Problem 1

Suppose two matrices are similar, then they have the same rational form. Therefore, the characteristic polynomial and minimal polynomial (which are the product of invariant factor and the largest invariant factor, respectively) are obviously the same Now suppose the minimal polynomial and the characteristic polynomial are the same. In 2×2 case, there are only two cases. 1. char polynomial equals to min polynomial, which implies that there are only 1 invariant factor and hence the rational form is the same. This means the two matrices are similar. 2. char polynomial does not equal to min polynomial, this means the only other invariant factor is char polynomial divided by min polynomial, which again implies same rational form and therefore similarity. In 3×3 case, there are 3 cases. 1. char polynomial equals to min polynomial, which by the previous logic, implies similarity. 2. Min polynomial is of degree 2, then the only other invariant factor is the char polynomial divided by min polynomial, which again implies similarity as the previous case 2. 3. min polynomial is of degree 1, then we have 3 identical invariant factor of degree 1. This implies all invariant factor must be equal to the min polynomial and therefore we again obtain the same rational form, which implies similarity.

For $M_4(F)$, let A be the matrix with 2 jordan blocks of size 2 with diagonal entries λ, let B be the matrix with 3 jordan blocks of size $2, 1, 1$ respectively with diagonal entries λ . Then the min polynomial of A is $(x - \lambda)^2$, which is the same as that of B. But A and B are obviously not similar as they have different Jordan form.

Problem 2

We have

$$M - \lambda I = \begin{bmatrix} -\lambda & -1 & -1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{bmatrix}$$

Hence $det(M - \lambda I) = -\lambda^3 + \lambda = -\lambda(\lambda - 1)(\lambda + 1)$. Hence we have 3 eigenvalues $-1, 0, 1$. But this means the Jordan canonical form is simply

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the minimal polynomial, which correspond to the largest invariant factor, can only be $(\lambda - 1)(\lambda + 1)\lambda = \lambda^3 - \lambda$. Hence the rational canonical form is

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 3

All of them have 4 duplicate eigenvalues 1 since they are upper triangular. $rank(A - I) = 2$ $rank(B - I) = 1$ $rank(C - I) = 2$.

Hence there are 3 Jordan blocks of size 1 for B, and 2 Jordan blocks of size 1 for A, C. Now we only consider the following:

$$(A - I)^2 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (C - I)^2 = 0$$

$rank((A - I)^2) = 1$ $rank((C - I)^2) = 0$ Hence A has 1 Jordan block of size 2, but C has 2 Jordan blocks of size 2. Hence A, B, C are not similar to one another.

Problem 4

$A^4 - I = 0$. The corresponding $x^4 - 1 = (x^2 - 1)(x^2 + 1) = 0$. In degree 2 case, either the min polynomial (with degree 2) equals to the char polynomial or the char polynomial is the square of min polynomial (with degree 1), as shown in problem 1

(a) Full factorization is $(x - 1)(x + 1)(x^2 + 1)$. Hence the min polynomial cannot be degree 1. It has to be degree 2 and the available choices are $x^2 - 1$ and $x^2 + 1$. But also note that if a the char polynomial is $x^2 - 1$, then $A^2 = I$, which contradicts its order. Hence the only similar class is the one associated with $x^2 + 1$. (b) Similarly, the full factorization is $(x - 1)(x + 1)(x + i)(x - i)$. Again the min polynomial has to have degree 2. There are 6 choices $(1, -1), (1, i), (1, -i), (-1, i), (-1, -i), (i, -i)$. $(1, -1)$ failed for the same reason as in (a). Hence we have 5 choices.

(c) The full factorization is $(x-1)^4$. Hence, the min polynomial can take either $(x-1)$ or $(x-1)^2 = x^2-1$. These two cases both fail as the char polynomial will always be x^2-1.

Problem 5

(a) $det(A-\lambda I) = det((A-\lambda I)^T) = det(A^T - \lambda I)$ Hence charpoly$(A^T) = I$. Since the minimal polynomial g is the smallest degree monic polynomial that satisfy $g(A) = 0$, $g(A^T) = 0^T = 0$. g must be the minimal polynomial for A^T as well as otherwise there will be g' of smaller degree than g that satisfy $g'(A) = 0$, which is a contradiction.

(b) If $f = g$, then the rational form of A is C_f and hence A is similar to C_f. If A is similar to C_f, then the eigenspace spans F^n by Cayley-Hamilton. This means the largest invariant factor is the only invariant factor and therefore $f = g$.

(c) Both A_h and A_h^T has minimal polynomial h. Therefore, they are similar.

(d) the rational form of A^T is similar to the transpose of the rational form of A (by simply transposing the equation $A = PC_fP^{-1}$). By (c), since a companion block and its transpose produce the same invariant factor, the rational form of A is similar to the rational form of A^T. Therefore, A and A^T are similar.

Problem 6

(a) The square J^2 is still a diagonal matrix, hence all eigenvalues in J^2 must be λ^2.