MA 105 Calculus II

Week 3

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Change of variables in $\mathbb{R}^2$

Let Ω be an open subset of $\mathbb{R}^2$ and $h : \Omega \to \mathbb{R}^2$ be an one-one transformation denoted by

$$h(u, v) := (h_1(u, v), h_2(u, v)), \quad \forall\, (u, v) \in \Omega.$$

We now want to make a general change of coordinates given by

$$x = h_1(u, v), \quad y = h_2(u, v).$$

What conditions do we need on h to be able to do a change of coordinates?

Can we compute the area of the image of a rectangle in the u-v plane?

Suppose we have a change in coordinates given by linear functions composed with translations (such functions are called affine linear functions):

$$x = au + bv + t_1 \quad \text{and} \quad y = cu + dv + t_2.$$

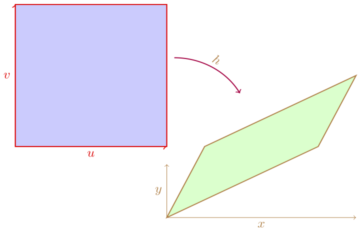
A linear change of coordinates

How does the area of the image of a rectangle under this map compare with the area of the original rectangle?

First, let us write down the affine map in a more compact notation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Clearly, a rectangle $[0,1] \times [0,1]$ in the u-v plane is mapped to a parallelogram in the x-y plane. The vertices of the parallelogram are given by (t_1, t_2), $(a + t_1, c + t_2)$, $(b + t_1, d + t_2)$ and $(a + b + t_1, c + d + t_2)$.

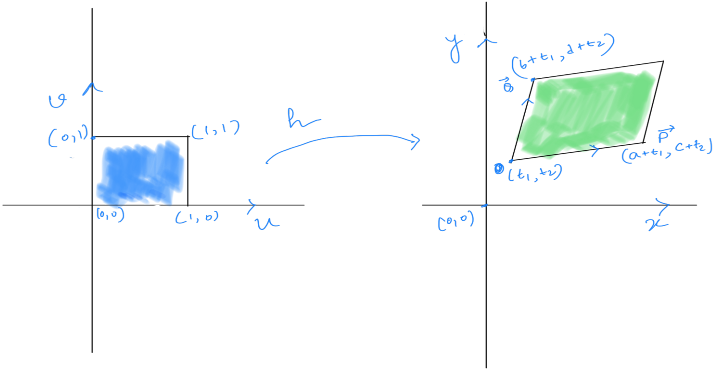


How does one compute the area of this parallelogram?

The area element for a change of coordinates

This is given by the absolute value of the cross product of the vectors,

$$(a, c, 0) \times (b, d, 0) = (ad - bc) \cdot \mathbf{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \mathbf{k}.$$



The area element for a change of coordinates

Let us now suppose that we have a general (not linear any more) change of coordinates given by $x = h_1(u, v)$ and $y = h_2(u, v)$.
How does the area of a rectangle in the u-v plane change? In order to compute the change we need to know the partial derivatives exist.
Let us assume h is a one-one continuously differentiable function .
Noting

$$\Delta x = h_1(u+\Delta u, v+\Delta v) - h_1(u, v), \quad \Delta y = h_2(u+\Delta u, v+\Delta v) - h_2(u, v),$$

and using Taylor's theorem for functions of two variables we see that

$$\Delta x \sim \frac{\partial h_1}{\partial u}\Delta u + \frac{\partial h_1}{\partial v}\Delta v \text{ and } \Delta y \sim \frac{\partial h_2}{\partial u}\Delta u + \frac{\partial h_2}{\partial v}\Delta v.$$

Using our previous notation, we can write

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

The Jacobian

You may recognize the matrix

$$J(h) = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}$$

that appears in the preceding formula. The derivative matrix for the function $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is called the Jacobian.

In a neighbourhood of the point (u_0, v_0), the function h and the function $J(h)$, behave very similarly (that is, they are the same upto the first order terms - use Taylor's theorem!). In fact, the derivative matrix is the *linear approximation* to the function h, at least in a neighbourhood of a point, say (u_0, v_0).

In particular, it is easy to see how the area of a small rectangle changes under h, since we have already done so in the case of a linear map. It simply changes by the (absolute value of) determinant of J!

Change of Variables Formula

Theorem (Change of Variables Formula)

- *Let* D *be a closed and bounded subset of* $\mathbb{R}^2$ *such that* ∂D *has content zero. Let* $f : D \to \mathbb{R}$ *be continuous.*
- *Suppose* Ω *is an open subset of* $\mathbb{R}^2$ *and* $h : \Omega \to \mathbb{R}^2$ *is a one-one differentiable function such that* $h := (h_1, h_2)$, *where* h_1 *and* h_2 *have continuous partial derivatives in* Ω *and* $det(J(h)(u, v)) \neq 0$ *for all* $(u, v) \in \Omega$.
- *Let* $D^* \subset \Omega$ *be such that* $h(D^*) = D$.

Then D^* *is a closed and bounded subset of* Ω, *and* ∂D^* *is of content zero. Moreover,* $f \circ h : D^* \to \mathbb{R}$ *is continuous, and*

$$\int\int_D f(x, y)\, dxdy = \int\int_{D^*} (f \circ h)(u, v)|det(J(h)(u, v))|\, dudv.$$

Notation

Often we write $x = x(u, v)$ and $y = y(u, v)$. In this case we use the notation $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$, for the Jacobian determinant.

Let D be a region in the xy plane and D^* a region in the uv plane such that $\phi(D^*) = D$. Then

$$\int \int_D f(x, y)dxdy = \int \int_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv.$$

Remark: Note what we get in the familiar case of polar coordinates: We have $x = r\cos\theta$, $y = r\sin\theta$ and

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r,$$

which is what we obtained previously.

How to choose the change of variables

- Aim: Find h such that a rectangle D^* in $u - v$ plane is getting mapped to the given area D in the xy plane. If D^* is not a rectangle, at least try to have it in the form of the elementary region Type 1 or Type 2.
- Presumably, the boundary D^* in $u - v$ plane should go to the boundary of D in $x - y$ plane.
- The non-vanishing Jacobian determinant of h assures that the properties of D^* is preserved under the transformation and D has similar properties as of D^*.
- In some cases, h can be chosen in a way such that the expression of the integrand becomes simpler after the change of variables.

Example

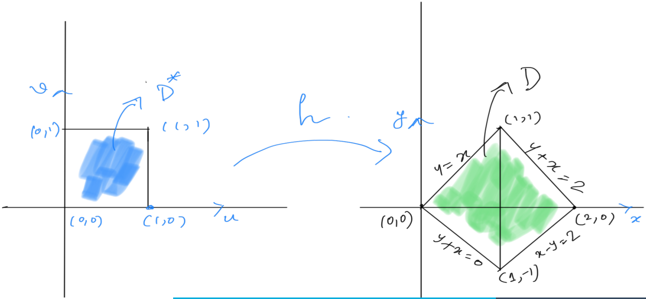
Example: Evaluate the integral

$$\iint_D (x^2 - y^2)dxdy$$

where D is the square with vertices at $(0,0)$, $(1,-1)$, $(1,1)$ and $(2,0)$.

Solution: Note D is the region in $x-y$ plane bounded by lines $y=x$, $y+x=0$, $x-y=2$ and $y+x=2$.
Put

$$x = u + v, \quad y = u - v,$$



Example Contd.

Then the rectangle

$$D^* = \left\{(u, v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1,\ \ 0 \leq v \leq 1\right\}$$

in the uv-plane gets mapped to D, in the xy-plane.
Further,

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2.$$

$$\int\int_D (x^2 - y^2)\, dxdy = \int\int_{D^*} (4uv) \times 2\, dudv$$

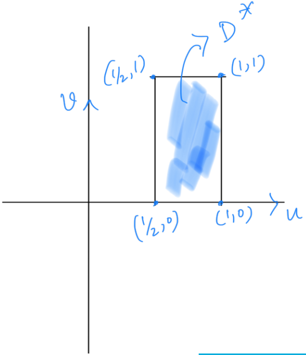
$$= 8\left(\int_0^1 udu\right)\left(\int_0^1 vdv\right) = 2.$$

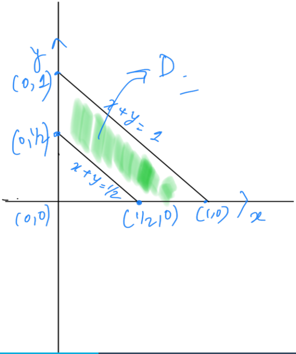
Example

Example: Let D be the region in the first quadrant of the xy-plane bounded by the lines $x + y = \frac{1}{2}$ and $x + y = 1$. Find $\iint_D dA$ by transforming it to $\iint_{D^*} du dv$, where $u = x + y, v = \frac{y}{x+y}$.

Solution: Put

$$x = u(1 - v), \;\; y = uv.$$





Example Contd.

Then the rectangle $D^* = \{(u,v) \in \mathbb{R}^2 \mid \frac{1}{2} \leq u \leq 1, \quad 0 \leq v \leq 1\}$ in the uv-plane gets mapped to D in the xy-plane. Further,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix} = u \neq 0.$$

Hence,

$$\text{Area}(D) = \int\int_D dA = \int\int_{D^*} |u| du dv$$
$$= \left(\int_{\frac{1}{2}}^{1} \frac{u^2}{2} du\right)\left(\int_0^1 dv\right) = \frac{3}{4}.$$

The change of variables formula in three variables

In three variables, we once again have a formula for a change of variables. The formula has the same form as in the two variable case:

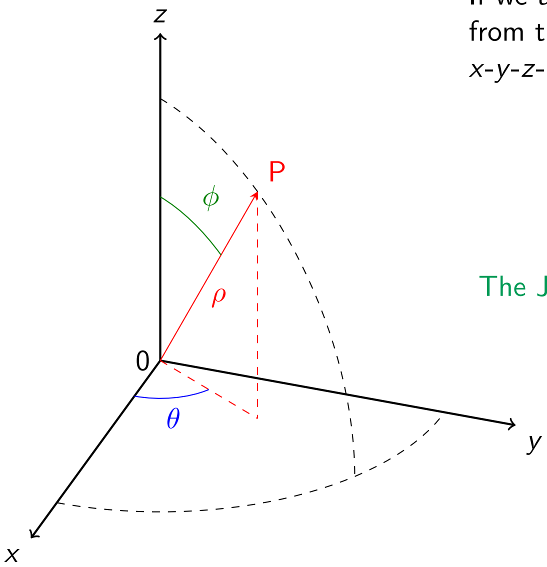
$$\iiint_P f(x,y,z)dxdydz = \iiint_{P^*} g(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw,$$

where $h(P^*) = P$. If the change in coordinates is given by $h = (h_1, h_2, h_3)$, the function g is defined as $g = f(h_1, h_2, h_3)$. The expression

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

is just the Jacobian determinant for a function of three variables.

Spherical Coordinates



If we use (ρ, θ, ϕ) what is the map from these coordinates to the x-y-z-planes?

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi. \end{aligned}$$

The Jacobian determinant is

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}x = \rho^2 \sin\phi.$$

Example

Example: It should be much easier computing the volume of the unit sphere now. Let $W = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$.

Then $W^* = \{(\rho,\theta,\phi) \in \mathbb{R}^3 \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$.
Then,

$$\begin{aligned}
\iiint_W dxdydz &= \iiint_{W^*} \rho^2 \sin\phi \; d\rho d\theta d\phi \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \; d\rho d\theta d\phi \\
&= \frac{2\pi}{3} \int_0^{\pi} \sin\phi \; d\phi = \frac{4\pi}{3}.
\end{aligned}$$

Cylindrical coordinates in formulae

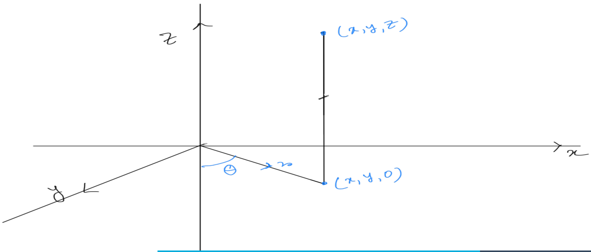
We can also consider a generalization of the polar coordinates. In this case, we use the change of transformation from (r, θ, z) coordinates to $P = (x, y, z) \in \mathbb{R}^3$ given by

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z.$$

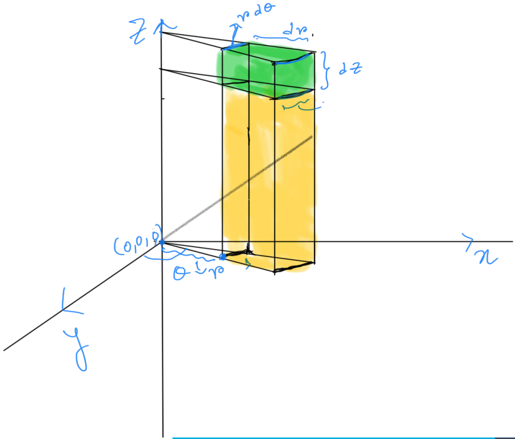
Here $r \geq 0$ and $0 \leq \theta \leq 2\pi$ and the (r, θ, z) are

It is very easy to see that

$$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = r.$$



The good thing about our convention is that θ means the same thing in both the cylindrical and spherical coordinate systems as well as in the (two-dimensional) polar coordinate system, and r means the same thing in both the cylindrical and (two-dimensional) polar coordinate systems.



Example

Evaluate $\int\int\int_W z^2(x^2+y^2)\,dxdydz$, where W is the cylindrical region determined by $x^2+y^2 \le 1$ and $-1 \le z \le 1$.

Solution. The region W is described in cylindrical coordinates as W^*

$$W^* = \{(r,\theta,z) \mid 0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad -1 \le z \le 1\}.$$

$$\begin{aligned}\int\int\int_W z^2(x^2+y^2)\,dxdydz &= \int_{z=-1}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} z^2 r^2\, r\, drd\theta dz \\ &= \int_{-1}^{1} \frac{2\pi}{4} z^2\,dz = \frac{\pi}{3}.\end{aligned}$$

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the Euclidean space defined by

$$\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n) \mid x_j \in \mathbb{R}; \quad \forall\, j = 1, 2, \cdots, n\},$$

equipped with the norm

$$\|x\| := \Big(\sum_{j=1}^{n} |x_j|^2\Big)^{\frac{1}{2}}.$$

- Any real number is called a scalar.
- For $n \in \mathbb{N}$, any element from $\mathbb{R}^n$ is called vector. Note this means elements of $\mathbb{R}$ can be thought of both as a scalar and vector. To avoid confusion we will talk about vectors in $\mathbb{R}^n$ for $n > 1$.

Basic structure:

For any $x := (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $y := (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and any $a \in \mathbb{R}$:

$$x + y := (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \in \mathbb{R}^n, \quad \text{sum of two elements in} \quad \mathbb{R}^n$$

$$ax := (ax_1, ax_2, \cdots, ax_n) \in \mathbb{R}^n, \quad \text{Scalar multiplication.}$$

Scalar fields and Vector fields

Let D be a subset of $\mathbb{R}^n$.

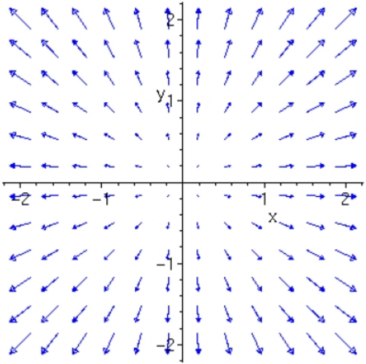
Definition: A scalar field on D is a map $f : D \to \mathbb{R}$.

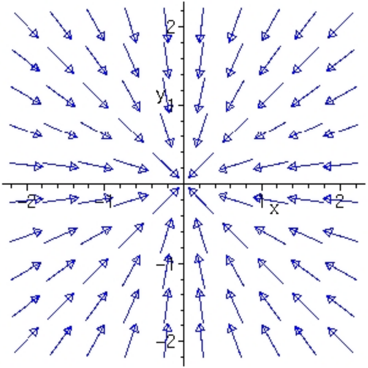
Definition A vector field on D is a map $\mathbf{F} : D \to \mathbb{R}^n$. We choose $n \geq 2$.

- A scalar field associates a number to each point of D, whereas a vector field associates a vector (of the same space) to each point of D.
- The temperature at a point on the earth is a scalar field.
- The velocity field of a moving fluid, a field describing heat flow, the gravitational field, the magnetic field etc are examples of various vector fields.

Vector fields: Examples

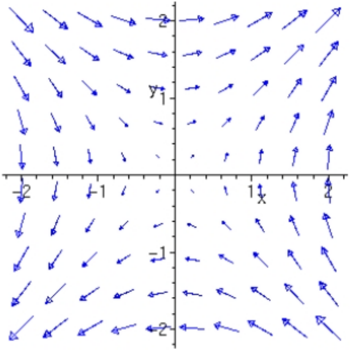
$$F_1(x, y) = (2x, 2y) \qquad F_2(x, y) = \left(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$



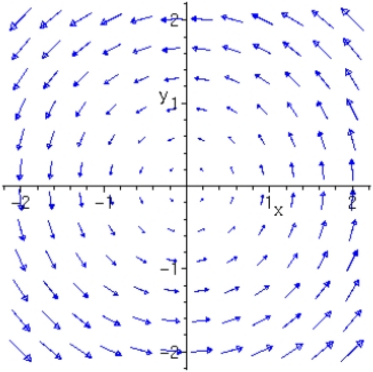


Vector fields: Examples

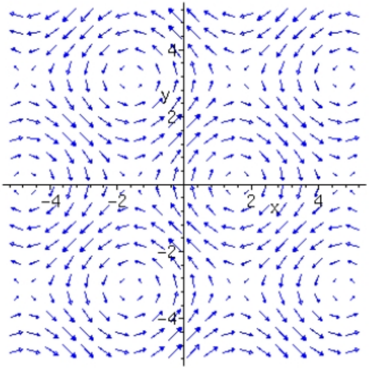
$F_3(x, y) = (y, x)$



$F_4(x, y) = (-y, x)$

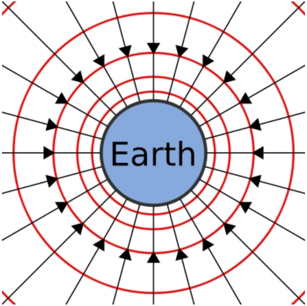


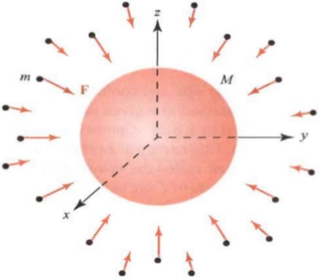
$$F_5(x, y) = (\sin y, \cos x)$$



The vector fields also occur in nature. Some of this you may have seen in MA 109 as well.

Gravitation fields





The first figure describes the gravitational field of the earth whereas the second one describes that of a body with mass M. The red lines denote the direction of the force exerted on the small particles around the body.

Del operator on Functions

We will assume from now on that our vector fields are smooth wherever they are defined.

One important class of vector fields are those that are given by the gradient of a scalar function. We will study these in some detail later.

The del operator on functions: We define the del operator restricting ourselves to the case $n = 3$:

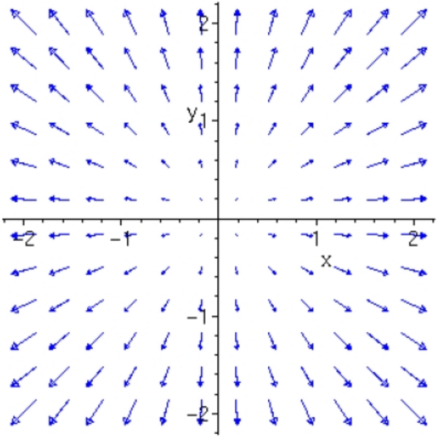
$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give a gradient vector field :

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

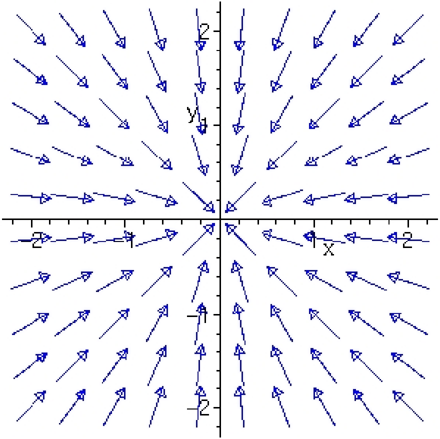
Thus the del operator takes scalar functions to vector fields.

Gradient fields



$$F_1(x, y) = (2x, 2y) = \nabla(x^2 + y^2)$$

Gradient fields



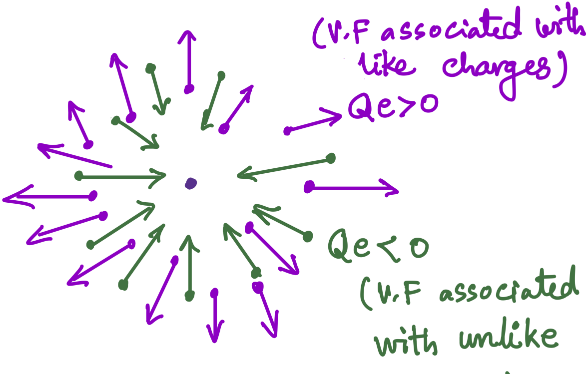
$$F_2(x, y) = \left(\frac{-x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) = \nabla \left(-\sqrt{x^2 + y^2} \right)$$

Gradient Vector fields

Coulomb's law says that the force acting on a charge e at a point r due to a charge Q at the origin is

$$F = -\nabla V$$

where $V = \epsilon Qe/r$ is the potential. For like charges $Qe > 0$ force is repulsive and for unlike charges $Qe < 0$ the force is attractive.

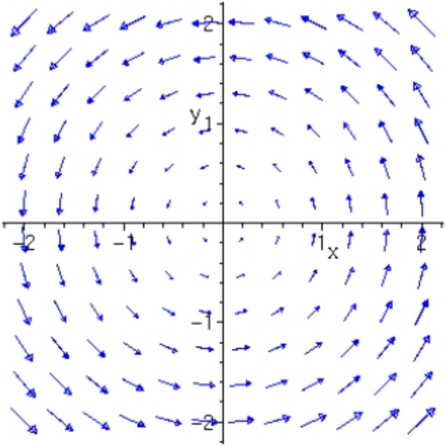


Such a force is called conservative. Conservative forces are important as work done along a path will be only dependent on the end points.

Several of the examples we have seen turn out to be gradient vector fields. The natural question to ask is which vector field is a gradient field.

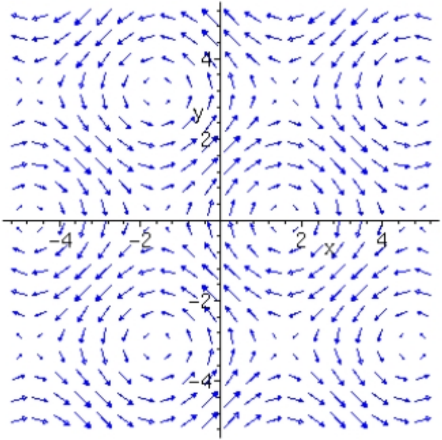
There is a neat answer to the above question, which we will see later. Not all vector fields will turn out to be gradient vector field.

Not gradient fields



$F_4(x, y) = (-y, x)$, this vector field is not ∇f for any f.

Not gradient fields



$F_5(x, y) = (\sin y, \cos x)$, this vector field is not ∇f for any f.

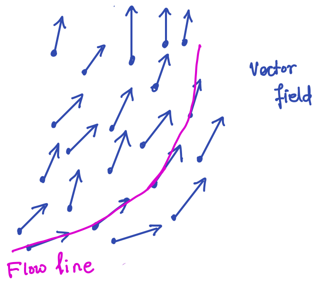
Flow lines for vector field

Vector fields also arise as the tangent vectors to the fluid flow.
Or conversely, given a vector field we can talk about its flow lines.

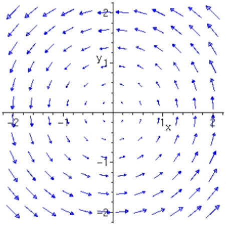
Definition If $\mathbf{F}$ is a vector field defined from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, a flow line or integral curve is a path i.e., a map $\mathbf{c} : [a, b] \to D$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)), \quad \forall t \in [a, b].$$

In particular, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}$.



Example: Show that $c(t) = (\cos t, \sin t)$ is a flow line for the vector field $F(x, y) = -y\mathbf{i} + x\mathbf{j}$. Does it have other flow lines? Can you guess by looking at the vector field?



Finding the flow line for a given vector field involves solving a system of differential equations, if $c(t) = (x(t), y(t), z(t))$ then

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)) \\ y'(t) &= Q(x(t), y(t), z(t)) \\ z'(t) &= R(x(t), y(t), z(t)), \end{aligned}$$

where the vector field is given by $F = (P, Q, R)$.

Such questions are dealt with in MA108.

Curve and path

Recall a path in $\mathbb{R}^n$ is a continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^n$.

A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c}$ in $\mathbb{R}^n$.

Both the curve and path are denoted by the same symbol $\mathbf{c}$.

- Let $n = 3$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, for all $t \in [a, b]$. The path $\mathbf{c}$ is continuous iff each component x, y, z is continuous. Similarly, $\mathbf{c}$ is a C^1 path, i.e., continuously differentiable if and only if each component is C^1.
- A path $\mathbf{c}$ is called closed if $\mathbf{c}(a) = \mathbf{c}(b)$.
- A path $\mathbf{c}$ is called simple if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ in $[a, b]$ other than $t_1 = a$ and $t_2 = b$ endpoints.
- If we write $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in vector notation, the tangent vector to $\mathbf{c}(t)$ is $\mathbf{c}'(t)$, i.e.,

$$\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

- If a C^1 curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$, the curve is called a regular or non-singular parametrised curve.

Examples of curves

• Let $\mathbf{c}(t) = (\cos 2\pi t, \sin 2\pi t)$ where $0 \leq t \leq 1$. This is a simple closed C^1 (actually smooth) curve.

• Let $\mathbf{c}(t) = (t, t^2)$ where $-1 \leq t \leq 5$ is a simple curve but not closed.

• Let $\mathbf{c}(t) = (\sin(2t), \sin t)$ where $-\pi \leq t \leq \pi$. It traces out a figure 8. It is not a simple but a closed C^1 curve.

• Let $\mathbf{c}(t) = (t^3, t)$ where $-1 \leq t \leq 1$ for some real numbers a, b is a part of the graph of the function $y = x^{1/3}$. This is simple but not a closed curve. Though the function $y = x^{\frac{1}{3}}$ is not a smooth function at origin, but this parametrization is regular.

Work done along a curve

• Recall from Physics, that work done by a particle on which force **F** is applied is given by the **F**.ds where ds is the displacement.

• If this is in one variable it is just the product and given by dot-product when it is in 2D or 3D space. This idea works when the displacement is straight line.

• If the particle is moving along a curve **c** then locally the curve can be approximated by a straight line.

• For a path $\mathbf{c} : [0,1] \to \mathbb{R}^n$ for $n = 2$ or 3 if $\Delta t = t_2 - t_1$ is *very very* small then

$$\Delta s = \mathbf{c}(t_2) - \mathbf{c}(t_1) = \mathbf{c}'(\widehat{t})(t_2 - t_1)$$

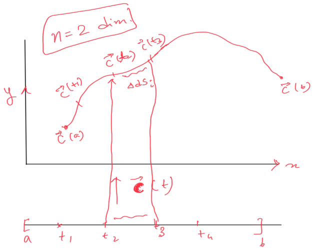
for some $\widehat{t} \in [t_1, t_2]$ by mean value theorem.

• Then work done will have to be computed over these small intervals $[t_i, t_{i+1}]$ for $i = 1, \ldots, n$.

Work done along a curve contd.

- Total work done $= \sum_{i=1}^{n} \mathbf{F}(\widehat{t_i}) \cdot \mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)$
$= \sum_{i=1}^{n} (\mathbf{F}(\widehat{t_i}) \cdot \mathbf{c}'(\widehat{t_i}))(t_{i+1} - t_i)$.
Does this remind you of something? Riemann sum: The limit of these Riemann sums as the length of the subintervals tends to zero, if it exists, is defined to be the line integral of the vector field $\mathbf{F}$ over the curve $\mathbf{c}$ and is denoted by

$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$



Line integrals of vector fields

Assume that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, for $n = 1, 2$, is continuous and the curve $\mathbf{c} : [a, b] \to D$ is C^1.

Then we define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

If $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, we see that

$$\begin{aligned}
&\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\
&= \int_a^b \left(F_1(\mathbf{c}(t)) \frac{dx(t)}{dt} + F_2(\mathbf{c}(t)) \frac{dy(t)}{dt} + F_3(\mathbf{c}(t)) \frac{dz(t)}{dt} \right) dt.
\end{aligned}$$

Because of the form of the right hand side the line integral is sometimes written as

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b F_1 dx + F_2 dy + F_3 dz.$$

The expression on the right hand side is just alternate notation for the line integral. It does not have any independent meaning.

Examples

Example 1: Evaluate

$$\int_{\mathbf{c}} x^2 dx + xydy + dz,$$

where $\mathbf{c} : [0, 1] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (t, t^2, 1)$.

Solution: Let $\mathbf{c}(t) = (t, t^2, 1)$.
Let $\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) = (x^2, xy, 1)$.

Thus $F_1(t, t^2, 1) = t^2$, $F_2(t, t^2, 1) = t^3$ and $F_3(t, t^2, 1) = 1$.

We have $\mathbf{c}'(t) = (1, 2t, 0)$, hence

$$(F_1(t, t^2, 1), F_2(t, t^2, 1), F_3(t, t^2, 1)) \cdot \mathbf{c}'(t) = t^2 + 2t^4 + 0.$$

$$\int_{\mathbf{c}} x^2 dx + xydy + dz = \int_0^1 (t^2 + 2t^4)dt = 11/15.$$

Examples

Example 2 (Marsden, Tromba, Weinstein): Find the work done by the force field $\mathbf{F} = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$ around the loop $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$.
Solution: The work done is given by

$$\begin{aligned} W &= \int_{\mathbf{c}} \mathbf{F} \cdot ds \\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)).\mathbf{c}'(t)\ dt \\ &= \int_0^{2\pi} (\cos t + \sin t) dt \\ &= (\sin t - \cos t)|_0^{2\pi} = 0 \end{aligned}$$

Integrating along successive paths

It is easy to see that if $\mathbf{c}_1$ is a path joining two points P_0 and P_1 and $\mathbf{c}_2$ is a path joining P_1 and P_2 and $\mathbf{c}$ is the union of these paths (that is, it is a path from P_0 to P_2 passing through P_1), which is C^1 then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

This property follows directly from the corresponding property for Riemann integrals:

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt,$$

where c is a point between a and b.

This allows us to define integration over peicewise differentiable curves for example the perimeter of a square.

Let the curve $\mathbf{c}$ be a union of curves $\mathbf{c}_1, \ldots, \mathbf{c}_n$. We often write this as $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \ldots \mathbf{c}_n$, where end point of $\mathbf{c}_i$ is the starting point of $\mathbf{c}_{i+1}$ for all $i = 1, \ldots, n-1$.

Then we can define

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds := \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \ldots + \int_{\mathbf{c}_n} \mathbf{F} \cdot ds.$$

Divide the curve $\mathbf{c}$ at a point p into two curves $\mathbf{c}_1$ and c_2. Then there it is easy to verify that $\int_{\mathbf{c}} \mathbf{F} \cdot = \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \int_{\mathbf{c}_2} \mathbf{F} \cdot ds$.

Let $\mathbf{c}$ be a curve on $[a, b]$ and $\widetilde{\mathbf{c}}(t) = \mathbf{c}(a + b - t)$, that is the curve $\widetilde{\mathbf{c}}$ traversed in the reverse direction and is denoted by $-\mathbf{c}$. What is $\int_{\mathbf{c}} \mathbf{F} \cdot ds + \int_{-\mathbf{c}} \mathbf{F} \cdot ds$?

$\int_c \mathbf{F} \cdot ds = -\int_{-c} \mathbf{F} \cdot ds$ (use change of variables formula).