Eigenfunctions of the Fourier transform

Mark Reeder

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1 Fourier transform of $e^{-(x/a)^2}$

The (complex) Fourier transform $\mathcal{F}[f]$ of a function $f(x)$ is defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx.$$

We first derive the Fourier transform

$$\mathcal{F}\left[e^{-(x/a)^2}\right] = \frac{a}{\sqrt{2}}\cdot e^{-(a\omega/2)^2}, \tag{1}$$

where $a > 0$ is a constant. Regard this transform as a function of ω:

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x/a)^2}\cdot e^{-i\omega x}\, dx,$$

and differentiate:

$$\frac{d}{d\omega}F(\omega) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{i}\int_{-\infty}^{\infty} xe^{-(x/a)^2}\cdot e^{-i\omega x}\, dx,$$

and now integrate by parts, with

$$u = e^{-i\omega x}, \qquad dv = xe^{-(x/a)^2}, \qquad du = -i\omega e^{-i\omega x}, \qquad v = -(a^2/2)e^{-(x/a)^2},$$

to get

$$\frac{d}{d\omega}F(\omega) = \frac{1}{\sqrt{2\pi}}\cdot\frac{-a^2\omega}{2}\cdot\int_{-\infty}^{\infty} e^{-(x/a)^2}\cdot e^{-i\omega x}\, dx = \frac{-a^2}{2}\cdot\omega F(\omega).$$

Solving this differential equation for $F(\omega)$ we get

$$F(\omega) = F(0)\cdot e^{-(a\omega/2)^2}.$$

So it remains to determine $F(0)$. This is essentially the Gaussian integral:

$$F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x/a)^2}\, dx,$$

which can be computed as follows. The trick is to compute the square of $F(0)$ and use polar coordinates:

$$F(0)^2 = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x/a)^2}\, dx\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x/a)^2}\cdot e^{-(y/a)^2}\, dx\, dy = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-(r/a)^2} r\, dr\, d\theta = \frac{1}{2\pi}\cdot 2\pi\cdot\frac{a^2}{2} = \frac{a^2}{2},$$

so we get

$$F(0) = \frac{a}{\sqrt{2}},$$

hence

$$F(\omega) = \frac{a}{\sqrt{2}} \cdot e^{-(a\omega/2)^2},$$

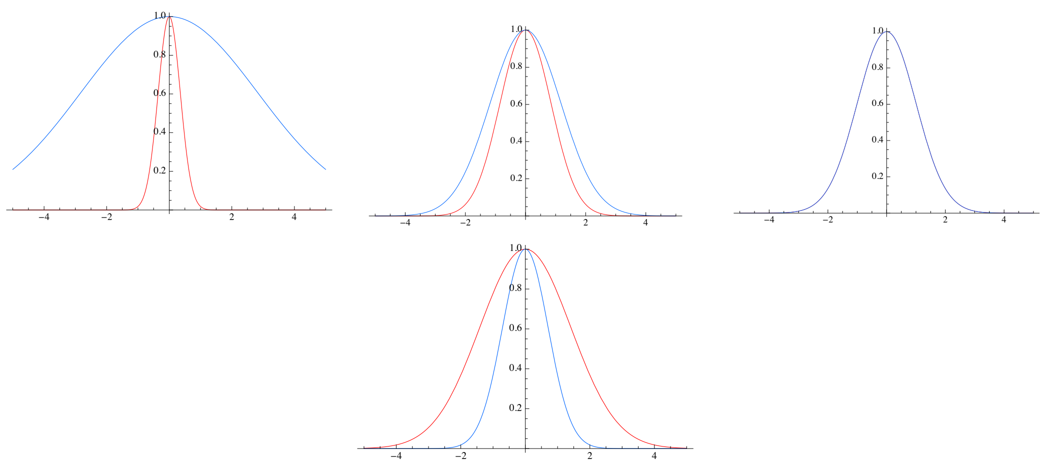
as claimed in (1).

If $a = \sqrt{2}$ then

$$F(\omega) = \mathcal{F}\left[e^{-x^2/2}\right] = e^{-\omega^2/2},$$

so the function $e^{-x^2/2}$ is its own Fourier transform.

In general, a smaller a gives a bigger spike in the graph of $f(x)$ and a more spread-out graph of $F(\omega)$. And vice-versa. Here are the superimposed graphs of $f(x) = e^{-(x/a)^2}$ (red) and $F(\omega) = e^{-(a\omega/2)^2}$ (blue) for $a = .5, 1.2, \sqrt{2}, 2$.



2 The differential operator viewpoint

The calculation of $\mathcal{F}[e^{-x^2}]$ in the first section is illuminated by a more algebraic point of view. First, let us modify the definition of $\mathcal{F}$ so that it takes functions of x to functions of the same variable x:

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-ixy}\, dy.$$

Now the derivative formula becomes

$$\mathcal{F}[f'(x)] = ixf(x). \tag{2}$$

On the other hand if we first apply $\mathcal{F}$ and then differentiate, we get

$$\frac{d}{dx}\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-iy)f(y)e^{-ixy}\, dy = -i\mathcal{F}[xf],$$

so

$$\mathcal{F}[f]' = -i\mathcal{F}[xf] \tag{3}$$

Consider the differential operator

$$\mathcal{D} = \frac{d}{dx} + x, \qquad \mathcal{D}[f] = f' + xf.$$

Using formulas (2) and (3), we have

$$\mathcal{F}[\mathcal{D}[f]] = \mathcal{F}[f' + xf] = ix\mathcal{F}[f] + i\mathcal{F}[f]' = i\mathcal{D}[\mathcal{F}[f]].$$

Thus $\mathcal{F}$ and $\mathcal{D}$ are linear operators such that

$$\mathcal{F}\mathcal{D} = i\mathcal{D}\mathcal{F}. \tag{4}$$

If f is a solution of the differential equation $\mathcal{D}[f] = 0$, then (4) implies that $\mathcal{F}[f]$ is also a solution. However, every solution of $\mathcal{D}[f] = 0$ is of the form

$$f(x) = Ce^{-x^2/2},$$

where $C = f(0)$ is a constant. Therefore there is a constant C such that

$$\mathcal{F}[e^{-x^2/2}] = Ce^{-x^2/2}.$$

To find C we evaluate at 0:

$$\mathcal{F}[e^{-x^2/2}](0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du = 1.$$

Thus, the equality

$$\mathcal{F}[e^{-x^2/2}] = e^{-x^2/2} \tag{5}$$

follows from commutation formula (4) and the Gaussian integral.

3 Eigenvectors of the Fourier Transform

In terms of linear algebra, equation (5) asserts that $e^{-x^2/2}$ is an eigenvector of the operator $\mathcal{F}$, with eigenvalue $= 1$, on the complex vector space V of functions of the form $p(x)e^{-x^2/2}$, where $p(x)$ is a polynomial. The vector space V is infinite dimensional, but note that V is a union of finite dimensional subspaces

$$V = \bigcup_{n=0}^{\infty} V_n,$$

where $V_n = \{p(x)e^{-x^2/2} : \deg p(x) \leq n\}$ is a subspace of dimension

$$\dim V_n = n + 1.$$

Note that

$$\mathcal{D}[p(x)e^{-x^2/2}] = p'(x)e^{-x^2/2},$$

so $V_n = \ker \mathcal{D}^{n+1}$ is the subspace of functions in V killed by $\mathcal{D}^{n+1}$.

The function

$$\frac{d^n}{dx^n} e^{-x^2/2}$$

belongs to V_n and **Hermite polynomials** $He_n(x)$ are defined by

$$He_n(x)e^{-x^2/2} = (-1)^n \frac{d^n}{dx^n} e^{-x^2/2}. \tag{6}$$

On the homework, you showed that

$$\mathcal{F}[x^n e^{-x^2/2}] = (-i)^n He_n(x)e^{-x^2/2} \tag{7}$$

so $\mathcal{F}[V_n] = V_n$ for each n. We have found the eigenvector $e^{-x^2/2} \in V_0$; we now find a basis of eigenvectors for $\mathcal{F}$ in V_n for $n \geq 0$.

Define another operator $\iota : V \to V$ by $\iota[f](x) = f(-x)$. The Fourier inversion theorem becomes the operator equation

$$\mathcal{F}^2 = \iota.$$

It follows that $\mathcal{F}^4 = I$ and the eigenvalues of $\mathcal{F}$ on V are powers of $i = \sqrt{-1}$. More precisely, $\mathcal{F}$ has order two on the even functions in V and order four on the odd functions in V. So the functions with $\mathcal{F}$-eigenvalues ± 1 are even and those with $\mathcal{F}$-eigenvalues $\pm i$ are odd.

In fact, $He_n = x^n +$ terms of lower degree, so equation (7) shows that if $\mathcal{F}$ has an eigenvector ψ which lies in V_n but not in V_{n-1}, then the eigenvalue must be $(-i)^n$. Hence for each $n \geq 0$ there is a unique (up to scalar) function $\psi_n(x) \in V_n$ of the form

$$\psi_n(x) = p_n(x)e^{-x^2/2},$$

where $p_n(x)$ is a polynomial of degree n, such that

$$\mathcal{F}[\psi_n] = (-i)^n \psi_n.$$

For example, we have

$$\psi_0 = e^{-x^2/2}, \qquad \psi_1 = 2xe^{-x^2/2}, \qquad \psi_2 = (4x^2-2)e^{-x^2/2}, \qquad \psi_3 = (8x^3-12x)e^{-x^2/2}, \qquad \psi_4 = (16x^4-48x^2+12)e^{-x^2/2}. \tag{8}$$

To find ψ_n in general we consider a new differential operator

$$\mathcal{D}_2 = \frac{d^2}{dx^2} - x^2.$$

Using equations (2) and (3) we find that $\mathcal{D}_2$ commutes with $\mathcal{F}$:

$$\mathcal{D}_2\mathcal{F} = \mathcal{F}\mathcal{D}_2.$$

It follows that $\mathcal{F}$ preserves each eigenspace of $\mathcal{D}_2$. That is, if $\mathcal{D}_2[\psi] = \lambda\psi$ for some $\lambda \in \mathbb{C}$ then $\mathcal{D}_2\mathcal{F}[\psi] = \lambda\mathcal{F}[\psi]$. We take $\lambda = -(2n+1)$, and consider the equation $\mathcal{D}_2\psi = -(2n+1)\psi$, or in other words,

$$\psi'' + (2n+1-x^2)\psi = 0. \tag{9}$$

This has a two dimensional solution space, but only one solution (up to scalar) lies in V. For $\psi(x) = p(x)e^{-x^2/2}$ is a solution of (9) in V exactly when

$$p'' - 2xp' + 2np = 0. \tag{10}$$

Writing $p = \sum c_k x^k$, equation (10) is equivalent to the recurrence formula

$$c_{k+2} = \frac{2(k-n)}{(k+2)(k+1)} c_k.$$

So (10) has a unique polynomial solution $p(x)$ of degree n. Taking $c_n = 2^n$ (to make the coefficients integers) $p(x)$ is given by the Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{n!}{(n-2m)!} (2x)^{n-2k}.$$

This is slightly different than the Hermite polynomial He_n defined in (6) above. The two kinds of Hermite polynomials are related by:

$$He_n(x) = 2^{-n/2} H_n(x/\sqrt{2}).$$

For example, the polynomials $H_0, \ldots, H_4$ are the coefficients of $e^{-x^2/2}$ in the list (8).

Now, since $\mathcal{F}$ preserves the one-dimensional space of solutions of (9) in V, we have

$$\mathcal{F}[H_n(x)e^{-x^2/2}] = \lambda H_n(x)e^{-x^2/2}$$

for some constant λ. But then $\lambda = (-i)^n$ because it is an eigenvalue of $\mathcal{F}$ on a function in V_n which is not in V_{n-1}. Hence

$$\psi_n(x) = H_n(x)e^{-x^2/2} = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}$$

is the desired eigenfunction of $\mathcal{F}$ and $\{\psi_0, \psi_1, \ldots, \psi_n\}$ is a basis of V_n consisting of $\mathcal{F}$-eigenvectors.