Computational Analysis of Felix Baumgartner's 2012 Balloon Jump

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Abstract

Using Python, as well as Numpy, Matplotlib, and Scipy, Felix Baumgartner's jump was modeled with various methods. These are then compared with each other to determine the best approach. Variable drag, together with Euler's method of numerical integration was found to be the most accurate method. He was shown to have reached mach 1.24, and also to have taken at 4 minutes and 5 seconds to reach the ground.

1 Introduction

Felix Baumgartner fell from a helium balloon, at a height of 39045m, to the ground. In this time, he reached a maximum speed of 373 ms^{-1}. It has been confirmed that at some point, Baumgartner did break the sound barrier.[1] That is to say he was moving faster than sound propagates through the air. Modelling this accurately will require use of Newton's second law.

2 Theory

As mentioned, the fall will be though of as a free fall, occurring in earth's atmosphere. An assumption is made that all significant movement is in the negative y direction, i.e. vertically downwards. For an object in free fall in the atmosphere, drag is proportional to the square of the velocity of the particle, assuming the object is somewhat large. That is to say, $F_{drag} = -kv_y^2\hat{\boldsymbol{v}}_y$.

For any given object, this constant k is defined as:

$$k = \frac{C_d \rho_0 A}{2} \tag{1}$$

where C_d is the drag coefficient of the object.[2] This is taken to be 1.15, as this is a typical value for a skydiver, ρ_0 is the air density at sea level, for $20°C$ and regular atmospheric pressure, taken to be $1.2kg\,m^{-3}$. A is the cross sectional area of Baumgartner. This can be used to derive an equation of motion for Baumgartner,

$$m\frac{d^2y}{dt^2} = -kv_y^2\hat{\boldsymbol{v}}_y - mg \tag{2}$$

However, to apply variable air resistance, a new definition of k needs to be used, namely

$$k = \frac{C_d \rho_0 A}{2} e^{\frac{-y}{h}} \tag{3}$$

which models an exponential decay of air resistance with respect to distance from the ground.

3 Method

A Python program was created to solve (2) for various times. Initially, an analytical solution was used. The solutions to which are

$$y = y_0 - \frac{m}{k} \ln \left(\cosh \left(\sqrt{\frac{kg}{m}} t \right) \right), \quad v_y = -\sqrt{\frac{mg}{k}} \tanh \left(\sqrt{\frac{kg}{m}} t \right) \tag{4}$$

Next, Euler's method was employed to give these equations

$$y_{n+1} = y_n + \Delta t \, v_{y(n)} \hat{\boldsymbol{v}}_{y(n)}, \quad v_{y(n+1)} = v_{y(n)} - \Delta t \left(g + \frac{k}{m} v_{y(n)}^2 \hat{\boldsymbol{v}}_{y(n)} \right) \tag{5}$$

This Python program was then modified to account for variable air resistance, from (2). This equation was applied to both (4) and (5), to varying success. Next, the mach number, the speed of sound, at each y value was calculated. Velocity was then plotted as a function of mach to determine if Baumgartner was moving, at any point, faster than the speed of sound. The speed of sound was calculated through

$$v_s = \sqrt{\frac{\gamma RT}{M}} \tag{6}$$

where γ is the adiabatic index of dry air (roughly 1.4)[3], R is the molar gas constant, M is the molar mass of dry air, and T is the temperature at a given altitude. Finally, the variance in acceleration due to gravity because of distance was taken into account, as in reality it does vary with the square of distance

$$a_{grav} = \frac{GM_{earth}}{r^2} \tag{7}$$

4 Results

4.1 Analytical vs. Euler Method

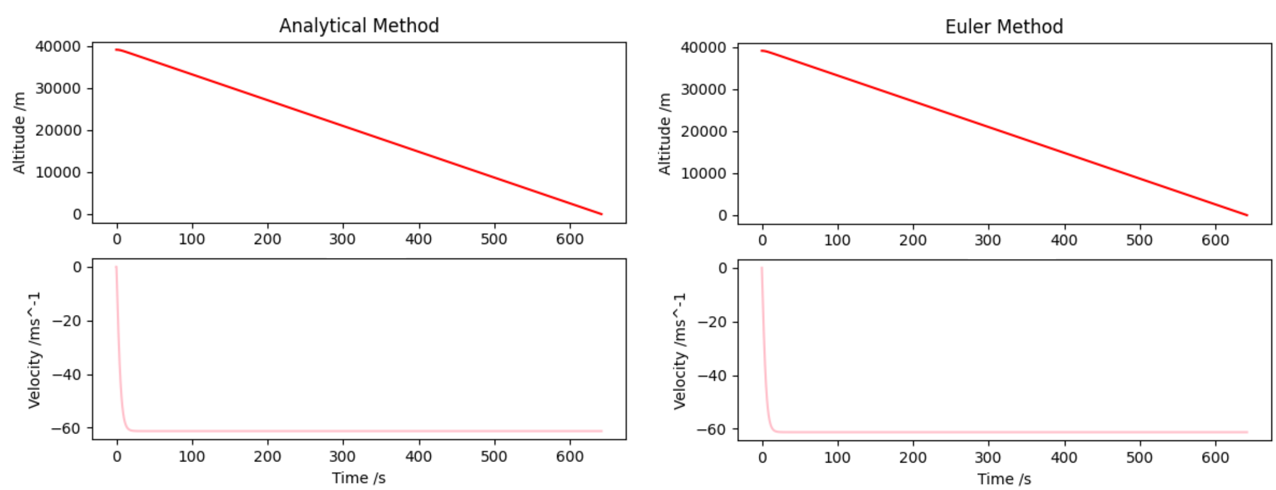

Figure 1: Analytical vs Euler Method for $\Delta t = 0.15s$. There is little difference between them, as Δt is small

Figure 1 shows both of these situations, there is a large constant air resistance, as if he were constantly at ground level. This means that it takes 641 seconds for him to reach the ground, much longer than the 259 seconds recorded. It can also be noted he reaches terminal velocity quickly, within 15 seconds of starting to fall, which is unrealistic for this altitude.

4.2 Validity of Euler Method

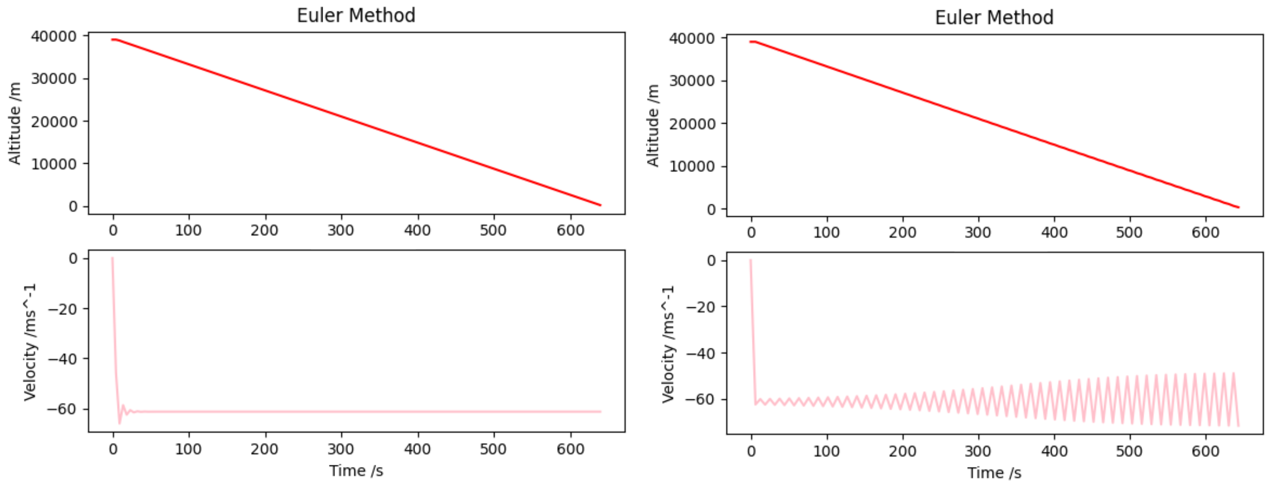

Figure 2: Euler Method for $\Delta t = 4.7$ and $6.4s$. A lack of smoothness, and increase in inaccuracy can be seen.

Increasing Δt has a noticeable effect in Figure 2. The smaller 4.7s data has a noticeable sharpness to it, suggesting a low resolution. However, the larger 6.4s data starts to oscillate around a value, with a significant variance of about $\pm 20ms^{-2}$ near the end. This is a signification deviation from reality.

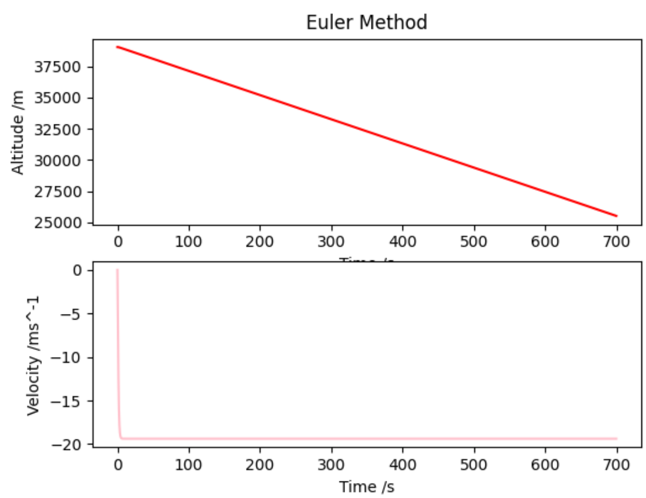

Figure 3: Euler Method with $\left(\frac{k}{m}\right)_{new} = 10\left(\frac{k}{m}\right)_{original}$ Increasing this quantity causes a decrease in terminal velocity.

In Figure 3, a tenfold increase in the quantity k/m causes the terminal velocity to decrease to roughly a third, to $20ms^{-1}$. So, making sure this quantity is well estimated will be important, as drag is a significant factor.

4.3 Variable Drag

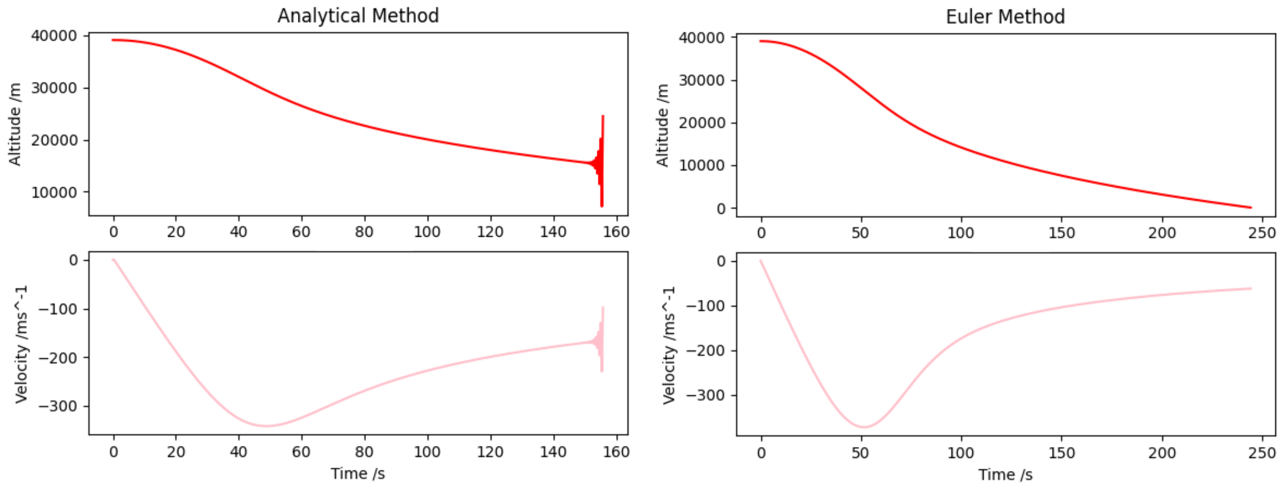

Figure 4: Analytical and Euler method with variable resistance. The analytical method displays strange characteristics due to non-linearity.

In Figure 4, the variable resistance causes some effects. (2) suggests that the ODE describing the motion is non-linear, and due to this variable resistance being a function of y, the exact solution will be of a different form. It is interesting to note, however, it is still accurate up to $y = 18000m$, where it become unstable. As expected, the variable resistance implies a lower resistance higher, which in turn implies a higher terminal velocity. The highest velocity is now $373ms^{-1}$, larger than the previous $19ms^{-1}$ terminal velocity.

4.4 Supersonic Free Fall

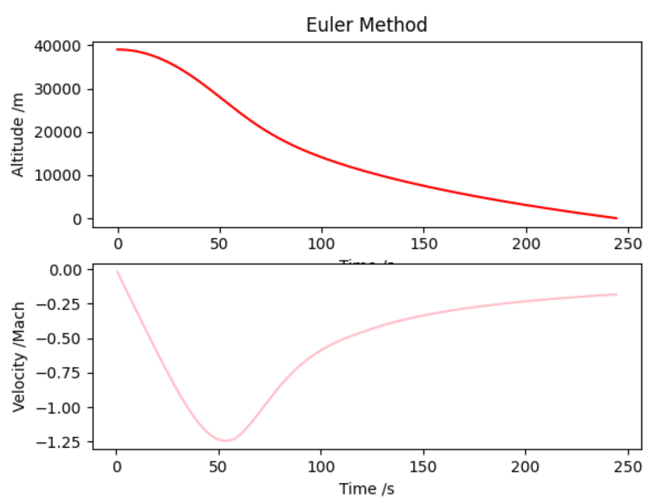

Figure 5: Euler Method, except in terms of speed of sound at each given altitude.

After taking into account the speed of sound, it was found that he did break the sound barrier, and reach a maximum of mach 1.24, at $t = 51.5s$.

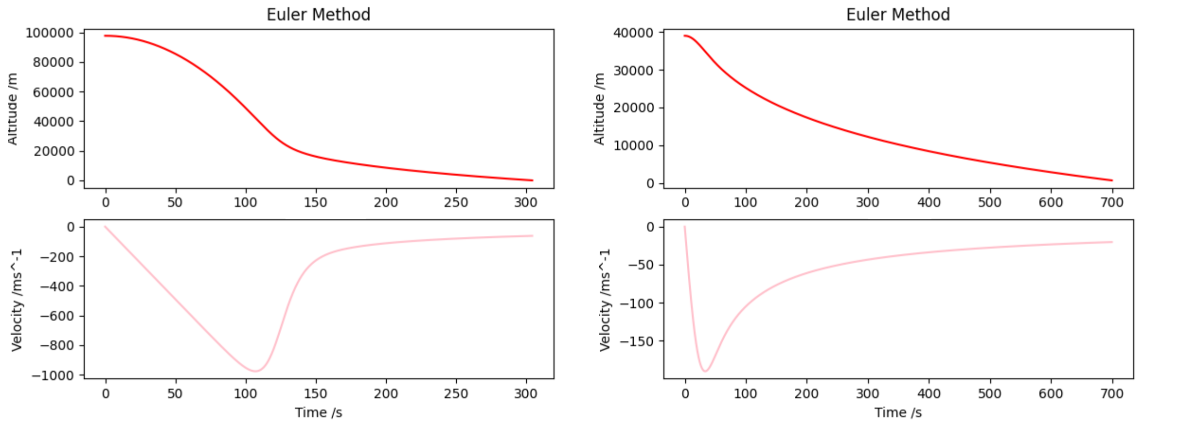

Figure 6: Left: Theoretical flight-path if he started from $y = 100km$. Right: Quantity $C_d A/m$ increased tenfold.

From Figure 6, we see that if he started from much higher, he would have reached a terminal velocity of roughly $v = 1000ms^{-1}$. This would likely result in his death. When $C_d A/m$ is increased, we see a decrease in terminal velocity, and an increase in total flight time. A tenfold increase roughly caused the terminal velocity to reduce to $\frac{1}{10}$ of the original value

4.5 Variable Gravity

Variable Gravity was also implemented. This created a graph very similar to that of Figure 5. However, the total time to reach the ground was $t = 244.99999s$ against the constant gravity $t = 244.29999s$, an increase of $0.7s$.

5 Discussion

Overall, it seems that this solution to modelling the fall is quite accurate. The final time taken to reach the ground, $t = 244.99999s$, is only a 5% error from the true time. This model also suggested he did manage to go supersonic, which is what happened in reality. The modelling of drag is somewhat incomplete, as the cross sectional area of Baumgartner will have changed over time (his angle with respect to his movement will change, causing his cross section to change). Also, it is not fully known how much his suit weighed. He was modelled as an ellipse in the program, while this is not totally accurate, it should have provided a fairly accurate area. The initial result from the analytical solution is quite accurate, but only for constant drag. This would be the ideal solution for situations where drag is near constant, as it is much more computationally efficient than the Euler method.

It was also shown that the affect of the change in gravity was quite small, only about 0.3% of the total error. However, this wasn't very complex to model.

5.1 Improvements

To improve this model further, an analytical solution for variable drag would be preferable, as this would decrease computing time. Apart from that, a function that describes Baumgartner's cross sectional area with respect to his movement would further increase the accuracy of this model, as at times this did change (i.e. he started tumbling half-way through). Computational fluid dynamics would be a good solution to solve this problem, as the complexities of fluid dynamics would be more well described.

Apart from this, it was noticed in the Results section that the size of Δt mattered a lot in the overall accuracy of the model. So, decreasing it as far as possible would further increase the accuracy. Another implementation, the modified Euler method, would provide some increase in accuracy, as it takes the 'average' between two calculated points.

The assumption in equation (3) creates inaccuracy, the air density at a given altitude in reality does not follow an inverse exponential. A more accuracy, piece-wise function, or lookup table, could improve on this.

Another issue is that in reality, Baumgartner used a parachute for the last section of the flight. This was not accounted for in this model, hence why the flight time the model gives is somewhat shorter than the real value. Possibly, a new term could be added to the air resistance function, so that it has a small 'jolt' for the last 30 seconds.

6 Conclusion

Through this model, we have shown and quantified the effect of various assumptions, to do with air density and gravity. A fairly accurate model for free fall in earths atmosphere has been produced, using the relatively simple Euler method. Baumgartner was found to have reached the speed of sound, and in fact a max speed of mach 1.24. It took him 4 minutes and 5 seconds with the model to reach the ground from his original altitude, a similar number to the 4 minutes and 19 seconds it took in real life.

References

[1] Guinness World Records. Felix baumgartner: First person to break sound barrier in freefall, 2014. Available at https://www.guinnessworldrecords.com/records/hall-of-fame/felix-baumgartner-first-person-to-break-sound-barrier-in-freefall.

[2] F.R.S. G. K. Batchelor. An introduction to fluid dynamics. *Cambridge University Press*, page 500, 1963.

[3] A.R.; Coppens A.B.; Sanders J.V. (2000). Kinsler, L.E.; Frey. Fundamentals of acoustics. *New York: John Wiley Sons*, page 483, 2000.