EE555

midterm-III

1) $\dot{x}_1 = -x_1 + b x_2$ (i) $\quad V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{b}\int_0^{x_1} g(y)\,dy$ (iii)

$\dot{x}_2 = b x_1 - g(x_1) - x_2 + u$ (ii) $\quad x_1 g(x_1) \geq 0, \; g(0) = 0$

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot \begin{bmatrix} -x_1 + b x_2 \\ b x_1 - g(x_1) - x_2 + u \end{bmatrix} = \begin{bmatrix} x_1 + \frac{g(x_1)}{b} & x_2 \end{bmatrix} \begin{bmatrix} -x_1 + b x_2 \\ b x_1 - g(x_1) - x_2 + u \end{bmatrix}$$

$$= -x_1^2 - \frac{x_1 g(x_1)}{b} + b x_1 x_2 + x_2 \cancel{g}(x_1) + b x_2 x_1 - x_2 \cancel{g}(x_1) - x_2^2 + u x_2$$

The $V(x)$ has to be P.D to check stability of the System. So,

$$V(x) = \underbrace{\frac{1}{2}x_1^2}_{>0} + \underbrace{\frac{1}{2}x_2^2}_{>0} + \frac{1}{b}\underbrace{\int_0^{x_1} g(y)\,dy}_{>0}$$

In that case, $\underline{b > 0}$

$$\dot{V}(x) = -x_1^2 - x_2^2 + 2b x_1 x_2 + x_2 u - \underbrace{\frac{x_1 g(x_1)}{b}}_{>0} \leq -x_1^2 - x_2^2 + 2b x_1 x_2 + x_2 u$$

$$= -x_1^2 + 2b x_1 x_2 - \theta b^2 x_2^2 - (1-\theta b^2) x_2^2 + x_2 u \leq -\frac{1}{\theta}x_1^2 - \frac{1}{\theta}(x_1 - \theta b x_2)^2 - |x_2|\left(x_2(1-\theta b^2) - u\right)$$

In that case, we define $p(u) = \frac{u}{(1-\theta b^2)}$ and require $\|x\|_\infty \geq p(|u|)$

- $|x_2| > \frac{|u|}{1-\theta b^2}$ (x)

$$\dot{V}(x) \leq -\frac{x_1^2}{\theta} - \overbrace{\frac{1}{\theta}(x_1 - \theta b x_2)^2}^{>0} - \underbrace{|x_2|\overbrace{\left(x_2(1-\theta b^2) - u\right)}^{>0}}_{>0} \overset{?}{\leq} -\frac{1}{\theta}x_1^2$$

To satisfy that and to say the system is ISS (Input-to-State Stable), relation (x) has to hold. So, $0 < b < \frac{1}{\sqrt{\theta}}$

- $x_2 < \frac{u}{1-\theta b^2}$. We note that this implies $|x_1| \geq \frac{|u|}{1-\theta b^2}$ ($\acute{J}$)

$$\dot{V}(x) \leq -\theta b^2 x_1^2 + \theta b x_1 x_2 - x_2^2 - (1-\theta b^2)x_1^2 + x_2 u \leq -\frac{x_2^2}{\theta} - \frac{1}{\theta}\overbrace{(\theta b x_1 - x_2)^2}^{>0} - (1-\theta b^2)x_1^2$$

$$\leq -\frac{x_2^2}{\theta} - (1-\theta b^2)x_1^2 + x_2 u \leq -\frac{x_2^2}{\theta} - (1-\theta b^2)x_1^2 + \frac{u^2}{(1-\theta b^2)}$$

$$\leq -\frac{x_2^2}{\theta} - \left(x_1^2(1-\theta b^2) - \frac{u^2}{(1-\theta b^2)} \right)$$

One can see from relation ($\acute{J}$), $x_1^2(1-\theta b^2) - \frac{u^2}{1-\theta b^2}$ is always positive. So,

$$\leq -\frac{x_2^2}{\theta} - \left(x_1^2(1-\theta b^2) - \frac{u^2}{(1-\theta b^2)} \right) \leq -\frac{x_2^2}{\theta}$$ ISS is satisfied also

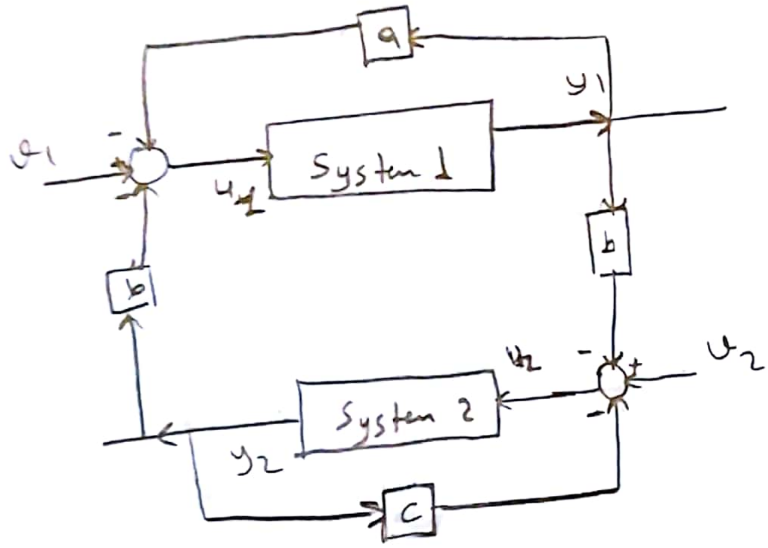
for same relation. So, we obtained only one boundary for b which is

$$0 < b \leq \frac{1}{\sqrt{\theta}} \text{ //}$$

2) a) $u = -Qy + v$, Q is $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and Positive Definite

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} ay_1 + by_2 \\ by_1 + cy_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



b) System 1 , System 2

$u_1^T y_1 \geq \dot{V}_1$ $\quad$ $u_2^T y_2 \geq \dot{V}_2$

$u = -Qy + v$ where

$Q = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$-ay_1^2 - by_1y_2 + v_1y_1 \geq \dot{V}_1$ (passive)

$-by_1y_2 - cy_2^2 + v_2y_2 \geq \dot{V}_2$ (passive)

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} -ay_1 - by_2 + v_1 \\ -by_1 - cy_2 + v_2 \end{bmatrix}$$

$\underbrace{v_1y_1 + v_2y_2}_{v^Ty} - (\underbrace{ay_1^2 + 2by_1y_2 + cy_2^2}_{y^TQy}) \geq \underbrace{\dot{V}_1 + \dot{V}_2}_{\dot{V} \text{ (overall storage function)}}$ (passive)

$v^Ty - y^TQy \geq \dot{V} \rightarrow v^Ty - \max(\lambda_1, \lambda_2)\, y^Ty \geq \dot{V}$

eigenvalues of the Q matrix

One can see that, the system is finite-gain L_2-stable and its upper bound is $\gamma \leq \frac{1}{\max(\lambda_1, \lambda_2)}$ //

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(c) As one can see from part (b), the overall system is also passive..

Stability

If $V(x)$ (storage function) is Positive Definite and the System is passive (which is proved in part-b) then the origin is stable.

Asymptotic Stability

In our overall system, passivity equation can be written as:

$$v^T y \geq \dot{V} + \frac{1}{\delta} y^T y \quad \text{(Output strictly stable)}$$

Lemma:
If the system is output strictly passive and zero-state detectable, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable.

proved ✓

The system is zero-state detectable if we can find a neighborhood D around the origin such that for any trajectory starting in D

$$v \equiv 0 \text{ and } y \equiv 0 \Rightarrow \lim_{t \to \infty} x(t) = 0$$

which means for zero-input the output is zero and states goes to origin. If we satisfy that, our system will be asymptotically stable.

dd System 1

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -z_1^3 + u_1$$
$$y_1 = z_2$$

System 2

$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = -z_3^3 + u_2$$
$$y_2 = z_4$$

Before checking the L_2-stability of the feedback connections, Storage functions have to be found for each system. The storage function has to be P.D or P.S.D, so it can be defined as for the systems:

$$V_1(z) = \frac{1}{4}z_1^4 + \frac{1}{2}z_2^2$$
$$V_2(z) = \frac{1}{4}z_3^4 + \frac{1}{2}z_4^2$$

(They found intuitively, I tried to eliminate indefinite parameters in $\dot{V}(x)$ and found $V(x)$ according to that.)

$u = -Qy + v$, as one can see Q is symmetric positive definite matrix which means that it is full rank $\longrightarrow$ well defined feedback connection

Moreover, we know that both system is passive from:

$$\dot{V}_1 = \frac{\partial V_1}{\partial z}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}z_1^3 & z_2\end{bmatrix}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \cancel{z_2 z_1^3} - \cancel{z_1^3 z_2} + u_1 z_2$$
$$= u_1 y_1 \quad , \text{So:}$$

$$\underline{u_1 y_1 \geq \dot{V}_1}$$
$$u_1^T y_1 \geq \dot{V}_1 \quad (\text{System 1 is passive})$$

$$\dot{V}_2 = \frac{\partial V_2}{\partial z}\begin{bmatrix}\dot{z}_3\\ \dot{z}_4\end{bmatrix} = \begin{bmatrix}z_3^3 & z_4\end{bmatrix}\begin{bmatrix}\dot{z}_3\\ \dot{z}_4\end{bmatrix} = \cancel{z_4 z_3^3} - \cancel{z_3^3 z_4} + u_2 z_4$$
$$= u_2 y_2, \text{ So :}$$

$$\underline{u_2 y_2 \geq \dot{V}_2}$$
$$u_2^T y_2 \geq \dot{V}_2 \quad (\text{System 2 is passive})$$

Both system passivity is proved. So, as indicated before in part b the overall system is L_2-stable.

For the asymptotic stability, we has to find zero-state detectable for both system so we can prove the origin is asymptotic stability for overall system

System 1

if $u_1 \equiv 0$ and $y_1 \equiv 0 \Rightarrow z_2 \equiv 0$ and $u_1 \equiv 0$. So,

$z_2 \equiv 0 \Rightarrow \dot{z}_2 = 0$ and $u_1 \equiv 0 \Rightarrow z_1 \equiv 0 \Rightarrow \dot{z}_1 = 0$

we found that

if $u_1 \equiv 0$ and $y_1 \equiv 0 \Rightarrow \lim\limits_{t \to \infty} z(t) = 0$, $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

System 1 is zero state detectable.

System 2

if $u_2 \equiv 0$ and $y_2 \equiv 0 \Rightarrow z_4 \equiv 0$ and $u_2 \equiv 0$. So;

$z_4 \equiv 0 \Rightarrow \dot{z}_4 = 0$ and $u_2 \equiv 0 \Rightarrow z_3 \equiv 0 \Rightarrow \dot{z}_3 = 0$

we found that

if $u_2 \equiv 0$ and $y_2 \equiv 0 \Rightarrow \lim\limits_{t \to \infty} z(t) = 0$, $z = \begin{bmatrix} z_3 \\ z_4 \end{bmatrix}$

System 2 is zero state detectable.

As we proved before in part (c) and stated in Theorem - Asymptotic Stability in Feedback connections:

If both systems are passive and zero state detectable the overall system will be asymptotic stable.

3) a)

System

$$\begin{bmatrix} \dot{q} \\ \dot{v} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} v \\ -g\tan\theta \\ \phi \\ 0 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{m\cos\theta} \\ 0 \\ L \end{bmatrix}}_{g(x)} \cdot u \qquad y = q = h(x)$$

$$dh = [1 \; 0 \; 0 \; 0] \qquad L_g h = dh \cdot g(x) = [1\,0\,0\,0]\begin{bmatrix} 0 \\ \frac{-1}{m\cos\theta} \\ 0 \\ L \end{bmatrix} = 0$$

$$L_f h = dh \cdot f(x) = [1\;0\;0\;0]\begin{bmatrix} v \\ -g\tan\theta \\ \phi \\ 0 \end{bmatrix} = v \qquad dL_f h = [0\;1\;0\;0]$$

$$L_g L_f h = dL_f h \; g(x) = [0\;1\;0\;0]\begin{bmatrix} 0 \\ \frac{-1}{m\cos\theta} \\ 0 \\ L \end{bmatrix} = -\frac{1}{m\cos\theta} \neq 0$$

The system's relative degree is $r = 2$ for $x_3 = \theta \in (-\pi/2, \pi/2)$.

b)

$$z = K(x) := \begin{bmatrix} h(x) \\ L_f h(x) \\ t_1(x) \\ t_2(x) \end{bmatrix} = \begin{bmatrix} q \\ v \\ vL + \frac{\phi}{m\cos\theta} \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 L + \frac{x_4}{mg x_3} \\ x_3 \end{bmatrix}$$

$h(x) = q$

$L_f h(x) = v$

$L_g t_1(x) = L_g t_2(x) = 0$

$$L_g t_1(x) = 0$$

$$\left[\frac{\partial t_1}{\partial x_1} \;\; \frac{\partial t_1}{\partial x_2} \;\; \frac{\partial t_1}{\partial x_3} \;\; \frac{\partial t_1}{\partial x_4}\right] g(x) = 0$$

$$-\frac{\partial t_1}{\partial x_2}\frac{1}{m\cos\theta} + \frac{\partial t_1}{\partial x_4} \cdot L = 0$$

$$\frac{\partial t_1}{\partial x_2} = L \;, \quad \frac{\partial t_1}{\partial x_4} = \frac{1}{m\cos\theta} \rightarrow t_1(x) = vL + \frac{\phi}{m\cos\theta}$$

$$= x(2)L + \frac{x_4}{m\cos x_3}$$

If we choose $t_2(x) = \theta$,

$L_g t_2(x) = 0$ is satisfied. We have to check whether transformed coordinates' Jacobian is nonsingular or not to have diffeomorphism.

$$\frac{\partial K}{\partial x} = \begin{bmatrix} \frac{\partial k_1}{\partial x_1} & \frac{\partial k_1}{\partial x_2} & \frac{\partial k_1}{\partial x_3} & \frac{\partial k_1}{\partial x_4} \\ \frac{\partial k_2}{\partial x_1} & \frac{\partial k_2}{\partial x_2} & \frac{\partial k_2}{\partial x_3} & \frac{\partial k_2}{\partial x_4} \\ \frac{\partial k_3}{\partial x_1} & \frac{\partial k_3}{\partial x_2} & \frac{\partial k_3}{\partial x_3} & \frac{\partial k_3}{\partial x_4} \\ \frac{\partial k_4}{\partial x_1} & \frac{\partial k_4}{\partial x_2} & \frac{\partial k_4}{\partial x_3} & \frac{\partial k_4}{\partial x_4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & L & \frac{-x_4 \sin x_3}{M\cos x_3} & \frac{1}{M\cos x_3} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ is nonsingular for $\theta : (-\pi/2, \pi/2)$

Byrnes - Isidori Canonical Form

$$b(z) = -L_f^2 h = -[0\ 1\ 0\ 0]\, f(x) = -g\tan\theta$$

$$a(z) = L_g L_f h(x) = \frac{-1}{M\cos\theta}\, u$$

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ b(z) + a(z)u \end{bmatrix}$$

$$= \begin{bmatrix} z_2 \\ -g\tan z_4 - \frac{u}{M\cos z_4} \end{bmatrix} //$$

$$z_3 = \eta_1 = \theta$$

$$\begin{bmatrix} \dot\eta_1 \\ \dot\eta_2 \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}\left(\dot v L + \frac{\phi}{M\cos\theta}\right) \\ \frac{d}{dt}(\theta) \end{bmatrix} = \begin{bmatrix} \dot v L + \frac{(\dot\phi M\cos\theta + M\sin\theta\,\dot\theta\,\phi)}{(M\cos\theta)^2} \\ \dot\theta = \phi \end{bmatrix} //$$

$\dot\theta = \phi \longrightarrow$ from z vector we know that

$$z_3 = z_2 L + \frac{\phi}{M\cos z_4} \longrightarrow \phi = (z_3 - z_2 L)\, M\cos z_4$$

$$\dot z_4 = \dot\theta = (z_3 - z_2 L)\, M\cos z_4$$

$$\dot\eta_1 = \dot v L + \frac{\dot\phi}{M\cos\theta} + \frac{M\sin\theta(\dot\theta\,\phi)}{(M\cos\theta)^2} = -gL\tan z_4 - \cancel{\frac{Lu}{M\cos\theta}} + \cancel{\frac{Lu}{M\cos\theta}} + \frac{\dot z_4^{\,2} M\sin z_4}{(M\cos z_4)^2}$$

$$= -gL\tan z_4 + \frac{\dot z_4^{\,2}\sin z_4}{M\cos^2 z_4}$$

$$\begin{bmatrix} \dot\eta_1 \\ \dot\eta_2 \end{bmatrix} = \begin{bmatrix} -gL\tan z_4 + \frac{\dot z_4^{\,2}\sin z_4}{M\cos^2 z_4} \\ (z_3 - z_2 L)\, M\cos z_4 \end{bmatrix} //$$

c) To exactly linearize the nonlinear system between new input signal u and output signal y,

$$y = \hat{x}_1 = \hat{z}_1 \longrightarrow \ddot{y} = \ddot{\hat{x}}_1 = \ddot{\hat{z}}_1 = \dot{\hat{z}}_2 = v$$

(desired value)

$$u = \frac{-b(z) + \ddot{y}}{a(z)}$$

$$= (-g \tan z_4 + v)(M \cos z_4) \leftarrow \text{state feedback}$$

d)

$$u = (-g \tan z_4 + v)(M \cos z_4) = (-g \tan x_3 + v)(M \cos x_3)$$

$$x_2 = v, \quad x_3 = \theta$$

e) We want to stabilize the position of the helicopter at $q = 0$

$$v = \dot{\hat{z}}_2 = K_p(q_{desired} - z_1) + K_d(v_{desired} - z_2)$$

So, the u has to be applied:

$$u = \frac{-b(z) + v}{a(z)} = (-g \tan z_4 + v)(M \cos z_4)$$

So, our $\dot{z} = A z$ → will be Hurwitz

f) If $\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix}$ (the internal dynamics) are asymptotically stable (which is can be found by generating appropriate Lyapunov function) can the system will be stable because of the linearization & can be controlled.

g) As indicated in part-b or $\frac{\partial K}{\partial x}$ is nonsingular and does not depend on q values just like equation of motion. As a result, we can generate a controller similar to part-e, only $q_{desired}$ value will be equal to 5.