Binomial Random Variables

Suppose that n *independent sub-experiments (or trials) are performed.*

Each of which results in either a "success" with probability p

Or a "failure" with probability $1 - p$.

If X is the total number of successes that occur in n trials,

Then X is said to be a binomial random variable with parameters n and p.

Binomial Random Variables

Suppose that $n = 3$ *and that we are interested in the probability that* X *is* equal to 2.

That is, we are interested in the probability that 3 independent trials, each of which is a success with probability p, *will result in a total of 2 successes*. The sample space for X =2 success will be

$$(s, s, f), (s, f, s), (f, s, s)$$

$$P(s, f, s) = P(S_1 F_2 S_3) = P(S_1)P(F_2)P(S_3)$$

$$= p(1-p)p \quad \textit{By Independence}$$

Binomial Random Variables

Since each of the 3 outcomes that result in a total of 2 successes consists of 2 successes and 1 failure,

it follows in a similar fashion that each occurs with probability

$$p^2(1 - p).$$

Therefore, the probability of a total of 2 successes in the 3 trials **is**

$$\mathbf{3p^2(1 - p).}$$

Binomial Random Variables

Consider now the general case in which we have *n independent trials. Let X denote* the number of successes, then

$$P\{X = i\} = \text{total of } i \textit{ successes.}$$

Since this outcome will have a total of i successes and (n − i) failures, it follows from the independence of the trials that its probability will be

$$p^i(1 - p)^{n-i}$$

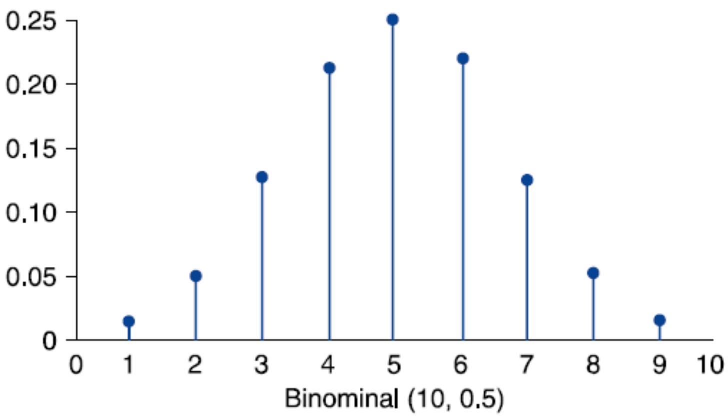
Total number of outcomes with exactly i successes and (n − i) failures for n events is

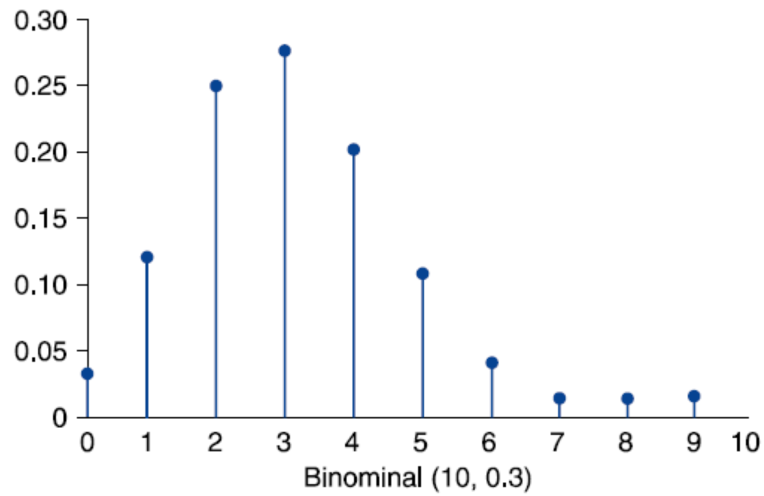
$$\boldsymbol{n!/[i!(n-i)!]}$$

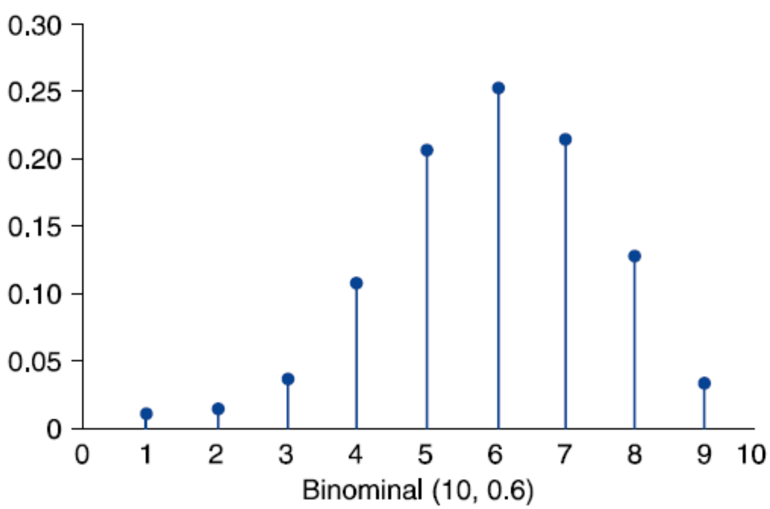
$$P\{X = i\} = \frac{n!}{i!\,(n-i)!}p^i(1-p)^{n-i}$$

Binomial Random Variables

The probabilities of three binomial random variables with respective parameters *n = 10, p = 0.5, n = 10, p = 0.3, and n = 10, p = 0.6 are*







Binomial Random Variables

Three fair coins are flipped. If the outcomes are independent, determine the probability that there are a total of *i heads, for i = 0, 1, 2, 3.*

If we let X denote the number of heads ("successes"), then X is a binomial random variable with parameters $n = 3$, $p = 0.5$. By the preceding we have

$$P\{X = 0\} = \frac{3!}{0!\,3!}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$P\{X = 1\} = \frac{3!}{1!\,2!}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = 3\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$P\{X = 2\} = \frac{3!}{2!\,1!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = 3\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$P\{X = 3\} = \frac{3!}{3!\,0!}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

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Binomial Random Variables

Suppose that a particular trait (such as eye color or handedness) is determined by a single pair of genes, and suppose that *d represents a dominant gene and r a recessive gene. A person with the pair of genes (d, d) is said to be pure dominant, one with the pair (r, r) is said to be pure recessive, and one with the* pair (*d, r) is said to be hybrid. The pure dominant and the hybrid are alike in* appearance. Their offspring receives one gene from each parent, and this gene is equally likely to be either of the parent's two genes.

(a) What is the probability that the offspring of two hybrid parents has the opposite (recessive) appearance?
(b) Suppose two hybrid parents have 4 offsprings. What is the probability 1 of the 4 offspring has the recessive appearance?

(a) The offspring will have the recessive appearance if it receives a recessive gene from each parent. By independence, the probability of this is (1/2)(1/2) = 1/4.

(b) Assuming the genes obtained by the different offspring are independent (which is the common assumption in genetics), it follows from part (a) that the number of offspring having the recessive appearance is a binomial random variable with parameters $n = 4$ *and* $p = 1/4$. *Therefore, if* X *is the* number of offspring that have the recessive appearance, then

$$P\{X = 1\} = \frac{4!}{1!\,3!}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3 = 27/64$$

Binomial Random Variables

Suppose that X *is a binomial random variable with parameters n and p, and* suppose we want to calculate the probability that X *is less than or equal to* some value j*. In principle, we could compute this as follows:*

$$P\{X \leq j\} = \sum_{i=0}^{j} P\{X = i\} = \sum_{i=0}^{j} \frac{n!}{i!\,(n-i)!} p^i (1-p)^{n-i}$$

Binomial Random Variables

Expected Value and Variance of a Binomial Random Variable

A binomial (n, p) *random variable* X *is equal to the number of successes in* n independent trials when each trial is a success with probability p.

$$X = \sum_{i=1}^{n} X_i$$

$$P\{X_i = 1\} = p \quad \text{and} \quad P\{X_i = 0\} = 1 - p$$

$$E[X_i] = p \quad \text{and} \quad \mathrm{Var}(X_i) = p(1-p)$$

Therefore, using the fact that the expectation of the sum of random variables is equal to the sum of their expectations, we see that

$$E[X] = np$$

Also, since the variance of the sum of independent random variables is equal to the sum of their variances, we have

$$\mathrm{Var}(X) = np(1-p)$$

Binomial Random Variables

A communications channel transmits the digits 0 and 1. Because of static, each digit transmitted is independently incorrectly received with probability 0.1. Suppose an important single-digit message is to be transmitted. To reduce the chance of error, the string of digits 0 0 0 0 0 is to be transmitted if the message is 0 and the string 1 1 1 1 1 is to be transmitted if the message is 1. The receiver of the message uses “majority rule” to decode; that is, she decodes the message as 0 if there are at least 3 zeros in the message received and as 1 otherwise.

(a) For the message to be incorrectly decoded, how many of the 5 digits received would have to be incorrect?

(b) What is the probability that the message is incorrectly decoded?

a) Three

b) Assume random variable is the transmission of incorrect signal with *p* of 0.1 and *n* =5

Now calculate the probability for X =3, X=4 and X=5 and then add them for the answer

Hypergeometeric Random Variables

Suppose that *n batteries are to be randomly selected from a bin of N batteries*

Out of which *Np are functional and the other N(1 − p) are defective.*

The random variable X, equal to the number of functional batteries in the sample is then said to be a

Hypergeometric random variable with parameters *n, N, p*.

Hypergeometeric Random Variables

Hypergeometric random variable with parameters n, N, p

Number of Trial = n , total size N and

probability of successful trial i = p

What distinguishes X from a binomial random variable is that these trials are not independent.

Hypergeometeric Random Variables

For instance, suppose that two batteries are to be withdrawn from a bin of five. Only one is functional and the others defective.

(That is, $n = 2, N = 5, p = 1/5.$)

Then the probability that the second battery withdrawn is functional is

$$= 1/5$$

However, if the first one withdrawn is functional, then the conditional probability that the second one is functional is 0 (since when the second battery is chosen all four remaining batteries in the bin are defective).

That is, when the selections of the batteries are made without replacing the previously chosen ones, the trials are not independent, so *X is not a* binomial random variable.

Hypergeometeric Random Variables

By using the result that each of the *n trials is a success with probability p, it can be* shown that the expected number of successes is *np. That is,*

$$E[X] = np$$

$$\mathrm{Var}\,(X) = \frac{N - n}{N - 1}\, np\,(1-p)$$

Thus, whereas the expected value of the hypergeometric random variable with parameters *n, N, p is the same as that of the binomial random variable with* parameters *n, p, its variance is smaller than that of the binomial by the factor* $(N - n)/(N - 1)$.

Hypergeometeric Random Variables

If 6 people are randomly selected from a group consisting of 12 men and 8 women, then the number of women chosen is a hypergeometric random variable with parameters $n = 6, N = 20, p = 8/20 = 0.4.$

Its mean and variance are

$$E[X] = np = 6(0.4) = 2.4$$

$$\text{Var (X)} = \frac{\text{N - n}}{\text{N - 1}}\, np\,(1\text{-}p) = (14/19)6(0.4)(0.6) \sim 1.061$$

Similarly, the number of men chosen is a hypergeometric random variable with parameters $n = 6, N = 20, p = 0.6$

Hypergeometeric Random Variables

suppose that 20 batteries are to be randomly chosen from a bin containing 10,000 batteries of which 90 percent are functional. In this case, no matter which batteries were previously chosen each new selection will be defective with a probability that is approximately equal to 0.9. For instance, the first battery selected will be functional with probability 0.9. If the first battery is functional then the next one will also be functional with probability $8999/9999 \approx .89999$, whereas if the first battery is defective then the second one will be functional with probability $9000/9999 \approx .90009$.

A similar argument holds for the other selections, and thus we may conclude that when N is large in relation to *n, then the n trials are approximately independent, which* means that X *is approximately a binomial random variable.*

Problem Set on P 249

Poisson Random Variables

Consider Binomial random variables with n independent trials, each of which results in either a success with probability p or a failure with probability $1 - p$.

If the number of trials is large and the probability of a success on a trial is small, then the total number of successes will be approximately a **Poisson random variable** with parameter $\lambda = np$.

Examples:

1. The number of misprints on a page of a book
2. The number of people in a community who are at least 100 years old
3. The number of people entering a post-office on a given day

Poisson Random Variables

A Random Variable X that takes one of the values of 0, 1, 2,.. is said to be a POSSION RANDOM VARIABLE with parameter λ,

if for some positive value of λ, its probabilities are given by

$$P\{X = i\} = (c\,\lambda^i) / i!\,,\quad i = 0, 1, \ldots$$

Where c is constant and its value depend upon λ

$$c = e^{-\lambda}$$

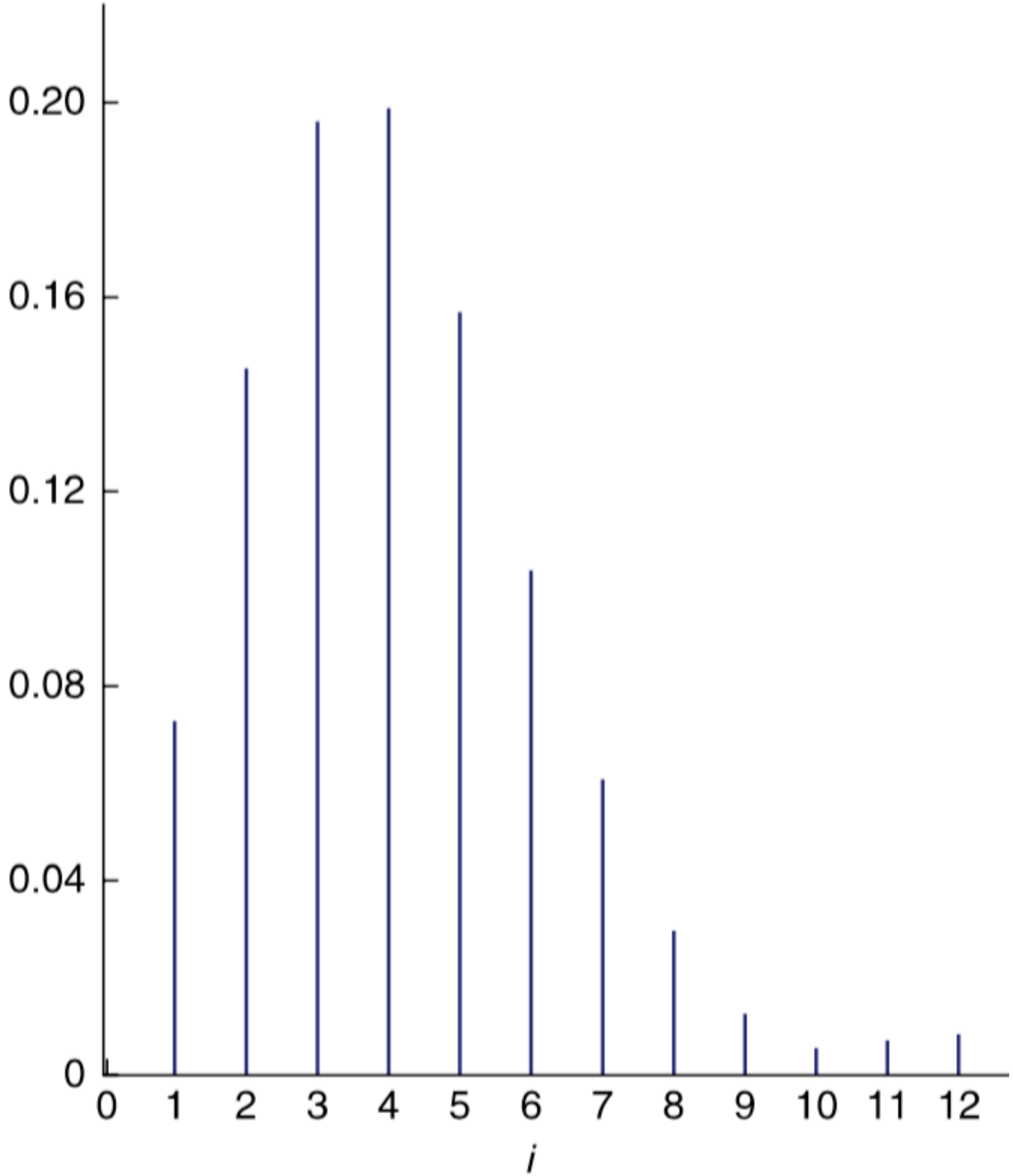
$$P\{X = i\} = (e^{-\lambda}\,\lambda^i) / i!\,,\quad i = 0, 1, \ldots$$

If X is POSSION RANDOM VARIABLE with parameter λ, $\lambda > 0$, then

$$E[X] = \lambda \quad \textit{and} \quad \mathrm{Var}(X) = \lambda$$

Poisson Random Variables

Probabilities of a Poisson random variable with λ = 4.



Poisson Random Variables

If *X is a Poisson random variable with parameter* $\lambda = 2$*, find* $P\{X = 0\}$.

$$P\{X = 0\} = (e^{-2}\, 2^{0}) \,/\, 0!$$

Now $2^0 = 1$ **and** $0! = 1$**, therefor**

$$P\{X = 0\} = e^{-2} = 0.1353$$

Poisson Random Variables

Suppose that items produced by a certain machine are independently defective with probability 0.1. What is the probability that a sample of 10 items will contain at most 1 defective item? What is the Poisson approximation for this probability?

If we let X denote the number of defective items, then X is a binomial random variable with parameters $n = 10$, $p = 0.1$.

Thus the desired probability is:

$$P\{X = 0\} + P\{X = 1\} = {}^{10}C_0\,(0.1)^0(0.9)^{10} + {}^{10}C_1\,(0.1)^1(0.9)^9$$
$$= 0.7361$$

Since $np = 10(0.1) = 1$, the Poisson approximation yields the value

$$P\{X=0\}+P\{X=1\}= e^{-1}+e^{-1} = 0.7358$$

Thus, even in this case, where n is equal to 10 (which is not that large) and p is equal to 0.1 (which is not that small), the Poisson approximation to the binomial probability is quite accurate.

Poisson Random Variables

Suppose the average number of accidents occurring weekly on a particular highway is equal to 1.2. Approximate the probability that there is at least one accident this week.

Let X *denote the number of accidents. Because it is reasonable to suppose that* there are a large number of cars passing along the highway, each having a small probability of being involved in an accident, the number of such accidents should be approximately a Poisson random variable.

That is, if X *denotes the* number of accidents that will occur this week, then X *is approximately a Poisson* random variable with mean value $\lambda = 1.2$. The desired probability is now obtained as follows:

$$P\{X > 0\} = 1 - P\{X = 0\} = 1 - [e^{-1.2}(1.2)^0 / 0!]$$

$$= 1 - e^{-1.2} = 0.6988$$