MA 105 Part II Tutorial Sheet 5 :
Green's theorem, October 30, 2023

1. Verify Green's theorem in each of the following cases:

   (i) $F_1(x,y) = -xy$; $F_2(x,y) = xy$; $R: \ x \geq 0, \ 0 \leq y \leq 1 - x^2$;

   (ii) $F_1(x,y) = 2xy$; $F_2(x,y) = e^x + x^2$; where R is the triangle with vertices $(0,0), (1,0)$, and $(1,1)$.

2. Use Green's theorem to evaluate the integral $\oint_{\partial R} y^2 \, dx + x \, dy$, where

   (i) R is the square with vertices $(0,0), (2,0), (2,2), (0,2)$.

   (ii) R is the square with vertices $(\pm 1, \pm 1)$.

   (iii) R is the disc of radius 2 and center $(0,0)$ oriented clock-wise.

3. For a simple closed curve given in polar coordinates show using Green's theorem that the area enclosed is given by

$$A = \frac{1}{2} \oint_C r^2 d\theta.$$

   Use this to compute the area enclosed by the following curves:

   (i) The cardioid: $r = a(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi;.$

   (ii) The lemniscate: $r^2 = a^2 \cos 2\theta, ; \ -\pi/4 \leq \theta \leq \pi/4$.

4. Find the area of the following regions:

   (i) The area lying in the first quadrant of the cardioid $r = a(1 - \cos\theta)$.

   (ii) The region under one arch of the cycloid

$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}, \ 0 \leq t \leq 2\pi.$$

   (iii) The region bounded by the limacon on

$$r = 1 - 2\cos\theta, \ \ 0 \leq \theta \leq \pi/2$$

   and the two axes.

5. Let $D = \{(x,y) \in \mathbb{R}^2 \mid a^2 \leq x^2 + y^2 \leq b^2\}$, where $0 < a < b$. Evaluate

$$\int_{\partial D} x e^{-y^2} dx + [-x^2 y e^{-y^2} + 1/(x^2 + y^2)] dy,$$

   where ∂D is positively oriented.

6. Let C be a simple closed curve in the xy-plane. Show that

$$3I_0 = \oint_C x^3 dy - y^3 dx,$$

   where $I_0 = \frac{1}{3} \int \int_D r^2 \ dxdy$, D is the region enclosed by C. This I_0 is often called 'polar moment of inertia' of the region D.

7. If C is the line segment connecting (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1.$$

8. Let C be any counterclockwise closed curve in the plane and let $\mathbf{n}$ be the outward unit normal to the curve C. Compute $\oint_C \nabla(x^2 - y^2) \cdot \mathbf{n}ds$.

9. Let D be a region in $\mathbb{R}^2$ with boundary ∂D satisfying the hypothesis stated in the 'Green's theorem'. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function.

   (i) Show that $\nabla^2\phi = \mathrm{div}(\mathrm{grad}\,\phi)$, where the operator ∇^2 is defined by

   $$\nabla^2\phi(x, y) = \frac{\partial^2\phi}{\partial^2 x}(x, y) + \frac{\partial^2\phi}{\partial^2 y}(x, y).$$

   The operator ∇^2 is called 'Laplace operator'.

   (ii) Show that the Green's Identity holds:

   $$\iint_D \nabla^2\phi\, d(x, y) = \oint_{\partial D} \frac{\partial\phi}{\partial\mathbf{n}}\, ds,$$

   where $\mathbf{n}$ is the outward unit normal to the curve ∂D .
   (Hint. Use the divergence form of Green's theorem for the vector field $\mathbf{F} = \mathrm{grad}\,\phi$)

   (iii) Using the above identity, compute

   $$\oint_C \frac{\partial\phi}{\partial\mathbf{n}}\, ds$$

   for $\phi = e^x \sin y$, and D the triangle with vertices $(0, 0), (4, 2), (0, 2)$.

10. Let us consider the region $\Omega = \{(x, y) \mid x^2 + y^2 > 1\}$ and the vector field be defined on Ω. Evaluate the following line integrals where the loops are traced in the counter clockwise sense

   (i)

   $$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

   where C is any simple closed curve in Ω enclosing the origin.

   (ii)

   $$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

   where C is any simple closed curve in Ω not enclosing the origin.

   (iii) Let C be a smooth simple closed curve lying in Ω. Find

   $$\oint_C \frac{\partial(\ln r)}{\partial y} dx - \frac{\partial(\ln r)}{\partial x} dy.$$

11. Is there a vector field $\mathbf{G}$ in $\mathbb{R}^3$ such that

   i) curl $\mathbf{G}(x, y, z) = (x \sin y)\mathbf{i} + (cosy)\mathbf{j} + (z - xy)\mathbf{k}$.

   ii) curl $\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{j}$.

12. Show that any vector field defined of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}, \quad \text{in} \quad \mathbb{R}^3,$$

where f, g, h are differentiable functions, is irrotational, i.e., curl $\mathbf{F} = 0$.

13. Show that any vector field defined of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}, \quad \text{in} \quad \mathbb{R}^3,$$

where f, g, h are differentiable functions, is incompressible, i.e., div $\mathbf{F} = 0$.