Concordia University

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Department of Mechanical, Industrial and Aerospace Engineering

Fluid Mechanics

(Mech-6201)

A

Project report on

Numerical Solution of The Blasius Boundary Layer Equation over a Flat Plate

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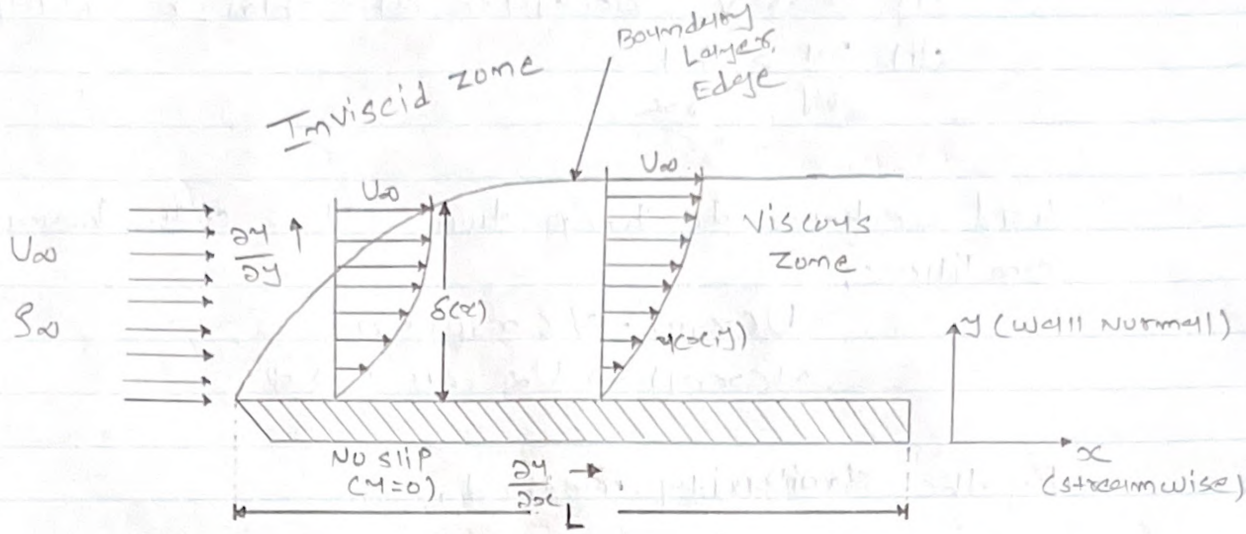
(1) Derive the 3^{rd} order, non-linear, Blasius equation, as shown in class. $F''' + \frac{1}{2} FF'' = 0$



Assumptions to be applied for blassius Solution,
(i) The flow is steady
(ii) 2-Dimension flow
(iii) Incompressible flow
(iv) Pressure gradient is zero due to constant free-stream velocity
(v) Free stream velocity (U_∞) is constant

The continuity equation is given by,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \dots \text{(i)}$$

x-momentum (N-S equation) equation is given by,

$$u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} = \nu \cdot \frac{\partial^2 u}{\partial y^2}$$

Here, u is the velocity component in the x-direction
v is the velocity component in the y-direction.

ν = kinematic viscosity.

Inside the boundary layer we are considering that
(i) $u >> v$ because of thin boundary layer &
(ii) $\frac{\partial u}{\partial y} >> \frac{\partial u}{\partial x}$

And we have to keep two PDE's with boundary conditions,

$$U(x,0) = V(x,0) = 0$$
$$u(x,y) \to U_\infty \text{ as } y \to \infty$$

By the similarity method,

lets take $\eta = \frac{y}{\delta}$ & $\frac{u}{U_\infty} = f(\eta)$

$$\therefore u = U_\infty . f(\eta)$$

Now, $\frac{\partial u}{\partial x} = U_\infty . f'(\eta) . \frac{\partial \eta}{\partial x}$

where, $\eta = \eta(x,y) = \frac{y}{\delta x}$

$$\therefore \frac{\partial u}{\partial x} = U_\infty . f' . \left(\frac{-y}{\delta^2}\right) . \delta'$$

$$\frac{\partial u}{\partial y} = U_\infty . f' . \frac{\partial \eta}{\partial y}$$
$$= U_\infty . f' . \frac{1}{\delta}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty . f'' . \frac{1}{\delta^2}$$

From the momentum equation we can write,

$$V = \frac{\nu \cdot \frac{\partial^2 u}{\partial y^2} - u \cdot \frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$$\therefore V = \frac{\nu \cdot \frac{U_\infty \cdot f''}{\delta^2} - U_\infty \cdot f \cdot \left[U_\infty \cdot f' \cdot \left(-\frac{y}{\delta^2} \right) \cdot \delta' \right]}{U_\infty \cdot f' \cdot \frac{1}{\delta}} \qquad (\because y = \eta \cdot \delta)$$

$$\therefore V = \frac{\nu}{\delta} \frac{f''}{f'} + U_\infty \cdot \eta \cdot f \cdot \delta'$$

From the continuity equation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

$$\therefore \frac{\nu}{\delta} \cdot \left[\frac{f''}{f'} \right]' \cdot \frac{1}{\delta} + \frac{U_\infty \cdot f \cdot \delta'}{\delta} + \cancel{\frac{U_\infty \cdot \eta \cdot f' \cdot \delta'}{\delta}} = \cancel{\frac{U_\infty \cdot \eta \cdot f' \cdot \delta'}{\delta}}$$

$$\therefore \frac{\nu}{\delta^2} \left[\frac{f''}{f'} \right]' = -\frac{U_\infty \cdot f \cdot \delta'}{\delta}$$

$$\therefore \frac{\left[\frac{f''}{f'} \right]'}{f} = -\frac{U_\infty \cdot \delta \cdot \delta'}{\nu} = k$$ where k is Arbitrary Constant

$$\Rightarrow -\frac{U_\infty}{\nu} \cdot \delta \cdot \frac{d\delta}{dx} = k$$

$$\Rightarrow \int \delta \cdot d\delta = -\int \frac{k \cdot \nu}{U_\infty} \cdot dx$$

$$\Rightarrow \frac{\delta^2}{2} = -\frac{k.v}{U_\infty}.x + C_1$$

Now, applying initial condition that,

$$x = 0 \rightarrow \delta = 0$$

So, we can get $C_1 = 0$

$$\therefore \delta = \sqrt{\frac{-2.k.v.x}{U_\infty}}$$

Here, we are choosing $k = -1/2$ which satisfy the above expression.

$$\therefore \delta(x) = \sqrt{\frac{v.x}{U_\infty}}$$

$$\therefore \delta(x) = \sqrt{\frac{v.x}{U_\infty}}.\frac{\sqrt{x}}{\sqrt{x}}$$

$$\therefore \delta(x) = \sqrt{\frac{\mu}{\rho.U_\infty.x}}.x$$

$$\therefore \frac{\delta(x)}{x} = \frac{1}{\sqrt{Re_x}}$$

Let, $F = \int f.d\eta \Rightarrow F' = f$

$\Rightarrow f' = F''$ & $f'' = F'''$

$$\therefore \frac{d}{d\eta}\left[\frac{F'''}{F''}\right] = -\frac{1}{2}F'$$

By Integrating, $\frac{F'''}{F''} = -\frac{1}{2}F$

$$\therefore F''' + \frac{1}{2}F.F'' = 0$$

(2) Explain the physical significance of each of the boundary conditions:

(a) $F(\eta = 0) = 0$

This condition says that at the plate at $y=0$ there is no flow, which means $F = \frac{df}{d\eta} = \frac{u}{U_\infty} = 0$

(b) $F'(\eta = 0) = 0$

This condition implies that if there is no flow at the plate surface then the velocity at the plate surface also becomes zero.

(c) $F'(\eta \to \infty) = 1$

This condition implies that velocity far away from the plate becomes U_∞. So as you move away from the plate velocity becomes the free stream velocity.

(3) Convert the Blasius equation into a set of coupled 1^{st} order ordinary Differential Equations [ODES]

$$F'(\eta) = G(\eta)$$
$$G'(\eta) = H(\eta)$$
$$H'(\eta) = -\tfrac{1}{2} F(\eta) H(\eta)$$

Blasius equation is given by,

$$F''' + \frac{1}{2} F F'' = 0$$

Here, as given $F'(\eta) = G(\eta)$

$$F''(\eta) = G'(\eta) = H(\eta)$$
$$F'''(\eta) = G''(\eta) = H'(\eta)$$

$$\therefore H'(\eta) + \frac{1}{2} F(\eta) \cdot H(\eta) = 0$$

The above equation refers to 1^{st} order ordinary Differential equation.

Use Euler's method for solving the coupled ODEs along with an iterative approach called the 'shooting algorithm' for determining the correct value of H(0), so that F'→1 as η→∞.

MATLAB CODE:

```
function E=Euler(f,a,b,f_0,h0,m)
clc;
clear all;
m=80; %itretion
a=0.0; %minimum value of eta
b=8;  %maximum value of eta (domain size)
delta_n=(b-a)/m; %step size
f_0=0.0;  %f(0)=0
df_0=0.0; %f'(0)=0
h0=0.3152; %changed by trial and error until f(b)=1.0
g(1)=df_0;
h(1)=h0;
f(1)=f_0;

%euler's method solution
for j=1:m;
 eta(j)=a+(delta_n*(j-1));
 f(j+1)=f(j)+(delta_n*g(j));
 g(j+1)=g(j)+(delta_n*h(j));
 h(j+1)=h(j)+(delta_n*(-0.5*f(j)*h(j)));
end

j=1:m;
eta=a:delta_n:b;
f(j)=f(j)';
g(j)=g(j)';
h(j)=h(j)';
disp('       eta       f        fprim     fdoubleprim ');
ans=[eta' f' g' h']

figure;
hold on;
grid on;
title('BLASIUS BOUNDARY LAYER SOLUTION FOR FLOW OVER FLAT PLATE')
xlabel('eta','FontSize',12,'FontWeight','bold','Color','R')
ylabel('f, fprim,
fdoubleprim','FontSize',12,'FontWeight','bold','Color','R');
plot(eta,h)
plot(eta,g)
plot(eta,f)
legend('fdoubleprim','fprim','f');
hold off;

end
```

RESULT:

eta	F	F‘	F’’
0	0	0	0.3152
0.1	0	0.0315	0.3152
0.2	0.0032	0.063	0.3152
0.3	0.0095	0.0946	0.3152
0.4	0.0189	0.1261	0.315
0.5	0.0315	0.1576	0.3147
0.6	0.0473	0.189	0.3142
0.7	0.0662	0.2205	0.3135
0.8	0.0882	0.2518	0.3124
0.9	0.1134	0.2831	0.311
1	0.1417	0.3142	0.3093
1.1	0.1731	0.3451	0.3071
1.2	0.2076	0.3758	0.3044
1.3	0.2452	0.4062	0.3013
1.4	0.2858	0.4364	0.2976
1.5	0.3295	0.4661	0.2933
1.6	0.3761	0.4955	0.2885
1.7	0.4256	0.5243	0.2831
1.8	0.4781	0.5526	0.277
1.9	0.5333	0.5803	0.2704
2	0.5914	0.6074	0.2632
2.1	0.6521	0.6337	0.2554
2.2	0.7155	0.6592	0.2471
2.3	0.7814	0.6839	0.2383
2.4	0.8498	0.7078	0.229
2.5	0.9206	0.7307	0.2192
2.6	0.9936	0.7526	0.2091
2.7	1.0689	0.7735	0.1987
2.8	1.1462	0.7934	0.1881
2.9	1.2256	0.8122	0.1773
3	1.3068	0.8299	0.1665
3.1	1.3898	0.8466	0.1556
3.2	1.4744	0.8621	0.1448
3.3	1.5607	0.8766	0.1341
3.4	1.6483	0.89	0.1236
3.5	1.7373	0.9024	0.1135
3.6	1.8276	0.9137	0.1036
3.7	1.9189	0.9241	0.0941

3.8	2.0113	0.9335	0.0851
3.9	2.1047	0.942	0.0765
4	2.1989	0.9497	0.0685
4.1	2.2938	0.9565	0.061
4.2	2.3895	0.9626	0.054
4.3	2.4858	0.968	0.0475
4.4	2.5826	0.9728	0.0416
4.5	2.6798	0.9769	0.0362
4.6	2.7775	0.9805	0.0314
4.7	2.8756	0.9837	0.027
4.8	2.9739	0.9864	0.0231
4.9	3.0726	0.9887	0.0197
5	3.1715	0.9907	0.0167

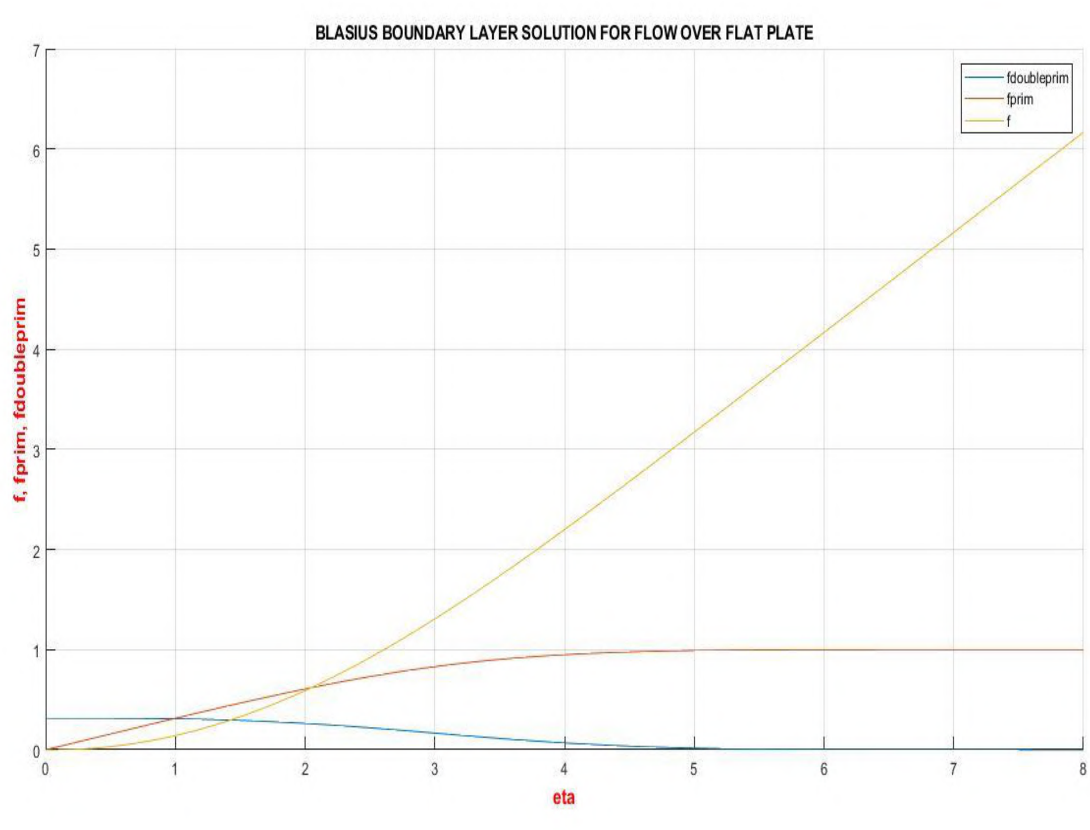
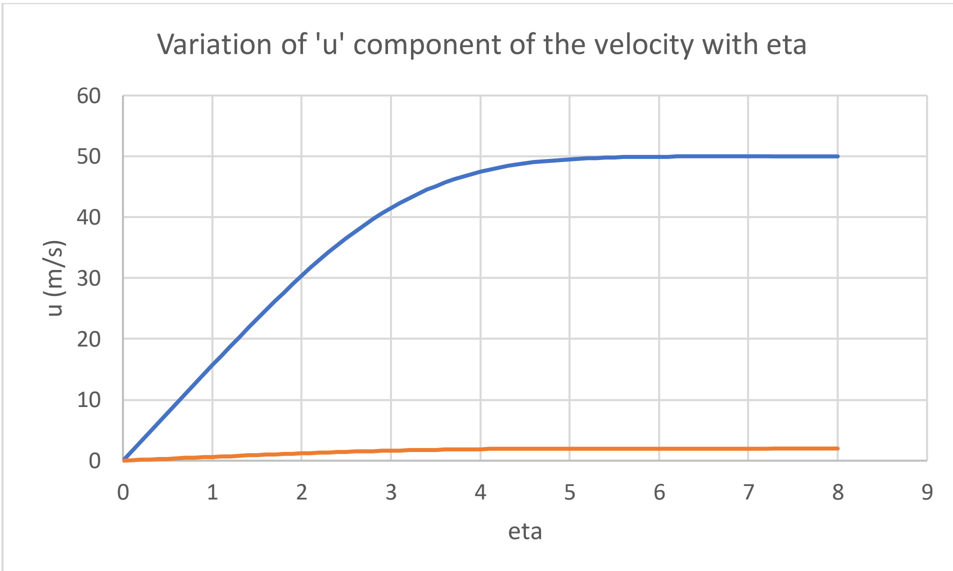


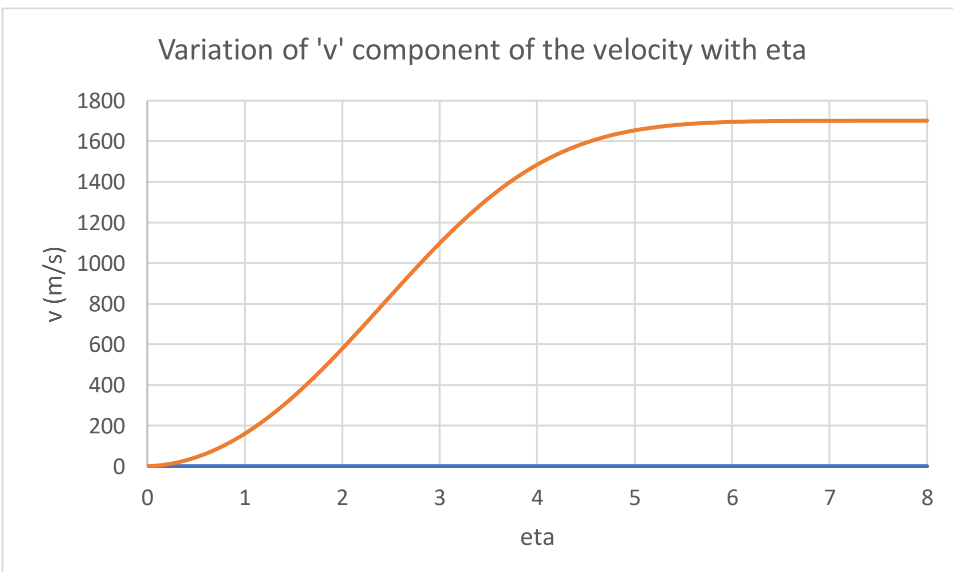
FIGURE:1 BLASIUS BOUNDARY LAYER SOLUTION FOR FLOW OVER FLAT PLATE

For both fluids, and at several x locations:

a) Plot The resultant velocity profiles within the Boundary layer (both x and y components).

At location x=0.7 m (Blue line curve for air and orange line curve for oil)





(b) Calculate the wall shear stress and friction co-efficient.

Given Data:

(a) $U_\infty = 50\, m/s$; $\rho_\infty = 1.2\, kg/m^3$; $\mu = 0.000017\, Pa\text{-}s$ (air)

(b) $U_\infty = 2\, m/s$; $\rho_\infty = 800\, kg/m^3$; $\mu = 0.0013\, Pa\text{-}s$ (oil)

From question (4) we obtained that $f''(0) = 0.3152$

For air, $$(\tau_w)_{air} = \frac{F''(0) \cdot \mu \cdot U_\infty}{\sqrt{\frac{\nu \cdot x}{U_\infty}}}$$ (Here, $x = 0.7\, m$ taken)

$$(\tau_w)_{air} = \frac{0.3152 \cdot 0.000017 \cdot 50}{\sqrt{\frac{0.000017 \times 0.7}{1.2 \times 50}}}$$

$$(\tau_w)_{air} = 0.6015\ N/m^2$$

For oil, $$(\tau_w)_{oil} = \frac{F''(0) \cdot \mu \cdot U_\infty}{\sqrt{\frac{\nu \cdot x}{U_\infty}}}$$

$$(\tau_w)_{oil} = \frac{0.3152 \cdot 0.0013 \cdot 2}{\sqrt{\frac{0.0013 \times 0.7}{800 \times 2}}}$$

$$(\tau_w)_{oil} = 1.086\ N/m^2$$

Now, changing x location 0.7 m to 0.5 m

$$(\tau_w)_{air} = \frac{0.3152 \times 0.000017 \times 50}{\sqrt{\frac{0.000017 \times 0.5}{1.2 \times 50}}} = 0.7118\ N/m^2$$

$$(\tau_w)_{oil} = \frac{0.3152 \times 0.0013 \times 2}{\sqrt{\frac{0.0013 \times 0.5}{800 \times 2}}} = 1.2858 \ N/m^2$$

Friction co-efficient :-

(a) at $x = 0.7$ m

$$f_{air} = \frac{2 \cdot F''(0)}{\sqrt{Re_x}}$$

$$Re_{x=0.7} = \frac{1.2 \times 50 \times 0.7}{0.000017} = 2470588.23$$

$$\therefore \sqrt{Re_{x=0.7}} = 1571.81$$

$$\therefore f_{air} = \frac{2 \times 0.3152}{1571.81}$$

$$\therefore f_{air} = 4.010 \times 10^{-4}$$

$$f_{oil} = \frac{2 \cdot F''(0)}{\sqrt{Re_x}}$$

$$Re_{x=0.7} = \frac{800 \times 2 \times 0.7}{0.0013} = 861538.46$$

$$\sqrt{Re_{x=0.7}} = 928.19$$

$$f_{oil} = \frac{2 \times 0.3152}{928.19}$$

$$f_{oil} = 6.7917 \times 10^{-4}$$

(b) at $x = 0.5$ m

$$Re_{x=0.5} = \frac{1.2 \times 50 \times 0.5}{0.000017} = 1764705.88$$

$$\sqrt{Re_{x=0.5}} = 1328.42$$

$$f_{air} = \frac{2 \times 0.3152}{1328.42}$$

$f_{air} = 4.7455 \times 10^{-4}$

$f_{oil} = \dfrac{2 \times 0.3152}{784.47}$

$Re_{x=0.5} = \dfrac{800 \times 2 \times 0.5}{0.0013}$

$= 615384.62$

$\sqrt{Re_{x=0.5}} = 784.47$

$f_{oil} = 8.0360 \times 10^{-4}$

(c) Calculate both the momentum and Displacement thicknesses.

→ Displacement thickness $\delta^* = 1.72 \sqrt{\dfrac{\nu x}{U_\infty}}$

(a) at $x = 0.7\,m$

$\delta^*_{air} = 1.72 \times \sqrt{\dfrac{0.000017 \times 0.7}{1.2 \times 50}}$

$\delta^*_{air} = 7.6599 \times 10^{-4}$

$\delta^*_{oil} = 1.72 \times \sqrt{\dfrac{0.0013 \times 0.7}{800 \times 2}}$

$\delta^*_{oil} = 1.2971 \times 10^{-3}$

(b) at $x = 0.5\,m$

$\delta^*_{air} = 1.72 \times \sqrt{\dfrac{0.000017 \times 0.5}{1.2 \times 50}} = 6.4738 \times 10^{-4}$

$$\delta^*_{oil} = 1.72\sqrt{\frac{0.0013 \times 0.5}{800 \times 2}}$$

$$\delta^*_{oil} = 1.0963 \times 10^{-3}$$

→ Momentum Thickness $\theta = 0.6304\sqrt{\frac{v \cdot x}{U_\infty}}$

(a) at $x = 0.7\,m$

$$\theta_{air} = 0.6304\sqrt{\frac{0.000017 \times 0.7}{1.2 \times 50}}$$

$$\theta_{air} = 2.8075 \times 10^{-4}$$

$$\theta_{oil} = 0.6304\sqrt{\frac{0.0013 \times 0.7}{800 \times 2}}$$

$$\theta_{oil} = 4.7542 \times 10^{-4}$$

(b) at $x = 0.5\,m$

$$\theta_{air} = 0.6304\sqrt{\frac{0.000017 \times 0.5}{1.2 \times 50}}$$

$$\theta_{air} = 2.3727 \times 10^{-4}$$

$$\theta_{oil} = 0.6304\sqrt{\frac{0.0013 \times 0.5}{800 \times 2}} = 4.0180 \times 10^{-4}$$

(d) Calculate the drag co-efficient if the plate has length L = 1m.

Drag Co-efficient $C_D(L) = \frac{4 \cdot F''(0)}{\sqrt{Re_L}}$

$(Re_L)_{air} = \frac{1.2 \times 50 \times 1}{0.000017} = 3529411.77$

$(Re_L)_{oil} = \frac{800 \times 2 \times 1}{0.0013} = 1230769.23$

$(C_D)_{air} = \frac{4 \times 0.3152}{1878.67} = 6.7111 \times 10^{-4}$

$(C_D)_{oil} = \frac{4 \times 0.3152}{1109.40} = 1.1365 \times 10^{-3}$

Comment on any differences that you see, in your results, between the 2 fluids.

From the above four sub questions, below shown the summary table.

		At x=0.5 m	At x=0.7 m
Wall shear stress (N/m^2)	AIR	0.711	0.6015
	OIL	1.2858	1.086
Friction coefficient	AIR	4.7455*E-4	4.010*E-4
	OIL	8.0360*E-4	6.7917*E-4
Momentum thickness	AIR	2.3727*E-4	2.8075*E-4
	OIL	4.0180*E-4	4.7542*E-4
Displacement thickness	AIR	6.4738*E-4	7.6599*E-4
	OIL	1.0963*E-3	1.2971*E-3
Length L=1m			
Drag coefficient	AIR	6.7111*E-4	
	OIL	1.1365*E-3	

From the value of the drag coefficient, it is observed different for both fluids and oil has higher drag coefficient then the air and we can say that it depends on the Reynolds number. For both fluids as the Reynolds number increases the overall boundary layer thickness decreases. So it is all depend on the Reynolds number at the end.