Derivation of the Poisson Distribution:

Assume 'n' occurences in interval 't'.

Assume t broken into interval δt sufficiently small that we have one occurence in each interval. Then by assumption 3. (with X = random variable representing the number of occurences in interval δt)

$$E(X, \delta t) = \lambda \delta t$$

As X can be either 0 or 1

$$E(X, \delta t) = 0 \times P(X=0; \delta t) + 1 \times P(X=1; \delta t)$$
$$= P(X=1; \delta t)$$

So the above becomes

$$P(X=1; \delta t) = \lambda \delta t$$

$$\Rightarrow \quad P(X=0; \delta t) = 1 - \lambda \delta t$$

Want to find $P(X=0)$, $P(X=1)$ etc.

Consider $P(X=0; t+\delta t)$. Then $X=0$ in interval

0 to t & $X=0$ in interval t to $t+\delta t$.

Independence (assumption 2) means

$$P(X=0;t+\delta t) = P(X=0;t) \times P(X=0;\delta t)$$

$$= P(X=0;t) \times (1-\lambda\delta t)$$

$$= P(X=0;t) - \lambda P(X=0;t)\,\delta t.$$

$$\Rightarrow P(X=0;t+\delta t) - P(X=0;t) = -\lambda P(X=0;t)\,\delta t.$$

$$\Rightarrow \frac{P(X=0;t+\delta t)-P(X=0;t)}{\delta t} = -\lambda P(X=0;t).$$

limit as $\delta t \to 0$ gives:

$$\frac{dP}{dt} = -\lambda P \Rightarrow \frac{1}{P}\frac{dP}{dt} = -\lambda.$$

$$\Rightarrow \ln P = -\lambda t + c \Rightarrow P = Ae^{-\lambda t}$$

For 0 ~~length~~ length interval we must have 0 events. So $P(X=0;0) = 1$.

$$\Rightarrow 1 = Ae^{-\lambda 0} \Rightarrow \underline{\underline{A=1}} \Rightarrow P(X=0;t) = e^{-\lambda t}$$

Now consider n events in interval $t+\delta t$.

There are two options:

1) $n-1$ events in interval t & 1 event in interval δt.

2) n events in interval t & 0 events in interval δt.

$$\Rightarrow P(X=n;t+\delta t) = P(X=n;t) \times P(X=0;\delta t) + P(X=n-1;t) \times P(X=1;\delta t).$$

$$\Rightarrow P(X=n;t+\delta t) = P(X=n;t) \times (1-\lambda\delta t) + P(X=n-1;t)\lambda\delta t.$$

$$\Rightarrow P(X=n;t+\delta t) = P(X=n;t) - \lambda P(X=n;t)\delta t + \lambda P(X=n-1;t)\delta t.$$

$$\Rightarrow \frac{P(X=n;t+\delta t) - P(X=n;t)}{\delta t} = -\lambda P(X=n;t) + \lambda P(X=n-1;t)$$

Taking the limit as $\delta t \to 0$ gives.

$$\frac{dP(X=n;t)}{dt} = -\lambda P(X=n;t) + \lambda P(X=n-1;t)$$

$$\Rightarrow \frac{dP(X=n;t)}{dt} + \lambda P(X=n;t) = \lambda P(X=n-1;t)$$

Integrating factor $e^{\int \lambda dt} = e^{\lambda t}$. Multiplying through gives.

$$e^{\lambda t}\frac{dP(X=n;t)}{dt} - \lambda e^{\lambda t} P(X=n;t).$$

$$= \lambda e^{\lambda t} P(X=n-1;t).$$

$$\Rightarrow \frac{d e^{-\lambda t} P(X=n;t)}{dt} = \lambda e^{\lambda t} P(X=n-1;t)$$

Given $n=1$ we have.

$$\frac{d e^{-\lambda t} P(X=1;t)}{dt} = \lambda e^{\lambda t} \times e^{-\lambda t} = \lambda$$

$$\Rightarrow e^{-\lambda t} P(X=1;t) = \lambda t + c$$

When $t=0$ $P(X=1;t)=0$ $\therefore$

$$e^{-\lambda \times 0} \times 0 = \lambda \times 0 + c. \Rightarrow c=0$$

$$\Rightarrow P(X=1;t) = \lambda t e^{\lambda t}$$

Induction $\Rightarrow$ $P(n;t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$

for $n=0,1,2,3,\dots$

Mean

$$P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda}$$

λ = average number of events in interval of concern.

$$E(X) = \sum_{x=0}^{\infty} x P(X=x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}.$$

$$\sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = 0 + \lambda e^{-\lambda} + \frac{2\lambda^2 e^{-\lambda}}{2!} + \frac{3\lambda^3 e^{-\lambda}}{3!} + \frac{4\lambda^4 e^{-\lambda}}{4!} + \ldots$$

$$= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right)$$

$$= \lambda e^{-\lambda} \times e^{\lambda} = \underline{\underline{\lambda}}.$$

Variance

$$E(X^2) = \sum_{x=0}^{\infty} x^2 P(X=x) = \sum_{x=0}^{\infty} x^2 \cdot \frac{\lambda^x e^{-\lambda}}{x!}.$$

$$\Rightarrow E(X^2) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x x^2}{x!}.$$

$$\sum_{x=0}^{\infty} \frac{\lambda^x x^2}{x!} = 0 + \lambda + \frac{\lambda^2 2^2}{2!} + \frac{\lambda^3 3^2}{3!} + \frac{\lambda^4 4^2}{4!}$$

$$\Rightarrow \sum_{x=0}^{\infty} \frac{\lambda^x x^2}{x!} = \underbrace{0 + \lambda}_{} + \frac{\lambda^2 \times 2^2}{2!} + \frac{\lambda^3 \times 3^2}{3!} + \frac{\lambda^4 4^2}{4!} + \frac{\lambda^5 5^2}{5!} +$$

$$= \lambda\left(1 + \frac{2\lambda}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \frac{5\lambda^4}{4!} + \dots\right.$$

For Poisson aiming for $Var(X) = \lambda$.

We know $E(X) = \lambda$ & $Var(X) = E(X^2) - E(X)^2$

$$= E(X^2) - \lambda^2$$

$\therefore$ Need $E(X^2) = \lambda^2 + \lambda$, So summation above needs to equal $e^\lambda(\lambda^2 + \lambda) = \lambda(\lambda e^\lambda + e^\lambda)$

Keep working with this in mind.

$$\lambda\left(1 + \frac{2\lambda}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \frac{5\lambda^4}{4!} + \dots\right.$$

$$= \lambda\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \dots\right.$$

$$+ \lambda + \frac{2\lambda^2}{2!} + \frac{3\lambda^3}{3!} + \frac{4\lambda^4}{4!} + \dots$$

$$= \lambda\left(e^\lambda + \lambda\left(1 + \frac{\lambda}{} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} +\right.\right.$$

$$= \lambda \left(e^{\lambda} + \lambda e^{\lambda} \right)$$

$$\Rightarrow E(X^2) = e^{-\lambda} \cdot \lambda \left(e^{\lambda} + \lambda e^{\lambda} \right) = \lambda (1 + \lambda)$$

$$\Rightarrow \underline{\underline{Var(X) = \lambda}}$$