MA 105 : Calculus

D1 - Lecture 13

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The Fundamental Theorem of Calculus Part II

Theorem 9: Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(t) = f(t)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a).$$

Note that this statement does not assume that the function $f(t)$ is continuous, and hence is stronger than the corollary we have just stated.

Proof: We can write:

$$g(b) - g(a) = \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})],$$

where $\{a = x_0 < x_1 < \cdots < x_n = b\}$ is an arbitrary partition of $[a, b]$.

Using the mean value theorem for each of the intervals $I_i = [x_{i-1}, x_i]$, we can write

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}).$$

for some $c_i \in (x_{i-1}, x_i)$.

Substituting this in the previous expression and using the fact that $g'(c_i) = f(c_i)$, we get

$$g(b) - g(a) = \sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})]. \qquad (*)$$

The calculation above is valid for any partition.

Since f is Riemann integrable on $[a, b]$, for a given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} [f(t_i)(x_i - x_{i-1})] - \int_a^b f(t)dt \right| < \epsilon$$

for any tagged partition (P, t) of $[a, b]$ having $\|P\| < \delta$ (using the first definition of Riemann integration).

Now, we construct a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ having $\|P\| < \delta$ and consider the tagging $t = \{c_i : x_{i-1} \le c_i \le x_i, 1 \le i \le n\}$ (of P) that we get from $(*)$ which we consider for this partition P having $\|P\| < \delta$.

Now, it follows from $(*)$ that

$$\left| g(b) - g(a) - \int_a^b f(t)dt \right| = \left| \sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})] - \int_a^b f(t)dt \right| < \epsilon.$$

Since the above inequality holds for all positive real number ϵ, we get

$$\left| g(b) - g(a) - \int_a^b f(t)dt \right| = 0,$$

that is,

$$\int_a^b f(t)dt = g(b) - g(a). \quad \square$$

FTC: Applications

Exercise 4.5. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a.

Solution: Consider $F(x) = \int_a^x f(t)dt,\ x \in \mathbb{R}$. Then $F'(x) = f(x)$. Note that

$$\int_{u(x)}^{v(x)} f(t)dt = \int_a^{v(x)} f(t)dt - \int_a^{u(x)} f(t)dt = F(v(x)) - F(u(x)).$$

Use the Chain rule to see that

$$\begin{aligned} \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x). \end{aligned}$$

Using this formula for $G(x) = \int_x^{x+p} f(t)dt$, we see that $G'(x) = 0$ for all $x \in \mathbb{R}$, and hence $G(x)$ is constant. Thus $\int_a^{a+p} f(t)dt$ has the same value for every real number a. $\square$

Mean Value Theorem for Integrals

Theorem 10 (Mean Value Theorem for Integrals): If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof: Since the function

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$ (by the Fundamental Theorem of Calculus), there is $c \in (a, b)$ such that

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b-a} = \frac{1}{b-a}\int_a^b f(x)dx$$

(by the Mean Value Theorem).

Thus

$$f(c)(b-a) = \int_a^b f(x)dx. \quad \square$$

The logarithmic function

Definition: The natural logarithmic function is defined on $(0, \infty)$ by

$$\ln x = \int_1^x \frac{1}{t} dt.$$

It is clear that $\ln 1 = 0$, $\ln x > 0$ for $x \in (1, \infty)$, and $\ln x < 0$ for $x \in (0, 1)$.

Theorem 11:

1. $\ln(xy) = \ln x + \ln y$
2. $\ln(\frac{x}{y}) = \ln x - \ln y$
3. $\ln(x^r) = r \ln x$, if r is a rational number.

Proof: (1). Let $f(t) = \ln(ty)$. Then, $f'(t) = \frac{1}{t}$. Therefore, by FTC - Part II, $\ln x = f(x) - f(1)$, that is, $\ln(xy) = \ln x + \ln y$. For (2), put $x = 1/y$ in (1) and get $\ln(1/y) = -\ln y$, and then use (1) again, for the product of x and $1/y$. (3) is clear if $r \in \mathbb{Z}$. Observe that $\ln x = \ln[(x^{1/q})^q] = q \ln(x^{1/q})$ for $q \neq 0 \in \mathbb{Z}$, which shows that $\ln(x^{1/q}) = (1/q) \ln x$. Now, if $r = p/q$ for $p, q \in \mathbb{Z}$, using the first case (for $p \in \mathbb{Z}$) we get $\ln(x^{p/q}) = (p/q) \ln x$. $\square$

The exponential function

Remark: $\ln x$ is increasing and concave (why?). Moreover, by IVT, there exists a number $e > 1$ such that $\ln e = 1$ (as $\exists x \in (1, \infty)$ such that $\ln x > 0$ (why?) and $\ln x^n = n \ln x \to \infty$ as $n \to \infty$).

It follows that $\ln x$ is a strictly increasing function whose range is full of $\mathbb{R}$. Therefore, it is invertible and has an inverse. We denote this by $\exp(x)$.

That is,

$$\exp(x) = y \iff \ln y = x$$

In particular, $\exp(0) = 1, \exp(1) = e$.

Since, $\ln(e^r) = r \ln e = r$, we get $e^r = \exp(r)$, when r is a rational number. Therefore, we define $e^x := \exp(x)$ for any $x \in \mathbb{R}$.

Laws of Exponents:

$e^{x+y} = e^x e^y,\ e^{x-y} = \frac{e^x}{e^y},\ (e^x)^r = e^{rx},$ if r is rational.

Proof: Use the laws of exponents for $\ln x$ (Theorem 11). $\square$

The exponential function

Theorem 12: $\frac{d}{dx}(e^x) = e^x$.

Proof: If f is differentiable with nonzero derivative, then f^{-1} is also differentiable and in this case, using the chain rule we get

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Thus, $\frac{d}{dx}(e^x) = \frac{1}{(\ln)'(e^x)} = \frac{1}{1/e^x} = e^x$. $\square$

Remark: Now, we can define a^x whenever $a > 0$ and $x \in \mathbb{R}$ as

$$a^x = e^{x \ln a}.$$

Exercise: Show that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$.

Solution: Let $f(x) = \ln x$. Then $f'(x) = 1/x$. Thus, $f'(1) = 1$. But,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1+h)}{h}.$$

Thus, by using the sequential criterion for limits, if we consider the sequence $\{1/n\}$ converging to 0, then

$$1 = f'(1) = \lim_{n\to\infty} \frac{f(1+(1/n))}{(1/n)} = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n.$$

Since the logarithmic function is continuous, we obtain that

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)$$

and hence $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. $\square$

Taylor series: Recall that if f is a $\mathcal{C}^\infty$ function on $\mathbb{R}$, then the Taylor series expansion of f about a is

$$f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots$$

Taylor Series for e^x

Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for a fixed x. If we choose $N > 2x > 0$, then for all $n > N$,

$$\frac{x^{n+1}}{(n+1)!} = \frac{x^n}{n!}\frac{x}{(n+1)} \leq \frac{x^n}{n!}\frac{x}{N} \leq \frac{x^n}{n!}\frac{1}{2}.$$

Thus, for $m \geq n > N$,

$$s_m - s_n = \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \leq \frac{x^{n+1}}{(n+1)!}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n}}\right)$$

and hence

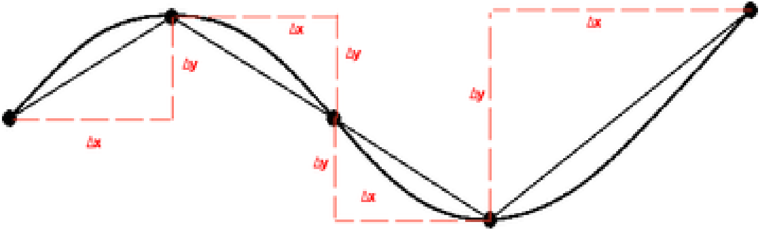
$$s_m - s_n \leq \frac{2x^{n+1}}{(n+1)!} \leq \frac{x^n}{n!}.$$

This shows that the sequence of partial sums of the Taylor series for e^x is Cauchy. Hence the series is convergent.

Does this Taylor series converge to e^x? Yes, as the Taylor's theorem insures that the remainder $R_n(x)$ associated to the function e^x converges to Zero. Therefore $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Arc length

The picture below and the discussion on the next slide are from Wikipedia (http://en.wikipedia.org/wiki/Arc_length).



See: http://en.wikipedia.org/wiki/File:Arclength-2.png

In the picture above, the curve $y = f(x)$ is being approximated by straight line segments which form the hypotenuses of the right angled triangles shown in the picture.

The formula for arc length

Let us denote the arc length of the curve $y = f(x)$ by S.

The length of any given hypotenuse in the previous slide is given by the Pythagorean Theorem: $\sqrt{\Delta x^2 + \Delta y^2}$.

Intuitively, the sum of the lengths of the n hypotenuses appears to approximate S:

$$S \sim \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i,$$

where "$\sim$" means approximately equal, $y_i = f(x_i)$, $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = f(x_i) - f(x_{i-1})$ corresponding to a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$.

The formula for arc length

Now, using the MVT for $y = f(x)$ on $[a, b]$ we get

$$S \sim \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i = \sum_{i=1}^{n} \sqrt{1 + (y'(t_i))^2}\, \Delta x_i$$

for some $t_i \in [x_{i-1}, x_i]$.

It follows that the arc length of the curve $y = f(x)$ (defined on $[a, b]$) is

$$S := \lim_{||P|| \to 0} \sum_{i=1}^{n} \sqrt{1 + (y'(t_i))^2}\, \Delta x_i = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

provided this limit exists which is equivalent of saying that the function $f'(x)$ is Riemann integrable on $[a, b]$.

Exercise 4.10. (ii) Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4.$$

Solution: The formula for the arc length of a curve $y = f(x)$ between the points $x = a$ and $x = b$ is given by

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

For the problem at hand this gives

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x}\, dx = \sqrt{2} \int_0^{\pi/4} \cos x\, dx = 1.$$

$\square$

Rectifiable curves

Not all curves have finite arc length! Here is an example of a curve with infinite arc length.

Example: Let $\gamma : [0,1] \to \mathbb{R}^2$ be the curve given by $\gamma(t) = (t, f(t))$, where

$$f(t) = \begin{cases} t\cos\left(\frac{\pi}{2t}\right), \text{ if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$$

If
http://math.stackexchange.com/questions/296397/nonrectifiable-curve
is correct, you should be able to check that this curve has infinite arc length. Try it as an exercise.

Notice that the curve above is given by a continuous function.
Curves for which the arc length S is finite are called rectifiable curves.
Exercise: Show that the graphs of piecewise $\mathcal{C}^1$ functions are rectifiable.

Convergence of Power series

We have already seen the convergence of a specific power series (namely, the Taylor series for e^x). There is a general test we can use to determine if a power series converges.

Theorem 13: Let $\sum_{n=0}^{\infty} a_n(x-b)^n$ be a power series about the point b. If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$$

for some $R \in \mathbb{R}$, the series converges in the interval $(b-R, b+R)$ to a smooth function. (if the limit is 0, the series converges on the whole real line).

Roughly speaking $|a_n|$ behaves like $1/R^n$ for large n. Hence, the terms in the power series can be bounded by $|(x-b)|^n/R^n$, and this latter (geometric) series converges in $(b-R, b+R)$. This argument can be made precise. Proving that the series is smooth is trickier and we will not get into it.

A convergent Taylor series (or more generally a convergent "power series") can be differentiated and integrated "term by term". That is, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

And similarly,

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n dx.$$

We will not be proving these facts but you can use them below.

Exercise 5: Using Taylor series write down a series for the integral

$$\int \frac{e^x}{x} dx.$$

Solution: We simply integrate term by term to get

$$\log x + x + \frac{x^2}{2 \cdot 2!} + \ldots = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}.$$

We can obtain Taylor series for the inverse trigonometric functions in this way. Indeed we could define the function $\arcsin x$ in this way:

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt.$$

Now we can use the binomial theorem for the integrand. Note that the binomial theorem for arbitrary real exponents is an example of Taylor series:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

It is not too hard to prove that the series on the right hand side above converges for $|x| < 1$. Applying the binomial theorem for $\alpha = -1/2$ to the integrand, we get

$$\arcsin x = \int_0^x \left(1 + \frac{1}{2}t^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^4 + \cdots\right) dt.$$

Integrating this term by term, you should verify that you get the series for $\arcsin x$ that you can derive directly from Taylor series.