Math 240 B. Winter 2020
Solution to Problems of HW #7
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1. Let X be a Hausdorff topological space. If X is also normal, then by Urysohn's lemma, the following (which is the conclusion of Urysohn's lemma) holds true:

If A, B are disjoint closed subsets of X, then there exists $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Suppose the above conclusion of Urysohn's lemma holds true for a Hausdorff space. Then $U = \{x \in X : f(x) < \frac{1}{2}\}$ and $V = \{x \in X : f(x) > \frac{1}{2}\}$ are disjoint open subsets of X such that $A \subseteq U$ and $B \subseteq V$. Hence, as X is Hausdorff, it is normal.

2. Let $\Sigma = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$. This is the class of open half-lines. Denote by $\mathcal{T}$ the Euclid topology of $\mathbb{R}^1$. Clearly, $\mathcal{T} \supseteq \Sigma$. Denote by $\mathcal{T}(\Sigma)$ the topology generated by Σ. We thus have $\mathcal{T} \supseteq \mathcal{T}(\Sigma)$. We show $\mathcal{T} \subseteq \mathcal{T}(\Sigma)$.

Clearly, $(-\infty,\infty) = (-\infty, 1) \cup (-1, \infty) \in \mathcal{T}(\mathcal{E})$

$\forall a, b \in \mathbb{R}$, $a < b$. $(a,b) = (-\infty, b) \cap (a, \infty) \in \mathcal{T}(\mathcal{E})$.

Hence all open intervals are in $\mathcal{T}(\mathcal{E})$

But any open subset $(\neq \phi)$ in $\mathbb{R}^1$ is a union of finite or countably many open intervals. Hence, any open subset of $\mathbb{R}^1$ is in $\mathcal{T}(\mathcal{E})$. Thus, $\mathcal{T}(\mathcal{E}) \supseteq \mathcal{T}$.

Finally, $\mathcal{T} = \mathcal{T}(\mathcal{E})$.

3. Suppose X is Hausdorff. Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X, $x, y \in X$ with $x \neq y$. Suppose $x_\alpha \to x$ and $x_\alpha \to y$. Since X is Hausdorff, there exist open sets U and V of X, $U \cap V = \phi$, $x \in U$, and $y \in V$. Since $x_\alpha \to x$, there exists $\alpha_1 \in A$ s.t. $x_\alpha \in U$ if $\alpha \gtrsim \alpha_1$. Since $x_\alpha \to y$ there exists $\alpha_2 \in A$ s.t. $x_\alpha \in V$ if $\alpha \gtrsim \alpha_2$. Let $\alpha_3 \gtrsim \alpha_1$ and $\alpha_3 \gtrsim \alpha_2$. Then $x_{\alpha_3} \in U \cap V$, a contradiction. Hence any net in X converges to at most one point.

Suppose X is not Hausdorff. Then there exist $x, y \in X$ with $x \neq y$ such that $U \cap V \neq \phi$ if U, V are open subsets of X and $x \in U$, $y \in V$. Let $A = \{(U, V) : U, V$ are open subsets of X, $x \in U$, and $y \in V\}$.

Note $(X, X) \in A$. Define in A $(U_1, V_1) \leq (U_2, V_2)$ if $U_1 \supseteq U_2$ and $V_1 \supseteq V_2$. For any $(U, V) \in A$, let $x_{(U,V)} \in U \cap V$. Then A is a directed set and $\langle x_{(U,V)} \rangle_{(U,V) \in A}$ is a net in X. We claim that this net converges to x and y. In fact, if $x \in U \subseteq X$ and U is open then $(U, X) \in A$. If $(U_1, V_1) \in A$ and $(U_1, V_1) \geq (U, X)$, i.e., $U_1 \subseteq U$, then $x_{(U_1, V_1)} \in U_1 \cap V_1 \subseteq U \cap X = U$. Hence $\langle x_{(U,V)} \rangle_{(U,V) \in A}$ converges to x. Similarly, the net converges to y. Therefore if any net in X can converge to only one limit, then X is Hausdorff.

4. Let $\mathcal{E} = \{ f^{-1}(G) : f \in \mathcal{F}, G \subseteq \mathbb{C}, G \text{ is open} \}$. Then the weak topology on X generated by $\mathcal{F}$ is the topology on X generated by $\mathcal{E}$. Denote this weak topology by $\mathcal{T}$. Then any member of $\mathcal{T}$ is a union of finite intersections of members of $\mathcal{E}$. Assume $x_\alpha \to x$. Let $f \in \mathcal{F}$. Let $G \subseteq \mathbb{C}$ be any open set such that $f(x) \in G$. Then, $f^{-1}(G) \in \mathcal{E} \subseteq \mathcal{T}$. Thus, $\langle x_\alpha \rangle_{\alpha \in A}$ is eventually in $f^{-1}(G)$. This means $\langle f(x_\alpha) \rangle_{\alpha \in A}$ is eventually in G. Hence $f(x_\alpha) \to f(x)$. Assume now $f(x_\alpha) \to f(x)$ $\forall f \in \mathcal{F}$. Let U be an open set of X (i.e., $U \in \mathcal{T}$) such that $x \in U$. Then, $\exists f_1, \cdots, f_m \in \mathcal{F}$ and open subsets $G_1, \cdots, G_m$ of $\mathbb{C}$, such that $x \in \bigcap_{j=1}^{m} f_j^{-1}(G_j)$

$\subseteq \mathcal{U}$. Hence $f_j(x) \in G_j$ $(1 \le j \le m)$. Since $f_j(x_\alpha) \to f_j(x)$ for each j $(1 \le j \le m)$, there exists $\alpha_j \in A$ such that $\alpha \succsim \alpha_j \Longrightarrow f_j(x_\alpha) \in G_j$. Let $\alpha_0 \succsim \alpha_j$ for all $j = 1, \dots, m$. Then, if $\alpha \succsim \alpha_0$, we have $f_j(x_\alpha) \in G_j$ $(1 \le j \le m)$. Hence $x_\alpha \in \bigcap_{j=1}^{m} f_j^{-1}(G_j) \subseteq \mathcal{U}$ if $\alpha \succsim \alpha_0$. Hence $x_\alpha \to x$.

5. Let X be a compact topological space and $F \subseteq X$ a closed subset. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of F. Since F^c is open, we have $X = F \cup F^c \subseteq (\bigcup_{\alpha \in A} U_\alpha) \cup F^c$. Since X is compact, there exist $\alpha_1, \dots, \alpha_n \in A$ such that $F \cup F^c \subseteq (\bigcup_{j=1}^{n} U_{\alpha_j}) \cup F^c$. Hence $F \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}$. Thus F is compact.

6. Let X be a sequentially compact topological space. Assume U_n $(n \in \mathbb{N})$ are open subsets of X and $X = \bigcup_{n=1}^{\infty} U_n$. We claim that there exists $N \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{N} U_n$.

If not: $X \ne \bigcup_{n=1}^{N} U_n$ $(\forall N \in \mathbb{N})$. Let $V_1 = U_1$, and $V_n = U_1 \cup \cdots \cup U_n$. Then all V_n $(n \in \mathbb{N})$ are open subsets of X, $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots$, and $X = \bigcup_{n=1}^{\infty} V_n$. Moreover, $\forall N \in \mathbb{N}$: $\bigcup_{n=1}^{N} U_n = \bigcup_{n=1}^{N} V_n$. Hence $X \ne \bigcup_{n=1}^{N} V_n$ $(\forall N \in \mathbb{N})$. We may assume that $V_1 \ne \phi$ (otherwise choose the first $V_n \ne \phi$ and relabel

$V_n, V_{n+1}, \cdots$ as $V_1, V_2, \cdots$.) Let $x_1 \in V_1$ and $n_1 = 1$. There exists $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $V_{n_2} \setminus V_{n_1} \neq \phi$. (If such n_2 does not exist, then all $V_n = V_{n_1}$ which is impossible, since $X \neq \bigcup_{n=1}^{N} V_n$ for any $N \in \mathbb{N}$). Choose some $x_2 \in V_{n_2} \setminus V_{n_1}$. Suppose we have $x_1, \cdots, x_k \in X$ such that $x_j \in V_{n_j} \setminus V_{n_{j-1}}$ $(j = 2, \cdots, k)$. For the same reason, $\exists n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $V_{n_{k+1}} \setminus V_{n_k} \neq \phi$. So, choose some $x_{n+1} \in V_{n_{k+1}}$. By induction, we have a sequence $\{x_k\}_{k=1}^{\infty}$ in X. This sequence would not have a subsequence converging to some point in X.

In fact, if $x_{k_j} \to x \in X$, then $x \in X = \bigcup_{n=1}^{\infty} V_n$. Hence, $\exists n$ such that $x \in V_n$. Thus $\{x_{k_j}\}_{j=1}^{\infty}$ should be eventually in V_n. But this is impossible as $x_{k_j} \in V_{k_j} \setminus V_{k_j - 1}$ $(\forall j \in \mathbb{N})$. So, if $k_j - 1 \geq n$, i.e., $k_j \geq n+1$, which is possible as $k_j \to \infty$ $(j \to \infty)$, then $x_{k_j} \in V_{k_j} \setminus V_n$ and $x_{k_j} \notin V_n$ (for all j s.t. $k_j \geq n+1$). Hence $x_{k_j} \not\to x$.

Therefore, there exists $N \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{N} U_n$. Hence X is countably compact.

7 (1) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X. Assume it has no cluster points. Define $U_n = \overline{\{x_j : j \geq n\}}^c$ $(n=1,2,\cdots)$. Then each U_n is an open subset of X and $U_n \subseteq U_{n+1}$ $(n=1,2,\cdots)$. Let $x \in X$. Since x is not a cluster point of $\{x_n\}_{n=1}^{\infty}$, there exists an open subset V_x of X such that $x \in V_x$ and there exists $N=N(x)$ so that $x_j \notin V_x$ if $j \geq N$. Thus $\{x_N, x_{N+1}, \cdots\} \subseteq V_x^c$ and $\overline{\{x_N, x_{N+1}, \cdots\}} \subseteq V_x^c$. Hence $x \in V_x \subseteq \overline{\{x_N, x_{N+1}, \cdots\}}^c = U_N$. Hence $X = \bigcup_{n=1}^{\infty} U_n$. Since X is countably compact, there is $m \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{m} U_n = U_m$ (as $U_n \uparrow$). Hence $\phi = X^c = U_m^c = \overline{\{x_m, x_{m+1}, \cdots\}}$. This is impossible. So $\{x_n\}_{n=1}^{\infty}$ must have at least one cluster point.

(2) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X. By Part (1), $\{x_n\}_{n=1}^{\infty}$ has a cluster point $x \in X$. Since X is first countable, there exist countably many open sets $\{V_n\}_{n=1}^{\infty}$ that form a neighborhood base at x. We may assume that $V_1 \supseteq V_2 \supseteq \cdots$. Since x is a cluster point of $\{x_n\}_{n=1}^{\infty}$ and V_1 is an open neighborhood of x, there exists $x_{n_1} \in V_1$. For the same reasons, $\{x_n\}_{n=1}^{\infty}$ is frequently in V_2. So, there exists $x_{n_2} \in V_2$ with $n_2 > n_1$. By induction we obtain a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \in V_k$ $(\forall k \in \mathbb{N})$

Now, for any open set U of X with $x \in U$, there exists $k_0 \in \mathbb{N}$ such that $V_{k_0} \subseteq U$. Since $V_k \subseteq V_{k_0}$ if $k \geq k_0$, we have $x_{n_k} \in V_k \subseteq V_{k_0} \subseteq U$ $\forall k \geq k_0$. Thus, $x_{n_k} \to x$. Hence, X is sequentially compact.

8. Suppose $f(X) \subseteq \bigcup_{n=1}^{\infty} U_n$ where U_n $(n \in \mathbb{N})$ are open subsets of Y. Then $X \subseteq f^{-1}(\bigcup_{n=1}^{\infty} U_n)$ $= \bigcup_{n=1}^{\infty} f^{-1}(U_n)$. Each $f^{-1}(U_n)$ is an open subset of X, since f is continuous. Since X is countably compact there exists $N \in \mathbb{N}$ such that $X \subseteq \bigcup_{n=1}^{N} f^{-1}(U_n)$, i.e., $X \subseteq f^{-1}(\bigcup_{n=1}^{N} U_n)$. Hence, $f(X) \subseteq \bigcup_{n=1}^{N} U_n$. Thus $f(X)$ is also countably compact.