Math 240B: Real Analysis, Winter 2020

Final Exam

Name ______________________ **ID number** ______________________

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Total**
Score									

INSTRUCTIONS
(Please Read Carefully!)

- This is an open-book and open-note exam. You can check out any references (books, notes, etc.). But you are not allowed to discuss any part of the exam with any other people; and you are not allowed to copy solutions that have possibly already existed in any form (e.g., in a book, online, etc.) except those of your own.
- There are 8 problems of total 200 points. To get credit, you must show your work. Partial credit will be given to partial answers.
- You can cite any results (theorems, lemmas, propositions, etc.) in our own textbook that have been covered in the class except that you are instructed to prove some of the results. Results stated in exercise problems in the textbook or in the assigned homework may be cited if you provide a proof.
- Please turn in your exam, **including this cover page with your name and ID number**, before or on **11:00 am, Friday, March 20, 2020**. You can
  - o email the instructor (bli@math.ucsd.edu, bli@ucsd.edu) the PDF of your typed out or scanned solution (including this cover page); (Cell phone images will not be accepted as those are often hard to read and also are of large data.) or
  - o staple your solution sheets and this cover page together (in order) and slip them in the instructor's office. In this case, please email the instructor immediately to notify the submission of your exam.
- **Late exams will not be accepted.**
- Please email the instructor if you have any questions.

1. (25 points) Let H be a real Hilbert space. Let $x_k \in H$ $(k = 1, 2, \dots)$ and $x \in H$. Prove that $\|x_k - x\| \to 0$ if and only if $x_k \to x$ weakly in H and $\|x_k\| \to \|x\|$.

2. (25 points) Let X be a vector space on $\mathbb{C}$ and $\{p_n\}_{n=1}^{\infty}$ a countable family of semi-norms on X. Assume that $p_n(x) = 0$ for all $n \in \mathbb{N}$ imply that $x = 0$ in X.
   (1) Define $\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} p_n(x - y)/[1 + p_n(x - y)]$ for any $x, y \in X$. Prove that $\rho : X \times X \to \mathbb{R}$ is a translation-invariant metric.
   (2) Prove that the locally convex topology on X defined by the family of semi-norms $\{p_n\}_{n=1}^{\infty}$ is the same as the topology defined by the metric ρ in Part (1).

3. (25 points) Let X be a real normed vector space. Let $x_1, \dots, x_n$ be linearly independent vectors in X. Assume for each j $(1 \le j \le n)$ that $\|x_j\| = 1$ and that $\inf\{\|x_j - y_j\| : y_j \in E_j\} \ge 1$, where $E_j = \operatorname{Span}\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$. Prove that there exist $f_1, \dots, f_n$ in X^* such that $\|f_j\| = 1$ $(j = 1, \dots, n)$ and $f_j(x_k) = \delta_{jk}$ $(j, k = 1, \dots, n)$, where $\delta_{jk} = 1$ if $j = k$ and 0 otherwise.

4. (25 points) Let $0 = x_0^{(k)} < x_1^{(k)} < \cdots x_{k-1}^{(k)} < x_k^{(k)} = 1$ $(k = 1, 2, \dots)$ and $0 \ne A_j^{(k)} \in \mathbb{R}$ $(j = 0, \dots, k;\ k = 1, 2, \dots)$. Denote by $C([0, 1])$ the Banach space over $\mathbb{R}$ of all real-valued continuous functions on $[0, 1]$ with the maximum norm. Define

$$I[f] = \int_0^1 f(x)\, dx \quad \text{and} \quad I_k[f] = \sum_{j=0}^{k} A_j^{(k)} f\left(x_j^{(k)}\right) \quad (k = 1, 2, \dots) \qquad \forall f \in C([0, 1]).$$

   (1) Assume $\sup_{k \ge 1} \sum_{j=0}^{k} \left|A_j^{(k)}\right| < \infty$ and $\lim_{k \to \infty} I_k[p] = I[p]$ for any polynomial p. Prove that $\lim_{k \to \infty} I_k[f] = I[f]$ for $f \in C([0, 1])$. (You are allowed to use the classical Weierstrass Theorem stating that the set of polynomials is dense in $C([0, 1])$.)
   (2) For any $k \ge 1$, I_k is a linear functional on $C([0, 1])$. Prove $\|I_k\| = \sum_{j=0}^{k} \left|A_j^{(k)}\right|$.
   (3) Assume $\lim_{k \to \infty} I_k[f] = I[f]$ for any $f \in C([0, 1])$. Prove $\sup_{k \ge 1} \sum_{j=0}^{k} \left|A_j^{(k)}\right| < \infty$.

5. (25 points) Let $C([0, 1])$ be the Banach space over $\mathbb{R}$ of all real-valued continuous functions on $[0, 1]$ with the maximum norm. For each $n \in \mathbb{N}$, let E_n be the set of all $f \in C([0, 1])$ for which there exists $x_0 \in [0, 1]$ (depending on f) such that $|f(x) - f(x_0)| \le n|x - x_0|$ for all $x \in [0, 1]$.
   (1) Prove that each E_n is nowhere dense in $C([0, 1])$. (See a hint of Exercise 42 on page 165 of the textbook.)
   (2) Denote by $\mathcal{N}$ the set of nowhere differentiable functions in $C([0, 1])$. Prove that the complement of $\mathcal{N}$ in $C([0, 1])$ is of the first category in $C([0, 1])$.

6. (25 points) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $1 < p < \infty$. Suppose $f_k \in L^p(\mu)$ $(k = 1, 2, \dots)$ are such that $\sup_{k \ge 1} \|f_k\|_{L^p(\mu)} < \infty$ and $f_k \to f$ in $L^1(\mu)$ for some $f \in L^1(\mu)$. Prove that $f \in L^p(\mu)$ and $f_k \to f$ in $L^q(\mu)$ for any $q \in (1, p)$.

7. (25 points) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $1 < p < \infty$ and $q = p/(p - 1)$. Denote $L^p(\mu)$ the real Banach space of L^p-integrable (real-valued) functions.

Let $F \in [L^p(\mu)]^*$. Prove there exists $g \in L^q(\mu)$ such that

$$F(f) = \int_X fg\, d\mu \qquad \forall f \in L^p(\mu),$$

as outlined in the following steps:

(1) Define $\nu : \mathcal{M} \to \mathbb{R}$ by $\nu(E) = F(\chi_E)$ for any $E \in \mathcal{M}$. Show that ν is a signed measure on $(X, \mathcal{M})$ with $\nu \ll \mu$ and that there exists $g \in L^1(\mu)$ such that $\nu(E) = \displaystyle\int_E g\, d\mu$ for any $E \in \mathcal{M}$. Let Σ denotes the set of all simple functions on $(X, \mathcal{M})$. Show that $F(f) = \displaystyle\int_X fg\, d\mu$ for any $f \in \Sigma$.

(2) (This is the key and most involved step; cf. the proof of Theorem 6.14. Note here the set up is much simpler.) Denote

$$M_q(g) = \sup \left\{ \left| \int_X fg\, d\mu \right| : f \in \Sigma \text{ and } \|f\|_p = 1 \right\}.$$

Show that $M_q(g) \le \|F\|$. Let $f : X \to \mathbb{R}$ be $\mathcal{M}$-measurable, bounded, and $\|f\|_p = 1$. By using the approximations by simple functions, show that $\left| \displaystyle\int_X fg\, d\mu \right| \le M_q(g)$. By using the approximations by simple functions, show that $\|g\|_q \le M_q(g) < \infty$. This implies that $g \in L^q(\mu)$.

(3) Finally show by citing some existing result that

$$F(f) = \int_X fg\, d\mu \qquad \forall f \in L^p(\mu).$$

8. (25 points) Let X be a locally compact Hausdorff space, K a nonempty compact subset of X, and $\{U_j\}_{j=1}^n$ an open cover of K. Here is an outline of the proof that there exist $h_j \in C_c(X, [0,1])$ $(j = 1, \ldots, n)$ such that $\operatorname{supp}(h_j) \subseteq U_j$ for each j and $\sum_{j=1}^n h_j = 1$ on K. Please give reasons for each of the steps.

(1) For any $x \in K$, there exists a compact neighborhood N_x of x such that $N_x \subseteq U_j$ for some j. Why?
(2) There are finitely many x_k $(k = 1, \ldots, m)$ such that $K \subseteq \cup_{i=1}^m N_{x_i}$. Why?
(3) For each j $(1 \le j \le n)$, let F_j be the union of those N_{x_i}'s that are subsets of U_j. Then F_j is a compact subset of U_j. Why?
(4) By Urysohn's lemma, for each j, there exists $g_j \in C_c(X, [0,1])$ such that $g_j = 1$ on F_j and $\operatorname{supp}(g_j) \subseteq U_j$. Show that $\sum_{j=1}^n g_n \ge 1$ on K.
(5) Let $G = \{x \in X : \sum_{j=1}^n g_j(x) > 0\}$. Why G is open in X? Why $K \subseteq G$?
(6) By Urysohn's lemma again, there exists $f \in C_c(X, [0,1])$ such that $f = 1$ on K and $\operatorname{supp}(f) \subseteq G$. Let $g_{n+1} = 1 - f$. Show that $\sum_{j=1}^{n+1} g_j > 0$ on X.
(7) Define $h_j = g_j / (\sum_{k=1}^{n+1} g_k)$ $(j = 1, \ldots, n)$. Show that all h_j satisfy the desired properties.