Problem 25

The 12th term of the Fibonacci sequence is the first term to contain three digits.

What is the first term in the Fibonacci sequence to contain 1000 digits?

Solution

By brute force:

In[75]:=
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Module[{n = 1}, While[IntegerLength[Fibonacci@n] < 1000, n++]; n]
```

Out[75]= 4782

By using logs:

In[77]:=
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Simplify[Fibonacci[n] // FunctionExpand, n ∈ Integers]
```

Out[77]=
$$\frac{-\left(-\frac{2}{1+\sqrt{5}}\right)^n + \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n}{\sqrt{5}}$$

In[101]:=
$$\mathrm{Log10}\left[\frac{-\left(-\frac{2}{1+\sqrt{5}}\right)^n + \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n}{\sqrt{5}}\right] ==$$
$$\mathrm{Log10}\left[-\left(-\frac{2}{1+\sqrt{5}}\right)^n + \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n\right] - \frac{1}{2}\mathrm{Log10}[5] \geq 999$$

Out[101]=
$$\frac{\mathrm{Log}\left[\frac{-\left(-\frac{2}{1+\sqrt{5}}\right)^n + \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n}{\sqrt{5}}\right]}{\mathrm{Log}[10]} == -\frac{\mathrm{Log}[5]}{2\,\mathrm{Log}[10]} + \frac{\mathrm{Log}\left[-\left(-\frac{2}{1+\sqrt{5}}\right)^n + \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n\right]}{\mathrm{Log}[10]} \geq 999$$

In[102]:=
$$\mathrm{Log10}\left[-\left(-\frac{2}{1+\sqrt{5}}\right)^n + \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n\right] \geq 999 + 1/2\,\mathrm{Log10}[5] \;//\; \mathrm{Simplify}$$

Out[102]=
$$\mathrm{Log}[100]\,\mathrm{Log}\left[-\left(-\frac{2}{1+\sqrt{5}}\right)^n + \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n\right] \geq \mathrm{Log}[10]\,(\mathrm{Log}[5] + 999\,\mathrm{Log}[100])$$

But the negative term in the log is going to 0 really quickly:

In[89]:=
$$-\left(-\frac{2}{1+\sqrt{5}}\right)^n \;/.\; n \to 100 \;//\; N$$

Out[89]= -1.26251×10^{-21}

So we'll ignore it.

$$n\,\mathrm{Log}\left[\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)\right] \geq \mathrm{Log}[10]\left(999 + \frac{\mathrm{Log}[5]}{2\,\mathrm{Log}[10]}\right)$$

In[103]:= $n \geq \text{Log}[10] \dfrac{\left(999 + \dfrac{\text{Log}[5]}{2\,\text{Log}[10]}\right)}{\text{Log}\left[\left(\dfrac{1}{2}\left(1 + \sqrt{5}\right)\right)\right]}$ // N

Out[103]= $n \geq 4781.86$

So $n == 4782$.