MA 105 Part II Week 1

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September 25 - 29, 2023

Welcome to MA 105 Part II!

There is a total of 50 marks to be earned in this part of the course. The following breakup is tentative*.

Quiz	10 marks
Final	40 marks
Total	50 marks

Academic Honesty: It is obligatory on your part to be honest and not to violate the academic integrity of the Institute. Any form of academic dishonesty, including, but not limited to cheating, plagiarism, submitting as one's own the same or substantially similar work of another, will not be tolerated, and will invite the harshest possible penalties as per institute norms.

Disclaimer: The instructors reserve the right to modify the schedules and procedures announced in this syllabus. Any such change will be announced in the class. It is the responsibility of the student to keep informed of such details.

Course objectives

Calculus can be broadly divided into two parts: differential calculus and integral calculus. This course will be focused on integral calculus of several variables and vector analysis, mainly,

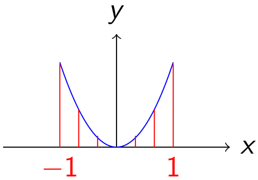
- double and triple integration, Jacobians and change of variables.
- parametrisation of curves, vector fields, line integrals.
- parametrisation of surfaces and surface integrals.
- gradient of functions, divergence and curl of vector fields, theorems of Green, Gauss, and Stokes and their applications.

References:

1. [MR] Debanjana Mitra and Ravi Raghunathan, *Lecture slides for MA 105*.
2. [MTW] J.E Marsden, A. J. Tromba, A. Weinstein. *Basic Multivariable Calculus*, South Asian Edition, Springer (2017).
3. [CJ] R. Courant and F. John, *Introduction to Calculus and Analysis, Volumes 1 and 2*, Springer-Verlag (1989).
4. [Apo] T.M. Apostol, *Calculus, Volumes 1 and 2*, 2nd ed., Wiley (2007).

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $a, b \in \mathbb{R}$.

- The area enclosed by the graph of a non-negative function over the region of the interval is $\int_a^b f(t)\, dt$.



The area in the figure on the left is $\int_{-1}^{1} x^2 \, dx = 2/3$.

- A partition of the interval $[a, b]$ is a set of points $P = \{a = x_0 \leq x_1 \leq \ldots x_n = b\}$ for some $n \in \mathbb{N}$.
- The lower Darboux integral and upper Darboux integral of f are $L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\}$, and $U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\}$, respectively.
- When $L(f) = U(f)$ then f is Darboux integrable and $\int_a^b f := L(f) = U(f)$.

- Tagged partition: partition P with a set of points $t = \{t_1, \dots t_n\}$, $t_j \in [x_{j-1}, x_j]$ for all $j = 1, \dots, n$. Define $S(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$ and define the *norm* of a partition P as $\|P\| = \max_j\{|x_j - x_{j-1}|\}, \quad 1 \le j \le n$.
- $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* if for some $S \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $|S(f, P, t) - S| < \epsilon$, whenever $\|P\| < \delta$. The Riemann integral of f is then S.
- Theorem: The Riemann integral exists iff the Darboux integral exists. Further, the two integrals are equal.
- Unlike the Darboux integral, Riemann integral can be computed as a limit: clearly advantageous in computations!
- $f : [a, b] \to \mathbb{R}$ is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.
- For computing integrals, we use the Fundamental theorem of calculus. If $f : [a, b] \to \mathbb{R}$ is continuous and $f = g'$ for some continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b), then
$$\int_a^b f = g(b) - g(a).$$

Integrating functions on two variables

Any *closed, bounded rectangle R* in $\mathbb{R}^2$:

$$R = [a, b] \times [c, d],$$

the Cartesian product of two closed intervals $[a, b]$ and $[c, d]$.

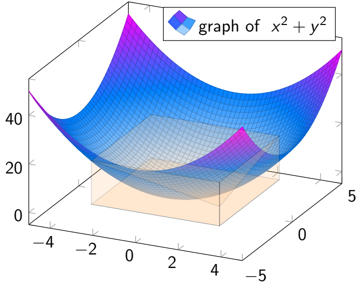
Consider a real valued function f defined on R i.e.,

$$f : R \subset \mathbb{R}^2 \to \mathbb{R}.$$

- Graph of f: The subset $\{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in R\}$ in $\mathbb{R}^3$ is called the graph of f.
- Contour line: Fix $c \in \mathbb{R}$. Then the set $\{(x, y, c) \in \mathbb{R}^3 \mid f(x, y) = c, (x, y) \in R\}$ in $\mathbb{R}^3$ is called the contour line of f. It is the intersection of the graph of f by the horizontal plane $z = c$ in $\mathbb{R}^3$. In other words, it is the image of the c-level set of f.

In particular, let $f(x,y) = x^2 + y^2$, for all $(x,y) \in \mathbb{R}^2$. We can use contour lines to draw the graph of this function by drawing $f(x,y) = c$ for varying values of c.

We want to compute volume of the region below the graph of f over the rectangle $[-3,3] \times [-3,3]$. The volume of the figure in the shaded region is $V := \{(x,y,z) \mid (x,y) \in [-3,3] \times [-3,3], \quad 0 \leq z \leq f(x,y)\}$.



The integral of the non-negative function f over $[-3,3] \times [-3,3]$ can be defined as the volume V; $\int\int_{[-3,3]\times[-3,3]} f(x,y)\, dxdy :=$ Volume of V.

Example: Let $g(x, y) = \alpha$, for some non-zero constant $\alpha \in \mathbb{R}$. Then for any rectangle $[a, b] \times [c, d]$ it is easy to see that
$\int \int_{[a,b]\times[c,d]} g(x, y)\, dxdy = (b - a)(d - c)\alpha$.

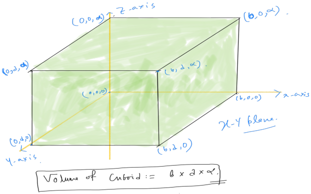


Figure: Cuboid: $[0, b] \times [0, d] \times [0, \alpha]$

Clearly for $f(x, y) = x^2 + y^2$, the computing of the volume is not that simple and we want to be able to define integral for all bounded functions instead of only non-negative ones.

Partitions for rectangles

Partition of R: A partition P of a rectangle $R = [a, b] \times [c, d]$ is the Cartesian product of a partition P_1 of $[a, b]$ and a partition P_2 of $[c, d]$. Let

$$P_1 = \{x_0, x_1, \cdots x_m\}, \quad \text{with} \quad a = x_0 < x_1 < x_2 < \cdots < x_m = b\},$$

$$P_2 = \{y_0, y_1, \cdots y_n\}, \quad \text{with} \quad c = y_0 < y_1 < y_2 < \cdots < y_n = d\},$$

and $P = P_1 \times P_2$ be defined by

$$P = \{(x_i, y_j) \mid i \in \{0, 1, \cdots m\}, \quad j \in \{0, 1, \cdots, n\}\}.$$

The points of P divide the rectangle R into *nm non-overlapping sub-rectangles* denoted by

$$R_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad \forall\, i = 0, \cdots m-1, \quad j = 1, \cdots, n-1.$$

Note $R = \cup_{i,j} R_{ij}$.

Example: Let P_1 denote a partition of $[-3, 3]$ into 3 equal intervals and P_2 the partition of $[-3, 3]$ into 2 equal intervals. Describe the rectangles in the partition $P_1 \times P_2$.

Note $P_1 = \{-3, -1, 1, 3\}$ and $P_2 = \{-3, 0, 3\}$ and thus $[-3, 3] \times [-3, 3]$ is devided into 6 sub-rectangles $R_{00} = [-3, -1] \times [-3, 0]$,
$R_{01} = [-3, -1] \times [0, 3]$, $R_{10} = [-1, 1] \times [-3, 0]$, $R_{11} = [-1, 1] \times [0, 3]$,
$R_{20} = [1, 3] \times [-3, 0]$, $R_{21} = [1, 3] \times [0, 3]$.

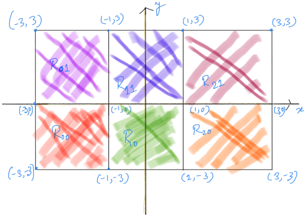


Figure: Partition of $[-3, 3] \times [-3, 3]$

Partitions for rectangles: continued

The area of each R_{ij}: $\Delta_{ij} := (x_{i+1} - x_i) \times (y_{j+1} - y_j)$, for all $i = 0, \cdots, m-1$, $j = 0, \cdots, n-1$.

Norm of the partition P:
$\|P\| := \max\{(x_{i+1} - x_i), (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$.

Question: Why do we not define the norm by
$\max\{(x_{i+1} - x_i) \times (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$?

Darboux integral

Let $f : R \to \mathbb{R}$ be a bounded function where R is a rectangle . Let $m(f) = \inf\{f(x,y) \mid (x,y) \in R\}$, $M(f) = \sup\{f(x,y) \mid (x,y) \in R\}$. For all $i = 0, 1, \cdots, m-1, \quad j = 0, 1, \cdots, n-1$, let,
$m_{ij}(f) := \inf\{f(x,y) \mid (x,y) \in R_{ij}\}$, and
$M_{ij}(f) := \sup\{f(x,y) \mid (x,y) \in R_{ij}\}$.

Lower double sum: $L(f,P) := \displaystyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} m_{ij}(f)\Delta_{ij}$, and

Upper double sum: $U(f,P) := \displaystyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} M_{ij}(f)\Delta_{ij}$,

Note that for any partition P of R

$$m(f)(b-a)(d-c) \leq L(f,P) \leq U(f,P) \leq M(f)(b-a)(d-c).$$

Lower Darboux integral: $L(f) := \sup\{L(f,P) \mid P \text{ is any partition of } R\}$.

Upper Darboux integral:$U(f) := \inf\{U(f,P) \mid P \text{ is any partition of } R\}$.

Note $L(f) \leq U(f)$.

Darboux integral continued

Definition (Darboux integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Darboux integrable* if $L(f) = U(f)$. The Double integral of f is the common value $U(f) = L(f)$ and is denoted by

$$\int\int_R f, \quad \text{or} \quad \int\int_R f(x,y)dA, \quad \text{or} \quad \int\int_R f(x,y)dxdy.$$

Theorem (Riemann condition)

Let $f : R \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$ there is a partition P_ϵ of R such that

$$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon.$$

Recall the Dirichlet function for one variable:

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Is f integrable over $[0,1]$? Ans. No!

Exercise: Check the integrability of bivariate Dirichlet function over $[0,1] \times [0,1]$

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

Riemann Integral

Riemann integral: Let P be any partition of a rectangle $R = [a, b] \times [c, d]$. We define a tagged partition (P, t) where

$$t = \{t_{ij} \mid t_{ij} \in R_{ij}, \quad i = 0, 1, \ldots m-1, \quad j = 0, 1, \ldots n-1\}.$$

The *Riemann sum* of f associated to (P, t) is defined by

$$S(f, P, t) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(t_{ij})\Delta_{ij} \text{ where, } \Delta_{ij} = (x_{i+1} - x_i)(y_{j+1} - y_j)$$

Definition (Riemann integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Riemann integrable* if there exists a real number S such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|S(f, P, t) - S| < \epsilon,$$

for every tagged partition (P, t) satisfying $\|P\| < \delta$ and S is the value of Riemann integral of f.

Riemann Integral contd.

- For any rectangle $R \subseteq \mathbb{R}^2$, let $f : R \to \mathbb{R}^2$ be bounded. The Darboux integrability and Riemann integrability are equivalent.
- A function $f : R \to \mathbb{R}^2$ is called integrable on R if (Darboux or) Riemann integrability condition holds on R.

Examples: Let $R = [a, b] \times [c, d]$.

• The constant function is integrable.

• The projection functions $p_1(x, y) = x$ and $p_2(x, y) = y$ are both integrable on any rectangle $R \subset \mathbb{R}^2$. Why?

• Let $f : R \to \mathbb{R}$ be defined as $f(x, y) = \phi(x)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function. Is f integrable? What is $\int\int_R f\, dxdy$?

Regular partitions

However, the current definition isn't truly helpful in making computations. We define *regular* partitions.

The regular partition of R of order any $n \in \mathbb{N}$ is defined by $x_0 = a$ and $y_0 = c$, and for $i = 0, 1, \cdots, n-1, \quad j = 0, 1, \cdots, n-1$,

$$x_{i+1} = x_i + \frac{b-a}{n}, \quad y_{j+1} = y_j + \frac{d-c}{n}.$$

We take $t = \{t_{ij} \in R_{ij} \mid i, j \in \{0, 1, \cdots, n-1\}\}$ any arbitrary tag.
To check the integrability of a function f, it is enough to consider a sequence of regular partitions P_n of R.

Theorem

A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij},$$

tends to the same limit $S \in \mathbb{R}$ as $n \to \infty$, for any choice of tag t.

An Example

Example: Let $f(x, y) = x^2 + y^2$. Is it a continuous function on $\mathbb{R}^2$?
Ans. Yes!
Suppose the function is integrable on $[0, 1] \times [0, 1]$. Compute the integral using the theorem.

Let $R = [0, 1] \times [0, 1]$ and P_n be a regular partition. Then for tag $t = \{(\frac{i}{n}, \frac{j}{n}) \mid i = 0, \ldots n-1, j = 0, \ldots, n-1\}$,

$$S(f, P_n, t) = (\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (\frac{i}{n})^2 + (\frac{j}{n})^2) \frac{1}{n^2}.$$

Compute $\lim_{n \to \infty} S(f, P_n, t)$. How would you go about it?

Conventions

Based on our definition, we make the following convention: Let $a, b, c, d \in \mathbb{R}$

- If $a = b$ or $c = d$, then $\int\int_{[a,b]\times[c,d]} f(x,y)dxdy := 0$.
- If $a < b$ and $c < d$:
  $\int\int_{[b,a]\times[c,d]} f(x,y)dxdy := -\int\int_{[a,b]\times[c,d]} f(x,y)dxdy$,
  $\int\int_{[a,b]\times[d,c]} f(x,y)dxdy := -\int\int_{[a,b]\times[c,d]} f(x,y)dxdy$,
  $\int\int_{[b,a]\times[d,c]} f(x,y)dxdy := \int\int_{[a,b]\times[c,d]} f(x,y)dxdy$.

Properties of integrals over rectangles

Domain Additivity Property: $f : R \to \mathbb{R}$ is a bounded function. Partition R into finitely many non-overlapping sub-rectangles. Then f is integrable on R iff it is integrable on each sub-rectangle. When it exists, the integral of f on R is the sum of the integrals of f on the sub-rectangles.

Algebraic properties :

Let $R := [a, b] \times [c, d]$. Let f and g are integrable on R.

- If f is defined as $f(x, y) = \alpha \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$ then $\int\int_R f = \alpha A(R)$ where A is the area of R.
- The function $f + g$ is integrable, and $\int\int_R f + g = \int\int_R f + \int\int_R g$.
- For all $\alpha \in \mathbb{R}$, αf is integrable and $\int\int_R \alpha f = \alpha \int\int_R f$.
- If $f(x, y) \leq g(x, y)$ for all $(x, y) \in R$, then $\int\int_R f \leq \int\int_R g$.
- $|f|$ is integrable and $|\int\int_R f| \leq \int\int_R |f|$.
- The function $f.g$ is integrable.
- If $\frac{1}{f}$ is well defined and bounded on R, then $\frac{1}{f}$ is integrable on R.

All these follow by applying the definition and properties of limits. An immediate consequence is that all polynomial functions are integrable.

Calculating integrals

While we have now given a reasonable definition of the integral for functions of two variables, actually calculating integrals using the definition proves much too complicated. After all, even in the one variable method, integrating any but the simplest functions using the definition of the Riemann integral is more or less impossible. Instead, we proved the fundamental theorem of calculus and used the fact that the integral and the antiderivative were the same in order to evaluate the integrals of various standard functions.

The key idea is to reduce integration in two variables to integrating in one variable (but doing it twice, that is, iteratedly).
In fact, this idea goes back all the way to Archimedes, but was perhaps first extensively used by Cavalieri, a student of Galileo (note that this was before Newton and Liebniz developed the Fundamental Theorem of Calculus).

Iterated Integrals

For $f : R \to \mathbb{R}$ we can define the iterated integrals as follows. We can first define functions of y and x respectively as follows, provided the integrands below are Riemann integrable as functions of one variable:

$$h(y) = \int_a^b f(x,y)dx \quad \text{and} \quad g(x) = \int_c^d f(x,y)dy.$$

We then consider the integrals

$$\int_c^d h(y)dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy \quad \text{and}$$

$$\int_a^b g(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx.$$

These integrals (if they exist) are called iterated integrals

If you think about it, it is not obvious that either of the integrals above should be equal to the double integral, but, in fact, they will be in the most common situations we encounter.

Fubini theorem and Iterated integrals

Theorem

Let $R := [a, b] \times [c, d]$ and $f : R \to \mathbb{R}$ be integrable. Let I denote the integral of f on R.

1. *If for each $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists, then the iterated integral $\int_a^b (\int_c^d f(x, y)\, dy)\, dx$ exists and is equal to I.*
2. *If for each $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)\, dx$ exists, then the iterated integral $\int_c^d (\int_a^b f(x, y)\, dx)\, dy$ exists and is equal to I.*

As a consequence, if f is integrable on R and if both iterated integrals exist in 1. and 2. in above theorem, then

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = I = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Sketch of the proof

The proof is using Riemann condition. Since f is double integrable over R, for any given $\epsilon > 0$, there exists a partition $P_\epsilon = \{(x_i, y_j) \mid i = 0, 1, \cdots k-1, \quad j = 0, \cdots n-1\}$ of R such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Assume for each fixed $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists. Define

$$A(x) := \int_c^d f(x, y)\, dy, \quad \forall\, x \in [a, b].$$

Claim: The function A is integrable over $[a, b]$. Note that $m(f)(d-c) \leq A(x) \leq M(f)(d-c)$ for all $x \in [a, b]$ and hence A is bounded. Also by domain additivity, $A(x) = \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} f(x, y)\, dy$, for all $x \in [a, b]$.

Thus for each fixed $i \in \{0, \cdots k-1\}$, for $x \in [x_i, x_{i+1}]$, we obtain

$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1} - y_j) \leq A(x) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1} - y_j).$$

Sketch of the proof continued

Denoting $m_i(A) := \inf\{A(x) \mid x \in [x_i, x_{i+1}]\}$ and $M_i(A) := \sup\{A(x) \mid x \in [x_i, x_{i+1}]\}$, we have

$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1} - y_j) \leq m_i(A) \leq M_i(A) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1} - y_j).$$

Multiplying by $(x_{i+1} - x_i)$ and summing over $i = 0, \ldots, k-1$, we obtain

$$L(f, P_\epsilon) \leq \sum_{i=0}^{k-1} m_i(A)(x_{i+1} - x_i) \leq \sum_{i=0}^{k-1} M_i(A)(x_{i+1} - x_i) \leq U(f, P_\epsilon).$$

and it yields that there exists a partition $P_1 := \{x_0, \cdots, x_{k-1}\}$ of $[a, b]$ such that

$$U(A, P_1) - L(A, P_1) < \epsilon.$$

Thus the function of A is integrable and

$$\int\int_R f\,dx\,dy = \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x, y)\,dy\right) dx.$$

Remarks on Fubini's theorem

The both iterated integrals may exist but the function f may not be double integrable.

Example 1: $R := [0,1] \times [0,1]$, $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

Compute both the iterated integrals. Are they same? Is f integrable?

The function f may be double integrable. But one of the iterated integrals may not exist (check Tutorial problems).

Let R be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be a continuous function. Then both iterated integrals of f exist and are equal to the double integral of f over R.

Examples:

Example : Find the integral of $f(x,y) = x^2 + y^2$ on the rectangle $[0,1] \times [0,1]$ if it exists.

Solution: Check the integrability of f using the definition. Let us now compute the integral using iterated integrals.

$$
\begin{aligned}
\int\int_{[0,1]\times[0,1]} x^2 + y^2 \ dxdy &= \int_0^1 \int_0^1 x^2 + y^2 \ dxdy \\
&= \int_0^1 \frac{x^3}{3} + xy^2 \Big|_0^1 \\
&= \int_0^1 (\frac{1}{3} + y^2) \\
&= \frac{y}{3} + \frac{y^3}{3} \Big|_0^1 = \frac{2}{3}
\end{aligned}
$$

Example (Marsden, Tromba and Weinstein page 288): Compute $\int\int_R \sin(x+y)dxdy$, where $R = [0,\pi] \times [0, 2\pi]$.

Solution:

$$\begin{aligned}\int\int_R \sin(x+y)dxdy &= \int_0^{2\pi}\left[\int_0^{\pi}\sin(x+y)dx\right]dy \\ &= \int_0^{2\pi}[-\cos(x+y)|_{x=0}^{\pi}]dy \\ &= \int_0^{2\pi}[\cos y - \cos(y+\pi)]dy \\ &= [\sin y - \sin(y+\pi)]|_{y=0}^{2\pi} = 0\end{aligned}$$

Example (Marsden, Tromba and Weinstein, page 289): If D is a plate defined by $1 \leq x \leq 2, 0 \leq y \leq 1$ (measured in centimeters), and the mass density $\rho(x,y) = ye^{xy}$ grams per square centimeter. Find the mass of the plate.

Solution: The total mass of the plate is got by integrating over the rectangular region covered by D:

$$\begin{aligned} \int\int_D \rho(x,y)dxdy &= \int_0^1 \int_0^2 ye^{xy}dxdy = \int_0^1 (e^{xy}|_{x=1}^2 dy \\ &= \int_0^1 (e^{2y} - e^y)dy = \frac{e^2}{2} - e + \frac{1}{2} \end{aligned}$$

Special case Let $\phi : [a,b] \to \mathbb{R}$ and $\psi : [c,d] \to \mathbb{R}$ be Riemann integrable. Define $f(x,y) := \phi(x)\psi(y)$, for all $(x,y) \in R = [a,b] \times [c,d]$. Then f is integrable on R and

$$\int\int_R f(x,y)\,dx\,dy = \Big(\int_a^b \phi(x)\,dx\Big)\Big(\int_c^d \psi(y)\,dy\Big).$$

Example Let $0 < a < b$ and $0 < c < d$ and $r \geq 0$ and $s \geq 0$. Denote $R = [a,b] \times [c,d]$. Compute $\int\int_R x^r y^s\,dx\,dy$.

Cavalieri's Principle

The volumes of two solids are equal if the areas of their corresponding cross sections are equal.

The Slice Method

Cavalieri's basic idea is that we can find the volume of a given solid by slicing it into thin cross sections, calculating the areas of the slices and then adding up these areas.

Let S be a solid and P_x be a family of planes perpendicular to the x-axis with x as x-coordinate such that
1. S lies between P_a and P_b,
2. the area of the slice of S cut by P_x is $A(x)$.

Then the volume of S is given by

$$\int_a^b A(x)dx.$$

Applying this to the solid graph of $z = f(x, y)$ above a rectangle R in the plane, we see that we get exactly the second of our iterated integrals.

Thus Cavalieri's principle is actually a generalization of the method of iterated integrals. Note that in order to apply the principle we do not require the solid to necessarily lie above a rectangular region in the plane.

Cavalieri's principle is particularly useful in computing the volumes of solids of revolution. These are obtained by taking a region B lying between the lines $x = a$ and $x = b$ on the x-axis and the graph of a function $y = f(x)$ and rotating it through an angle 2π around the x-axis.

Solids of revolution

In this case, we can easily compute the cross-sectional area $A(x)$, since each cross section is nothing but a disc. The radius of the circle is nothing but $f(x)$. Hence, the area $A(x)$ is given by

$$A(x) = \pi[f(x)]^2,$$

and the volume V of the solid is given by

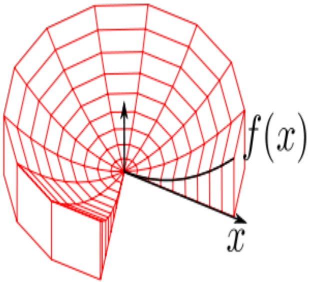
$$V = \pi \int_a^b [f(x)]^2 dx.$$

Solids of revolution may also arise by rotating the graph of a function $f(x)$. around the y-axis. In this case, we can follow the procedure above, replacing x by y and the function $f(x)$ by its inverse.

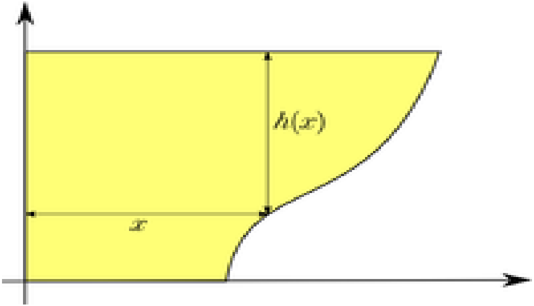
The shell method

There is another way to compute the volume of a solid of revolution obtained by rotating the graph of a function around the y-axis. It is called the shell method.

In this case, rather than slicing the solid by cross sections, we view the solid as being made of cylindrical shells.



The shell method continued



As you can see, from the picture above, the radius of the cylindrical shell above the point $(x, f(x))$ is x and the height is $h(x)$. (The point is $h(x)$ can be determined: it is constant initially and later $= f(b) - f(x)$.) Hence its surface area is $2\pi x h(x)$. To get the volume we must integrate, and this yields

$$2\pi \int_a^b x h(x) dx.$$

The washer method

This is a variant on the previous methods. Sometimes we have have calculate the volume of a solid of revolution which is hollow, where the shape of the hollow part of the solid is also given as a solid of revolution. Thus, we can think of the solid as being obtained by rotating the region that lies between the graphs of two functions $f_1(x)$ and $f_2(x)$ on an interval $[a, b]$ around an axis. If we are rotating around the x-axis, we get

$$\pi \int_a^b [f_2(x)^2 - f_1(x)^2] dx.$$

When we use the shell method, we get the formula

$$2\pi \int_a^b x[h_2(x) - h_1(x)] dx.$$

Above, we assume that $f_2(x)$ lies further away from the axis of rotation than $f_1(x)$. This method of calculating the volumes of hollow solids of revolution is called the washer method.

Exercise 4.15: A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

Solution: We may describe the desired volume as the difference of the volume of the sphere of radius 2 and a certain hollow solid of revolution.

Let us use the slice method first.

The hollow solid of revolution may be described as being obtained by rotating the region between the line $x = \sqrt{3}$ and $x = \sqrt{4 - y^2}$ around the y-axis. The two curves intersect at the points $(\sqrt{3}, \pm 1)$.

The volume of the hollow solid is given by

$$\int_{-1}^{1} \pi x^2 dy - \pi(\sqrt{3})^2 2] = 2\pi[\int_{0}^{1} (4 - y^2) dy - 3] = \frac{4}{3}\pi.$$

The volume of the sphere is $\frac{32}{3}\pi$. Hence the required volume is

$$\frac{32}{3}\pi - \frac{4}{3}\pi = \frac{28}{3}\pi.$$

We could also use the shell method to solve this problem. In the case we will get

$$\begin{aligned} 32\pi/3 - \int_{\sqrt{3}}^{2} 2\pi x(2y)dx \quad &= 32\pi/3 - 4\pi \textstyle\int_{\sqrt{3}}^{2} x\sqrt{4-x^2}dx \\ &= 32\pi/3 - 4\pi(1/3) = 28\pi/3 \end{aligned}$$

Existence of integrals on $R = [a, b] \times [c, d]$ - Part I

All our statements so far depend on f being integrable on R. Is there any characterisation to determine if f is integrable?
Let $f : R \to \mathbb{R}$ be bounded. 'f is monotonic in each of two variables' means that for each fixed x, $f(x, y)$ is a monotonic function in y variable and similarly, for each fixed y, $f(x, y)$ is a monotonic function in x variable.

Theorem

If f is bounded and monotonic in each of two variables, then f is integrable on R.

Again the proof follows by using Riemann condition.

Example: Let $f(x, y) := [x + y]$, for all $(x, y) \in R$, where $[u]$ means the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Since f is monotonic in each of two variables, f is integrable on R.
However, the previous condition is not that common.

Surely what worked in one variable should work here. In fact, a proof similar to the case of one variable will show the following theorem.

Existence of integrals on $R = [a,b] \times [c,d]$ - Part II

Theorem

If a function $f : R \to \mathbb{R}$ is bounded and continuous on R except possibly finitely many points in R, then f is integrable on R.

Example. Let $R := [-1,1] \times [-1,1]$,

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)}, & (x,y) \in R, \quad (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

What are points of discontinuity for f on R?

In the one variable case, we saw that a bounded function with at most a finite number of discontinuities on a closed bounded interval is Riemann integrable.

The reason that a finite number of discontinuities do not matter is that points have length zero. What might be the analogous result in two variables?

In other words what sets have "zero area"?

A bounded subset E of $\mathbb{R}^2$ has 'zero area' if for every $\epsilon > 0$, there are finitely many rectangles whose union contains E and the sum of whose areas is less than ϵ.

It turns out graph of a continuous function, that is, set of the form $\{(x, \phi(x)) \mid x \in [a, b]\}$ for a continuous function $\phi : [a, b] \to [c, d]$ has 'zero area' or has *content zero.*

Theorem

If a function f is bounded and continuous on a rectangle $R = [a, b] \times [c, d]$ except possibly along a finite number of graphs of continuous functions, then f is integrable on R.

Example: Let $R = [0, 1] \times [0, 1]$ and

$$f(x, y) = \begin{cases} 1, & 0 \leq x < y, \quad y \in [0, 1], \\ 0, & y \leq x \leq 1, \quad y \in [0, 1]. \end{cases}$$

Is f integrable over R?

Theorem

(Slightly more general) Given a rectangle R and a bounded function $f : R \to \mathbb{R}$, the function is integrable over R if the points of discontinuity of f is a set of 'content zero'.

However the converse of the above statement is not true. There are integrable functions whose points of discontinuity is not a set of 'content zero'. (Check Tutorial)

Counter example: Bivariate Thomae function: $f : [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} 1, & \text{if} \quad x = 0, \quad y \in \mathbb{Q} \cap [0,1], \\ \frac{1}{q}, & x, y \in \mathbb{Q} \cap [0,1] \quad \text{and} \quad x = \frac{p}{q}, \\ & \qquad p, q \in \mathbb{N} \quad \text{are relatively prime}, \\ 0, & \text{otherwise.} \end{cases}$$