MA 105 : Calculus

D1 - Lecture 10

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Smooth functions and Taylor series

Given a smooth function $f(x)$ on on an open interval $I \subseteq \mathbb{R}$, we can write down its associated Taylor polynomials $P_n(x)$ around any point a in I.

Here are some natural questions that arise. Let us take $a = 0$ in what follows.

Question 1. When $x = 0$, obviously $P_n(0) = f(0)$ for all n. Do the Taylor polynomials $P_n(x)$ (around 0, say) always converge as $n \to \infty$ for $x \neq 0, x \in I$? at least for all x in some sub-interval $(c, d) \ni 0$?

Question 2. If $P_n(x)$ converges as $n \to \infty$, does it necessarily converge to $f(x)$?

We will answer the second question.

Smooth but not approximated by Taylor polynomials

The standard example is the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

Notice that $f^{(k)}(0) = 0$ for all $k \geq 0$. Hence $P_n(x) = 0$ for all $n \geq 0$. Hence, $\lim_{n\to\infty} P_n(x) = 0$. Thus the Taylor polynomials $P_n(x)$ around 0 converge to 0 for any $x \in \mathbb{R}$.

But obviously, they do not converge to the value of the function, since $f(x) > 0$ if $x > 0$.

In this case, the Taylor series does a very poor job of approximating the function. Indeed, the remainder term $R_n(x) = f(x)$ for all $x > 0$.

Thus, when we use Taylor series to approximate a function in an interval I, we must make sure that $R_n(x) \to 0$ as $n \to \infty$, for all $x \in I$.

L'Hôpital's rule

Suppose f and g are $\mathcal{C}^1$ functions in an interval I containing 0. By the MVT, for $x \in I$,

$$f(x) = f(0) + f^{(1)}(c_1)x \quad \text{and} \quad g(x) = g(0) + g^{(1)}(c_2)x$$

for $0 < c_1, c_2 < x$. If $f(0) = g(0) = 0$,

$$\lim_{x\to 0} f(x)/g(x) = \lim_{x\to 0} f^{(1)}(c_1)x/g^{(1)}(c_2)x = \lim_{x\to 0} f^{(1)}(c_1)/g^{(1)}(c_2).$$

But $f^{(1)}$ and $g^{(1)}$ are continuous functions and as $x \to 0$, $c_1, c_2 \to 0$. Hence,

$$\lim_{x\to 0} f^{(1)}(c_1)/g^{(1)}(c_2) = f^{(1)}(0)/g^{(1)}(0).$$

If the functions are in $\mathcal{C}^n$, and $f^{(k)}(0) = g^{(k)}(0) = 0$ for all $k < n$, we can apply the MVT repeatedly (or we can apply Taylor's theorem directly) to get $f^{(n)}(0)/g^{(n)}(0)$ as the limit.

Partitions

Definition: Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collections of points

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}.$$

We can think of the points of the partition as dividing the original interval $I = [a, b]$ into sub-intervals $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq n$.

Indeed, $I = \cup_j I_j$ and if two sub-intervals intersect, they have at most one point in common. Hence, the notation "partition".

Definition: A partition $P' = \{a = x'_0 < x'_1 < \cdots < x'_m = b\}$ is said to be a refinement of the partition P if for each $x_i \in P$, there exists an $x'_j \in P'$ such that $x_i = x'_j$.

Intuitively, a refinement P' of a partition P will break some of the sub-intervals in P into smaller sub-intervals.

Any two partitions P_1 and P_2 have a common refinement $P = P_1 \cup P_2$.

Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \cdots < x_{b-1} < x_n = b\}$ and a bounded function $f : [a, b] \to \mathbb{R}$, we define two associated quantities.

First, we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n.$$

Definition: We define the Lower sum as

$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

In case the words “infimum” and “supremum” bother you, you can think “minimum” and “maximum most of time since we will usually be dealing with continuous functions on $[a, b]$.

The Darboux integrals

For any partition $P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ of $[a, b]$, $m_j \leq M_j$, $\forall\ 1 \leq j \leq n$ and hence

$$L(f, P) \leq U(f, P).$$

Since the function f is bounded on $[a, b]$, there exists $m, M \in \mathbb{R}$ such that

$$m \leq f(x) \leq M$$

for all $x \in [a, b]$ and hence

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a)$$

for any partition P of $[a, b]$.

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\},$$

where the supremum is taken over all partitions of $[a, b]$.

The Darboux integrals

Similarly, the upper Darboux integral of f is defined by

$$U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\},$$

and again the infimum is over all partitions of $[a, b]$.

(This time there is no escaping inf and sup!)

If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\, dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

One basic example

In order to illustrate what we are saying we will take the following basic example. Let $[a, b] = [0, 1]$ and let $f(x) = x$.

One of the most natural partitions of an interval is a partition that divides the interval into sub-intervals of equal length.

For $[0, 1]$, this is

$$P_n = \{0 < 1/n < 2/n < \cdots < (n-1)/n < 1\}.$$

On the interval $I_j = [\frac{j-1}{n}, \frac{j}{n}]$, where does the function $f(x) = x$ take its minimum?

Clearly, the minimum $m_j = \frac{j-1}{n}$ is attained at $\frac{j-1}{n}$ and the maximum $M_j = \frac{j}{n}$ at $\frac{j}{n}$. And finally, $\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$, for all $1 \leq j \leq n$.

An example of a refinement of P_n is P_{2n}, or, more generally, P_{kn} for any natural number k.

The $L(f, P_n)$ and $U(f, P_n)$ for $f(x) = x$ on $[0, 1]$

Let us calculate $L(f, P_n)$ and $U(f, P_n)$ for the example we gave in the last slide.

$$L(f, P_n) = \sum_{j=1}^{n} \frac{(j-1)}{n} \cdot \frac{1}{n} = \sum_{j=0}^{n-1} \frac{j}{n^2}.$$

This can be evaluated explicitly:

$$L(f, P_n) = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} - \frac{1}{2n}.$$

Similarly, we can check that

$$U(f, P_n) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \sum_{j=1}^{n} \frac{j}{n^2} = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} + \frac{1}{2n}.$$

Can we conclude that the Darboux integral is $\frac{1}{2}$ by letting $n \to \infty$? Unfortunately, no, as of now. But we will see soon that the function $f(x) = x$ is Darboux integrable on any finite interval.

An example of a function that is not Darboux integrable

Here is a function that is not Darboux integrable on $[0, 1]$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It should be clear that no matter what partition one takes the infimum on any sub-interval in the partition will be 0 and the supremum will be 1.

From this, one can see immediately that

$$L(f, P) = 0 \neq 1 = U(f, P),$$

for every P, and hence that $L(f) = 0 \neq 1 = U(f)$.