MA 105 Calculus II

Week 4

Mayukh Mukherjee

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

October 16 - 20, 2023

Different parametrisations of the same path

Example 1: Let $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then $c_2(t) = (\cos 2t, \sin 2t)$ for $0 \leq t \leq \pi$, the paths are different as a function but the curves traversed are the same.

Example: 2: Here is an example of a simple path with three different parametrisations with the same domain.
Take the straight line segment between $(0, 0, 0)$ and $(1, 0, 0)$.

Here are three different ways of parametrising it:

$$\{t, 0, 0)\}, \quad \{(t^2, 0, 0)\} \quad \text{and} \quad \{(t^3, 0, 0)\},$$

where $0 \leq t \leq 1$.

Reparametrisation

Let $\mathbf{c} : [a, b] \to \mathbb{R}^n$ be a path which is non-singular, that is, $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$.

- Suppose we now make change of variables $t = h(u)$, where h is $\mathcal{C}^1$ diffeomorphism (this means that h is bijective, $\mathcal{C}^1$ and so is its inverse) from $[\alpha, \beta]$ to $[a, b]$. We let $\gamma(u) = \mathbf{c}(h(u))$.
- We will assume that $h(\alpha) = a$ and $h(\beta) = b$.
- Then γ is called a reparametrisation of $\mathbf{c}$.
- Because h is a C^1 diffeomorphism, γ is also a C^1 curve.

The line integral of a vector field $\mathbf{F}$ along γ is given by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\alpha^\beta \mathbf{F}(\gamma(u)) \cdot \gamma'(u) du = \int_\alpha^\beta \mathbf{F}(c(h(u)) \cdot \mathbf{c}'(h(u)) h'(u) du,$$

where the last equality follows from the chain rule. Using the fact that $h'(u)du = dt$, we can change variables from u to t to get

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_\mathbf{c} \mathbf{F} \cdot d\mathbf{s}.$$

Orientation of Curves

For given two points P and Q on $\mathbb{R}^n$ for $n = 2, 3$, and a path connecting them, we can ask whether the path is traversed from P to Q or from Q to P?

Since a path from P to Q is a mapping $\mathbf{c} : [a, b] \to \mathbb{R}^n$ with $\mathbf{c}(a) = P$ and $\mathbf{c}(b) = Q$, (or vice-versa), it allows us to determine the direction in which the path is traversed. This direction of the path is called its Orientation.

If the reparametrisation $\gamma(\cdot) = \mathbf{c}(h(\cdot))$ preserves the orientation of $\mathbf{c}$, then

$$\int_\gamma \mathbf{F}.\mathbf{ds} = \int_\mathbf{c} \mathbf{F}.\mathbf{ds}.$$

If the reparamtrisation reverses the orientation, then

$$\int_\gamma \mathbf{F}.\mathbf{ds} = -\int_\mathbf{c} \mathbf{F}.\mathbf{ds}.$$

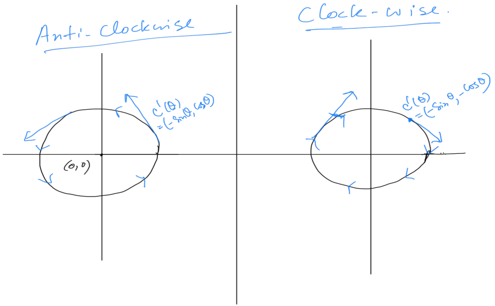
Curves on plane

Let us consider the paths lying in $\mathbb{R}^2$, namely, Planar curves.

For a simple closed planar curve, we get a choice of direction- clockwise or anti-clockwise.

Ex. $\gamma(\theta) = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi]$. This is a circle with direction anti-clockwise.

Set $\gamma_1(\theta) = (\cos(\theta), -\sin(\theta))$, $\theta \in [0, 2\pi]$. It is circle with clockwise direction.



The argument just made shows that the line integral is independent of the choice of parametrisation - it depends only on the image of the non-singular parametrised path.

- A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrised path in the given direction. Often the line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F}.\mathbf{ds}$ or by $\int_C F_1 dx + F_2 dy + F_3 dz$.
- To evaluate $\int_C \mathbf{F}.\mathbf{ds}$, choose a convenient parametrisation $\mathbf{c}$ of C traversing C in the given direction and then

$$\int_C \mathbf{F}.\mathbf{ds} := \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$

- '$\oint_C$' means the line integral over a closed curve C.

The arc length parametrisation

There is a natural choice of parametrisation we can make which is useful in many situations. This is the parametrisation by arc length.

Recall the length of a curve $\mathbf{c}$ for a path $\mathbf{c} : [a, b] \to \mathbb{R}^3$, called its arc length, is given by

$$\ell(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\| \; dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \; dt.$$

We now set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $s'(t) = \|\mathbf{c}'(t)\|$. Why? Fundamental theorem of Calculus.

It is easy to see that s is a strictly increasing differentiable function. Let $h : [0, \ell(\mathbf{c})] \to [a, b]$ be its inverse. Then it is differentiable and its derivative is not vanishing. Define $\tilde{\mathbf{c}}(u) := \mathbf{c}(h(u))$ for $u \in [0, \ell(\mathbf{c})]$. This is called the arc length parametrisation.

Let $h(u) = t \in [a, b]$ or $s(t) = u$.
Note that

$$\begin{aligned} \frac{d\tilde{\mathbf{c}}(u)}{du} &= \mathbf{c}'(h(u))h'(u) \\ &= \mathbf{c}'(h(u))\frac{1}{s'(h(u))} \\ &= \mathbf{c}'(t)\frac{1}{\|\mathbf{c}'(t)\|} \end{aligned}$$

Using the reparametrisation theorem we get that

$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = \int_{\tilde{\mathbf{c}}} \mathbf{F}.\mathbf{ds}.$$

Note,

$$
\begin{aligned}
\int_{\tilde{\mathbf{c}}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\tilde{\mathbf{c}}(u)) \cdot \tilde{\mathbf{c}}'(u) du \\
&= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u))) \cdot \frac{\mathbf{c}'(h(u))}{\|\mathbf{c}'(h(u))\|} du \\
&= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u)) \cdot \mathbf{T}(h(u)) \; du
\end{aligned}
$$

where $\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t\|}$ is the unit tangent vector along the curve.

In other words, the line integral is nothing but the (Riemann) integral of the tangential component of $\mathbf{F}$ with respect to arc length.

Note for this reparametrisation we need to assume $\mathbf{c}$ is a non singular curve.

To this end, as before we set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $ds = \|\mathbf{c}'(t)\| dt$.

Integrals of scalar functions along path: Let $f : D \to \mathbb{R}$ be a continuous scalar function and $\mathbf{c} : [a, b] \longrightarrow D$ be a non-singular path. Then the path integral of f along $\mathbf{c}$ is defined by

$$\int_{\mathbf{c}} f \, ds := \int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\| \, dt.$$

Example. Find the circumference of the circle in $\mathbb{R}^2$ whose center is at origin and radius is r, for some $r > 0$.

Ans. Check $\int_{\mathbf{c}} f \, ds$ for $f = 1$ and $\mathbf{c}(t) = (r \cos t, r \sin t)$, for $t \in [0, 2\pi]$.

Characterization of gradient fields

Theorem (Variant of fundamental theorem of calculus)

Let $n = 2, 3$ and let $D \subset \mathbb{R}^n$.

1. *Let $\mathbf{c} : [a, b] \to D \subset \mathbb{R}^n$ be a smooth path.*
2. *Let $f : D \to \mathbb{R}$ be a differentiable function and let ∇f be continuous on $\mathbf{c}$.*

Then $\displaystyle\int_{\mathbf{c}} \nabla f . \mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$.

Proof. From definition, it follows that

$$\int_{\mathbf{c}} \nabla f . \mathbf{ds} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

Now the integrand on the right hand side is nothing but the directional derivative of f in the direction of $\mathbf{c}(t)$. Hence, we obtain

$$\int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_a^b \frac{d}{dt} f(\mathbf{c}(t)) dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

- Suppose the vector field $\mathbf{F}$ is a continuous conservative field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function f. Then for any smooth path $\mathbf{c}$, we have
$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$
- This shows that the value of the line integral of a conservative field depends only on the value of the function at the end points of the curve, not on the curve itself.

Definition

The line integral of a vector field $\mathbf{F}$ is independent of path in a domain if for any $\mathbf{c}_1$ and $\mathbf{c}_2$ paths in D with the same initial and terminal points,
$$\int_{\mathbf{c}_1} \mathbf{F}.\mathbf{ds} = \int_{\mathbf{c}_2} \mathbf{F}.\mathbf{ds}.$$

Equivalently, the line integral of $\mathbf{F}$ is independent of path in D if for any closed curve $\mathbf{c}$ (why?)
$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = 0.$$

Examples

Example Find the work done by the gravitational field $\mathbf{F}(x, y, z) = -\frac{mMG}{|\mathbf{r}(x,y,z)|^3}\mathbf{r}(x, y.z)$, in moving a particle with mass m and position vector $\mathbf{r}(x, y, z) = (x, y, z)$ from $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C.

Ans Since the gravitational field is a conservative field and $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$, where

$$f(x, y, z) = \frac{mMG}{|\mathbf{r}(x, y, z)|} = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}.$$

Using the Fundamental theorem for line integrals, the work done is

$$W = \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = f(\mathbf{c}(b) - f(\mathbf{c}(a)) = f(2, 2, 0) - f(3, 4, 12) = mMG(\frac{1}{2\sqrt{2}} - \frac{1}{13}),$$

where $\mathbf{c} : [a, b] \to \mathbb{R}$, a parametrisation of curve C with $\mathbf{c}(a) = (3, 4, 12)$ and $\mathbf{c}(b) = (2, 2, 0)$.

Example Evaluate $\int_C y^2 dx + x dy$, where

1. $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$,
2. $C = C_2$ is the part of parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Are the line integrals along C_1 and C_2 same?

Ans 1.) Consider parametrisation for C_1,
$\mathbf{c}_1(t) = (5t-5, 5t-3), \quad t \in [0,1]$. Thus $\mathbf{c}_1'(t) = (5,5)$ for all $t \in [0,1]$.
So, $\mathbf{F}(\mathbf{c}_1(t)) = ((5t-3)^2, 5t-5)$ and

$$\int_{C_1} y^2 dx + x dy = \int_0^1 \left[(5t-3)^2.5 + (5t-5).5\right] dt = -\frac{5}{6}.$$

2. Consider parametrisation for C_2, $\mathbf{c}_2(t) = (4-t^2, t), \quad t \in [-3,2]$. Thus $\mathbf{c}_2'(t) = (-2t, 1)$ for all $t \in [-3,2]$. So, $\mathbf{F}(\mathbf{c}_2(t)) = (t^2, 4-t^2)$ and

$$\int_{C_2} y^2 dx + x dy = \int_{-3}^2 \left[t^2(-2t) + (4-t^2)\right] dt = 40\frac{5}{6}.$$

Line integrals along C_1 and C_2 are Not same! Though the endpoints of C_1 and C_2 are same!

Conservative vector fields

In general, the line integral of a vector field depends on the path.

Fundemental theorem of calculus for line integrals yields that the line integral of a conservative field is independent of path in D.

What about the converse?

We will now prove the converse to our previous assertion under some assumption on D.

Definition: A subset D of $\mathbb{R}^n$ is called connected if it cannot be written as a disjoint union of two non-empty subsets $D_1 \cup D_2$, with $D_1 = D \cap U_1$ and $D_2 = D \cap U_2$, where U_1 and U_2 are open sets.

Definition: A subset of D of $\mathbb{R}^n$ is said to be path connected if any two points in the subset can be joined by a path (that is the image of a continuous curve) inside D.

In $\mathbb{R}^n$ we can show that an open subset is connected if and only if it is path connected. So it is sufficient to assume region of the vector field is open and connected.

Examples

Example. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < a^2\}$ is path-connected.

Ans. If $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ are in D, then which path lying in D can be defined connecting P and Q?

Path connected implies connected.

Example. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \cup \{(2,2)\}$ is connected in $\mathbb{R}^2$?
Ans No. (Why?)

Example. $D = \{(x, \sin(\frac{1}{x}) \in \mathbb{R}^2 \mid x \in (0,1]\} \cup \{(0,0)\}$ is connected in $\mathbb{R}^2$ but not path-connected.

Theorem

Let $\mathbf{F} : D \to \mathbb{R}^3$ be a continuous vector field on a connected open region D in $\mathbb{R}^3$. If the line integral of $\mathbf{F}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field in D.

Proof: Let the line integral of $\mathbf{F}$ be path-independent in D, where D is an open, connected set of $\mathbb{R}^n$, for $n = 3$.
Goal: Find a differentiable function $V : D \to \mathbb{R}$ such that

$$\mathbf{F}(x, y, z) = \nabla V(x, y, z), \quad \text{for all} \quad (x, y, z) \in D.$$

We construct such V in the following way.
Step 1 Let $P_0 = (x_0, y_0, z_0)$ be a fixed point in D. Let $P = (x, y, z)$ be an arbitrary point in D. We define

$$V(x, y, z) = \int_{\mathbf{c}_P} \mathbf{F} \cdot d\mathbf{s}, \quad \text{for all} \quad (x, y, z) \in D,$$

where $\mathbf{c}_P : [a, b] \to D$ is any path from P_0 to P.
Since D is path connected, there always exists a path from P_0 to any point $P \in D$. Hence V is defined on the whole of D.

Since the line integral of $\mathbf{F}$ is path-independent in D, $V(x, y, z)$ does not depend on which path we took from P_0 to P and hence is well-defined.

The proof of theorem contd.

Step 2 It remains to show that $\mathbf{F} = \nabla V$.

Let $\mathbf{F} = (F_1, F_2, F_3)$. Then we have to show

$$\frac{\partial V}{\partial x} = F_1, \quad \frac{\partial V}{\partial y} = F_2, \quad \frac{\partial V}{\partial z} = F_3, \quad \text{on} \quad D.$$

Evaluate $\frac{\partial V}{\partial x}(x, y, z) = \lim_{h\to 0} \frac{V(x+h,y,z)-V(x,y,z)}{h}$ for all $(x, y, z) \in D$.

From definition of V,

$$V(x + h, y, z) = \int_{\mathbf{c}_{P_h}} \mathbf{F}.\mathbf{ds},$$

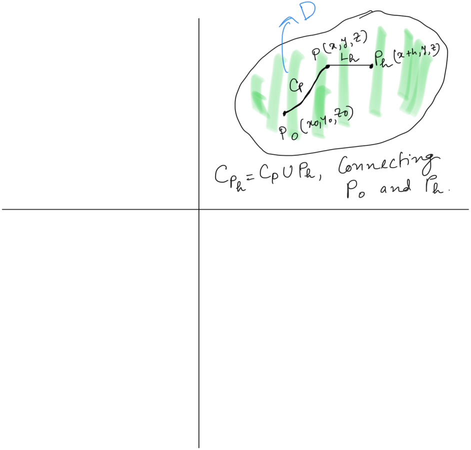
where $P_h = (x + h, y, z)$ and $\mathbf{c}_{P_h}$ is any path joining P_0 and P_h in D.

Choose $\mathbf{c}_{Ph}$ conveniently: Since D is open, for a given $P = (x, y, z) \in D$, there exists a disk contained in D with center P containing points $P_h = (x + h, y, z)$ for all h such that h is small enough. Thus for all h with $|h|$ suitable small, the straight line $\mathbf{L}_h$ joining P and P_h lies in D, where

$$\mathbf{L}_h(t) = (x + th, y, z) \quad \forall\, 0 \le t \le 1.$$

The proof of theorem contd.

We choose the path $\mathbf{c}_{P_h}$ from P_0 to P_h as the union of the two paths $\mathbf{c}_P$ from P_0 to P and the straith line $\mathbf{L}_h$ from P to P_h.



The proof of theorem contd.

From the property of line integrals we mentioned earlier

$$\int_{\mathbf{c}_{P_h}} \mathbf{F} \cdot \mathbf{d}s = \int_{\mathbf{c}_P} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{L}_h} \mathbf{F} \cdot \mathbf{d}s.$$

Hence it yields

$$V(x+h, y, z) = V(x, y, z)$$

$$+ \int_0^1 (F_1(x+th, y, z), F_2(x+th, y, z), F_3(x+th, y, z)) \cdot (h, 0, 0) dt$$

Thus

$$\frac{\partial V}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{V(x+h, y, z) - V(x, y, z)}{h} = \lim_{h \to 0} \int_0^1 F_1(x+th, y, z) dt.$$

Due to the continuity of F_1,

$$\lim_{h \to 0} \int_0^1 F_1(x+th, y, z) dt = F_1(x, y, z).$$

The proof of theorem contd.

Hence we get

$$\frac{\partial V}{\partial x}(x,y,z) = F_1(x,y,z), \quad \forall\,(x,y,z) \in D.$$

We can similarly show that

$$\frac{\partial V}{\partial y} = F_2 \quad \text{and} \quad \frac{\partial V}{\partial z} = F_3, \quad \text{on} \quad D.$$

This proves our theorem.

In summary, for a given continuous vector field $\mathbf{F}$ in $\mathbb{R}^n$ defined on D, an open, path connected subset of $\mathbb{R}^n$, the vector field $\mathbf{F}$ is a conservative field if and only if the line integral of $\mathbf{F}$ in D is independent of path in D.

Examples Contd.

Example Determine whether or not the vector field

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}, \quad \text{in} \quad \mathbb{R}^2 \backslash \{(0,0)\},$$

is conservative.
Ans Check for the closed curve $\mathbf{c} = (\cos t, \sin t)$, $t \in [0, 2\pi]$, the line integral of $\mathbf{F}$ along $\mathbf{c}$?

$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = \int_0^{2\pi} (\sin t)(\sin t) + (\cos t)(\cos t)\, dt = 2\pi.$$

so, $\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} \neq 0$, though $\mathbf{c}$ is a closed curve, and hence $\mathbf{F}$ cannot be conservative field.

However, the equivalent formulation of conservative field and the path independency of the line integral of the vector field may not be always useful to determine if a vector field conservative.

Necessary condition for conservative fields

Theorem

- *For $n = 2$, if $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ is a conservative vector field, where F_1 and F_2 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^2$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D.$$

- *For $n = 3$, if $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ is a conservative vector field, where F_1, F_2, F_3 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^3$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D.$$

The theorem follows from a direct calculation using the fact that $\mathbf{F} = \nabla V$ and using the properties of the mixed partial derivatives of V.

Example Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}, \quad \text{in} \quad \mathbb{R}^2$$

is conservative.
Ans Here $F_1(x, y) = x - y$ and $F_2(x, y) = x - 2$. Then

$$\frac{\partial F_1}{\partial y} = -1, \quad \text{and} \quad \frac{\partial F_2}{\partial x} = 1.$$

So by previous theorem, **F** cannot be a conservative field.

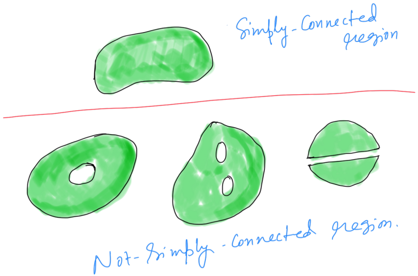
What about the converse of the theorem?
The converse is partially true under some additional hypothesis on D.
However, it is often a convenient method verifying if a vector field is conservative.

Simply connected domain

Definition

A subset D of $\mathbb{R}^n$ for $n = 2, 3$, is simply connected, if D is a connected region such that any simple closed curve lying in D encloses a region that is in D.

Basically, a simply-connected region contains no hole and cannot consist of two separate pieces.



Sufficient condition for conservative field

Theorem

Let $n = 2, 3$ and let D be an open, simply connected region in $\mathbb{R}^n$.

1. *For $n = 2$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on D satisfying*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D,$$

   Then $\mathbf{F}$ is a conservative field.

2. *For $n = 3$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is such that F_1, F_2 and F_3 have continuous first order partial derivatives on D satisfying*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D,$$

   Then $\mathbf{F}$ is a conservative field.

We postpone the proof (Green's theorem!)

Example. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}, \quad \text{in} \quad \mathbb{R}^2$$

is conservative.

Ans Note that the region $\mathbb{R}^2$ is open and simply-connected and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable.

Let $F_1(x, y) = (3 + 2xy)$ and $F_2(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial F_1}{\partial y}(x, y) = 2x = \frac{\partial F_2}{\partial x}.$$

Thus using the previous theorem, we conclude that $\mathbf{F}$ is a conservative field.

How to find a potential function f such that $\mathbf{F} = \nabla f$, for above example?

Example contd.

Let $\mathbf{F} = \nabla f$, then $\frac{\partial f}{\partial x}(x,y) = F_1(x,y)$ and $\frac{\partial f}{\partial y}(x,y) = F_2(x,y)$.
Step 1 Fixing y, solve the ODE with respect to x-variable:

$$\frac{\partial f}{\partial x}(x,y) = F_1(x,y).$$

Integrating with respect to x in both side, we get

$$f(x,y) = \int_0^x F_1(s,y)\,dx + c(y) = 3x + x^2y + c(y).$$

Step 2 Determine the $c(y)$ using $\frac{\partial f}{\partial y}(x,y) = F_2(x,y)$. Differentiating $f(x,y)$ with respect to y,

$$\frac{\partial f}{\partial y}(x,y) = x^2 + c'(y),$$

and it has to be equal to $F_2(x,y)$.
so, $x^2 + c'(y) = x^2 - 3y^2$ and thus $c'(y) = -3y^2$. Now solving this ODE with respect to y variable:

$$c(y) = -y^3 + K,$$

In summary, for a given vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 3$:

1. If $\mathbf{F}$ is a continuous, conservative vector field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function, then the line integral of $\mathbf{F}$ along any path C from P to Q in D given by
$$\int_C \mathbf{F}.\mathbf{ds} = f(Q) - f(P),$$
and it only depends on the value of f, the potential function, at the initial and terminal points of the path.
2. Let $\mathbf{F}$ be a continuous field and let D be an open connected set in $\mathbb{R}^n$. Then $\mathbf{F}$ is a conservative field if and only if the line integral of $\mathbf{F}$ is path-independent in D.
3. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is a C^1 conservative vector field on an open region D, then $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on D. Similar result holds in $\mathbb{R}^3$.
4. Let D be an open, simply connected region in $\mathbb{R}^2$ and let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be C^1 on D. Then $\mathbf{F}$ is conservative in D if and only if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on D. Similar result holds in $\mathbb{R}^3$.

• If $\mathbf{F}$ is a vector field defined from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, a flow line or integral curve is a path i.e., a map $\mathbf{c} : [a, b] \to D$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)), \quad \forall t \in [a, b].$$

In particular, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}$. • Finding the flow line for a given vector field involves solving a system of differential equations,

Recall a path in $\mathbb{R}^n$ is a continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^n.$ A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c}$ in $\mathbb{R}^n.$ Both the curve and path are denoted by the same symbol $\mathbf{c}$.

• Let $n = 3$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, for all $t \in [a, b].$ The path $\mathbf{c}$ is continuous iff each component x, y, z is continuous. Similarly, $\mathbf{c}$ is a C^1 path, i.e., continuously differentiable if and only if each component is C^1.
• A path $\mathbf{c}$ is called closed if $\mathbf{c}(a) = \mathbf{c}(b)$.
• A path $\mathbf{c}$ is called simple if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ in $[a, b]$ other than $t_1 = a$ and $t_2 = b$ endpoints.
• If a C^1 curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$, the curve is called a regular or non-singular parametrised curve.

Line integrals of vector fields

• Assume that the vector field $\mathbf{F}$ is continuous and the curve $\mathbf{c}$ is C^1.

Then we define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

• If $\mathbf{c}_1$ is a path joining two points P_0 and P_1, $\mathbf{c}_2$ is a path joining P_1 and P_2 and $\mathbf{c}$ is the union of these paths (that is, it is a path from P_0 to P_2 passing through P_1), then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

Here $\mathbf{c}$, the union of two C^1 paths $\mathbf{c}_1$ and $\mathbf{c}_2$ is need not be C^1 but picewise C^1. The line integral of a continuous vector field is defined along piecewise C^1 curves.

• Let the curve $\mathbf{c}$ be a union of curves $\mathbf{c}_1, \ldots, \mathbf{c}_n$. We often write this as $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \ldots \mathbf{c}_n$, where end point of $\mathbf{c}_i$ is the starting point of $\mathbf{c}_{i+1}$ for all $i = 1, \ldots, n-1$.

Then we can define

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds := \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \ldots + \int_{\mathbf{c}_n} \mathbf{F} \cdot ds.$$

• Let $\mathbf{c}$ be a curve on $[a, b]$ and $-\mathbf{c}(t) = \mathbf{c}(b + a - t)$, that is the curve $\mathbf{c}$ traversed in the reverse direction. Then $\int_{\mathbf{c}} \mathbf{F} \cdot ds + \int_{-\mathbf{c}} \mathbf{F} \cdot ds = 0$.
• Let $\mathbf{c}(t) : [t_1, t_2] \to \mathbb{R}^n$ be a path which is non-singular, that is, $\mathbf{c}'(t) \neq 0$ for all $t \in [t_1, t_2]$.

- Suppose we now make change of variables $t = h(u)$, where h is $\mathcal{C}^1$ diffeomorphism (this means that h is bijective, $\mathcal{C}^1$ and so is its inverse) from $[u_1, u_2]$ to $[t_1, t_2]$.
- We let $\gamma(u) = \mathbf{c}(h(u))$. Then γ is called a reparametrisation of $\mathbf{c}$. We will assume that $h(u_i) = t_i$ for $i = 1, 2$
- Since a path between P and Q is a mapping $\mathbf{c} : [a, b] \to \mathbb{R}^n$ with $\mathbf{c}(a) = P$ and $\mathbf{c}(b) = Q$, (or vice-versa), it allows us to determine the direction in which the path is traversed. This direction of the path is called its Orientation.
- If the reparametrisation $\gamma = \mathbf{c}(h)$ preserves the orientation of $\mathbf{c}$, then $\int_\gamma \mathbf{F}.\mathbf{ds} = \int_\mathbf{c} \mathbf{F}.\mathbf{ds}$.
- If the reparamtrisation reverses the orientation, then $\int_\gamma \mathbf{F}.\mathbf{ds} = -\int_\mathbf{c} \mathbf{F}.\mathbf{ds}$.
- Let $f : D \to \mathbb{R}$ be a continuous scalar function and $\mathbf{c} : [a, b] \longrightarrow D$ be a non-singular path. Then the path integral of f along $\mathbf{c}$ is defined by $\int_\mathbf{c} f\, ds := \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt$.

Green's Theorem

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\left(D \cup \partial D\right) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions.*

Then

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a double integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.

Example: Let C be the circle of radius r oriented in the counterclockwise direction, and let $F_1(x,y) = -y$ and $F_2(x,y) = x$. Evaluate

$$\int_C F_1(x,y)dx + F_2(x,y)dy.$$

Solution: Let D denote the disc of radius r. Then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2$. Hence, by Green's theorem

$$\int_C F_1(x,y)dx + F_2(x,y)dy = \iint_D 2dxdy = 2\pi r^2.$$

Also by the direct calculation, denoting $\mathbf{F} = (F_1, F_2)$, check
$\int_C \mathbf{F}.\mathbf{ds} = \int_C F_1(x,y)dx + F_2(x,y)dy = ?$.

Examples.

Example. Compute the line integral $\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy$, where C is the circle in $\mathbb{R}^2$ with origin at $(2,0)$ and radius 1.

Can compute directly using definition of line integral! But is there any better way?

Use Green's theorem: Set $F_1(x,y) = ye^{-x}$ and $F_2(x,y) = (\frac{1}{2}x^2 - e^{-x})$, for all $(x,y) \in D$, where $D = \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 \leq 1\}$. Using Green's theorem,

$$\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy = \int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy.$$

Now see

$$\int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy = \int\int_D xdxdy,$$

and derive the double integral using polar coordinates: Check!

$$\int\int_D xdxdy = 2\pi.$$

Area of a region

Can the area of a region enclosed be expressed as a line integral?

If C is a positively oriented curve that bounds a region D , then the area $A(D)$ is given by (Why?)

$$A(D) = \frac{1}{2} \int_C x dy - y dx.$$

Note if $F_1(x, y) = -\frac{y}{2}$ and $F_2(x, y) = \frac{x}{2}$, for all $(x, y) \in D$, then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, and hence $A(D) := \int\int_D 1\, dxdy = \int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dxdy$.

By Green's theorem,

$$\int_D \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \int_C F_1\, dx + F_2\, dy = \frac{1}{2} \int_C x dy - y dx,$$

Thus $A(D) = \frac{1}{2} \int_C x dy - y dx$.

Also note for $F_1 \equiv 0$ and $F_2(x, y) = x$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, thus $A(D) = \int_C x\, dy$.

Further for $F_1(x, y) = -y$ and $F_2 \equiv 0$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, thus $A(D) = -\int_C y\, dx$.

Example: Let us use the formula above to find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: We parametrise the curve C by $\mathbf{c}(t) = (a\cos t, b\sin t)$, $0 \le t \le 2\pi$. By the formula above, we get

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_C xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)dt - (b\sin t)(-a\sin t)dt \\
&= \frac{1}{2}\int_0^{2\pi} abdt = \pi ab.
\end{aligned}$$

Suppose we are given a simple positively oriented closed curve $C : (r(t), \theta(t))$ in polar coordinates. Thus for $t \in [a, b]$ $x(t) = r(t)\cos(\theta(t))$ and $y(t) = r(t)\sin(\theta(t))$ and using chain rule formula:

$$\frac{dx}{dt}(t) = \cos(\theta(t))\frac{dr}{dt}(t) - r(t)\sin\theta(t)\frac{d\theta}{dt}(t),$$
$$\frac{dy}{dt}(t) = \sin(\theta(t))\frac{dr}{dt}(t) + r(t)\cos\theta(t)\frac{d\theta}{dt}(t).$$

Then, by the area formula above, we know that the area enclosed by C is given by

$$\begin{aligned}
&\frac{1}{2}\int_C xdy - ydx := \frac{1}{2}\int_C \left(x(t)\frac{dy}{dt}(t) - y(t)\frac{dx}{dt}(t)\right) dt \\
&= \frac{1}{2}\int_a^b r(t)\cos\theta(t)\sin\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r^2(t)\cos^2\theta(t)\frac{d\theta}{dt}dt \\
&-\frac{1}{2}\int_a^b r(t)\sin\theta(t)\cos\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r(t)^2\sin^2\theta(t)\frac{d\theta}{dt}dt \\
&= \frac{1}{2}\int r^2 d\theta.
\end{aligned}$$

Exercise: Find the area of the cardioid $r = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$.

Solution: Using the formula we have just derived, the desired area is simply

$$\frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2 d\theta = a^2\int_0^{2\pi} -2\cos\theta + \frac{\cos 2\theta}{2} + \frac{3}{2} d\theta$$
$$= \frac{3a^2\pi}{2}.$$