Indian Institute of Technology, Bombay | Chemical Engineering
CL205: AI & DS | Announced Quiz 1, 24 Aug 2024

Instructions: Time Duration: 55 mins; Maximum Points: 20
Important: Please show your intermediate steps. Just writing final answer is not enough.

1. A student takes a multiple-choice exam. Suppose for each question he either knows the answer or gambles and chooses an option at random. Further suppose that if he knows the answer, the probability of a correct answer is 1, and if he gambles this probability is $\frac{1}{4}$. To pass, students need to answer at least 60% of the questions correctly. The student has "studied for a minimal pass" i.e., with probability 0.6 he knows the answer to a question. Given that he answers a question correctly, what is the probability that he actually knows the answer? [6 points]

2. The probability density function of a continuous random variable X is given by:

$$f_X(x) = \begin{cases} cx + 3, & \text{for } -3 \leq x \leq -2 \\ 3 - cx, & \text{for } 2 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) Compute c.
   (b) Compute the distribution function of X.

   [2+4=6 points]

3. Consider two tosses of a fair coin.

   (a) Write down the sample space Ω for the experiment i.e. write down the elementary outcomes.
   (b) Let us say we are interested only in the number of heads in the two tosses. Define a corresponding σ-algebra. Note: It is not the power set of Ω.
   (c) Associate probabilities to all the events in the sigma-algebra obtained in previous step. Note: each elementary outcome in Ω is equally likely.
   (d) Define random variable $X(\omega)$ to be the number of heads appearing in ω. Obtain the probability distribution function for X.

   [2+2+2+2=8 points]

Problem 1 Solution:
Let K be the event that the student knows the answer, and C be the event that the student answers correctly. We need to find the conditional probability $P(K \mid C)$, which is the probability that the student knows the answer given that they answered correctly.
We are given:

- $P(K) = 0.6$ (the probability that the student knows the answer),
- $P(C \mid K) = 1$ (the probability that the student answers correctly given that they know the answer),
- $P(C \mid K^c) = \frac{1}{4}$ (the probability that the student answers correctly given that they don't know the answer and choose randomly).

We need to calculate $P(K \mid C)$ using Bayes' theorem:

$$P(K \mid C) = \frac{P(C \mid K) \cdot P(K)}{P(C)}$$

To find $P(C)$, we use the law of total probability:

$$P(C) = P(C \mid K) \cdot P(K) + P(C \mid K^c) \cdot P(K^c)$$

Substitute the given values:

$$P(C) = (1) \cdot 0.6 + \left(\frac{1}{4}\right) \cdot 0.4 = 0.6 + 0.1 = 0.7$$

Now, substitute $P(C)$ into Bayes' theorem:

$$P(K \mid C) = \frac{1 \cdot 0.6}{0.7} = \frac{0.6}{0.7} = \frac{6}{7}$$

Thus, the probability that the student actually knows the answer given that they answered correctly is:

$$\boxed{\frac{6}{7}}$$

Problem 2 Solution:

1. We know that,

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

   Since, other than the intervals $[-3, -2]$ and $[2, 3]$, the probability density is zero everywhere else, we have,

$$\int_{-3}^{-2} f_X(x)dx + \int_{2}^{3} f_X(x)dx = 1 \tag{1}$$

   On solving we get,

$$\boxed{c = 1}$$

2. As,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)$$

   When $x \in (-\infty, -3)$,

$$F_X(x) = \int_{-\infty}^{x} 0 \ dx = 0$$

   When $x \in [-3, -2]$,

$$F_X(x) = \int_{-\infty}^{-3} 0 \ dx + \int_{-3}^{x} (x + 3)dx = \left[\frac{x^2}{2} + 3x\right]_{-3}^{x} = \frac{x^2}{2} + 3x + \frac{9}{2} = \frac{(x + 3)^2}{2}$$

When $x \in (-2, 2)$,

$$F_X(x) = F_X(-2) + \int_{-2}^{x} 0 \ dx = \frac{1}{2}$$

When $x \in [2, 3]$,

$$F_X(x) = F_X(2) + \int_{2}^{x} (3 - x)dx = \frac{1}{2} + \left[3x - \frac{x^2}{2}\right]_2^x = 3x - \frac{x^2}{2} - \frac{7}{2}$$

And finally when $x \in (3, \infty)$,

$$F_X(x) = F_X(3) = 1$$

Problem 3 Solution:

1. The elementary outcomes are as follows, $\omega_1 = \{H, H\}, \omega_2 = \{H, T\}, \omega_3 = \{T, H\}, \omega_4 = \{T, T\}$. The sample space is then $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

2. The respective events classified on the basis of heads appearing in the two tosses are as follows,
$A_1 = 0$ Head $= \{\omega_4\}$
$A_2 = 1$ Head $= \{\omega_2, \omega_3\}$
$A_3 = 2$ Head $= \{\omega_1\}$
$\sigma -$ Algebra $\mathcal{F} = \{\{\omega_1\}, \{\omega_4\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_2, \omega_3\}, \Omega, \phi\}$

3. Since $P(\{\omega_1\}) = P(\{\omega_2\}) = P(\{\omega_3\}) = P(\{\omega_4\}) = 0.25$ the probabilities in $\mathcal{F}$ are as follows, $P(\{\omega_1\}) = 0.25$,$P(\{\omega_4\}) = 0.25$,$P(\{\omega_4, \omega_1\}) = 0.5$,$P(\{\omega_2, \omega_3\}) = 0.5$,$P(\{\omega_2, \omega_3, \omega_1\}) = 0.75$,$P(\{\omega_2, \omega_3, \omega_4\}) = 0.75$,$P(\{\Omega\}) = 1$,$P(\{\phi\}) = 0$.

4. The number of heads appearing in an individual outcome ω are 0, or 1, or 2. Then, $X(\omega_1) = 2$,$X(\omega_4) = 0$,$X(\omega_2) = X(\omega_3) = 1$.

$$F_X(\xi) = \begin{cases} 0, & \xi > 0 \\ 0.25, & 0 \leq \xi < 1 \\ 0.75, & 1 \leq \xi < 2 \\ 1, & 2 \leq \xi \end{cases}$$