MA 105 : Calculus

D1 - Lecture 7

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The Mean Value Theorem

The Mean Value Theorem (MVT) is a special case of Rolle's theorem.

Theorem 15: Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable in (a, b). Then there is a point x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0).$$

Proof: Apply Rolle's Theorem to the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

$\square$

Applications of the MVT

Here is an application of the MVT which you have probably always taken for granted:

Theorem 16: If f satisfies the hypotheses of the MVT, and further $f'(x) = 0$ for every $x \in (a, b)$, f is a constant function.

Indeed, if $f(c) \neq f(d)$ for some two points $c < d$ in $[a, b]$,

$$0 \neq \frac{f(d) - f(c)}{d - c} = f'(x_0),$$

for some $x_0 \in (c, d)$, by the MVT. This contradicts the hypothesis. □

Consider Exercise 2.6: Let f be continuous on $[a, b]$ and differentiable on (a,b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

Solution: Split the interval $[a, b]$ into two pieces: $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ - and apply the MVT to each interval.

Darboux's Theorem

Theorem 17: Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If c, d, $c < d$ are points in (a, b), then for every u between $f'(c)$ and $f'(d)$, there exists an x in $[c, d]$ such that $f'(x) = u$.

Proof: We can assume, without loss of generality, that $f'(c) < u < f'(d)$, otherwise we can take $x = c$ or $x = d$.

Define $g(t) = ut - f(t)$. This is a continuous function on $[c, d]$ (not only on $[c, d]$ but also on (a, b) and differentiable at all the points in (a, b)) and hence, by Theorem 11 must attain its supremum (also the infimum but we will consider only the supremum).

Since $g'(c) = u - f'(c) > 0$ (as g is differentiable at all the points in (a, b), it is also differentiable at c and d) and by the definition of the derivative of a function, for $\epsilon = \frac{g'(c)}{2} > 0$, $\exists \delta > 0$ such that

$$\left| \frac{g(c+h) - g(c)}{h} - g'(c) \right| < \epsilon$$

whenever $|h| < \delta$.

Proof continued...

That is,

$$g'(c) - \epsilon < \frac{g(c+h) - g(c)}{h} < g'(c) + \epsilon$$

whenever $|h| < \delta$.

In particular, for $0 < h < \delta$ such that $c + h \in (c, d)$,

$$g(c+h) - g(c) > h(g'(c) - \epsilon) = h\frac{g'(c)}{2} > 0,$$

that is,

$$g(c+h) > g(c)$$

and hence $g(c)$ cannot be the $\sup_{x \in [c,d]} g(x)$.

Since $g'(d) = u - f'(d) < 0$, for $\epsilon = -\frac{g'(d)}{2} > 0$, $\exists \delta > 0$ such that

$$\left| \frac{g(d+h) - g(d)}{h} - g'(d) \right| < \epsilon$$

whenever $|h| < \delta$.

Proof continued...

That is,

$$g'(d) - \epsilon < \frac{g(d+h) - g(d)}{h} < g'(d) + \epsilon$$

whenever $|h| < \delta$.

In particular, for $-\delta < h < 0$ such that $d + h \in (c, d)$,

$$g(d+h) - g(d) > h(g'(d) + \epsilon) = h\frac{g'(d)}{2} > 0,$$

that is,

$$g(d+h) > g(d)$$

and hence $g(d)$ cannot be the $\sup_{x \in [c,d]} g(x)$.

Thus there exists $x \in (c, d)$ where g attains its supremum, and hence by Fermat's Theorem, $g'(x) = 0$ which yields $f'(x) = u$. $\square$

Continuity of the first derivative

We have just seen that the derivative satisfies the IVP. Can we find a function which is differentiable but for which the derivative is not continuous?

Here is the standard example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

This function will be differentiable at 0 but its derivative will not be continuous at that point. In order to see this you will need to study the function in Exercise 1.13(ii). This will show that $f'(0) = 0$. On the other hand, if we use the product rule when $x \neq 0$ we get

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

which does not go to 0 as $x \to 0$.

Back to maxima and minima

We will assume that $f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable on (a, b).

A point x_0 in (a, b) such that $f'(x_0) = 0$, is often called a stationary point.

We will assume further that $f'(x)$ is differentiable at x_0, that is, that the second derivative $f''(x_0)$ exists. We formulate the Second Derivative Test below.

Theorem 18: With the assumptions above:

1. If $f''(x_0) > 0$, the function has a local minimum at x_0.
2. If $f''(x_0) < 0$, the function has a local maximum at x_0.
3. If $f''(x_0) = 0$, no conclusion can be drawn.

The proof of the Second Derivative Test

Proof: The proofs are straightforward. For instance, to prove the first part we observe that

$$0 < f''(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h)}{h} \qquad (\text{since } f'(x_0) = 0)$$

that is, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f''(x_0) - \epsilon < \frac{f'(x_0 + h)}{h} < f''(x_0) + \epsilon$$

whenever $|h| < \delta$. Now, by putting $\epsilon = \frac{f''(x_0)}{2}$ in the above inequality we get

$$0 < \frac{f''(x_0)}{2} < \frac{f'(x_0 + h)}{h}$$

whenever $|h| < \delta$. It follows that for $|h| < \delta$,

$$f'(x_0 + h) < 0 \quad \text{if } h < 0, \text{ and } f'(x_0 + h) > 0 \quad \text{if } h > 0.$$

It follows that $f(x)$ is decreasing to the left of x_0 and increasing to the right of x_0 (why?).

The proof of the Second Derivative Test continued...

Hence, x_0 must be a local minimum. A similar argument yields the second case. $\square$

If the third case of the theorem above occurs, the function may be changing from concave to convex. In this case x_0 is called a point of inflection. An example of this phenomenon is given by $f(x) = x^3$ at $x = 0$.

Concavity and convexity

Let I denote an interval (open or closed or half-open).

Definition: A function $f : I \to \mathbb{R}$ is said to be concave (or sometimes concave downwards) if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

for all x_1 and x_2 in I and $t \in [0, 1]$. Similarly, a function is said to be convex (or concave upwards) if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

By replacing the $\geq$ and $\leq$ signs above by strict inequalities we can define strictly concave and strictly convex functions.

Note that if $f(x)$ is a concave function, $-f(x)$ is a convex function, so it is enough to study either the convex or the concave functions.

Examples of concave and convex functions

Here are some examples of convex functions.

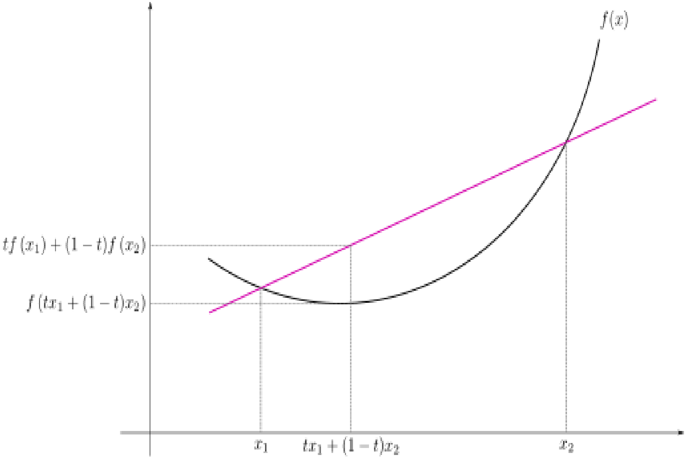
1. $f(x) = x^2$ on $\mathbb{R}$.
2. $f(x) = x^3$ on $[0, \infty)$.
3. $f(x) = e^x$ on $\mathbb{R}$.

Examples of concave functions include

1. $f(x) = -x^2$ on $\mathbb{R}$.
2. $f(x) = x^3$ on $(-\infty, 0]$
3. $f(x) = \log x$ on $(0, \infty)$.

For a convex function f and point $c \in (x_1, x_2)$, the point $(c, f(c))$ always lies below the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Convexity illustrated graphically



Properties of Convex functions

Convex functions have many nice properties. For instance, it is easy to show that convex functions are continuous (do this!) (Hint: Show that, for $x_1 < x_2 < x_3$, $[f(x_2) - f(x_1)]/[x_2 - x_1] \leq [f(x_3) - f(x_1)]/[x_3 - x_1] \leq [f(x_3) - f(x_2)]/[x_3 - x_2]$). More is true.

Exercise 1. Every convex function f (on a bounded interval) is Lipschitz continuous (cf. Exercise 1.16 with $\alpha = 1$), that is, there exists $M > 0$ such that $|f(x+h) - f(x)| \leq M|h|$, for all $x, x+h$ inside the domain of the function f. (Can you think of a convex function which is not Lipschitz continuous? How about the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = x^2$?; note that this function is Lipschitz continuous on any bounded interval).

In fact, much more is true. A convex function is actually differentiable at all but at most countably many points.

A differentiable function is convex if and only if its derivative is monotonically increasing. Moreover, if a function is both differentiable and convex, it is continuously differentiable, that is, its derivative is continuous (feel free to try proving these facts).