Double inverted pendulum stabilization around equilibrium point $\theta_1 = \theta_2 = 0$

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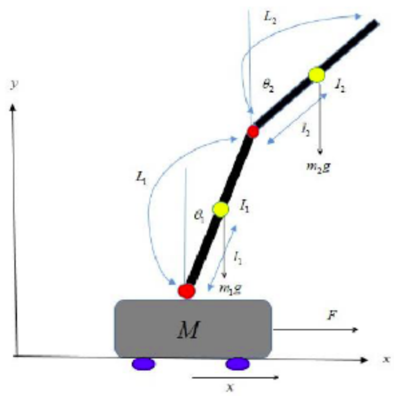

System parameters:

- L_1, L_2–lenghts of bottom and top pendulum $l_i = \frac{L_i}{2}$
- m_1, m_2– masses of the lower and upper pendulum, respectively
- $I_i = \frac{1}{3}mL_i^2$ – inertia moment
- θ_1, θ_2– angles of deviation from vertical of the lower and upper pendulum, respectively (state variables)
- M – mass of cartpole
- x – position of cartpole (state vatiable)
- F – applied force (control variable)
- $g = 9.81$ – gravitational constant

Equations of moution:

Lagrange Function L:

$$L = T - V \tag{1}$$

T – kinetic energy
V– potential energy

$$T_1 = \frac{1}{2}M\dot{x}^2 \tag{2}$$

$$T_2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}\left(m_1\left(\frac{L_1}{2}\right)^2 + I_1\right)\dot{\theta}_1^2 + m_1\frac{L_1}{2}\dot{x}\dot{\theta}_1\cos(\theta_1) \tag{3}$$

$$T_3 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2L_1^2\dot{\theta}_1^2 + \frac{1}{2}\left(m_2\left(\frac{L_2}{2}\right)^2 + I_2\right)\dot{\theta}_2^2 + m_2L_1\dot{x}\dot{\theta}_1\cos(\theta_1) + m_2\frac{L_2}{2}\dot{x}\dot{\theta}_2\cos(\theta_2) + m_2L_1\frac{L_2}{2}\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \tag{4}$$

$$V_1 = 0,\ \ V_2 = m_1g\frac{L_1}{2}\cos(\theta_1),\ \ V_3 = m_2g\left(L_1\cos(\theta_1) + \frac{L_2}{2}\cos(\theta_2)\right) \tag{5}$$

$$L=\sum_{i=1}^{3}(T_i-V_i) \tag{6}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=F,\ \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=0,\ \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}=0 \tag{7}$$

$$l_i=\frac{L_i}{2},\ i=1,2$$

Finally we get:

$$\left(M+m_1+m_2\right)\ddot{x}+\left(m_1l_1+m_2L_1\right)\left(cos(\theta_1)\ddot{\theta}_1-sin(\theta_1)\dot{\theta}_1^2\right)+m_2l_2\left(cos(\theta_2)\ddot{\theta}_2-sin(\theta_2)\dot{\theta}_2^2\right)=F \tag{8}$$

$$\left(m_1l_1+m_2L_1\right)cos(\theta_1)\ddot{x}+\left(m_1l_1^2+m_2L_1^2+I_1\right)\ddot{\theta}_1+m_2l_2L_1\left(cos(\theta_1-\theta_2)\ddot{\theta}_2+sin(\theta_1-\theta_2)\dot{\theta}_2^2\right)-g\left(m_1l_1+m_2L_1\right)sin(\theta_1)=0 \tag{9}$$

$$m_2l_2cos(\theta_2)\ddot{x}+m_2L_1l_2\left(cos(\theta_1-\theta_2)\ddot{\theta}_1-sin(\theta_1-\theta_2)\dot{\theta}_1^2\right)+\left(m_2l_2^2+I_2\right)\ddot{\theta}_2-gm_2l_2sin(\theta_2)=0 \tag{10}$$

Linearize system around equilibrium point:

$$\theta_1\approx\theta_2\approx 0$$

$$cos(\theta_i)\approx 1,\ \ sin(\theta_i)\approx\theta_i$$

$$\theta_1-\theta_2\approx 0$$

$$\dot{\theta}_1\approx\dot{\theta}_2\approx 0$$

$$\begin{pmatrix} \left(M+m_1+m_2\right) & \left(m_1l_1+m_2L_1\right) & m_2l_2 \\ \left(m_1l_1+m_2L_1\right) & \left(m_1l_1^2+m_2L_1^2+I_1\right) & m_2l_2L_1 \\ m_2l_2 & m_2L_1l_2 & \left(m_2l_2^2+I_2\right) \end{pmatrix}\begin{pmatrix}\ddot{x}\\ \ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 \\ 0 & -g\left(m_1l_1+m_2L_1\right) & 0 \\ 0 & 0 & -gm_2l_2 \end{pmatrix}\begin{pmatrix}x\\ \theta_1\\ \theta_2\end{pmatrix}=\begin{pmatrix}F\\ 0\\ 0\end{pmatrix} \tag{11}$$

Control Design:
In the state space our equations transforms to

$$\dot{X}=AX+BU$$

$$A=M^{-1}N,\ \ B=M^{-1}F$$

$$M=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(M+m_1+m_2\right) & \left(m_1l_1+m_2L_1\right) & m_2l_2 \\ 0 & 0 & 0 & \left(m_1l_1+m_2L_1\right) & \left(m_1l_1^2+m_2L_1^2+I_1\right) & m_2l_2L_1 \\ 0 & 0 & 0 & m_2l_2 & m_2L_1l_2 & \left(m_2l_2^2+I_2\right) \end{pmatrix}$$

$$N=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g\left(m_1l_1+m_2L_1\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & -gm_2l_2 & 0 & 0 & 0 \end{pmatrix},\ \ F=\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix},\ \ X=\begin{pmatrix}x\\ \theta_1\\ \theta_2\\ \dot{x}\\ \dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}$$

Then we use LQR control

$$J=\int xQx+uRu$$

$$Q=E,\ E-6\times 6\ identity\ matrix$$

$$R=1$$

$$u=-KX$$