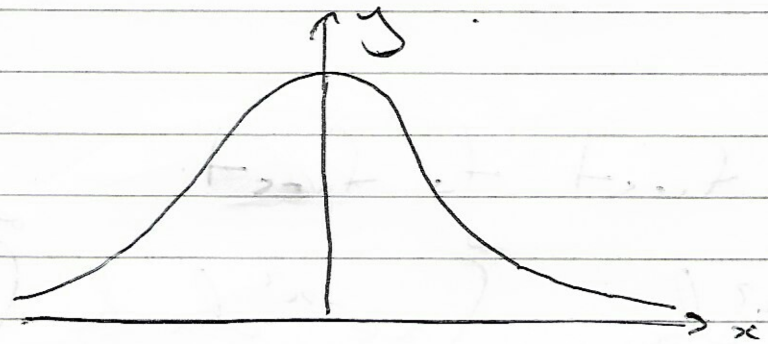
1

The Normal Distribution from First Principles

Based on curve $y = e^{-x^2}$



Symmetric about the y axis.

$\frac{dy}{dx} = -2xe^{-x^2}$. $x = 0 \Rightarrow \frac{dy}{dx} = 0$.

So curve flat when $x = 0$.

To generalise consider stretching in the x direction & the y direction then translating. E.g.

$$f(x) = ae^{-(b(x-c))^2}$$

Then we need to find a, b and c such that:

$$\text{(A)} \quad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(2) (b) $\int_{-\infty}^{\infty} x f(x)\, dx = \mu$.

(c) $\int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2 = \sigma^2$.

Useful first to find:

$$\underset{(1)}{\int_{-\infty}^{\infty} e^{-x^2}\, dx}, \quad \underset{(2)}{\int_{-\infty}^{\infty} x e^{-x^2}\, dx} \;\&\; \underset{(3)}{\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx}$$

(1) let $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$.

then $I = \int_{-\infty}^{\infty} e^{-y^2}\, dy$.

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \times \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} \times e^{-y^2}\, dx\, dy.$$

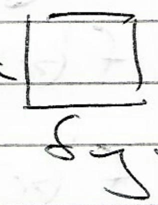
$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

Let

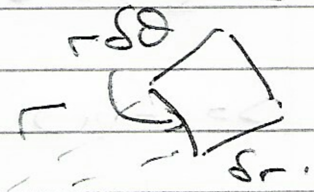
③ $x = r\cos\theta$ $\quad y = r\sin\theta \Rightarrow r^2 = x^2 + y^2$.

E.g. swap to polar coordinates. Then

$$\delta A = \delta x\,\delta y.$$



or $\delta A = r\,\delta r\,\delta\theta$.



So in polar coordinates we have.

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r \; dr\,d\theta.$$

$$\Rightarrow I^2 = \int_0^{2\pi} \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} d\theta.$$

$$\Rightarrow I^2 = \int_0^{2\pi} \left(0 - -\frac{1}{2}\right) d\theta \Rightarrow I^2 = \int_0^{2\pi} \frac{1}{2}\,d\theta.$$

$$\Rightarrow I^2 = \left[\frac{\theta}{2}\right]_0^{2\pi} \Rightarrow I^2 = \pi \Rightarrow \underline{\underline{I = \sqrt{\pi}}}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

② let $I = \int_{-\infty}^{\infty} x e^{-x^2}\,dx$ then

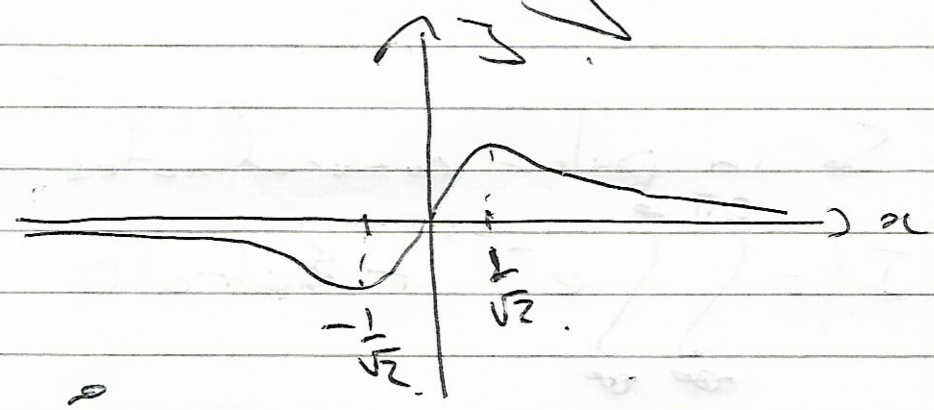
(4) let $f(x) = xe^{-x^2}$. then $f(-x) = -f(x)$.

e.g. $f(x)$ is an odd function.

$$f'(x) = e^{-x^2} + x \times -2xe^{-x^2} = e^{-x^2}(1-2x^2)$$

$$f'(x) = 0 \Rightarrow 1-2x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

So $f(x)$ looks something like.



Clearly $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = 0$.

Finally (3) $I = \int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$.

Can be integrated by parts. Let.

$u = x \qquad \frac{dv}{dx} = xe^{-x^2}$

$\frac{du}{dx} = 1 \qquad v = -\frac{1}{2}e^{-x^2}$

$$\text{So } I = \left[-\frac{1}{2}e^{-x^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 1 \times -\frac{1}{2}e^{-x^2}\,dx$$

(5) $\Rightarrow I = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

So $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

<u>Recall</u> (A) $\int_{-\infty}^{\infty} f(x)\,dx = 1$:

$$\Rightarrow a\int_{-\infty}^{\infty} e^{-b^2(x-c)^2}\,dx = 1.$$

let $y = b(x-c)$ $\Rightarrow dy = b\,dx$.

$\Rightarrow dx = \frac{dy}{b}$

& integral becomes.

$$a\int_{-\infty}^{\infty} e^{-y^2}\cdot\frac{dy}{b} = 1.$$

$$\Rightarrow \frac{a}{b}\int_{-\infty}^{\infty} e^{-y^2}\,dy = 1. \Rightarrow \frac{a}{b}\sqrt{\pi} = 1.$$

$$\Rightarrow a\sqrt{\pi} = b.$$

Recall (B) $\int_{-\infty}^{\infty} x f(x)\,dx = \mu$.

$$\Rightarrow a\int_{-\infty}^{\infty} x e^{-b^2(x-c)^2}\,dx = \mu.$$

(6) Again substituting $y = b(x-c)$

$\Rightarrow dx = \frac{dy}{b}$.

Then the integral becomes:

$$a\int_{-\infty}^{\infty}\left(\frac{y}{b}+c\right)e^{-y^2}\frac{dy}{b} = \mu.$$

$$\Rightarrow \frac{a}{b}\int_{-\infty}^{\infty}\left(\frac{y}{b}+c\right)e^{-y^2}dy = \mu.$$

$$\Rightarrow \frac{a}{b^2}\int_{-\infty}^{\infty} y e^{-y^2}dy + \frac{ac}{b}\int_{-\infty}^{\infty} e^{-y^2}dy = \mu.$$

$$\Rightarrow \frac{ac}{b}\times\sqrt{\pi} = \mu \quad \Rightarrow \underline{\underline{c = \mu}}$$

<u>Recall (C)</u>

$$\sigma^2 = \underbrace{\int_{-\infty}^{\infty} x^2 f(x)\,dx}_{\cup} - \mu^2.$$

$$a\int_{-\infty}^{\infty} x^2 e^{-b^2(x-c)^2}dx = I$$

(7) Substituting $y = b(x-c) \Rightarrow dy = b\,dx$.

$$\Rightarrow \frac{dy}{b} = dx$$

$\Rightarrow \frac{y}{b} + c = x$. Then I becomes

$$a\int_{-\infty}^{\infty} \cancel{x}\left(\frac{y}{b}+c\right)^2 e^{-y^2}\,\frac{dy}{b}.$$

$$\Rightarrow I = \frac{a}{b}\int_{-\infty}^{\infty}\left(\frac{y^2}{b^2} + \frac{2cy}{b} + c^2\right)e^{-y^2}\,dy$$

$$= \frac{a}{b} \times \frac{1}{b^2}\,\frac{\sqrt{\pi}}{2} + \frac{ac^2}{b}\sqrt{\pi}$$

$$= \frac{a}{b^3} \times \frac{\sqrt{\pi}}{2} + c^2 = \frac{a}{b^3} \times \frac{\sqrt{\pi}}{2} + \mu^2.$$

$$\sigma^2 = I - \mu^2 \Rightarrow \sigma^2 = \frac{a}{b^3} \times \frac{\sqrt{\pi}}{2}.$$

$$b^3\, 2\sigma^2 = a\sqrt{\pi} \quad \& \quad a\sqrt{\pi} = b.$$

$$\Rightarrow a^3\pi^{\frac{3}{2}}\, 2\sigma^2 = a\sqrt{\pi} \Rightarrow a^2 = \frac{1}{2\sigma^2\pi}.$$

$$\Rightarrow a = \frac{1}{\sigma\sqrt{2\pi}} \Rightarrow b = \frac{1}{\sigma\sqrt{2}}.$$

(8) $\Rightarrow f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$