A Unified Framework for the Fractional Fourier Transform

Gianfranco Cariolaro, *Member, IEEE*, Tomaso Erseghe, Peter Kraniauskas, and Nicola Laurenti

***Abstract*—The paper investigates the possibility for giving a general definition of the fractional Fourier transform (FRT) for all signal classes [one-dimensional (1-D) and multidimensional, continuous and discrete, periodic and aperiodic]. Since the definition is based on the eigenfunctions of the ordinary Fourier transform (FT), the preliminary conditions is that the signal domain/periodicity be the same as the FT domain/periodicity. Within these classes, a general FRT definition is formulated, and the FRT properties are established. In addition, the multiplicity (which is intrinsic in a fractional operator) is clearly developed. The general definition is checked in the case in which the FRT is presently available and, moreover, to establish the FRT in new classes of signals.**

***Index Terms*—Eigenfunctions, Fourier transform, fractional Fourier transform, groups, multiplicative groups.**

NOMENCLATURE

$\mathbb{R}$	Field of real numbers.
$\mathbb{C}$	Field of complex numbers.
$\mathbb{Z}$	Ring of integer numbers.
$\mathbb{N}_0$	Set of natural numbers (including 0).
$\lfloor x \rfloor$	Largest integer less than or equal to x.
$(k)_N$	Non-negative remainder of k divided by N.
FT	(Ordinary) Fourier transform.
FRT	Fractional Fourier transform.
CFRT	Chirp-type fractional Fourier transform.
WFRT	Weighted-type fractional Fourier transform.
GS	Generating sequence.

I. INTRODUCTION

THE FRACTIONAL Fourier transform (FRT) emerged [1], [2] as an extension of the Fourier transform, which, here, is referred to as the *ordinary* FT and is one of the most commonly used tools in signal processing. The FRT has recently been introduced in bulk optics [3]–[5], as well as in fiber optics [6], [7], as a fundamental tool for optical information processing.

This paper investigates the possibilities for giving a general definition of the FRT that is valid for all signal classes. We start by inspecting the available definitions for both the ordinary FT and the FRT. In the theory of one-dimensional (1-D) signals, we encounter four classes:

1) continuous-time aperiodic signals;
2) continuous-time periodic signals;
3) discrete-time aperiodic signals;
4) discrete-time periodic signals.

It is well known that the FT is defined for all four classes. In contrast, the definitions proposed for the FRT are limited to classes 1) and 4) because, for these two classes, the signal and its FT have the same domain/periodicity format, that is, the time domain and the frequency domain are identical. Considering the general case, we conclude that a full answer to this investigation can be found in the Topology of Groups, where signals and their FT's (and possibly their FRT's) are defined on groups.

Having established "where" the FRT can be defined, we investigate "how" it can be done. Our general approach is based on *eigenfunctions* of the ordinary FT, which is the same approach Namias [1] used to introduce the FRT for continuous-time signals. It not only permits us to define the FRT for all those signal classes in which it has a meaning but to clearly state the multiplicity of FRT definitions within a same class as well. We will arrive at the customary definition, which, here, is called the *chirp* FRT, as well as at another FRT form originally derived in connection with the DFT [8], [9], which, here, is called *weighted* FRT, as particular cases.

The paper is organized as follows. In Section II, we examine the available definitions of the ordinary FT to then investigate in Section III the classes of signals in which an FRT definition is allowed. In Section IV, we give the general definition of the FRT using the eigenfunction approach and, in Section V, discuss FRT multiplicity. In the final sections, we discuss a few special cases of the FRT as application examples.

The multiplicity of FRT definitions is examined in greater detail and unified in separate papers: in [10] for continuous-time signals and in [11] for periodic discrete-time signals.

We wish to point out that the Group Topology results used in this paper are expressed in a language suitable for the nonspecialist reader.

II. SIGNAL CLASSES THAT ADMIT AN FT DEFINITION

The unified formulation of the ordinary FT is a preliminary to that of the FRT.

A. Customary Definitions of the FT

The ordinary FT is well defined for the four classes of 1-D signals, where it takes different but equivalent forms. The four

Manuscript received August 15, 1997; revised April 7, 1998. The associate editor coordinating the review of this paper and approving it for publication was Prof. Moeness G. Amin.

G. Cariolaro and N. Laurenti are with the Dipartimento di Elettronica ed Informatica, Università di Padova, Padova, Italy.

T. Erseghe is with Snell and Wilcox Ltd., Petersfield, U.K.

P. Kraniauskas is an independent consultant in Southampton, U.K.

Publisher Item Identifier S 1053-587X(98)08705-4.

1053–587X/98$10.00 © 1998 IEEE

TABLE I
FT FORMS FOR 1-D SIGNALS

Signal class		*FT and inverse FT*	*domain/periodicity*	
			time	*frequency*
1)	aperiodic	$S(f) = \int_{-\infty}^{+\infty} s(t)\, e^{-j2\pi ft}\, dt$	$\mathbb{R}/\mathbb{O}$	$\mathbb{R}/\mathbb{O}$
	continuous-time	$s(t) = \int_{-\infty}^{+\infty} S(f)\, e^{j2\pi ft}\, df$		
2)	periodic	$S(kF) = \int_{0}^{T_p} s(t)\, e^{-j2\pi kFt}\, dt$	$\mathbb{R}/Z(T_p)$	$Z(F)/\mathbb{O}$
	continuous-time	$s(t) = \sum_{k=-\infty}^{+\infty} F\, S(kF)\, e^{j2\pi kFt}$	$T_p = 1/F$	
3)	aperiodic	$S(f) = \sum_{n=-\infty}^{+\infty} T\, s(nT)\, e^{-j2\pi fnT}$	$Z(T)/\mathbb{O}$	$\mathbb{R}/Z(F_p)$
	discrete-time	$s(nT) = \int_{0}^{F_p} S(f)\, e^{j2\pi fnT}\, df$		$F_p = 1/T$
4)	periodic	$S(kF) = \sum_{n=0}^{N-1} T\, s(nT)\, e^{-j2\pi kFnT}$	$Z(T)/Z(T_p)$	$Z(F)/Z(F_p)$
	discrete-time	$s(nT) = \sum_{k=0}^{N-1} F\, S(kF)\, e^{j2\pi kFnT}$	$T_p = NT$	$F_p = NF$

definitions were collected in Table I in a form that is most symmetric for comparison [12]. In each case, we have also listed both the domain D and the periodicity P, where the latter represents a subset of the domain, such that

$$s(t-p) = s(t) \quad \forall p \in P.$$

This is valid even for aperiodic signals, by letting $P = \{0\} \triangleq \mathbb{O}$, where $\mathbb{O}$ is the trivial group.

Aperiodic continuous-time signals are defined on the real line $\mathbb{R}$. Their FT's have the same domain $\mathbb{R}$ and are also aperiodic. For periodic continuous-time signals, it is customary to consider the Fourier series expansion

$$s(t) = \sum_{k=-\infty}^{+\infty} S_k e^{j2\pi kFt}$$

where the Fourier coefficients $S_k = (1/T_p) \int_0^{T_p} s(t) e^{-j2\pi kFt}\, dt$ provide the frequency representation of $s(t)$. The form in Table I is obtained by letting $S(kF) = T_p S_k$ so that the Fourier transform turns out to be defined on the discrete set $Z(F) = \{kF | k \in \mathbb{Z}\}$. It is significant that the nature of the FT domain is determined not only by the signal domain but also by the *periodicity* $Z(T_p)$, i.e. the subset of $\mathbb{R}$ for which the signal verifies the shift-invariance property. In class 3) the situation is reversed with respect to class 2); the signal is of discrete time, and its FT is a periodic continuous-frequency function. For the sake of generality and symmetry, we consider an arbitrary spacing T for the signal domain, whereas in the literature, T is commonly normalized to 1. In class 4), both the signal and its FT are discrete and periodic. The FT for this class is known as the discrete Fourier transform (DFT). Note that if in Table I we set either $T = 1/N$ so that $F = 1$ or $T = 1/\sqrt{N}$ so that $F = 1/\sqrt{N}$, we find the usual forms of the DFT.

The FT is also defined for multidimensional signals, usually for continuous-argument signals (defined on $\mathbb{R}^m$), and for discrete-argument signals (defined on a lattice). The expressions of the FT and inverse FT can be composed from the 1-D expressions.

The above overview not only reveals "strong similarities" between FT expressions for different signal classes but also opens several questions, such as the following:

1) Why is the FT definition in 1-D limited to only four signal classes?
2) Why does the FT kernel have to be a complex exponential?
3) What is the general rule for finding the frequency domain?

The answers to these and similar questions are found in the Topology of Groups. We now outline the *topological approach* to the FT using the (simplified) language of signal theory.

B. Unified FT Definition

The relationships in Table I, as well as several others, can be obtained from a general formulation of Group Topology

[13], [14], whichis expressed as

$$\mathcal{F}\colon\ S(f)=\int_G dt s(t)\psi(f,t) \qquad f\in\hat{G}$$
$$\mathcal{F}^{-1}\colon\ s(t)=\int_{\hat{G}} df S(f)\psi^*(f,t) \qquad t\in G \tag{1}$$

where $s(t), t\in G$ denotes a *signal* defined as a complex function on an Abelian group G, $S(f)$ is its FT defined on the *dual* group $\hat{G}$, $\psi(f,t)$ is the kernel defined on $\hat{G}\times G$, and the integrals are Haar integrals.

Given a group G, the kernel $\psi(f,t)$ is uniquely determined[1] by the functional equation

$$\psi(f,t_1+t_2)=\psi(f,t_1)\psi(f,t_2) \tag{2}$$

with the constraint $|\psi(f,t)|=1$, where "+" represents the group operation.

In general, G is a quotient group $G=D/P$, where D is an Abelian group, and P is a subgroup. In terms of signal language, D represents the domain and P the periodicity. The latter states the shift-invariance property of the signal, namely, $s(t-p)=s(t), p\in P$. When P is the trivial group $\mathbb{O}=\{0\}$ consisting of the identity element of D, the periodicity degenerates into aperiodicity, and the notation $D/\mathbb{O}$ is shortened to D.

The dual group $\hat{G}$ itself is typically a quotient group $\hat{G}=D_f/P_f$, where D_f represents the frequency domain and P_f the periodicity of the FT. $\hat{G}$ is related to the *reciprocal* group J^*, which is defined [13, p. 35] by

$$J^* \triangleq \{f|\psi(f,t)=1, t\in J\} \tag{3}$$

for every subgroup J of D. Then

$$D_f=P^*,\quad P_f=D^* \tag{4}$$

which means that the domain of the FT is given by the reciprocal of the signal periodicity, and the FT periodicity is given by the reciprocal of the signal domain.

The above definitions do not hold on every quotient group G but only for the subclass of *locally compact Abelian* (LCA) groups, where both the Haar integral and the FT can be defined.

Verification: We can verify that the general definitions (1) lead to the expressions listed in Table I whenever the quotient group G is generated by the LCA subgroups of $\mathbb{R}$, which is the additive group of real numbers. A theorem of Group Topology [15, p. 12] states that the only LCA subgroups of $\mathbb{R}$ are $\mathbb{R}$ itself, $Z(T)$ for every $T>0$, and the trivial group $\mathbb{O}$, so that the only possible quotient groups are those listed in Table I. This explains why, in 1-D, FT definitions exist only for four signal classes.

When "+" represents ordinary addition, the function equation (2), with the constraint $|\psi(f,t)|=1$, leads to a solution of the form

$$\psi(f,t)=e^{-j2\pi ft}$$

or some equivalent form.[2] From (3), we have that $\mathbb{R}^*=\mathbb{O}, \mathbb{O}^*=\mathbb{R}$ and $Z(T)^*=Z(1/T)$, which lead to the frequency domain/periodicity structures listed in Table I.

To complete the verification, we need to know the expressions of the Haar integral for the four 1-D cases. From textbooks on Group Topology (see, e.g. Higgins [14, p. 84]), we find that the Haar integral for $G=\mathbb{R}=\mathbb{R}/\mathbb{O}$ is given by the Lebesgue integral extended to $\mathbb{R}$, for $G=\mathbb{R}/Z(T_p)$ by the Lebesgue integral over one period, for $G=Z(T)=Z(T)/\mathbb{O}$ by the summation over $Z(T)$, and for $G=Z(T)/Z(T_p)$ by the summation limited to one period.[3] The same expressions apply for the inverse FT, thus completing the verification.

For reference, we add a nonstandard example of the unified FT definition. If we let $G=\mathbb{R}_{\rm pos}$, which is the multiplicative group of positive real numbers, we find that the FT kernel becomes $\psi(f,t)=\exp(-j2\pi\log f\log t)$ and that $\hat{\mathbb{R}}_{\rm pos}=\mathbb{R}_{\rm pos}$. The Haar integral of $s(t), t\in\mathbb{R}_{\rm pos}$ is now given by the Lebesgue integral of $s(t)/t$ over $(0,\infty)$ [14, p. 88], so that the FT and the inverse FT on $\mathbb{R}_{\rm pos}$ are given by

$$S(f)=\int_0^\infty s(t)e^{-j2\pi\log f\log t}\frac{dt}{t}, \qquad f\in\mathbb{R}_{\rm pos} \tag{5a}$$
$$s(t)=\int_0^\infty S(f)e^{j2\pi\log f\log t}\frac{df}{f}, \qquad t\in\mathbb{R}_{\rm pos}. \tag{5b}$$

For an m-dimensional signal defined on a Cartesian product $G=G_1\times\cdots\times G_m$, the parameters for the FT definition can be built up from the parameters of the component groups. Specifically, the dual group is $\hat{G}=\hat{G}_1\times\cdots\times\hat{G}_m$, and the kernel is $\psi(f_1,\cdots,f_m;t_1,\cdots,t_m)=\psi_1(f_1,t_1)\cdots\psi_m(f_m,t_m)$, where ψ_i is the kernel on G_i. In addition, the Haar integral is given by a "natural" composition of Haar integrals on the component groups [14, p. 78].

C. Rules Common to All FT's

The unified FT has essentially the same rules as the FT on $\mathbb{R}$. For instance, the rule for convolving two signals $x(t)$ and $y(t), t\in G$ is

$$x*y(t)=\int_G du x(t-u)y(u)\xrightarrow{\mathcal{F}}X(f)Y(f)$$

and Parseval's theorem takes the highly symmetric form

$$\int_G dt x(t)y^*(t)=\int_{\hat{G}} df X(f)Y^*(f). \tag{6}$$

For each LCA group G, an *impulse function* $\delta_G(t)$ can be defined as the inverse FT of the function $S(f)=1$, namely

$$\delta_G(t)=\int_{\hat{G}} df\psi^*(f,t), \qquad t\in G.$$

An equivalent definition states that $\delta_G(t)$ is the *identity element of convolution*, i.e., $\delta_G*s=s$ for every signal s defined on G. A theorem of Group Topology states that $\delta_G(t)$ is an ordinary function if and only if the group G is discrete [13, p. 6]; otherwise, $\delta_G(t)$ becomes a distribution. In fact, for

[1] Up to an isomorphism.

[2] For instance, $e^{j2\pi ft}$, $e^{-j\omega t}$, $e^{j\omega t}$.

[3] The Haar integral is unique, up to a constant factor. In Table I, the constant is chosen to be T (F in the frequency domain) to give symmetric expressions.

$G = \mathbb{R}$, we know that the impulse function is given by the delta function, whereas for the discrete group $G = Z(T)$, it is given by the Kronecker delta (which is an ordinary function).

For later use, we sketch the effects of a repeated application of the FT operator $\mathcal{F}$ defined by (1), namely

$$\underset{G}{s(t)} \xrightarrow{\mathcal{F}} \underset{\hat{G}}{S(f)} \xrightarrow{\mathcal{F}} \underset{G}{s(-t)} \xrightarrow{\mathcal{F}} \underset{\hat{G}}{S(-f)} \xrightarrow{\mathcal{F}} \underset{G}{s(t)}. \quad (7)$$

This graph summarizes several properties that hold for any LCA group G (and can be verified for the four cases of Table I). It states that for any given signal $s(t), t \in G$, a first application of $\mathcal{F}$ gives the FT $S(f), f \in \hat{G}$, a second gives the "reflected" version $s(-t)$ of the original signal, a third gives the reflected version $S(-f)$ of the FT, and a fourth application recreates the original signal. This is due to an integral property of the kernel, namely

$$\int_{\hat{G}} df \psi(f,t)\psi(f,t') = \int_{\hat{G}} df \psi(f,t+t') = \delta_G(t+t'). \quad (8)$$

Similar integral equations can be found for subsequent applications of $\mathcal{F}$ in (7). However, the graph also reveals some other properties. If the first application changes G into $\hat{G}$, the second changes $\hat{G}$ into $\hat{\hat{G}}$. In the graph, we have set $\hat{\hat{G}} = G$, which is permitted by Pontriagin's duality theorem, which states that the dual of the dual group is the original group [16].

III. Signal Classes that Admit an FRT Definition

A. Existing Definitions

One type of FRT was first developed in the context of *continuous-time* signals, i.e., for signals defined on the real line $\mathbb{R}$. Different versions can be found in the literature, of which the most common can be expressed as (see, e.g., [17])

$$S_a(f) = K_a e^{j\pi B_a f^2} \int_{-\infty}^{+\infty} s(t) e^{j\pi B_a t^2} e^{-j2\pi C_a f t}\, dt \quad (9)$$

where

$$B_a = \cot\left(\frac{\pi}{2}a\right), \quad C_a = \csc\left(\frac{\pi}{2}a\right)$$
$$K_a = |C_a|^{1/2} e^{j(\pi/2)(a - \operatorname{sgn}[\sin((\pi/2)a)])}. \quad (10)$$

For such a continuous signal $s(t)$, this FRT turns out to be a *continuous-frequency function* $S_a(f)$, where f is the "frequency" associated with the fractional domain, and $a \in \mathbb{R}$ is the "fraction." Owing to the chirp factors in its kernel, we call this the *chirp-type* FRT.

Having established the 1-D FRT on $\mathbb{R}$, two-dimensional (2-D) and multidimensional extensions are easily obtained. In fact, it is the 2-D extension that is commonly used in optics [6], [18], [19]. Specifically, the FRT of a 2-D signal $s(t_1, t_2)$, which is defined on $\mathbb{R}^2$, is a 2-D function $S_{a_1 a_2}(f_1, f_2)$, defined again on $\mathbb{R}^2$, where a_1 and a_2 are the "fractions."

Another form of FRT was considered in connection with *discrete-time* signals and the DFT [8], [20]. For a discrete sequence s_n with DFT S_k, this FRT can be expressed in the form

$$S_k^{(a)} = p_0(a)s_k + p_1(a)S_k + p_2(a)s_{-k} + p_3(a)S_{-k} \quad (11)$$

where p_i are "weights" given by

$$p_i(a) = \frac{1}{4}\frac{1 - e^{-j2\pi a}}{1 - e^{-j(\pi/2)(a-i)}}, \qquad i = 0,1,2,3. \quad (12)$$

Inspection of (11) shows that $S_k^{(a)}$ is a weighted combination of the original sequence s_k, its DFT S_k, and their reflected-index versions s_{-k} and S_{-k}. For this reason, we will refer to this transform as a *weighted-type* FRT.

It is worth pointing out that the sequence s_n must be periodic, with some period N, and that the same period is found in the corresponding DFT and in its fractional version. We conclude that the discrete FRT found in the literature has been considered for the class of *periodic discrete-time signals* and that the corresponding FT and FRT are *periodic discrete-frequency functions*. Note that if we rewrite (11) with the notation of Table I as

$$S_a(kF) = p_0(a)s(kF) + p_1(a)S(kF) + p_2(a)s(-kF)$$
$$+ p_3(a)S(-kF) \quad (13)$$

where the spacings F and T, and, hence, the periods NF and NT, are equal, and since $F = 1/(NT)$, we have

$$F = T = \frac{1}{\sqrt{N}}, \qquad NF = NT = \sqrt{N}. \quad (14)$$

Later in the paper, we will show that this transform can also be considered for signals on $\mathbb{R}$ in the form

$$S_a(f) = p_0(a)s(f) + p_1(a)S(f) + p_2(a)s(-f)$$
$$+ p_3(a)S(-f) \quad (15)$$

which has been introduced by Shih [21] and more recently generalized in [22] and [23].

A different DFT-based FRT form for discrete sequences has been presented in [24] using fractional roots of unity in the DFT kernel as

$$S_k^{(a)} = \sum_{n=0}^{N-1} s_n e^{-j2\pi kna} \quad (16)$$

and proves to be very useful in solving several application problems related to the DFT. However, we will see in Section IV that this cannot be considered as a fractional operator in a strict sense.

At this point, it is natural to ask the following question: Why has the FRT not been considered for discrete-time aperiodic signals? We saw that the FT is well defined for this class and that it is a periodic continuous-frequency function. The difference with the two previous cases is that the FT class is different from the signal class. This would appear to be a fundamental obstacle for defining an FRT for this class.

B. Domain/Periodicity Constraints for FRT Definitions

Reconsidering the two cases for which the FRT definition is currently available, we observe that *the class of the signal is the same as that of its FT*, namely, the class of complex functions defined on $\mathbb{R}$ in the first case and the class of discrete and periodic complex functions, with the same period, in the second. Moreover, in each case, the class of the FRT, for any fraction a, is the same as that of the signal and of its ordinary FT. In fact, in the context of the FRT, both the signal and its ordinary FT can be viewed as particular cases of the FRT, with $a = 0$ and $a = 1$, respectively.

Using the language of Group Topology, the above properties can be stated as follows: The quotient group $G = D/P$ on which the signal is defined coincides with the dual group $\hat{G} = D_f/P_f = P^*/D^*$, that is, the quotient group is *self-dual*

$$\hat{G} = G. \tag{17}$$

This common group also represents the domain for the FRT.

This would appear to be the preliminary constraint to impose for the FRT definition and we shall develop this statement.

C. Self-Dual Groups

An important result from Group Topology relates the nature of dual groups by the table [25, p. 101], [13, p. 9]

G continuous	$\hat{G}$ continuous
G compact	$\hat{G}$ discrete
G discrete	$\hat{G}$ compact
G finite	$\hat{G}$ finite.

This corresponds exactly to the classification of Table I but has general validity. Hence, only the class of finite groups and that of continuous groups are suitable candidates for FRT definitions.

In the class of quotient groups generated by $\mathbb{R}$ (see Table I), the only self-dual groups are

$$\mathbb{R} = \mathbb{R}/\mathbb{O} \quad \text{and} \quad Z(1/\sqrt{N})/Z(\sqrt{N})$$

for every natural N. Note that the latter is consistent with the considerations leading to (14). Another example of 1-D self-dual groups is $\mathbb{R}_{\text{pos}}$, which is the multiplicative group of positive real numbers.

In the class of multidimensional groups, self-dual groups can be obtained as Cartesian products of 1-D self-dual groups. Thus, if G_1 and G_2 are self-dual, we find that $G = G_1 \times G_2$ is self-dual since $\hat{G} = \hat{G}_1 \times \hat{G}_2 = G_1 \times G_2 = G$. Hence, in the classes of groups generated by $\mathbb{R}^2$, we find that

$$\mathbb{R}^2, \quad \mathbb{R} \times Z(1/\sqrt{N})/Z(\sqrt{N}) \quad \text{and}$$
$$Z(1/\sqrt{N_1})/Z(\sqrt{N_1}) \times Z(1/\sqrt{N_2})/Z(\sqrt{N_2})$$

are self-dual.

However, it is possible to find self-dual groups that are not described by a Cartesian product. As an example, the well-known quincunx lattice $Z_Q(d_1, d_2)$ of television scanning [26], [27] is defined as the subgroup of $\mathbb{R}^2$ consisting of the points (md_1, nd_2), where m and n are both even or both odd integers. This group cannot be expressed as a Cartesian product. The reciprocal of $Z_Q(d_1, d_2)$ is the quincunx lattice $Z_Q(F_1, F_2)$ with $F_1 d_1 = F_2 d_2 = 1/2$. Hence, the quotient group

$$Z_Q(1/\sqrt{2N_1}, 1/\sqrt{2N_2})/Z_Q(\sqrt{N_1/2}, \sqrt{N_2/2})$$

is self-dual.

IV. General FRT Definition

Following Namias [1], [23], we give a completely general definition. Let $\mathcal{F}^a$ be the operator generating the FRT, i.e., the operator that for any given "fraction" $a \in \mathbb{R}$ maps a signal $s(t)$ into the FRT $S_a(f)$. This operator must also

1) be linear;
2) verify the FT condition

$$\mathcal{F}^1 = \mathcal{F}$$

3) for every choice of a and b have the *additive* property

$$\mathcal{F}^{a+b} = \mathcal{F}^a \mathcal{F}^b.$$

Operators that do not meet the above requirements can not be considered fractional Fourier transforms, as is the unfortunate case with the operator (16), which fails to meet postulate 3). [As a matter of fact, it can be shown that the composition of two transformations (16) is not in general a transformation of the kind (16) itself].

These FRT constraints confer certain fundamental properties to the $\mathcal{F}^a$ operator, such as the *marginal conditions*

$$\mathcal{F}^0 = \mathcal{F}^4 = \mathcal{I}, \quad \mathcal{F}^2 = \mathcal{I}_-, \quad \mathcal{F}^3 = \mathcal{F}_-$$

where

- $\mathcal{I}$ identity operator;
- $\mathcal{I}_-$ reflection operator;
- $\mathcal{F}_-$ ordinary inverse FT.

The above conditions also imply *periodicity*, with period 4

$$\mathcal{F}^{a+4} = \mathcal{F}^a.$$

A. FRT on Self-Dual Groups

In this section, we give a general FRT definition for the signal on a *self-dual* group G. From property 1), we find that the FRT must have the Haar integral form

$$\mathcal{F}^a\colon \quad S_a(f) = \int_G dt\, s(t)\psi_a(f,t), \qquad f \in G \tag{18}$$

where $a \in \mathbb{R}$ is the "fraction," and $\psi_a(f,t)$ is the fractional kernel. The above constraints can also be written in terms of the kernel $\psi_a(f,t)$. Specifically, the FT property 2) leads to

$$\psi_1(f,t) = \psi(f,t) \tag{19}$$

and the additive property 3) is equivalent to the integral equation

$$\psi_{a+b}(f,t) = \int_G du\, \psi_b(f,u)\psi_a(u,t), \qquad f,t \in G. \tag{20}$$

Moreover, the solution for $a = 0$ will be $\delta_G(f-t), \delta_G(f+t)$ for $a = 2$ and $\psi^*(f,t)$ for $a = 3$ and will be periodic in a with period 4.

A possible approach for obtaining one particular variety of kernels is based on eigenfunctions of the ordinary FT. For this reason, we first outline the eigenstructure of the FT.

B. Eigenfunctions of the Ordinary FT

A signal $\varphi(t), t \in G$ is an *eigenfunction* of Fourier transformation if its FT is proportional to the signal itself, that is

$$\varphi(\lambda) \xrightarrow{\mathcal{F}} \mu\varphi(\lambda) \tag{21}$$

where μ is called the *eigenvalue*. Clearly, an eigenfunction implies that G is self–dual; therefore, $\varphi(\lambda)$ and its FT can be compared, as in (21).

We note that eigenfunctions and eigenvalues have tight constraints imposed by the sequence (7). In fact, if $\varphi(t)$ is an eigenfunction with eigenvalue μ, repeated application of the operator $\mathcal{F}$ gives

$$\varphi(t) \xrightarrow{\mathcal{F}} \mu\varphi(t) \xrightarrow{\mathcal{F}} \mu^2\varphi(t) \xrightarrow{\mathcal{F}} \mu^3\varphi(t) \xrightarrow{\mathcal{F}} \mu^4\varphi(t).$$

From (7), however, we know that $\mathcal{F}^2$ gives the flipped signal, and $\mathcal{F}^4$ returns the original signal so that $\mu^2\varphi(t) = \varphi(-t)$, and $\mu^4\varphi(t) = \varphi(t)$. From the above relations, we have $\mu^4 = 1$, and hence, the only possible eigenvalues are the *fourth roots of unity*

$$\mu \in \{1, -j, -1, j\}. \tag{22}$$

We also find that $\mu^2 = \pm 1$ so that the eigenfunctions can only be *even* or *odd* functions.

It is very easy to generate eigenfunctions on self–dual groups. For instance, if $s(t), t \in G$ is an arbitrary signal, and $S(f), f \in G$ is its FT, it is easy to see that the function

$$\varphi(t) = s(t) + \mu^{-1}S(t) + \mu^{-2}s(-t) + \mu^{-3}S(-t) \qquad t \in G \tag{23}$$

is an eigenfunction with eigenvalue $\mu \in \{1, j, -1, -j\}$, as was pointed out in [28] for the particular case $\mu = 1$ and in [29] for the general case.

The class of FT eigenfunctions is wider than we would expect. In particular, we wish to find a set of *orthonormal* eigenfunctions $\varphi_n(t), n \in \mathcal{N}$, i.e., a set with the property

$$\int_G dt\varphi_m(t)\varphi_n^*(t) = \delta_{mn} \tag{24}$$

which is also *complete*. Such a complete set of functions permits a signal expansion of the form

$$s(t) = \sum_{n\in\mathcal{N}} S_n\varphi_n(t), \qquad t \in G \tag{25}$$

whose coefficients S_n are given by

$$S_n = \int_G dts(t)\varphi_n^*(t). \tag{26}$$

Thus, if μ_n is the eigenvalue corresponding to the eigenfunction $\varphi_n(t)$, the sum (25) leads to a similar expansion of the Fourier transform

$$\boxed{S(f) = \sum_{n\in\mathcal{N}} \mu_n S_n\varphi_n(f), \qquad f \in G.} \tag{27}$$

Inserting (26), we find

$$S(f) = \sum_{n\in\mathcal{N}} \mu_n\varphi_n(f) \int_G dts(t)\varphi_n^*(t) \tag{28}$$

which, compared with the top part of (1), leads to an expansion of the FT kernel as

$$\psi(f,t) = \sum_{n\in\mathcal{N}} \mu_n\varphi_n(f)\varphi_n^*(t), \qquad f,t \in G. \tag{29}$$

In the Appendix, we prove the following.

Theorem 1: For every self-dual LCA group G, there exists a *complete orthonormal* set of *real* eigenfunctions $\varphi_n(t), n \in \mathcal{N}$, where $\mathcal{N}$ could be $\mathbb{N}_0 \triangleq \{0, 1, 2, \cdots\}$ (or a multidimensional extension thereof) or possibly a finite set if G itself is finite.

This justifies writing the FT kernel expansion in the form (29). The properties of real eigenfunctions simplify some of the formulas and add certain symmetries of the FRT. In particular, they assure the *Hermitian symmetry* of the FRT with respect to the fraction, as will be seen in Section VI. Hereafter, we assume that the real form was chosen.

C. FRT Definition

We are ready to formulate a general definition of the FRT on an arbitrary LCA self-dual group G. We saw that a complete set of orthonormal eigenfunctions $\varphi_n(t), n \in \mathcal{N}$ with corresponding eigenvalues μ_n allow us to write the kernel of the ordinary FT in the form (29). Now, to define the kernel of the FRT, it is sufficient to replace the eigenvalues μ_n in (29) by their ath power μ_n^a, that is

$$\psi_a(f,t) = \sum_{n\in\mathcal{N}} \mu_n^a\varphi_n(f)\varphi_n^*(t), \qquad f,t \in G. \tag{30}$$

The same substitution in (27) leads to an equivalent expansion for the FRT [see (18) and (25)]

$$S_a(f) = \sum_{n\in\mathcal{N}} \mu_n^a S_n\varphi_n(f), \qquad f \in G \tag{31}$$

where S_n are the signal coefficients given by (26). Comparison with (27) shows that $\varphi_n(t)$ is also an eigenfunction of the FRT with fraction a and that μ_n^a in (31) takes the meaning of eigenvalue for that fraction. In fact, we have the following.

Theorem 2: The kernel defined by (30) satisfies both the FT condition (19) and the additive property (20).

Proof: The FT condition is obvious. The key to the second assertion is the additive property of the power to a real number, namely

$$\mu_n^{a+b} = \mu_n^a\mu_n^b \tag{32}$$

combined with the orthonormality condition (24). In fact, using (30), we find

$$\begin{aligned}\int_G du \psi_b(f,u)\psi_a(u,t) &= \sum_{m\in\mathcal{N}}\sum_{n\in\mathcal{N}} \mu_n^a \mu_n^b \varphi_m(f)\varphi_n^*(t) \\ &\quad \cdot \int_G du \varphi_m(u)\varphi_n^*(u) \\ &= \sum_{m\in\mathcal{N}} \mu_m^a \mu_m^b \varphi_m(f)\varphi_m^*(t) \\ &= \psi_{a+b}(f,t)\end{aligned}$$

which proves the additive property (20). □

To summarize, using the eigenfunctions of the ordinary FT, we have defined the FRT over an arbitrary LCA group G for any "fraction" a. The kernel is given by (30), and the FRT expression is given by (31) in terms of the signal coefficients S_n. In Section VII, we shall see that this general definition includes the existing definitions (9) and (11) as particular cases for $G = \mathbb{R}$ and $G = Z(1/\sqrt{N})/Z(\sqrt{N})$, respectively. First, we discuss an ambiguity of the FRT definition, which is inherent in any fractional operator definition.

V. Multiplicity of FRT Definitions

The choice of complete orthonormal set of eigenfunctions (24) on G allows some degree of freedom, yielding a different kernel (30) and a different FRT for each choice. When $G = \mathbb{R}$, however, a preferred set of eigenfunctions exists (38a), based on the optical interpretation of the FT, where such functions are the modes of propagation in graded index media. In contrast, when $G = Z(1/\sqrt{N})/Z(\sqrt{N})$, there is no such customary choice.

Another fundamental reason for the *nonuniqueness* of FRT definitions on any self-dual group G lies in the fact that the real power of a complex number μ_n^a is not unique. Considering (22), we can write

$$\mu_n = e^{-j(\pi/2)h_n} \quad \text{and} \quad h_n \in \{0,1,2,3\} \tag{33}$$

where h_n is uniquely related to μ_n. Then, the possible values of μ_n^a are given by

$$\mu_n^a = e^{-j(\pi/2)(h_n+4k_n)a}, \qquad k_n \in \mathbb{Z} \tag{34}$$

where $k_n, n \in \mathcal{N}$ is an arbitrary sequence of integers. If a is rational, the possible number of different values is finite; otherwise, it can be infinite.

To remove the ambiguity in the FRT definition (30), for any n in (34), we need to choose a particular value of k_n. By the additive property (32), this must not depend on a. We shall call $k_n, n \in \mathcal{N}$ a *generating sequence* (GS) for the FRT. On the given group G, different GS's k_n will lead to different versions of the FRT, each consistent with the additive property and the marginal conditions.

Considering that μ_n^a is an eigenvalue of an FRT, we can say that by choosing a GS, we choose the eigenvalues for that particular FRT. Some consistent examples of GS's are

$$\begin{aligned}&k_n = 0, \quad k_n = n, \quad k_n = (n)_4, \quad k_n = \lfloor n/4 \rfloor \\ &k_n = (\lfloor n/4 \rfloor)_4\end{aligned}$$

where $n \in \mathcal{N}, (\cdot)_N$ denotes modulo N, and $\lfloor \cdot \rfloor$ denotes integer part.

Later, we will see that the available FRT definitions (9) and (15) can be obtained on $\mathbb{R}$ using the set of Hermite–Gauss functions with eigenvalues

$$\text{A)} \quad e^{-j(\pi/2)na}, \qquad \text{B)} \quad e^{-j(\pi/2)(n)_4 a}$$

respectively. The corresponding FRT's can be obtained by the respective GS's

$$\text{A)} \quad k_n = \lfloor n/4 \rfloor, \qquad \text{B)} \quad k_n = 0. \tag{35}$$

Fig. 1 shows the pattern of the four eigenvalues of the ordinary FT and the eigenvalues of the FRT's generated by A) and B), evaluated for the rational "fractions" $a = 1/4$ and $a = 6/17$.

Note that the eigenfunction approach not only permits a universal FRT definition but also gives a simple and clear framework for discussing ambiguity and for finding all possible versions. A comprehensive discussion on FRT multiplicity is carried out in [10] and in [11].

VI. Properties Common to All FRTS

Some of the common properties result from the constraints seen in the previous section. In each case, once the ambiguity in μ_n^a has been resolved by choosing a specific GS, the FRT expression becomes unique, the kernel is determined by (30), and we can derive the properties of the particular FRT. As outlined by Almeida [17] for $G = \mathbb{R}$, the properties are quite different from those of the ordinary transform, apart from the following.

Inverse FRT: From the additive property, we find that the operator $\mathcal{F}^a$ followed by the operator $\mathcal{F}^{-a}$ gives the identity. Hence, the recovery of a signal $s(t)$ from its FRT $S_a(f)$ is obtained by applying to $S_a(f)$ the FRT with fraction $-a$.

Eigenfunctions: The FRT definition (31) shows that any FRT eigenfunction is also an FT eigenfunction. The difference lies in the eigenvalues, which are $\mu_n \in \{1, -j, -1, j\}$ for the FT and μ_n^a for the FRT.

Even-Odd Symmetries: The fact that the FT eigenfunctions are either even or odd assures that

$$s(-t) \xrightarrow{\mathcal{F}^a} S_a(-f)$$

which implies that every *even* (*odd*) signal has an *even* (*odd*) FRT.

Parseval's Theorem: The Parseval relation for any a once again has the form

$$\int_G dt |s(t)|^2 = \int_{\hat{G}} df |S_a(f)|^2 \tag{36}$$

and is assured by having chosen an orthonormal set of eigenfunctions with unit-amplitude eigenvalues.

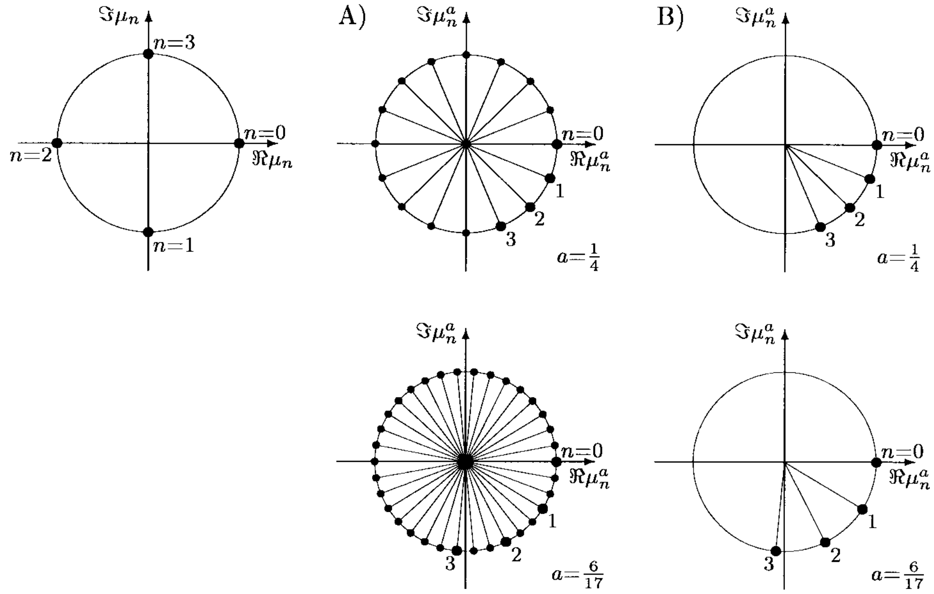


Fig. 1. Eigenvalues of the ordinary FT; example of powers of the eigenvalues for choice A) and choice B).

Hermitian Symmetry: If the eigenfunctions are chosen orthonormal and real, the kernel satisfies

$$\psi_{-a}(f,t) = \psi_a^*(f,t) \tag{37}$$

which makes it a *unitary* operator. For a real signal $s(t), t \in G$, (37) gives

$$S_{-a}(f) = S_a^*(f)$$

which allows the limitation of the study to $0 < a < 2$.

Identity Function: As in the ordinary case, we find that

$$\delta_G(t-t_0) \xrightarrow{\mathcal{F}^a} \psi_a(f,t_0)$$

where the impulse function $\delta_G(t)$ on G can be defined as the inverse FRT of $\psi_a(f,0)$.

All these properties are independent of the choice for the complete orthonormal set of eigenfunctions and for the eigenvalues, apart from the Hermitian symmetry, which can be obtained only by choosing *real* eigenfunctions.

VII. FRT Examples

We now derive the fractional transforms on $\mathbb{R}$ for the choices A) and B) of (35). Choice A) is the most important from an applications point of view and is well known on groups $\mathbb{R}$ and $\mathbb{R}^2$. Its kernel contains *chirp* factors (hence chirp-type FRT or CFRT). Choice B) has so far been applied to discrete groups but can be extended to any self-dual group. It produces a fractional transform that is a *weighted* combination of the signal, its ordinary transform, and their flipped versions (hence weighted FRT or WFRT).

A. Chirp-Type FRT on $\mathbb{R}$

Here, we require explicit knowledge of a complete orthonormal set of eigenfunctions and their corresponding eigenvalues. On $\mathbb{R}$, we find [1], [29] the set of Hermite–Gauss functions

$$\varphi_n(t) = \frac{\sqrt[4]{2}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi}t)e^{-\pi t^2}, \qquad n \in \mathbb{N}_0 \tag{38a}$$

where $H_n(t)$ are nth-order Hermite polynomials, and the corresponding eigenvalues are $\mu_n = e^{-j(\pi/2)n}$. Choice A) in (35) then gives

$$\mu_n^a = e^{-j(\pi/2)na}. \tag{38b}$$

Inserting (38) into (30) and summing over n leads to the expression for the kernel [1]

$$\psi_a(f,t) = K_a e^{j\pi B_a(f^2+t^2)} e^{-j2\pi C_a ft} \tag{39}$$

where K_a, B_a, and C_a are given by (10). This is precisely the kernel of (9); therefore, the chirp-type fractional transform on $\mathbb{R}$ takes the familiar form

$$\boxed{S_a(f) = K_a e^{j\pi B_a f^2} \int_{-\infty}^{+\infty} s(t) e^{j\pi B_a t^2} e^{-j2\pi C_a ft}\, dt.} \tag{40}$$

An investigation of the above expressions reveals, as was pointed out in [20] and [30], that the FRT operation can be performed using the ordinary FT through the following steps (see Fig. 2) involving two chirp modulations.

1) Obtain the signal $y(t) = s(t)e^{j\pi B_a t^2}$.
2) Perform the ordinary transform on $y(t)$, giving $Y(C_a f)$.
3) Scale the frequency variable, obtaining $Y(f)$.
4) Complete the operation with $S_a(f) = K_a e^{j\pi B_a f^2} Y(f)$.

As an example, following this approach, we find that the fractional transform of $s(t) = \text{rect}\,(t)$ has the expression

$$\begin{aligned} S_a(f) = \frac{K_a}{\sqrt{2|B_a|}} e^{-j\pi f^2/B_a} \{[C(f_+) - C(f_-)] \\ + j\,\text{sgn}\,(B_a)[S(f_+) - S(f_-)]\} \end{aligned} \tag{41}$$

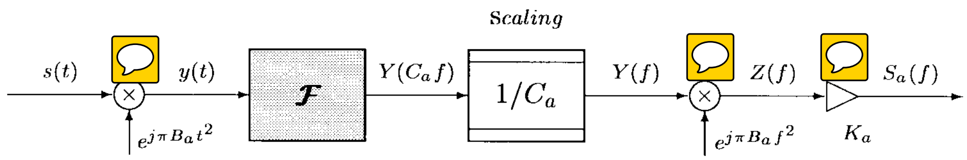


Fig. 2. Decomposition of CFRT on ℝ.

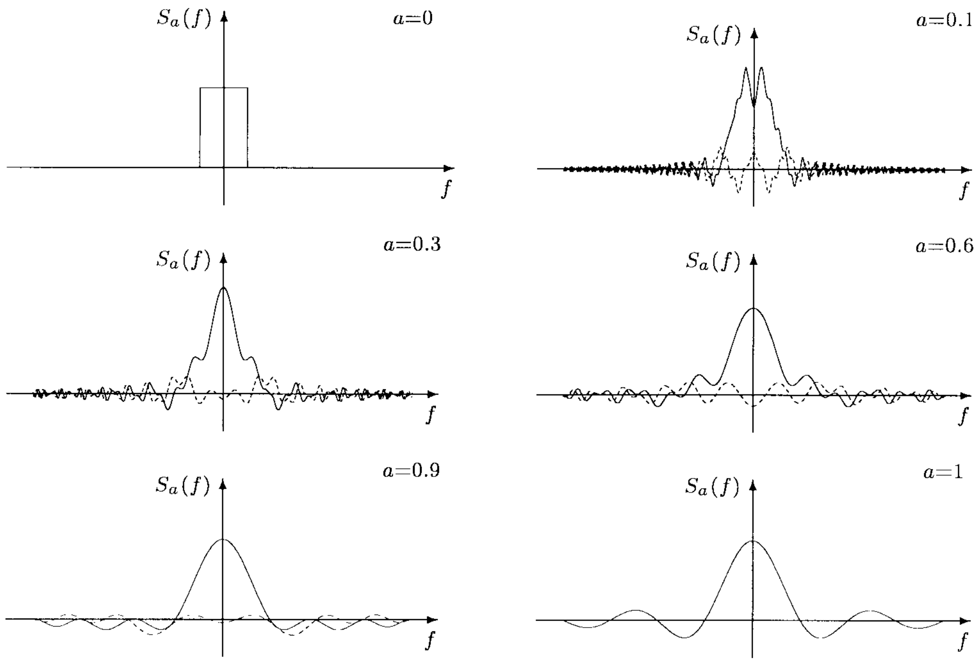


Fig. 3. Example of CFRT on ℝ for several values of a. Complex transforms are plotted with the real part in solid and the imaginary part in dashed lines.

where $C(x) + jS(x) \triangleq \int_0^x e^{j(\pi/2)y^2}\,dy$ is the Fresnel integral with real part $C(x)$ and imaginary part $S(x)$, and $f_\pm = \pm\sqrt{|B_a|/2} - \operatorname{sgn}(B_a)\sqrt{2/|B_a|}C_a f$. This transform is illustrated in Fig. 3 for several values of a. For small values the fractional transform resembles the signal and approaches the ordinary transform $S(f) = \operatorname{sinc}(f)$ as $a \to 1$.

B. Weighted-Type Transform

Choice B) does not require an explicit knowledge of a complete orthonormal set of eigenfunctions of the ordinary Fourier transform. In fact, taking into account that $\mu_n \in \{1, e^{-j(\pi/2)a}, e^{-j\pi a}, e^{-j(3/2)\pi a}\}$, (30) can be written as

$$\psi_a(f,t) = \beta_0(f,t) + e^{-ja(\pi/2)}\beta_1(f,t) + e^{-ja\pi}\beta_2(f,t) + e^{-ja(3/2)\pi}\beta_3(f,t) \quad (42)$$

which separates the dependence on a from that on f and t. From the explicit expressions of the kernel for $a = 0, 1, 2, 3$ (see Section IV), we obtain a system of four equations, whose solution leads to the functions $\beta_i(f,t)$ (see [21], where this form of FRT was first introduced). The resulting kernel is

$$\psi_a(f,t) = p_0(a)\delta(f-t) + p_1(a)\psi(f,t) + p_2(a)\delta(f+t) + p_3(a)\psi^*(f,t) \quad (43)$$

where $p_i(a)$ are given by (12). The additive property and the marginal conditions are now guaranteed by the structure of the weights $p_i(a)$. The weighted FRT is thus defined as a linear combination of the signal, its ordinary transform, and their reflected versions (Fig. 4) and can be written as

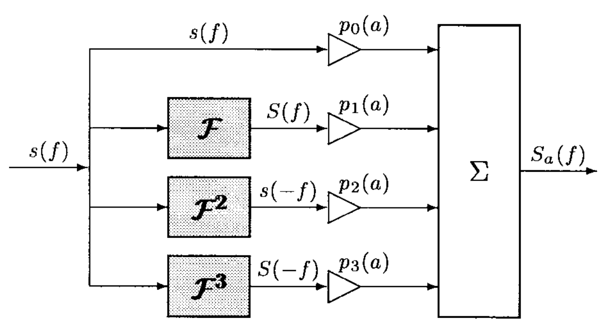


Fig. 4. Interpretation of the WFRT.

$$\boxed{S_a(f) = p_0(a)s(f) + p_1(a)S(f) + p_2(a)s(-t) + p_3(a)S(-f)} \quad (44)$$

where $s(t)$ is the signal, $S(f)$ its ordinary transform, etc. Based on the preceding results, this fractional operator can also be described by

$$\boxed{\mathcal{F}^a = \sum_{i=0}^{3} p_i(a)\mathcal{F}^i}. \quad (45)$$

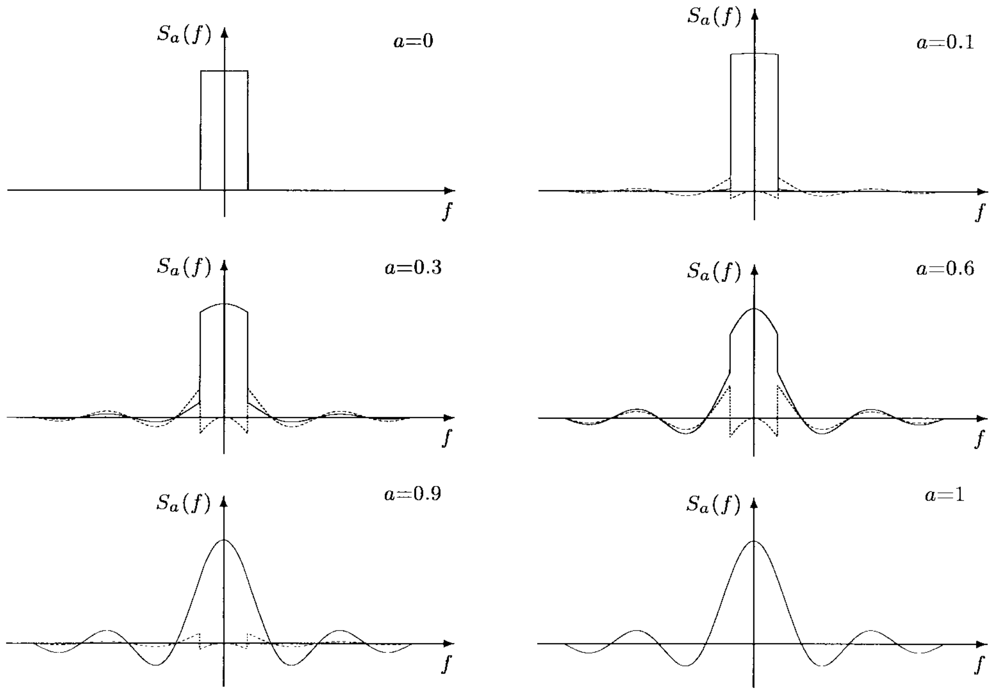


Fig. 5. WFRT of order $4L = 4$ of the rectangular signal for several values of a. Complex transforms are plotted with the real part in solid and the imaginary part in dashed lines.

The fractional operator is thus related by means of the weights to the integer powers of $\mathcal{F}$.

This definition leads to quite different results from those of the CFRT version. For instance, the earlier signal $s(t) = \text{rect}\,(t)$ has a weighted fractional transform

$$S_a(f) = [p_0(a) + p_2(a)]\,\text{rect}\,(f) + [p_1(a) + p_3(a)]\,\text{sinc}\,(f)$$

illustrated in Fig. 5 for some values of a. Again, the fractional transform properties depart from those of the ordinary form, apart from Parseval's theorem (36). For instance, the fractional transform of the shifted signal $y(t) = s(t - t_0)$ becomes

$$\begin{aligned} Y_a(f) = &p_0(a)s(f - t_0) + p_1(a)S(f)e^{-j2\pi f t_0} \\ &+ p_2(a)s(-f + t_0) + p_3(a)S(-f)e^{j2\pi f t_0}. \end{aligned}$$

Remark: Although we derived the weighted FRT on the group $\mathbb{R}$, the result stated by (44) holds for every self-dual LCA group, and the weights do not depend on the group. When G is the 1-D self-dual group $Z(1/\sqrt{N})/Z(\sqrt{N})$, (44) yields the FRT form found in [8], [9].

C. Generalization of the WFRT

The WFRT was obtained by choosing the eigenvalues $e^{j(\pi/2)(n)_4 a}$, which are at most four distinct eigenvalues for any fraction a (see Fig. 1). This leads to a FRT with four weights. More generally, we can choose $4L$ distinct eigenvalues and obtain a weighted FRT with $4L$ weights [22]. For instance, by choosing the eigenvalues

$$e^{-j(\pi/2)(n)_{16} a}$$

the fractional operator becomes [10]

$$\mathcal{F}^a = \sum_{i=0}^{15} p_i^{(16)}(a)\mathcal{F}^{4i/16} \tag{46}$$

where the weights become

$$p_i^{(16)}(a) = \frac{1}{16}\frac{1 - e^{-j(\pi/2)16a}}{1 - e^{-j(\pi/2)(a-i)}}, \qquad i = 0, 1, \cdots, 15.$$

A consequence of (46) is that the FRT can be calculated for all fractions a as the weighted sum of the FRT's with fractions $a = 0, 1/4, 2/4, \cdots, 15/4$.

Even this generalized WFRT can be considered on any LCA group, in particular on $G = \mathbb{R}$, where it can be compared with both the CFRT and the ordinary WFRT with four weights. As an example, Fig. 6 illustrates these three different FRT's of the signal $s(t) = \text{rect}\,(t)$ for the same fraction $a = 0.3$. We see that the 16-weight WFRT resembles the CFRT more than the four-weight WFRT. In fact, it is mentioned in [22] and [23] and shown in [10] that the CFRT can be regarded as the limit of the $4L$-weight WFRT as $L \to \infty$.

VIII. FRT ON MULTIPLICATIVE GROUPS

As a nonstandard application of our general definition, we derive the chirp form of the FRT on the multiplicative group of positive real numbers $\mathbb{R}_{\text{pos}}$.

As on any other group, the FRT on this group could be obtained independently by

1) providing a set of orthonormal eigenfunctions and corresponding eigenvalues;
2) choosing a GS k_n;
3) calculating the kernel by using (30) with $\mu_n^a = e^{-j(\pi/2)(h_n + 4k_n)a}$.

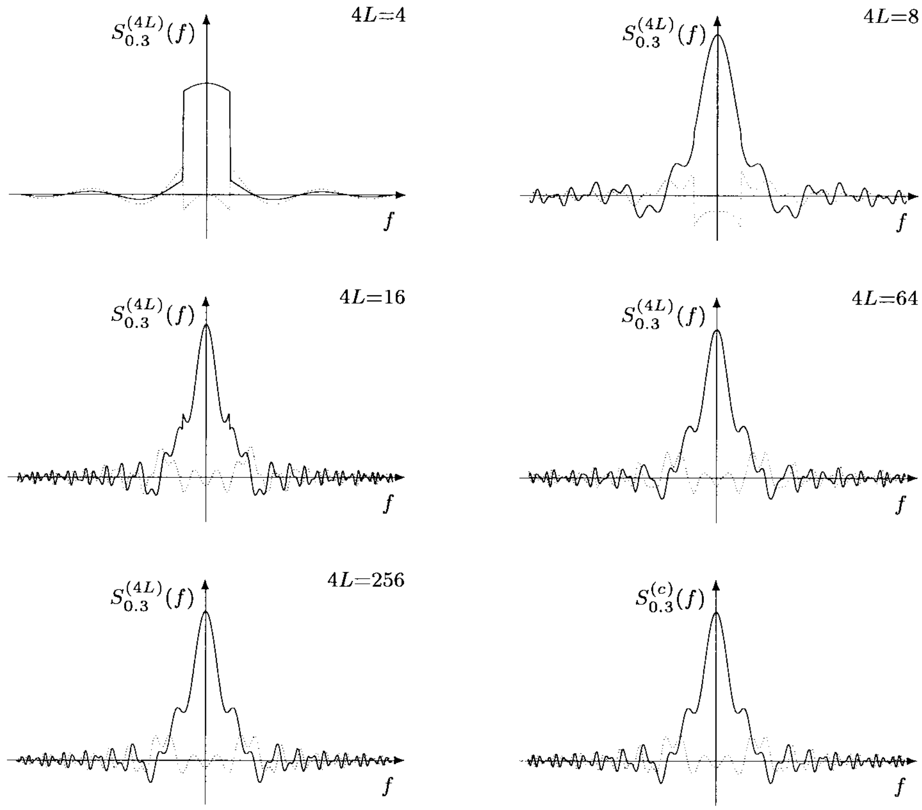


Fig. 6. Standard WFRT with $4L = 4, 8, 16, 64$, and 246 weights, compared with the CFRT of the signal $s(t) = \mathrm{rect}\,(t)$ and the fraction $a = 0.3$. Complex transforms are plotted with the real part in solid and the imaginary part in dotted lines.

However, having established the FRT on $\mathbb{R}$, it is more expedient to exploit the *isomorphism* relating the two groups (see [12] and the Appendix)

$$\exp\colon \quad \mathbb{R} \to \mathbb{R}_{\mathrm{pos}}$$

which is a one-to-one mapping of elements of the additive group $\mathbb{R}$ onto elements of the multiplicative group $\mathbb{R}_{\mathrm{pos}}$ and, moreover, of any function $g(u), u \in \mathbb{R}$ into a function

$$\tilde{g}(v) = g(\log v). \tag{47}$$

Hence, we can obtain the kernel of the chirp FRT on $\mathbb{R}_{\mathrm{pos}}$ from (39), namely

$$\psi_a(f,t) = K_a e^{j\pi B_a((\log f)^2 + (\log t)^2)} e^{-j2\pi C_a \log f \log t}.$$

Now, if we know a chirp-FRT $S_a(f)$ defined on $\mathbb{R}$ of a signal $s(t), t \in \mathbb{R}$, using (47), we find that on $\mathbb{R}_{\mathrm{pos}}$, the chirp-FRT $\tilde{S}_a(f)$ of $\tilde{s}(t) = s(\log t)$ is given by $S_a(\log f)$. Thus, reconsidering the example of Section VII, where $s(t) = \mathrm{rect}\,(t)$, we find that the FRT of $\tilde{s}(t) = \mathrm{rect}\,(\log t)$ is obtained from (41) by substituting $\log f$ for f. The illustration of Fig. 3 is also valid for $\tilde{S}_a(f)$, provided that the frequency axis has a logarithmic scale.

Of course, in $\mathbb{R}_{\mathrm{pos}}$, the weighted FRT is still given by (44), where on a multiplicative group, $-t$ and $-f$ must be replaced by $1/t$ and $1/f$, respectively.

IX. Conclusions

In this paper, we presented a general approach for defining the FRT, discussed its multiplicity, and found a place for the definitions found in the literature. Thus, for a class of signals where the FRT can be defined, we are able to build many different FRT's, some of which may be useful in applications, whereas others may not. For instance, the chirp-FRT has received much attention in applications, whereas the weighted FRT would appear to be less important.

Another problem is the relationship between the different types of FRT on a given group. For example, although the chirp-FRT and the weighted-FRT can be obtained by the same eigenfunction approach, they are very different (compare Figs. 2 and 3 with Figs. 4 and 5). Their relationship is not obvious but can be seen to be of some relevance and is the subject of a separate paper [10].

Appendix
Proof of Theorem 1

We prove that for any self-dual LCA group G, we can find a countable set $\{\varphi_n(t)\}, n = 1, 2, \cdots$ of real orthonormal eigenfunctions of the FT that spans the signal space on G. The proof is based on the isomorphism between LCA groups.

An *isomorphism* α: $G_1 \mapsto G_2$ is a mapping that preserves the group structure, i.e., $\alpha(t_1+t_2) = \alpha(t_1)+\alpha(t_2)$, and is also bijective. A theorem in Group Topology [25, p. 98] states that the most general LCA group G is isomorphic to the Cartesian

TABLE II
SOME DFT EIGENVECTORS

N	μ	$\mathbf{w}_n$
1	1	$[\,1\,]$
2	$1/-1$	$\left[\frac{1}{\sqrt{2}\sqrt{2-\sqrt{2}}}\ ;\ \frac{\sqrt{2}-1}{\sqrt{2}\sqrt{2-\sqrt{2}}}\right]$ $\quad \left[\frac{1}{\sqrt{2}\sqrt{2+\sqrt{2}}}\ ;\ -\frac{\sqrt{2}+1}{\sqrt{2}\sqrt{2+\sqrt{2}}}\right]$
3	$1/-1$	$\left[\frac{1}{\sqrt{3-\sqrt{3}}}\ ;\ \frac{\sqrt{3}-1}{2\sqrt{3-\sqrt{3}}}\ ;\ \frac{\sqrt{3}-1}{2\sqrt{3-\sqrt{3}}}\right]$ $\quad \left[\frac{1}{\sqrt{3+\sqrt{3}}}\ ;\ -\frac{\sqrt{3}+1}{2\sqrt{3+\sqrt{3}}}\ ;\ -\frac{\sqrt{3}+1}{2\sqrt{3+\sqrt{3}}}\right]$
	j	$\left[0\ ;\ \frac{1}{\sqrt{2}}\ ;\ -\frac{1}{\sqrt{2}}\right]$
4	$1/1$	$\left[\frac{\sqrt{3}}{2}\ ;\ \frac{1}{2\sqrt{3}}\ ;\ \frac{1}{2\sqrt{3}}\ ;\ \frac{1}{2\sqrt{3}}\right]$ $\quad \left[0\ ;\ \frac{1}{\sqrt{3}\sqrt{2}}\ ;\ -\frac{2}{\sqrt{3}\sqrt{2}}\ ;\ \frac{1}{\sqrt{3}\sqrt{2}}\right]$
	$-1/-j$	$\left[\frac{1}{2}\ ;\ -\frac{1}{2}\ ;\ -\frac{1}{2}\ ;\ -\frac{1}{2}\right]$ $\quad \left[0\ ;\ \frac{1}{\sqrt{2}}\ ;\ 0\ ;\ -\frac{1}{\sqrt{2}}\right]$
5	1	$\left[\frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}}\ ;\ \frac{\sqrt{5}-1}{2\sqrt{2}\sqrt{5-\sqrt{5}}}\ ;\ \frac{\sqrt{5}-1}{2\sqrt{2}\sqrt{5-\sqrt{5}}}\ ;\ \frac{\sqrt{5}-1}{2\sqrt{2}\sqrt{5-\sqrt{5}}}\ ;\ \frac{\sqrt{5}-1}{2\sqrt{2}\sqrt{5-\sqrt{5}}}\right]$
	1	$\left[0\ ;\ \frac{1}{2}\ ;\ -\frac{1}{2}\ ;\ -\frac{1}{2}\ ;\ \frac{1}{2}\right]$
	-1	$\left[\frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}}\ ;\ -\frac{\sqrt{5}+1}{2\sqrt{2}\sqrt{5+\sqrt{5}}}\ ;\ -\frac{\sqrt{5}+1}{2\sqrt{2}\sqrt{5+\sqrt{5}}}\ ;\ -\frac{\sqrt{5}+1}{2\sqrt{2}\sqrt{5+\sqrt{5}}}\ ;\ -\frac{\sqrt{5}+1}{2\sqrt{2}\sqrt{5+\sqrt{5}}}\right]$
	j	$\left[0\ ;\ \frac{\sqrt{5-\sqrt{5}}}{2\sqrt[4]{10}\sqrt{\sqrt{10}-\sqrt{5+\sqrt{5}}}}\ ;\ \frac{\sqrt{\sqrt{10}-\sqrt{5+\sqrt{5}}}}{2\sqrt[4]{10}}\ ;\ -\frac{\sqrt{\sqrt{10}-\sqrt{5+\sqrt{5}}}}{2\sqrt[4]{10}}\ ;\ -\frac{\sqrt{5-\sqrt{5}}}{2\sqrt[4]{10}\sqrt{\sqrt{10}-\sqrt{5+\sqrt{5}}}}\right]$
	$-j$	$\left[0\ ;\ \frac{\sqrt{\sqrt{10}-\sqrt{5-\sqrt{5}}}}{2\sqrt[4]{10}}\ ;\ \frac{\sqrt{5+\sqrt{5}}}{2\sqrt[4]{10}\sqrt{\sqrt{10}-\sqrt{5-\sqrt{5}}}}\ ;\ -\frac{\sqrt{5+\sqrt{5}}}{2\sqrt[4]{10}\sqrt{\sqrt{10}-\sqrt{5-\sqrt{5}}}}\ ;\ -\frac{\sqrt{\sqrt{10}-\sqrt{5-\sqrt{5}}}}{2\sqrt[4]{10}}\right]$

product of LCA groups of $\mathbb{R}$

$$G \sim \mathbb{R}^m \times (\mathbb{R}/\mathbb{Z})^n \times \mathbb{Z}^r \times \mathbb{Z}/Z(N_1) \times \cdots \times \mathbb{Z}/Z(N_s) \quad (48)$$

where $m, n, r, s, N_1, \cdots, N_s$ are suitable natural numbers, and $\sim$ denotes isomorphism.

Keeping in mind that we are looking for eigenfunctions, the class must be confined to self-dual groups so that (48) reduces to

$$G \sim \mathbb{R}^m \times \mathbb{Z}/Z(N_1) \times \cdots \times \mathbb{Z}/Z(N_s).$$

Moreover, eigenfunctions for multidimensional groups can be obtained by composition from those for 1-D groups so that we can limit our search to the cases

$$G \sim \mathbb{R} \quad \text{and} \quad G \sim \mathbb{Z}/Z(N).$$

Case $G \sim \mathbb{R}$: We have seen that on $\mathbb{R}$, a set of real orthonormal eigenfunctions is given by the Hermite–Gauss functions φ_n, as defined in (38a). Now, if α: $\mathbb{R} \mapsto G$ is a direct isomorphism and α^{-1}: $G \mapsto \mathbb{R}$ is its inverse isomorphism, a complete orthonormal set of eigenfunctions is given by $\{\tilde{\varphi}_n\}, n \in \mathbb{N}_0$ with

$$\tilde{\varphi}_n(u) \triangleq \varphi_n(\alpha^{-1}(u)) \text{ with the same eigenvalues } e^{-j(\pi/2)n}.$$

For instance, if $G = \mathbb{R}_{\text{pos}}$ the isomorphism is

$$\exp\text{: } \mathbb{R} \mapsto \mathbb{R}_{\text{pos}}, \quad \log\text{: } \mathbb{R}_{\text{pos}} \mapsto \mathbb{R}$$

and one class of orthonormal eigenfunctions is obtained from

$$\tilde{\varphi}_n(u) = \varphi_n(\log u) = \frac{\sqrt[4]{2}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi} \log u) e^{-\pi(\log u)^2}.$$

Case $G \sim \mathbb{Z}/Z(N)$: In this case, the space of signals $s(t)$ on G is of finite dimensionality (the dimensionality being N) and isomorphic to the space of complex N-ples $\mathbb{C}^N$. If we think of the N-ples as column vectors, the FT of the signal $s(t)$ represented by the N-ple $\boldsymbol{s}$ is represented by the vector $\boldsymbol{S} = \boldsymbol{W}\boldsymbol{s}$, where $\boldsymbol{W} \in \mathbb{C}^{N \times N}$ is the DFT matrix.

Through this representation, the eigenfunctions of the FT on G correspond to the eigenvectors of $\boldsymbol{W}$. It can be seen [31] that the eigenvalues $1, -j, -1, j$ have the following multiplicities:

$$m(1) = \left\lfloor \frac{N}{4} \right\rfloor + 1, \quad m(-j) = \left\lfloor \frac{N+1}{4} \right\rfloor$$
$$m(-1) = \left\lfloor \frac{N+2}{4} \right\rfloor, \quad m(j) = \left\lfloor \frac{N-1}{4} \right\rfloor.$$

Since $\boldsymbol{W}$ is unitary, the existence of N orthonormal complex eigenvectors of $\boldsymbol{W}$ is assured by a fundamental result in Matrix theory (see for example [32, pp. 100–102]). We have thus obtained a complete set of complex orthonormal eigenfunctions of the FT on G. To obtain a set of *real* eigenfunctions, we make use of the fact that the complex conjugate and the real and imaginary parts of an eigenfunction of the FT on G are again eigenfunctions, with the same eigenvalue μ. In fact, if $S(f) = \mu s(f)$, by taking conjugates of both sides in the second of (1), and keeping in mind that $\hat{G} = G$, we obtain

$$s^*(t) = \int_G df \mu^* s^*(f) \psi(f,t).$$

Since $|\mu| = 1$, we have $\mu^* = 1/\mu$ so that

$$\mu s^*(t) = \int_G df s^*(f) \psi(f,t).$$

which, compared with the top part of (1), states that $\mu s^*(f)$ is the Fourier transform of $s^*(t)$. Regarding the real part $\Re s = (s + s^*)/2$ and the imaginary part $\Im s = (s - s^*)/(2j)$, the result follows from the linearity of the FT.

Therefore, given a complex basis $v_1(t), \cdots, v_r(t)$ for the eigenspace V_μ, we can find a real orthonormal basis $u_1(t), \cdots, u_r(t)$ from the sequence of $2r$ real eigenfunctions $\Re v_1(t), \cdots, \Re v_r(t), \Im v_1(t), \cdots, \Im v_r(t)$ through the Gram–Schmidt procedure. By joining the orthonormal bases of the four eigenspaces, we eventually obtain a real orthonormal basis for the N-dimensional signal space on G. Such bases, for $N = 1, \cdots, 5$, are given in Table II.

Incidentally, a basis for $\mathbb{C}^N$ made up of eigenvectors of the DFT is derived in [31] (this is not explicitly indicated). Although not stated there, such a basis is real. It is not orthonormal, but the authors claim an orthonormal basis can be obtained. Indeed, this can be easily accomplished by orthonormalizing the bases of the single eigenspaces.

References

[1] V. Namias, "The fractional order Fourier transform and its applications to quantum mechanics," *J. Inst. Math. Appl.*, vol. 25, pp. 241–265, 1980.

[2] A. C. McBride and F. H. Kerr, "On Namias's fractional Fourier transform," *IMA J. Appl. Math.*, vol. 39, pp. 159–175, 1987.

[3] A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," *J. Opt. Soc. Amer. A.*, vol. 10, no. 10, pp. 2181–2186, Oct. 1993.

[4] L. M. Bernardo and O. D. D. Soares, "Fractional Fourier transforms and imaging," *J. Opt. Soc. Amer. A.*, vol. 11, no. 10, pp. 2622–2626, Oct. 1994.

[5] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier optics," *J. Opt. Soc. Amer. A.*, vol. 12, no. 4, pp. 743–751, Apr. 1995.

[6] D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transforms and their optical implementation: I," *J. Opt. Soc. Amer. A.*, vol. 10, no. 9, pp. 1875–1881, Sept. 1993.

[7] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier transforms and their optical implementation: II," *J. Opt. Soc. Amer. A.*, vol. 10, no. 12, pp. 2522–2531, Dec. 1993.

[8] B. W. Dickinson and K. Steiglitz, "Eigenvectors and functions of the discrete Fourier transform," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 25–31, 1982.

[9] H. M. Ozaktas, B. Barshan, D. Mendlovic, and L. Onural, "Convolution, filtering, and multiplexing in fractional Fourier domains and their relation to chirp and wavelet transforms," *J. Opt. Soc. Amer. A.*, vol. 11, no. 2, pp. 547–559, Feb. 1994.

[10] G. Cariolaro, T. Erseghe, P. Kraniauskas, N. Laurenti, "Multiplicity of fractional Fourier transforms and their relationships," submitted for publication.

[11] G. Cariolaro and P. Kraniauskas, "The fractional discrete Fourier transform," in progress.

[12] G. Cariolaro, *La Teoria Unificata dei Segnali*. Torino, Italy, UTET, 1996.

[13] W. Rudin, *Fourier Analysis on Groups*. New York: Interscience, 1962.

[14] P. J. Higgins, *Introduction to Topological Groups*. London, U.K.: Cambridge Univ. Press, 1974.

[15] N. Bourbaki, *General Topology*, Parts 1–2. Paris, France: Hermann, 1966.

[16] L. Pontriagin, *Topological Groups*. Princeton, NJ: Princeton Univ. Press, 1946.

[17] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Processing*, vol. 42, pp. 3084–3091, Nov. 1994.

[18] H. M. Ozaktas and D. Mendlovic, "Fourier transforms of fractional order and their optical interpretation," *Opt. Commun.*, vol. 101, pp. 163–169, 1993.

[19] D. Mendlovic, Y. Bitran, R. G. Dorsch, A. W. Lohmann, "Optical fractional correlation: Experimental results," *J. Opt. Soc. Amer. A*, vol. 12, no. 8, pp. 1665–1670, Aug. 1995.

[20] H. M. Ozaktas, O. Arikan, M. A. Kutay, G. Bozdaği, "Digital computation of the fractional Fourier transform," *IEEE Trans. Signal Processing*, vol. 44, pp. 2141–2150, Sept. 1996.

[21] C.-C. Shih, "Fractionalization of Fourier transform," *Opt. Commun.*, no. 118, pp. 495–498, Aug. 1, 1995.

[22] S. Liu, J. Zhang, and Y. Zhang, "Properties of the fractionalization of a Fourier transform," *Opt. Commun.*, no. 133, pp. 50–54, Jan. 1, 1997.

[23] S. Liu, J. Jiang, Y. Zhang, and J. Zhang, "Generalized fractional Fourier transforms," *J. Phys. A: Math. Gen.*, vol. 30, pp. 973–981, 1997.

[24] D. H. Bailey and P. N. Swarztrauber, "The fractional Fourier transform and applications," *SIAM Rev.*, vol. 33, no. 3, pp. 389–404, Sept. 1991.

[25] A. Weil, *L'Integration dans les Groupes Topologiques*. Paris, France: Hermann, 1940.

[26] G. J. Tonge, "The television scanning process," *Soc. Motion Picture Television Eng. J.*, vol. 93, pp. 657–666, July 1984.

[27] E. Dubois, "The sampling and reconstruction of time-varying imagery with application in video systems," *Proc. IEEE*, vol. 73, pp. 502–523, Apr. 1985.

[28] M. J. Caola, "Self-Fourier functions," *J. Phys. A: Math. Gen.*, vol. 24, pp. L1143–L1144, 1991.

[29] G. Cincotti, F. Gori, and M. Santarsiero, "Generalized self-Fourier functions," *J. Phys. A: Math. Gen.*, vol. 25, pp. L1191–L1194, 1992.

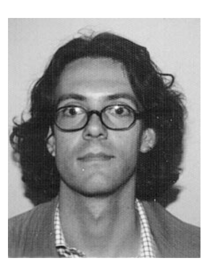
[30] A. I. Zayed, "On the relationship between the Fourier and fractional Fourier transforms," *IEEE Signal Processing Lett.*, vol. 3, Dec. 1996.

[31] J. H. McClellan and T. W. Parks, "Eigenvalue and eigenvector decomposition of the discrete Fourier transform," *IEEE Trans. Audio Electroacoust.*, vol. AU-20, pp. 66–74, Mar. 1972.

[32] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

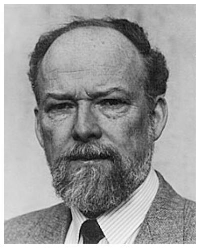
Gianfranco Cariolaro (M'66) was born in 1936 and received the degree in electrical engineering from the University of Padova, Padova, Italy, in 1960. He received the Libera Docenza in electrical communications in 1968 from the same university.

He was appointed Full Professor in 1975, and presently, he is Professor of Electrical Communications and Signal Theory at the University of Padova. His main researches are in the fields of data transmission, image, digital television, multicarrier modulation systems (OFDM), and deep space communications. He is author of several books including *Unified Signal Theory* (Torino, Italy: UTET, 1980).



Tomaso Erseghe was born in Valdagno, Italy, in 1972. He received the degree in telecommunication engineering from the University of Padova, Padova, Italy, in 1996 with a degree thesis on the fractional Fourier transform.

He is currently working as a Research and Development Engineer at Snell and Wilcox, a British broadcast equipment manufacturer. His research areas include image restoration, motion compensation, and the fractional Fourier transform.



Peter Kraniauskas was born in Lithuania in 1939. He received the Electromechanical Engineering degree from the University of Buenos Aires, Buenos Aires, Argentina, in 1962 and the Ph.D. degree from the University of Newcastle upon Tyne, Newcastle upon Tyne, U.K., in 1974.

After 20 years in industry, including research and development posts with Xerox Research and the Recal Electronics Group, he became an Independent Engineering Consultant to teach signal and systems fundamentals in high-tech industry and research establishments. He is the author of the text book *Transforms in Signals and Systems* (Wokingham, U.K.: Addision-Wesley, 1992), whose graphical approach he is currently extending to multidimensional signals.

Nicola Laurenti was born in 1970 in Adria, Italy. He received the laurea degree in electrical engineering in 1995 from the University of Padova, Padova, Italy. He is currently working toward the Ph.D. degree at the same university.

His research interests include the theory of projections, multicarrier modulation systems, multidimensional signal theory, and the fractional Fourier transform.