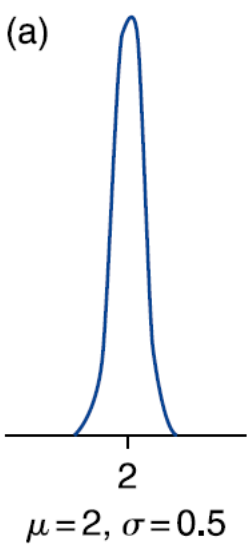
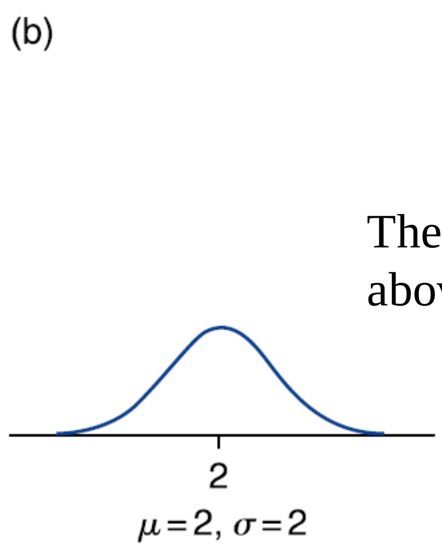
Continuous Normal Random Variable

The most important type of random variable is the normal random variable. The probability density function of a normal random variable X is determined by two parameters: the expected value and the standard deviation of X.

$$\mu = E[X] \ and \ \sigma = SD(X)$$

The normal probability density function is a bell-shaped density curve that is symmetric about the value μ. Its variability is measured by σ.

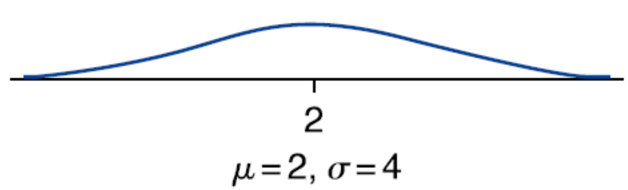




(c)

$$P\{X < \mu\} = P\{X > \mu\} = 1/2$$

The height of the curve above point x on the abscissa is $\dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$

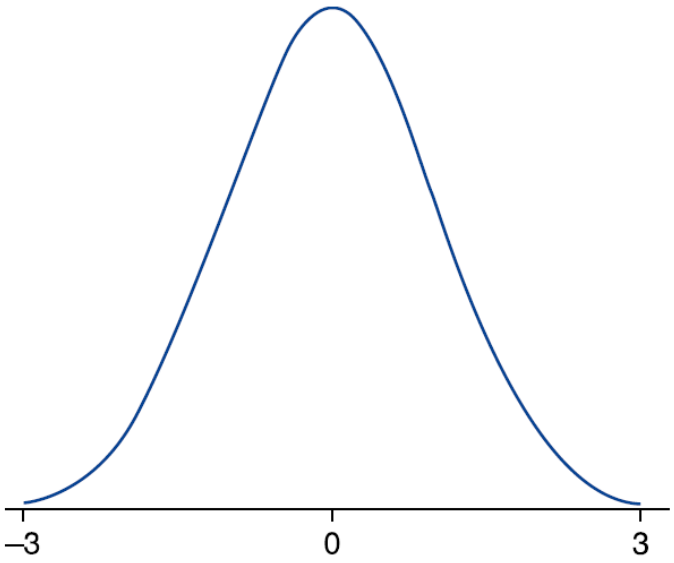


Standard Continuous Normal Random Variable

A normal random variable having mean value 0 and standard deviation 1 is called

a ***standard normal random variable***, *and its density curve is called the standard*

normal curve. Z is used to represent a standard normal random variable.



Z can be used to determine probabilities concerning an arbitrary normal random

variable by relating them to probabilities about the standard normal random variable.

Standard Continuous Normal Random Variable

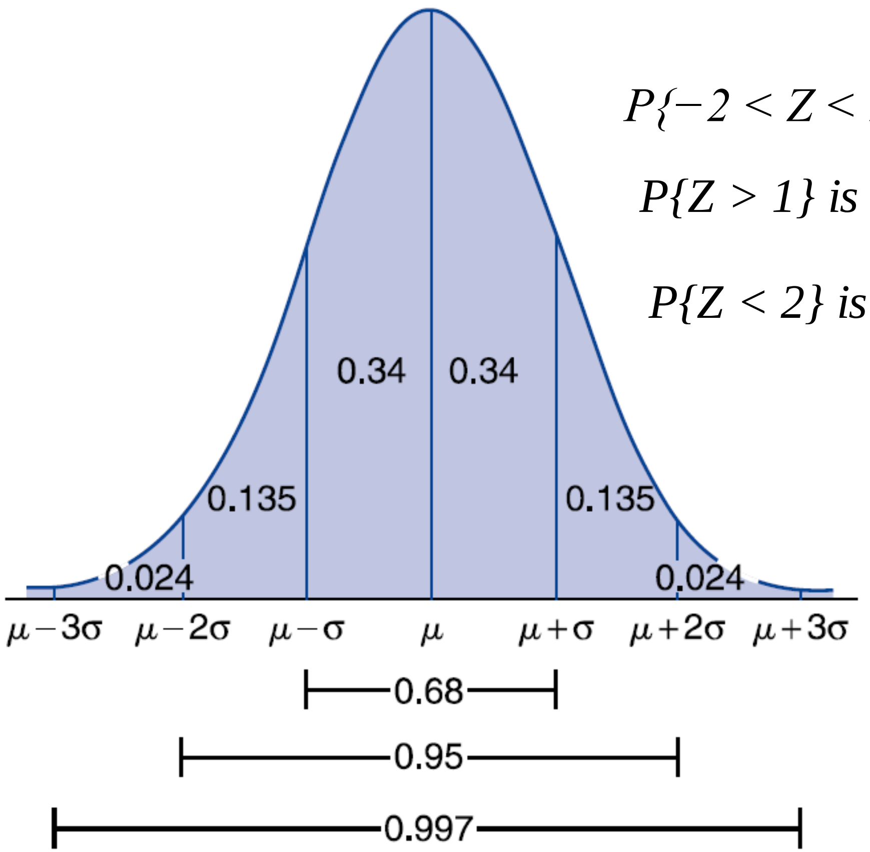
Approximation Rule

A normal random variable with mean μ and standard deviation σ will be

Between $\mu - \sigma$ and $\mu + \sigma$ with approximate probability 0.68

Between $\mu - 2\sigma$ and $\mu + 2\sigma$ with approximate probability 0.95

Between $\mu - 3\sigma$ and $\mu + 3\sigma$ with approximate probability 0.997



$P\{-2 < Z < 2\}$ *is approximately* = 0.95

$P\{Z > 1\}$ *is approximately* = 0.16

$P\{Z < 2\}$ *is approximately* = 0.975

Normal Random Variables

Variable X *is a normal random variable with standard deviation 3.* If the probability that X *is less than 16 is 0.84, then the expected value* of X *is approximately* 13

Variable X *is a normal random variable with expected value 100. If the* probability that X *is greater than 90 is 0.84, then the standard deviation* of X *is approximately* 10

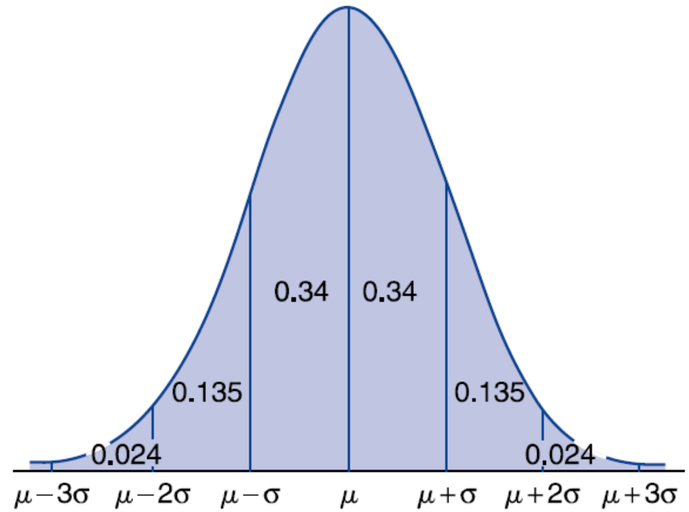
If X *is normal with expected value 100 and standard deviation 2, and* Y *is normal with expected value 100 and standard deviation 4, is* X *or* is Y *more likely (probability) to*

Exceed 104 Y

Exceed 96 X

Exceed 100 Both

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998



$$P\{Z < x\}$$

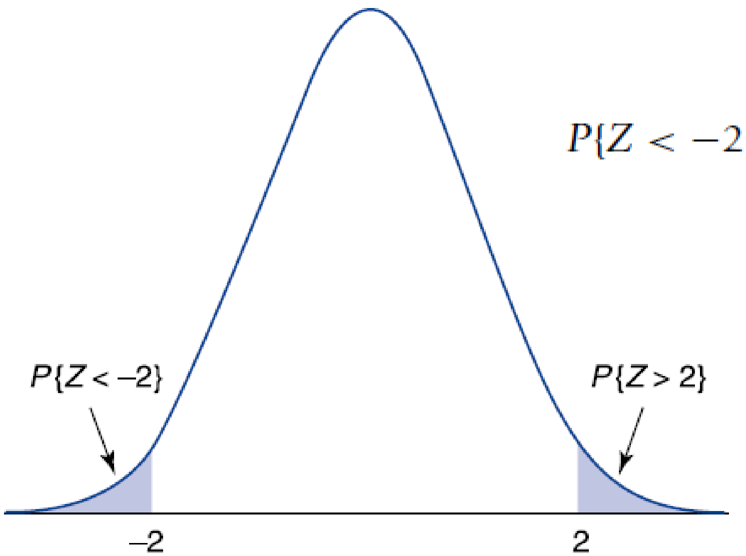
$$P\{Z < 1.22\} = 0.8888$$

$$P\{Z \leq 2\} + P\{Z > 2\} = 1$$

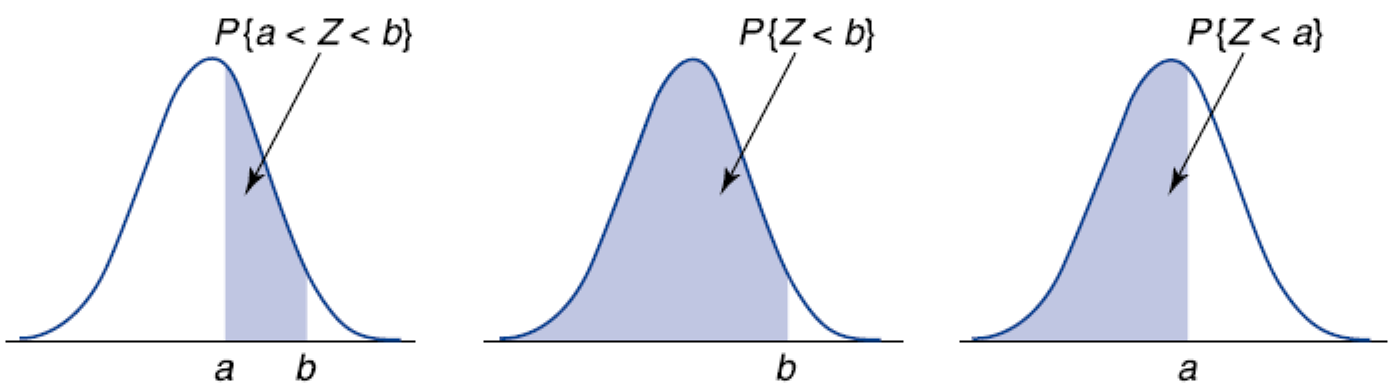
$$\text{So } P\{Z>2\} = 1 - P\{Z \leq 2\}$$
$$= 1 - 0.9772$$
$$= 0.0228$$

$$P\{Z < -x\} = P\{Z > x\}$$
$$= 1 - P\{Z < x\}$$

Standard Continuous Normal Random Variable



$$P\{Z < -2\} = P\{Z > 2\}.$$



$$P\{a < Z < b\} = P\{Z < b\} - P\{Z < a\}.$$

Standard Continuous Normal Random Variable

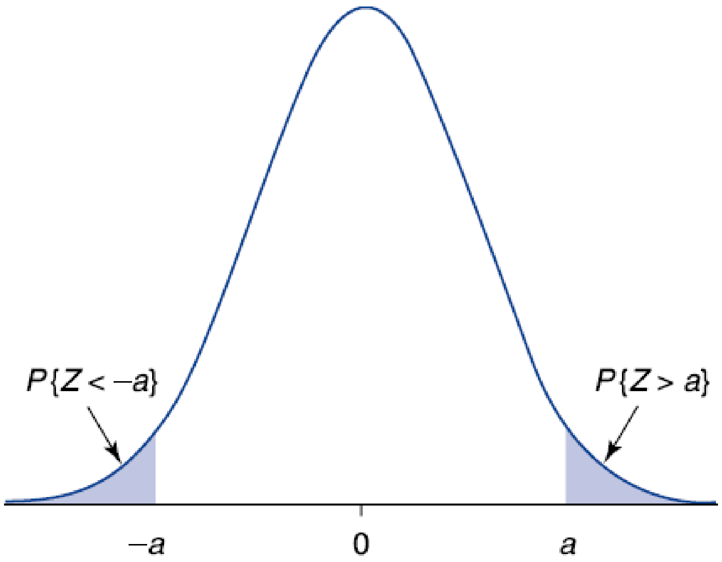


FIGURE 6.9

$P\{Z > a\} = P\{Z < -a\}$.

Let a be positive and consider $P\{|Z| > a\}$, the probability that a standard normal is, in absolute value, larger than a. Since $|Z|$ will exceed a if either $Z > a$ or $Z < -a$, we see that

$$\begin{aligned} P\{|Z| > a\} &= P\{Z > a\} + P\{Z < -a\} \\ &= 2P\{Z > a\} \end{aligned}$$

where the last equality uses the symmetry of the standard normal density curve (Fig. 6.9). ■

Standard Continuous Normal Random Variable

Prove $P\{-a < Z < a\} = 2P\{Z < a\} - 1$

$$P\{-a < Z < a\} = P\{Z < a\} - P\{Z < -a\}$$

$$= P\{Z < a\} - P\{Z > a\}$$

$$= P\{Z < a\} - [1 - P\{Z < a\}]$$

$$= 2P\{Z < a\} - 1$$

Normal Random Variables

Conversion of Normal Probabilities to Standard Normal

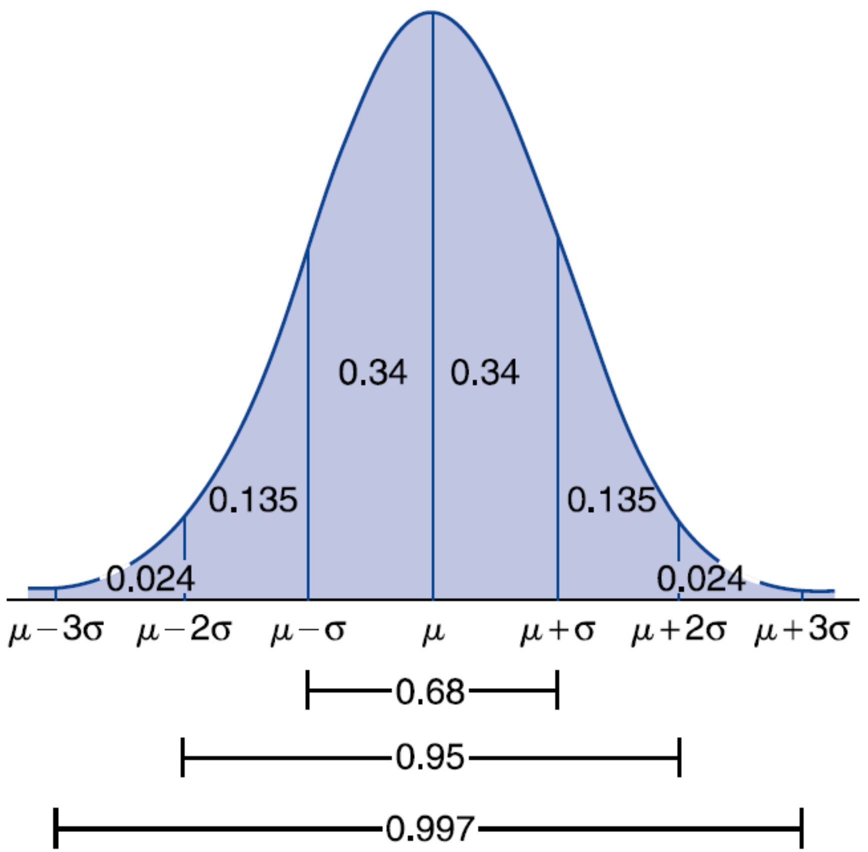
Let *X be a normal random variable with mean μ and standard deviation σ. We* can determine probabilities concerning *X by using the fact that the variable Z* defined by

$$Z = \frac{X - \mu}{\sigma}$$

That is, if we *standardize a normal random* variable by subtracting its mean and then dividing by its standard deviation, the resulting variable has a standard normal distribution.

Sample	Weight	Normalized	Shoe Size	Normalized
A	10	-1.2	1	-1.2
B	20	-0.4	2	-0.4
C	30	0.4	3	0.4
D	40	1.2	4	1.2
Mean	25	0	2.5	0
SD	12.9	1	1.3	1

Normal Random Variables



Normal Random Variables

We can compute any probability statement about X *by writing an equivalent statement* in terms of $Z = (X - \mu)/\sigma$ *and then making use of Table.*

For instance, suppose we want to compute $P\{X < a\}$. *Since* $X < a$ *then*

$$\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}$$

$$P\,(X < a) = P\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right)$$

$$P\,(X < a) = P\left(Z < \frac{a-\mu}{\sigma}\right)$$

Normal Random Variables

IQ examination scores for sixth-graders are normally distributed with mean value 100 and standard deviation 14.2.

(a) What is the probability a randomly chosen sixth-grader has a score greater than 130?

(b) What is the probability a randomly chosen sixth-grader has a score between 90 and 115?

$$\left(Z = \frac{X - 100}{14.2}\right)$$

$$P\,(X > 130) = P\left(\frac{X - 100}{14.2} > \frac{130 - 100}{14.2}\right)$$

$$P\,(X > 130) = P\left(Z > \frac{130 - 100}{14.2}\right) = 0.017$$

$$\frac{90 - 100}{14.2} < \frac{X - 100}{14.2} < \frac{115 - 100}{14.2} \quad = -0.7042 < Z < 1.056$$

$$P\{90<X<115\} = P\{-0.7042 < Z < 1.056\} = P\{Z<1.056\} - P\{Z < -0.7042\}$$

$$= 0.612$$

Normal Random Variables

Let X be normal with mean μ and standard deviation σ. Find

(a) $P\{|X - \mu| > \sigma\}$
(b) $P\{|X - \mu| > 2\sigma\}$
(c) $P\{|X - \mu| > 3\sigma\}$

Solution

The statement $|X - \mu| > a\sigma$ is, in terms of the standardized variable $Z = (X - \mu)/\sigma$, equivalent to the statement $|Z| > a$. Using this fact, we obtain the following results.

(a)
$$\begin{aligned} P\{|X - \mu| > \sigma\} &= P\{|Z| > 1\} \\ &= 2P\{Z > 1\} \\ &= 2(1 - 0.8413) \\ &= 0.3174 \end{aligned}$$

(b)
$$\begin{aligned} P\{|X - \mu| > 2\sigma\} &= P\{|Z| > 2\} \\ &= 2P\{Z > 2\} \\ &= 0.0456 \end{aligned}$$

(c)
$$\begin{aligned} P\{|X - \mu| > 3\sigma\} &= P\{|Z| > 3\} \\ &= 2P\{Z > 3\} \\ &= 0.0026 \end{aligned}$$

Normal Random Variables

The fact that $Z = (X - \mu)/\sigma$ *is a standard normal random variable follows from* the fact that if one either adds or multiplies a normal random variable by a constant, then the resulting random variable remains normal.

The sum of independent normal random variables is also a normal random variable

Suppose X and Y are independent normal random variables with means μ_x and μ_y and standard deviations σ_x and σ_y, respectively. Then $X + Y$ is normal with mean

$$E[X + Y] = \mu_x + \mu_y$$

and standard deviation

$$\mathrm{SD}(X + Y) = \sqrt{\sigma_x^2 + \sigma_y^2}$$

Normal Random Variables

Suppose the amount of time a light bulb works before burning out is a normal random variable with mean 400 hours and standard deviation 40 hours. If an individual purchases two such bulbs, one of which will be used as a spare to replace the other when it burns out, what is the probability that the total life of the bulbs will exceed 750 hours?

We need to compute the probability that $X + Y > 750$, where X is the life of the bulb used first and Y is the life of the other bulb. Variables X and Y are both normal with mean 400 and standard deviation 40. In addition, we will suppose they are independent, and so $X + Y$ is also normal with mean 800 and standard deviation $\sqrt{40^2 + 40^2} = \sqrt{3200}$. Therefore, $Z = (X + Y - 800)/\sqrt{3200}$ has a standard normal distribution. Thus, we have

$$
\begin{aligned}
P[X + Y > 750] &= P\left\{\frac{X + Y - 800}{\sqrt{3200}} > \frac{750 - 800}{\sqrt{3200}}\right\} \\
&= P\{Z > -0.884\} \\
&= P\{Z < 0.884\} \\
&= 0.81
\end{aligned}
$$

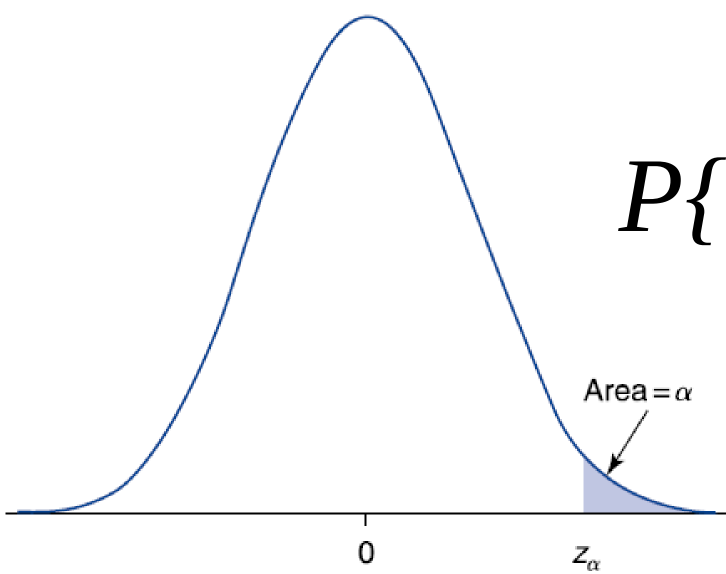
Therefore, there is an 81 percent chance that the total life of the two bulbs exceeds 750 hours. ■

Percentile of Normal Random Variables

For any α between 0 and 1, we define $\boldsymbol{Z_\alpha}$ *to be that value for which*

$$P\{Z > \boldsymbol{Z}_\alpha\} = \alpha$$

In words, the probability that a standard normal random variable is greater than $\boldsymbol{Z_\alpha}$ *is equal to* α



$$P\{Z > z_\alpha\} = \alpha$$

Normal Random Variables

We can determine the value of z_α *by using Table*.

For instance, suppose we want to find $z_{0.025}$. *Since*

$$P\{Z < z_{0.025}\} = 1 - P\{Z > z_{0.025}\} = 1\text{-}0.025 = 0.975$$

we search in Table for the entry 0.975, and then we find the value *x that* corresponds to this entry. Since the value 0.975 is found in the row labelled 1.9 and the column labelled 0.06, we see that

$$z_{0.025} = 1.96$$

That is, 2.5 percent of the time a standard normal random variable will exceed 1.96.

Since 97.5 percent of the time a standard normal random variable will be less than 1.96, we say that 1.96 is the 97.5 percentile of the standard normal distribution.

In general, since $100(1 - \alpha)$ percent of the time a standard normal random variable will be less than z_α, *we call* z_α *the* $100(1 - \alpha)$ *percentile of the standard normal* distribution.

Normal Random Variables

Suppose now that we want to find $z_{0.05}$. *If we search Table 6.1 for the value 0.95*, we do not find this exact value. Rather, we see that

$$P\{Z < 1.64\} = 0.9495$$

and

$$P\{Z < 1.65\} = 0.9505$$

Therefore, it would seem that $z_{0.05}$ *lies roughly halfway between 1.64 and 1.65*, and so we could approximate it by 1.645. In fact, it turns out that, to three decimal places, this is the correct answer, and so

$$z_{0.05} = 1.645$$

For all other values of α, we can use Table to find z_α *by searching for the entry* that is closest to $1 - \alpha$.

$$z_{0.25} = 75^{\text{th}} \text{ Percentile}$$

Normal Random Variables

Find $z_{0.80}$

$P\{Z > z_{0.80}\} = 0.80$

Now the value of $z_{0.80}$ *will be negative (why is this?),*

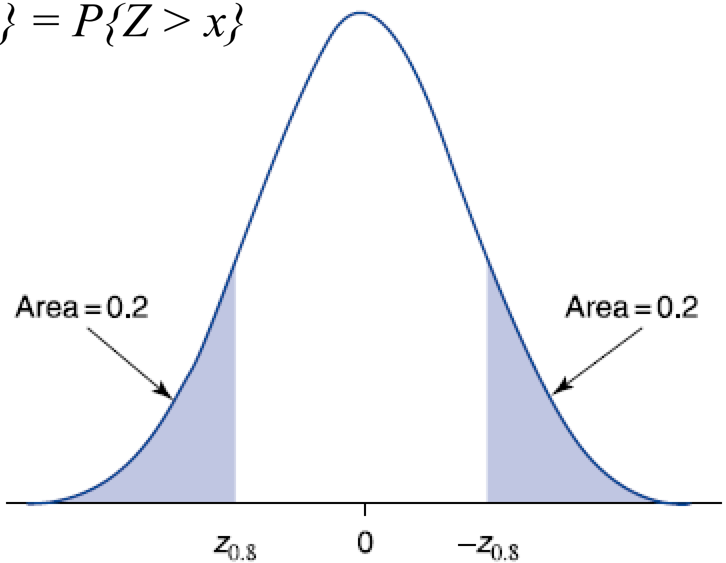
and so it is best to write the equivalent equation

$P\{Z < -z_{0.80}\} = 0.80$ *Because* $P\{Z < -x\} = P\{Z > x\}$

$-z_{0.80} \approx 0.84$

and so

$z_{0.80} \approx -0.84$



$P\{Z < -z_{0.80}\} = 0.80$

Normal Random Variables

We can easily obtain the percentiles of any normal random variable by converting to the standard normal. For instance, suppose we want to find the value *x for* which

$$P\{X < x\} = 0.95$$

when *X is normal with mean 40 and standard deviation 5. By writing the* inequality $X < x$ *in terms of the standardized variable* $Z = (X - 40)/5$, *we see* that

$$\begin{aligned} 0.95 &= P\{X < x\} \\ &= P\left\{\frac{X-40}{5} < \frac{x-40}{5}\right\} \\ &= P\left\{Z < \frac{x-40}{5}\right\} \end{aligned}$$

But $P\{Z < z_{0.05}\} = 0.95$, and so we obtain $\quad P\{Z < z_\alpha\} = 1 - P\{Z > z_\alpha\}$

$$\frac{x-40}{5} = z_{0.05} = 1.645$$

and so the desired value of x is

$$x = 5(1.645) + 40 = 48.225$$