MA 105 : Mid - Semester Review (D1 and D4)

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General Advice

1. Concentrate on understanding the statements of the theorems. You will not be asked to reproduce long proofs.
2. When trying to understand a definition, make sure you know plenty of examples.
3. When trying to understand a theorem, make sure you know counter-examples to the conclusion of the theorem when you drop some of the hypotheses.
4. In general, the statement of the theorem is more important than its proof. And examples are more important than theorems!

Limits of sequences

1. Learn the definition.
2. When proving a fact/theorem/etc. about some limit being l start with an $\epsilon > 0$ and find an N so that the sequence x_n you are dealing with satisfies
$$|x_n - l| < \epsilon,$$
for every $n > N$.
3. To prove that a sequence does not converge you have to show that no real number can be a limit. Thus you must take an arbitrary l and find some fixed $\epsilon > 0$ - this ϵ can be chosen to your convenience so that $|a_n - l| > \epsilon$ for infinitely many n.
4. Theorems to remember for showing that limits exist: the sum, difference, product and quotient and the Sandwich Theorem. In this case you will already know that some sequence has a limit and deduce that another sequence has a limit by comparing it to the known one.

Theorems that abstractly guarantee that the limit of a sequence exists:

A monotonically increasing sequence bounded above converges.

Every Cauchy sequence converges. It is a good idea to know the definition of a Cauchy sequence. However, you will not be asked questions on Cauchy sequences.

Unless we explicitly mention that you must use the ϵ-N definition to prove that a limit exists, you do not have to. You may use the rules for limits and other theorems instead. You can use simple facts without proving them: e.g. $\lim_{n\to\infty} \frac{1}{n^{\alpha}} = 0$ if $\alpha > 0$.

Question-1: If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} \sqrt{a_n} = 0$ using epsilon-N definition.

Answer: Fix $\epsilon > 0$. We want to find an N such that $n \geq N \implies |\sqrt{a_n}| < \epsilon$. Since $\lim_{n\to\infty} a_n = 0$, for our ϵ^2, the square of ϵ that we fixed earlier, there exists N_1 such that

$$n \geq N_1 \implies |a_n| < \epsilon^2.$$

Then for $N = N_1$ we have

$$n \geq N \implies |\sqrt{a_n}| < \epsilon.$$

Limits of functions

Limits of functions should be treated like limits of sequences.

Do not use $\epsilon - \delta$ to prove something unless you are asked to.

The ideas behind proving or disproving the existence of limits are the same as for sequences (of course, there is no analogue of monotonic bounded sequences or Cauchy sequences).

You can use the basic limits you learnt in 11th/12th standard like $\lim_{x \to 0} \sin x/x = 1$.

Continuity

Of course, you have to know the definition. Again, unless asked do not use $\epsilon - \delta$. You may use basic facts about limits of functions to prove what you want.
The basic theorems are:

1. A continuous function on a closed bounded interval is bounded and attains its infimum and supremum.

2. Continuous functions have the IVP.

   The sum, difference, product etc. of continuous functions is continuous. The composition of continuous functions is continuous.

Differentiation

Know the definition. Again, here you can use the basic facts about limits.

The basic theorems are:

1. Fermat's Theorem.
2. Rolle's theorem and the MVT.
3. Darboux's theorem.

Know the basic examples and counter-examples: a function that is continuous but not differentiable, a function that is differentiable but not continuously differentiable.

Question-2: Show that $x^3 - 10x + 4$ has three real roots.

Answer: Let $f(x) = x^3 - 10x + 4$. Then $f_x = 3x^2 - 10$ which has two roots, namely, $\pm\sqrt{10/3}$.

By the second derivative test we find that $-\sqrt{10/3}$ is a local maximum for f and $\sqrt{10/3}$ is a local minimum. By a simple computation it follows that

$$f(-\sqrt{10/3}) > 0 > f(\sqrt{10/3}).$$

By the IVP of f, there exists a zero of f in the interval $(-\sqrt{10/3}, \sqrt{10/3})$.

Since the given function is a cubic (that is of odd degree and the highest coefficient is positive), $f(x) \to \pm\infty$ as $x \to \pm\infty$, hence again by IVP we get two more zeros of f in the intervals $(-\infty, -\sqrt{10/3})$ and $(\sqrt{10/3}, \infty)$.

Question-3: Show that the function $f(x) = x^4 + 3x + 1$ has exactly one zero in the interval $[-2, -1]$.

Answer: By observing that $f(-2) > 0$ and $f(-1) < 0$, we conclude by IVP that f has a zero in the interval $[-2, -1]$.

Further, the derivative, $f'(x) = 4x^3 + 3$, is non-zero on $[-2, -1]$, so by Roll's theorem, f has no more zeros in the given interval.

Maxima, minima, convex, concave

Remember that the definitions of maxima, minima, concavity, convexity, inflection points etc. have nothing to do with differentiation.

If the function is (twice) differentiable then one can apply the various derivative tests. Otherwise, one can't.

Note that the existence of maxima and minima usually follows from the fact that we are dealing with continuous functions on a closed bounded interval.

Remember the difference between supremum and maximum (and of course, between infimum and minimum - know the relevant examples).

Taylor's theorem: Know how to compute the Taylor polynomials. Know the form of the Remainder term. Recall that there are smooth functions for which the Taylor series about a point converges but does not converge to the function ($e^{-1/x}$).

Question-4: Find the first three terms of the Taylor series of the function $1/x^2$ at 1.

Answer: If the Taylor series of the function f at $x = a$ is $\sum_{n=0}^{\infty} a_n(x-a)^n$, then $a_n = \dfrac{f^{(n)}(a)}{n!}$. Using these notations, for $f(x) = 1/x^2$ and $a = 1$, we get $a_0 = 1$, $a_1 = -2$ and $a_2 = 3$.

Exercise-3.10: Using the Taylor's theorem show that e is an irrational number.

Answer: By using the Taylor series expansion of e^x, we obtain that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Note that $\frac{1}{n!} < \frac{1}{2^{n-1}}$, for all $n \geq 3$ and hence

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} < 1+1+\frac{1}{2}+\sum_{n=3}^{\infty} \frac{1}{2^{n-1}} = 1+\sum_{n=0}^{\infty} \frac{1}{2^n} = 1+\frac{1}{1-1/2} = 1+2 = 3.$$

Thus, $e < 3$. Note that $e = 2.71828$ **(approximately)**.

By the Taylor's theorem (applied to $a = 0$ and $x = 1$) we know that

$$e - \sum_{k=0}^{n} \frac{1}{k!} =: R_n = e^{\alpha} \frac{1}{(n+1)!}$$

for some α between 0 and 1.

Since $e < 3$, $R_n < \frac{3}{(n+1)!}$ (since $e^{\alpha} < e$).

Now suppose e is a rational number $\frac{c}{d}$, where c and d are positive integers (as $e > 2$) and have no common factors.

For $n = d$, we see that $d!R_d$ is an integer. On the other hand, using the estimate for R_d that we have obtained using Taylor's Theorem,

$$0 < d!R_d < \frac{d! \times 3}{(d+1)!} = \frac{3}{d+1} \leq 1$$

if $d \geq 2$. This is a contradiction to the fact that $d!R_d$ is an integer. Thus, e cannot be a rational number.

What if $d = 1$ in the above proof? In that case, take any $n \geq 2$.

Integration

Remember what partitions and tagged partitions are.

Recall the definitions of the (Darboux) lower sums, upper sums, lower integrals, upper integrals and Riemann sums.

Learn the equivalent definitions of the Riemann integral.

Basic fact: Bounded functions on closed intervals with at most a finite number of discontinuities are Riemann/Darboux integrable.

The Fundamental Theorem of calculus.

Question-5: For the function $f(x) = 2x$ and the partition

$$P_n = \left\{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\right\}$$

of $[0, 1]$ find the lower sum, $L(f, P_n)$, upper sum, $U(f, P_n)$. Compute $\sup_n L(f, P_n)$ and $\inf_n U(f, P_n)$.

Answer: A simple computation shows that

$$L(f, P_n) = \frac{n-1}{n} \quad \text{and} \quad U(f, P_n) = \frac{n+1}{n}.$$

Thus,

$$\sup_n L(f, P_n) = \sup_n \left(\frac{n-1}{n}\right) = 1$$

and

$$\inf_n U(f, P_n) = \inf_n \left(\frac{n+1}{n}\right) = 1.$$

Question-6: Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{i^2+n^2}$ by identifying it as a Riemann sum for a certain continuous function on a certain interval and with respect to a certain partition.

Answer: We observe that

$$S_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(i/n)^2+1}.$$

Thus, the given sum is the Riemann sum for the function $\dfrac{1}{x^2+1}$ over the interval $[0,1]$ with respect to the partition $0 < \dfrac{1}{n} < \dfrac{2}{n} < \cdots < \dfrac{n-1}{n} < 1$. Since the function $f : [0,1] \to \mathbb{R}$ defined as $f(x) = \frac{1}{x^2+1}$ is continuous (on the closed and bounded interval $[0,1]$), it is Riemann integrable and

$$\int_0^1 \frac{1}{x^2+1} dx = \pi/4.$$

Thus, by the definition of the Riemann integral, for $\epsilon > 0$, there exists $\delta > 0$ such that

$$|R(P, t) - \pi/4| < \epsilon$$

whenever (P, t) is a tagged partition of $[0, 1]$ with $||P|| < \delta$.

Thus, if we take $N \in \mathbb{N}$ such that $N > \frac{1}{\delta}$, then for $n \geq N$, $||(P_n, t)|| = \frac{1}{n} \leq \frac{1}{N} < \delta$ and hence $|S_n - \pi/4| < \epsilon$, whenever $n \geq N$. That is,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{i^2 + n^2} = \pi/4.$$

Multivariable Calculus

Functions from $\mathbb{R}^2/\mathbb{R}^3 \to \mathbb{R}$: recall what limits and continuity mean for these functions.

Remember what partial and directional derivatives are. A function may have both partial derivatives or even all directional derivatives and still not be continuous (look at the various examples in Tutorial sheet 5- especially 5.1-5.5).

Learn the definition of differentiability for functions of 2 and 3 variables. Learn the definition of the derivative matrix.

The main point is that if a function has continuous partial derivatives in the neighbourhood of a point, then it is differentiable at that point.

Question-7: Using the $\epsilon - \delta$ definition of the limit show that $f(x) + g(y)$ is a continuous function of two variables if $f(x)$ and $g(x)$ are continuous functions of one variable.

Answer: Let $(a, b) \in \mathbb{R}^2$ and ϵ be positive. Since f is continuous at a and g is continuous at b, for $\epsilon/2$ we get $\delta_f > 0$ and $\delta_g > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \epsilon/2$$

and

$$|y - b| < \delta_g \implies |g(y) - g(b)| < \epsilon/2.$$

Now define $\delta = \min\{\delta_f, \delta_g\}$, then it is clear that

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |x - a| < \delta_f \text{ and } |y - b| < \delta_g$$

and hence

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies \left|\big(f(x)+g(y)\big)-\big(f(a)+g(b)\big)\right| < \epsilon.$$

The gradient

The gradient controls a lot of information about the function and its graph.

If we take $\nabla f \cdot u$ for a unit vector u, we get $\nabla_u f$, the directional derivative of f which measures the rate of change of f in the direction u.

The function $f(x, y, z)$ increases fastest in the direction of the gradient.

If $f(x, y, z) = c$ is a surface, $\nabla f(x_0, y_0, z_0)$ represents a vector normal to the surface at a point (x_0, y_0, z_0) on the surface. We can thus easily derive the equation of the tangent plane to the surface.

Make sure you have understood the Chain rule.

Question-8: Find the tangent plane to the implicit surface $x^2 + y^2 - z^2 = 18$ at the point $(3, 5, -4)$.

Answer: The tangent plane to an implicit surface $f(x, y, z) = b$ at (x_0, y_0, z_0) is given by

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0.$$

Using this formula, we get the required tangent plane to be the one given by

$$6(x - 3) + 10(y - 5) + 8(z + 4) = 0.$$

All the best for the Mid-semester Exams.