MA 105 : Calculus

D1 - Lecture 1

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Aims of the course

First, welcome to IIT Bombay.

- To briefly review the calculus of functions of one variable and to teach the calculus of functions of several variables.

For details about the syllabus, tutorials, assignments, quizzes, exams and procedures for evaluation please refer to the course booklet which is available on moodle: https://moodle.iitb.ac.in

The emphasis of this course will be on the underlying ideas and methods rather than intricate problem solving (though there will be some of that too). The aim is to get you to think about calculus, in particular, and mathematics in general.

Syllabus

- Convergence of sequences and series, power series.
- Review of limits, continuity, differentiability.
- Mean value theorem, Taylor's theorem, maxima and minima.
- Riemann integrals, fundamental theorem of calculus, improper integrals, applications to area, volume.
- Partial derivatives, gradient and directional derivatives, chain rule, maxima and minima, Lagrange multipliers.
- Double and triple integration, Jacobians and change of variables formula.
- Parametrization of curves and surfaces, vector fields, line and surface integrals.
- Divergence and curl, theorems of Green, Gauss, and Stokes.

Sequences

Definition: A sequence in a set X is a function $f : \mathbb{N} \to X$, that is, a function from the set of natural numbers to X.

In this course X will usually be a subset of (or equal to) $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$, though we will also have occasion to consider sequences of functions sometimes. In later mathematics courses X may be the set of complex numbers $\mathbb{C}$, vector spaces (whatever those maybe), the set of continuous functions on an interval $\mathcal{C}([a, b])$ or other sets of functions.

Rather than denoting a sequence by a function, it is often customary to describe a sequence by listing the first few elements

$$a_1, a_2, a_3, \ldots$$

or, more generally by describing the n^{th} term a_n.

Note that we write a_n rather than $a(n)$. When we want to talk about the sequence as a whole we sometimes write $\{a_n\}_{n=1}^{\infty}$, but more often we once again just write a_n.

Examples of sequences

1. $a_n = n$ (here we can take $X = \mathbb{N} \subset \mathbb{R}$ if we want and f is just the identity function).
2. $a_n = \frac{1}{n}$ (here we can take $X = \mathbb{Q} \subset \mathbb{R}$ if we want, where $\mathbb{Q}$ denotes the set of rational numbers, or we can take $X = \mathbb{R}$ itself).
3. $a_n = \sin\left(\frac{1}{n}\right)$ (here the values taken by a_n are irrational numbers, so it is best to take $X = \mathbb{R}$).
4. $a_n = \frac{n!}{n^n}$.
5. $a_n = n^{1/n}$.
6. $s_n = \sum_{i=0}^{n} r^i$, for some r such that $0 \leq r < 1$.
7. $a_n = \left(n^2, \frac{1}{n}\right)$ (here $X = \mathbb{R}^2$ or $X = \mathbb{Q}^2$).
8. $f_n(x) = \cos nx$ (here X is the space of continuous functions on any interval $[a, b]$ or even on $\mathbb{R}$).
9. $s_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$, or writing it out $s_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$. Once again X is a space of functions, for instance the space of continuous functions on $\mathbb{R}$.

Monotonic sequences

For the moment we will concentrate on sequences in $\mathbb{R}$.

Definition: A sequence is said to be a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition: A sequence is said to be a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A monotonic sequence is one that is either monotonically increasing or monotonically decreasing.

From the examples in the previous slide, Example 1 ($a_n = n$) is a monotonically increasing sequence, Example 2 ($a_n = 1/n$) is a monotonically decreasing sequence, while Example 3 ($a_n = \sin\left(\frac{1}{n}\right)$) is also monotonically decreasing. How about Examples 4 and 5?

In Example 4 we notice that if $a_n = \frac{n!}{n^n}$,

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}} = a_n \times \frac{(n+1)n^n}{(n+1)^{(n+1)}} \leq a_n,$$

so the sequence is monotonically decreasing.

Eventually monotonic sequences

In Example 5 ($a_n = n^{1/n}$), we note that

$$a_1 = 1 < 2^{1/2} = a_2 < 3^{1/3} = a_3,$$

(raise both a_2 and a_3 to the sixth power to see that $2^3 < 3^2$).

However, $3^{1/3} > 4^{1/4} > 5^{1/5}$. So what do you think happens as n gets larger?

In fact, $a_{n+1} \leq a_n$, for all $n \geq 3$. Prove this fact as an exercise. Such a sequence is called an eventually monotonically decreasing sequence, that is, the sequence becomes monotonically decreasing after some stage. One can similarly define eventually monotonically increasing sequences.

For any fixed non-negative value of r, Example 6 ($s_n = \sum_{j=0}^{n} r^j$) gives a monotonically increasing sequence, while for any fixed non-negative value of x, the sequence $s_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$ in Example 9 also gives a monotonically increasing sequence.

Preliminaries

While all of you are familiar with limits, most of you have probably not worked with a rigorous definition. We will be more interested in limits of functions (which is what arise in the differential calculus), but limits of sequences are closely related to the former, and occur in their own right in the theory of Riemann integration.

So what does it mean for a sequence to tend to a limit? Let us look at the sequence $a_n = 1/n^2$. We wish to study the behaviour of this sequence as n gets large. Clearly as n gets larger and larger, $1/n^2$ gets smaller and smaller and seems to approach the value 0, or more precisely

the distance between $1/n^2$ and 0 becomes smaller and smaller.

In fact (and this is the key point), by choosing n large enough, we can make the distance between $1/n^2$ and 0 smaller than any prescribed quantity.

Let us examine the above statement, and then try and quantify it.

More precisely:

The distance between $1/n^2$ and 0 is given by $|1/n^2 - 0| = 1/n^2$.

Suppose I require that $1/n^2$ be less that 0.1 (that is, 0.1 is my prescribed quantity). Clearly, $1/n^2 < 1/10$ for all $n > 3$.

Similarly, if I require that $1/n^2$ be less than $0.0001(= 10^{-4})$, this will be true for all $n > 100$.

We can do this for any number, no matter how small. If $\epsilon > 0$ is any number,

$$1/n^2 < \epsilon \iff 1/\epsilon < n^2 \iff n > 1/\sqrt{\epsilon}.$$

In other words, given any $\epsilon > 0$, we can always find a natural number N (in this case, any $N > 1/\sqrt{\epsilon}$) such that for all $n > N$, $|1/n^2 - 0| < \epsilon$.

The rigorous definition of a limit

Motivated by the previous example, we define the limit as follows.

Definition: A sequence a_n tends to a limit ℓ, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - \ell| < \epsilon$$

whenever $n > N$.

This is what we mean when we write

$$\lim_{n \to \infty} a_n = \ell.$$

Equivalently, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit ℓ. If we just want to say that the sequence has a limit without specifying what that limit is, we simply say $\{a_n\}_{n=1}^{\infty}$ converges, or that it is convergent.

A sequence that does not converge is said to diverge, or to be divergent.