MA 105 Calculus II

Week 5

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A proof of Green's theorem for regions of special type

We give a proof of Green's theorem when the region D is both of type 1 and type 2 .

Examples: Rectangles, Discs are examples of such region.

Assume that D is of Type 1

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

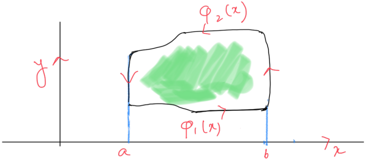
for some continuous functions ϕ_1 and ϕ_2.

Also assume there exist two continuous functions ψ_1 and ψ_2 such that D can be written as Type 2:

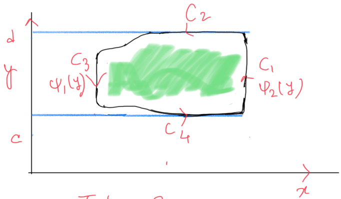
$$D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \quad \psi_1(y) \leq x \leq \psi_2(y)\}.$$

The proof follows two main steps:

- Double integrals can be reduced to iterated integrals.
- Then the fundamental theorem of calculus can be applied to the resulting one-variable integrals.



Type 1.



Type 2.

The proof of Green's theorem, contd.

Step 1: Using the fact that D is a region of Type 2,

$$\iint_D \frac{\partial F_2}{\partial x} = \int_{\partial D} F_2 dy.$$

Step 2: Using the fact that D is a region of Type 1,

$$-\iint_D \frac{\partial F_1}{\partial y} = \int_{\partial D} F_1 dx.$$

Then combining the both equalities, we get our result.
Since D is a region of Type 2, it gives

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y) dxdy.$$

Using the Fundamental Theorem of Calculus we get

$$\int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y) dxdy = \int_c^d F_2(\psi_2(y), y) - F_2(\psi_1(y), y) dy$$

The proof of Green's theorem contd.

Now let us calculate $\int_{\partial D} F_2 dy$. Note that ∂D can be written as union of four curves C_1, C_2, C_3 and C_4 such that

On C_1: $C_1 = \{(\psi_2(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction upwards. So,

$$\int_{C_1} F_2\, dy = \int_c^d F_2(\psi_2(y), y)\, dy.$$

On C_3: $C_3 = \{(\psi_1(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction downwards. So,

$$\int_{C_3} F_2\, dy = -\int_{-C_3} F_2\, dy = -\int_c^d F_2(\psi_1(y), y)\, dy.$$

On C_2 and C_4: $C_2 = \{(x, d) \mid \psi_1(d) \leq x \leq \psi_2(d)\}$ going from right to left and $C_4 = \{(x, c) \mid \psi_1(c) \leq x \leq \psi_2(c)\}$ going from left to right. In particular, they are vertical lines and y is constant along these lines. Thus, for any parametrisation of C_2 and C_4, $\frac{dy}{dt} = 0$, and

$$\int_{C_2} F_2\, dy = 0 = \int_{C_4} F_2\, dy.$$

The proof of Green's theorem contd.

$$\int_{\partial D} F_2 dy = \int_{C_1} F_2\, dy + \int_{C_2} F_2\, dy + \int_{C_3} F_2\, dy + \int_{C_4} F_2\, dy,$$

and using previous results, we obtain

$$\int_{\partial D} F_2 dy = \int_c^d F_2(\psi_2(y), y)\, dy - \int_c^d F_2(\psi_1(y), y)\, dy,$$

and thus

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_{\partial D} F_2 dy.$$

Similarly, using that D can be written as a region of Type 1, we get

$$\iint_D \frac{\partial F_1}{\partial y} dxdy = -\int_{\partial D} F_1 dx.$$

Subtracting the two equations above, we get

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_{\partial D} F_1 dx + \int_{\partial D} F_2 dy.$$

A more general case

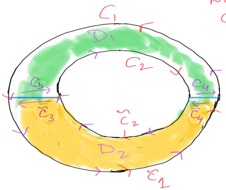
How does one proceed in general, that is for more general regions which may not be of both type 1 and type 2. We can try proceeding as follows:

- Break up D into smaller regions each of which is of both type 1 and type 2 but so that any two pieces meet only along the boundary.
- Apply Green's theorem to each piece.
- Observe that the line integrals along the interior boundaries cancel, leaving only the line integral around the boundary of D.

$D = D_1 \cup D_2$, $\partial D = C_1 \cup C_2$.

$\partial D_1 = C_1 \cup C_3 \cup C_2 \cup C_4$, $\partial D_2 = \tilde{C}_1 \cup \tilde{C}_4 \cup \tilde{C}_2 \cup \tilde{C}_3$

Note:
$C_3 = -\tilde{C}_3$
$C_4 = -\tilde{C}_4$



Del operator on vector fields

The del operator operates on vector fields as in two different ways. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ we define the curl of $\mathbf{F}$:

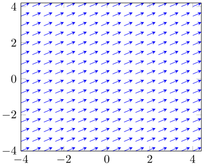
$$\operatorname{curl}\mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

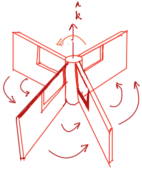
It is often written as a determinant;

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

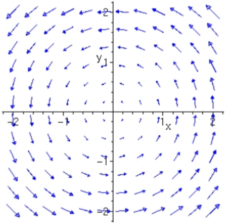
Curl as a measure of rotation

Curl of a vector field is measuring the extent to which the field rotate a particle. For instance ,

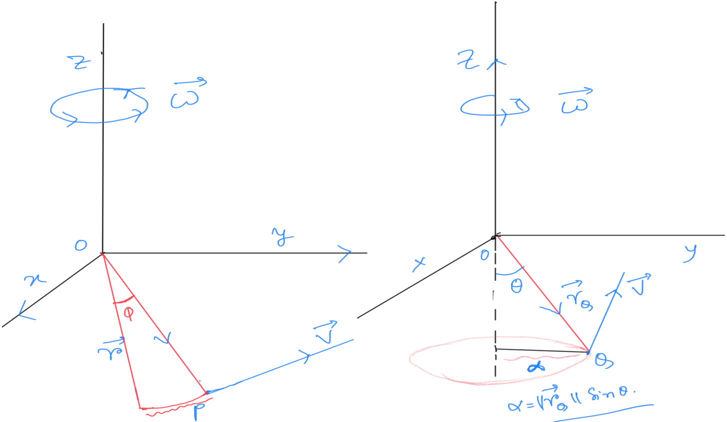




Imagine putting a small paddle wheel as shown in the above figure at any point in the plane with the vector field acting on it and visualize how it will rotate. Clearly in this example it will not rotate.



Angular velocity



$$\vec{V} = \vec{\omega} \times \vec{r}.$$

$$\alpha = \|\vec{r}_\Theta\| \sin\theta.$$

Angular velocity

Consider a solid body B rotating around the z-axis on the xy-plane.

Let $\mathbf{v}$ denote the velocity vector, $\mathbf{w}$ the angular velocity vector at a point $\mathbf{r}$ in B. Note $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed. Further, $\|v\| = \|w\|\|r\| \sin\theta$ where θ is the angle made by $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ with the axis of rotation.

Then $\mathbf{v} = \mathbf{w} \times \mathbf{r} = -\omega y\mathbf{i} + \omega x\mathbf{j}$. Check?

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\mathbf{w}.$$

Thus, the curl of velocity is twice the angular velocity.

If a vector field $\mathbf{F}$ represents the flow of a fluid, then the value of $\nabla \times \mathbf{F}$ at a point is twice the rotation vector of a rigid body that rotates as the fluid does near that point. In particular, $\nabla \times \mathbf{F} = 0$ at a point P means that the fluid is free from the rigid rotations at P.

The curl free vector field is called irrotational field.

The curl of a gradient

Suppose that $\mathbf{F} = \nabla f$ for some scalar function f and f is $\mathcal{C}^2$. Then

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k}.$$

Clearly,

$$\nabla \times \mathbf{F} = 0.$$

In particular, this gives a criterion for deciding whether a vector field arises as the gradient of a function. This gives that curl$\mathbf{F} = 0$ is a necessary condition for any smooth vector field $\mathbf{F}$ to be the gradient field.

Is the condition $\nabla \times \mathbf{F} = 0$ sufficient for $\mathbf{F}$ to be a gradient field?

Recall that we have previously looked at the vector field

$$\mathbf{F} = \frac{y}{x^2+y^2} \cdot \mathbf{i} + \frac{-x}{x^2+y^2} \cdot \mathbf{j},$$

Exercise 1: Check that $\nabla \times \mathbf{F} = 0$.
Can you express $\mathbf{F}$ as the gradient of a suitable scalar function? Ans. No!

We can conclude that the Image of ∇ operator on scalar functions defined on $D \subset \mathbb{R}^3$ is a proper subset of

$$\ker(\text{curl}) = \{\mathbf{F} \text{ is a vector field on } D \mid \text{curl}\mathbf{F} = 0\}.$$

Definition (Scalar curl:) If $\mathbf{F} := (F_1, F_2)$ (a vector field in $\mathbb{R}^2$), then we define the curl of F by thinking of it as a vector field in $\mathbb{R}^3$ on the x-y plane with $F_3 = 0$.

$$\operatorname{curl} \mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}.$$

The function $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$ is called the scalar curl of $\mathbf{F}$.

We can now state a vector valued version of Green's theorem using curl.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be a C^1 vector field on an open connected region D with ∂D be positively oriented. Then

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{ds} = \iint_D (\operatorname{curl} F \cdot \mathbf{k}) \; dxdy.$$

Note that given a gradient vector field $\mathbf{F} = \nabla f$, its curl will be 0. Why?

Other forms of Green's theorem in $\mathbb{R}^2$

Under the hypothesis on the region D and the functions F_1 and F_2 as stated in Green's theorem, we have

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

We assume that ∂D can be parametrised by a single curve - otherwise break up the curve into parametrisable pieces.

Let ∂D be a non-singular, positively oriented curve in $\mathbb{R}^2$, parametrized by $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$. Then the unit tangent to the curve $\mathbf{c}$ and the unit outward normal to the curve are denoted by

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \quad \mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}, \quad \forall\, t \in [a, b].$$

Other form Green's theorem

As consequences of Green's theorem in $\mathbb{R}^2$, we have following results:

Curl form or Tangential form

$$\int_{\partial D} \mathbf{F}.\mathbf{T} ds = \int\int_{D} (\text{curl}\mathbf{F}).\mathbf{k} dxdy.$$

The Stokes theorem is a 3-dimensional version of the above result.

Outline of its proof: Considering $\mathbf{F} = (F_1, F_2, 0)$ and noting that $ds = \|c'(t)\| dt$, we get

$$\int_{\partial D} \mathbf{F}.\mathbf{T} ds = \int_{\partial D} \mathbf{F}(\mathbf{c}(t)).\mathbf{c}'(t) dt = \int_{\partial D} F_1 dx + F_2 dy.$$

Now using Green's theorem and noting $\text{curl}\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$, the result follows.

Conservative field and its curl in $\mathbb{R}^2$

Theorem

1. *Let Ω be an open, simply connected region in $\mathbb{R}^2$.*
2. *if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on Ω.*

Then $\mathbf{F}$ is a conservative field in Ω if and only if

$$\mathit{curl}\,\mathbf{F} = 0, \quad \text{in} \quad \Omega.$$

Outline of the proof: Let the assumptions on Ω and $\mathbf{F}$ in the statement hold.

- If $\mathbf{F}$ is C^1 and a conservative field, i.e., $\mathbf{F} = \nabla f$, for some f is C^2. Then a direct calculation gives curl $F = 0$.
- Now conversely, if $\mathbf{F}$ is C^1 and curl $\mathbf{F} = 0$ on Ω. Then by Green's theorem we can show that the line integral of F over any simple closed curve is 0. That is, the line integral of F in Ω is path independent. Hence the result follows.

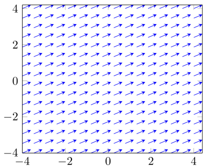
The divergence of a vector field

The del operator can be made to operate on vector fields to give a scalar function as follows.

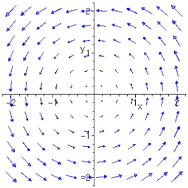
Definition: Let $\mathbf{F} = (F_1, F_2, F_3)$ be a vector field. The divergence of $\mathbf{F}$ is the scalar function defined by

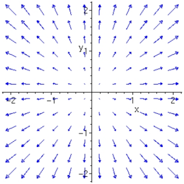
$$\operatorname{div} \mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

One way to interpret divergence of a velocity vector field at a point P as the amount of fluid flowing in versus the amount of fluid flowing out.



If $\mathbf{F}$ is a constant vector field then at any point what is flowing in is flowing out and the divergence is 0.





Is the divergence for this vector field 0?

This should have non-zero divergence. But what is it measuring?

Physical interpretation

If $\mathbf{F}$ is the velocity field of a fluid, the divergence of $\mathbf{F}$ gives the rate of expansion of the volume of the fluid per unit volume as the volume moves with the flow. In the case of planar vector fields we get the corresponding rate of expansion of area.

Example : $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. The flow lines of this vector field point radially outward from the origin, so it is clear that the fluid is expanding as it flows. This is reflected in the fact that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0.$$

Example : If we look at the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$, we see that $\nabla \cdot \mathbf{F} = -2$. This is consistent with the fact that the flow lines of the vector field all point towards the origin, and the fluid is getting compressed.

Example : $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. In this case the fluid is moving counterclockwise around the origin - so it is neither being compressed, nor is it expanding. One checks easily that $\nabla \cdot \mathbf{F} = 0$.

The change in area in a flow

Let us assume the vector field $\mathbf{v} = (u, v)$ represents the velocity field of a fluid in $\mathbb{R}^2$. Let us compute the rate of change of unit area of the fluid as it flows along the curve.

We assume that we start at time $t = 0$ at $P = (x, y)$. Let the point evolve under the velocity field $\mathbf{v}$ to a point (X, Y) at time t. In particular,

$$X = X(x, y, t), Y = Y(x, y, t).$$

The change of variables formula tells us how an elementary area changes. Computing the Jacobian determinant for mapping $h(x, y) = (X, Y)$

$$J(x, y, t) = \begin{vmatrix} \frac{\partial X}{\partial x}(x, y, t) & \frac{\partial X}{\partial y}(x, y, t) \\ \frac{\partial Y}{\partial x}(x, y, t) & \frac{\partial Y}{\partial y}(x, y, t) \end{vmatrix}.$$

Now computing $\frac{\partial J(x,y,t)}{\partial t}$,

$$\begin{aligned} \frac{\partial J}{\partial t} &= \frac{\partial^2 X}{\partial x \partial t}\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial x}\frac{\partial^2 Y}{\partial y \partial t} - \left(\frac{\partial^2 X}{\partial y \partial t}\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial^2 Y}{\partial x \partial t}\right) \\ &= (\nabla \cdot \mathbf{v})J. \end{aligned}$$

Divergence free is area preserving

Putting $\frac{\partial X}{\partial t}(x,y,t) = u(X(x,y,t), Y(x,y,t))$ and $\frac{\partial Y}{\partial t}(x,y,t) = v(X(x,y,t), Y(x,y,t))$, $\frac{\partial J}{\partial t}$ is equal to

$$\frac{\partial J}{\partial t} = (\nabla \cdot \mathbf{v})J.$$

Thus, $\nabla \cdot \mathbf{v} = 0$ if and only if J is independent of t. Since at $t = 0$, $J(x,y,0) = 1$, $J(x,y,t) = J(x,y,0) = 1$, for all t. There is no change of coordinates and hence Jacobian is trivial.
Clearly, $J = 1$ means that there is no change in the area,

$$Area(D) = \iint_D dXdY = \iint_{D'} |J(x,y)| dxdy = \iint_{D'} dxdy = Area(D').$$

The divergence free vector field is called incompressible field.

The divergence of any curl is zero. In other words, if $\mathbf{G}$ is a $\mathcal{C}^2$ vector field,

$$\operatorname{div}(\operatorname{curl}\mathbf{G}) = \nabla\cdot(\nabla\times\mathbf{G}) = 0.$$

Qn : If $\nabla\cdot\mathbf{F} = 0$, does it imply that $\mathbf{F} = \nabla\times\mathbf{G}$ for some vector field $\mathbf{G}$?

This question is related to the topological properties to of the domain of the vector field as in the case of when a curl free vector field is a gradient vector field. We will be able to show that this is the case when the domain is $\mathbb{R}^n$ for $n = 2, 3$. We postpone it for later.

Next, we mention the Divergence theorem in $\mathbb{R}^2$:
Let ∂D be a non-singular, positively oriented curve in $\mathbb{R}^2$, parametrized by $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$. Then the unit tangent to the curve $\mathbf{c}$ and the unit outward normal to the curve are denoted by

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \quad \mathbf{n}(t) = \mathbf{T}(t)\times\mathbf{k}, \quad \forall\, t\in[a, b].$$

Divergence form of Green's theorem

Divergence form or Normal form:

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int\int_D \mathrm{div}\mathbf{F} dxdy.$$

Gauss's divergence theorem is a 3-dimensional analogue of the above.

Outline of its proof: Since $\mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}$, for all $t \in [a, b]$, using the definition of $\mathbf{c}(t)$, we get $\mathbf{n}(t) = \left(\frac{y'(t)}{\|\mathbf{c}'(t)\|}, \frac{-x'(t)}{\|\mathbf{c}'(t)\|}, 0\right)$. Thus, for $\mathbf{F} = (F_1, F_2, 0)$, using $ds = \|\mathbf{c}'(t)\| dt$

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int_{\partial D} \left[F_1(\mathbf{c}(t))\frac{y'(t)}{\|\mathbf{c}'(t)\|} - F_2(\mathbf{c}(t))\frac{x'(t)}{\|\mathbf{c}'(t)\|}\right] ds$$
$$= \int_{\partial D} [F_1(\mathbf{c}(t))y'(t) - F_2(\mathbf{c}(t))x'(t)] dt = \int_{\partial D} F_1\, dy - F_2\, dx.$$

Now by Green's theorem, we get

$$\int_{\partial D} F_1\, dy - F_2\, dx = \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dxdy = \iint_D \mathrm{div}\mathbf{F} dxdy.$$

Physical interpretation of Divergence theorem

We can interpret the above theorem in the context of fluid flow. If $\mathbf{F}$ represents the flow of a fluid, then the left hand side of the divergence theorem represents the net flux of the fluid across the boundary ∂D. On the other hand, the right hand side represents the integral over D of the rate $\nabla \cdot \mathbf{F}$ at which fluid area is being created. In particular if the fluid is incompressible (or, more generally, if the fluid is being neither compressed nor expanded) the net flow across ∂D is zero.

We can talk about volume analogously in the three dimensional case after proving Stokes theorem.

For further studies on curl and divergence of a vector field with physical applications, you can check this source:
https://math.libretexts.org/Bookshelves/Calculus/Book

Surfaces : Definition

A curve is a "one-dimensional" object. Intuitively, this means that if we want to describe a curve, it should be possible to do so using just one variable or parameter.

To do line integration, we further required some extra properties of the curve - that it should be $\mathcal{C}^1$ and non-singular.

We will now discuss the two dimensional analog, namely, surfaces. In order to describe a surface, which is a two-dimensional object, we clearly need two parameters.

Definition

Let D be a path connected subset in $\mathbb{R}^2$. A parametrised surface is a continuous function $\mathbf{\Phi} : D \to \mathbb{R}^3$.

This definition is the analogue of what we called paths in one dimension and what are often called parametrized curves.

Geometric parametrised surfaces

As with curves and paths, we will distinguish between the surface $\mathbf{\Phi}$ and its image. Similarly, the image $S = \mathbf{\Phi}(D)$ will be called the geometric surface corresponding to $\mathbf{\Phi}$.

Note that for a given $(u, v) \in D$, $\mathbf{\Phi}(u, v)$ is a vector in $\mathbb{R}^3$. Each of the coordinates of the vector depends on u and v. Hence we write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are scalar functions on D.

The parametrized surface $\mathbf{\Phi}$ is said to be a smooth parametrized surface if the functions x, y, z have continuous partial derivatives in a open subset of $\mathbb{R}^2$ containing D.

Examples

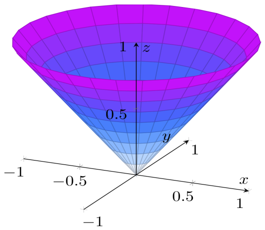
Example 1: Graphs of real valued functions of two independent variables are parametrised surfaces.
Let $f(x, y)$ be a scalar function and let $z = f(x, y)$, for all $(x, y) \in D$, where D is a path connected region in $\mathbb{R}^2$. We can define the parametrised surface $\mathbf{\Phi}$ by

$$\mathbf{\Phi}(u, v) = (u, v, f(u, v)), \quad \forall\, (u, v) \in D.$$

More specifically, we have $x(u, v) = u$, $y(u, v) = v$ and $z(u, v) = f(u, v)$.

Example 2: Consider the cylinder, $x^2 + y^2 = a^2$. Then this is parametrized surface defined by $\mathbf{\Phi} : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$ defined as
$\mathbf{\Phi}(u, v) = (a \cos u, a \sin u, v)$.

Example 3: Consider the sphere of radius a,
$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Is it a parametrized surface? Recall using spherical coordinates we can represent it using the following parametrization, $\mathbf{\Phi} : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ defined as
$\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v)$.



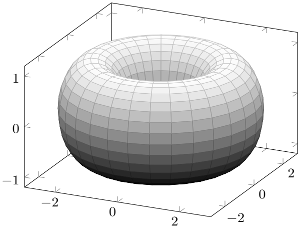
Example 4: The graph of $z = \sqrt{x^2 + y^2}$ can also be parametrized. We use the idea that at each value of z we get a circle of radius z. We can describe the cone as the parametrized surface $\mathbf{\Phi} : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^3$ as $\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u)$.

Example 5: If we have parametrized curve on the z-y-plane $(0, y(u), z(u))$ which we rotate around z-axis, we can parametrise it as follows:

$$x = y(u) \cos v, \quad y = y(u) \sin v, \quad \text{and} \quad z = z(u).$$

Here $a \leq u \leq b$ if $[a, b]$ is the domain of the curve, and $0 \leq v \leq 2\pi$.

Surfaces of revolution around the z-axis



For instance we can parametrize a torus by taking a circle in the y-z plane with center $(0, a, 0)$ of radius b. This is given by the curve $(0, a + b\cos u, b\sin u)$.

Then the parametrization of the torus is then $\mathbf{\Phi}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.

Parametrised surfaces are more general than graphs of functions.

Tangent vectors for a parametrised surface

Let $\mathbf{\Phi}(u, v)$ be a smooth parametrised surface. If we fix the variable v, say $v = v_0$, we obtain a curve $\mathbf{c}(u, v_0)$ that lies on the surface. Thus

$$\mathbf{c}(u) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

Since this curve is C^1 we can talk about its tangent vector at the point u_0. This is given by

$$\mathbf{c}'(u_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

We can *define* the partial derivative of a vector valued function as

$$\mathbf{\Phi}_u(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0) := \mathbf{c}'(u_0).$$

Similarly, by fixing u and varying v we obtain a curve $\mathbf{l}(u_0, v)$ and we can set

$$\mathbf{\Phi}_v(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0) := \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

The tangent plane

Let for any given point on the surface, $P_0 = (x_0, y_0, z_0) := \mathbf{\Phi}(u_0, v_0)$ for some $(u_0, v_0) \in D$.
The two tangent vectors $\mathbf{\Phi}_u(u_0, v_0)$ and $\mathbf{\Phi}_v(u_0, v_0)$ at P_0 define a plane. We call this plane as the tangent plane to the surface at P_0.

The normal to this plane at P_0, $\mathbf{n}(u_0, v_0) = \mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)$.

Thus for a given point $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ in $\mathbb{R}^3$ the equation of the tangent plane is given by

$$\mathbf{n}(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

provided $\mathbf{n} \neq 0$.

In particular, if $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0) = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the equation of the tangent plane at P_0 is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Let us find the equation of the tangent plane at points on the various parametrised surfaces we have already looked at.

Example 1: Let D be a path-connected subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a C^1 function. The surface given by the graph of the function $z = f(x, y)$ is parametrized by $\mathbf{\Phi}(x, y) = (x, y, f(x, y))$. In this case, at $P_0 = \mathbf{\Phi}(x_0, y_0)$ for $(x_0, y_0) \in D$,

$$\mathbf{\Phi}_x(x_0, y_0) = \mathbf{i} + \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_y(x_0, y_0) = \mathbf{j} + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{k}.$$

Hence,

$$\mathbf{n}(x_0, y_0) = \mathbf{\Phi}_x(x_0, y_0) \times \mathbf{\Phi}_y(x_0, y_0) = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right).$$

Thus the equation of the tangent plane is

$$(x - x_0, y - y_0, z - z_0) \cdot \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right) = 0;$$

which yields,

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Tangent Plane: Examples

Example 2: Let us consider a cylinder parametrized as

$$\mathbf{\Phi}(u,v) = (a\cos u, a\sin u, v), \quad \forall (u,v) \in [0,2\pi]\times[0,h],$$

where $a > 0$. Then

$$\mathbf{\Phi}_u(u,v) \times \mathbf{\Phi}_v(u,v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos u, a\sin u, 0).$$

Since this is non-zero on $[0,2\pi]\times[0,h]$ for any $h > 0$, we can define the tangent plane to $\mathbf{\Phi}$ at any point $P_0 = (x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ as

$$(a\cos u_0, a\sin u_0, 0).(x - x_0, y - y_0, z - z_0) = 0.$$

Now using $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0) = (a\cos u_0, a\sin u_0, v_0)$, we get the equation for the tangent plane to $\mathbf{\Phi}$ at P_0 is

$$(\cos u_0)x + (\sin u_0)y = a.$$

Example 3: The sphere: $x^2 + y^2 + z^2 = a^2$, for some $a > 0$. Let us consider the parametrization

$$\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v), \quad \forall (u, v) \in [0, 2\pi] \times [0, \pi].$$

Check $\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_u(u, v) = (a \sin v)\mathbf{\Phi}(u, v)$, for all $(u, v) \in [0, 2\pi] \times [0, \pi]$.

Note for $(u_0, v_0) \in [0, 2\pi] \times (0, \pi)$, $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_u(u_0, v_0) \neq (0, 0, 0)$ and the tangent plane at $P_0 = \mathbf{\Phi}(u_0, v_0)$ is

$$(\sin v_0 \cos u_0)x + (\sin v_0 \sin u_0)y + (\cos v_0)z = a.$$

Example 4: This was the example of the right circular cone. The parametric surface was given by

$$\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u), \quad (u, v) \in [0, \infty) \times [0, 2\pi].$$

In this case we get

$$\mathbf{\Phi}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j},$$

where $\mathbf{n}(u, v) = \mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v) = (-u \cos v, -u \sin v, u)$.

For any $(u_0, v_0) \in (0, \infty) \times [0, 2\pi]$, $\mathbf{n}(u_0, v_0) \neq (0, 0, 0)$ and the tangent plane check

$$(\cos v_0)x + (\sin v_0)y = z.$$

Note that if $(u, v) = (0, 0)$, then $\mathbf{n}(0, 0) = 0$, so the tangent plane is not defined at the origin. However, it is defined at any other point.

Non-singular surfaces

In analogy with the situation for curves, we will call $\mathbf{\Phi}$ a regular or non-singular parametrised surface if $\mathbf{\Phi}$ is C^1 and $\mathbf{\Phi}_u \times \mathbf{\Phi}_v \neq 0$ at all points.

As we just saw, the right circular cone is not a regular parametrised surface.

For a regular surface parametrized by $\mathbf{\Phi} : D \to \mathbb{R}^3$, the unit normal $\hat{\mathbf{n}}$ to the surface at any point $P_0 = \mathbf{\Phi}(u_0, v_0)$ is defined by

$$\hat{\mathbf{n}}(u_0, v_0) := \frac{\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)}{\|\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)\|}.$$

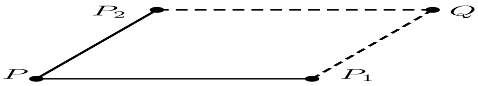
Surface Area

Let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a smooth parametrized surface, where E is a path-connected, bounded subset of $\mathbb{R}^2$ having a non-zero area. Also assume ∂E, the boundary of E, is of content zero.

Let $(u, v) \in E$. For $h, k \in \mathbb{R}$ with $|h|, |k|$ small, assuming $\mathbf{\Phi}$ is C^1 we can get the following approximations;

$$P := \mathbf{\Phi}(u, v), \quad P_1 := \mathbf{\Phi}(u + h, v) \approx \mathbf{\Phi}(u, v) + h\, \mathbf{\Phi}_u(u, v),$$

$$P_2 := \mathbf{\Phi}(u, v + k) \approx \mathbf{\Phi}(u, v) + k\, \mathbf{\Phi}_v(u, v), \quad Q := \mathbf{\Phi}(u + h, v + k).$$



Area of the parallelogram with sides PP_1 and PP_2

$$= \|(P_1 - P) \times (P_2 - P)\| \approx \|\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v)\|\, |h||k|.$$

In view of this approximation, we define

$$\text{Area}(\mathbf{\Phi}) := \iint_E \|(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u,v)\|\, du\, dv.$$

Since the subset E of $\mathbb{R}^2$ is bounded with boundary ∂E which is of content zero and the function $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$ is continuous on E, the integral in the definition of Area($\mathbf{\Phi}$) is well-defined.

In analogy with the differential notation $ds = \|\gamma'(t)\| dt$, we introduce the following differential notation:

$$dS = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\, dudv.$$

Thus $\text{Area}(\mathbf{\Phi}) := \iint_E dS$.

Examples

• Graph of a function: Given a subset E of $\mathbb{R}^2$ have an area, $f : E \to \mathbb{R}$ be a smooth function, and $\mathbf{\Phi}(u,v) = (u, v, f(u,v))$ for $(u,v) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|(-f_u, -f_v, 1)\| dudv \\ &= \iint_E \sqrt{1 + f_u^2 + f_v^2}\, dudv \end{aligned}$$

Example: Let $E := [0, 2\pi] \times [0, h]$, $\mathbf{\Phi}(\theta, z) := (a\cos\theta, a\sin\theta, z)$, and $\mathbf{\Psi}(\theta, z) := (a\cos 2\theta, a\sin 2\theta, z)$ for $(\theta, z) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|\mathbf{\Phi}_\theta \times \mathbf{\Phi}_z\| d\theta dz = \iint_E a\, d\theta dz = 2\pi a\, h, \\ \text{Area}(\mathbf{\Psi}) &= \iint_E \|\mathbf{\Psi}_\theta \times \mathbf{\Psi}_z\| d\theta dz = \iint_E 2a\, d\theta dz = 4\pi a\, h. \end{aligned}$$

We note that $\mathbf{\Psi}(E) = \mathbf{\Phi}(E)$, but $\text{Area}(\mathbf{\Psi}) = 2\,\text{Area}(\mathbf{\Phi})$.

Example: Let $E := [0, \pi] \times [0, 2\pi]$, and $\mathbf{\Phi}(\varphi, \theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|\mathbf{\Phi}_\varphi \times \mathbf{\Phi}_\theta\| d\varphi d\theta = \iint_E a^2 \sin\varphi\, d\varphi d\theta \\ &= \int_0^{2\pi} \left(\int_0^{\pi} a^2 \sin\varphi\, d\varphi \right) d\theta = 4\pi a^2. \end{aligned}$$

Let C be a smooth curve in $\mathbb{R}^2 \times \{0\}$ given by $\boldsymbol{\gamma}(t) := (x(t), y(t))$, $t \in [\alpha, \beta]$. If C lies on or above the x-axis, and C is revolved about the x-axis, then it generates a surface parametrized by

$$\boldsymbol{\Phi}(t,\theta) := (x(t),\ y(t)\cos\theta,\ y(t)\sin\theta) \quad \text{for } (t,\theta) \in E,$$

where $E := [\alpha, \beta] \times [0, 2\pi]$. For all $(t, \theta) \in E$,

$$\begin{aligned}(\boldsymbol{\Phi}_t \times \boldsymbol{\Phi}_\theta)(t,\theta) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta \\ 0 & -y(t)\sin\theta & y(t)\cos\theta \end{vmatrix} \\ &= (y(t)y'(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta).\end{aligned}$$

By the Fubini theorem, we obtain

$$\begin{aligned}\text{Area}(\boldsymbol{\Phi}) &= \iint_E \sqrt{y(t)^2 y'(t)^2 + x'(t)^2 y(t)^2}\, d(t,\theta) \\ &= 2\pi \int_\alpha^\beta y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt,\end{aligned}$$

Note: $\boldsymbol{\Phi}$ is non-singular $\iff$ $\boldsymbol{\gamma}$ is non-singular and $y(t) \neq 0$ for $t \in [\alpha, \beta]$.

The area vector of an infinitesimal surface element

We see that $\mathbf{\Phi}$ takes the small rectangle R to the parallelogram given by the vectors $\mathbf{\Phi}_u \Delta u$ and $\mathbf{\Phi}_v \Delta v$.

It follows that the 'area vector' $\Delta \mathbf{S}$ of this parallelogram is

$$\Delta \mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) \Delta u \Delta v.$$

Thus the surface 'area vector' is to be thought of as a vector pointing in the direction of the normal to the surface and in differential notation:

$$d\mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v)\, du\, dv.$$

The magnitude of the surface 'area vector' is given by

$$dS = \|d\mathbf{S}\| = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\, du\, dv.$$

If the parametric surface $\mathbf{\Phi}$ is non-singular, we can write

$$d\mathbf{S} = \hat{\mathbf{n}} dS,$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface.

The magnitude of the area vector

It remains to compute the magnitude dS. To do this we must find $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$. Writing this out in terms of x, y and z, we see that

$$d\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv.$$

$$dS = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

where $\frac{\partial(y,z)}{\partial(u,v)}$, $\frac{\partial(x,z)}{\partial(u,v)}$, $\frac{\partial(x,y)}{\partial(u,v)}$ are the determinant of corresponding Jacobian matrix. For example

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v},$$

$$\frac{\partial(x,z)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

The surface area integral

Because of the calculations we have just made, the surface area is given by the double integral

$$\iint_S dS = \iint_E \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

The area is nothing but the integral of the constant function 1 on the surface S. We integrate any bounded scalar function $f : S \to \mathbb{R}$:

$$\iint_S fdS = \iint_E f(x,y,z)\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

provided the R.H.S double integral exists. If Σ is a union of parametrised surfaces S_i that intersect only along their boundary curves, then we can define

$$\iint_\Sigma fdS = \sum_i \iint_{S_i} fdS.$$

The surface integral of a vector field

Let $\mathbf{F}$ be a bounded vector field (on $\mathbb{R}^3$) such that the domain of $\mathbf{F}$ contains the non-singular parametrised surface $\mathbf{\Phi} : E \to \mathbb{R}^3$. Then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\mathbf{\Phi}(u, v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv,$$

provided the R.H.S double integral exists. This can also be written more compactly as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

which is the surface integral of the scalar function given by the normal component of $\mathbf{F}$ over S.

Examples

(i) Let a subset E of $\mathbb{R}^2$ have an area, and let $f : E \to \mathbb{R}$ be a smooth function. Let the smooth parametrized surface $\varPhi : E \to \mathbb{R}^3$ represent the graph of f, and let $\boldsymbol{F} : \boldsymbol{\Phi}(E) \to \mathbb{R}^3$ be a continuous vector field. If $\boldsymbol{F} := (P, Q, R)$, then

$$\iint_{\boldsymbol{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_E (-P\, f_x - Q\, f_y + R) d(x, y)$$

since $d\boldsymbol{S} = (\boldsymbol{\Phi}_x \times \boldsymbol{\Phi}_y) dxdy = (-f_x, -f_y, 1) dxdy$.

Using above result, let $E := [0, 1] \times [0, 1]$, $f(x, y) := x + y + 1$ for $(x, y) \in E$. If $\boldsymbol{F}(x, y, z) := (x^2, y^2, z)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\begin{aligned} \iint_{\varPhi} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_E \big(- x^2 - y^2 + (x + y + 1)\big) d(x, y) \\ &= \int_0^1 \left(\int_0^1 (x + y + 1 - x^2 - y^2) dy \right) dx \\ &= = \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}. \end{aligned}$$

Examples Contd.

(ii) Let $E := [0, 2\pi] \times [0, h]$, and $\mathbf{\Phi}(u, v) := (a\cos u, a\sin u, v)$ for $(u, v) \in E$. If $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\boldsymbol{S} = \iint_E (a^2 \cos u \sin u + v\, a \sin u + 0) du dv = 0,$$

since $d\boldsymbol{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv = (a\cos u, a\sin u, 0) du dv$.