Math 240A, Fall 2019
Solution to Problems of HW #7
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1. (1) Suppose E is ν-null. Let $X = P \cup N$ be a Hahn decomposition for ν, where $P, N \in \mathcal{M}$ are disjoint, P is ν-positive and N ν-negative. Then, $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$ for any $A \in \mathcal{M}$, and $\nu = \nu^+ - \nu^-$ with $\nu^+ \perp \nu^-$ is the Jordan decomposition of ν. Since E is ν-null, $\nu(E \cap P) = 0$, $\nu(E \cap N) = 0$. Thus, $|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N) = 0$.

Suppose now $|\nu|(E) = 0$. Then $\nu^+(E) = 0$, $\nu^-(E) = 0$. Thus, for any $F \in \mathcal{M}$, $F \subseteq E$, $0 \le \nu^+(F) \le \nu^+(E) = 0$, So, $\nu^+(F) = 0$. Similarly, $\nu^-(F) = 0$. Thus, $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. Hence, E is ν-null.

(2) Suppose $\nu \perp \mu$. Then $\exists E, F \in \mathcal{M}$, $X = E \cup F$, $E \cap F = \phi$, E is ν-null and F is μ-null. By (1), $|\nu|(E) = 0$ and $|\mu|(F) = 0$. Hence, $0 \le \nu^+(E) \le |\nu|(E) = 0$ i.e., $\nu^+(E) = 0$. Similarly, $\nu^-(E) = 0$. Since ν^+, ν^- are positive measures E is ν^+-null and ν^--null. Thus, $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Conversely, assume $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Then, there exist $E^+, E^-, F^+, F^- \in \mathcal{M}$, $X = E^+ \cup F^+$, $E^+ \cap F^+ = \phi$, E^+ is ν^+-null, i.e., $\nu^+(E^+) = 0$, F^+ is μ-null, i.e., by (1), $|\mu|(F^+) = 0$. Similarly, $X = E^- \cup F^-$, $E^- \cap F^- = \phi$, $\nu^-(E^-) = 0$ and $|\mu|(F^-) = 0$. Let $F = F^+ \cup F^- \in \mathcal{M}$ and $E = E^+ \cap E^- \in \mathcal{M}$. Since $E^+ \cap F^+ = \phi$ and $E^- \cap F^- = \phi$, $E \cap F = (E^+ \cap E^-) \cap (F^+ \cup F^-)$

$= (E^+ \cap E^- \cap F^-) \cup (E^+ \cap E^- \cap F^-) = \phi$. Since $E^+ = F^{+c}$, $E^- = F^{-c}$,

$$E \cup F = (E^+ \cap E^-) \cup (F^+ \cup F^-) = (F^{+c} \cap F^{-c}) \cup (F^+ \cup F^-)$$
$$= (F^+ \cup F^-)^c \cup (F^+ \cup F^-) = X.$$

Moreover, $|\mu|(F) \le |\mu|(F^+) + |\mu|(F^-) = 0$. So, F is μ-null. $|\nu|(E) = \nu^+(E) + \nu^-(E) \le \nu^+(E^+) + \nu^-(E^-) = 0$. So E is ν-null. Thus, $\mu \perp \nu$.

2. (1) Let $X = P \cup N$ be a Hahn decomposition for ν. P is ν-positive, N ν-negative, $P \cap N = \phi$. Then $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Since $E \cap P \in \mathcal{M}$ and $E \cap P \subseteq E$,
$\nu^+(E) \le \sup\{\nu(F) : F \in \mathcal{M} \text{ and } F \subseteq E\}$. On the other hand, if $F \in \mathcal{M}$, $F \subseteq E$, then

$$\nu(F) = \nu^+(F) - \nu^-(F) \le \nu^+(F) = \nu^+(F \cap P) \le \nu^+(F) \le \nu^+(E)$$

Hence, $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subseteq E\}$.

Similarly, $\nu^-(E) = -\nu(E \cap N) \le -\inf\{\nu(F) : F \in \mathcal{M}, F \subseteq E\}$

If $F \in \mathcal{M}$ and $F \subseteq E$, then

$$\nu(F) = \nu^+(F) - \nu^-(F) \ge -\nu^-(F) \ge -\nu^-(E).$$

Hence $\nu^-(E) \ge -\inf\{\nu(F) : F \in \mathcal{M} \text{ and } F \subseteq E\}$. Finally,

$$\nu^-(E) = \inf\{\nu(F) : F \in \mathcal{M} \text{ and } F \subseteq E\}$$

(2) If $E = \bigcup_{j=1}^n E_j$ with $E_j \in \mathcal{M}$ $(j = 1, \dots, n)$ disjoint, then

$$\sum_{j=1}^n |\nu(E_j)| \le \sum_{j=1}^n |\nu|(E_j) = |\nu|\left(\bigcup_{j=1}^n E_j\right) = |\nu|(E).$$

Hence,
$$|\nu|(E) \ge \sup\left\{\sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \dots, E_n \in \mathcal{M} \text{ disjoint}, E = \bigcup_{j=1}^n E_j\right\}.$$

Now, let $X = P \cup N$ be a Hahn decomposition for ν. $P, N \in \mathcal{M}$, $P \cap N = \phi$. P, N: ν-positive, ν-negative, respectively. Set $E_1 = P \cap E$, $E_2 = N \cap E$. Then

$E = E_1 \cup E_2$ and $|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E\cap P) - \nu(E\cap N)$
$\leq |\nu(E\cap P)| + |\nu(E\cap N)| = |\nu(E_1)| + |\nu(E_2)|$. Hence
$|\nu|(E) \leq \sup\{\sum_{j=1}^n |\nu(E_j)| : n\in\mathbb{N}, E_1,\dots,E_n \in \mathcal{M}$, disjoint,
and $E = \bigcup_{j=1}^n E_j\}$.
The two inequalities imply the equality as desired.

3. (1) Let $\nu = \nu^+ - \nu^-$ with $\nu^+ \perp \nu^-$ be the Jordan decomposition for ν. Then $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$.
If $f \in L^1(\nu)$, then $f \in L^1(\nu^+) \cap L^1(\nu^-)$. Hence
$$\int |f|\, d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^- < \infty, \text{ i.e., } f \in L^1(|\nu|).$$
Conversely, $f \in L^1(|\nu|) \Longrightarrow \int |f| d|\nu| < \infty$. Thus
$\int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^- < \infty$ and $f \in L^1(\nu^+) \cap L^1(\nu^-)$
$= L^1(\nu)$.

(2) By (1), $f \in L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-) \Longleftrightarrow f \in L^1(|\nu|)$
$$\left|\int f d\nu\right| = \left|\int f d\nu^+ - \int f d\nu^-\right| \leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|.$$

(3). Let $E \in \mathcal{M}$. If $f : X \to \mathbb{C}$ (or $\overline{\mathbb{R}}$) is measurable and $|f| \leq 1$ then $\left|\int_E f d\nu\right| \leq \int_E |f| d|\nu| \leq \int_E d|\nu| = |\nu|(E)$.
Let $X = P \cup N$ be a Hahn decomposition with P, N positive, negative sets for ν, disjoint. Let $f = \chi_P - \chi_N$. then $|f| = 1$ and $\int_E f d\nu = \int_E \chi_P d\nu - \int_E \chi_N d\nu$
$= \nu(E\cap P) - \nu(E\cap N) = \nu^+(E) + \nu^-(E) = |\nu|(E)$.
Hence $|\nu|(E) = \sup\{\left|\int_E f d\nu\right| : |f| \leq 1\}$.

4. (1) $\forall E \in \mathcal{M}$: $|\nu(E)| \le \int_E |f| d\mu \le \int_X |f| d\mu < \infty$, since $f \in L^1(\mu)$.
So, $\nu(E) \ne \pm\infty$. Clearly $\nu(\phi) = 0$. Let $E_j \in \mathcal{M}$ $(j = 1, 2, \cdots)$ be disjoint, and let $E = \bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$. Then,

$$\sum_{j=1}^{\infty} \int_X |\chi_{E_j} f| d\mu = \int_X \sum_{j=1}^{\infty} |\chi_{E_j} f| d\mu = \int_X \left(\sum_{j=1}^{\infty} \chi_{E_j}\right) |f| d\mu$$
$$= \int_X \chi_{\bigcup_{j=1}^{\infty} E_j} |f| d\mu = \int_X \chi_E |f| d\mu = \int_E |f| d\mu < \infty.$$

Consequently

$$\nu(E) = \int_E f d\mu = \int_X \chi_E f d\mu = \int_X \chi_{\bigcup_{j=1}^{\infty} E_j} f d\mu$$
$$= \int_X \left(\sum_{j=1}^{\infty} \chi_{E_j}\right) f d\mu = \int_X \sum_{j=1}^{\infty} (\chi_{E_j} f) d\mu$$
$$= \sum_{j=1}^{\infty} \int_X \chi_{E_j} f d\mu = \sum_{j=1}^{\infty} \int_{E_j} f d\mu = \sum_{j=1}^{\infty} \nu(E_j).$$

Hence ν is a signed measure.

(2) $P = \{f \ge 0\}$, $N = \{f < 0\}$. $X = P \cup N$ is a Hahn decomposition for ν, as P, N disjoint, and they are positive, negative for ν, respectively.

$$\nu^+(E) = \nu(E \cap P) = \int_{E \cap P} f d\mu = \int_{E \cap P} |f| d\mu = \int_{E \cap P} |f| d\mu = \int_E f^+ d\mu,$$
$$\nu^-(E) = -\nu(E \cap N) = -\int_{E \cap N} f d\mu = \int_{E \cap N} |f| d\mu = \int_E f^- d\mu.$$

$|\nu|(E) = \nu^+(E) + \nu^-(E)$. So, $\nu^\pm(E) = \int_E f^\pm d\mu$, ~~[illegible]~~ i.e. $d\nu^\pm = f^\pm d\mu$, and $d|\nu| = |f| d\mu$.

5. (1) $\Rightarrow$ (2) Let $E \in \mathcal{M}$. Suppose $\nu \ll \mu$. If $\mu(E) = 0$ then for a Hahn decomposition $X = P \cup N$ (disjoint) for ν (P, N: positive, negative for ν, respectively) we have $\mu(E \cap P) = 0$, $\mu(E \cap N) = 0$. But $\nu \ll \mu$. So, $\nu^+(E) = \nu(E \cap P) = 0$, $\nu^-(E) = \nu(E \cap N) = 0$. Hence $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$, i.e., $|\nu| \ll \mu$.

$(2) \Rightarrow (3)$. If $E \in \mathcal{M}$ and $\mu(E)=0$, then $|\nu|(E)=0$ by (2). Hence $\nu^+(E) \leq \nu^+(E)+\nu^-(E)=|\nu|(E)=0$. Similarly, $\nu^-(E)=0$. So, $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

$(3) \Rightarrow (1)$. Let $E \in \mathcal{M}$ and $\mu(E)=0$. Then, by (3), $\nu^+(E)=\nu^-(E)=0$. Hence $|\nu(E)| = |\nu^+(E)-\nu^-(E)|$ $\leq |\nu^+(E)|+|\nu^-(E)|=0$. i.e., $\nu \ll \mu$.

6. Let $\varepsilon>0$. Since $f_n \to f$ in $L^1(\mu)$, there exists $N \in \mathbb{N}$ such that $\int_X |f_n - f|\,d\mu < \varepsilon/2$ $\forall n > N$. Since all $f, f_1, \dots, f_N \in L^1(\mu)$, there exists $\delta>0$ such that for any $E \in \mathcal{M}$ with $\mu(E)<\delta$,

$$\int_E |f|\,d\mu < \frac{\varepsilon}{2},\quad \int_E |f_j|\,d\mu < \varepsilon \quad (j=1,\dots,N).$$

Therefore, for any $n \in \mathbb{N}$ since $\int_E |f_n|\,d\mu$ $\leq \int_X |f_n-f|\,d\mu + \int_E |f|\,d\mu$, we have

$$\int_E |f_n|\,d\mu \leq \max\Big\{\int_E |f_1|\,d\mu, \dots, \int_E |f_N|\,d\mu, \int_X |f_n-f|\,d\mu + \int_E |f|\,d\mu\Big\} < \varepsilon.$$

Hence $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable.

7. (1) If $E \in \mathcal{M}$ and $\mu(E)=0$, then $E=\phi$. Hence $m(\phi)=0$. i.e., $m \ll \mu$.

Suppose $\exists f \in L^1(\mu)$ such that $dm = f\,d\mu$. Let $x \in X=[0,1]$ and $E=\{x\} \in \mathcal{B}_{[0,1]}$. Then

$$0 = m(E) = \int_E f\,d\mu = \int_{\{x\}} f\,d\mu = f(x)\mu(\{x\}) = f(x).$$

So $f \equiv 0$ on $[0,1]$. But then $1=m(X)=\int_X f\,d\mu = 0$. a contradiction. So, no $f \in L^1(\mu)$ will satisfy $dm = f\,d\mu$.

(2) Suppose $\mu = \lambda + \rho$ for some signed measures λ and ρ on X, with $\lambda \perp m$ and $\rho \ll m$. Since $\lambda \perp m$, there exist $A, B \in \mathcal{B}_{[0,1]}$, disjoint, such that $X = A \cup B$, A is null for λ and $m(B) = 0$. We have $A \neq \phi$, for otherwise, $A = \phi$, $1 = m(X) = m(A) + m(B) = 0$. This is impossible. Now, let $y \in A$. $E = \{y\}$. Then $m(E) = 0$. Since $\rho \ll m$, $\rho(E) = 0$. Thus, $1 = \mu(E) = \lambda(E) + \rho(E) = \lambda(E)$. But A is λ-null. So $\lambda(E) = 0$, a contradiction. So, μ permits no Lebesgue decomposition with respect to m.

8. Since $\lambda = \mu + \nu$ and $\nu \ll \mu$, we have that $\lambda \ll \mu$ and that $d\lambda = d\mu + d\nu$. Since $\mu \ll \lambda$ and $f = \frac{d\nu}{d\lambda}$, we have
$$1 = \frac{d\lambda}{d\lambda} = \frac{d\mu + d\nu}{d\lambda} = \frac{d\mu}{d\lambda} + f$$

Thus $1 - f = \frac{d\mu}{d\lambda}$. We show that $f < 1$ μ-a.e., or equivalently, λ-a.e. If there existed $E \in \mathcal{M}_\xi$ with $\mu(E) > 0$ (equivalently $\lambda(E) > 0$) such that $f \geq 1$ on E. Then, by the fact that $1 - f = \frac{d\mu}{d\lambda}$,
$$\int_E (1-f)\, d\lambda = \int_E \frac{d\mu}{d\lambda}\, d\lambda = \int_E d\mu = \mu(E) > 0.$$

But, $\int_E (1-f)\, d\mu \leq 0$ since $1 - f \leq 0$ on E. This is a contradiction. Thus, $f < 1$ μ-a.e. Hence
$$\frac{1}{1-f} = \frac{d\lambda}{d\mu} \quad \mu\text{-a.e. (or equivalently, } \lambda\text{-a.e.)}$$
But $f = \frac{d\nu}{d\lambda} \geq 0$ since $\nu, \lambda \geq 0$. Thus, $\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu}$
$$= f \Big/ \frac{d\mu}{d\lambda} = \frac{f}{1-f}.$$

9. Note that $\nu = \mu|_{\mathcal{N}}$ is a finite measure on $(X, \mathcal{N})$. Define $\lambda : \mathcal{N} \to \mathbb{C}$ by $\lambda(E) = \int_E f\,d\mu = \int_E f\,d\nu$ for any $E \in \mathcal{N}$ where $f \in L^1(\nu)$ since $f \in L^1(\mu)$ and $\nu = \mu|_{\mathcal{N}}$. If $E \in \mathcal{N}$ and $\nu(E) = \mu(E) = 0$, then $\lambda(E) = 0$. Thus $\lambda \ll \nu$. By the Lebesgue-Radon-Nikodym Theorem, $\exists\, g \in L^1(\nu)$ such that $d\lambda = g\,d\nu$, i.e., $\lambda(E) = \int_E g\,d\nu$, hence $\int_E f\,d\mu = \int_E g\,d\nu$ $\forall E \in \mathcal{N}$. In particular g is $\mathcal{N}$-measurable.

Suppose $h \in L^1(\nu)$ and $\int_E g\,d\nu = \int_E h\,d\nu = \int_E f\,d\mu$ $\forall E \in \mathcal{N}$. Then, setting $u = g - h \in L^1(\nu)$, we have $\int_E u\,d\nu = 0$ $\forall E \in \mathcal{N}$. Hence $u = 0$, i.e., $g = h$, ν-a.e.

10. By Proposition 3.13, $\exists\, f \in L^1(|\nu|)$ such that $|f| = 1$ $|\nu|$-a.e. and $d\nu = f\,d|\nu|$. Since $\nu(X) = |\nu|(X)$, we get $|\nu|(X) = \nu(X) = \int_X f\,d|\nu| = \int_X \mathrm{Re} f\,d|\nu| + i\int_X \mathrm{Im} f\,d|\nu|$.

Since $|\nu|(X)$ is a real number, $\int_X \mathrm{Im} f\,d|\nu| = 0$, and
$$\int_X \mathrm{Re} f\,d|\nu| = |\nu|(X) = \int_X d|\nu|.$$
Consequently, $\int_X (1 - \mathrm{Re} f)\,d|\nu| = 0$. But $|f| = 1$ $|\nu|$-a.e. So, $\mathrm{Re} f \le 1$ $|\nu|$-a.e. and $1 - \mathrm{Re} f \ge 0$ $|\nu|$-a.e. Thus, by $\int_X (1 - \mathrm{Re} f)\,d|\nu| = 0$, we have $1 - \mathrm{Re} f = 0$ $|\nu|$-a.e. Since $|f| = 1$ $|\nu|$-a.e., $|f| = \sqrt{(\mathrm{Re} f)^2 + (\mathrm{Im} f)^2}$, we have $\mathrm{Im} f = 0$ $|\nu|$-a.e. and $f = \mathrm{Re} f = 1$ $|\nu|$-a.e. Finally, $d\nu = f\,d|\nu| = d|\nu|$ and $|\nu| = \nu$.