# YOUNG SYMMETRIZER MODULES

Let $U = \mathbf{C}^2$ and let $d > 0$ be an integer.

Let $A$ be the algebra $\bigoplus_{n\ge 0}(\mathrm{Sym}^n U)^{\otimes d}$ and $B = \bigoplus_{n\ge 0} D^d(\mathrm{Sym}^n U)$ the subalgebra of $S_d$-invariants. Then $A$ is a free $B$-module and we explain how to decompose it.

For each partition $\lambda$ of $d$, we let

$$M_\lambda = \bigoplus_{n\ge 0} \mathbf{S}_\lambda(\mathrm{Sym}^n U)$$

be the corresponding module of covariants.

First, we record characters of $\mathbf{GL}(U)$ as $q^i$ for the function $\begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} \mapsto t_1^i t_2^j$ independent of what $j$ is. So the character of $\mathrm{Sym}^n U$ is $1 + q + q^2 + \cdots + q^n = [n+1]_q$. Then the equivariant Hilbert series of $A$ and $B$ are

$$\mathrm{H}_A(t) = \sum_{n\ge 0} [n+1]^d t^n,$$
$$\mathrm{H}_B(t) = \sum_{n\ge 0} h_d(1, q, \dots, q^n) t^n$$

where $h_d$ is the $d$th homogeneous symmetric polynomial. The first one simplifies by an identity of Carlitz as follows (see [C], though it's not stated in this exact form, but one that can be transformed to it). Given a permutation $\sigma \in S_n$, $\sigma$ has a descent at $i$ if $\sigma(i) > \sigma(i+1)$. We let $\mathrm{des}(\sigma)$ be the number of descents, and $\mathrm{maj}(\sigma)$ be the sum of the descents. Then Carlitz's identity says

$$\mathrm{H}_A(t) = \frac{\sum_{\sigma\in S_d} t^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}}{(1-t)(1-qt)\cdots(1-q^d t)}.$$

For the second formula, we have the identity

$$h_d(1, q, \dots, q^n) = \begin{bmatrix} n+d \\ d \end{bmatrix}_q,$$

and so

$$\mathrm{H}_B(t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^d t)}.$$

(This can be proven using $q$-Pascal identity.) Since $B$ is free over $A$, the quotient

$$\frac{\mathrm{H}_A(t)}{\mathrm{H}_B(t)} = \sum_{\sigma\in S_d} t^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}$$

is the Hilbert series for the minimal generators for $A$ as a $B$-module.

Next, given a standard Young tableau $T$, we say $T$ has a descent at $i$ if $i+1$ appears in a lower row than $i$. We define $\mathrm{des}(T)$ to be the set of descents of $T$ and $\mathrm{maj}(T)$ to be the sum of descents of $T$. The RSK algorithm gives a bijection $\sigma \mapsto (P(\sigma), Q(\sigma))$ between

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permutations and pairs of standard Young tableaux of the same shape. For our purposes, we just need to know that [St, Lemma 7.23.1]

$$\mathrm{des}(\sigma) = \mathrm{des}(Q(\sigma)).$$

In particular, let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableau of shape $\lambda$ and let $f^\lambda = |\mathrm{SYT}(\lambda)|$. Then we have

$$\sum_{\sigma \in S_d} t^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)} = \sum_{|\lambda|=d} f^\lambda \sum_{T \in \mathrm{SYT}(\lambda)} t^{\mathrm{des}(T)} q^{\mathrm{maj}(T)}.$$

Our next goal is to show that this is compatible with the decomposition

$$A = \bigoplus_{|\lambda|=d} M_\lambda^{\oplus f_\lambda}.$$

**Proposition 0.1.** *Let $\lambda$ be a partition of $d$. We have*

$$\sum_{n \geq 0} s_\lambda(1, q, \dots, q^n) t^n = \frac{\sum_{T \in \mathrm{SYT}(\lambda)} t^{\mathrm{des}(T)} q^{\mathrm{maj}(T)}}{(1-t)(1-qt)\cdots(1-q^d t)}.$$

*Proof.* By [St, Proposition 7.19.12], we have

$$s_\lambda(1, q, \dots, q^n) = \sum_{T \in \mathrm{SYT}(\lambda)} \begin{bmatrix} n - \mathrm{des}(T) + d \\ d \end{bmatrix}_q q^{\mathrm{maj}(T)}.$$

Now sum over all $n \geq 0$:

$$\begin{aligned} \sum_{n \geq 0} s_\lambda(1, q, \dots, q^n) t^n &= \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} \sum_{n \geq 0} \begin{bmatrix} n - \mathrm{des}(T) + d \\ d \end{bmatrix}_q t^n \\ &= \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} \sum_{n \geq \mathrm{des}(T)} \begin{bmatrix} n + d \\ d \end{bmatrix}_q t^{n + \mathrm{des}(T)} \\ &= \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} \frac{t^{\mathrm{des}(T)}}{(1-t)(1-qt)\cdots(1-q^d t)}, \end{aligned}$$

which is what we wanted. $\square$

The above identity actually works for any skew partition $\lambda/\mu$.

**Corollary 0.2.** *As a $B$-module, $M_\lambda$ has one generator for every SYT $T$ of shape $\lambda$ and it lies in degree* $\mathrm{des}(T)$. *Furthermore, the character of the generators in degree $i$ as an* $\mathbf{GL}(U)$*-module is* $\sum_{T,\ \mathrm{des}(T)=i} q^{\mathrm{maj}(T)}$.

It would be interesting to have some way to evaluate the sum appearing in the corollary. It is studied in [K] (see also references in that paper). There they are concerned with unimodality of the sum and say that it follows from a result of Kirillov-Reshetikhin. Namely, on p.7 it is stated that

$$q^{\binom{d}{2}} \sum_{T,\ \mathrm{des}(T)=i} q^{-\mathrm{maj}(T)} = \sum_{T,\ \mathrm{des}(T)=i} q^{\mathrm{charge(T)}},$$

which is I suppose why Jerzy and Andrei's note mention Kostka polynomials.

But note that unimodality also follows easily from the interpretation as the character of some $\mathbf{SL}_2$-representation.

## References


[C] L. Carlitz, A Combinatorial Property of $q$-Eulerian Numbers, *American Mathematical Monthly* **82**, (1975), no. 1, 51–54.

[K] William J. Keith, Families of major index distributions: closed forms and unimodality, arXiv:1808.01362v1.

[St] Richard Stanley, *Enumerative Combinatorics Vol. 2*