Problem Set E

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3.7.6

a) $R = (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y < 1)$. $S = (\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + y < 1)$.

b) $\neg R = (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x+y \geq 1)$. $\neg S = (\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(x+y \geq 1)$.

c) The statement R is true because for any $x \in \mathbb{R}$, we can fix some $y \in \mathbb{R} : y = -x$ to satisfy R. The statement S is false because if $y \in \mathbb{R}$ is fixed, it is possible to find $x \in \mathbb{R} : x = |y| + 2$.

3.7.8

b) Truth table for $P \Rightarrow (P \Rightarrow Q)$:

P	Q	$P \Rightarrow Q$	$P \Rightarrow (P \Rightarrow Q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

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d) Truth table for $(P \Rightarrow Q) \Rightarrow (P \wedge Q)$:

P	Q	$P \Rightarrow Q$	$P \wedge Q$	$(P \Rightarrow Q) \Rightarrow (P \wedge Q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

f) Truth table for $[P \vee (\neg Q)] \Rightarrow [Q \wedge (\neg P)]$:

P	Q	$\neg P$	$\neg Q$	$P \vee (\neg Q)$	$Q \wedge (\neg P)$	$[P \vee (\neg Q)] \Rightarrow [Q \wedge (\neg P)]$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

3.7.10

a) If $P \vee (Q \Rightarrow (\neg R))$ is *false*, then P must be false since if it was true, the "or" part of the given statement would evaluate the entire statement to true. Similarly, the statement $Q \Rightarrow (\neg R)$ must also be false by the same logic. Since this is also confirmed to be false, we can deduce that Q and R are both true, because the negation of any implication in the form $A \Rightarrow B$ is $A \wedge \neg B$; if we substitute $A = Q, B = \neg R$, we see that $A \wedge \neg B$ is in fact $Q \wedge R$, meaning that both Q and R are true. Therefore, P is false while Q and R are true.

b) We are given that $[(P \wedge Q \vee R] \Rightarrow (R \vee S)$ is *false*. This means its negation is true: $[(P \wedge Q \vee R] \wedge (\neg R \wedge \neg S)$. From this point we clearly see that to prevent contradiction, R and S are false; meaning that $P \wedge Q$ must be true, making both P and Q true.

3.7.22

a) The negation of $\alpha = (\exists M \in \mathbb{Z})(\forall x \in \mathbb{R})(x^2 \leq M)$ is $\beta = (\forall M \in \mathbb{Z})(\exists x \in \mathbb{R})(x^2 > M)$. The given statement, α, is false because for any fixed integer, it is always possible to pick $x = |M| + 100$ in order to demonstrate a contradiction.

b) The negation of $\alpha = (\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(|x-y| = |x| - |y|)$ is $\beta = (\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(|x-y| \neq |x| - |y|)$. The given statement, α, is true since $y = 0$ satisfies α.

c) Statement $\alpha = (\forall x \in \mathbb{R})[(x-6)^2 = 4 \implies x = 8]$, $\quad \neg\alpha = (\exists x \in \mathbb{R})[(x-6)^2 = 4 \wedge x \neq 8]$. The statement α is false, since its hypothesis implies that $x = 8$, which does not include the alternate solution $x = 4$. Since one solution for the given equation in α's hypothesis is not included, this statement (α) must evaluate to false.

d) Statement $\xi = (\forall x, y \in \mathbb{R})(x^2 - y^2 = 9 \implies |x| \geq 3)$, $\quad \neg\xi = (\exists x, y \in \mathbb{R})(x^2 - y^2 = 9 \wedge |x| < 3)$. The statement ξ is true since the term y^2 is always non-negative, and so we have $x^2 - y^2 \leq x^2$. Since this means that x^2 must also then satisfy $x^2 \geq 9 = x^2 - y^2$, this is the case when $|x| \geq 3$ as suggested.

e) Statement $\upsilon = (\forall x \in \mathbb{R})[(x-1)(x-3) = 3 \implies x - 1 = 3 \vee x - 3 = 3]$, $\neg\upsilon = (\exists x \in \mathbb{R})[(x-1)(x-3) = 3 \wedge x - 1 \neq 3 \wedge x - 3 \neq 3]$. Statement υ is false since $x = 0$ can be offered as a counterexample.

3.7.24

a)

b) The statement P is telling us that for any two consecutive integers, their product is *at least* zero. To determine whether this statement is correct, let us consider the following possible cases:

$$\text{Case 1: } x = 0 \vee x - 1 = 0 \implies x(x-1) = 0$$

$$\text{Case 2: } x \neq 0 \wedge x - 1 \neq 0 \implies (x > 0 \wedge x - 1 > 0) \vee (x < 0 \wedge x - 1 < 0)$$

Notice that in either of the above cases, $x(x-1) \geq 0$. Therefore, P is true.