Math 240A Fall 2019

Solution to Problems of HW#2

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1. All $E\cup F$, $E\cap F$, $E\setminus F$, $F\setminus E \in \mathcal{M}$. Since

$$E\cup F = (E\cap F)\cup(E\setminus F)\cup(F\setminus E)$$

is a disjoint union, we have

$$\mu(E\cup F) = \mu(E\cap F) + \mu(E\setminus F) + \mu(F\setminus E).$$

Moreover,

$$\mu(E) = \mu(E\setminus F) + \mu(E\cap F),$$

$$\mu(F) = \mu(F\setminus E) + \mu(E\cap F).$$

Therefore,

$$\mu(E)+\mu(F) = \mu(E\setminus F)+\mu(F\setminus E)+\mu(E\cap F)+\mu(E\cap F)$$

$$= \mu(E\cup F)+\mu(E\cap F). \qquad \square$$

2. $\mu_E(\phi) = \mu(\phi\cap E) = \mu(\phi) = 0$

If $A_j \in \mathcal{M}$ $(j=1,2,\dots)$ are disjoint, then all $A_j\cap E \in \mathcal{M}$ $(j=1,2,\dots)$ and they are disjoint.

Hence,

$$\mu_E\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\left(\bigcup_{j=1}^{\infty} A_j\right)\cap E\right) = \mu\left(\bigcup_{j=1}^{\infty}(A_j\cap E)\right)$$

$$= \sum_{j=1}^{\infty}\mu(A_j\cap E) = \sum_{j=1}^{\infty}\mu_E(A_j).$$

Thus, $\mu_E$ is a measure. $\square$

3 (1) Since $\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n$ and $\bigcap_{n=k}^{\infty} E_n$ increases as $k$ increases, it follows from the continuity from below of a measure, and the fact that $\bigcap_{n=k}^{\infty} E_n \subseteq E_k$ $(k \in \mathbb{N})$, that

$$\mu(\liminf_{n\to\infty} E_n) = \mu(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n)$$
$$= \lim_{k\to\infty}\mu(\bigcap_{n=k}^{\infty} E_n) \le \liminf_{k\to\infty}\mu(E_k).$$

(2) We have $\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n$. The sequence $\bigcup_{n=k}^{\infty} E_n \in \mathcal{M}_\xi$ $(k=1,2,\cdots)$ decreases and $\mu(\bigcup_{n=1}^{\infty} E_n) < \infty$. Thus, by the continuity of measure (from above) of a measure and the fact that $\bigcup_{n=k}^{\infty} E_n \supseteq E_k$ for each $k \in \mathbb{N}$ that

$$\mu(\limsup_{n\to\infty} E_n) = \mu(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n)$$
$$= \lim_{k\to\infty}\mu(\bigcup_{n=k}^{\infty} E_n) \ge \limsup_{k\to\infty}\mu(E_k). \quad \square$$

4. (1) Let $\mu$ be a measure. If $E_j \in \mathcal{M}_\xi$ $(j=1,2,\cdots)$ increases, then $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty}(E_j \setminus E_{j-1})$ (with $E_0 = \phi$), and the right-hand side is a disjoint union. Hence

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \mu(\bigcup_{j=1}^{\infty}(E_j \setminus E_{j-1}))$$
$$= \sum_{j=1}^{\infty}\mu(E_j \setminus E_{j-1}) = \sum_{j=1}^{\infty}[\mu(E_j) - \mu(E_{j-1})]$$
$$= \lim_{n\to\infty}\sum_{j=1}^{n}[\mu(E_j) - \mu(E_{j-1})] = \lim_{n\to\infty}\mu(E_n).$$

Assume $\mu$ is continuous from below (and is finitely additive). Let $A_j \in \mathcal{M}$ ($j=1,2,\cdots$) be disjoint. Since $\bigcup_{k=1}^{n} A_k$ increases as $n$ increases, we have

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} A_j\right) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) = \sum_{n=1}^{\infty} \mu(A_n).$$

Hence, $\mu$ is a measure.

(2) Suppose $\mu$ is a measure. Let $E_j \in \mathcal{M}$ ($j=1,2,\cdots$) be such that $E_1 \supseteq E_2 \supseteq \cdots$. Define $F_j = E_1 \setminus E_j$ ($j=1,2,\cdots$). Then, $F_j \in \mathcal{M}$ and $F_j$ increases. Note that $\mu(E_1) = \mu(F_j) + \mu(E_j)$ and that $\mu(E_1) \le \mu(X) < \infty$. It then follows from Par (1) that

$$\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j) = \lim_{j\to\infty} \left[\mu(E_1) - \mu(E_j)\right]$$
$$= \mu(E_1) - \lim_{j\to\infty} \mu(E_j).$$

But, $\bigcup_{j=1}^{\infty} F_j = \bigcup_{j=1}^{\infty} (E_1 \setminus E_j) = E_1 \setminus \left(\bigcap_{j=1}^{\infty} E_j\right)$. Hence,

$$\mu(E_1) = \mu\left(\bigcap_{j=1}^{\infty} E_j\right) + \mu\left(\bigcup_{j=1}^{\infty} F_j\right)$$
$$= \mu\left(\bigcap_{j=1}^{\infty} E_j\right) + \mu(E_1) - \lim_{j\to\infty} \mu(F_j)$$

Therefore $$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(F_j),$$

i.e., $\mu$ is continuous from above.

Suppose $\mu$ is continuous from above (and is finitely additive). Let $A_j \in \mathcal{M}$ ($j=1,2,\cdots$) be disjoint. Since $\bigcap_{j=1}^{n} A_j^c$ decreases with $n$ and $\mu(A_1^c) \le \mu(X) < \infty$, we have

$$\mu(X) - \mu\left(\bigcup_{j=1}^{\infty} A_j\right)$$
$$= \mu\left(X \setminus \left(\bigcup_{j=1}^{\infty} A_j\right)\right)$$
$$= \mu\left(\bigcap_{j=1}^{\infty} A_j^c\right)$$
$$= \lim_{n\to\infty} \mu\left(\bigcap_{j=1}^{n} A_j^c\right) \quad [\text{continuity from above}]$$
$$= \lim_{n\to\infty} \mu\left(\left(\bigcup_{j=1}^{n} A_j\right)^c\right)$$
$$= \mu(X) - \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} A_j\right)$$
$$= \mu(X) - \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) \quad [\text{finite additivity}]$$
$$= \mu(X) - \sum_{j=1}^{\infty} \mu(A_j).$$

Hence, $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ and $\mu$ is a measure.

$\square$

5. Suppose $\mu(E_i \cap E_j) = 0$ for any $i, j \in \mathbb{N}$ with $i \neq j$.

We show by induction on $n \in \mathbb{N}$ that

$$\mu\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mu(E_j) \quad \forall n \in \mathbb{N},\ n \geq 2.$$

For $n=2$, we have

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2).$$

Assume $\mu\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mu(E_j)$. We have

$$\mu\left(\bigcup_{j=1}^{n+1} E_j\right) = \mu\left(\left(\bigcup_{j=1}^{n} E_j\right) \cup E_{n+1}\right)$$
$$= \mu\left(\bigcup_{j=1}^{n} E_j\right) + \mu(E_{n+1}) + \mu\left(\left(\bigcup_{j=1}^{n} E_j\right) \cap E_{n+1}\right).$$
$$= \sum_{j=1}^{n} \mu(E_j) + \mu(E_{n+1}) + \mu\left(\bigcup_{j=1}^{n} (E_j \cap E_{n+1})\right)$$
$$= \sum_{j=1}^{n+1} \mu(E_j).$$

since $0 \leq \mu\left(\bigcup_{j=1}^{n} (E_j \cap E_{n+1})\right) \leq \sum_{j=1}^{n} \mu(E_j \cap E_{n+1}) = 0$,

which implies that $\mu\left(\bigcup_{j=1}^{n} (E_j \cap E_{n+1})\right) = 0$.

We have now for any $n \in \mathbb{N}$ that

$$\sum_{j=1}^{\infty} \mu(E_j) \geq \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \geq \mu\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mu(E_j)$$

$$\longrightarrow \sum_{j=1}^{\infty} \mu(E_j) \text{ as } n \to \infty.$$

Hence, $\sum_{j=1}^{\infty} \mu(E_j) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$.

Assume now $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$, $\forall i,j \in \mathbb{N}, i \neq j$.
Let $B = \bigcup_{\substack{n=1 \\ n \neq i,j}}^{\infty} E_n \in \mathcal{M}$. Then $\mu(B) \leq \sum_{\substack{n=1 \\ n \neq i,j}}^{\infty} \mu(E_n)$. Thus,

$$\sum_{k=1}^{\infty} \mu(E_k) = \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu(E_i \cup E_j \cup B)$$

$$\leq \mu(E_i) + \mu(E_j) + \mu(B)$$

$$\leq \mu(E_i) + \mu(E_j) + \sum_{\substack{n=1 \\ n \neq i,j}}^{\infty} \mu(E_n)$$

Thus,

$$\mu(B) = \sum_{\substack{n=1 \\ n \neq i,j}}^{\infty} \mu(E_n).$$

Similarly,

$$\sum_{n=1}^{\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu((E_i \cup E_j) \cup B)$$

$$\leq \mu(E_i \cup E_j) + \mu(B) \leq \mu(E_i \cup E_j) + \sum_{\substack{n=1 \\ n \neq i,j}}^{\infty} \mu(E_n).$$

Therefore,

$$\mu(E_i) + \mu(E_j) \leq \mu(E_i \cup E_j).$$

But $\mu(E_i \cup E_j) \leq \mu(E_i) + \mu(E_j)$. Hence,

$$\mu(E_i \cup E_j) = \mu(E_i) + \mu(E_j).$$

Consequently

$$\mu(E_i \cap E_j) = \mu(E_i) + \mu(E_j) - \mu(E_i \cup E_j) = 0. \quad \square$$

6. Let $E \in \mathcal{M}$ be such that $\mu(E) = \infty$. Since $\mu$ is $\sigma$-finite, $\exists A_j \in \mathcal{M}$ with $\mu(A_j) < \infty$ $(j = 1, 2, \dots)$ such that $X = \bigcup_{j=1}^{\infty} A_j$. Thus

$$\infty = \mu(E) = \mu(E \cap X) = \mu(E \cap (\bigcup_{j=1}^{\infty} A_j))$$
$$= \mu(\bigcup_{j=1}^{\infty} (E \cap A_j)) \le \sum_{j=1}^{\infty} \mu(E \cap A_j)$$

For each $j \ge 1$, $\mu(E \cap A_j) \le \mu(A_j) < \infty$. Thus, $\exists j_0 \ge 1$ such that $\mu(E \cap A_{j_0}) > 0$, for otherwise all $\mu(E \cap A_j) = 0$ and we have a contradiction that $\infty = 0$.

Let $F = E \cap A_{j_0} \subseteq E$. $F \in \mathcal{M}$. $0 < \mu(F) < \infty$. $\square$

7. Let $E \in \mathcal{M}$ be such that $\mu(E) = \infty$. Define

$$\mathcal{A} = \{ F \in \mathcal{M} : F \subseteq E \text{ and } 0 < \mu(F) < \infty \}.$$

Since $\mu$ is ~~$\sigma$-~~ semi-finite, $\mathcal{A} \ne \phi$. Let

$$\alpha = \sup\{ \mu(F) : F \in \mathcal{A} \} > 0.$$

It sufficies to show that $\alpha = \infty$.

Assume $\alpha < \infty$. There exist $F_k \in \mathcal{A}$ $(k = 1, 2 \cdots)$ such that $\mu(F_k) \to \alpha$ as $k \to \infty$. Let

$$H_k = \bigcup_{j=1}^{k} F_j \in \mathcal{A} \quad (\forall k \ge 1).$$

$H_k$ increases and $\mu(H_k) \to \alpha$. Set $H = \bigcup_{k=1}^{\infty} H_k$. Clearly $H \subseteq E$ since each $H_k \subseteq E$. Moreover

$$0 < \mu(H) = \mu(\bigcup_{k=1}^{\infty} H_k) = \lim_{k\to\infty} \mu(H_k) = \alpha < \infty.$$

Hence $H \in \mathcal{A}$.

Note that $E \setminus H \subseteq E$ and $\mu(E \setminus H) = \infty$ as $\mu(E) = \infty$ and $\mu(H)$ is finite. By the semi-finiteness of $\mu$, $\exists\, K \subseteq E \setminus H \subseteq E$ such that $0 < \mu(K) < \infty$. Hence $K \in \mathcal{A}$ and $K \cap H = \phi$. Now, $K \cup H \in \mathcal{A}$ as $K \cup H \subseteq E$ and $\mu(K \cup H) = \mu(K) + \mu(H) < \infty$. Thus,

$$\alpha \geq \mu(K \cup H) = \mu(K) + \mu(H) > 0 + \alpha = \alpha,$$

leading to a contradiction. $\square$

8. Let $E \subseteq X$. Clearly,

$$\sum_{j=1}^{\infty} \mu^*(E \cap A_j) \geq \mu^*\left(\bigcup_{j=1}^{\infty}(E \cap A_j)\right) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right).$$

Let $A = \bigcup_{j=1}^{\infty} A_j$. $A$ is $\mu^*$-measurable since each $A_j$ is. Thus,

$$\mu^*(E \cap A) = \mu^*(E \cap A \cap A_1) + \mu^*(E \cap A \cap A_1^c)$$

[since $A_1$ is $\mu^*$-measurable]

$$= \mu^*(E \cap A_1) + \mu^*\left(E \cap \left(\bigcup_{j=2}^{\infty} A_j\right)\right)$$ [all $A_j$'s are disjoint]

$$= \mu^*(E \cap A_1) + \mu^*\left(E \cap \left(\bigcup_{j=2}^{\infty} A_j\right) \cap A_2\right)$$

$$+ \mu^*\left(E \cap \left(\bigcup_{j=2}^{\infty} A_j\right) \cap A_2^c\right)$$ [since $A_2$ is $\mu^*$-measurable]

$$= \mu^*(E \cap A_1) + \mu^*(E \cap A_2) + \mu^*\left(E \cap \left(\bigcup_{j=3}^{\infty} A_j\right)\right)$$

[all $A_j$'s are disjoint]

$$= \cdots$$

$$= \mu^*(E \cap A_1) + \cdots + \mu^*(E \cap A_n) + \mu^*\left(E \cap \left(\bigcup_{j=n+1}^{\infty} A_j\right)\right)$$

$$\geq \sum_{j=1}^{n} \mu(E \cap A_j) \quad (n = 1, 2, \ldots)$$

Thus, $\mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) = \mu^*(E \cap A) \geq \sum_{j=1}^{\infty} \mu(E \cap A_j)$.

Finally, $\mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) = \sum_{j=1}^{\infty} \mu(E \cap A_j)$. $\square$

9 (1) By definition

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(A_j) : \text{all } A_j \in \mathcal{A},\ \bigcup_{j=1}^{\infty} A_j \supseteq E\right\}.$$

Thus for $\varepsilon > 0$, $\exists\, A_j \in \mathcal{A}$ ($j = 1, 2, \cdots$) such that $\bigcup_{j=1}^{\infty} A_j \supseteq E$ and $\mu^*(E) + \varepsilon \geq \sum_{j=1}^{\infty} \mu_0(A_j)$. Let $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}_\sigma$. Since $\mu^*|_{\mathcal{A}} = \mu_0$, we have

$$\mu^*(A) = \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon.$$

(2) Assume there exists $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq B$ and $\mu^*(B \setminus E) = 0$. We show that $E$ is $\mu^*$-measurable. Let $F \subseteq X$. Since $E \subseteq B$ and $\mu^*(B \setminus E) = 0$, we have by the disjoint union $F \setminus E = (F \setminus B) \cup (B \setminus E)$ that $\mu^*(F \setminus E) \leq \mu^*(F \setminus B) + \mu^*(B \setminus E) = \mu^*(F \setminus B) = \mu^*(F \cap B^c)$. Consequently,

$$\begin{aligned}
\mu^*(F) &= \mu^*((F \cap E) \cup (F \cap E^c)) \\
&\leq \mu^*(F \cap E) + \mu^*(F \cap E^c) \\
&\leq \mu^*(F \cap B) + \mu^*(F \setminus E) \qquad [E \subseteq B]. \\
&\leq \mu^*(F \cap B) + \mu^*(F \cap B^c) \\
&= \mu^*(F), \qquad [B \text{ is } \mu^*\text{-measurable}]
\end{aligned}$$

where the last step follows from the fact that $B$ is $\mu^*$-measure, as all members in $\mathcal{A}$ are $\mu^*$-measurable, and thus all members in $\mathcal{A}_\sigma$ are $\mu^*$-measurable, and finally all members in $\mathcal{A}_{\sigma\delta}$ are $\mu^*$-measurable. Consequently,

$$\mu^*(F \cap E) + \mu^*(F \cap E^c) = \mu^*(F),$$

and $E$ is $\mu^*$-measurable.

Assume now $E$ is $\mu^*$-measurable. By Part (1), $\forall j \in \mathbb{N}$, $\exists A_j \in \mathcal{A}_\sigma$ such that $E \subseteq A_j$ and

$$\mu^*(A_j) \le \mu^*(E) + 1/j \quad (j = 1, 2, \cdots).$$

Let $B = \bigcap_{j=1}^{\infty} A_j \in \mathcal{A}_{\sigma\delta}$. Clearly $E \subseteq B$, and $B$ is $\mu^*$-measurable. Thus

$$\mu^*(E) \le \mu(B) \le \mu(A_j) \le \mu^*(E) + \frac{1}{j} \quad \forall j \ge 1.$$

Sending $j \to \infty$ we get $\mu^*(E) = \mu^*(B) < \infty$ (since $\mu^*(E) < \infty$). Since both $E$ and $B$ are $\mu^*$-measurable, and $\mu^*$ is a measure on the $\sigma$-algebra of all $\mu^*$-measurable sets, we have by the disjoint union $B = E \cup (B \setminus E)$ that

$$\mu^*(B) = \mu^*(E) + \mu^*(B \setminus E).$$

But $\mu^*(B) = \mu^*(E) < \infty$. Hence $\mu^*(B \setminus E) = 0$. $\square$

10. (1) We claim that $E \subseteq A \cup B^c$. Let $x \in E$. If $x \notin B^c$, then $x \in B$. Hence $x \in E \cap B = E \cap A \subseteq A$. So, $x \in A \cup B^c$. Thus $E \subseteq A \cup B^c$. Now

$$\mu(X) = \mu^*(X) = \mu^*(E) \le \mu^*(A \cup B^c) = \mu^*((A^c \cap B)^c)$$
$$= \mu(X) - \mu(A^c \cap B)$$

Hence $\mu(A^c \cap B) = \mu(B \setminus A) = 0$. Similarly $\mu(B^c \cap A) = 0$.

Thus since $\mu(A \cup B) = \mu(A) + \mu(B \setminus A) = \mu(A)$

and $\mu(A \cup B) = \mu(B) + \mu(A \setminus B) = \mu(B) + \mu(B^c \cap A)$

$$= \mu(B).$$

we obtain $\mu(A) = \mu(B)$.

(2) Clearly $\mathcal{M}_E$ is not empty. If $A \cap E \in \mathcal{M}_E$ where $A \in \mathcal{M}$, then $E \setminus (A \cap E) = A^c \cap E \in \mathcal{M}_E$ since $A^c = X \setminus A \in \mathcal{M}$. If $A_j \cap E \in \mathcal{M}_E$ with $A_j \in \mathcal{M}$ $(j=1,2,\cdots)$, then $\bigcup_{j=1}^{\infty}(A_j \cap E) = (\bigcup_{j=1}^{\infty} A_j) \cap E \in \mathcal{M}_E$ as $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$. Thus, $\mathcal{M}_E$ is a $\sigma$-algebra.

We have $\overset{=\nu(\phi \cap E)}{\nu(\phi)} = \mu(\phi) = 0$. Let $A_j \cap E \in \mathcal{M}_E$ $(j=1,2,\cdots)$ with all $A_j \in \mathcal{M}$, and $A_j \cap E$ are pairwise disjoint. (Note that $A_j$ $(j=1,2,\cdots)$ may not be pairwise disjoint.) Denote $A = \bigcup_{i=1}^{\infty}\bigcup_{j=i+1}^{\infty}(A_i \cap A_j) = \bigcup_{\substack{i,j=1 \\ i \neq j}}^{\infty}(A_i \cap A_j)$. Define $B_n = A_n \setminus A \in \mathcal{M}$ $(n=1,2,\cdots)$. We have for any $i,j$, $i \neq j$, that

$$B_i \cap B_j = (A_i \setminus A) \cap (A_j \setminus A) = A_i \cap A_j \cap A^c$$
$$= A_i \cap A_j \cap \bigcap_{\substack{k,l=1 \\ k \neq l}}^{\infty}(A_k^c \cap A_l^c) \subseteq A_i \cap A_j \cap (A_i^c \cap A_j^c) = \phi.$$

Moreover, $B_n \cap E \subseteq A_n \cap E$, $(n=1,2,\cdots)$.
If $x \in A_n \cap E$, then $x \in A_n$ and $x \in E$. But $A_n \cap E$ and $A_k \cap E$ are disjoint if $k \neq n$. Hence $x \notin A_k$ for any $k \neq n$. Hence $x \notin A$. Thus, $x \in A_n \setminus A = B_n$. Hence $A_n \cap E \subseteq B_n \cap E$, and so $A_n \cap E = B_n \cap E$. Now,

$$\nu\left(\bigcup_{j=1}^{\infty}(A_j \cap E)\right) = \nu\left(\bigcup_{j=1}^{\infty}(B_j \cap E)\right) = \nu\left(\left(\bigcup_{j=1}^{\infty} B_j\right) \cap E\right)$$
$$= \mu\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \mu(B_j) = \sum_{j=1}^{\infty} \nu(B_j \cap E)$$
$$= \sum_{j=1}^{\infty} \nu(A_j \cap E).$$

Thus, $\nu$ is a measure on $\mathcal{M}_E$. $\square$