Negative Binomial Distributions

Binomial but with the exception that the trials will be repeated until a fixed number of successes occur.

Therefore, instead of the probability of x successes in n trials, where n is fixed;

we are now interested in the probability that the k^{th} success occurs on the x^{th} trial.

Experiments of this kind are called negative binomial experiments.

Negative Binomial Distributions

As an illustration, consider the use of a drug that is known to be effective in 60% of the cases where it is used.

The drug will be considered a success if it is effective in bringing some degree of relief to the patient.

We are interested in finding the probability that the fifth patient to experience relief is the seventh patient to receive the drug during a given week.

Designating a success by S and a failure by F, a possible order of achieving the desired result is

SFSSSFS, which occurs with probability

$$(0.6)(0.4)(0.6)(0.6)(0.6)(0.4)(0.6) = (0.6)^5(0.4)^2.$$

Negative Binomial Distributions

We could list all possible orders by rearranging the F's and S's except for the last outcome, which must be the fifth success.

The total number of possible orders is equal to the number of partitions of the first six trials into two groups with 2 failures assigned to the one group and 4 successes assigned to the other group.

This can be done in

$\binom{6}{4}$ **= 15 mutually exclusive ways.**

Hence, if X represents the outcome on which the fifth success occurs, then

$$P(X = 7) = \binom{6}{4} (0.6)^5(0.4)^2 = 0.1866$$

Negative Binomial Distributions

If repeated independent trials can result in a success with probability p and a failure with probability q = 1 − p,

then the probability distribution of the random variable X, the number of the trial on which the k^th success occurs, is

$$b^*(x;k,p) = \binom{x-1}{k-1} (p)^k (q)^{x-k} \quad x = k, k+1, k+2, \ldots..$$

Negative Binomial Distributions

In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner.

Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

(a) What is the probability that team A will win the series in 6 games?

(b) What is the probability that team A will win the series?

$$b^*(6; 4, 0.55) = \binom{5}{3} 0.554^{4} (1 - 0.55)^{6-4} = 0.1853$$

P(team A wins the championship series) is

$$b^*(4; 4, 0.55) + b^*(5; 4, 0.55) + b^*(6; 4, 0.55) + b^*(7; 4, 0.55)$$

$$= 0.0915 + 0.1647 + 0.1853 + 0.1668 = 0.6083.$$

Geometric Distributions

If we consider the special case of the negative binomial distribution where k = 1, we have a probability distribution for the number of trials required for a single success Then

$$b^*(x;k,p)=\binom{x-1}{k-1}\ (p)^k(q)^{x-k}\quad x = k, k+1, k+2,\ldots..$$

$$b^*(x; 1,p) = pq^{x-1},\ \ x= 1, 2, 3, \ldots.$$

Geometric Distributions

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$,

then the probability distribution of the random variable X, the number of the trial on which the first success occurs, is called the Geometric Distribution and denoted its value by $g(x;p)$

$$g(x; p) = pq^{x-1}, \quad x = 1, 2, 3, \ldots$$

Geometric Distributions

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Using the geometric distribution with $x = 5$ and $p = 0.01$, we have

$$g(5; 0.01) = (0.01)(0.99)^4 = 0.0096.$$

Review

If X is a binomial random variable with parameters n and p

$$P\{X = i\} = \frac{n!}{i!\,(n-i)!} p^i (1-p)^{n-i}$$

$$b(x;n,p) = \binom{n}{x} (p)^x (q)^{n-x}$$

$$E[X] = np$$

$$\mathrm{Var}(X) = np(1-p)$$

Review

For a Hypergeometric random variable (*trials are not independent*) with parameters n, N, p

$$E[X] = np$$

$$\text{Var}(X) = \frac{N - n}{N - 1}\, np(1-p)$$

When N *is large in relation to* n, *a hypergeometric random variable with parameters* n, N, p *approximately has a binomial distribution with parameters* n and p.

Review

For a Poisson random variable with parameter λ if for some positive value λ its probabilities are given by

$$P\{X = i\} = e^{-\lambda} \lambda^i / i!, \quad i = 0, 1, \ldots$$

$$E[X] = \lambda \quad \textit{and} \quad \mathrm{Var}(X) = \lambda$$

Review

Negative Binomial Distributions: k^{th} success occurs on the x^{th} trial

If repeated independent trials can result in a success with probability p and a failure with probability q = 1 − p, then the probability distribution of the random variable X, the number of the trial on which the k^{th} success occurs, is

$$b^*(x;k,p) = \binom{x-1}{k-1}\ (p)^k (q)^{x-k} \quad x = k, k+1, k+2,\ldots..$$

Geometric Distributions: The number of the trial on which the first success occurs

$$g(x; p) = pq^{x-1}, \quad x = 1, 2, 3, \ldots .$$

A large chain retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 3%.

(a) The inspector randomly picks 20 items from a shipment. What is the probability that there will be at least one defective item among these 20?

(b) Suppose that the retailer receives 10 shipments in a month and the inspector randomly tests 20 devices per shipment. What is the probability that there will be exactly 3 shipments each containing at least one defective device among the 20 that are selected and tested from the shipment?

Denote by X the number of defective devices among the 20. Then X follows a $b(x; 20, 0.03)$ distribution. Hence,

$$P(X \geq 1) = 1 - P(X = 0) = 1 - b(0; 20, 0.03)$$

$$= 1 - (0.03)^0(1 - 0.03)^{20-0} = 0.4562.$$

b) In this case, each shipment can either contain at least one defective item or not. Hence, testing of each shipment can be viewed as a trial with p = 0.4562 from part (a). Assuming independence from shipment to shipment and denoting by Y the number of shipments containing at least one defective item, Y follows another binomial distribution b(y; 10, 0.4562). Therefore,

$$P(Y = 3) = \binom{10}{3} 0.4562^3 (1 - 0.4562)^7 = 0.1602.$$

It is conjectured that an impurity exists in 30% of all drinking wells in a certain rural community. In order to gain some insight into the true extent of the problem, it is determined that some testing is necessary. It is too expensive to test all of the wells in the area, so 10 are randomly selected for testing.

(a) Using the binomial distribution, what is the probability that exactly 3 wells have the impurity, assuming that the conjecture is correct?

(b) What is the probability that more than 3 wells are impure?

(a) We require $b(3; 10, 0.3) = 0.2668$.

(b) We require $1-[b(0; 10, 0.3) + b(1; 10, 0.3) + b(2; 10, 0.3) + b(3; 10, 0.3)] = 0.3504$

Twelve people are given two identical speakers, which they are asked to listen to for differences, if any. Suppose that these people answer simply by guessing. Find the probability that three people claim to have heard a difference between the two speakers.

We require $b(3; 12, 0.5) = 0.0537$

Suppose that airplane engines operate independently and fail with probability equal to 0.4. Assuming that a plane makes a safe flight if at least one-half of its engines run, determine whether a 4-engine plane or a 2- engine plane has the higher probability for a successful flight.

Probability of 2 or more of 4 engines operating when $p = 0.6$ is

$$P(X \geq 2) = 1 - P(X = 1) - P(x = 0) = 0.8208,$$

and the probability of 1 or more of 2 engines operating when $p = 0.6$ is

$$P(X \geq 1) = 1 - P(X = 0) = 0.8400.$$

The 2-engine plane has a slightly higher probability for a successful flight when $p = 0.6$.