MA 105 Part II Tutorial Sheet 3 : Change of variables, Line integrals, October 16, 2023

I Multiple integrals and change of variables

2. Using a suitable change of variables, evaluate the integral $\int\int_D y dx dy$, where D is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.
4. Use cylindrical coordinates to evaluate $\int\int\int_W (x^2 + y^2) dz dy dx$, where

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq x \leq 2, \quad -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \quad \sqrt{x^2 + y^2} \leq z \leq 2\}.$$

6. Find $\iiint_F \dfrac{1}{(x^2 + y^2 + z^2)^{n/2}} dV$, where F is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.
7. Evaluate the integral

$$\iint_D (x - y)^2 \sin^2(x + y) d(x, y),$$

where D is the parallelogram with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

8. Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Find $\displaystyle\iint_D dx dy$ by transforming it to $\displaystyle\iint_E du dv$, where $x = \dfrac{u}{v}$, $y = uv$, $v > 0$.
9. Using suitable change of variables, evaluate the following:

   i.
   $$I = \iiint_D (z^2 x^2 + z^2 y^2) dx dy dz$$
   where D is the cylindrical region $x^2 + y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

   ii.
   $$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dx dy dz$$
   over the region enclosed by the unit sphere in $\mathbb{R}^3$.

II Vector analysis and line integrals

1. Let f, g be differentiable functions on $\mathbb{R}^2$. Show that
   A. $\nabla(fg) = f\nabla g + g\nabla f$;
   B. $\nabla f^n = n f^{n-1} \nabla f$;
   C. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.
2. Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:
   (i) $\nabla(r^n) = n r^{n-2} \mathbf{r}$ for any integer n.
   (ii) $\mathbf{a} \cdot \nabla \left(\dfrac{1}{r}\right) = -\left(\dfrac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.
   (iii) $\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \left(\dfrac{1}{r}\right)\right) = \dfrac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \dfrac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.

3. Calculate the line integral of the vector field
$$\mathbf{F}(x,y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$
from $(-1,1)$ to $(1,1)$ along $y = x^2$.
4. Calculate the line integral of
$$\mathbf{F}(x,y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$
once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.
Remark Often line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F}.\mathbf{ds}$. A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.
To evaluate $\int_C \mathbf{F}.\mathbf{ds}$, choose a convenient parametrization $\mathbf{c}$ of C traversing C in the given direction and then
$$\int_C \mathbf{F}.\mathbf{ds} := \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$
'$\oint_C$' means the line integral over a closed curve C.
5. Calculate the value of the line integral
$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$$
where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.
6. Calculate
$$\oint_C ydx + zdy + xdz$$
where C is the intersection of two surfaces $z = xy$ and $x^2+y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.
7. Let the curve C be given by $x^2 + y^2 = 1, z = 0$. Let $\mathbf{c}_1$ be a parametrization defined by $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Find the line integral of $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j}$ along this curve. Also find the line integral along the curve parametrized by $\mathbf{c}_2(t) = (\cos t, -\sin t)$, for $t \in [0, \pi]$.
8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle: $x^2+y^2 = 1$. Is this also true for a force field $\mathbf{F}(x,y,z) = \alpha(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, for some constant α.
9. Let $C : x^2 + y^2 = 1$. Find
$$\oint_C \text{grad } (x^2 - y^2) \cdot \mathbf{ds}.$$
10. Evaluate
$$\int_C \text{grad } (x^2 - y^2) \cdot \mathbf{ds},$$
where C is $y = x^3$, joining $(0,0)$ and $(2,8)$.
11. Compute the line integral
$$\oint_C \frac{dx + dy}{|x| + |y|}$$
where C is the square with vertices $(1,0), (0,1), (-1,0)$ and $(0,-1)$ traversed once in the counter clockwise direction.
12. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0,0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.