Indian Institute of Technology Roorkee

Spring Semester 2023-24

MAI-102 (Mathematics II)

Assignment 4

Topics: Concept of probability, random variables and their probability distributions, expectation, moments and moment generating functions, Chebyshev's inequality.

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(1) Let A, B and C be three events associated with an experiment such that $P(A) = 0.7, P(B) = 0.5, P(C) = 0.4, P(\overline{A} \cap B) = P(B \cap C) = P(\overline{A} \cap \overline{C}) = 0.2$, and $P(A \cup (B \cap C)) = 0.8$. What is the probability that at least one of A, B and C occurs?

(2) One integer is chosen at random from the integers $1, 2, \ldots, 1000$. What is the probability that the chosen integer is divisible by 6 or 8?

(3) A problem is given to two students A and B. The (conditional) probability that A can solve the problem given that B can solve it is $\frac{3}{7}$ and the (conditional) probability that B can solve the problem given that A cannot solve it is $\frac{1}{7}$. If the probability that B can solve the problem is $\frac{1}{10}$, then what is the probability that A can solve it?

(4) There are three coins in a box. One of the coins is a two-headed coin (one head on each side), another is a fair coin, and the third one is a biased coin that comes up heads 75 percent of the time.
   (a) If a coin is drawn from the box at random and flipped, what is the probability of getting a head?
   (b) If a coin drawn at random from the box shows head when flipped, what is the probability that the coin drawn was the two-headed coin?

(5) Suppose a certain computer program operates using either of two subroutines A and B, depending on the problem. It is known that subroutine A is used 40 percent of the time and B is used 60 percent of the time. If A is used, then there is a 75% chance that the program will run before its time limit is exceeded; and if B is used there is a 50% chance that it will do so.
   (a) What is the probability that the program will run without exceeding the time limit?
   (b) If for some random problem, it is known that the program has exceeded its time limit, what is the probability that subroutine B was used?

(6) Suppose that A and B are two independent events associated with an experiment. If the probability that A or B occurs equals 0.6, while the probability that A occurs equals 0.4, determine the probability that B does not occur.

(7) A problem is given to three students whose probabilities of solving (independently) it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. What is the probability that (a) only one of them solves the problem (b) the problem is solved?

(8) A coin is known to come up heads three times as often as tails. This coin is tossed three times. Let X be the number of heads that appear. Write the probability mass function (pmf) and the cumulative distribution function (cdf) of X.

(9) Consider an urn which contains slips of paper each with one number $1, 2, \ldots, 100$ on it. Suppose there are i slips with the number i on them for $i = 1, 2, \ldots, 100$. The slips are identical except for the numbers. Suppose one slip is drawn at random and let X be the number on the slip. Then

(a) Find the cdf of X. (b) Compute $P(25 < X \leq 50)$.

(10) A fair coin is tossed four times. Let X denote the number of times a head is followed immediately by a tail. Find the pmf of X.

(11) Suppose a random variable X assumes only four values with probabilities $\frac{\alpha+3\beta}{4\alpha}$, $\frac{\alpha-\beta}{4\alpha}$, $\frac{\alpha-4\beta}{4\alpha}$, and $\frac{\alpha+2\beta}{4\alpha}$, where $\alpha > 0$. Determine all such possible α, β.

(12) Let the probability density function (pdf) of a continuous random variable X be

$$f(x) = \begin{cases} \alpha x \ , & 0 \leq x < 1, \\ \alpha \ , & 1 \leq x < 2, \\ 3\alpha - \alpha x \ , & 2 \leq x \leq 3, \\ 0 & \text{otherwise} \ . \end{cases}$$

(a) Determine α, and the cdf of X.
(b) If x_1, x_2 and x_3 are 3 independent observations from X, what is the probability that exactly one of these 3 observations is greater than 1.5?

(13) Suppose a continuous random variable X has the pdf

$$f(x) = \begin{cases} kx^2 e^{-\frac{x}{2}} \ , & x > 0, \\ 0 \ , & \text{elsewhere}, \end{cases}$$

where k is a suitable constant. (a) Determine the cdf of X, (b) Compute the probability that X is at most 10 given that X is at least 5.

(14) (a) Two distinct integers are chosen at random and without replacement from the first six positive integers. Let X be the absolute value of the difference of these two numbers. Compute the mean and the variance of X.
(b) Let a random variable X has the pdf $f(x) = \frac{3}{8}x^2$, $0 < x < 2$, zero elsewhere. Consider a random rectangle whose sides are X and $2 - X$. Determine the expected value of the area of the rectangle.

(15) Five devices are subjected to successive reliability tests. Each device is tested only if the preceding one passes the reliability test. Determine the expected number of reliability tests done if the probability of passing the reliability test is 0.9 for each device.

(16) Let X be a mixed type (continuous and discrete) random variable with the cdf

$$F(x) = \begin{cases} 0 \ , & x < 0, \\ (x+1)/6 \ , & 0 \leq x < 1, \\ \frac{7}{12} \ , & 1 \leq x < 2, \\ 1 \ , & x \geq 2 \ . \end{cases}$$

Determine the mean and the variance of X.

(17) Let X be a random variable with the pdf $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$. Determine the moment generating function (mgf) of X.

(18) (a) The moment generating function (mgf) of random variable X is $M_X(t) = (1-t)^{-3}$, $|t| < 1$. Find the first four moments of X about origin. Also determine the mean and variance of X.
(b) The mgf of a discrete random variable X is $\frac{2e^t}{3-e^t}$, $t < \log 3$. Determine the pmf of X.
(c) Given that the mgf of a random variable X is $M_X(t) = e^{3t+8t^2}$. Find the mgf of the random variable $Z = \frac{1}{4}(X - 3)$. Use it to determine the mean and variance of Z.

(19) (a) If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$, use Chebyshev's inequality to find (a) a lower bound for $P[-2 < X < 8]$ (b) an upper bound for $P[|X-3| \geq 7]$.

(b) Does there exist a random variable X with mean 5 and variance 9 such that $P[-1 < X < 11] = 0.6$? Explain.

(20) Let X be a discrete random variable with pmf $P[X=-1] = \frac{1}{8}$, $P[X=0] = \frac{3}{4}$, $P[X=1] = \frac{1}{8}$, zero elsewhere. Verify that Chebyshev's inequality gives an exact bound for $P[|X-\mu| \geq 2\sigma]$.

ANSWERS

(1) 0.9

(2) $\frac{1}{4}$

(3) $\frac{3}{5}$

(4) (a) $\frac{3}{4}$ (b) $\frac{4}{9}$

(5) (a) $\frac{3}{5}$ (b) $\frac{3}{4}$

(6) $\frac{2}{3}$

(7) (a) $\frac{11}{24}$ (b) $\frac{3}{4}$

(8) Pmf:

x	:	0	1	2	3
$f(x)$	:	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{27}{64}$	$\frac{27}{64}$

Cdf:

$$F(x) = \begin{cases} 0\ , & \text{for} \quad x < 0 \\ 1/64\ , & \text{for} \quad 0 \leq x < 1 \\ 10/64\ , & \text{for} \quad 1 \leq x < 2 \\ 37/64\ , & \text{for} \quad 2 \leq x < 3 \\ 1\ , & \text{for} \quad x \geq 3\ . \end{cases}$$

(9) (a) Cdf:

$$F(x) = \begin{cases} 0\ , & \text{for} \quad x < 1, \\ \frac{\lfloor x \rfloor (\lfloor x \rfloor + 1)}{10100}\ , & \text{for} \quad 1 \leq x < 100, \\ 1\ , & \text{for} \quad x \geq 100. \end{cases}$$

(b) $\frac{19}{101}$

(10) Pmf:

x	:	0	1	2
$f(x)$	:	$\frac{5}{16}$	$\frac{10}{16}$	$\frac{1}{16}$

(11) $\alpha > 0, \quad -\frac{\alpha}{3} < \beta < \frac{\alpha}{4}$.

(12) (a) $\alpha = \frac{1}{2}$

Cdf:

$$F(x) = \begin{cases} 0\ , & \text{for} \quad x < 0 \\ x^2/4\ , & \text{for} \quad 0 \leq x < 1 \\ (2x-1)/4\ , & \text{for} \quad 1 \leq x < 2 \\ (6x - x^2 - 5)/4\ , & \text{for} \quad 2 \leq x < 3 \\ 1\ , & \text{for} \quad x \geq 3\ . \end{cases}$$

(b) $\frac{3}{8}$

(13) (a) Cdf:

$$F(x) = \begin{cases} 0\ , & \text{for} \quad x < 0 \\ 1 - \left(\frac{x^2+4x+8}{8}\right) e^{-\frac{x}{2}}\ , & \text{for} \quad x \geq 0 \end{cases}$$

(b) $1 - \frac{148}{53} e^{-\frac{5}{2}}$

(14) (a) Mean $X = \frac{7}{3}$, $\quad$ $\text{Var}(X) = \frac{14}{9}$ $\quad$ (b) $\frac{3}{5}$

(15) 4.0951

(16) Mean $X = \frac{7}{6}$, $\quad$ $\mathrm{Var}(X) = \frac{11}{18}$

(17) $M_X(t) = \frac{1}{1-t^2}$, $\quad |t| < 1$

(18) (a) $\mu'_1 = 3$, $\quad \mu'_2 = 12$, $\quad \mu'_3 = 60$, $\quad \mu'_4 = 360$, $\quad$ Mean $X = 3$, $\quad$ $\mathrm{Var}(X) = 3$

(b) $p(x) = 2\left(\frac{1}{3}\right)^x$, $x = 1, 2, 3, \ldots$

(c) $M_Z(t) = e^{\frac{t^2}{2}}$, $\quad$ Mean $Z = 0$, $\quad$ $\mathrm{Var}(Z) = 1$

(19) (a) (a) $\frac{21}{25}$, $\quad$ (b) $\frac{4}{49}$

(b) No