MA 105 : Calculus

D1 - Lecture 3

Sandip Singh

Department of Mathematics

Autumn 2023, IIT Bombay, Mumbai

The proof of the product rule

We wish to prove that $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

Suppose $\lim_{n\to\infty} a_n = \ell_1$ and $\lim_{n\to\infty} b_n = \ell_2$. We need to show that $\lim_{n\to\infty} a_n b_n = \ell_1 \ell_2$.

Let $\epsilon > 0$ be an arbitrary real number. We need to show that there exists $N \in \mathbb{N}$ such that $|a_n b_n - \ell_1 \ell_2| < \epsilon$, whenever $n > N$. Notice that

$$\begin{aligned} |a_n b_n - \ell_1 \ell_2| &= |a_n b_n - a_n \ell_2 + a_n \ell_2 - \ell_1 \ell_2| \\ &= |a_n (b_n - \ell_2) + (a_n - \ell_1) \ell_2| \\ &\leq |a_n||b_n - \ell_2| + |a_n - \ell_1||\ell_2|, \end{aligned}$$

where the last inequality follows from the triangle inequality.

So in order to guarantee that the left hand side is less than ϵ, we must ensure that the two terms on the right hand side together add up to less than ϵ.

In fact, we make sure that each term on right hand side is less than $\epsilon/2$.

The proof of the product rule, continued

Since a_n is convergent, it is bounded (by the lemma, we have just proved). Hence, there is an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Given the positive real numbers $\frac{\epsilon}{2|\ell_2|+1}$ and $\frac{\epsilon}{2M}$, there exist N_1 and N_2 such that

$$|a_n - \ell_1| < \frac{\epsilon}{2|\ell_2| + 1}, \quad n \geq N_1 \quad \text{and} \quad |b_n - \ell_2| < \frac{\epsilon}{2M}, \quad n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. If $n > N$, then both the inequalities above hold. Hence, we have

$$|a_n||b_n - \ell_2| \leq M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2} \quad \text{and} \quad |a_n - \ell_1||\ell_2| \leq \frac{\epsilon}{2|\ell_2| + 1} \cdot |\ell_2| < \frac{\epsilon}{2}.$$

Now, it follows that

$$|a_n b_n - \ell_1 \ell_2| \leq |a_n||b_n - \ell_2| + |a_n - \ell_1||\ell_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $n > N$, which is what we needed to prove. $\square$

The proofs of the other rules for limits are similar to the one we proved above. Try them as exercises.

A guarantee for convergence

As we mentioned earlier, proving that a limit exists is hard because we have to guess what its value might be and then prove that it satisfies the definition.

The following theorem guarantees the convergence of a sequence without knowing the limit beforehand.

Definition: A sequence a_n is said to be bounded above (resp. bounded below) if $a_n \leq M$ (resp. $a_n \geq m$) for some $M \in \mathbb{R}$ (resp. $m \in \mathbb{R}$).

A sequence that is bounded both above and below is obviously bounded (maximum of $|m|$ and $|M|$ works as a bound for $|a_n|$).

Theorem 3: A monotonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges.

Remarks on Theorem 3

Theorem 3 clearly makes things very simple in many cases. For instance, if we have a monotonically decreasing sequence of positive numbers, it must have a limit, since 0 is always a lower bound!

Can we guess what the limit of a monotonically increasing sequence a_n bounded above might be?

It will be the supremum or least upper bound (lub) of the sequence. This is the number, say M, which has the following properties:

1. $a_n \leq M$ for all n and
2. If M_1 is such that $a_n < M_1$ for all n, then $M \leq M_1$. In other words, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_N > M - \epsilon$.

The point is that a sequence bounded above may not have a maximum but will always have a supremum. As an example, take the sequence $1 - 1/n$. Clearly there is no maximal element in the sequence, but 1 is its supremum.

Another monotonic sequence

Let us look at Exercise 1.5.(i) which considers the sequence

$$a_1 = 3/2 \quad \text{and} \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

Since a_n is positive for all n,

$$\begin{aligned} a_{n+1} \leq a_n \quad &\iff \tfrac{1}{2}\left(a_n + \tfrac{2}{a_n}\right) \leq a_n \\ &\iff a_n^2 + 2 \leq 2a_n^2 \\ &\iff \sqrt{2} \leq a_n. \end{aligned}$$

On the other hand,

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{2} \quad \text{(Why is this true? } \quad A.M. \geq G.M.\text{)}$$

so $a_{n+1} \geq \sqrt{2}$ for all $n \geq 1$ and $a_1 > \sqrt{2}$ is given.

Hence, $\{a_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence, bounded below by $\sqrt{2}$. By Theorem 3, it converges.

More remarks on limits

Exercise 1. What do you think is the limit of the above sequence (Refer to the supplement to Tutorial 1)?

Exercise 2. More generally, what is the limit of a monotonically decreasing sequence bounded below? How can you describe it?

This number is called the infimum or greatest lower bound (glb) of the sequence.

Theorem 3 can be proved by using the fact that the set of real numbers has the least upper bound property: every nonempty subset of real numbers having an upper bound has the least upper bound. You are now encouraged to prove Theorem 3 using the ϵ - N definition of convergence.

The proof of the least upper bound property of the set of real numbers more or less involves understanding what a real number is. It is related to the notion of Cauchy sequences (again, refer to the supplement to Tutorial 1).

Cauchy sequences

As we saw last time, it is not easy to tell whether a sequence converges or not because we have to first guess what the limit might be and then try and prove that the sequence actually converges to this limit. For a monotonic sequence, things are slightly better since we only need to bound the sequence.

There is another very useful notion which allows us to decide whether the sequence converges by looking only at the terms of the sequence itself. We describe this below.

Definition: A sequence a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon,$$

for all $m, n > N$.

Theorem 4: Every Cauchy sequence in $\mathbb{R}$ converges.

Cauchy sequences: Some Remarks

Remark 1: One can now check the convergence of a sequence just by looking at the sequence itself!

One can easily check the converse:

Theorem 5: Every convergent sequence is Cauchy.

Remark 2: Remember that when we defined sequences we defined them to be functions from $\mathbb{N}$ to X, for any set X. So far we have only considered $X = \mathbb{R}$, but as we said earlier we can take other sets, for instance, subsets of $\mathbb{R}$.

For instance, if we take $X = \mathbb{R} \setminus \{0\}$, Theorem 4 is not valid. The sequence $1/n$ is a Cauchy sequence in this X but obviously does not converge in X.

If we take $X = \mathbb{Q}$, the example given in 1.5.(i) ($a_1 = 3/2$ and $a_{n+1} = (a_n + 2/a_n)/2$) is a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$.

Thus Theorem 4 is really a theorem about real numbers.

The completeness of $\mathbb{R}$ and more remarks on limits

A set in which every Cauchy sequence converges is called a complete set. Thus, Theorem 4 is sometimes rewritten as

Theorem 4': The set of real numbers is complete.

An important remark: If we change finitely many terms of a sequence, it does not affect the convergence and boundedness properties of a sequence.

If it is convergent, the limit will not change. If it is bounded, it will remain bounded though the supremum may change.

Thus, an eventually monotonically increasing sequence bounded above will converge (formulate the analogue for decreasing sequences).

Bottomline: From the point of view of the limit, only what happens for large N matters.