Math 240C: Real Analysis, Spring 2020

Homework Assignment 6

Due 12:00 noon, Monday, May 18, 2020

1. Let $f \in C^1(\mathbb{R})$ be such that $f' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\lim_{|x|\to\infty} x[f(x)]^2 = 0$. Prove the following:
   (1)
   $$\left[\int |f(x)|^2 dx\right]^2 \le 4 \int |xf(x)|^2 dx \int |f'(x)|^2 dx;$$
   (2) **(Heisenberg's Inequality)** For any $b,\ \beta \in \mathbb{R}$,
   $$\int (x-b)^2 |f(x)|^2 dx \int (\xi - \beta)^2 |\hat{f}(\xi)|^2 d\xi \ge \frac{\|f\|_2^4}{16\pi^2}.$$
   (See Exercise 18 on page 255.)

2. Let $f \in L^1(\mathbb{R}^2)$ be radial, i.e., there exists $g : [0, \infty) \to \mathbb{R}$ such that $f(x) = g(|x|)$ for all $x \in \mathbb{R}^2$. Prove that $\hat{f}$ is also radial. (Note that this result is true for $\mathbb{R}^n$ for a general $n \ge 1$. See Exercise 22 on page 256. Here, for $n = 2$, you can use the polar coordinates and change of variables.)

3. Let $0 < r < 1$. Consider the Poisson kernel on $\mathbb{T}$: $P_r(x) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k x}$.
   (1) Prove that
   $$P_r(x) = \frac{1 - r^2}{1 + r^2 - 2r \cos 2\pi x}.$$
   (2) Let $f \in L^1(\mathbb{T})$ and define $A_r f(x) = \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}(k) e^{2\pi i k x}$. Prove that $A_r f = f * P_r$.

4. Let $a_k \in \mathbb{C}$ $(k = 0, 1, \dots)$, $S_n = \sum_{k=0}^{n} a_k$ $(n = 0, 1, \dots)$, and $\sigma_m = (m+1)^{-1} \sum_{n=0}^{m} S_n$ $(m = 0, 1, \dots)$.
   (1) Show that $\sigma_m = (m+1)^{-1} \sum_{k=0}^{m} (m + 1 - k) a_k$.
   (2) Assume $\lim_{n\to\infty} S_n$ exists. Show $\lim_{m\to\infty} \sigma_m$ exists, and the two limits are equal.
   (3) Show that the series $\sum_{k=0}^{\infty} (-1)^k$ diverges but is Abel and Cesàro summable to $1/2$.

5. Let $\sigma_m f$ denote the Cesàro means of the Fourier series of f given by (8.39) on page 261.
   (1) Denote by $F_m = (m+1)^{-1} \sum_{k=0}^{m} D_k$, where D_k is the kth Dirichlet kernel. Prove that $\sigma_m f = f * F_m$. (F_m is called the mth Fejér kernel.)
   (2) Prove that
   $$F_m(x) = \frac{\sin^2 (m+1)\pi x}{(m+1) \sin^2 \pi x}.$$

6. (1) Let D_n denote the nth Dirichlet kernel. Show that $\|D_n\|_1 > (4/\pi^2) \sum_{j=1}^{n} 1/j$.
   (2) Denote by $S_m f$ the mth symmetric partial sum of the Fourier series of $f \in L^1(\mathbb{T})$. Prove that the set of all $f \in C(\mathbb{T})$ such that the sequence $\{S_m f(0)\}$ converges is meager in $C(\mathbb{T})$.

   (See Exercises 34 and 35 on page 269.)