Machine Learning Assignment # 1
Universität Bern

Due date: 11/10/2017

Late submissions will incur a penalty. Submit your answers in ILIAS (as a pdf or as a picture of your written notes if the handwriting is very clear). Submission instructions will be provided via email. You are not allowed to work with others.

Linear algebra review **[Total 100 points]**

Solve each of the following problems and show all the steps of your working.

1. Use the definition of trace to show that $trAB = trBA$, where $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times m}$. **[10 points]**

2. Consider the matrix $G = A^\top A$, where $A \in \mathbb{R}^{m \times n}$. Show that G is positive semi-definite. **[10 points]**

3. Show that a positive definite matrix is non-singular. **[10 points]**

4. Show that if (λ_i, x_i) are the i-th eigenvalue and i-th eigenvector of a non-singular and symmetric matrix $A \in \mathbb{R}^{n \times n}$, then $(\frac{1}{\lambda_i}, x_i)$ are the i-th eigenvalue and i-th eigenvector of A^{-1}.
   Hint: use the eigendecomposition of A. **[20 points]**

5. Given two sets of vectors $\{x_1, ...x_n\} \subset \mathbb{R}^n$ and $\{y_1, ..., y_n\} \subset \mathbb{R}^n$, show that rank $\left[\sum_{i=1}^m x_i y_i^\top\right] \leq m$.
   Hint: First show that the square matrix $x_i y_i^\top$ has rank 1. **[20 points]**

6. Show that rank$(A) \leq \min\{m, n\}$, where $A \in \mathbb{R}^{m \times n}$. **[10 points]**

7. In each of the following cases, state whether the matrix A is guaranteed to be singular or not. Justify your answer in each case.
   (a) $A \in \mathbb{R}^{m \times n}$ is a full rank matrix. **[4 points]**
   (b) $|A| = 0$. **[4 points]**
   (c) A is an orthogonal matrix. **[4 points]**
   (d) A has no eigenvalue equal to zero. **[4 points]**
   (e) A is a symmetric matrix with non-negative eigenvalues. **[4 points]**