MA 105 : Calculus

D1 - Lecture 4

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Series

Given a sequence a_n of real numbers, we can construct a new sequence, namely the sequence of partial sums s_n:

$$s_1 = a_1, \ s_2 = a_1 + a_2, \ s_3 = a_1 + a_2 + a_3, \ldots$$

More precisely, we have the sequence

$$s_n = \sum_{k=1}^{n} a_k$$

which is called the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$.

For example, we can define $a_n = r^{n-1}$, for some $r \in \mathbb{R}$ and in this case the series $\sum_{k=1}^{\infty} a_k$ is the geometric progression $\sum_{k=0}^{\infty} r^k$ for which the n-th partial sum $s_n = \sum_{k=0}^{n-1} r^k$.

We say that the series $\sum_{k=1}^{\infty} a_k$ converges if the sequence of the corresponding n-th partial sum converges.

When does the series $\sum_{k=0}^{\infty} r^k$ converge?

Infinite series - a rigorous treatment

Let us recall what we mean when we write, for $|r| < 1$,

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

Another way of writing the same expression is

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

The precise meaning is the following. Form the partial sums

$$s_n = \sum_{k=1}^{n} ar^{k-1}.$$

These partial sums $s_1, s_2, \ldots s_n, \ldots$ form a sequence and by $\sum_{k=1}^{\infty} ar^{k-1} = a/(1-r)$, we mean $\lim_{n\to\infty} s_n = a/1-r$.

So when we speak of the sum of an infinite series, what we really mean is the limit of its partial sums.

Convergence of the geometric series

So to justify our formula we should show that $\lim_{n\to\infty} s_n = a/1 - r$, that is, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon,$$

for all $n > N$. In other words we need to show that

$$\left| \frac{a(1-r^n)}{1-r} - \frac{a}{1-r} \right| = \left| \frac{ar^n}{1-r} \right| < \epsilon$$

if n is chosen large enough. The case $a = 0$ is trivial, so we assume $a \neq 0$. Since $\lim_{n\to\infty} r^n = 0$ (as $|r| < 1$), there exists $N \in \mathbb{N}$ such that $|r|^n < (1-r)\epsilon/|a|$ for all $n > N$, so for this N, if $n > N$,

$$\left| s_n - \frac{a}{1-r} \right| = \left| \frac{ar^n}{1-r} \right| = \frac{|a||r|^n}{(1-r)} < \frac{|a|}{(1-r)} \frac{(1-r)\epsilon}{|a|} = \epsilon.$$

This shows that the geometric series converges to the given expression. $\square$

Sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$

Most of our definitions for sequences in $\mathbb{R}$ are actually valid for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$. Indeed, the only thing we really need to define the limit is the notion of distance. Thus if we replace the modulus function $| \quad |$ on $\mathbb{R}$ by the distance functions in $\mathbb{R}^2$ and $\mathbb{R}^3$, all the definitions of convergent sequences and Cauchy sequences remain the same.

For instance, a sequence $a(n) = (a(n)_1, a(n)_2)$ in $\mathbb{R}^2$ is said to converge to a point $\ell = (\ell_1, \ell_2)$ (in $\mathbb{R}^2$) if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{(a(n)_1 - \ell_1)^2 + (a(n)_2 - \ell_2)^2} < \epsilon$$

whenever $n > N$. A similar definition can be made in $\mathbb{R}^3$ using the distance function on $\mathbb{R}^3$.

Theorems 2 (the Sandwich Theorem) and 3 (about monotonic sequences) don't really make sense for $\mathbb{R}^2$ or $\mathbb{R}^3$ because there is no ordering on these sets, that is, it doesn't really make sense to ask if one point on the plane or in space is less than the other.

The completeness of other spaces

Theorem 4, however, makes perfect sense - one can define Cauchy sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$ exactly as before, using the distance functions - and indeed, remains valid in $\mathbb{R}^2$ and $\mathbb{R}^3$.

So $\mathbb{R}^2$ and $\mathbb{R}^3$ are complete sets too (but $\mathbb{Q}^2$ and $\mathbb{Q}^3$ are not).

Finally, to emphasis that only the notion of distance matters for such questions we can define a distance function on $X = \mathcal{C}([a, b])$, the set of continuous functions from $[a, b]$ to $\mathbb{R}$, as follows:

$$\operatorname{dist}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Then, Cauchy and convergent sequences in X can be defined as before, and we can prove (maybe next semester) that X is complete.

Remark: We **do not** define the distance between the continuous functions f and g as $\inf_{x \in [a,b]} |f(x) - g(x)|$ to make sure that the distance between two **distinct** functions is **nonzero** and it is zero only when they are identical.

The rigorous definition of a limit of a function

Since we have already defined the limit of a sequence rigorously, it will not be hard to define the limit of a real valued function $f : (a, b) \to \mathbb{R}$.

Definition: A function $f : (a, b) \to \mathbb{R}$ is said to tend to (or converge to) a limit ℓ at a point $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - \ell| < \epsilon$$

for all $x \in (a, b)$ satisfying $0 < |x - x_0| < \delta$. In this case, we write

$$\lim_{x \to x_0} f(x) = \ell,$$

or $f(x) \to \ell$ as $x \to x_0$ which we read as "$f(x)$ tends to ℓ as x tends to x_0".

This is just the rigorous way of saying that the distance between $f(x)$ and ℓ can be made as small as one pleases by making the distance between x and x_0 sufficiently small.

A subtle point and the rules for limits

Notice that in the definition above, the point x_0 can be one of the end points a or b.

Thus the limit of a function at a point may exist even if the function is not defined at that point.

The rules and formulæ for limits of functions are the same as those for sequences and can be proved in almost exactly the same way.

If $\lim_{x \to x_0} f(x) = \ell_1$ and $\lim_{x \to x_0} g(x) = \ell_2$, then

1. $\lim_{x \to x_0} f(x) \pm g(x) = \ell_1 \pm \ell_2$.
2. $\lim_{x \to x_0} f(x)g(x) = \ell_1 \ell_2$.
3. $\lim_{x \to x_0} f(x)/g(x) = \ell_1/\ell_2$, provided $\ell_2 \neq 0$

As before, implicit in the formulæ is the fact that the limits on the left hand side exist. We prove the first rule below.

The proof of the addition formula for limits

Proof: We first show that $\lim_{x \to x_0} f(x) + g(x) = \ell_1 + \ell_2$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to x_0} f(x) = \ell_1$ and $\lim_{x \to x_0} g(x) = \ell_2$, there exist δ_1, δ_2 in $\mathbb{R}$ such that

$$|f(x) - \ell_1| < \frac{\epsilon}{2} \quad (\text{for } 0 < |x - x_0| < \delta_1)$$

$$\text{and} \quad |g(x) - \ell_2| < \frac{\epsilon}{2} \quad (\text{for } 0 < |x - x_0| < \delta_2).$$

If we choose $\delta = \min\{\delta_1, \delta_2\}$ and if $0 < |x - x_0| < \delta$ then both the above inequalities hold. Thus, if $0 < |x - x_0| < \delta$, then

$$\begin{aligned} |f(x) + g(x) - (\ell_1 + \ell_2)| &= |f(x) - \ell_1 + g(x) - \ell_2| \\ &\leq |f(x) - \ell_1| + |g(x) - \ell_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which is what we needed to prove.

If we replace $g(x)$ by $-g(x)$, we get the second part of the first rule. $\square$

The Sandwich Theorem(s) for limits of functions

Theorem 5: As $x \to x_0$, if $f(x) \to \ell_1$, $g(x) \to \ell_2$ and $h(x) \to \ell_3$ for functions f, g, h on some interval (a, b) such that $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then

$$\ell_1 \leq \ell_2 \leq \ell_3.$$

As before, we have a second version.

Theorem 6: Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell$ and if $g(x)$ is a function satisfying $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then $g(x)$ converges to a limit as $x \to x_0$ and

$$\lim_{x \to x_0} g(x) = \ell$$

Once again, note that we do not assume that $g(x)$ converges to a limit in this version of the theorem - we get the convergence of $g(x)$ for free.

Some examples

Let us look at Exercise 1.11. We will use this exercise to explore a few subtle points.

Let $c \in [a, b]$ and $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x \to c} f(x) = 0$. Prove or disprove the following statements.

(i) $\lim_{x \to c}[f(x)g(x)] = 0$.

(ii) $\lim_{x \to c}[f(x)g(x)] = 0$, if g is bounded. ($g(x)$ is said to be bounded on (a, b) if there exists $M > 0$ such that $|g(x)| \leq M$ for all $x \in (a, b)$).

(iii) $\lim_{x \to c}[f(x)g(x)] = 0$, if $\lim_{x \to c} g(x)$ exists.

Before getting into proofs, let us guess whether the statements above are true or false.

(i) false

(ii) true

(iii) true.

(i) Notice that $g(x)$ is not given to be bounded - if this was not obvious before, you should suspect that such a condition is needed after looking at part (ii). So the most natural thing to do is to look for a counter-example to this statement by taking $g(x)$ to be an unbounded function. What is the simplest example of an unbounded function $g(x)$ on an open interval?

How about $g(x) = \frac{1}{x}$ on $(0, 1)$?

What would a candidate for $f(x)$ be - what is the simplest example of a function $f(x)$ which tends to 0 for some value of c in $[0, 1]$.

$f(x) = x$, and $c = 0$ is a pretty simple candidate.

Clearly $\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} 1 = 1 \neq 0$, which shows that (i) is not true in general.

Exercise 1: Can you find a counter-example to (i) with c in (a, b) (that is, c should not be one of the end points)? (Hint: Can you find an unbounded function on a closed interval $[a, b]$?)

Let us move to part (ii).

Suppose $g(x)$ is bounded on (a, b). This means that there is some real number $M > 0$ such that $|g(x)| \leq M$.

Let $\epsilon > 0$. We would like to show that there exists a $\delta > 0$ such that

$$|f(x)g(x) - 0| = |f(x)g(x)| < \epsilon,$$

if $0 < |x - c| < \delta$.

Since $\lim_{x \to c} f(x) = 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon/M$ for all $0 < |x - c| < \delta$.

It follows that

$$|f(x)g(x)| = |f(x)||g(x)| < \frac{\epsilon}{M} \cdot M = \epsilon$$

for all $0 < |x - c| < \delta$, and this is what we had to show.

Part (iii) follows immediately from the product rule, but can one deduce part (iii) from (ii) instead?

Hint: Think back to the lemma on convergent sequences that we proved in Lecture 2: Every convergent sequence is bounded. What is the analogue for functions which converge to a limit at some point? Indeed, you can easily show the following:

Lemma 7: Let $f : (a, b) \to \mathbb{R}$ be a function such that $\lim_{x \to c} f(x)$ exists for some $c \in [a, b]$. If $c \in (a, b)$, there exists an (open) interval $I = (c - \eta, c + \eta) \subset (a, b)$ such that $f(x)$ is bounded on I. If $c = a$, then there is a half-open interval $I_1 = (a, a + \eta)$ such that $f(x)$ is bounded on I_1. Similarly if $c = b$, there exists a half-open interval $I_2 = (b - \eta, b)$ such that $f(x)$ is bounded on I_2.

The proof of the lemma above is almost the same as the lemma for convergent sequences. Basically, replace "N" by "δ" in the proof.

If one applies the Lemma above to $g(x)$, we see that $g(x)$ is bounded in some (possibly) smaller interval $(c - \eta, c + \eta)$. Now apply part (ii) to this interval to deduce that (iii) is true.