目录

1 柱坐标系和球坐标系

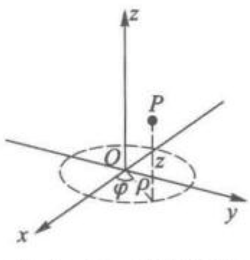
1.1 柱坐标系

柱坐标系相当于把直角坐标系中的 x、y 换为二维极坐标 ρ、φ，同时保留 z 轴. 柱坐标变量 $u_1 = \rho$、$u_2 = \varphi$、$u_3 = z$ 与直角坐标变量 x、y、z 的变换关系如下：

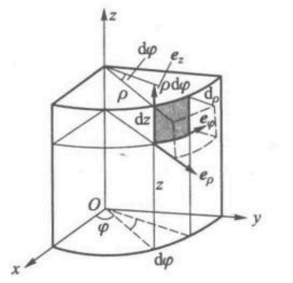
$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \quad \text{或} \begin{cases} \rho = \sqrt{x^2+y^2} \\ \tan\varphi = \frac{y}{x} \\ z = z \end{cases}$$

柱坐标系三个变量的范围：

$$0 \le \rho < +\infty, \quad 0 \le \varphi < 2\pi, \quad -\infty < z < +\infty$$



(a) 柱坐标系



(b) 柱坐标系的面元与体积元

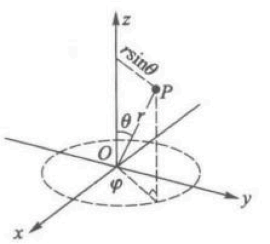
1.2 球坐标系

球坐标系的三个坐标变量是矢径的长度 r、径矢与 z 轴的夹角 θ, 和径矢在 xy 平面上的投影与 x 轴的夹角 φ. 球坐标变量比 $u_1 = r, u_2 = \theta, u_3 = \varphi$ 与直角坐标变量 x, y, z 的变换关系如下：

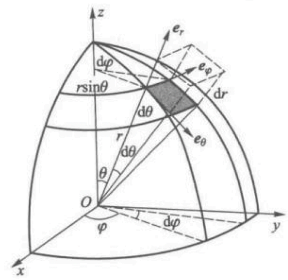
$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \quad \text{或} \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \tan\varphi = \frac{y}{x} \end{cases}$$

球坐标系三个变量的范围：

$$0 \le r < +\infty, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi$$



(c) 球坐标系



(d) 球坐标系的面元与体积元

2 矢量分析提要

2.1 哈密顿算子或称耐普拉算子

直角坐标

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

柱坐标

$$\nabla = \frac{\partial}{\partial \rho}\vec{e_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\vec{e_\varphi} + \frac{\partial}{\partial z}\vec{e_z}$$

球坐标

$$\nabla = \frac{\partial}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\vec{e_\varphi}$$

算符 $\nabla\cdot\nabla$ 常写作 ∇^2，叫作拉普拉斯算符.

推导过程

柱坐标

$$\begin{aligned}
\because \vec{e_\rho} &= \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j} \\
\vec{e_\varphi} &= -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j} \\
\therefore \begin{pmatrix} \vec{e_\rho} \\ \vec{e_\varphi} \\ \vec{e_z} \end{pmatrix} &= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} \\
\begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} &= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{e_\rho} \\ \vec{e_\varphi} \\ \vec{e_z} \end{pmatrix} \\
\text{又} \because \frac{\partial}{\partial \rho} &= \frac{\partial x}{\partial \rho}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \rho}\frac{\partial}{\partial z} \\
&= \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \varphi} &= \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \varphi}\frac{\partial}{\partial z}
\end{aligned}$$

$$= -\rho\sin\varphi\frac{\partial}{\partial x} + \rho\sin\varphi\frac{\partial}{\partial y}$$

$$\therefore \begin{pmatrix} \frac{\partial}{\partial\rho} \\ \frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\frac{1}{\rho}\sin\varphi & 0 \\ \sin\varphi & \frac{1}{\rho}\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\rho} \\ \frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\therefore \nabla = \left(\cos\varphi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial\varphi}\right)(\cos\varphi\vec{e_\rho} - \sin\varphi\vec{e_\varphi})$$

$$+\left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\varphi\frac{\partial}{\partial\varphi}\right)(\sin\varphi\vec{e_\rho} + \cos\varphi\vec{e_\varphi}) + \frac{\partial}{\partial z}\vec{e_z}$$

化简得

$$\nabla = \frac{\partial}{\partial\rho}\vec{e_\rho} + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\vec{e_\varphi} + \frac{\partial}{\partial z}\vec{e_z}$$

球坐标

$$\because \vec{e_r} = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$$

$$\vec{e_\varphi} = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}\ (\vec{k}\text{分量为}0)$$

$$\vec{e_\theta} = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k}$$

$$\therefore \begin{pmatrix} \vec{e_r} \\ \vec{e_\theta} \\ \vec{e_\varphi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

$$\begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \vec{e_r} \\ \vec{e_\theta} \\ \vec{e_\varphi} \end{pmatrix}$$

$$\text{又} \because \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z}$$

$$= \sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z}$$

$$= r\cos\theta\cos\varphi\frac{\partial}{\partial x} + r\cos\theta\sin\varphi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\varphi}\frac{\partial}{\partial z}$$

$$= -r\sin\theta\sin\varphi\frac{\partial}{\partial x} + r\sin\theta\cos\varphi\frac{\partial}{\partial y}$$

$$\therefore \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \frac{1}{r}\cos\theta\cos\varphi & -\frac{\sin\varphi}{r\sin\theta} \\ \sin\theta\sin\varphi & \frac{1}{r}\cos\theta\sin\varphi & \frac{\cos\varphi}{r\sin\theta} \\ \cos\theta & -\frac{1}{r}\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}$$

$$\therefore \nabla = \left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)(\sin\theta\cos\varphi\vec{e_r} + \cos\theta\cos\varphi\vec{e_\theta} - \sin\varphi\vec{e_\varphi})$$
$$+ \left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)(\sin\theta\sin\varphi\vec{e_r} + \cos\theta\sin\varphi\vec{e_\theta} + \cos\varphi\vec{e_\varphi})$$
$$+ \left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)(\cos\theta\vec{e_r} - \sin\theta\vec{e_\theta})$$

化简得

$$\nabla = \frac{\partial}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e_\varphi}$$

2.2 场

标量场：空间各点存在着一个标量 φ，它的数值是空间位置的函数.

矢量场：空间各点存在着一个矢量 $\vec{E}$，它的大小和方向是空间位置的函数.

2.3 标量场梯度

2.3.1 定义

梯度：一个空间位置函数的的变化率. 它沿方向微商最大的方向，数值上等于这个最大的方向微商 $\frac{\partial\varphi}{\partial n}$，其中沿 Δn 方向的方向微商为

$$\frac{\partial\varphi}{\partial n} = \lim_{\Delta n\to 0}\frac{\Delta\varphi}{\Delta n}$$

Δn 的方向是两等值面间最短的位移矢量.

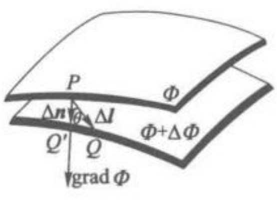


图 1: 标量场的梯度

梯度意义：空间某点标量场函数的最大变化率，刻画了标量场的空间分布特征. 已知梯度即可求沿任一方向的方向导数.

标量场的梯度是个矢量场.

2.3.2 坐标表示式

直角坐标

$$\nabla\Phi=\frac{\partial\Phi}{\partial x}\vec{i}+\frac{\partial\Phi}{\partial y}\vec{j}+\frac{\partial\Phi}{\partial z}\vec{k}$$

柱坐标

$$\nabla\Phi=\frac{\partial\Phi}{\partial\rho}\vec{e_\rho}+\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}\vec{e_\varphi}+\frac{\partial\Phi}{\partial z}\vec{e_z}$$

球坐标

$$\nabla\Phi=\frac{\partial\Phi}{\partial r}\vec{e_r}+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\vec{e_\varphi}$$

2.4 矢量场通量和散度

2.4.1 定义

通量：矢量场通过一个截面的面积分为通量

$$\Phi_A=\iint_S \boldsymbol{A}\cdot d\boldsymbol{S}=\iint_S A\cos\theta dS$$

式中 θ 为 A 与面元 dS 的法线之间的夹角.

散度：令 S 为一闭合曲面，包含体积为 ΔV. 当体积趋向于空间某点 P，$\Delta V\to 0$，Φ_A 也趋于零. 若两者之比有一极限，则这极限值为矢量场在 P 点的散度，记作 $div\,\boldsymbol{A}$ 或 $\nabla\cdot\boldsymbol{A}$.

$$\nabla\cdot\boldsymbol{A}=\lim_{\Delta V\to 0}\frac{\Phi_A}{\Delta V}=\lim_{\Delta V\to 0}\frac{\oiint_S \boldsymbol{A}\cdot d\boldsymbol{S}}{\Delta V}$$

散度的意义：用来描述空间某一范围内场的发散或汇聚，具有局域性质.

矢量场的散度是个标量场.

2.4.2 散度的坐标表示式

直角坐标

$$\nabla\cdot\boldsymbol{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

柱坐标

$$\nabla\cdot\boldsymbol{A}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}$$

球坐标

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^2A_r)\right]+\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\theta)\right]+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

2.5 矢量场的环量和旋度

2.5.1 定义

环量：矢量场 $\boldsymbol{A}$ 沿闭合回路 L 的线积分为环量

$$\Gamma_A = \oint_L \boldsymbol{A} \cdot d\boldsymbol{l}$$

旋度：令 ΔS 为一闭合曲线包围的面积。当回路缩小至空间某点 P，$\Delta S \to 0$，Γ_A 也趋于零. 若两者之比有一极限，则这极限值为矢量场在 P 点的旋度.$\boldsymbol{A}$ 的旋度记作 $curl\,\boldsymbol{A}$ 或 $rot\,\boldsymbol{A}$，或 $\nabla \times \boldsymbol{A}$.

$$(\nabla \times \boldsymbol{A})_n = \lim_{\Delta S \to 0} \frac{\Delta \Gamma_A}{\Delta S} = \lim_{\Delta S \to 0} \frac{\oint_L \boldsymbol{A} \cdot d\boldsymbol{l}}{\Delta S}$$

旋度的意义：$\Gamma_A = 0$ 表明在区域内无涡旋状态，场线不闭合.$\Gamma_A \neq 0$ 表明在区域内存在涡旋状态，场线闭合.

矢量场的旋度是个矢量场.

2.5.2 旋度的坐标表示式

直角坐标

$$\begin{aligned}\nabla \times \boldsymbol{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}\end{aligned}$$

柱坐标

$$\begin{aligned}\nabla \times \boldsymbol{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e_\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\vec{e_\varphi} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\vec{e_z} \\ &= \begin{vmatrix} \vec{e_\rho} & \vec{e_\varphi} & \vec{e_z} \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho}\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & A_\varphi & A_z \end{vmatrix}\end{aligned}$$

球坐标

$$\begin{aligned}\nabla \times \boldsymbol{A} &= \left[\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial \varphi}\right)\right]\vec{e_r} + \left[\frac{1}{r}\,\frac{1}{\sin\theta}\,\frac{\partial A_r}{\partial \varphi} - \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\varphi)\right)\right]\vec{e_\theta} \\ &\quad + \left[\frac{1}{r}\,\frac{\partial(rA_\theta)}{\partial r} - \frac{1}{r}\,\frac{\partial A_r}{\partial \theta}\right]\vec{e_\varphi} \\ &= \begin{vmatrix} \vec{e_r} & \vec{e_\theta} & \vec{e_\varphi} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \\ A_r & A_\theta & A_\varphi \end{vmatrix}\end{aligned}$$

2.6 矢量的定理

高斯定理：矢量场通过任意闭合曲面 S 的通量等于它包含的体积 V 内的散度的积分

$$\oiint_S \boldsymbol{A} \cdot d\boldsymbol{S} = \iiint_V \nabla \cdot \boldsymbol{A} dV$$

斯托克斯定理：矢量场在任意闭合回路 L 上的环量等于以它为边界的曲面 S 上旋度的积分

$$\oint_L \boldsymbol{A} \cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S}$$

2.7 一些定理

$$\nabla(fg) = f(\nabla g) + (\nabla f)g$$
$$\nabla \times (\nabla f) = 0$$
$$\nabla \cdot (\nabla \times \vec{f}) = 0$$
$$\nabla \times (\nabla \times \vec{f}) = \nabla(\nabla \cdot \vec{f}) - \nabla^2 \vec{f}$$

3 静电场