Math 240A: Real Analysis, Fall 2019

Homework Assignment 5

Due Monday, November 4, 2019

Unless otherwise stated, we assume that $(X, \mathcal{M}, \mu)$ is a measure space.

1. Let $f_n \in L^1(\mu)$ $(n = 1, 2, \dots)$ and assume $f_n \to f$ uniformly on X for some measurable function f. Assume further that $\mu(X) < \infty$. Prove $f \in L^1(\mu)$ and $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$. What if $\mu(X) = \infty$?

2. Prove the generalized Dominated Convergence Theorem: If $f, g, f_n, g_n \in L^1(\mu)$ $(n = 1, 2, \dots)$, $f_n \to f$ a.e. and $g_n \to g$ a.e., $|f_n| \le g_n$ on X $(n = 1, 2, \dots)$, and $\int g_n \, d\mu \to \int g \, d\mu$, then, $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

3. Assume $f, f_n \in L^1(\mu)$ $(n = 1, 2, \dots)$ and $f_n \to f$ a.e. Prove that $\int |f_n - f| \, d\mu \to 0$ if and only if $\int |f_n| \, d\mu \to \int |f| \, d\mu$.

4. Let $f \in L^1(m)$ and $F(x) = \int_{-\infty}^{x} f(t) \, dt$ $(x \in \mathbb{R})$. Prove that F is continuous on $\mathbb{R}$.

5. Calculate the following limits with justification:
   (1) $\lim_{n\to\infty} \int_0^\infty (1 + x/n)^{-n} \sin(x/n) dx$;
   (2) $\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx$.

6. Let $f, f_n \in L^1(\mu)$ $(n = 1, 2, \dots)$. Prove that $f_n \to f$ in measure if and only if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x \in X : |f_n(x) - f(x)| \ge \epsilon\}) < \epsilon$ for all $n \ge N$.

7. Assume $f_n \to f$ in measure. Prove the following:
   (1) If $f_n \ge 0$ on X $(n = 1, 2, \dots)$, then $\int f \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu$;
   (2) If there exists $g \in L^1(\mu)$ such that $|f_n| \le g$ $(n = 1, 2, \dots)$ then $f_n \to f$ in $L^1(\mu)$.

8. Let $E_n \in \mathcal{M}$ be such that $\mu(E_n) < \infty$ $(n = 1, 2, \dots)$, $f \in L^1(\mu)$, and $\chi_{E_n} \to f$ in $L^1(\mu)$. Prove that there exists $E \in \mathcal{M}$ such that $f = \chi_E$ a.e. on X.

9. Suppose $f_n \to f$ and $g_n \to g$, both in measure. Prove that $f_n + g_n \to f + g$ in measure. Prove also that $f_n g_n \to fg$ in measure provided additionally that $\mu(X) < \infty$. What if $\mu(X) = \infty$?

10. Assume μ is σ-finite. Let f_n $(n = 1, 2, \dots)$ be measurable and $f_n \to f$ a.e. Prove that there exist $E_j \in \mathcal{M}$ $(j = 1, 2, \dots)$ such that $\mu((\cup_{j=1}^\infty E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j.