MA105 : Final Exam (17:30 - 19:30 on 16/11/2023)

Total Marks: 40

Division _ _ / Tutorial Batch _ _ Roll Number: _ _ _ _ _ _ _ _ _ _ _ _

Name: _ $A =$ $B =$

- Fill in the numbers "A" and "B" above as follows:
  If the last digit a of your roll number satisfies $0 \leq a \leq 4$, let $A = a + 5$. If $5 \leq a \leq 9$, let $A = a$.
  If the second-last digit b of your roll number satisfies $0 \leq b \leq 4$, let $B = b + 5$. If $5 \leq b \leq 9$, let $B = b$. Thus $5 \leq A, B \leq 9$.
  Example: Your Roll number is 23B0092. Then $A = 7$ and $B = 9$.
  You must use these values of A and B below. Using the wrong value of A or B in even one question may lead to the loss of full marks in this exam.

Write the answers of the appropriate Questions **only** in the box provided below the questions.
You will get full (respectively, partial) marks in the multiple choice/selection questions below only if you select all (respectively, some but not all) of the TRUE statements and only the TRUE statements (that is, if you select a FALSE statement, you will get ZERO mark in that question).

(1) (1 + 1= 2 marks) Write down the area between the circles $x^2 + y^2 = Ax$ and $x^2 + y^2 = Bx$ as a double integral.
Solution: If $B > A$, then the answer is

$$\int_0^B \int_{-\sqrt{Bx-x^2}}^{\sqrt{Bx-x^2}} dydx - \int_0^A \int_{-\sqrt{Ax-x^2}}^{\sqrt{Ax-x^2}} dydx$$

Similarly for the case $A > B$.
The area is

$$\frac{\pi}{4}(B^2 - A^2).$$

(2) (2 marks) Find the flow line for the vector field $(Ax, By, (A + B)z)$ passing through the origin.
Solution: The flow line is given by $\gamma(t) = (0, 0, 0),\ t \in \mathbb{R}$.

(3) (2 marks) Rewrite the integral

$$\int_0^1 \int_0^{Ax} \int_0^{A^2-y^2} f(x, y, z)\ dzdydx$$

as an iterated integral with order of integration $dxdydz$.

Solution: The rewritten integral is given by

$$\int_0^{A^2}\int_0^{\sqrt{A^2-z}}\int_{y/A}^{1} f(x,y,z)\ dxdydz.$$

(4) (3 marks) Let W be the set defined by $x^2+y^2+z^2 \le A^2$ and $y \le x$. Find the flux of $(x^3-Bx, y^3+Bxy, z^3-Bzx)$ out of W.
Solution:

$$\frac{2\pi}{15}(9-5B).$$

(5) (2 marks) If C is a simple closed curve in $\mathbb{R}^3$, and $\mathbf{v}$ is the constant vector $(A,B,0)$, what is the value of $\int_C \mathbf{v}.d\mathbf{s}$?
Solution:

$$\int_C \mathbf{v}.d\mathbf{s} = 0.$$

(6) (3 marks) Let S be the surface of the tetrahedron whose vertices are $(A,0,0)$, $(0,A,0)$, $(0,0,2A)$ and the origin. Evaluate $\int\int_S f\ dS$, where $f(x,y,z) = Bxz$.
Solution:

$$\frac{5}{12}A^4B.$$

(7) (2 + 1 + 2 = 5 marks) Let $\mathbf{n}$ be the outer unit normal (normal away from the origin) of the parabolic shell

$$S: Ax^2+y+Az^2 = B,\ y \ge 0,$$

and let

$$\mathbf{F} = \left(-z+\frac{1}{2+x}, \tan^{-1}y, x+\frac{1}{4+z}\right).$$

Then $\mathbf{n}$ is

$\boxed{}$

$\mathrm{Curl}\ \mathbf{F}$ is

$\boxed{}$

The value of

$$\int\int_S \mathrm{Curl}\ \mathbf{F}.d\mathbf{S}$$

is

$$\boxed{}$$

Solution:

$$\mathbf{n} = \left(\frac{2Ax}{\sqrt{1+4A^2(x^2+z^2)}}, \frac{1}{\sqrt{1+4A^2(x^2+z^2)}}, \frac{2Az}{\sqrt{1+4A^2(x^2+z^2)}} \right).$$

$$\text{Curl } \mathbf{F} = (0, -2, 0)$$

$$\int\int_S \text{Curl } \mathbf{F}.d\mathbf{S} = -\frac{2\pi B}{A}.$$

(8) (1+ 2 = 3 marks) Let S be the surface $z = \sin x \sin y$ with upward pointing normals and where $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$. Find Curl $\mathbf{F}$, where $\mathbf{F} = (5z - y^2, x\cos z, xy)$.

$$\boxed{\text{Curl } \mathbf{F} = }$$

Find the surface integral of Curl $\mathbf{F}$ over S.
Solution:

$$\text{Curl } \mathbf{F} = (x(1+\sin z), 5-y, \cos z + 2y)).$$

$$\int\int_S \text{Curl } \mathbf{F}.d\mathbf{S} = \pi^3 + \pi^2.$$

(9) (2 marks) Consider the vector field $\mathbf{F}$ on $\mathbb{R}^2$, given by

$$\mathbf{F}(x, y) = (Ae^x(x\cos x + \sin x) - By, Be^y(y\sin y - \cos y) + Ax).$$

Evaluate $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$, where γ is the unit circle with anti-clockwise orientation.
Solution:

$$(A+B)\pi.$$

(10) (5 marks) Let $\mathbf{F}$ be a vector field defined by

$$\mathbf{F}(x, y) = \begin{cases} \frac{e^x(x\cos y + y\sin x) - x}{x^2+y^2}\mathbf{i} + \frac{e^x(x\sin y - y\cos y) + y}{x^2+y^2}\mathbf{j} & (x, y) \neq (0, 0) \\ \mathbf{i} & (x, y) = (0, 0) \end{cases}.$$

Which of the following statements is/are TRUE?

(A) $\mathbf{F}$ is a conservative vector field.
(B) $\mathbf{F}$ is not a conservative vector field.
(C) The line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$ is non-zero for any simple closed curve containing the origin.

(D) The line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$ is non-zero for any simple closed curve in $\mathbb{R}^2$.
Solution: We have given full points to everyone in this problem.

(11) (3 marks) Find an equation for the plane through the origin such that the circulation of the flow field $\mathbf{F} = Az\mathbf{i} + Bz\mathbf{j} + y\mathbf{k}$ around the circle of intersection of the plane with the sphere $x^2 + y^2 + z^2 = 4$ is maximum.
Solution:

$$(1 - B)x + Ay = 0.$$

(12) (2 + 3 = 5 marks) Let S be the paraboloid $z = x^2 + y^2, z \leq A$. Let the orientation on S be such that it induces anti-clockwise orientation on the boundary circle when viewed from a high point upon the positive z-axis. Find an orientation preserving parametrization of S.
Evaluate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (Axe^y, Bze^x - Ae^y, x^2 + y^2)$.
Solution:

$$\Phi(x, y) = \{(x, y, x^2 + y^2) : (x, y) \in B(0, \sqrt{A})\}.$$

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \frac{\pi A^2}{2}.$$

(13) (3 marks) Let $\mathbf{F}$ be a C^2 vector field on $\mathbb{R}^2$. For any point $P \in \mathbb{R}^2$, suppose that $\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s}$, where γ_1 and γ_2 are paths with initial point (A, B) and end point P. Which of the following statements is/are TRUE?
(A) $\nabla \times \mathbf{F}$ is zero except the point (A, B).
(B) $\nabla \times \mathbf{F} = 0$.
(C) $\nabla \cdot \mathbf{F} = 0$.
(D) $\nabla \cdot \mathbf{F}$ is zero at (A, B).
(E) $\nabla \cdot \mathbf{F}$ can be non-zero at every point.
Solution: (B), (E). We have awarded 1 mark for one correctly chosen option.