MA 105 : Calculus

D1 - Lecture 9

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The Taylor Series: Some Notation

We will first introduce some notation.

The space $\mathcal{C}^k(I)$ will denote the space of k times continuously differentiable functions on an interval I, for some fixed $k \in \mathbb{N}$, that is, the space of functions for which k derivatives exist and such that the k-th derivative is a continuous functions.

The space $\mathcal{C}^\infty(I)$ will consist of functions that lie in $\mathcal{C}^k(I)$ for every $k \in \mathbb{N}$. Such functions are called smooth or infinitely differentiable functions.

From now on we will denote the k-th derivative of a function $f(x)$ by $f^{(k)}(x)$.

Our aim will be to enlarge the class of functions we understand using the polynomials as stepping stones.

The Taylor polynomials

Given a function $f(x)$ which is n times differentiable at some point x_0 in an interval I, we can associate to it a family of polynomials $P_0(x), P_1(x), \ldots, P_n(x)$ called the Taylor polynomials of order $0, 1, \ldots, n$ at x_0 as follows.

We let $P_0(x) = f(x_0)$,

$$P_1(x) = f(x_0) + f^{(1)}(x_0)(x - x_0),$$

$$P_2(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2,$$

we can continue in this way to define

$$P_n(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Taylor's Theorem

The Taylor polynomials are rigged exactly so that the order n Taylor polynomial has the same first n derivatives at the point x_0 as the function $f(x)$ has, that is, $P^{(k)}(x_0) = f^{(k)}(x_0)$ for all $0 \leq k \leq n$, where $f^{(0)}(x_0) = f(x_0)$ by convention.

Taylor's Theorem says that we can recover a lot of information about the function from the Taylor polynomials.

Theorem 19: Let $f \in \mathcal{C}^n(I)$ for some open interval I containing a, and suppose that $f^{(n+1)}$ exists on this interval. Then for each $b \neq a \in I$, there exists c between a and b such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

where P_n denotes the Taylor polynomial of order n at a.

It is customary to denote the function $f(b) - P_n(b)$ by $R_n(b)$.

Taylor's Theorem gives us a simple formula for $R_n(b)$. If we can make $R_n(b)$ small, we can approximate our function $f(x)$ by a polynomial.

The proof of Taylor's theorem

Proof: Consider the function

$$F(x) = f(b) - f(x) - f^{(1)}(x)(b-x) - \frac{f^{(2)}(x)}{2!}(b-x)^2 - \cdots - \frac{f^{(n)}(x)}{n!}(b-x)^n.$$

Clearly $F(b) = 0$, and

$$F^{(1)}(x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!} \tag{1}$$

Observe that $F(a) = f(b) - P_n(b)$ which need not be zero in general but it is zero if f is a polynomial of degree less or equal to n (as in this case $P_n(b) = f(b)$) and in this case $f^{(n+1)}(c) = 0$ for all c and the theorem gets proved!

We compute $F(a)$ by using the Rolle's Theorem. For, we consider

$$g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^{n+1} F(a)$$

(this is similar to the method by which we proved the MVT using Rolle's Theorem), and we see that $g(b) = g(a) = 0$.

Applying Rolle's Theorem we see that there is a c between a and b such that $g'(c) = 0$.

This yields

$$F^{(1)}(c) = -(n+1)\left(\frac{(b-c)^n}{(b-a)^{n+1}}\right)F(a) \tag{2}$$

We can eliminate $F^{(1)}(c)$ using (1). This gives

$$-(n+1)\left(\frac{(b-c)^n}{(b-a)^{n+1}}\right)F(a) = -\frac{f^{(n+1)}(c)(b-c)^n}{n!},$$

from which we obtain

$$f(b) - P_n(b) = F(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

This proves what we want. $\square$

Remarks on Taylor's Theorem and some examples

Remark 1: When $n = 0$ in Taylor's Theorem we get the MVT. When $n = 1$, Taylor's Theorem is called the Extended Mean Value Theorem.

Remark 2: The Taylor polynomials are nothing but the partial sums of the Taylor Series associated to a $\mathcal{C}^\infty$ function about (or at) the point a:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(b-a)^k.$$

We can show that this series converges to $f(b)$ provided we know that the difference $f(b) - P_n(b) = R_n(b)$ can be made less than any $\epsilon > 0$ when n is sufficiently large. We will see how to do this for certain simple functions like e^x or $\sin x$.

The Taylor series for e^x

Let us show that the Taylor series for the function e^x about the point 0 is a convergent series for any value of $x = b \geq 0$ and that it converges to the value e^b (a similar proof works for $b < 0$).

In this case, at any point a, $f^{(n)}(a) = e^a$, so at $a = 0$ we obtain $f^{(n)}(0) = 1$. Hence the series about 0 is

$$\sum_{k=0}^{\infty} \frac{b^k}{k!}.$$

If we look at $R_n(b) = e^b - P_n(b)$, we obtain

$$|R_n(b)| = \frac{e^c b^{n+1}}{(n+1)!} \leq \frac{e^b b^{n+1}}{(n+1)!},$$

since $c \leq b$. (In case when $b < 0$, we get $b \leq c \leq 0$ and $e^c < 1$).

As $n \to \infty$ this clearly goes to 0. This shows that the Taylor series of e^b converges to the value of the function at each real number b.

Appendix: The ratio test for the convergence of a series

Theorem: Let $\sum_{k=1}^{\infty} a_k$ be an infinite series and let

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L.$$

Then, there are three possibilities:

1. If $L < 1$, then the above series is convergent.
2. If $L > 1$, then the above series is divergent.
3. If $L = 1$, then the test is inconclusive.

Proof: Exercise (Hint: 1. The geometric series $\sum_{k=1}^{\infty} a_N r^{k-1}$ is convergent for $|r| < 1$. 2. If $L > 1$, then $\lim_{k\to\infty} a_k \neq 0$. 3. Try to find an example of a convergent series for which $L = 1$. Also, find an example of a divergent series for which $L = 1$).

Defining functions using Taylor series

Instead of finding the Taylor series of a given function we can reverse the process and define functions using convergent series.

Thus, one can define the function e^x as

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

In this case, we have to first show that the series on the right hand side converges for a given value of x, in which case the definition above makes sense.

We show the convergence of such series by showing that they are Cauchy series, that is, the sequence of the n-th partial sum of the series is a Cauchy sequence. This means that we do not have to guess a value of the limit.

One can use the ratio test too to conclude the convergence of such series!

Power series

As we have explained in the previous slide, the "correct" (both from the point of view of proofs and of computation) way to define a function like e^x is via convergent series involving non-negative integer powers of x.

Such series are called power series and such functions should be viewed as the natural generalizations of polynomials.

The nice thing about power series is that once we know that they converge in some interval $(a - r, a + r)$ around a, it is not hard to show that the functions that they define are continuous functions.

In fact, it is not too hard to show that they are smooth functions (that is, that all their derivatives exist).

Thus when functions are given by convergent power series, we can automatically conclude they are smooth. This is the advantage of defining functions in this way.

Computing the values of functions

A calculator or a computer program calculates the values of various common functions like trigonometric polynomials and expressions in $\log x$ and e^x by using Taylor series.

The great advantage of Taylor series is that one can estimate the error since we have a simple formula for the error which can be easily estimated.

For instance, for the function $\sin x$, the n-th derivative is either $\pm \sin x$ or $\pm \cos x$, so in either case $|f^{(n)}(x)| \leq 1,\ \forall n \geq 1$.

Hence,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

If we take $x = 1$, and we want to compute $\sin 1$ to an error of less than 10^{-16}, we need only make sure that $(n+1)! > 10^{16}$, which is achieved when $n \geq 21$.

Computing the values of $\sin x$ for general $x \in \mathbb{R}$

First, remember that $\sin x$ is periodic, so we only have to look at the values of x between $-\pi$ and π.

But we can do better, because $\sin(-x) = -\sin x$. So we only have to bother about the interval $[0, \pi]$.

We can do still better! Once we know $\sin x$ in $[0, \pi/2]$, we can easily figure out what it is in $[\pi/2, \pi]$.

So finally, it is enough to find the desired value of n for $x \in [0, \pi/2]$.

Computing the values of $\sin x$

We know that the remainder term satisfies

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Hence, we need

$$\frac{|x|^{n+1}}{(n+1)!} < 10^{-16}.$$

We know that it is enough to look at $0 \leq x \leq \pi/2$. Let us be a little careless and allow $x \leq 2$ (so we won't get the best possible n, maybe).

We already know that $1/(n+1)! < 10^{-16}$ if $n \geq 21$. Now $|x|^{22} \leq 2^{22}$. If we take $n = 31$, we see that $|x|^{32} \leq 2^{22} \cdot 2^{10}$,

$$1/(n+1)! = 1/32! < 10^{-16} \cdot 10^{-10} \cdot 2^{-10}.$$

Hence $|R_n(x)| = \frac{|x|^{n+1}}{(n+1)!} < 10^{-16}$ for $n = 31$. $\square$