Summary Sheet

Please detach and do not turn in.

Notation

- (model) $X \sim \mathbb{P}_\theta$ with values in $\mathcal{X}$ (the sample space), where $\theta \in \Omega$ (the parameter space)
- (loss and risk) A choice of loss function $L(\theta, d)$ defines a $R(\theta, \delta) = \mathbb{E}_\theta[L(\theta, \delta(X))]$, where δ is a given estimator

Unbiasedness

- We say that a statistic T is complete if the only function f such that $\mathbb{E}_\theta[f(T(X)] = 0$ for all θ is $f \equiv 0$.

Fact 1. *Assume T is a complete and sufficient statistic. Then for every function $g(\theta)$ that admits an unbiased estimator there is a (unique) UMVU given by the unique unbiased estimator that is a function of T. This is true for squared error loss, and any other convex loss.*

Fact 2 (Cramér-Rao information bound)**.** *Suppose that $\mathbb{P}_\theta$ has density f_θ with respect to some (dominating) measure on $\mathcal{X}$. Under some regularity conditions, for an estimator δ such that $\mathbb{E}_\theta[\delta(X)] = g(\theta)$, we have $\mathrm{Var}_\theta[\delta(X)] \geq [\dot{g}(\theta)]^2/I(\theta)$, where $I(\theta) = \mathbb{E}_\theta[(\dot{f}_\theta(X)/f_\theta(X))^2] = \mathrm{Var}_\theta[\dot{f}_\theta(X)/f_\theta(X)] = -\mathbb{E}_\theta[\partial^2 \log f_\theta(X)]$.*

Location equivariance

- Model: $X = (X_1, \ldots, X_n) \sim f(x_1 - \xi, \ldots, x_n - \xi)$ where f is a *given* density on $\mathbb{R}^n$ and $\xi \in \mathbb{R}$ is unknown.
- The transformations of interest are shifts of the form $x \mapsto x + a\mathbf{1}$ where $a \in \mathbb{R}$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.
- We want to estimate ξ using a loss of the form $L(\xi, d) = \rho(d - \xi)$, and restrict ourselves to *equivariant* estimators.

Fact 3. *Let δ_0 be any equivariant estimator with finite risk. Suppose $v^*(y) = \arg\min_{v \in \mathbb{R}} \mathbb{E}_0[\rho(\delta_0(X) - v) \mid Y = y]$ is well defined, where $Y = (Y_1, \ldots, Y_{n-1})$ with $Y_i = X_i - X_n$. Then $\delta^*(x) = \delta_0(x) - v^*(y)$ is MRE.*

Fact 4. *Under squared error loss, the MRE estimator may be expressed as $\delta^*(x) = \int_{-\infty}^{\infty} v f(x - v\mathbf{1}) \mathrm{d}v / \int_{-\infty}^{\infty} f(x - v\mathbf{1}) \mathrm{d}v$.*

Fact 5. *Under squared error loss, if an UMVU estimator exists and is equivariant, then it is MRE.*