Problem Set J

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5.6.20

$$f : A \to \mathbb{R}, \quad g : A \to \mathbb{R}. \quad g(x) = 3 \cdot [f(x)]^2 + 1$$

Claim:

$$(\forall a,\ b,\ p,\ q \in \mathbb{R})\big[(g(a) = g(b) \implies a = b) \implies (f(p) = f(q) \implies p = q)\big]$$

Proof:

$$\begin{aligned} & (g(a) = g(b) \implies a = b) \\ \Rightarrow \quad & (g(p) = g(q) \implies p = q) \\ \Rightarrow \quad & (3 \cdot [f(p)]^2 + 1 = 3 \cdot [f(q)]^2 + 1 \implies p = q) \\ \Rightarrow \quad & (f(p) = f(q) \implies p = q) \end{aligned}$$

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5.6.26

Claim:

$$\boxed{\text{The composition of two injections is an injection.}}$$

Proof:

We want to show:

$$(\forall f,\ g)(f \text{ injective},\ g \text{ injective})(f(g(a)) = f(g(b)) \implies a = b)$$

$$\begin{aligned} & f(g(a)) = f(g(b)) \\ \implies & g(a) = g(b) \\ \implies & a = b \end{aligned}$$

■

5.6.38

We know that $\mathbb{Z}$ is *countable*, and its power set $P(\mathbb{Z})$ has a strictly greater cardinality than $|\mathbb{Z}|$, hence $P(\mathbb{Z})$ is uncountable. If it was countable, this would mean it has the same cardinality as $|\mathbb{N}| = |\mathbb{Z}|$, a contradiction of the theorem $(\forall \text{ Set } X)(|P(X)| > |X|)$. $P(\mathbb{Z})$ is trivially not finite since its cardinality is infinite $\because$ it is lower-bounded by $|\mathbb{Z}|$.

We know that $\mathbb{R}$ is uncountable. We also know that $|\mathbb{R}| = |(0,1)|$ from lectures. We can now form a bijection $f : (0,1) \to (2,3) \quad , \quad f(x) = x + 2$ hence showing $|(2,3)| = |(0,1)| = |\mathbb{R}|$, $\therefore (2,3)$ must be uncountable.

By the Prime Number Theorem, there are infinitely many prime numbers $\therefore$ they are not finite. In addition, $\lim_{x\to\infty} \pi(x)$, representing the number of positive primes is upper-bounded by $|\mathbb{N}|$. We can list the negatives of all such primes p simply as $-p$, hence it maintains its cardinality. $\therefore$ the number of prime numbers is countable ($\because$ it does not exceed the cardinality of naturals, it can't be uncountable).

Lemmas:
(1) There are infinitely many rationals in $[0,1]$.
(2) $(\forall \text{ Sets } A, B)\big(|A \cap B| \leq \min(|A|, |B|)\big)$
Notice that (1) $\implies$ ($\mathbb{Q} \cap [0,1]$ is not finite). It cannot be uncountable since it is upperbounded by $|\mathbb{Q}|$ $\therefore$ it must be countable.

Trivially, $\mathbb{N} \cap (-\infty, 1000) = \{1, 2, 3 \cdots 997, 998, 999\}$. This is obviously finite since we can assign a *finite* value to $|\mathbb{N} \cap (-\infty, 1000)|$. Id est we can start counting the elements of the given set and finish counting at some point in time precisely due to its finite cardinality.

5.6.40

$$|A| \neq |B|$$

In the context of MAT102, we say a set is "countable" iff it has the *same cardinality* as $\mathbb{N}$. We observe there are 3 cases for cardinality:

a) If A is countable, it is possible that $|B|$ is finite. Exempli gratia: $B = \{1\}$.
b) If A is uncountable, $|B|$ be may still be finite. Exempli gratia: $B = \{2\}$.

5.6.48

$B = \{f : X \to \{0,1\}\}$. We are producing a bijection $\quad h : P(X) \to B$. In order to do this, we produce an injection h followed by a surjection h.

For injectivity, we want $(\forall X_1,\ X_2 \in P(X))\,[(f_1 : X_1 \to \{0,1\} = f_2 : X_2 \to \{0,1\}) \implies X_1 = X_2]$.

$$\begin{aligned} & f_1 = f_2 \\ \implies & (\forall x_1 \in X_1)(x_1 \in X_2) \wedge (\forall x_2 \in X_2)(x_2 \in X_1) \\ \implies & X_1 = X_2 \end{aligned}$$

For surjectivity, we want $h(P(X)) = B$. Due to injectivity, it suffices to show $|P(X)| = |B|$. Alternately we can also show $\forall f \in B, \quad \exists X \in P(X), \quad h(X) = f$.

Let $f \in B$ be given. Since X is nonempty, we know $P(X)$ always has at least two elements; which are in particular, $\phi \wedge X$. Since h is one-to-one, it maps its elements to at least two distinct functions, each of which map X to f's entire codomain consisting of $0 \wedge 1$.

Another way of thinking about this is:

$$|P(X)| = 2^{|X|} = |B| \ (\because\ f \text{ maps to two outputs, yielding cardinality } 2^{|X|})$$

...as required for bijection.

6.4.2

a) True	c) True	e) False
b) False	d) True	f) True

6.4.4

a) *True Claim*: $d|a \wedge d|b \implies d|(a+b)$.

Proof:

$$\begin{aligned} &\frac{a}{d} \in \mathbb{Z} \wedge \frac{b}{d} \in \mathbb{Z} \\ \implies &\frac{a}{d} + \frac{b}{d} \in \mathbb{Z} \\ \implies &\frac{a+b}{d} \in \mathbb{Z} \\ \implies &d|(a+b) \end{aligned}$$

$\blacksquare$

b) *False Claim*: $d|(a+b) \implies d|a \wedge d|b$

Counterexample: $d = 3, \quad a = 2, \quad b = 1.$

c) *False Claim*: $d|(a+b) \implies d|a \vee d|b$

Counterexample: $d = 3, \quad a = 2, \quad b = 1.$

d) *True Claim*: $d|(a+b) \implies (d|a \wedge d|b) \vee (d \not| \, a \wedge d \not| \, b)$

Proof:

For the sake of contradiction:

$$\begin{aligned} & \neg\text{Claim} = \underline{d|(a+b) \wedge (d|a \vee d|b)} \wedge (d \not| \, a \vee d \not| \, b) \\ \text{Part (c)} \quad \implies & \underline{d|(a+b) \wedge \neg(d|a \vee d|b)} \text{ is True} \\ \implies & \neg\text{Claim is False} \\ \implies & \text{Claim is True} \end{aligned}$$

$\blacksquare$

6.4.8

$$\exists a, b, c \in \mathbb{Z}, \quad a^2 + b^2 = c^2$$

Claim: $2|c \implies 2|a \wedge 2|b$

Proof:

Lemma: The product of evens is even, the product of odds is odd.

$$\begin{aligned} & a^2 + b^2 = c^2 - b^2 \\ & \implies a^2 = (c+b)(c-b) \\ \because \ (c\ even), (Lemma) \qquad & \implies \text{(a even} \wedge \text{b even)} \vee \text{(a odd} \wedge \text{b odd)} \end{aligned}$$

For the sake of contradiction, assume a, b are odd. Then, $(\exists p, q, r \in \mathbb{Z})(2p + 1 = a)(2q + 1 = b)(2r = c)$

$$\begin{aligned} & a^2 + b^2 = c^2 \\ & \implies (2p+1)^2 + (2q+1)^2 = 4r^2 \\ & \implies 4p^2 + 4p + 4q^2 + 4q + 2 = 4r^2 \\ & \implies 4p^2 + 4p + 4q^2 + 4q - 4r^2 = 2 \\ \text{Notice} \qquad & 4|(4p^2 + 4p + 4q^2 + 4q - 4r^2) \\ \text{But } \because \qquad & 4 \nmid 2 \quad \text{...a contradiction} \end{aligned}$$

By our lemma, we know that both a, b must be both even or both odd. Since we contradicted the case where a, b are odd, we conclude they must both be even.

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