Introducing the Fractional Fourier Transform

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1 Introduction: Eigenfunction Analysis

We showed in class that the Fourier transform preserves the L_2 norm of functions it acts on.[1] This means we can think of the Fourier transform as a map from $L_2[-\infty, \infty]$ to itself. Indeed, we even talked about the Fourier transform as a basis change (albeit, the bases were of distributions, so perhaps our space is more accurately that of "well tempered" distributions on L_2, which is a superset of functions in L_2, but this is a digression which I don't fully understand). Now, this means we can compose the Fourier transform with itself. Thus we can define $\mathcal{F}^n\{f(t)\} = F_n(u)$ by

$$\mathcal{F}^n\{f(t)\} \equiv \underbrace{(\mathcal{F} \circ \mathcal{F} \circ \cdots \circ \mathcal{F})}_{n}\{f(t)\} = F_n(u)$$

The idea of the fractional Fourier transform is to generalize this notion to non-integer n. When we do this, we want certain properties to hold, such as the additive property. That is, we demand that

$$\mathcal{F}^\alpha \circ \mathcal{F}^\beta = \mathcal{F}^{\alpha+\beta}$$

If we wanted to calculate the fractional power of a matrix, we would diagonalize it, and then raise the eigenvalues to said fractional power.[2] The same can be done with the Fourier transform. As has been established, the Fourier transform has an orthogonal (countable) basis of eigenfunctions $\{\Psi_n(t)\}$, where $\mathcal{F}\{\Psi_n(t)\} = \lambda_n \Psi_n(u)$.[1] Computing the fractional Fourier transform (henceforth referred to as the FRFT) of one of these basis vectors is trivial. We have that

$$\mathcal{F}^\alpha\{\Psi_n\} = \lambda_n^\alpha \Psi_n$$

We can essentially think of this as our definition of the FRFT, since defining a linear transformation on the basis of a space defines the transformation on the whole space (with

[1] Actually, we showed that it didn't, but the normalized Fourier transform does, and that is what we consider here

[2] If the diagonal decomposition is not unique, then there is quite the issue of finding the best diagonal decomposition, but that's an issue for another paper.

this definition, it is easy to see that our additive property holds). Now, to calculate the FRFT of an arbitrary function $f(t)$, we must first express it in terms of the eigenbasis,

$$f = \sum_{1}^{\infty} A_n \Psi_n$$

and then use linearity to apply the FRFT to each term:

$$\mathcal{F}^{\alpha}\{f\}(u) = \sum_{n=1}^{\infty} A_n \lambda_n^{\alpha} \Psi_n(u)$$

Since the eigenbasis is orthogonal, we can compute the basis transformation very simply with just the inner product:

$$A_n = \int_{-\infty}^{\infty} f(t)\Psi_n(t)dt$$

Now we have

$$\begin{aligned}
\mathcal{F}^{\alpha}\{f\}(u) &= \sum_{n=1}^{\infty} A_n \lambda_n^{\alpha} \Psi_n(u) && (1)\\
&= \sum_{n=1}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\Psi_n(t)dt \right) \lambda_n^{\alpha} \Psi_n(u) && (2)\\
&= \sum_{n=1}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\lambda_n^{\alpha}\Psi_n(t)\Psi_n(u)dt \right) && (3)\\
&= \int_{t=-\infty}^{\infty} f(t) \left[\sum_{n=1}^{\infty} \lambda_n^{\alpha}\Psi_n(t)\Psi_n(u) \right] dt && (4)
\end{aligned}$$

Then, if we define $B_{\alpha}(t,u) = \sum_{n=1}^{\infty} \lambda_n^{\alpha}\Psi_n(t)\Psi_n(u)$, we have expressed the FRFT as an integral transformation:

$$\mathcal{F}^{\alpha}\{f(t)\}(u) = \int_{-\infty}^{\infty} f(t)B_{\alpha}(t,u)dt$$

Thus, finding an elementary expression for the FRFT comes down to finding these functions $B_{\alpha}(t,u)$. This can be done since the eigenfunctions $\Psi_n(t)$ are well known. However, the computation is not terribly fun.[1] However, without doing any computation, we know certain things about these functions and the FRFT:

1. $B_{\alpha}(t,u)$ is symmetric with respect to u and t.
2. For $\alpha = \pi/2$ we must have the Fourier transform. Thus $B_{\pi/2}(t,u) = e^{-iut}$.
3. For $\alpha = 0$ we must have the identity transform. Thus $B_0(t,u) = \delta(t-u)$
4. The FRFT preserves L_2 norm.

2 Constructing The Kernel

The family of function $B_\alpha(t,u)$ entirely determines the transformation. They are called the transformation kernel. To visualize these functions, we move to the time-frequency plane.[3] The kernel of the identity transformation is the set of delta functions. We can think of $\delta(t-t_0)$ as a vertical line in this space that intersects the t axis at t_0. At time $t = t_0$, the function goes crazy and has every mode equally represented. Everywhere else it is zero. On the other hand, the kernel functions for the Fourier transform are the functions $e^{i\omega t}$. These functions are horizontal lines in the frequency-time plane, since their frequency is constant through time. Thus to go from one transformation kernel to the other, is a rotation of $\pi/2$ in the time-frequency plane. Thus, to define the fractional Fourier transform, we can specify the kernel functions as a rotation of the vertical lines through some arbitrary angle α in the time-frequency plane.

Before we proceed, we should make concrete what we mean by frequency of a function. For a function of the form $Ae^{if(t)}$, we can think of the "frequency at time t" as $f'(t)$: the rate at which the phase is changing with time. Now lets consider a generic kernel function for the FRFT. It is any Euclidean line in the phase-plane, which we can consider as a rotation of a vertical line of distance u from the origin through angle α. The slope of the line is $\tan(\alpha - \pi/2) = -\cot(\alpha)$. Now through basic trigonometry (or just by definition), we find that the w-intercept, is at $u\csc(\alpha)$. Thus, we have a function which at time $t = 0$ has a frequency of $u\csc(\alpha)$. Thus, $f'(0) = u\csc(\alpha)$, but the frequency changes at a rate of $-\cot(\alpha)$ with respect to time. Thus, if we write our function as $Ae^{if(t)}$, then we know that

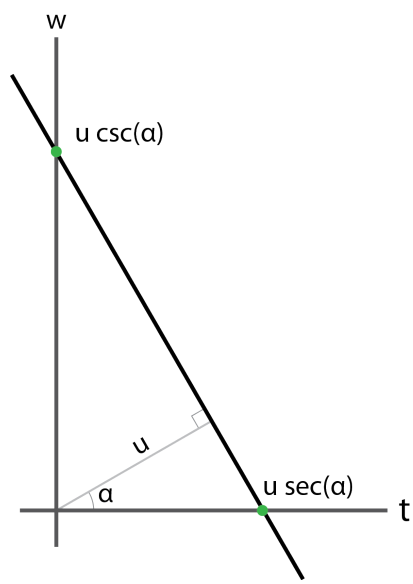


Figure 1: The phase-plane representation of a kernel function of $\mathcal{F}^\alpha$.

$$f'(0) = u\csc(\alpha) \tag{5}$$
$$f''(t) = -\cot(\alpha) \tag{6}$$

The simplest solution here is that $f(t) = u\csc(\alpha)t - \cot(\alpha)t^2/2$. Thus, we have that our

[3] A rigorous treatment of this plane is pretty subtle, and we will not give one here, but we can think of it as a kind of short-time Fourier transform

kernel functions are of the form:

$$B_\alpha(t,u) = C(u,\alpha)e^{i\left(u\csc(\alpha)t - \cot(\alpha)t^2/2\right)}$$

Where $C(u,\alpha)$ is a normalization constant. We need to figure out what this constant is. We know from our definition of $B_\alpha(t,u)$ that it is symmetric in t and u. To make it so, without messing with our conditions on $f(t)$, we must have a factor of $e^{i\cot(\alpha)u^2/2}$, which takes care of the u portion of our normalization constant and we have

$$B_\alpha(t,u) = C(\alpha)e^{i\left(u\csc(\alpha)t - \cot(\alpha)(t^2+u^2)/2\right)}$$

Now we must calculate what $C(\alpha)$ (henceforth just C) is. The strategy for finding this is to assume $C = 1$, and then see how much the norm of a function changes by transforming with respect to this kernel. That is,

$$\left(\frac{1}{C}\mathcal{F}^\alpha\{f(t)\}\middle|\,\frac{1}{C}\mathcal{F}^\alpha\{f(t)\}\right) = \frac{1}{|C|^2}(f|f)$$

where $(\cdot|\cdot)$ is the inner product.[4] So, we have,

$$\frac{1}{|C|^2}(f|f) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{i\left(u\csc(\alpha)t-\cot(\alpha)(t^2+u^2)/2\right)}dt\right]\overline{\left[\int_{-\infty}^{\infty}\bar{f}(\tau)e^{i(u\csc(\alpha)\tau-\cot(\alpha)(\tau^2+u^2)/2)}d\tau\right]}du \tag{7}$$

$$= \int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty}\bar{f}(\tau)\int_{-\infty}^{\infty} e^{i(ut\csc(\alpha)-cot(\alpha)(t^2+u^2)/2)}e^{-i(u\tau\csc(\alpha)-cot(\alpha)(\tau^2+u^2)/2)}dud\tau dt \tag{8}$$

$$= \int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty}\bar{f}(\tau)\int_{-\infty}^{\infty} e^{i(u(t-\tau)\csc(\alpha)-cot(\alpha)(t^2-\tau^2)/2)}dud\tau dt \tag{9}$$

$$= \int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty}\bar{f}(\tau)e^{-cot(\alpha)(t^2-\tau^2)/2)}\int_{-\infty}^{\infty} e^{iu(t-\tau)\csc(\alpha)}dud\tau dt \tag{10}$$

Now, using the fact that

$$\int_{-\infty}^{\infty} e^{i\omega t}d\omega = 2\pi\delta(t)$$

[4]The standard inner product is define $(f|g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$. Because of this conjugation, we uncover the magnitude of C this way, but not the phase.

we simply make the substitution $\omega = u\csc(\alpha) \implies du = d\omega sin(\alpha)$ and obtain

$$\frac{1}{|C|^2}(f|f) = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} \bar{f}(\tau) e^{-cot(\alpha)(t^2-\tau^2)/2)} 2\pi\sin(\alpha)\delta(t-\tau)d\tau dt \tag{11}$$

$$= 2\pi\sin(\alpha)\int_{-\infty}^{\infty} f(t)\bar{f}(t)dt \tag{12}$$

$$= 2\pi\sin(\alpha)(f|f) \tag{13}$$

$$|C| = \sqrt{\frac{1}{2\pi\sin(\alpha)}} \tag{14}$$

Now, all that remains is to find the phase of C. We know that as $\alpha \to 0$ we want $B_\alpha(t,u)$ to approach a delta function. To do this we will exploit the "well known"[5] fact that

$$\lim_{\epsilon\to 0}\sqrt{\frac{1}{2\pi i\epsilon}}e^{-\frac{i}{2\epsilon}t^2} = \delta(t)$$

Now, as $\alpha \to 0$ we have that $\csc(\alpha) = \cot(\alpha) = 1/\alpha$, and of course $\sin(\alpha) = \alpha$. Thus,

$$\lim_{\alpha\to 0}\sqrt{\frac{1}{2\pi\sin(\alpha)}}e^{iut\csc(\alpha)-i\cot(\alpha)\left(\frac{t^2}{2}+\frac{u^2}{2}\right)} = \lim_{\alpha\to 0}\sqrt{\frac{1}{2\pi\alpha}}e^{\frac{iut}{\alpha}-i\frac{t^2}{2\alpha}-i\frac{u^2}{2\alpha}} \tag{15}$$

$$= \lim_{\alpha\to 0}\sqrt{\frac{1}{2\pi\alpha}}e^{\frac{i}{2\alpha}(2ut-t^2-u^2)} \tag{16}$$

$$= \lim_{\alpha\to 0}\sqrt{\frac{1}{2\pi\alpha}}e^{-\frac{i}{2\alpha}(u-t)^2} \tag{17}$$

$$= \sqrt{i}\delta(u-t) \tag{18}$$

We need to divide by $\sqrt{i} = e^{pi/4}$ for $\alpha = 0$. However, we have exactly the phase we want for $\alpha = \pi/2$. Thus, we need the phase adjustment to be a function of α that is $-\pi/4$ for $\alpha = 0$ and 0 for $\alpha = \pi/2$. To maintain the additive property of the FRFT, this must happen linearly. Thus, our phase adjustment is $e^{i(\alpha/2-\pi/4)} = \sqrt{e^{i(\alpha-\pi/2)}}$. Thus, we finally have that

$$C = \sqrt{\frac{e^{i(\alpha-\pi/2)}}{2\pi\sin(\alpha)}} = \sqrt{\frac{1-i\cot(\alpha)}{2\pi}}$$

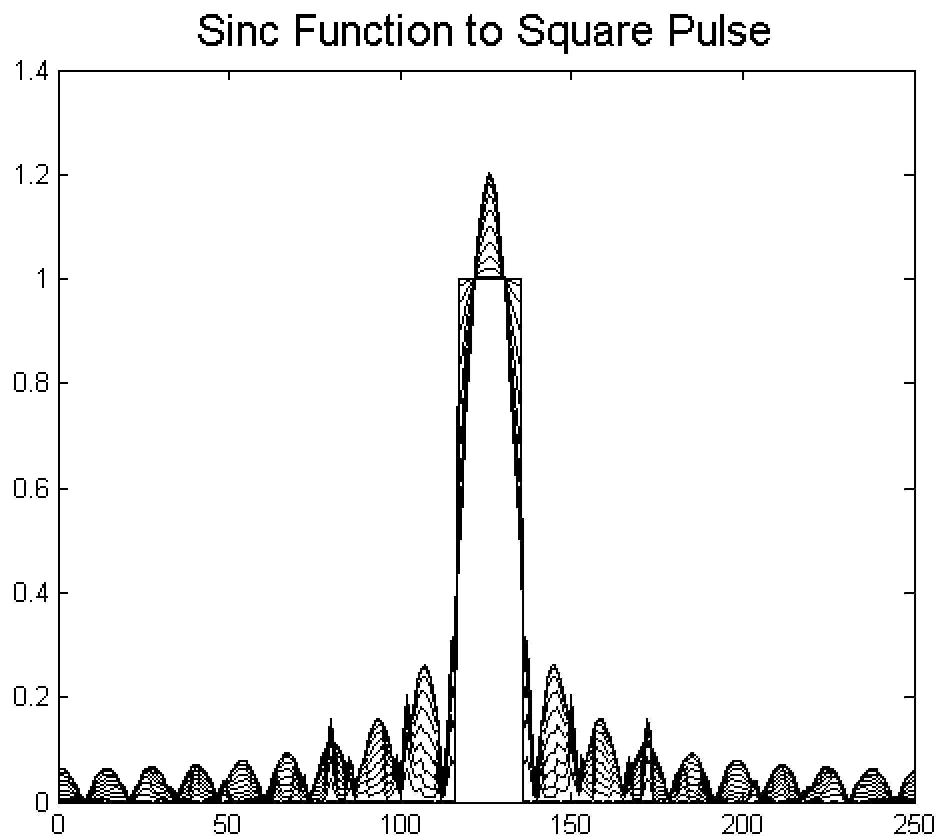
And we finally have our FRFT:

$$\mathcal{F}^\alpha\{f(t)\}(u) = \sqrt{\frac{1-\cot(\alpha)}{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i(ut\csc(\alpha)-\cot(\alpha)(t^2+u^2)/2)}dt$$

[5] Hah!

3 The Discrete Fractional Fourier Transform (briefly)

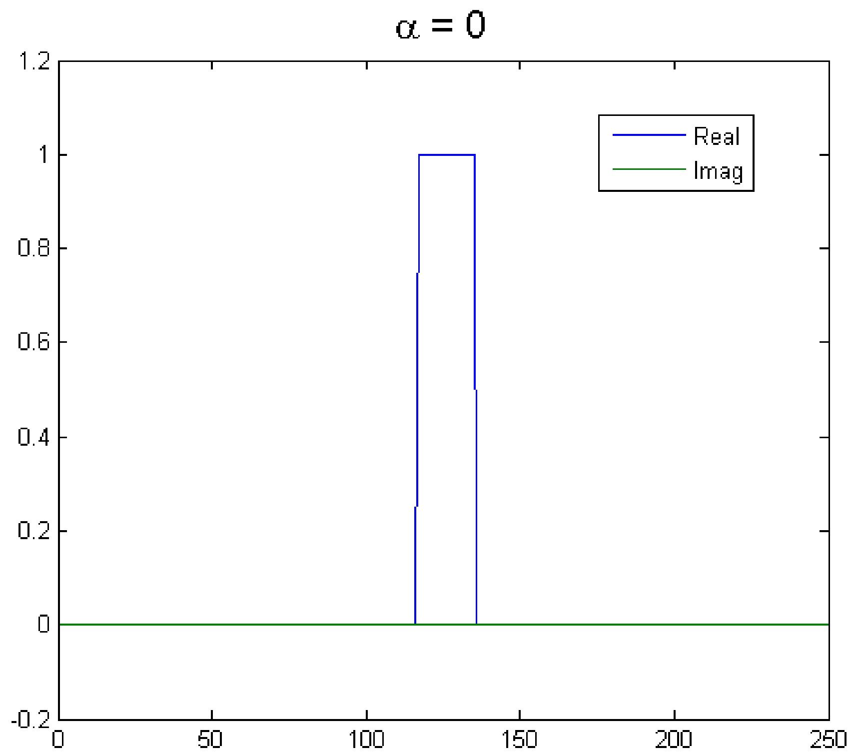
The main difficulty with the discrete case is that the DFT matrix has many different eigen-decompositions. Choosing an appropriate orthogonal eigenbasis is the entirety of [2]. Now essentially using their techniques, I wrote a matlab script to continuously vary α and plot the result in real-time. I plot some the more fun results here and throughout the document.



4 Derivative With Respect to α

Given our integral transform

$$\mathcal{F}^{\alpha}\{f(t)\} \int_{-\infty}^{\infty} f(t) B_{\alpha}(t, u) dt$$



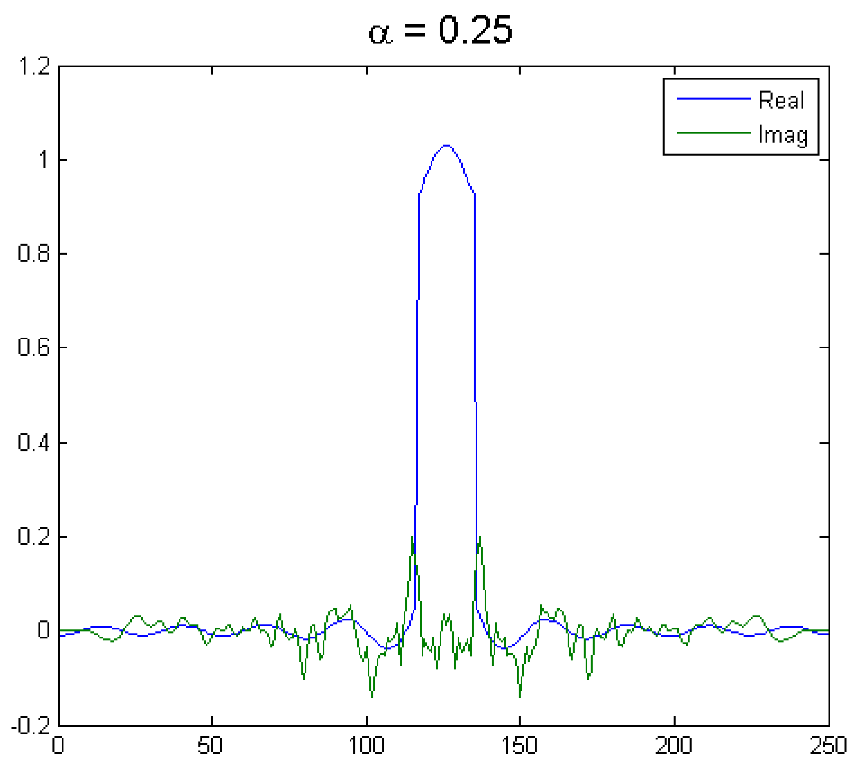
we wish to investigate how the output functions changes with respect to the parameter α, for a fixed input function $f(t)$. Instead of writing $B_\alpha(t,u)$ we could just as well write $B(\alpha,t,u)$. Now, assuming convergence of the FRFT is well behaved, we can combine indefinite integrals, yielding

$$\frac{d}{d\alpha}\mathcal{F}^{\alpha}\{f(t)\} = \lim_{\epsilon\to 0}\frac{\int_{-\infty}^{\infty} f(t)B(\alpha+\epsilon,t,u)dt - \int_{-\infty}^{\infty} f(t)B(\alpha,t,u)dt}{|\epsilon|} \tag{19}$$

$$= \lim_{\epsilon\to 0}\frac{\int_{-\infty}^{\infty} f(t)(B(\alpha+\epsilon,t,u) - B(\alpha,t,u))dt}{|\epsilon|} \tag{20}$$

$$= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t)\frac{(B(\alpha+\epsilon,t,u) - B(\alpha,t,u))}{|\epsilon|}dt \tag{21}$$

$$= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t)\frac{\partial}{\partial\alpha}B(\alpha,t,u)dt \tag{22}$$



Now, letting $A(\alpha) = \sqrt{\frac{1-i\cot(\alpha)}{2\pi}}$, we have

$$\frac{\partial}{\partial\alpha}B(\alpha,t,u) = \frac{\partial}{\partial\alpha}A(\alpha)e^{-iut\csc(\alpha)+i\left(\frac{u^2}{2}+\frac{t^2}{2}\right)\cot(\alpha)} \tag{23}$$

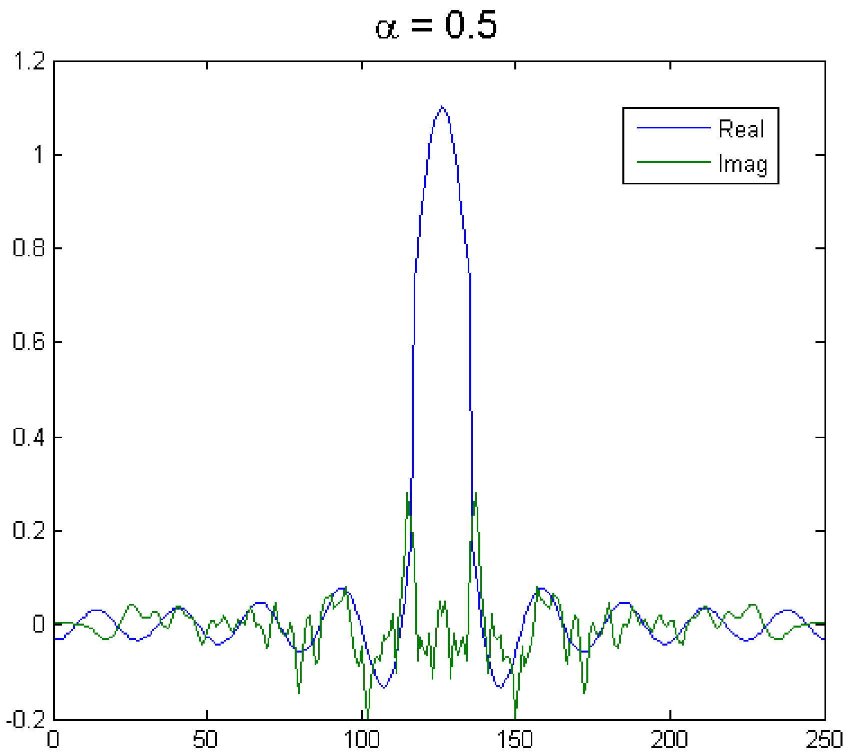
$$= A(\alpha)\left(iut\cot(\alpha)\csc(\alpha) - i\left(\frac{u^2}{2}+\frac{t^2}{2}\right)\csc(\alpha)^2\right)e^{junk} + A'(\alpha)e^{junk} \tag{24}$$

$$= \left(iut\cot(\alpha)\csc(\alpha) - i\left(\frac{u^2}{2}+\frac{t^2}{2}\right)\csc(\alpha)^2 + \frac{A'(\alpha)}{A(\alpha)}\right)B(\alpha,t,u) \tag{25}$$

$$\tag{26}$$

Furthermore, $A'(\alpha) = \frac{i}{2\sqrt{2\pi}}(1-i\cot(\alpha))^{-1/2}\csc(\alpha)^2$ so

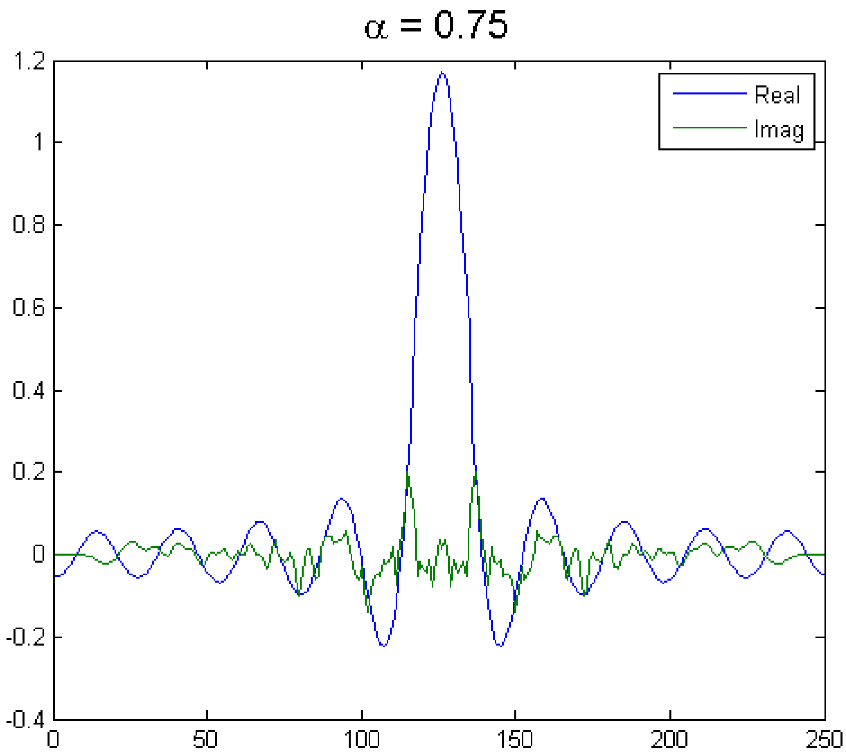
$$\frac{A'(\alpha)}{A(\alpha)} = i\frac{1}{2}\frac{\csc(\alpha)^2}{1-\cot(\alpha)}$$



Now, we collect terms and get

$$\frac{d}{d\alpha}\mathcal{F}^{\alpha}\{f(t)\} = \left(\frac{i}{2}\frac{\csc(\alpha)^2}{1-\cot(\alpha)} - i\frac{u^2}{2}\csc(\alpha)^2\right)\mathcal{F}^{\alpha}\{f(t)\} \\ + iu\cot(\alpha)\csc(\alpha)\mathcal{F}^{\alpha}\{tf(t)\} - i\csc(\alpha)^2\mathcal{F}^{\alpha}\{f(t)t^2/2\} \quad (27)$$

This is obviously horrendously messy, but there are still some interesting takeaways. First, it is generally true that the coefficients on the FRFT's in that expression diverge for $\alpha = 0$. Thus, while $\mathcal{F}^{\alpha}$ is continuous in α (see [3]) this result shows that it is often not differentiable at $\alpha = 0$. Furthermore, we express the derivative with respect to α in terms of the transform of $f(t)$, $tf(t)$ and $t^2f(t)/2$. The reason this might be interesting is that it implies that while a function $f(t)$ my be Fourier transformable, for the derivative to be defined, the the function $t^2f(t)$ must also be Fourier transformable, which is certainly not the case in general, since the Fourier transform of t^2 is not convergent but the Fourier transform of $1/2$ is. Thus, what I have derived implies that there are functions, $f(t) = \textit{constant}$ being a prime example, for which the FRFT is nowhere differentiable, despite being continuous.

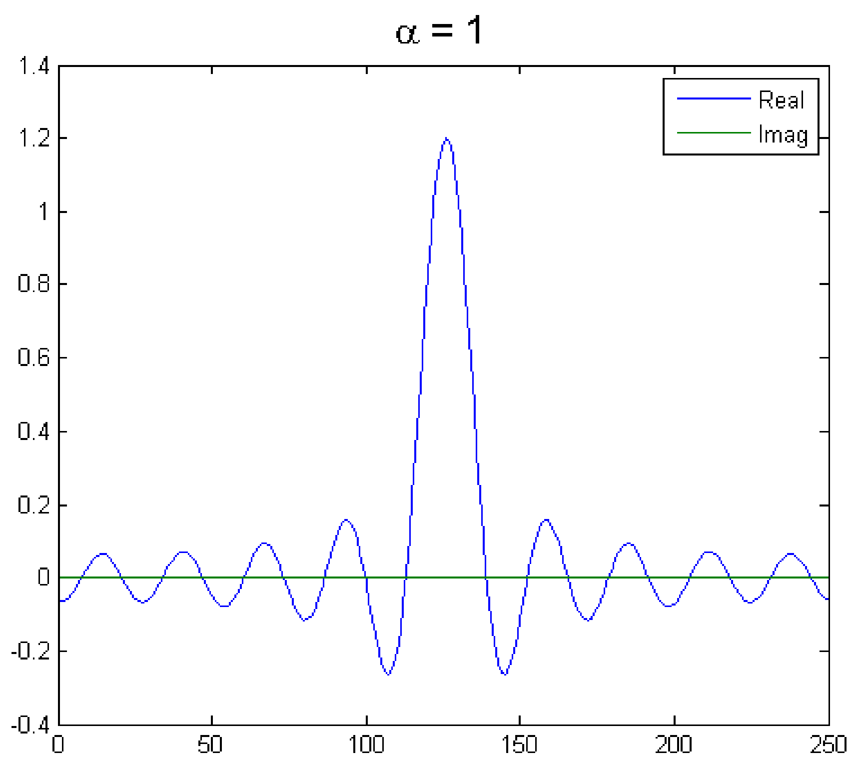


References

[1] Mark Reeder, *Eigen Functions of the Fourier Transform.* 2013

[2] agatay Candan, Student Member, IEEE, M. Alper Kutay, Member, IEEE, and Haldun M. Ozaktas, *The Discrete Fractional Fourier Transform* IEEE TRANSACTIONS ON SIGNAL PROCESSING, VOL. 48, NO. 5, MAY 2000

[3] Ozaktas, Kutay, Mendlovic *Introduction to the Fractional Fourier Transform and Its Applications*



α = 1
Real
Imag
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
50
100
150
200
250