Linear Algebra Review Assignment 1

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1.)

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}$$

$$tr(AB) = (a_{11}b_{11}+...+a_{1n}b_{n1})+(a_{21}b_{12}+...+a_{2n}b_{n2})+...+(a_{m1}b_{1m}+...+a_{mn}b_{nm})$$

$$= (b_{11}a_{11} + b_{12}a_{21} + ... + b_{1m}a_{m1}) + ... + (b_{n1}a_{1n} + ... + b_{nm}a_{mn})$$

$= tr(BA)$This is valid because of the associativity and commutativity in $\mathbb{R}$.

2.)

$$G = A^T A$$

We know from linear algebra:

$$(Ax)^T = x^T A^T$$

$$x^T A^T Ax = (Ax)^T (Ax)$$

Let Ax be y.

$$y^T y = \sum_{i=0}^{n} y_i y_i = \sum_{i=0}^{n} y_i^2 \succeq 0$$

3.)
Let's assume singularity :

$$\exists x \in Ker(A) \neq 0 : Ax = 0$$

$$=> x^T Ax = x^T 0 = 0$$

what is a contradiction to the definition of positive definite:

$$x^T Ax > 0$$

$$=> \nexists x \in Ker(A) \neq 0 : Ax = 0$$

$$=> Ker(A) = \{0\}$$

A is nonsingular.

4.)

$$A \in \mathbb{R}^{nxn}\ Ax_i = \lambda_i x_i$$

For the following we use the knowledge that A is regular and symmetrical, in other words eigendecomposable where no eigenvalue is zero.

$$A = QDQ^{-1}\ A^{-1} = QD^{-1}Q^{-1}$$

where D is diagonal.

$$D_{ii}^{-1} = \lambda_i^{-1} = \frac{1}{\lambda_i}$$

$$A^{-1}x_i = \frac{1}{\lambda_i}x_i\Box$$

5.)

$$A = x_i y_i^T \in \mathbb{R}^{nxm}$$

$$A_{ij} = x_i x_j$$

$$col_i = \left\{ \begin{pmatrix} x_1 y_i \\ \vdots \\ x_n y_i \end{pmatrix} \right\}$$

All columns are linearly dependent from one column since column i can be retrieved by col_j / y_j * y_i

$$\begin{pmatrix} x_1 y_i \\ \vdots \\ x_n y_i \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 y_j \\ \vdots \\ x_n y_j \end{pmatrix}$$

$$\Longrightarrow A = \sum_{i=0}^{m} x_i y_i^T \Longrightarrow col_i = \begin{pmatrix} x_{11} y_{1i} \\ \vdots \\ x_{1n} y_{1i} \end{pmatrix} + ... + \begin{pmatrix} x_{m1} y_{mi} \\ \vdots \\ x_{mn} y_{mi} \end{pmatrix}$$

$$A = \sum_{i=0}^{m} G_i : G_i = x_i y_i^T$$

This means that A is a linar combination of m matrices each of rank 1. Therefore the rank cannot exceed m: rank(A) <= m

6.)

We know from linear algebra that the column rank is the same as the row rank. If proof for this well known lemma is needed, then you can find it in the jpeg picture. Following from this the rank cannot exceed the amount of BOTH rows AND columns and must therefore be the minimum of these.

7.)

a) We know that the rank even though it's full cannot exceed the minimum of n and m. So we're gonna have either linear dependent rows or columns. So there exists a vector v and coefficients c (not all zero) with

$$c_1 v_1 + ... + c_r v_r = 0$$

$$\implies Mc = c_1 v_1 + ... + c_r v_r = 0$$

We also know that $M0 = 0$. So if M^{-1}existed the following two things would be given:

$$0 = M^{-1}0 \, , \, c = M^{-1}0$$

But it is given that c cannot be zero and therefore M^{-1}cannot exist.

b) Singular, since there are rows in this matrix that are linearly dependent and we know that a matrix is regular if all rows are linarly independent.

c) orhtogonal matrices have the property that all rows and columns are linarly independent. Therefore it is clear that this matrix must also be regular.

d) Suppose A is square matrix and has an eigenvalue of 0. For the sake of contradiction, lets assume A is invertible.

Consider, Av=λv, with λ=0 means there exists a non-zero vv such that Av=0. This implies Av=0v⇒Av=0 For an invertible matrix A, Av=0 implies v=0. So, Av=0=A*0. Since v cannot be 0,this means A must not have been one-to-one. Hence, our contradiction, A must not be invertible. So we know that a matrix is invertible if and only if it has no eigenvalue equal to zero.

e) As we stated before, as long as the matrix has no eigenvalue equal to zero it is regular. In all other cases it is singular.