $f_n \longrightarrow f$ in $L^p(\mu)$,

$$\mu(\{|f_n-f|\ge\varepsilon\}) = \int_{\{|f_n-f|\ge\varepsilon\}} 1\,d\mu$$

$$\le \int_{\{|f_n-f|\ge\varepsilon\}} \frac{|f_n-f|^p}{\varepsilon^p}\,d\mu \le \frac{1}{\varepsilon^p}\int |f_n-f|^p d\mu$$

$$= \frac{1}{\varepsilon^p}\|f_n-f\|_p^p \to 0 \text{ as } n\to\infty.$$

Hence $f_n\to f$ in measure. Consequently, $\{f_n\}_{n=1}^\infty$ has a subsequence converging to f μ-a.e.

(2) If $\|f_n-f\|_p \not\to 0$ then $\exists\, \varepsilon_0>0$ and a subseq. $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that $\|f_{n_k}-f\|_p \ge \varepsilon_0$ for all $k\ge 1$. Since $f_n\to f$ in measure, $f_{n_k}\to f$ in measure. Thus, $\{f_{n_k}\}_{k=1}^\infty$ has a farther subsequence $\{f_{n_{k_j}}\}_{j=1}^\infty$ such that $f_{n_{k_j}}\to f$ μ-a.e. Since all $|f_n|\le g$ on X and $g\in L^p(\mu)$ the Dominated Convergence Theorem implies that $\|f_{n_{k_j}}-f\|_p\to 0$. But $\|f_{n_{k_j}}-f\|_p \ge \varepsilon_0$ $(j=1,2,\cdots)$. That is a contradiction. Hence $\|f_n-f\|_p\to 0$.

6. If $\|f_n - f\|_p \to 0$ then $\big| \|f_n\|_p - \|f\|_p \big| \le \|f_n - f\|_p \to 0$.
Assume $f_n \to f$ μ-a.e. and $\|f_n\|_p \to \|f\|_p$.
Let $F_n = |f_n - f|^p$, $F = 0$, $G_n = |f_n|^p + |f|^p$, $G = 2|f|^p$.
Then all $F_n, G_n, F, G \in L^1(\mu)$. $F_n \to F$ a.e. $G_n \to G$ a.e. $0 \le F_n \le G_n$, and $\int G_n d\mu \to \int G d\mu$.
By the generalized Dominant Convergence Theorem (Cf. Exercise 20 on page 59 of the textbook, or Problem 2 of Homework Assignment 5, last quarter), we then have $\int F_n d\mu \to \int F d\mu$, i.e., $\int |f_n - f|^p d\mu = \|f_n - f\|_p^p \to 0$ as $n \to \infty$.

7. Clearly $T : L^p(\mu) \to L^p(\mu)$ is linear. Moreover, if $f \in L^p(\mu)$, then

$$\|Tf\|_p = \|fg\|_p \le \|g\|_\infty \|f\|_p.$$

Hence T is also bounded and $\|T\| \le \|g\|_\infty$.
Assume μ is semifinite. Assume also $\|g\|_\infty > 0$ (otherwise $\|T\| \ge \|g\|_\infty = 0$.) $\forall \varepsilon \in (0, \|g\|_\infty)$. Let $A_\varepsilon = \{ |g| \ge \|g\|_\infty - \varepsilon \}$. Then $0 < \mu(A_\varepsilon) \le \infty$. Since μ is semifinite, $\exists B_\varepsilon \subseteq A_\varepsilon$, B_ε is measurable, and $0 < \mu(B_\varepsilon) < \infty$. Define

$$f = \frac{\overline{\operatorname{sgn} g}}{\mu(B_\varepsilon)} \chi_{B_\varepsilon}.$$

Then $\|f\|_p = 1$. and

$$\|Tf\|_p = \|fg\|_p = \left(\int_{B_\varepsilon} \frac{|g|^p}{\mu(B_\varepsilon)} d\mu \right)^{1/p} \ge \|g\|_\infty - \varepsilon.$$

Hence, $\|T\| \ge \|g\|_\infty - \varepsilon$ and $\|T\| \ge \|g\|_\infty$.
Finally, $\|T\| = \|g\|_\infty$.

8. Since $(L^p(\mu))^* = L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, we need to show that for any $g \in L^q(\mu)$,

$$\int f_n g \, d\mu \longrightarrow \int f g \, d\mu.$$

Let $\varepsilon > 0$. Since $g \in L^q(\mu)$, there exists $\delta > 0$ such that $\int_E |g|^q d\mu < \varepsilon^q$ if $E \in \mathcal{M}$ and $\mu(E) < \delta$. Let $E_k = \{\frac{1}{k} < |g| < k\}$ $(k = 1, 2, \cdots)$. Then $E_k \uparrow \{0 < |g| < \infty\}$. Hence, the Monotone Convergence Theorem implies that

$$\int_{E_k} |g|^q d\mu = \int_X |g|^q \chi_{E_k} d\mu \rightarrow \int_X |g|^q d\mu.$$

Consequently, $\exists$ some $k \geq 1$ such that $\int_{X \setminus E_k} |g|^q d\mu < \varepsilon^q$. Clearly $\mu(E_k) < \infty$ as otherwise, $\mu(E_k) = \infty$,

$$\infty > \int_X |g|^q d\mu \geq \int_{E_k} |g|^q d\mu \geq \frac{1}{k} \mu(E_k) = \infty,$$

impossible.

By Egoroff's theorem, $\exists A \in \mathcal{M}$, $A \subseteq E_k$, and $\mu(E_k \setminus A) < \delta$ such that $f_n \longrightarrow f$ uniformly on A.

Now, letting $C = \sup_{n \geq 1} \|f_n\|_p \in [0, \infty)$, we have

$$\left|\int (f_n - f) g \, d\mu\right| \leq \int_{X \setminus E_k} |f_n - f| |g| d\mu + \int_{E_k \setminus A} |f_n - f| |g| d\mu + \int_A |f_n - f| |g| d\mu$$

$$\leq \|f_n - f\|_{L^p(X \setminus E_k)} \left(\int_{X \setminus E_k} |g|^q\right)^{\frac{1}{q}} + \|f_n - f\|_p \left(\int_{E_k \setminus A} |g|^q d\mu\right)^{\frac{1}{q}} + \left(\int_A |f_n - f|^p d\mu\right)^{\frac{1}{p}} \|g\|_q$$

$$\leq 2(C + \|f\|_p)\, \varepsilon + \left(\int_A |f_n - f|^p d\mu\right)^{\frac{1}{p}} \|g\|_q$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$