MA 105 : Calculus

D1 - Lecture 14

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Functions with range contained in $\mathbb{R}$

We will now be interested in studying functions $f : \mathbb{R}^m \to \mathbb{R}$, when $m = 2, 3$.

We have already mentioned how limits of such functions can be studied in the first few lectures.

Before doing this in detail, however, we will study certain other features of functions in two and three variables.

The most basic thing one needs to understand about a function is the domain on which it is defined. Very often a function is given by a formula which makes sense only on some subset of $\mathbb{R}^m$ and not on the whole of $\mathbb{R}^m$.

When studying functions of two or more variables given by formulae it makes sense to first identify this subset, which is sometimes called the natural domain of the function, and to describe it geometrically if possible.

Exercise 5.1: Find the natural domains of the following functions:
(i) $f(x, y) = \frac{xy}{x^2 - y^2}$

Clearly this function is defined whenever the denominator is not zero, in other words when $x^2 - y^2 \neq 0$.

The natural domain is thus

$$\mathbb{R}^2 \setminus \{(x, y) \,|\, x^2 - y^2 = 0\},$$

that is, $\mathbb{R}^2$ minus the pair of straight lines with slopes ± 1.

(ii) $f(x, y) = \log(x^2 + y^2)$

This function is defined whenever $x^2 + y^2 \neq 0$, in other words, in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Level curves and contour lines

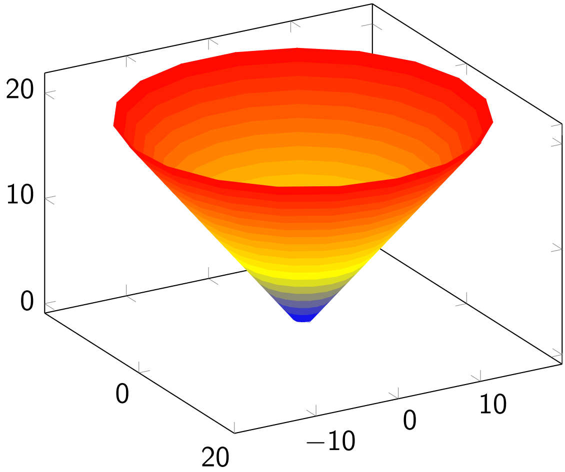
The second thing one should do with a function from $\mathbb{R}^2 \to \mathbb{R}$ is to study its range. This is done in different ways.

One way is to study the level sets of the functions. These are the sets of the form $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$, where c is a constant. The level set "lives" in the xy-plane, and in this case the level set is called level curve.

One can also plot (in three dimensions) the surface $z = f(x, y)$. By varying the value of c in the level curves one can get a good idea of what the surface looks like.

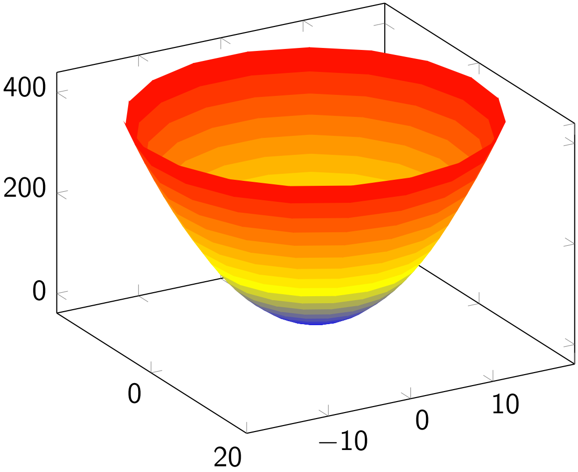
When one plots the $f(x, y) = c$ for some constant c one gets a curve. Such a curve is usually called a contour line (the contour "lives" in the $z = c$ plane).

I have a couple of pictures in the next two slides to illustrate the point.



This is the graph of the function $z = \sqrt{x^2 + y^2}$ lying above the xy-plane. It is a right circular cone.

The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.The curves given by $z = f(x, 0)$ and $z = f(0, y)$ give pairs of straight lines in the planes $y = 0$ and $x = 0$.



This is the graph of the function $z = x^2 + y^2$ lying above the xy-plane. It is a paraboloid of revolution.

The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.The curves $z = f(x, 0)$ or $z = f(0, y)$ give parabolæ lying in the planes $y = 0$ and $x = 0$. Exercise 5.2.(ii).

Limits

We have already said what it means for a function of two or more variables to approach a limit.

We simply have to replace the absolute value function on $\mathbb{R}$ by the distance function on $\mathbb{R}^m$.

We will do this in two variables. The three variable definition is entirely analogous.

Recall that for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, $x - y = (x_1 - y_1, x_2 - y_2)$ and the distance between x and y is given by

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

We will denote by U a set in $\mathbb{R}^2$.

Definition: A function $f : U \to \mathbb{R}$ is said to tend to a limit ℓ as $x = (x_1, x_2)$ approaches $c = (c_1, c_2)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - \ell| < \epsilon,$$

whenever $0 < \|x - c\| < \delta$. What set is $\{x \in \mathbb{R}^2 : 0 < \|x - c\| < \delta\}$?

Continuity

Before talking about continuity we remark the following. In the plane $\mathbb{R}^2$ it is possible to approach the point c from infinitely many different directions - not just from the right and from the left.

In fact, one may not even be approaching the point c along a straight line!

Hence, to say that a function from $\mathbb{R}^2$ to $\mathbb{R}$ possesses a limit is actually imposing a strong condition - for instance, the limits along all possible curves leading to the point must exist and all these (infinitely many) limits must be equal.

Once we have the notion of a limit, the definition of continuity is just the same as for functions of one variable.

Definition: The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

Exercise 1: Formulate the definition of a limit and of continuity for functions from $\mathbb{R}^3$ to $\mathbb{R}$.

The rules for limits and continuity

Exercise 2: Show that the projection functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ defined as $f_1(x, y) = x$ and $f_2(x, y) = y$ for all $(x, y) \in \mathbb{R}^2$ are continuous functions, using the ϵ, δ definition of continuity (Hint: does $\delta = \epsilon$ work?)

The rules for addition, subtraction, multiplication and division of limits remain valid for functions of two variables (or three variables for that matter). Nothing really changes in the statements or the proofs.

Using these rules, we can conclude, as before, that the sum, difference, product and quotient of continuous functions are continuous (as usual we must assume that the denominator of the quotient is non zero). In certain cases Exercise 2 may help in determining the continuity of many functions at given points.

Remark: Since the sequential criterion for limit and continuity holds for functions of several variables (see the Remark following the proof of Theorem 12 in slides of Lecture 6), we often use it for proving the nonexistence of limit and discontinuity of functions.

Continuity through examples

Once again, we emphasise that continuity at a point c is a very powerful condition (since the existence of a limit is implicit).

Exercise 5.3.(i) asks whether the function

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^6+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Solution: Let us look at the sequence of points $z_n = (\frac{1}{n}, \frac{1}{n^3})$, which converges to $(0,0)$ as $n \to \infty$. Clearly $f(z_n) = \frac{1}{2}$ for all n, so

$$\lim_{n\to\infty} f(z_n) = \frac{1}{2} \neq 0 = f(0,0).$$

This shows that f is not continuous at 0.

But does the limit of $f(x,y)$ at $(0,0)$ exist?

Iterated limits

When evaluating a limit of the form $\lim_{(x,y)\to(a,b)} f(x,y)$ one may naturally be tempted to let x go to a first, and then let y go to b.

Does this give the limit in the previous sense?

Exercise 5.5: Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$ be defined as

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}.$$

We have

$$\lim_{x\to 0} \lim_{y\to 0} \frac{x^2y^2}{x^2y^2 + (x-y)^2} = \lim_{x\to 0} 0 = 0.$$

Similarly, one has $\lim_{y\to 0} \lim_{x\to 0} f(x,y) = 0$.

However, choosing $z_n = (\frac{1}{n}, \frac{1}{n})$, shows that $f(z_n) = 1$ for all $n \in \mathbb{N}$. Now choose $z_n = (\frac{1}{n}, \frac{1}{2n})$ to see that the limit does not exist.

Partial derivatives

As before, U will denote a subset of $\mathbb{R}^2$. Given a function $f : U \to \mathbb{R}$, we can fix one of the variables and view the function f as a function of the other variable alone. We can then take the derivative of this one variable function.

To make things precise, fix x_2.

Definition: The partial derivative of $f : U \to \mathbb{R}$ with respect to x_1 at the point (a, b) is defined by

$$\frac{\partial f}{\partial x_1}(a, b) := \lim_{x_1 \to a} \frac{f((x_1, b)) - f((a, b))}{x_1 - a} = \lim_{t \to 0} \frac{f((a + t, b)) - f((a, b))}{t}.$$

Similarly, one can define the partial derivative with respect to x_2.

In this case the variable x_1 is fixed and f is regarded only as a function x_2:

$$\frac{\partial f}{\partial x_2}(a, b) := \lim_{x_2 \to b} \frac{f((a, x_2)) - f((a, b))}{x_2 - b} = \lim_{t \to 0} \frac{f((a, b + t)) - f((a, b))}{t}.$$