MA 105 : Calculus

D1 - Lecture 12

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The main theorem for Riemann integration

The main theorem of Riemann integration is the following:

Theorem 5: Let $f : [a, b] \to \mathbb{R}$ be a function that is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

In particular, continuous functions on closed and bounded intervals are Riemann integrable.

In fact, one can allow even countably many points of discontinuities and the Theorem will remain true.

Exercise 1: Those of you who have an extra interest in the course should think about trying to prove both Theorem 5 and the extension to countably many discontinuities (Warning: there is one crucial fact about continuous functions that we have not covered that you will have to discover for yourself).

Properties of the Riemann integral

From the definition of the Riemann integral we can easily prove the following properties. We assume that f and g are Riemann integrable. Then

$$\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt,$$

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt,$$

for any constant $c \in \mathbb{R}$, and finally if $f(t) \leq g(t)$ for all $t \in [a, b]$, then

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt.$$

Implicit in the properties above is the fact that if f and g are Riemann integrable, then so are $f + g$ and cf.

It is not hard to prove either of the properties. One needs only to use the corresponding properties for inf and sup.

Proving the properties of the integral

Observe that on any interval $[x_{j-1}, x_j]$,

$$\inf_{[x_{j-1},x_j]} f + \inf_{[x_{j-1},x_j]} g \leq (f+g)(x) = f(x)+g(x) \leq \sup_{[x_{j-1},x_j]} f + \sup_{[x_{j-1},x_j]} g.$$

Hence

$$\inf_{[x_{j-1},x_j]} f + \inf_{[x_{j-1},x_j]} g \leq \inf_{[x_{j-1},x_j]} (f+g) \leq \sup_{[x_{j-1},x_j]} (f+g) \leq \sup_{[x_{j-1},x_j]} f + \sup_{[x_{j-1},x_j]} g.$$

It follows that for any partitions P_1, P_2 and $P = P_1 \cup P_2$ of $[a, b]$,

$$L(f, P_1) + L(g, P_2) \leq L(f, P) + L(g, P) \leq L(f + g, P) \leq L(f + g)$$

and hence

$$\sup_{P_1} L(f, P_1) + \sup_{P_2} L(g, P_2) \leq L(f + g).$$

That is,

$$L(f) + L(g) \leq L(f + g).$$

Proof continues...

Similarly, for any partitions P_1, P_2 and $P = P_1 \cup P_2$ of $[a, b]$,

$$U(f+g) \leq U(f+g, P) \leq U(f, P) + U(g, P) \leq U(f, P_1) + U(g, P_2)$$

and hence

$$U(f+g) \leq \inf_{P_1} U(f, P_1) + \inf_{P_2} U(g, P_2).$$

That is,

$$U(f+g) \leq U(f) + U(g).$$

It follows from the above two inequalities (for the lower and upper Darboux integrals) that

$$L(f) + L(g) \leq L(f+g) \leq U(f+g) \leq U(f) + U(g).$$

Since f, g are integrable, we get

$$\int_a^b f(t)dt + \int_a^b g(t)dt \leq L(f+g) \leq U(f+g) \leq \int_a^b f(t)dt + \int_a^b g(t)dt$$

Proof continues...

and hence

$$L(f+g) = U(f+g) = \int_a^b f(t)dt + \int_a^b g(t)dt$$

that is, $\int_a^b [f(t)+g(t)]dt$ exits and

$$\int_a^b [f(t)+g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt.$$

Also, observe that

$$\sup_{[x_{j-1},x_j]}(cf) = c \sup_{[x_{j-1},x_j]} f \text{ and } \inf_{[x_{j-1},x_j]}(cf) = c \inf_{[x_{j-1},x_j]} f, \text{ for } c \geq 0.$$

Therefore, for $c \geq 0$ (when $c < 0$, we apply these observations to the function $(-c)f$ and use the fact that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$),

$$\inf_P U(cf,P) = c \inf_P U(f,P), \quad \sup_P L(cf,P) = c \sup_P L(f,P).$$

Proof continues...

It follows that

$$U(cf) = c\int_a^b f(t)dt \quad \text{and} \quad L(cf) = c\int_a^b f(t)dt$$

and hence $L(cf) = U(cf)$, which shows that cf is integrable on $[a, b]$ and

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt.$$

In case when $f(t) \geq g(t)$ for all $t \in [a, b]$, we get $f(t) - g(t) \geq 0$ for all $t \in [a, b]$ and hence $L(f - g) \geq 0$ and it follows that

$$\int_a^b [f(t) - g(t)]dt \geq 0$$

that is,

$$\int_a^b f(t)dt \geq \int_a^b g(t)dt. \qquad \square$$

Products of Riemann Integrable Functions

Theorem 6: Let $f : [a, b] \to [m, M]$ be a Riemann integrable function and let $\phi : [m, M] \to \mathbb{R}$ be a continuous function. Then the function $\phi \circ f$ (defined as $\phi \circ f(x) = \phi(f(x))$) is Riemann integrable on $[a, b]$.

The above theorem has the following interesting corollaries.

Corollary 1: Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions which are Riemann integrable on $[a, b]$. Then $f \cdot g$, $|f|$ and f^n (for any positive integer n) are Riemann integrable.

Proof: Exercise (Hint: $f \cdot g = \frac{1}{4}[(f + g)^2 - (f - g)^2]$).

Corollary 2: If $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function and $[c, d] \subseteq [a, b]$. Then the function $g : [c, d] \to \mathbb{R}$ defined as $g(x) = f(x)$ for all $x \in [c, d]$ is Riemann integrable.

Proof: Exercise (Hint: First show that the characteristic function $\chi_{[c,d]}$ is integrable on $[a, b]$ and then show that $f \cdot \chi_{[c,d]}$ is integrable on $[a, b]$ by using Corollary 1, and $\int_a^b f(t)\chi_{[c,d]}(t)dt = \int_c^d g(t)dt$.)

Another property of the Riemann Integral

Theorem 7: Suppose f is Riemann integrable on $[a, b]$ and $c \in [a, b]$. Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proof: First we note that if $c = a$ or $c = b$, there is nothing to prove.

Next, if $c \in (a, b)$ we proceed as follows. If P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$, then $P_1 \cup P_2 = P'$ is obviously a partition of $[a, b]$. Thus, partitions of the form $P_1 \cup P_2$ constitute a subset of the set of all partitions of $[a, b]$. For such partitions P', we have

$$L(f, P') = L(f, P_1) + L(f, P_2).$$

Let us denote by $L(f)_{[a,c]}$ (resp. $L(f)_{[c,b]}$) the Darboux lower integral of f on the interval $[a, c]$ (resp. $[c, b]$).

If we take the supremum over all partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ we get

$$\sup_{P'} L(f, P') = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

Now the supremum on the left hand side is taken only over partitions P' having the special form $P_1 \cup P_2$. Hence it is less than or equal to $\sup_P L(f, P)$ where this supremum is taken over all partitions P. We thus obtain

$$L(f)_{[a,c]} + L(f)_{[c,b]} \leq L(f).$$

On the other hand, for any partition $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$, we can consider the partition $P' = P \cup \{c\}$. This will be a refinement of the partition P and can be written as a union of two partitions P_1 of $[a, c]$ and P_2 of $[c, b]$.

By the property for refinements for Darboux sums we know that $L(f, P) \leq L(f, P')$.

Thus, given any partition P of $[a, b]$, there is a refinement P' which can be written as the union of two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively, and by the above inequality,

$$\sup_P L(f, P) \leq \sup_{P'} L(f, P'),$$

where the first supremum is taken over all partitions of $[a, b]$ and the second only over those partitions P' which can be written as a union of two partitions as above. This shows that

$$L(f) \leq L(f)_{[a,c]} + L(f)_{[c,b]},$$

so, together with the previous inequality, we get

$$L(f) = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

By Corollary 2 of Theorem 6, $\int_a^c f(t)dt$ and $\int_c^b f(t)dt$ exist, and hence $L(f)_{[a,c]} = \int_a^c f(t)dt$ and $L(f)_{[c,b]} = \int_c^b f(t)dt$.

Since $L(f) = \int_a^b f(t)dt$, we obtain that

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt. \quad \square$$

The fundamental theorem of calculus: Motivation

The Fundamental Theorem of calculus allows us to relate the process of Riemann integration to the process of differentiation. Essentially, it tells us that integrating and differentiating are inverse processes. This is a tremendously useful theorem for several reasons.

It turns out that (Riemann) integrating even simple functions is much harder than differentiating them (if you don't believe me, try integrating $(\tan x)^3$ via Riemann sums!).

In practice, however, integration is what we need to do to solve physical problems. Usually, when we are studying the motion of a particle or a planet what we find is that the position of a particle, which is a function of time, satisfies some differential equation.

Solving the differential equation involves performing the inverse operation of taking some combination of derivatives. The simplest such inverse operation is taking the inverse of the first derivative, which the Fundamental Theorem says, is the same as integrating.

Calculating Integrals

Thus, calculating integrals is one of the basic things one needs to do for solving even the simplest physics and engineering problems.

The problem is that this is quite difficult to do.

Once we know the derivatives of some basic functions (polynomials, trigonometric functions, exponentials, logarithms) we can differentiate a wide class of functions using the rules for differentiation, especially the product and chain rules.

By contrast, the only rule for Riemann integration that can be proved from the basic definitions is the sum rule.

The Fundamental Theorem solves this problem (partially) because it allows us to deduce formalæ for the integrals of the products and the composition of functions from the corresponding rules for derivatives.

The Fundamental Theorem - Part I

Theorem 8: Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x),$$

for all $x \in (a, b)$.

Proof: We know that $f(t)$ is Riemann integrable on $[a, x]$ for any $x \in [a, b]$ because of Theorem 5 (every continuous function is Riemann integrable).

We show that

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

for $h > 0$. The same way one can prove the case $h < 0$.

By Theorem 7, we know that

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt,$$

for $x + h \in [a, b]$. Hence

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f(t)dt.$$

Let $m(h)$ and $M(h)$ be the constant functions given, respectively, by the infimum and supremum of the function f on $[x, x+h]$.

Then, $m(h) \leq f(t) \leq M(h)$, for all $t \in [x, x+h]$, and hence

$$m(h) \cdot h \leq \int_x^{x+h} f(t)dt \leq M(h) \cdot h.$$

Dividing by h and taking the limit gives

$$\lim_{h \to 0} m(h) \leq \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \leq \lim_{h \to 0} M(h).$$

We now show that $\lim_{h\to 0} m(h)$ exists and the value of this limit is $f(x)$. By a similar argument one can show that $\lim_{h\to 0} M(h)$ also exists and the value of this limit is also $f(x)$.

For, since $m(h) = \inf_{t\in[x,x+h]} f(t)$, for a given $\epsilon > 0$, there exists $s \in [0, h]$ such that

$$f(x+s) < m(h) + \epsilon/2 \Rightarrow f(x+s) - m(h) < \epsilon/2.$$

Now, we write

$$f(x) - m(h) = f(x) - f(x+s) + f(x+s) - m(h)$$

$$\Rightarrow |f(x) - m(h)| \leq |f(x) - f(x+s)| + |f(x+s) - m(h)|.$$

Since the function f is continuous, there exists $\delta > 0$ such that

$$|f(x+h) - f(x)| < \epsilon/2$$

whenever $|h| < \delta$.

Thus, we obtain that

$$|f(x)-m(h)| \leq |f(x)-f(x+s)|+|f(x+s)-m(h)| < \epsilon/2+\epsilon/2 = \epsilon$$

whenever $|h| < \delta$ (since $s \in [0, h]$) and hence

$$\lim_{h\to 0} m(h) = f(x).$$

Similarly, $\lim_{h\to 0} M(h) = f(x)$. Now, we go back to our inequality at the end of the last to the last slide.

By the Sandwich theorem for limits (use version 2), we see that the limit in the middle exists and is equal to $f(x)$, that is,

$$F'(x) = f(x).$$

This proves that $F(x)$ is differentiable on (a, b) and $F'(x) = f(x)$.

How to show that $F(x)$ is continuous on $[a, b]$? Can you show that $F(x)$ is continuous at the end points a, b? This I leave as an exercise. (Hint: $|\int_c^d f(t)dt| \leq \int_c^d |f(t)|dt$) $\square$

Keeping the notation as in the Theorem, we obtain

Corollary: $\int_c^d f(t)dt = \int_a^d f(t)dt - \int_a^c f(t)dt = F(d) - F(c)$ for any two points $c, d \in [a, b]$.