Math 240B: Real Analysis, Winter 2020

## Midterm Exam

**Name** ______________________ **ID number** ______________________

**Note:** This is a close-book and close-note exam. There are 4 problems of total 100 points. To get credit, you must show your work. Partial credit will be given to partial answers.

| Problem | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Score | | | | | |

1. (25 points) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^\infty(\mu)$. Let $E = \{x \in X : |f(x)| \le \|f\|_\infty\}$. Use the definition $\|f\|_\infty = \inf\{a \ge 0 : \mu(\{|f| > a\}) = 0\}$ to prove that $\mu(E^c) = 0$ and that $\|f\|_\infty = \sup_{x \in E} |f(x)|$.

Proof. Let $A = \{a \ge 0 : \mu(\{|f| > a\}) = 0\}$. By the definition of $\|f\|_\infty$, $\exists\, a_n \in A$ s.t. $a_n \downarrow \|f\|_\infty$. Hence,

$$E^c = \{|f| > \|f\|_\infty\} = \bigcup_{n=1}^{\infty} \{|f| > a_n\}, \text{ and}$$

$$\mu(E^c) \le \sum_{n=1}^{\infty} \mu(\{|f| > a_n\}) = 0.$$

Clearly, $\|f\|_\infty \ge \sup_{x \in E} |f(x)|$. If $\|f\|_\infty > \sup_{x \in E} |f(x)|$, then, there exists $\varepsilon \in (0, \|f\|_\infty)$ s.t. $\|f\|_\infty - \varepsilon > \sup_{x \in E} |f(x)|$. This implies that $E = \{x \in X : |f(x)| \le \|f\|_\infty - \varepsilon\}$. Consequently, $\mu(\{|f| > \|f\|_\infty - \varepsilon\}) = \mu(E^c) = 0$. The definition of $\|f\|_\infty$ then implies that $\|f\|_\infty \le \|f\|_\infty - \varepsilon$. This is impossible. Hence, $\|f\|_\infty = \sup_{x \in E} |f(x)|$.

2. (25 points) Prove the following:

(1) If $y_n \to y$ weakly in a Banach space $X$ (i.e., $f(y_n) \to f(y)$ for any $f \in X^*$), then $\sup_{n\geq 1} \|y_n\| < \infty$;

(2) If $x_n \to x$ strongly and $y_n \to y$ weakly in a Hilbert space $H$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

Proof. (1) Recall that each $z \in X$ defines $\hat{z} \in X^{**}$ by $\hat{z}(f) = f(z)$ $\forall f \in X^*$ and $\|\hat{z}\| = \|z\|$. Now, $y_n \to y$ weakly. So, $\lim_{n\to\infty} f(y_n) = f(y)$ $\forall f \in X^*$. Thus $\lim_{n\to\infty} \hat{y}_n(f) = y(f)$ $\forall f \in X^*$. This implies that $\sup_{n\geq 1} |\hat{y}_n(f)| < \infty$ $\forall f \in X^*$. By the Principle of Uniform Boundedness,

$$\sup_{n\geq 1} \|y_n\| = \sup_{n\geq 1} \|\hat{y}_n\| < \infty.$$

(2) We have $\langle x_n, y_n\rangle - \langle x, y\rangle$

$$= \langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle.$$

By the Cauchy-Schwarz inequality and (1),

$$|\langle x_n - x, y_n\rangle| \leq \|x_n - x\| \, \|y_n\| \leq \left(\sup_{k\geq 1} \|y_k\|\right) \|x_n - x\| \to 0.$$

Now, $f_x(z) = \langle z, x\rangle$ $(\forall z \in H)$ defines $f_x \in H^*$ and $\|f_x\| = \|x\|$. Since $y_n \to y$ weakly, $f_x(y_n) \to f_x(y)$, i.e., $\langle y_n, x\rangle \to \langle y, x\rangle$. Hence $\langle y_n - y, x\rangle \to 0$. i.e., $\langle x, y_n - y\rangle \to 0$.

Therefore, $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

3. (25 points) Let $X$ be a Banach space and $T \in L(X,X)$. Assume that $\|T\| < 1$. Show that $\sum_{n=0}^{\infty} T^n$ converges in $L(X,X)$. Moreover, if $S = \sum_{n=0}^{\infty} T^n$, then $S(I-T) = (I-T)S = I$, where $I : X \to X$ is the identity map.

Proof. If $A, B \in L(X,X)$ then $AB : X \to X$ is linear and $\|ABx\| = \|A(Bx)\| \le \|A\|\,\|Bx\| \le \|A\|\,\|B\|\,\|x\|$ $\forall x \in X$. Hence $\|AB\| \le \|A\|\,\|B\|$. (In particular, $AB \in L(X,X)$)

Similarly, if $n \ge 2$, then $T^n \in L(X,X)$, and $\|T^n\| \le \|T^{n-1}\|\,\|T\| \le \|T^{n-2}\|\,\|T\|^2 \le \cdots \le \|T\|^n$.

Now, $\|\sum_{j=n}^{m} T^j\| \le \sum_{j=n}^{m} \|T\|^j \xrightarrow[(m,n\to\infty)]{} 0$ since $\|T\| < 1$.

Since $L(X,X)$ is a Banach space, $\sum_{n=0}^{\infty} T^n$ converges in $L(X,X)$. Let $S = \sum_{n=0}^{\infty} T^n \in L(X,X)$. i.e.,

$\|\sum_{j=0}^{n} T^j - S\| \to 0$ as $n \to \infty$. It is easy to verify that

$$\left(\sum_{j=0}^{n} T^j\right)(I-T) = \sum_{j=0}^{n} T^j - \sum_{j=1}^{n+1} T^j = I - T^{n+1}.$$

Hence, $\|\left(\sum_{j=0}^{n} T^j\right)(I-T) - I\| = \|T^{n+1}\| \le \|T\|^{n+1} \to 0$.

Consequently,

$$\|S(I-T) - I\| \le \|(S - \sum_{j=0}^{n} T^j)(I-T)\| + \|\left(\sum_{j=0}^{n} T^j\right)(I-T) - I\|$$

$$\le \|S - \sum_{j=0}^{n} T^j\|\,\|I-T\| + \|\left(\sum_{j=0}^{n} T^j\right)(I-T) - I\|$$

$$\to 0 \quad \text{as } n \to \infty$$

i.e. $S(I-T) = I$. Similarly, $(I-T)S = I$.

4. (25 points) For each $n \in \mathbb{N}$, let $\phi_n \in C(\mathbb{R})$ be such that $\phi_n \geq 0$ on $\mathbb{R}$, $\phi_n(x) = 0$ if $|x| \geq 1/n$, and $\int_{\mathbb{R}} \phi_n(x)\,dx = 1$. For any $f \in L^1(\mathbb{R})$, define $J_n f : \mathbb{R} \to \mathbb{R}$ by

$$(J_n f)(x) = \int_R \phi_n(x-y) f(y)\,dy \qquad \forall x \in \mathbb{R}.$$

Prove the following:

(1) If $g \in C_c(\mathbb{R})$ (i.e., $g \in C(\mathbb{R})$ and $g = 0$ outside a finite interval) then $\cancel{g_n}\; J_n g \to g$ uniformly on $\mathbb{R}$;

(2) If $1 < p < \infty$ and $h \in L^p(\mathbb{R})$, then $\|J_n h\|_p \leq \|h\|_p$ for each $n$ and $J_n h \to h$ in $L^p(\mathbb{R})$.

Proof. (1) We have for any $x \in \mathbb{R}$ that

$$|J_n g(x) - g(x)| = \left| \int_{\mathbb{R}} \phi_n(x-y)\,[g(y) - g(x)]\,dy \right|$$

$$= \left| \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} \phi_n(x-y)\,[g(y) - g(x)]\,dy \right|$$

$$\leq \left( \max_{y' \in [x-\frac{1}{n},\, x+\frac{1}{n}]} |g(y') - g(x)| \right) \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} \phi_n(x-y)\,dy$$

$$\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} \phi_n(x-y)\,dy = \int_{\mathbb{R}} \phi_n(x-y)\,dy = \int_{\mathbb{R}} \phi_n(z)\,dz = 1.$$

Since $g$ is uniformly continuous,

$$\max_{y' \in [x-\frac{1}{n},\, x+\frac{1}{n}]} |g(y') - g(x)| \leq \max_{x', y' \in \mathbb{R},\ |x'-y'| \leq \frac{1}{n}} |g(x') - g(y')| \triangleq \omega_{\frac{1}{n}}(g)$$

and $\max_{x \in \mathbb{R}} |J_n g(x) - g(x)| \leq \omega_{\frac{1}{n}}(g) \longrightarrow 0.$

Hence $J_n g \to g$ uniformly on $\mathbb{R}$.

(2) Let $q = \frac{p}{p-1} \in (1, \infty)$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$|J_n h(x)|^p \leq \left( \int_{\mathbb{R}} \phi_n(x-y)\,|h(y)|\,dy \right)^p$$

$$= \left( \int_{\mathbb{R}} \phi_n(x-y)^{\frac{1}{q}}\, \phi_n(x-y)^{\frac{1}{p}}\, |h(y)|\,dy \right)^p$$

$$\overset{\text{Hölder}}{\leq} \left( \int_{\mathbb{R}} \phi_n(x-y)\,dy \right)^{p/q} \int_{\mathbb{R}} \phi_n(x-y)\,|h(y)|^p\,dy$$

$$= \int_{\mathbb{R}} \phi_n(x-y)\,|h(y)|^p\,dy \qquad \forall x \in \mathbb{R}.$$

By Fubini's Theorem,

$$\int_{\mathbb{R}} |J_n h(x)|^p dx \leq \int_{\mathbb{R}} |h(y)|^p \left( \int_{\mathbb{R}} \phi_n(x-y)\, dx \right) dy$$
$$= \int_{\mathbb{R}} |h(y)|^p dy$$

Hence, $\|J_n h\|_p \leq \|h\|_p$.

$\forall \varepsilon > 0$. $\exists \tilde{h} \in C_c(\mathbb{R})$ s.t. $\|\tilde{h} - h\|_p < \varepsilon$. We have

$$\|J_n h - h\|_p \leq \|J_n h - J_n \tilde{h}\|_p + \|J_n \tilde{h} - \tilde{h}\|_p + \|\tilde{h} - h\|_p$$
$$\leq 2\|h - \tilde{h}\|_p + \|J_n \tilde{h} - \tilde{h}\|_p$$
$$\leq 2\varepsilon + \|J_n \tilde{h} - \tilde{h}\|_p.$$

Since all $J_n \tilde{h}$ $(n \geq 1)$, $\tilde{h}$ vanish outside a single finite interval, and since $J_n \tilde{h} \to \tilde{h}$ uniformly on $\mathbb{R}$ (by (1)) we have $\|J_n \tilde{h} - \tilde{h}\|_p \to 0$.

Hence, $\limsup_{n\to\infty} \|J_n h - h\|_p \leq 2\varepsilon$.

Consequently, $\|J_n h - h\|_p \to 0$. i.e., $J_n h \to h$ in $L^p(\mathbb{R})$.