MA 105 : Calculus

D1 - Lecture 6

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Autumn 2023, IIT Bombay, Mumbai

The function $\sin \frac{1}{x}$

Let us look at Exercise 1.13 part (i).
Consider the function defined as $f(x) = \sin \frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$. The question asks if this function is continuous at $x = 0$.
How about $x \neq 0$? Why is $f(x)$ continuous? Because it is a composition of the sin function and a rational function $1/x$. Since both of these are continuous when $x \neq 0$, so is $f(x)$.

Let us look at the sequence of points $x_n = \frac{2}{(2n+1)\pi}$. Clearly $x_n \to 0$ as $n \to \infty$.
For these points, $f(x_n) = \pm 1$. This means that no matter how small I take my δ, there will be a point $x_n \in (0, \delta)$, such that $|f(x_n)| = 1$.

But this means that $|f(x) - f(0)| = |f(x)|$ cannot be made smaller than 1 no matter how small δ may be. Hence, f is not continuous at 0. The same kind of argument will show that there is no value that we can assign $f(0)$ to make the function $f(x)$ continuous at 0.

You can easily check that $f(x)$ has the IVP. However, we have proved that it is not continuous. So IVP $\not\Rightarrow$ continuity.

Sequential continuity

The preceding example showed that in order to demonstrate that a function, say $f(x)$, is not continuous at a point x_0 it is enough to find a sequence x_n tending to x_0 such that the value of the function $|f(x_n) - f(x_0)|$ remains large.

Suppose it is not possible to find such a sequence. Does that mean the function is continuous at x_0? The following theorem answers the question affirmatively.

Theorem 12: A function $f(x)$ is continuous at a point a if and only if for every sequence x_n converging to a, the sequence $f(x_n)$ converges to $f(a)$.

A function that satisfies the property that for every sequence $x_n \to a$, $\lim_{n\to\infty} f(x_n) = f(a)$ is said to be sequentially continuous.

The theorem says that sequential continuity and continuity are the same thing. Indeed, it is clear that a continuous function is necessarily sequentially continuous. It is the reverse that is harder to prove.

Proof of Theorem 12

Proof. Let $I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. Let $a \in I$.

$(\Rightarrow)$. Let f be continuous at a. That is, for a given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Since the sequence x_n converges to a, for the above $\delta > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - a| < \delta$ whenever $n \geq N$.

Hence $|f(x_n) - f(a)| < \epsilon$ whenever $n \geq N$.

$(\Leftarrow)$. We will show this part by using the method of contradiction.

For, let if possible, the function f is NOT continuous at a, that is, there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists $x_\delta \in I$ with $|x_\delta - a| < \delta$ and $|f(x_\delta) - f(a)| \geq \epsilon$.

Now, we find a sequence x_n converging to a, for which, $f(x_n)$ does not converge to $f(a)$.

Proof continued...

For $n \in \mathbb{N}$ and for the same ϵ, if we take $\delta_n = \frac{1}{n}$ then there exists $x_n \in I$ such that $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - f(a)| \geq \epsilon$.

It is now clear that the sequence x_n converges to a but for the ϵ above, there does not exist $N \in \mathbb{N}$ such that $|f(x_n) - f(x)| < \epsilon$, whenever $n \geq N$, that is, the sequence $f(x_n)$ does not converge to $f(a)$ which is a contradiction.

Hence the function f is continuous. $\square$

Remark: Theorem 12 (continuity is same as sequential continuity) goes through without any problems even when the range and/or domain of the function are/is in $\mathbb{R}^2$ or $\mathbb{R}^3$. Exactly the same proof works in this case. Note that we have not yet defined the continuity of functions having range and/or domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. You can try defining it and proving the above theorem in this case.

Differentiability: The definition

Recall that $f : (a, b) \to \mathbb{R}$ is said to be differentiable at a point $c \in (a, b)$ if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

In this case the value of the limit is denoted $f'(c)$ and is called the derivative of f at c. The derivative may also be denoted by $\frac{df}{dx}(c)$ or by $\frac{dy}{dx}|_c$, where $y = f(x)$.

In general, the derivative measures the rate of change of a function at a given point. Thus, if the function we are studying is the position of a particle on the x-coordinate, then $x'(t)$ is the velocity of the particle.

The slope of the tangent

If the function we are studying is the velocity $v(t)$ of the particle, then the derivative $v'(t)$ is the acceleration of the particle.

From the point of view of geometry, the derivative $f'(c)$ gives us the slope of the curve, that is, the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$.

This becomes particularly clear if we rewrite the derivative as the following limit:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

The expression inside the limit obviously represents the slope of a line passing through $(c, f(c))$ and $(x, f(x))$, and as x approaches c this line obviously becomes tangent to $y = f(x)$ at the point $(c, f(c))$.

Examples

Exercise 1.16: Let $f : (a, b) \to \mathbb{R}$ be a function such that

$$|f(x+h) - f(x)| \leq c|h|^{\alpha}$$

for all $x, x+h \in (a, b)$, where c is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f'(x)$ for $x \in (a, b)$.

Solution: By the Sandwich theorem

$$\lim_{h\to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leq c \lim_{h\to 0} |h|^{\alpha-1} = 0 \implies f'(x) = 0$$

($\lim_{h\to 0} |g(h)| = 0 \iff \lim_{h\to 0} g(h) = 0$). $\square$

Note: Functions that satisfy the property above for $\alpha > 0$ (not necessarily greater than 1) are said to be Lipschitz continuous with exponent α.

Calculating derivatives

As with limits, all of you are already familiar with the rule for calculating the derivatives of the sums, differences, products and quotients of differentiable functions. You should try and remember how to prove these.

You should also recall the chain rule (for $F(x) = f(g(x))$, $F'(x) = f'(g(x))g'(x)$) for calculating the derivative of the composition of functions and try to prove it as an exercise using the $\epsilon - \delta$ definition of a limit.

Exercise: Show that the differentiable functions are continuous.

Use the simple observation that

$$f(x) = f(a) + (x - a)\frac{f(x) - f(a)}{(x - a)}$$

and then apply the limit rule.

Maxima and minima

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function (you can think of X as an open, closed or half-open interval, for instance).

Definition: The function f is said to attain a maximum (resp. minimum) at a point $x_0 \in X$ if $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in X$.

Once again, I remind you that, in general, f may not attain a maximum or minimum at all on the set X. The standard example being $X = (0,1)$ and $f(x) = 1/x$ (can you find an example on the closed interval $[0,1]$?).

However, if X is a closed and bounded interval and f is a continuous function, Theorem 11 tells us that the maximum and minimum are actually attained. Theorem 11 is sometimes called the Extreme Value Theorem.

Maxima and minima and the derivative

If f has a maximum at the point x_0 and if it is also differentiable at x_0, we can reason as follows.

We know that $f(x_0 + h) - f(x_0) \leq 0$ for every $h \in \mathbb{R}$ such that $x_0 + h \in X$.

Hence, we see (one half of the Sandwich Theorem!) that when $h > 0$,

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

On the other hand, when $h < 0$, we get

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

Because f is assumed to be differentiable at x_0 we know that left and right hand limits must be equal. It follows that we must have $f'(x_0) = 0$. A similar argument shows that $f'(x_0) = 0$ if f has a minimum at the point x_0.

Local maxima and minima

The preceding argument is purely local. Before explaining what this means, we give the following definition.

Definition: Let $f : X \to \mathbb{R}$ be a function and x_0 be in X. Suppose there is a sub-interval $x_0 \in (c, d) \subset X$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (c, d)$, then f is said to have a local maximum (resp. local minimum) at x_0.

Sometimes we use the terms global maximum or global minimum instead of just maximum or minimum in order to emphasize the points are not just local maxima or minima. The argument of the previous slide actually proves the following:

Theorem 13: If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in X$, $f'(x_0) = 0$.

Proof: Exercise.

Rolle's Theorem

Theorem 13 is known as Fermat's theorem. It can be used to prove Rolle's Theorem.

Theorem 14: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function which is differentiable in (a, b) and $f(a) = f(b)$. Then there is a point x_0 in (a, b) such that $f'(x_0) = 0$.

Proof: Since f is a continuous function on a closed and bounded interval, Theorem 11 tells us that f must attain its minimum and maximum somewhere in $[a, b]$. If both the minimum and maximum are attained at the end points, f must be the constant function, in which case, we know that $f'(x) = 0$ for all $x \in (a, b)$. Hence, we can assume that at least one of the minimum or maximum is attained at an interior point x_0 and Theorem 13 shows that $f'(x_0) = 0$ in this case. □

One easy consequence: If $P(x)$ is a polynomial of degree n with n real roots, then all the roots of $P'(x)$ are also real. (How do we know that polynomials are differentiable?)

Problems centered around Rolle's Theorem

Exercise 2.3: Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose f is differentiable on (a, b). If $f(a)$ and $f(b)$ are of opposite signs and $f'(x) \neq 0$ for all $x \in (a, b)$, then there is a unique point x_0 in (a, b) such that $f(x_0) = 0$.

Solution: Since the Intermediate Value Theorem guarantees the existence of a point x_0 such that $f(x_0) = 0$, the real point of this exercise is the uniqueness.

Suppose there were two points $x_1, x_2 \in (a, b)$ such that $f(x_1) = f(x_2) = 0$. Applying Rolle's Theorem, we see that there would exist $c \in (x_1, x_2)$ such that $f'(c) = 0$ contradicting our hypothesis. This proves the exercise. $\square$

Let us look at Exercise 2.8(i): Find a function f which satisfies all the given conditions, or else show that no such function exists: $f''(x) > 0$ for all $x \in \mathbb{R}$ and $f'(0) = 1, f'(1) = 1$.

Solution: Apply Rolle's Theorem to $f'(x)$ to conclude that such a function cannot exist.