Linear Algebra Review Assignment 1

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1.)

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}$$

$$tr(AB) = (a_{11}b_{11}+...+a_{1n}b_{n1})+(a_{21}b_{12}+...+a_{2n}b_{n2})+...+(a_{m1}b_{1m}+...+a_{mn}b_{nm})$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}b_{ji} = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ji}a_{ij}$$

$$= (b_{11}a_{11} + b_{12}a_{21} + ... + b_{1m}a_{m1}) + ... + (b_{n1}a_{1n} + ... + b_{nm}a_{mn}) = tr(BA)$$

This is valid because of the associativity and commutativity in $\mathbb{R}$.

2.)

$$G = A^T A$$

We know from linear algebra:

$$(Ax)^T = x^T A^T$$

$$x^T G x = x^T A^T A x = (Ax)^T (Ax)$$

Let Ax be y.

$$y^T y = \textstyle\sum_{i=0}^{n} y_i y_i = \sum_{i=0}^{n} y_i^2 \succeq 0$$

3.)
Let's assume singularity :

$$\exists x \in Ker(A) \neq 0 : Ax = 0$$

$$=> x^T Ax = x^T 0 = 0$$

what is a contradiction to the definition of positive definite:

$$x^T Ax > 0$$

$$=> \nexists x \in Ker(A) \neq 0 : Ax = 0$$

$$=> Ker(A) = \{0\}$$

A is nonsingular.
4.)

$$A \in \mathbb{R}^{nxn}\ Ax_i = \lambda_i x_i$$

For the following we use the knowledge that A is regular and symmetrical, in other words eigendecomposable where no eigenvalue is zero.

$$A = QDQ^{-1},\ A^{-1} = QD^{-1}Q^{-1}$$

where D is diagonal. Now $D^{-1}$is the diagonal matrix to $A^{-1}$where the eigenvalues of A^{-1} are on the diagonal. We also know that the eigenvalue $\lambda_i$is on the matrix entry D_{ii}

$$D_{ii}^{-1} = \lambda_i^{-1} = \frac{1}{\lambda_i}$$

$$A^{-1}x_i = \frac{1}{\lambda_i} x_i \square$$

5.)

$$A = x_i y_i^T \in \mathbb{R}^{nxm}$$

$$A_{ij} = x_i x_j$$

$$col_i = \left\{ \begin{pmatrix} x_1 y_i \\ \vdots \\ x_n y_i \end{pmatrix} \right\}$$

All columns are linearly dependent from one column since column i can be retrieved by $col_j / y_j \cdot y_i$. Therefore the rank of this matrix is 1.

$$\begin{pmatrix} x_1 y_i \\ \vdots \\ x_n y_i \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 y_j \\ \vdots \\ x_n y_j \end{pmatrix}$$

$$\implies A = \sum_{i=0}^{m} x_i y_i^T \implies col_i = \begin{pmatrix} x_{11}y_{1i} \\ \vdots \\ x_{1n}y_{1i} \end{pmatrix} + ... + \begin{pmatrix} x_{m1}y_{mi} \\ \vdots \\ x_{mn}y_{mi} \end{pmatrix}$$

$$A = \sum_{i=0}^{m} G_i : G_i = x_i y_i^T$$

This means that A is a linar combination of m matrices each of rank 1. Therefore the rank cannot exceed m: $rank(A) \leq m$

6.)

We know from linear algebra that the column rank is the same as the row rank. Following from this the rank cannot exceed the amount of BOTH rows AND columns and must therefore be the minimum of these.

Proof Lemma:

$A \in R^{mxn}$. Let the row rank be r. Therefore the dimension of the row space is alos r. Now let $x_1, \ldots, x_r$be a basis of this row space. Now consider a linear homogeneous relation with these vectors and scalar coefficents $c_1, \ldots, c_r$.

$$0 = c_1 Ax_1 + \cdots + c_r Ax_r = A(c_1x_1 + \ldots c_r x_r) = Av$$

with $v = c_1x_1 + \cdots + c_r x_r$.Now v is obviously a linear combination of vectors of the row space in A, which implies that v belongs to this row space. Furthermore since $Av = 0$, v is orthogonal to every row vector of A. These facts together imply that v is orthogonal to itself, which means that $c_1x_1 + \ldots c_r x_r = 0$. Recall now that the $x_i$are a basis and therefore linearly independent. Therefore it must be true that $c_1 = \cdots = c_r = 0$. Therefore $Ax_1, \ldots, Ax_r$are also linearly independent. Now each $Ax_i$is a vector in the column space of A, so the $Ax_i$are linearly independent vectors in the column space and therefore the column rank must at least be as big as r. Now you can do this exact same proces with $A^\top$what leads to reverse equality and concludes this proof.

7.)

a) We know now that the rank, even though it's full, cannot exceed the minimum of n and m. So we're gonna have either linear dependent rows or columns. So there exists a vector v and coefficients c (not all zero) with

$$c_1v_1 + ... + c_r v_r = 0$$

$$\implies Mc = c_1v_1 + ... + c_r v_r = 0$$

We also know that $M0 = 0$. So if M^{-1}existed the following two things would be given:

$$0 = M^{-1}0 \; , \; c = M^{-1}0$$

But it is given that c cannot be zero and therefore M^{-1}cannot exist.

In the case $m = n$, the matrix is regular. Therefore singularity is not guaranteed.

b) Singular, since there are rows in this matrix that are linearly dependent and we know that a matrix is regular if all rows are linarly independent. Furthermore a lemma states that a matrix is singular iff $|A| \neq 0$

c) orhtogonal matrices have the property that all rows and columns are linarly independent. Therefore it is clear that this matrix must also be regular.

d) Suppose A is square matrix and has an eigenvalue of 0. For the sake of contradiction, lets assume A is invertible.

Consider, Av=λv, with λ=0 means there exists a non-zero vector v such that Av=0. This implies Av=0v⇒Av=0 For an invertible matrix A, Av=0 implies v=0. So, Av=0=A*0. Since v cannot be 0,this means A must not have been one-to-one. Hence, our contradiction, A must not be invertible. So we know that a matrix is invertible if and only if it has no eigenvalue equal to zero.

e) As we stated before, as long as the matrix has no eigenvalue equal to zero it is regular. In all other cases it is singular.