

Problem 51

By replacing the 1st digit of the 2-digit number $*3$, it turns out that six of the nine possible values: 13, 23, 43, 53, 73, and 83, are all prime.

By replacing the 3rd and 4th digits of $56**3$ with the same digit, this 5-digit number is the first example having seven primes among the ten generated numbers, yielding the family: 56003, 56113, 56333, 56443, 56663, 56773, and 56993. Consequently 56003, being the first member of this family, is the smallest prime with this property.

Find the smallest prime which, by replacing part of the number (not necessarily adjacent digits) with the same digit, is part of an eight prime value family.

Solution

We make an optimisation that we are ignoring numbers which differ from each other only in one place. This is because there are ten different numbers thus created, of which at least three are divisible by 3, so we can't possibly have eight primes. Similarly, we ignore numbers which are the same as each other in only one place, because then mod 3 the number is $(n-1)a+b$, but we need this to be non-0 for eight different a as a varies, and it can't be. We also ignore "differs everywhere" and "differs nowhere".

If we differ in the last place, then we're filling a gap in the last place; we can't have 0 or 5 filling the gap, so we need every single $\{1,2,3,4,6,7,8,9\}$ to make a prime if we fill in the differing places.

Hence we require a multiple of 3 digits to be varying together, if the last digit is varying: $__1_1$, $__2_2$, $__3_3$ has gone through 0,1,2 mod 3.

In summary, if it involves digit n then we need a multiple of 3 digits to be varying; we need $2..n-2$ digits varying.

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In[14]:= n = 6;
primes = IntegerDigits /@ Prime[Range[PrimePi[10n-1] + 1, PrimePi[10n]]];
subs = Join[Subsets[Range[n - 1], {2, n - 2}],
Append[#, n] & /@ Subsets[Range[n - 1], {2, n - 3, 3}]];

In[17]:= gathered = Function[{sub},
With[{numsWhichDifferInSpecifiedPlace = Select[primes, 1 == Length@
DeleteDuplicates@Extract[#, List /@ Complement[Range[n], sub]] &]},
With[{ans = Gather[numsWhichDifferInSpecifiedPlace,
Extract[#1, List /@ sub] == Extract[#2, List /@ sub] &]},
With[{selected = Select[ans, Length[#] == 8 &]}, If[Length[selected] > 0,
Print[selected], PrintTemporary["not found" <> ToString[sub]]]];
ans
]]] /@ subs;
Select[Flatten[gathered, 1], Length[#] == 8 &]

Out[18]:= {{1, 2, 1, 3, 1, 3}, {2, 2, 2, 3, 2, 3}, {3, 2, 3, 3, 3, 3}, {4, 2, 4, 3, 4, 3},
{5, 2, 5, 3, 5, 3}, {6, 2, 6, 3, 6, 3}, {8, 2, 8, 3, 8, 3}, {9, 2, 9, 3, 9, 3}}
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This took 52 seconds, with the answer printed out after about the 40th second. The answer is thus 121313, being part of the following family:

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In[19]:= FromDigits /@%[[1]]
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Out[19]= { 121 313, 222 323, 323 333, 424 343, 525 353, 626 363, 828 383, 929 393 }
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