MA 105 Part II (IIT Bombay) Tutorial Sheet 1 : Multiple integrals

1. (a) Let $R := [0,1] \times [0,1]$ and $f(x,y) := [x] + [y] + 1$ for all $(x,y) \in R$, where $[u]$ is the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Using the definition of integration over rectangles, show that f is integrable over R. Also, find its value.

   (b) Let $R := [0,1] \times [0,1]$ and $f(x,y) := (x+y)^2$ for all $(x,y) \in R$. Show that f is integrable over R and find its value using Riemann sum.

   (c) Let $R := [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be integrable. Show that $|f|$ is also integrable over R.

   (d) Check the integrability of the function f over $[0,1] \times [0,1]$;

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{ and } \quad y \quad \text{are rational numbers,} \\ -1 & \text{otherwise.} \end{cases}$$

   What do you conclude about the integrability of $|f|$?

2. (a) Sketch the solid bounded by the surface $z = \sin y$, the planes $x = -1$, $x = 0$, $y = 0$ and $y = \frac{\pi}{2}$ and the xy plane and compute its volume.

   (b) The integral $\int\int_R \sqrt{9 - y^2}\, dxdy$, where $R = [0,3] \times [0,3]$, represents the volume of a solid. Sketch the solid and find its volume.

3. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} 1 - 1/q & \text{if } x = p/q \quad \text{where} \quad p, q \in \mathbb{N} \quad \text{are relatively prime and } y \quad \text{is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

   Show that f is integrable but the iterated integrals do not always exist.

4. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1, \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

   Is f integrable over the rectangle? Do both iterated integrals exist? If they exist, do they have the same value?

5. For the following, write an equivalent iterated integral with the order of integration reversed and verify if their values are equal:

   (a) $\displaystyle\int_0^1 \Big(\int_0^1 \log[(x+1)(y+1)]\, dx \Big)\, dy.$

   (b) $\displaystyle\int_0^1 \Big(\int_0^1 (xy)^2 \cos(x^3)\, dx \Big)\, dy.$

6. (a) Let $R = [a,b] \times [c,d]$ and $f(x,y) = \phi(x)\psi(y)$ for all $(x,y) \in R$, where ϕ is continuous on $[a,b]$ and ψ is continuous on $[c,d]$. Show that

$$\int\int_R f(x,y)\, dxdy = \Big(\int_a^b \phi(x)\, dx \Big) \Big(\int_c^d \psi(y)\, dy \Big).$$

(b) Compute $\int\int_{[1,2]\times[1,2]} x^r y^s \, dxdy$, for any given $r \geq 0$ and $s \geq 0$.

(c) Compute $\int\int_{[0,1]\times[0,1]} xye^{x+y} \, dxdy$.

7. Evaluate the following integrals:

   (a) $\displaystyle\int\int_R (x+2y)^2 \, dxdy$, where $R = [-1,2] \times [0,2]$.

   (b) $\displaystyle\int\int_R \left[xy + \frac{x}{y+1}\right] dxdy$, where $R = [1,4] \times [1,2]$.

8. Consider the function f over $[-1,1] \times [-1,1]$:

$$f(x,y) = \begin{cases} x+y & \text{if} \quad x^2+y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the set of points at which f is discontinuous. Is f integrable over $[-1,1]\times[-1,1]$?