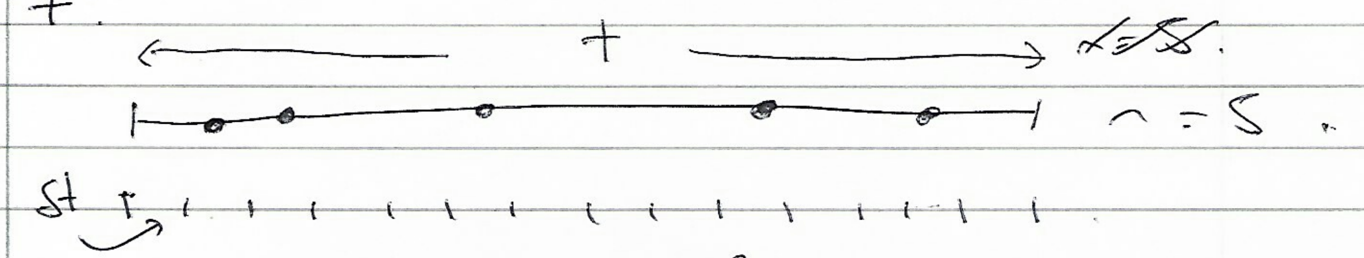
Derivation of the Poisson Distribution

Assume 'n' occurences in interval 't'.



Assume t broken down into intervals δt sufficiently small that ne one occurence in each interval.

Then by assumption 3 (with X = random variable representing the number of occurences in interval δt)

$$E(X, \delta t) = \lambda \delta t \quad (1)$$

As X can be either 0 or 1

$$E(X, \delta t) = 0 \times P(X=0; \delta t) + 1 \times P(X=1; \delta t)$$

$$= P(X=1; \delta t)$$

So (1) becomes:

$$P(X=1; \delta t) = \lambda \delta t$$

$$\Rightarrow P(X=0; \delta t) = 1 - \lambda \delta t$$

Want to find $P(X=0)$, $P(X=1)$ etc for general t.

Consider $P(X=0; t+\delta t)$. This means

$X=0$ in interval 0 to t & $X=0$ in interval t to $t+\delta t$.

The independence (assumption 2) means

$$P(X=0; t+\delta t) = P(X=0; t) \times P(X=0; \delta t)$$

$$= P(X=0;t) \times (1-\lambda\delta t).$$

$$= P(X=0;t) - \lambda P(X=0;t)\,\delta t$$

$$\Rightarrow \frac{P(X=0; t+\delta t) - P(X=0;t)}{\delta t} = -\lambda P(X=0;t)$$

limit as $\delta t \to 0 \Rightarrow \frac{dP}{dt} = -\lambda P$

$$\Rightarrow \frac{1}{P}\frac{dP}{dt} = -\lambda \Rightarrow \ln P = -\lambda t + c.$$

$$\Rightarrow P = Ae^{-\lambda t}$$

$1 = A \times 1$

For $t=0$ length interval $=0$ & we must have 0 events. So $P(X=0;0) = 1 \Rightarrow A=1$

$$P(X=0;t) = e^{-\lambda t} \quad -②$$

Now consider n events in interval $t+\delta t$. There are 2 options.

1) $n-1$ events in interval t & 1 event in interval δt

2) n events in interval t & 0 events in interval δt

$$\Rightarrow P(X=n; t+\delta t) = P(X=n; t) \times P(X=0; \delta t) + P(X=n-1; t) \times P(X=1; \delta t)$$

$$\Rightarrow P(X=n; t+\delta t) = P(X=n; t) \times (1-\lambda \delta t) + P(X=n-1; t) \times \lambda \delta t$$

$$= P(X=n; t) - \lambda P(X=n; t)\,\delta t + P(X=n-1; t) \times \lambda \delta t$$

$$\Rightarrow \frac{P(X=n; t+\delta t) - P(X=n; t)}{\delta t} = -\lambda P(X=n; t) + \lambda P(X=n-1; t)$$

Taking limit as $\delta t \to 0$ we get

$$\frac{dP(X=n)}{dt} = -\lambda P(X=n) + \lambda P(X=n-1)$$

$$\Rightarrow \frac{dP(X=n)}{dt} + \lambda P(X=n) = \lambda P(X=n-1)$$

$$I.F. = e^{\int \lambda\, dt} = e^{\lambda t}$$

Multiplying through gives

$$e^{\lambda t}\frac{dP(X=n)}{dt} + \lambda e^{\lambda t}P(X=n) = \lambda e^{\lambda t}P(X=n-1)$$

$$\Rightarrow \frac{d e^{\lambda t}P(X=n)}{dt} = \lambda e^{\lambda t}P(X=n-1) \quad — 3$$

$n=1$ we get.

$$\frac{d e^{\lambda t}P(X=1)}{dt} = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\Rightarrow e^{\lambda t}P(X=1) = \lambda t + c.$$

$t=0 \Rightarrow P(X=1) = 0$ So $c=0$.

$$P(X=1) = e^{-\lambda t}\lambda t$$

Induction $\Rightarrow P(X=n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$.

$n = 0, 1, 2, 3, 4, \ldots\ldots$

E.g. true for $n=0, n=1$ (previously seen). Assume true for $n-1$.

$$\frac{d e^{\lambda t}P(X=n)}{dt} = \lambda e^{\lambda t}\frac{(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!}$$

$$= \frac{\lambda^n t^{n-1}}{(n-1)!}.$$

Integrating gives:

$$e^{\lambda t} P(X=n) = \frac{\lambda^n t^n}{n!}$$

$$\Rightarrow P(X=n) = \frac{\lambda^n t^n}{n!} e^{-\lambda t}.$$

Mean

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

λ = average number of events per interval ~~of~~ T

$$E(X) = \sum_{x=0}^{\infty} x P(X=x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}.$$

$$\sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = 0 + \lambda e^{-\lambda} + \frac{2\lambda^2 e^{-\lambda}}{2!} + \frac{3\lambda^3 e^{-\lambda}}{3!} + \frac{4\lambda^4 e^{-\lambda}}{4!}.$$

$$= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right.$$

$$= \lambda e^{-\lambda} \times e^{\lambda} = \underline{\underline{\lambda}}.$$

Variance

$$E(X^2) = \sum_{x=0}^{\infty} x^2 P(X=x) = \sum_{x=0}^{\infty} x^2 \times \frac{\lambda^x e^{-\lambda}}{x!}$$

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$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x x^2}{x!}$$

Consider

$$\sum_{x=0}^{\infty} \frac{\lambda^x x^2}{x!} = 0 + \lambda + \frac{\lambda^2 2^2}{2!} + \frac{\lambda^3 3^2}{3!} + \frac{\lambda^4 4^2}{4!}$$

$$\Rightarrow RHS = \lambda \left(1 + \frac{2\lambda}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \right.$$

For Poisson we are aiming for $Var(X) = \lambda$. We know $E(X) = \lambda$ & $Var(X) = E(X^2) - E(X)^2 = E(X^2) - \lambda^2$.

$\therefore$ Need $E(X^2) = \lambda^2 + \lambda$.

$\Rightarrow$ Summation above needs to equal $e^{\lambda}(\lambda^2 + \lambda) = \lambda(\lambda e^{\lambda} + e^{\lambda})$ (*).

Keep working with this in mind.

$$\lambda \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} \right.$$

$$\lambda + \frac{2\lambda^2}{2!} + \frac{3\lambda^3}{3!} + \frac{4\lambda^4}{4!}$$

$$= \lambda \left(e^{\lambda} + \lambda \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \right.\right.$$

$= \lambda(e^{\lambda} + \lambda e^{\lambda})$ match (*) so $Var(X) = \lambda$