Set-theory

Unit-I

* A Fundamental place in the mainstream of mathematical thinking.

Set : A set is a collection of well defined objects.

eg: The collection of prime numbers

George Cantor

Symbols: $\in$, $\subset$, $\subseteq$, $\cup$ and $\cap$.

$\in \rightarrow$ The symbol should be used between an element and a set and it should not be used between two sets.

ex: If $A = \{1, 2\}$ and $B = \{1, \{1, 2\}, 3, 4\}$

then $A \in B$ ✗

Note:

* Elements of a set are listed only once.
* The order of listing the elements of the set does not change the set.

$\{1, 2, 3, 4, 5\} \rightarrow \{1, 5, 3, 4, 2\}$

Notation:

* A set is usually denoted by capital letters of the English Alphabets A, B, P, Q, X, Y, etc.
* The elements of a set is written culy brackets "{}".

* If x is an element of a set A or x belongs to A, we write $x \in A$.

* If x is not an element of a set A or x does not belongs to A, we write $x \notin A$.

consider the set $A = \{2, 3, 5, 7\}$

$2 \in A$, $3 \in A$, $6 \notin A$

Representation of a set

A set can be represented in any one of the following three ways or forms:

1) Descriptive Form - In descriptive form, a set is described in words.

e.g. The set of all vowels in English alphabets, the set of whole numbers.

2. Set builder Form or Rule form.

In a set builder form, all the elements are described by a rule.

ex: $A = \{x : x$ is a vowel in English alphabets$\}$

$B = \{x \mid x$ is a whole number$\}$

The symbol ':' or '|' stands for "such that."

3. Roster form or Tabular Form.

A set can be described by listing all the elements of the set.

ex: $A = \{a, e, i, o, u\}$, $B = \{0, 1, 2, 3, \ldots\}$

The three dots (...) is called ellipsis. It indicates that the pattern of the listed elements continues in the same order.

(...) → continue on.

Types of set:

1. EMPTY SET or NULL SET

A set consisting of no elements is called the empty set or null set or void set.

It is denoted by ϕ or $\{\}$.

ex: $A = \{x : x$ is an odd integer and divisible by $2\}$

$\therefore A = \{\}$ or ϕ

2. Singleton set

A set which has only one element, is called a singleton set.

ex: i) $A = \{x : 3 < x < 5; x \in N\}$

$A = \{4\}$

ii) The set of all even prime numbers

$B = \{2\}$

An empty set has no elements, so ϕ is a finite set.

3. **Finite set :** A set with finite number of elements is called a finite set.

ex: The set of family members, The set of students in BCA

4. **Infinite set**

A set which is not finite is called an infinite sets.

ex: {5, 10, 15, ...}, The set of stars in the sky.

CARDINAL NUMBER OF A SET:

When a set is finite, it is very useful to know how many elements it has.

* The number of elements in a set is called the cardinal number of the set.

The cardinal number of a set A is denoted by n(A).

ex: $A = \{1, 2, 3, 4, 5, 7, 9, 11\}$

$n(A) = 8$.

5) EQUIVALENT SETS

Two finite sets A and B are said to be equivalent if they contain the same number elements. It is written as $A \equiv B$.

If A and B are equivalent sets, then $n(A) = n(B)$.

ex: Consider $A = \{ball, bat\}$ and $B = \{Tamil, English\}$

Here A is equivalent to B, Because $n(A) = n(B) = 2$.

6) EQUAL SET

Two sets are said to be equal if they contain exactly the same elements, Otherwise they are said to be unequal.

In otherwords, two sets A and B are said to be equal, if

i) Every element of A is also an element of B.

(ii) Every element of B is also an element of A.

ex: $A = \{1, 2, 3, 4\}$ $B = \{4, 2, 3, 1\}$

Since A and B contain exactly the same elements, A and B are equal sets.

* If A and B are equal sets, we write $A = B$.

* If A and B are unequal sets, we write $A \neq B$.

*A set does not change, if one or more elements of the set are repeated.

ex: If given $A = \{a, b, c\}$ and $B = \{a, a, b, b, b, c\}$, then we write $B = \{a, b, c\}$. Since every element of A is also an element of B and every element of B is also an element of A, the sets A and B are equal.

*Equal sets are equivalent sets but equivalent sets need not be equal sets.

7) UNIVERSAL SET.

A universal set is a set which contains all the elements of all the sets under consideration and is usually denoted by U.

ex: If A = { Earth, Mars, Jupiter }, then the universal set U is the planet of Solar System.

Subset

* Let A be any two sets. If every element of A is also an element of B, then A is called a subset of B. We write $A \subseteq B$.

: $A \subseteq B$ is read as " A is a subset of B "

Thus $A \subseteq B$, if $a \in A$ implies $a \in B$.

* If A is not a subset of B, we write $A \nsubseteq B$.

Clearly if A is a subset of B, then $n(A) \leq n(B)$.

Since every element of A is also an element of B, the set B must have at least as many elements as A, thus $n(A) \leq n(B)$.

The other way is also true. Suppose that $n(A) > n(B)$, then A has more elements than B, and hence there is atleast one element in A that cannot be in B, so A is not a subset of B.

ex. (i) $\{1\} \subseteq \{1, 2, 3\}$

(ii) $\{2, 4\} \nsubseteq \{1, 2, 3\}$

Write all the Subsets of $A = \{a, b\}$.

soln $A = \{a, b\}$

Subsets of A are ϕ, $\{a\}$, $\{b\}$, $\{a, b\}$

* If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
* Empty set is a subset of every set.
* Every set is a subset of itself.

9) Propersubset

Let A and B be two sets. If A is a subset of B and $A \neq B$, then A is called a proper subset of B and we write $A \subset B$.

ex:- $A = \{1, 2, 5\}$ $B = \{1, 2, 3, 4, 5\}$

A is a proper subset of B ie, $A \subset B$

10. ## Disjoint set.

Two sets A and B are said to be disjoint if they do not have common elements

In other words, if $A \cap B = \phi$, the A and B are said to be disjoint sets.

eg: $A = \{a, b, c\}$ $B = \{1, 2, 3\}$

$$A \cap B = \phi$$

* If $A \cap B \neq \phi$, then A and B are said to be overlapping sets. Thus if two sets have atleast one common element, they are called overlapping sets.

11) ## Powerset:

The set of all subsets of a set A is called powerset of 'A'. It is denoted by P(A).

eg: $A = \{2, 3\}$

The subsets of A are ϕ, $\{2\}$, $\{3\}$, $\{2, 3\}$

The power set of A.

$P(A) = \{\phi, \{2\}, \{3\}, \{2, 3\}\}$

An important property.

i) If $n(A) = m$, then $n[P(A)] = 2^m$

ii) The number of proper subsets of a set A is $n[P(A)] - 1 = 2^m - 1$

* Every set has only one improper subset.

ex: $X = \{a, b, c, d, e, f\}$

$n(X) = 6$ → Number of elements.

$n[P(X)] = 2^6 = 64$ → subsets

$n[P(X)] - 1 = 2^6 - 1 = 64 - 1 = 63$ → number of proper subsets

Set Operations

John Venn → English Mathematician

Invented Venn diagram.

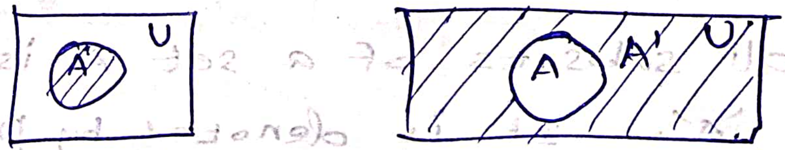
Operations On sets

1. Complement of a set.

The complement of a set A is the set of all elements of U (the universal set) that are not in A. It is denoted by A' or AC. In symbols

$A' = \{x : x \in U, x \notin A\}$

Venn diagram for complement of a set



A (Shaded region) A' (Shaded Region)

ex: U = {all boys in a class}

A = {boys who play cricket}

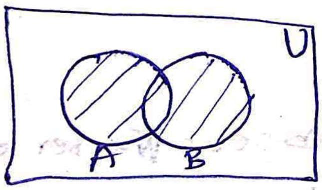
A' = {boy who do not play cricket}

* $(A')' = A$ * $U' = \phi$ * $\phi' = U$

2) Union of two sets

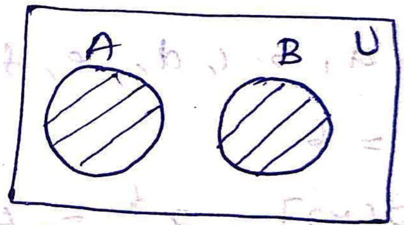
The union of two sets A and B is the set of all elements which are either in A or in B or in both. It is denoted by $A \cup B$ and read as A union B.

In symbol, $A \cup B = \{x : x \in A \text{ or } x \in B\}$



Sets A and B have common elements

$A \cup B$



Sets A and B are disjoint

$A \cup B = \phi$.

eg: $P = \{1, 2, 3, 4\}$ $Q = \{6, 7, 8, 9\}$

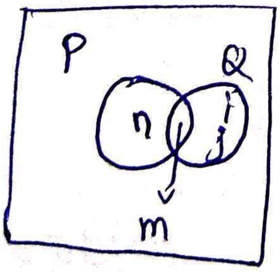
$P \cup Q = \{1, 2, 3, 4, 6, 7, 8, 9\}$

<u>Note</u> $A \cup A' = U$, $A \cup U = U$

$A \cup A = A$, $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $A \cup \phi = A$, $A \cup B = B \cup A$ (union of two sets is commutative) * $A \cup U = U$ where A is any subset of universal set U.

ex: $P = \{m, n\}$ $Q = \{m, i, j\}$

$P \cup Q = \{m, n, i, j\}$

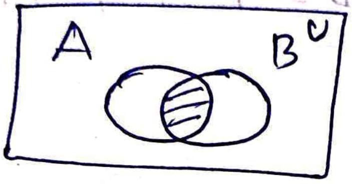


PUQ

3) Intersection of Two sets

The intersection of two sets A and B is the set of all elements common to both A and B. It is denoted by $A \cap B$ and read as A intersection B.

In symbol, $A \cap B = \{x : x \in A \text{ and } x \in B\}$



$A \cap B$

eg: $A = \{1, 2, 6\}$, $B = \{2, 3, 4\} \Rightarrow A \cap B = \{2\}$

Note: $A \cap A = A$, $A \cap \phi = \phi$, $A \cap U = A$ where A is any subset of universet U, $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $A \cap B = B \cap A$ (Intersection of two sets is Commutative).

$A \cap A' = \phi$

$A \cap B = B$ $A \cup B = A$ $B \subset A$

* If A and B are any two non empty sets such that $A \cup B = A \cap B$, then $A = B$.

* Let $n(A) = p$ and $n(B) = q$, then

a) Minimum of $n(A \cup B) = \max\{p, q\}$

b) maximum of $n(A \cup B) = p + q$

c) minimum of $n(A \cap B) = 0$

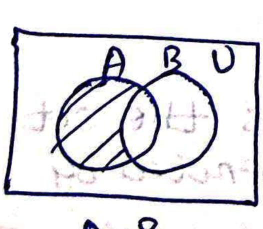
d) Maximum of $n(A \cap B) = \min\{p, q\}$

4) Difference of Two sets

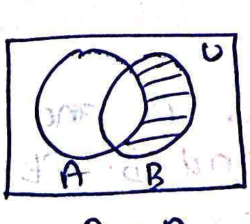
Let A and B be two sets, the difference of sets A and B is the set of all elements which are in A but not in B. It is denoted by $A - B$ or $A \setminus B$ and $A \sim B$ read as A difference B.

In symbol, $A - B = \{x : x \in A \text{ and } x \notin B\}$

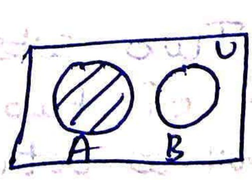
$B - A = \{y : y \in B \text{ and } y \notin A\}$.



A−B



B−A



A−B

eg: $A = \{-3, -2, 1, 4\}$, $B = \{0, 1, 2, 4\}$

$A - B = \{-3, -2\}$

$B - A = \{0, 2\}$

Note: $A' = U - A$, $A - B = A \cap B'$, $A - A = \phi$,

$A - \phi = A$, $A - B = B - A \Leftrightarrow A = B$,

$A - B = A$ and $B - A = B$ if $A \cap B = \phi$.

5) Symmetric Difference of sets.

The symmetric Difference of two sets A and B is the set $(A-B) \cup (B-A)$. It is denoted by $A \Delta B$. $A \oplus B$

$A \Delta B = \{x : x \in A - B \text{ or } x \in B - A\}$.

eg: $A = \{6, 7, 8, 9\}$ and $B = \{8, 10, 12\}$

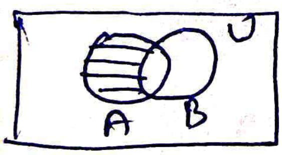
$A - B = \{6, 7, 9\}$, $B - A = \{10, 12\}$

$A \Delta B = (A - B) \cup (B - A) = \{6, 7, 9\} \cup \{10, 12\}$

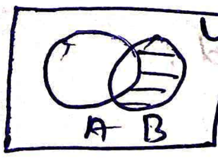
$= \{6, 7, 9, 10, 12\}$

$A \Delta B = (A - B) \cup (B - A)$

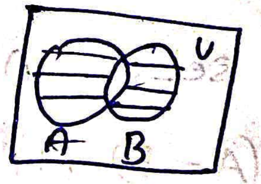
6 see last page



$A - B$



$B - A$



$(A - B) \cup (B - A)$

Note $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

$A \Delta A = \phi$, $A \Delta B = B \Delta A$, $A \Delta B = (A \cup B) - (A \cap B)$

$A \Delta B = \{x : x \in A \cup B \text{ and } x \notin A \cap B\}$

$A \cap (B \Delta C) = (A \cap B)\ (\Delta\ A \cap C)$

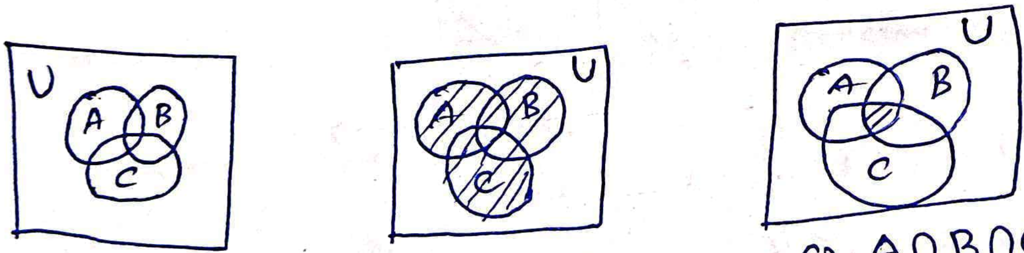
Operations on Sets

6. Union and Intersection in the case of three or more sets.

The operations of union and intersection can be defined for three or more sets in a similar manner. Thus:

$A \cup B \cup C = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in C\}$ and

$A \cap B \cap C = \{x \mid x \in A \text{ and } x \in B \text{ and } x \in C\}$.



a, Set A, B and C

b) $A \cup B \cup C$ is the shaded region

c) $A \cap B \cap C$ is the shaded region

Ex:- Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Then $A \cup B \cup C = \{1, 2, 3, 4, 6, 8, 5\}$ and $A \cap B \cap C = \{4\}$.

Properties of Set Operations

1. Properties of Union Operation.

Let A, B, C be subsets of U. Then

i) $A \cup A = A$ (Idempotent property)

ii) $A \cup \phi = A$

iii) $A \cup U = U$

iv) $A \cup A' = U$

v) $A \subseteq A \cup B$ and $B \subseteq A \cup B$

vi) $A \cup B = B \cup A$ (Commutative property)

vii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law).

2. Properties of Intersection Operation
For any three set A, B, C we have

(i) $A \cap A = A$

(ii) $A \cap B = A$ whenever $A \subseteq B$. In particular $A \cap U = A$.

(iii) $A \cap B = B$ whenever $B \subseteq A$. In particular $A \cap \phi = \phi$.

(iv) $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

(v) $A \cap B = B \cap A$

(vi) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative Law).

3. Properties of the Complement.

(i) The union of any set A and its complement A' is the universal set i.e., $A \cup A' = U$.

(ii) The intersection of any set A and its complement A' is the null set i.e., $A \cap A' = \phi$.

(iii) The complement of the universal set is the null set. i.e., $U' = \phi$.

(iv) The complement of the null set is the universal set. i.e., $\phi' = U$.

(v) The complement of the complement of a set A is the set A itself. i.e., $(A')' = A$.

(vi) $(A \cup B)' = A' \cap B'$
(vii) $(A \cap B)' = A' \cup B'$
De Morgan's Laws.

Distributive Law

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4) Properties of the difference operation.

i) Let A, B, C are sets and U be the Universal set.

Then a) $A' = U - A$

b) $A - B = A \cap B'$

c) $A - A = \phi$

d) $A - \phi = A$

e) $A - B = B - A$ if and only if $A = B$

f) $A - B = A$ if and only if $A \cap B = \phi$.

g) $A - B = \phi$ if and only if $A \subseteq B$.

5) Properties of Symmetric Difference..

$A \Delta B = \{(A - B) \cup (B - A)\}$

If A and B are sets, then

(i) $A \Delta A = \phi$

(ii) $A \Delta \phi = A$

(iii) $A \Delta B = B \Delta A$.

(iv) $A \Delta B = (A \cup B) - (A \cap B)$.

~~Verification of~~ the basic laws of Algebra of sets ~~by Venn Diagrams~~.

(i) Idempotent Laws $A \cup A = A$

$A \cap A = A$

(ii) Commutative Laws $A \cap B = B \cap A$

$A \cup B = B \cup A$

(iii) Associative Laws $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

(iv) Distributive Laws $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cap (A \cap C)$

(v) Absorption Laws $A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

(vi) De Morgan's Laws $(A \cup B)' = A' \cap B'$

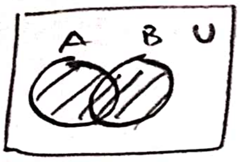
$(A \cap B)' = A' \cup B'$

(vii) Complement Laws $\phi' = U$; $U' = \phi$; $(A')' = A$;

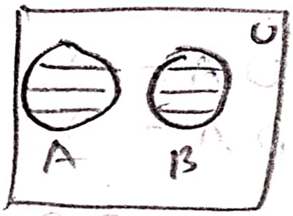
$A \cap A' = \phi$; $A \cup A' = U$.

Venn Diagram:-

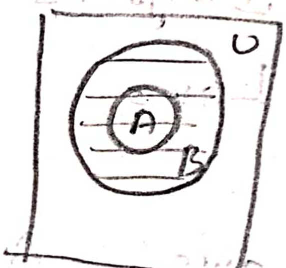
Venn Diagram is a method of representing the relationship between sets. A rectangle is drawn to represent the universal set U and sets are represented by circles. Elements of a set are represented by points within these circles, representing the set.



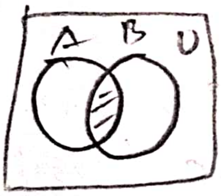
$A \cup B$



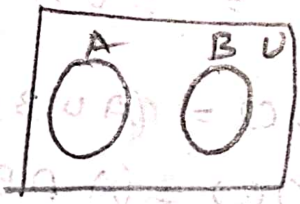
$A \cup B$



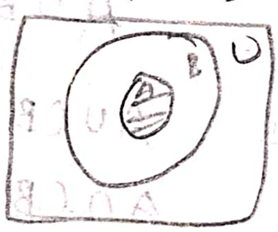
$A \cup B$



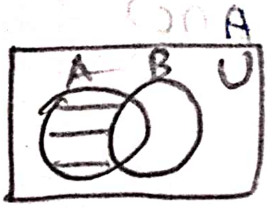
$A \cap B$



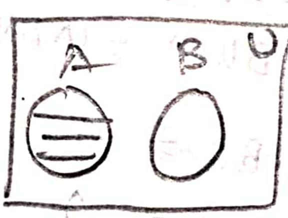
$A \cap B = \phi$



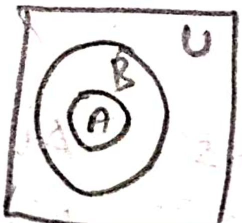
$A \cap B = A$



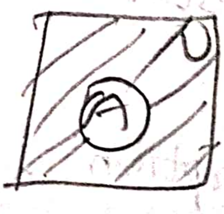
$A - B$



$A - B = A$



$A - B = \phi$



A'

Properties of Set Operations

1. Commutative Property.

For any two set A and B,

Commutative property of union of sets : $A \cup B = B \cup A$

Commutative property of intersection of sets : $A \cap B = B \cap A$.

Note:- $A \cup A = A$ and $A \cap A = A$ (Idempotent Laws)

$A \cup \phi = A$ and $A \cap U = A$ (Identity Laws)

$A - B \neq B - A$

2) Associative Property.

For any three sets A, B and C

(i) $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

Note: The set difference in general is not associative

i.e., $(A-B) - C \neq A - (B - C)$

But, if the sets A, B and C are mutually disjoint set then the set difference is associative

i.e, $(A-B) - C = A - (B - C)$

3) <u>Distributive property:</u> For any three sets A, B and C

i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [Intersection over Union]

ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ [Union over intersection]

De Morgan's Laws

1. De Morgan's Laws for Set Difference.

For any three sets A, B and C

(i) $A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$

2. De Morgan's Law for Complementation.

Let U be the universal set containing finite sets A and B. Then (i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Application on Cardinality of sets:

Inclusion – Exclusion principle

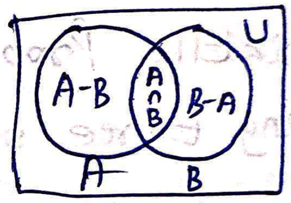
If A and B are two finite sets, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A - B) = n(A) - n(A \cap B)$

(iii) $n(B - A) = n(B) - n(A \cap B)$

(iv) $n(A') = n(U) - n(A)$.



Note:-

* $n(A \cap B) = n(A) + n(B) - n(A \cup B)$
* $n(U) = n(A) + n(A')$
* If A and B are disjoint sets then
  $n(A \cup B) = n(A) + n(B)$

For any three finite sets, A, B and C

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Absorption Law : $A \cup (A \cap B) = A$,

$$A \cap (A \cup B) = A$$

The principle of Duality

Every algebraic identity in set theory remains valid if we systematically interchange:

- Union ($\cup$) $\leftrightarrow$ Intersection ($\cap$)
- Universal set (U) $\leftrightarrow$ Empty set (ϕ)
- Complement (A' or A^c) remains the same

EX

i) Idempotent Law $A \cup A = A$ Dual $A \cap A = A$

ii) $A \cup U = U$ Dual $A \cap \phi = \phi$ (Identity Law)

iii) $A \cup B = B \cup A$ Dual $A \cap B = B \cap A$ (Commutative Law)

iv) $A \cup A' = U$ Dual $A \cap A' = \phi$ (complement Law)

v) $(A')' = A$ Dual $(A')' = A$ (complement Law)

vi) $(A \cup B)' = A' \cap B'$ Dual $(A \cap B)' = A' \cup B'$ (De Morgan's law).

Example Theorem

$(A \cup B)' \cap B = A' \cap B$

		Reason
Proof L.H.S	$= (A' \cap B) \cap B$	(Demorgan' law)
	$= A' \cap (B \cap B)$	(Associative Law)
	$= A' \cap B$	(Idempotent Law).
	$=$ R.H.S	

Dual Theorem

$(A \cap B')' \cup B = A' \cup B$

Proof (Dual Steps)

$$\begin{aligned} L.H.S &= (A' \cup B) \cup B \\ &= A' \cup (B \cup B) \\ &= A' \cup B \\ &= R.H.S. \end{aligned}$$