MA 105 : Calculus

D1 - Lecture 8

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Concavity and convexity

Let I denote an interval (open or closed or half-open).

Definition: A function $f : I \to \mathbb{R}$ is said to be concave (or sometimes concave downwards) if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

for all x_1 and x_2 in I and $t \in [0,1]$. Similarly, a function is said to be convex (or concave upwards) if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

By replacing the $\geq$ and $\leq$ signs above by strict inequalities we can define strictly concave and strictly convex functions.

Note that if $f(x)$ is a concave function, $-f(x)$ is a convex function, so it is enough to study either the convex or the concave functions.

Examples of concave and convex functions

Here are some examples of convex functions.

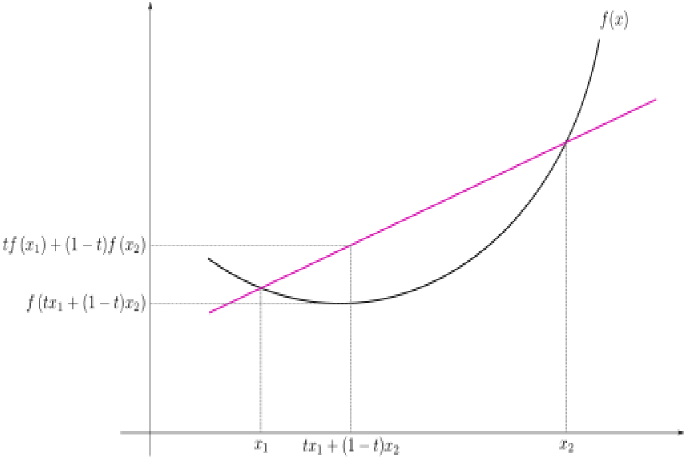
1. $f(x) = x^2$ on $\mathbb{R}$.
2. $f(x) = x^3$ on $[0, \infty)$.
3. $f(x) = e^x$ on $\mathbb{R}$.

Examples of concave functions include

1. $f(x) = -x^2$ on $\mathbb{R}$.
2. $f(x) = x^3$ on $(-\infty, 0]$
3. $f(x) = \log x$ on $(0, \infty)$.

For a convex function f and point $c \in (x_1, x_2)$, the point $(c, f(c))$ always lies below the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Convexity illustrated graphically



Properties of Convex functions

Convex functions have many nice properties. For instance, it is easy to show that convex functions are continuous (do this!) (Hint: Show that, for $x_1 < x_2 < x_3$, $[f(x_2) - f(x_1)]/[x_2 - x_1] \leq [f(x_3) - f(x_1)]/[x_3 - x_1] \leq [f(x_3) - f(x_2)]/[x_3 - x_2]$). More is true.

Exercise 1. Every convex function f (on a bounded interval) is Lipschitz continuous (cf. Exercise 1.16 with $\alpha = 1$), that is, there exists $M > 0$ such that $|f(x + h) - f(x)| \leq M|h|$, for all $x, x + h$ inside the domain of the function f. (Can you think of a convex function which is not Lipschitz continuous? How about the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = x^2$?; note that this function is Lipschitz continuous on any bounded interval).

In fact, much more is true. A convex function is actually differentiable at all but at most countably many points.

A differentiable function is convex if and only if its derivative is monotonically increasing. Moreover, if a function is both differentiable and convex, it is continuously differentiable, that is, its derivative is continuous (feel free to try proving these facts).

Convexity and the second derivative

It follows that a twice differentiable function on an interval will be convex if its second derivative is everywhere non-negative. If the second derivative is positive, the function will be strictly convex.

However, the converse of the second statement above is not true. Can you give a counter-example to the converse of the second statement?

How about $f(x) = x^4$?

Definition: A point of inflection x_0 for a function f is a point where the function changes its behavior from concave to convex (or vice-versa). At such a point $f''(x_0) = 0$, but this is only a necessary, not a sufficient condition. (Why?) If further, we also assume that the lowest order (≥ 2) of the non-zero derivatives is odd, then we get a sufficient condition.