MA 105 Supplementary reading

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Parametrized Curve

A **parametrized curve** or a **path** C in $\mathbb{R}^2$ is given by $(x(t), y(t))$, where $x, y : [\alpha, \beta] \to \mathbb{R}$ are continuous functions.

Here $[\alpha, \beta]$ is called the **parameter interval**.

We wish to define the 'length' of C.

Basic assumption: The (Euclidean) length of a line segment joining points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is equal to

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We shall assume that C is **smooth**, that is, the functions x, y are **continuously differentiable** on $[\alpha, \beta]$. This means that x, y are differentiable on $[\alpha, \beta]$, and their derivatives x', y' are continuous on $[\alpha, \beta]$.

Arc Length of a Smooth Curve

- Partition $[\alpha, \beta]$ into $\alpha = t_0 < t_1 < \cdots < t_n = \beta$.
- Let $P_i := \big(x(t_i), y(t_i)\big)$ for $i = 1, \ldots, n$, and draw the line segments joining P_0 to P_1, P_1 to P_2, ..., P_{n-1} to P_n.
- The sum of the lengths of these line segments is

$$\begin{aligned} & \sum_{i=1}^{n} \sqrt{\left(x(t_i) - x(t_{i-1})\right)^2 + \left(y(t_i) - y(t_{i-1})\right)^2} \\ = & \sum_{i=1}^{n} \sqrt{\left(x'(s_i)\right)^2 + \left(y'(u_i)\right)^2}\,(t_i - t_{i-1}), \end{aligned}$$

  for some $s_i,\ u_i \in (t_{i-1}, t_i)$ for $i = 1, \ldots, n$ by the MVT.

- We define the **arc length** of C by

$$\ell(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Special Cases

Special cases:

(i) Let a curve C be given by $y = f(x)$, $x \in [a, b]$.
Here $\alpha := a$, $\beta := b$, $x(t) := t$ and $y(t) := f(t)$ for $t \in [a, b]$. Suppose f is continuously differentiable on $[a, b]$. Then

$$\ell(C) := \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

(ii) Let a curve C be given by $x = g(y)$, $y \in [c, d]$.
Here $\alpha := c$, $\beta := d$, $x(t) := g(t)$ and $y(t) := t$ for $t \in [c, d]$. Suppose g is continuously differentiable on $[c, d]$. Then

$$\ell(C) := \int_c^d \sqrt{g'(y)^2 + 1}\, dy.$$

Arc Length in Polar coordinates

Let C be given by a polar equation $r = p(\theta)$, $\theta \in [\alpha, \beta]$. As a parametrized curve, C is given by $(x(\theta), y(\theta))$, where

$$x(\theta) := p(\theta)\cos\theta \quad \text{and} \quad y(\theta) := p(\theta)\sin\theta, \quad \theta \in [\alpha, \beta].$$

Suppose the function p is continuously differentiable on $[\alpha, \beta]$.

For $\theta \in [\alpha, \beta]$, we note that $\sqrt{x'(\theta)^2 + y'(\theta)^2}$ is equal to

$$\begin{aligned} & \sqrt{\left(p'(\theta)\cos\theta - p(\theta)\sin\theta\right)^2 + \left(p'(\theta)\sin\theta + p(\theta)\cos\theta\right)^2} \\ = \; & \sqrt{p(\theta)^2 + p'(\theta)^2}. \end{aligned}$$

Hence

$$\ell(C) := \int_\alpha^\beta \sqrt{p(\theta)^2 + p'(\theta)^2}\, d\theta.$$

Examples

(i) Let C be given by $y = x^2$, $x \in [0,1]$. Then

$$\begin{aligned}\ell(C) &= \int_0^1 \sqrt{1+(2x)^2}\,dx = \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du \\ &= \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln\left(2+\sqrt{5}\right).\end{aligned}$$

(Use Integration by Parts. Also, if $f(u) := \ln(u + \sqrt{1+u^2})$ for $u \in \mathbb{R}$, then note that $f'(u) = 1/\sqrt{1+u^2}$ for $u \in \mathbb{R}$, and so
$\int_0^x \sqrt{1+u^2}\,du = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left(x+\sqrt{1+x^2}\right)\right)$ for $x \in \mathbb{R}$.)

(ii) Let C be given by $x = (2y^6+1)/8y^2$, $y \in [1,2]$. Then

$$\int_1^2 \left(1 + \left(y^3 - \frac{1}{4y^3}\right)^2\right)^{1/2} dy = \int_1^2 \left(y^3 + \frac{1}{4y^3}\right) dy = \frac{123}{32}.$$

(iii) Let $a > 0$ and $\varphi \in [0, \pi]$. Let C denote the arc of a circle of radius a given by $x(\theta) := a\cos\theta$, $y(\theta) := a\sin\theta$ for $\theta \in [0, \varphi]$. Then C is given by the polar equation $r = p(\theta)$, where $p(\theta) = a$ for $\theta \in [0, \phi]$, and so

$$\ell\,(C) = \int_0^{\varphi} \sqrt{a^2 + 0^2}\, d\theta = a\,\varphi.$$

Hence the length of a circle of radius a is $\int_{-\pi}^{\pi} a\, d\theta = 2\pi a$.

(iv) Let C be given by $r = 1 + \cos\theta$ for $\theta \in [0, \pi]$. Then

$$\begin{aligned} \ell\,(C) &= \int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta = 2\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 4. \end{aligned}$$

(Note: $\cos(\theta/2) \geq 0$ for $\theta \in [0, \pi]$.)

Suppose C is a smooth parametrized curve in $\mathbb{R}^3$ given by $(x(t), y(t), z(t))$, where $x,\ y,\ z : [\alpha, \beta] \to \mathbb{R}$ are continuously differentiable functions on $[\alpha, \beta]$.

In analogy with the definition of the arc length of a curve in $\mathbb{R}^2$, we define the **arc length** of C by

$$\ell\,(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Example

Let C denote a **helix** in $\mathbb{R}^3$ given by $x(t) := a \cos t,\ \ y(t) := a \sin t,\ \ z(t) := b\, t,\ \ t \in [\alpha, \beta]$, where $a,\ b, \in \mathbb{R}$, $a > 0$ and $b \neq 0$. Then

$$\ell\,(C) = \int_{\alpha}^{\beta} \sqrt{(-a \sin t)^2 + (a \cos t)^2 + b^2}\, dt = (\beta - \alpha)\sqrt{a^2 + b^2}.$$

Surface of Revolution

A **surface of revolution** is generated when a curve C in $\mathbb{R}^2$ is revolved about a line L in $\mathbb{R}^2$.

First suppose the curve C is a slanted line segment P_1P_2 of length λ_2, and C does not cross L. Let d_1 and d_2 denote the distances of P_1 and P_2 from L with $d_1 \leq d_2$. Then the surface of revolution is a frustum F of a right circular cone with base radii d_1 and d_2, and slant height λ_2. We find its surface area.

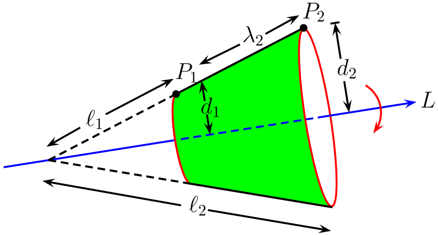


Figure: Frustum of a right circular cone

Consider a cone with base radius d and slant height ℓ. If we slit open this cone, we obtain a sector of a disk of radius ℓ.

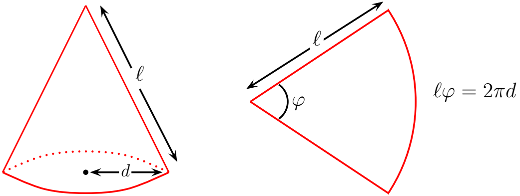


Figure: Right circular cone and sector of a disk

Since $\ell\varphi = 2\pi d$, the surface area of the cone is equal to

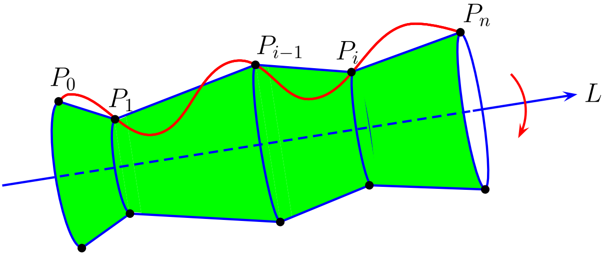
$$\frac{1}{2}\ell^2\varphi = \frac{1}{2}\ell^2\frac{2\pi d}{\ell} = \pi\ell d.$$

Hence the surface area of the frustrum F of the cone is $\pi\ell_2 d_2 - \pi\ell_1 d_1 = \pi(d_1 + d_2)(\ell_2 - \ell_1) = \pi(d_1 + d_2)\lambda_2$.

Now suppose C is parametrized by $(x(t), y(t))$, $t \in [\alpha, \beta]$.

- Partition $[\alpha, \beta]$ into $\alpha = t_0 < t_1 < \cdots < t_n = \beta$.
- Let $P_i := \big(x(t_i), y(t_i)\big)$ for $i = 0, 1, \ldots, n$, and draw the line segments P_0P_1, P_1P_2, …, $P_{n-1}P_n$.

Let $d_0, d_1, d_2, \ldots, d_n$ be the distances of $P_0, P_1, P_2, \ldots, P_n$ from the line L. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the lengths of the line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$. Suppose they don't cross L.



Fix $i \in \{1, \ldots, n\}$. When the line segment $P_{i-1}P_i$ is revolved about the line L, it generates a frustum F_i (of a right circular cone) whose surface area is $\pi(d_{i-1} + d_i)\lambda_i$.

Let $\rho(t)$ denote the distance of the point $(x(t), y(t))$ on the curve C from the line L. Then $d_i = \rho(t_i)$ for $i = 0, 1, \ldots, n$.

Thus the sum of the surface areas of the frustrums $F_1, \ldots, F_n$ is

$$\pi \sum_{i=1}^{n} \big(\rho(t_{i-1}) + \rho(t_i)\big)\lambda_i,$$

If the functions x' and y' are continuously differentiable on $[\alpha, \beta]$, then the length λ_i of the line segment $P_{i-1}P_i$ is given by

$$\begin{aligned} \lambda_i &= \sqrt{\big(x(t_i) - x(t_{i-1})\big)^2 + \big(y(t_i) - y(t_{i-1})\big)^2} \\ &= \sqrt{x'(s_i)^2 + y'(u_i)^2}\,(t_i - t_{i-1}) \end{aligned}$$

for some s_i, $u_i \in (t_{i-1}, t_i)$ for $i = 1, \ldots, n$ (by the MVT).

Area of Surface of Revolution

Let C be a smooth curve parametrized by $(x(t), y(t))$, $t \in [\alpha, \beta]$. Suppose the curve C does not cross the line L given by $ax + by + c = 0$. We define the **area of the surface** S generated by revolving C about the line L by

$$\text{Area}\,(S) := 2\pi \int_{\alpha}^{\beta} \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt,$$

where $\rho(t)$ is the distance of $(x(t), y(t))$ from the line L,

that is, $\rho(t) := |ax(t) + by(t) + c|/\sqrt{a^2 + b^2}$ for $t \in [a, b]$.

Note: Since the curve C does not cross the line L, the curve C lies entirely on one of the sides of the line L, that is,
either $ax(t) + by(t) + c \geq 0$ for all $t \in [\alpha, \beta]$,
or $ax(t) + by(t) + c \leq 0$ for all $t \in [\alpha, \beta]$.

Special Cases:

(i) Let the line L be the x-axis, and let the curve C be given by $y = f(x)$ for $x \in [a, b]$, where f is continuously differentiable. If $f \geq 0$ on $[a, b]$ or $f \leq 0$ on $[a, b]$, then

$$\text{Area}(S) = 2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2} dx.$$

(ii) Let the line L be the y-axis, and let the curve C be given by $x = g(y)$ for $y \in [c, d]$, where g is continuously differentiable. If $g \geq 0$ on $[c, d]$ or $g \leq 0$ on $[c, d]$, then

$$\text{Area}(S) = 2\pi \int_c^d |g(y)| \sqrt{1 + g'(y)^2} dy.$$

(iii) Let the line L be given by $\theta = \gamma$, where $\gamma \in (-\pi, \pi]$, and let the curve C be given by $r = p(\theta)$ for $\theta \in [\alpha, \beta]$, where p is continuously differentiable on $[\alpha, \beta]$. Suppose C does not cross L. Now the curve C is also given by $(p(\theta)\cos\theta, p(\theta)\sin\theta)$ for $\theta \in [\alpha, \beta]$.

Also, $\rho(\theta) = p(\theta)|\sin(\theta - \gamma)|$ for $\theta \in [\alpha, \beta]$.

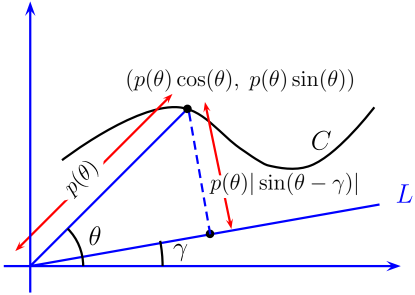


Figure: Distance of a point on a polar curve from a ray.

$$\text{Thus Area}(S) = 2\pi \int_\alpha^\beta p(\theta)|\sin(\theta - \gamma)|\sqrt{p(\theta)^2 + p'(\theta)^2}\,d\theta.$$

Examples

(i) Let S denote the surface generated by revolving the curve $y = (x^3/3) + (1/4x)$, $x \in [1, 3]$, about the line $y = -1$. Then

$$\begin{aligned}
\text{Area}(S) &= 2\pi \int_1^3 (y+1)\sqrt{1+(y')^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\sqrt{1+\left(x^2 - \frac{1}{4x^2}\right)^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\left(x^2 + \frac{1}{4x^2}\right)dx \\
&= 1823\pi/18.
\end{aligned}$$

(iii) Let $0 < b < a$ and let C denote the circle given by $(a + b\cos t, b\sin t)$, $t \in [-\pi, \pi]$. Let S denote the surface generated by revolving the curve C about the y-axis. Then $a + b\cos t > 0$ for all $t \in [-\pi, \pi]$, and so

$$
\begin{aligned}
\text{Area}\,(S) &= 2\pi \int_{-\pi}^{\pi} (a + b\cos t)\sqrt{(-b\sin t)^2 + (b\cos t)^2}\, dt \\
&= 2\pi b \int_{-\pi}^{\pi} (a + b\cos t) dt \\
&= 4\pi^2 ab.
\end{aligned}
$$

Note: S is in fact the surface of a torus in $\mathbb{R}^3$.

(iii) Let $a > 0$, and S denote the surface generated by revolving the semicircle $p(\theta) = a$, $\theta \in [0, \pi]$, about the x-axis. Then

$$
\text{Area}(S) = 2\pi \int_0^{\pi} a \sin\theta \sqrt{a^2 + 0^2}\, d\theta = 4\pi a^2.
$$

Note: S is in fact the sphere of radius a in $\mathbb{R}^3$.