Math 240B: Real Analysis, Winter 2020

Homework Assignment 6

Due Wednesday, February 26, 2020

1. Let A and B be two subsets of a topological space X. Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2. Let X be a topological space, U an open subset of X, and A a dense subset of X. Prove that $\overline{U} = \overline{U \cap A}$.

3. Prove that every separable metric space is second countable.

4. Prove that any metric space (X, ρ) is normal, i.e., for any disjoint closed sets A and B of X, there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

5. Let X and Y be two topological spaces. Let $f : X \to Y$ be given. Prove that the following are equivalent:

   (1) $f : X \to Y$ is continuous;
   (2) $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$;
   (3) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subseteq Y$.

6. Let X be a topological space and $A \subseteq X$ a closed subset. Assume that $g \in C(A)$ satisfies $g = 0$ on ∂A. Prove that the extension of g to X defined by $g(x) = 0$ for $x \in A^c$ is continuous.

7. Let X be a topological space and Y a Hausdorff space. Let f and g be continuous maps from X to Y. Prove the following:

   (1) The set $\{x \in X : f(x) = g(x)\}$ is closed subset of X;
   (2) If $f = g$ on a dense subset of X, then $f = g$ on all of X.

8. Prove the following:

   (1) If X_n $(n = 1, 2, \dots)$ are first countable topological spaces, then the product space $\prod_{n=1}^{\infty} X_n$ is also first countable;
   (2) If X_n $(n = 1, 2, \dots)$ are second countable topological spaces, then the product space $\prod_{n=1}^{\infty} X_n$ is also second countable.

9. Let X be a topological space, (Y, ρ) a complete metric space, and $\{f_n\}_{n=1}^{\infty}$ a sequence of maps from X to Y. Assume that $\sup_{x \in X} \rho(f_n(x), f_m(x)) \to 0$ as $m, n \to \infty$. Prove that there is a unique map $f : X \to Y$ such that $\sup_{x \in X} \rho(f_n(x), f(x)) \to 0$ as $n \to \infty$. Moreover, if each f_n is continuous, so is f.