CL205: AI & DS
Conditional Probability, Independence (Summary)

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Conditional Probability and Independence Ideas

- Consider probability space $(\Omega, \mathcal{F}, P)$.
- Let A and B represent two events in $\mathcal{F}$.
- Conditional Probability: Knowing that an event B has occurred sometimes forces us to reassess the probability of event A. The new probability is the conditional probability.
- Independence: If the conditional probability of A equals what the probability of A was before, then events A and B are called independent.

Conditional Probability Formally

- Conditional probability of event A given event B has occurred is defined as
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
provided $P(B) > 0$.
- The conditional probability function satisfies all the axioms of probability, and, thus, is a valid probability function in itself.

Multiplication Rule

- Multiplication Rule: For events A and B

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

- Further Implication: Consider event A, which can be expressed as

$$A = (A \cap B) \cup (A \cap B^*)$$

$\Rightarrow$ Probability of event A:

$$\begin{aligned} P(A) &= P(A \cap B) \cup P(A \cap B^*) \\ &= P(A \mid B)P(B) + P(A \mid B^*)P(B^*) \end{aligned}$$

Example

- Mad Cow Disease:
  Consider a test in which a cow is tested to determine infection with the "mad cow disease." It is known that, using the specified test, an infected cow has a 70% chance of testing positive, and a healthy cow just 10%. It is also known that 2% cows are infected. Find probability that an arbitrary cow tests positive.
- Event B : A randomly picked cow is infected; Event T: Test comes positive.

$$P(T \mid B) = 0.7 \text{ and } P(T \mid B^*) = 0.1$$

- Note: As no test is 100% accurate, most tests have the problem of false positives and false negatives. A false positive means that according to the test the cow is infected, but in actuality it is not. A false negative means an infected cow is not detected by the test.

Example Cont.

- Since $\Omega = B \cup B^*$

$$P(B) = 0.02 \Rightarrow P(B^*) = 0.98$$

- Problem is to find $P(T)$.
- Since $T = (T \cap B) \cup (T \cap B^*)$

$$P(T) = P(T \cap B) + P(T \cap B^*)$$
$$P(T) = P(T \mid B)P(B) + P(T \mid B^*)P(B^*)$$
$$= 0.7 \times 0.02 + 0.1 \times 0.98 = 0.112$$

- This is application of the law of total probability.

Law of Total Probability

- Computing a probability through conditioning on several disjoint events that make up the whole sample space.
- Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \ldots \cup C_m = \Omega$
- Then, the probability of an arbitrary event A can be expressed as:

$$\begin{aligned} P(A) &= P(A \cap C_1) + P(A \cap C_2) + \ldots + P(A \cap C_m) \\ &= P(A \mid C_1) P(C_1) + P(A \mid C_2) P(C_2) + \\ &\quad \ldots + P(A \mid C_m) P(C_m) \end{aligned}$$

Illustration: Law of Total Probability

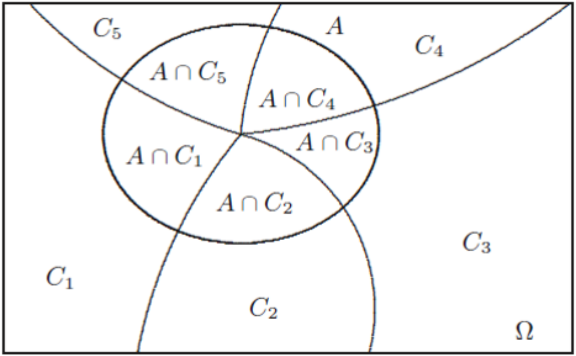

Figure: Law of Total Probability Illustration ($m = 5$) (Dekking et al., 2005)

Mad Cow Example (contd.)

- A more pertinent question about the mad cow disease test is the following:
  Suppose a cow tests positive; what is the probability it really has the mad cow disease?
- In mathematical terms, what is $P(B \mid T)$?

$$
\begin{aligned}
P(B \mid T) = \frac{P(T \cap B)}{P(T)} &= \frac{P(T \cap B)}{P(T \cap B) + P(T \cap B^*)} \\
&= \frac{P(T \mid B)P(B)}{P(T \mid B)P(B) + P(T \mid B^*)P(B^*)} \\
&= \frac{0.7 \times 0.02}{0.7 \times 0.02 + 0.1 \times 0.98} = 0.125
\end{aligned}
$$

(Dekking et al., 2005)

Mad Cow Disease Interpretation

- If we know nothing about a cow, we would say that there is a 2% chance it is infected.
- However, if we know it tested positive, then we can say there is a 12.5% chance the cow is infected.
- Finding $P(B \mid T)$ using $P(T \mid B)$ is an application of Bayes Rule derived by English clergyman Thomas Bayes in the 18th century.

Bayes Rule

- For events A and B:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

assuming $P(B) > 0$.

Bayes Rule and Law of Total Probability

- Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \ldots \cup C_m = \Omega$
- Then, the conditional probability of C_i, given an arbitrary event A, can be expressed as :

$$
\begin{aligned}
P\left(C_i \mid A\right) &= \frac{P\left(A \mid C_i\right) P\left(C_i\right)}{P\left(A \mid C_1\right) P\left(C_1\right) + \ldots\ldots + P\left(A \mid C_m\right) P\left(C_m\right)} \\
&= \frac{P\left(A \mid C_i\right) P\left(C_i\right)}{P(A)}
\end{aligned}
$$

Independence of Events

- An event A is independent of B if

$$P(A \mid B) = P(A)$$

- Result 1: A independent of $B \Leftrightarrow A^*$ independent of B

$$P(A^* \mid B) = 1 - P(A \mid B) = 1 - P(A) = P(A^*)$$

- Result 2: A independent of $B \Leftrightarrow P(A \cap B) = P(A)P(B)$.
  By application of the multiplication rule, if A is independent of B

$$P(A \cap B) = P(A \mid B)P(B) = P(A)P(B)$$

Independence (Cont.)

- Result 3 : A independent of $B \Leftrightarrow B$ independent of A

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

- To show that A and B are independent it suffices to prove just one of the following

$$P(A \mid B) = P(A)$$
$$P(B \mid A) = P(B)$$
$$P(A \cap B) = P(A)P(B)$$

  (A may be replaced by A^* and B may be replaced by B^*.)
- If one of the above statements holds, all of them are true.
- If two events are not independent, they are called dependent.

Independence: Multiple Events

- Events $A_1, A_2, \ldots, A_n$ are called independent if

$$P(A_i \cap A_j \cap \ldots . \cap A_m) = P(A_i)\,P(A_j) \ldots . P(A_m)$$

for every subset of events $A_i, A_j, ..., A_m;\ m \leq n$

- Note: If A and B are independent and B and C are independent, then it does NOT imply that A and C are independent.

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