MA 105 : Calculus

D1 - Lecture 15

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Directional Derivatives

The partial derivatives are special cases of the directional derivative. Let $v = (v_1, v_2)$ be a unit vector. Then v specifies a direction in $\mathbb{R}^2$.

Definition: The directional derivative of f in the direction v at a point $x = (x_1, x_2)$ is defined as

$$\nabla_v f(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = \lim_{t \to 0} \frac{f((x_1 + tv_1, x_2 + tv_2)) - f((x_1, x_2))}{t}.$$

It measures the rate of change of the function f at x along the path $x + tv$.

Observe that if we take $v = (1, 0)$ in the above definition, we obtain $\partial f / \partial x_1$, while $v = (0, 1)$ yields $\partial f / \partial x_2$.

Consider the function

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = 0 \text{ or if } x_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It should be clear to you that since this function is constant along the two axes,

$$\frac{\partial f}{\partial x_1}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_2}(0,0) = 0.$$

On the other hand, $f(x_1, x_2)$ is not continuous at the origin. (why?)

Thus, a function may have both partial derivatives (and, in fact, any directional derivative - see the next slide) but still not be continuous.

This suggests that for a function of two variables, just requiring that both partial derivatives exist is not a good or useful definition of "differentiability" (why?).

In the section on iterated limits, we studied the following function from Exercise 5.5:

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2} \quad \text{for} \quad (x,y) \neq (0,0).$$

Let us further set $f(0,0) = 0$. You can check that every directional derivative of f at $(0,0)$ exists and is equal to 0, except along $y = x$ (that is, along the unit vector $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$) when the directional derivative is not defined.

However, we have already seen that the function is not continuous at the origin since we have shown that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. For an example with directional derivatives in all directions see Exercise 5.3(i).

Conclusion: All directional derivatives may exist at a point even if the function is discontinuous at that point.

Let us go back and examine the notion of differentiability for a function $f(x)$ of one variable.

Suppose f is differentiable at the point x_0. When is a line passing through the point $(x_0, f(x_0))$ on the curve $y = f(x)$ is the tangent line to the curve?

Recall that the equation of a line passing through the point $(x_0, f(x_0))$ and having slope m is

$$y = g(x) = f(x_0) + m(x - x_0).$$

If we consider the difference
$f(x) - g(x) = f(x) - f(x_0) - m(x - x_0)$ and write $h = (x - x_0)$, we see that the difference can be rewritten as

$$f(x_0 + h) - f(x_0) - m \cdot h.$$

The above line is the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ on the curve if $m = f'(x_0)$, which is equivalent of saying that the above difference

$$f(x_0 + h) - f(x_0) - m \cdot h = o(h) = \varepsilon_1(h)h$$

where $\varepsilon_1(h)$ is a function of h that goes to 0 as h goes to 0, and in this case the function $o(h) = \varepsilon_1(h)h$ is a function of h "that goes to zero faster than h" (that is, $\lim_{h\to 0} \frac{o(h)}{h} = 0$).

The preceding idea generalises to two (or more) dimensions.

Let $f(x, y)$ be a function which has both partial derivatives at (x_0, y_0). In the two variable case we will consider the difference of $z = f(x, y)$ (defining a surface in $\mathbb{R}^3$) and $z = g(x, y)$ (defining the tangent plane to the surface $z = f(x, y)$ in $\mathbb{R}^3$).

Let us first determine how to find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0)$ on the surface. The tangent plane is the graph of the function

$$z = g(x, y) = z_0 + a(x - x_0) + b(y - y_0)$$

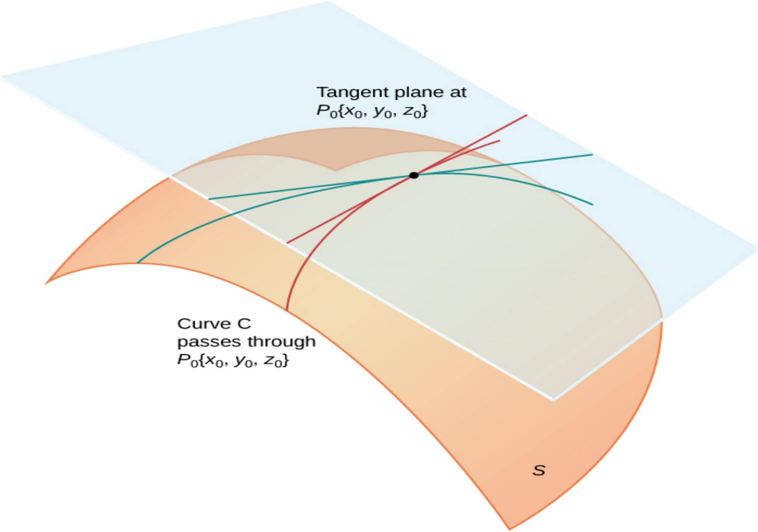
for some $a, b \in \mathbb{R}$.

Now, we determine the values of a and b, for which, the above plane $z = g(x, y)$, passing through the point $P = (x_0, y_0, z_0)$ *on the surface* $z = f(x, y)$, is the tangent plane to the surface.

If we fix the y variable as $y = y_0$ and treat $f(x, y)$ only as a function of x, we get a curve on the surface $z = f(x, y)$.

Similarly, if we treat $g(x, y)$ as function only of x (by fixing $y = y_0$), we obtain a line on the plane $z = g(x, y)$.

The tangent plane in a picture



https://openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations

The tangent line to the curve passing through (x_0, y_0, z_0) must be the same as the line passing through (x_0, y_0, z_0), and, in any event, their slopes (which are given by the derivatives of the curve $z = f(x, y_0)$ at $x = x_0$ and the line $z = g(x, y_0)$ at $x = x_0$, resp.) must be the same.

Since the above derivatives with respect to x are same as the partial derivatives with respect to x, we get

$$\frac{dz}{dx}(x_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial g}{\partial x}(x_0, y_0) = a.$$

Arguing in exactly the same way but fixing the x variable as $x = x_0$ and varying the y variable, we obtain

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial g}{\partial y}(x_0, y_0) = b.$$

Hence, the equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) (remember that $z_0 = f(x_0, y_0)$) is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0). \quad \square$$

A remark on tangent plane to the surface $z = f(x, y)$

Remark: Note that the general form of the equation of a plane passing thorough the point $P = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

for some $(a, b, c) \neq (0, 0, 0) \in \mathbb{R}^3$.

Since we assumed that the partial derivatives of the function $f(x, y)$ at (x_0, y_0) exist (and hence are finite real numbers), using the way we have determined the values of a and b in the last slides we get that the number c appearing in the above equation of the plane has to be a nonzero real number (as the derivative of the curve $z = f(x, y_0)$ at $x = x_0$ exists, the slope of the tangent line to this curve at (x_0, y_0) cannot be infinite) and then without loss of any generality c can be taken as -1.

It now follows from the above discussion that the general form of the equation of the tangent plane can be taken as

$$z = g(x, y) = z_0 + a(x - x_0) + b(y - y_0)$$

and the rest is given in the last couple of slides.

Differentiability for functions of two variables

We now define differentiability for functions of two variables by imitating the one variable definition, but using the "$o(h)$" version.

We let $(x, y) = (x_0, y_0) + (h, k) = (x_0 + h, y_0 + k)$.

Definition 1: A function $f : U \to \mathbb{R}$ is said to be differentiable at a point (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ exist and

$$\lim_{(h,k)\to(0,0)} \frac{\left|f(x_0 + h, y_0 + k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k\right|}{\|(h, k)\|} = 0.$$

We could rewrite this as

$$f((x_0, y_0) + (h, k)) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k = \varepsilon(h, k)\|(h, k)\|$$

where $\varepsilon(h, k)$ is a function that goes to 0 as $(h, k) \to (0, 0)$.

This form of differentiability now looks exactly like the one variable version case (put $o(h, k) = \varepsilon(h, k)\|(h, k)\|$). Can you guess the derivative $f'(x_0, y_0)$ of the function $f(x, y)$ at (x_0, y_0)?