Indian Institute of Technology, Bombay | Chemical Engineering
CL205: AI & DS | Tutorial 1, 2024

1. Consider an experiment with sample space consisting of three elementary outcomes i.e. $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Consider the following cases:

   (a) $\mathcal{F}_1$ is a collection of events Ω, ϕ, $\{\omega_1\}, \{\omega_2, \omega_3\}$.

   (b) $\mathcal{F}_2$ is a collection of events Ω, ϕ, $\{\omega_1\}, \{\omega_2\}$.

   For both the above cases, verify if $\mathcal{F}$ (i.e. $\mathcal{F}_1, \mathcal{F}_2$) is a valid σ-algebra. If not, add minimal number of sets to the collection to obtain a σ-algebra.

2. Consider an experiment of tossing a coin three consecutive times.

   (a) Write down the sample space of this experiment. Note: elementary outcome of the experiment will be of the form $\{a, b, c\}$ where a, b, c are either H (Head) or T (Tail).

   (b) Assume that each elementary outcome has the same probability (coin is fair). Then obtain the probability of the occurrence of the event A defined as $A = \{\text{heads in the first two tosses}\}$

3. Show that for two events A, B if $A \cap B = \emptyset$, then $P(A) \leq P(B^*)$ where * indicates complement. You can use ideas from set theory (unions, intersection, etc.) but ensure that while considering probabilities of events, the axioms of probability are satisfied.

4. If $A = \{2 \leq x \leq 5\}$ and $B = \{3 \leq x \leq 6\}$, then find $A \cup B, A \cap B$, and $(A \cup B) \cap (A \cap B)^*$.

5. Given two event sets $A, B \subset \Omega$, we can define *conditional probability* of event A given that event B has occurred, as:

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$$

   where $P(A|B)$ is the conditional probability that event A has occurred given that event B has occurred. It is assumed that $P(B) > 0$.
   Show that, for a specified B, conditional probabilities are indeed valid probabilities; that is they satisfy the probability axioms.

6. Random variable correctness illustration: Consider an experiment where the sample space is an interval on the real line: $\Omega = (0, 10]$. We are interested in distinguishing whether ω takes on a value in the interval $I_1 = (0, 5]$ or in $I_2 = (5, 10]$. The σ-algebra could then be: $\mathcal{F} = \{\emptyset, \Omega = (0, 10], I_1 = (0, 5], I_2 = (5, 10]\}$. Do the following:

   (a) Define random variable as: $\mathbf{x}(\omega) = \omega, \ \forall \omega \in (0, 10]$. Verify that this is not a suitable choice by checking the sets of the form $A = \{\omega : \mathbf{x}(\omega) \leq \xi\}$ for various values of ξ are not part of $\mathcal{F}$.

   (b) Define random variable as: $\mathbf{x}(\omega) = 5$ if $\omega \in I_1$ and $\mathbf{x}(\omega) = 10$ if $\omega \in I_2$. For this definition show that the sets $A = \{\omega : \mathbf{x}(\omega) \leq \xi\}$ for all values of $\xi \in \mathbb{R}$ are part of $\mathcal{F}$.

7. Consider a toss of a die with the sample space being $\Omega = \{1, 2, 3, 4, 5, 6\}$. Suppose that, for some reason, you are interested only in the occurrence of one of two events,

$$A_1 = \{\text{a 1 or 2 was thrown}\} = \{1, 2\}, \text{ and } A_2 = \{\text{a 3 was thrown}\} = \{3\}$$

   (a) Define the σ-algebra by taking appropriate complements, unions, intersections of A_1, A_2 and the sets resulting from these operations.

   (b) It is given that $P(A_1) = p_1$ and $P(A_2) = p_2$. Using this information, assign probabilities to all the elements of the σ-algebra $\mathcal{F}$ constructed in the above part.

(c) Define a random variable as

$$\mathbf{x}(\omega) = \begin{cases} 0, & \text{if } \omega \notin A_1 \text{ or } A_2 \\ 1, & \text{if } \omega \in A_1 \\ 2, & \text{if } \omega \in A_2 \end{cases}$$

Construct the probability distribution function $F_{\mathbf{x}}(\xi)$ for this random variable. Use $p_1 = 1/3, p_2 = 1/6$.

8. An experiment consists of observing the voltage $\mathbf{x}$ of the parity bit in a word in computer memory. If the bit is *on*, then $\mathbf{x} = 1$; if *off*, then $\mathbf{x} = 0$. Assume that the *off* state has probability q and *on* state has probability 1-q. Generate the probability distribution function $F_{\mathbf{x}}(\xi)$.

9. The probability distribution function $F_{\mathbf{x}}(\xi)$ of the random variable $\mathbf{x}$ is defined as,

$$F_{\mathbf{x}}(\xi) = \begin{cases} 0, & \xi < 0 \\ K(1 - e^{-2\xi}), & \xi \geq 0 \end{cases}$$

   (a) For what value of K is the function a valid probability distribution function?

   (b) With the above value of K, what is $P(2 < \mathbf{x} < \infty)$?

10. Find the probability mass function of a discrete random variable $\mathbf{x}$ whose probability distribution function is given as,

$$F_{\mathbf{x}}(\xi) = \begin{cases} 0, & \xi < 0 \\ 1/6, & 0 \leq \xi < 2 \\ 1/2, & 2 \leq \xi < 4 \\ 5/8, & 4 \leq \xi < 6 \\ 1, & \xi \geq 6 \end{cases}$$