Harmonische Trillingen

1. Harmonische oscillatie

$$\sum \vec{F} = m \cdot \vec{a}$$

$$m \frac{d^2x}{dt^2} = -k\,x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$\frac{k}{m}$ => eigenfrequentie ω_0^2

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \qquad (1)$$

Algemene methode:

$$x(t) = e^{\lambda t}$$

$$(1) \Rightarrow \lambda^2 e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0$$

$$\Rightarrow e^{\lambda t}(\lambda^2 + \omega_0^2) = 0$$

$$\Rightarrow \lambda = \pm \omega_0 i$$

Oplossingsruimte:

$$\left\{ e^{i\omega_0 t}, e^{-i\omega_0 t} \right\}$$

$$\Rightarrow x(t) = \underbrace{A e^{i\omega_0 t} + B e^{-i\omega_0 t}}_{\text{homogeen}}$$

$$= A(\cos(\omega_0 t) + i\sin(\omega_0 t))$$
$$+ B(\cos(-\omega_0 t) + i\sin(-\omega_0 t))$$

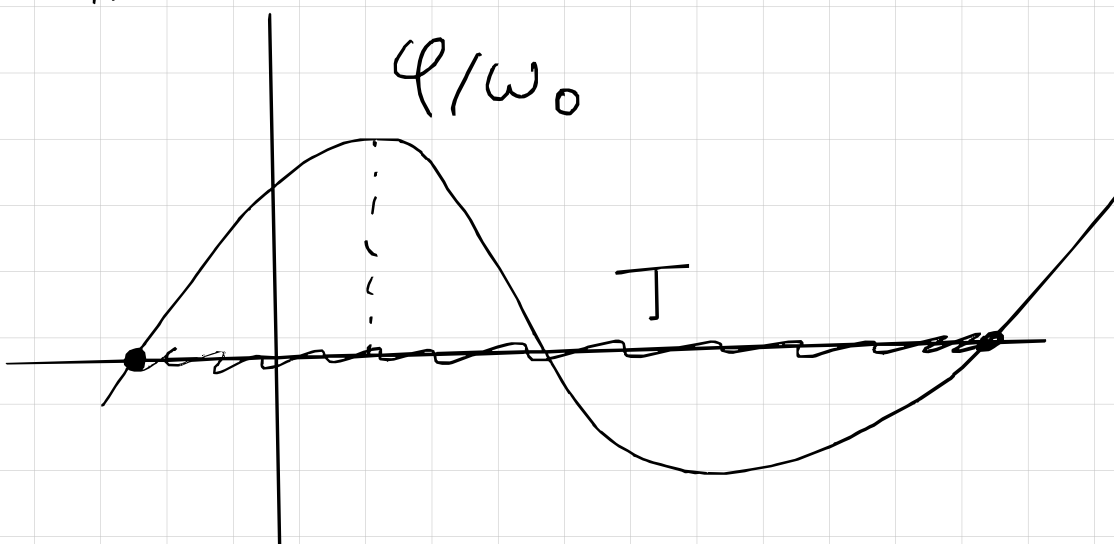
$\rightsquigarrow$ Reële ruimte:

Basisfuncties: $\{\cos\omega_0 t, \sin\omega_0 t\}$

$$\Rightarrow x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \quad (2)$$

$$\Rightarrow \equiv x(t) = C\cos(\omega_0 t - \varphi) \quad (3)$$

$$(3):\quad x(t) = C\cos(\omega_0 t - \varphi)$$
$$= C(\cos\omega_0 t\cos(\varphi) + \sin\omega_0 t\sin\varphi)$$
$$= \underbrace{C\cos\varphi}_{A}\cdot\cos(\omega_0 t) + \underbrace{C\sin\varphi}_{B}\cos(\omega_0 t)$$
$$A^2 + B^2 = C^2\cos^2\varphi + C^2\sin^2\varphi$$
$$= C^2$$
$$\frac{B}{A} = \tan\varphi$$



$$\left.\begin{aligned}\omega_0 T &= 2\pi\\ T &= \frac{2\pi}{\omega_0}\end{aligned}\right\} \Rightarrow \omega_0 = \frac{2\pi}{T}$$

2: gedempte harmonische trilling

Tweede wet van newton (laag Reynolds nummer)

$$m \frac{d^2x}{dt^2} = -kx - \gamma \frac{dx}{dt} \qquad (1)$$

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0 \qquad (2)$$

$$\rightarrow x(t) = e^{\lambda t}$$

$$(2) \rightarrow m\lambda^2 e^{\lambda t} + \gamma \lambda e^{\lambda t} + k e^{\lambda t} = 0$$

$$e^{\lambda t} \underbrace{(m\lambda^2 + \gamma\lambda + k)}_{=0} = 0$$

$$D = \sqrt{\gamma^2 - 4m \cdot k}$$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4m \cdot k}}{2m}$$

3 gevallen afhankelijk van D

2.1. zwakke demping
$(\gamma^2 - 4m \cdot k < 0)$

$$\lambda_{1,2} = \frac{-\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \underbrace{\frac{k}{m}}_{\omega_0^2}}$$

$$\left.\begin{aligned} &\frac{\gamma^2}{4m^2} - \frac{k}{m} < 0 \\ &\underbrace{-\frac{\gamma^2}{4m^2} - \frac{k}{m}}_{= \omega_1^2} > 0 \end{aligned}\right\} \Rightarrow \omega_1 < \omega_0$$

$$\Rightarrow \lambda_{1,2} = \frac{-\gamma}{2m} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}}$$

$$= \frac{-\gamma}{2m} \pm i\omega_1$$

$$\Rightarrow x(t) \quad e^{\frac{-\gamma}{2m} \pm i\omega}$$

$$\rightsquigarrow \left\{ e^{\frac{-\gamma}{2m} + i\omega}, e^{\frac{-\gamma}{2m} - i\omega} \right\}$$

$$\Rightarrow \quad x(t) = e^{-\frac{\gamma}{2m}t}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right)$$

$$x(t) = e^{-\frac{\gamma}{2m}t}\, C \cos(\omega_1 t - \varphi)$$

2.2. Sterke demping

$(\gamma^2 - 4mk > 0)$

$$\rightarrow \lambda_{1,2} = -\frac{\gamma}{2m} \pm \underbrace{\sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{\frac{m}{\omega_0^2}}}}_{\equiv \delta}$$

$$-\frac{\gamma}{2m} > \delta$$

$$\left(\frac{\gamma}{2m}\right)^2 - \omega_0^2 < \frac{\gamma}{2m}$$

$$\Rightarrow \lambda_1 < 0$$
$$\lambda_2 < 0$$

$$\lambda_2^- < \lambda_1^+ < 0$$

$$\Rightarrow x(t) = e^{-\frac{\gamma}{2m}t} e^{\pm \delta t}$$

$$x(t) = e^{-\frac{\gamma}{2m}t}\left(A e^{\delta t} + B e^{-\delta t}\right)$$

2.3. Kritisch demping

$$\gamma^2 - 4mk = 0 \Rightarrow \gamma^2 = 2\sqrt{mk}$$

$$\lambda_1 = \frac{-\gamma}{2m}$$

$$= -\frac{\sqrt{mk}}{m} = -\sqrt{\frac{k}{m}} = -\omega_0$$

$$\xrightarrow{\text{verz}} \left\{ e^{-\omega_0 t}, t e^{-\omega_0 t} \right\}$$

$$x(t) = Ae^{-\omega_0 t} + Bt e^{-\omega_0 t}$$

$$= (A + Bt)e^{-\omega_0 t}$$

$$\frac{dx}{dt} = Be^{-\omega_0 t} - \omega_0 e^{-\omega_0 t}(A+Bt)$$

$$t = 0 \rightarrow \begin{cases} x_0 = A \\ B_0 = x_0 \omega_0 \end{cases}$$

3. harmonische oscilatie met aandrijving

$$m \frac{d^2x}{dt^2} = -kx - \gamma \frac{dx}{dt} + F\cos(\omega t)$$

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = F\cos(\omega t) \quad (1)$$

$\longrightarrow$ niet homogeen

$\hookrightarrow$ 1) los homogene variant op

$\rightsquigarrow x^{hom}(t)$

2) zoek 1 particuliere oplossing

$x^{part}(t)$

$\Rightarrow x(t) = x^{hom}(t) + x^{part}(t)$

$$x^{part}(t) = A\sin(\omega t - \varphi)$$

Hoorcollege deel 2

$\rightsquigarrow$ x^{part} invullen in (1)

$$-mA\omega^2 \sin(\omega t - \varphi) + \gamma A \omega \cos(\omega t - \varphi) + kA \sin(\omega t - \varphi) = F \cos(\omega t)$$

$$\Rightarrow \underbrace{-mA\omega^2 \cos\varphi \sin\omega t + \gamma A \omega \sin\varphi \sin\omega t + kA \cos\varphi \sin\omega t}_{=0} + \underbrace{mA\omega^2 \sin\varphi \cos\omega t + \gamma A \omega \cos\varphi \cos\omega t - kA \sin\varphi \cos\omega t}_{=F}$$

Leen heeft het voor mij gedaan:

$$A = \frac{F}{\gamma^2 \omega^2 + m^2(\omega^2 - \omega_0^2)^2}$$

$$\tan\varphi = \frac{m(\omega^2 - \omega_0^2)}{\gamma \omega}$$

Rare punten : $\left.\begin{array}{l}\gamma = 0 \\ \omega = \omega_0\end{array}\right\} \to A \to \infty$

1ste wet van Kepler :

$$\sum \vec{F} = m \cdot \vec{a}$$

$$m\vec{a} \begin{cases} \hat{r} \to & m(\ddot{r} - r\dot{\theta}^2) \quad (1) \\ \hat{\theta} \to & m(2\dot{r}\dot{\theta} + r\ddot{\theta}) \quad (2) \end{cases}$$

$$\vec{F_G} = -G \frac{m_1 m_2}{r^2} \hat{r} = (1)$$

$$\underbrace{0}_{\text{geen enkele kracht op } \hat{\theta}} = (2)$$

$$\to \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right)$$

$$\frac{\cancel{m}}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0$$

$$\boxed{r^2 \frac{d\theta}{dt} = \frac{\overset{\text{cte}}{L}}{m}} \quad (3)$$

(1):

$$u = \frac{1}{r} \quad \text{of} \quad r = \frac{1}{u}$$

(3) $L = \frac{m\,\omega}{u^2}$

$L' = \frac{L}{m} = \frac{1}{u^2}\frac{d\theta}{dt}$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= \frac{-1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= -L'\frac{du}{d\theta} \quad (4)$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = -L'\frac{d}{dt}\left(\frac{du}{d\theta}\right)$$

$$= -L'\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\cdot\frac{d\theta}{dt}$$

$$= -L'\frac{d\theta}{dt}\cdot\frac{d^2u}{d\theta^2}$$

$$= -L'^2u^2\frac{d^2u}{d\theta^2} \quad (5)$$

terug (1):

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \vec{F_G}$$

$$\to -\left(\frac{1}{u}\frac{L'^2}{r^4}\right) = -u^3 L'^2$$

$$\frac{d^2 r}{dt^2} - u^3 L'^2 = \vec{F}_G \quad (6)$$

$$(5) \quad -L'^2 u^2 \frac{d^2 u}{d\theta^2} - u^3 L'^2 = -GMu^2$$

$$-L'^2 \frac{d^2 u}{d\theta^2} - L'^2 u = -GM$$

$$\frac{d^2 u}{d\theta^2} + u = \frac{GM}{L'^2}$$

$\to$ harmonische trilling

$\rightsquigarrow u(\theta) = A\cos\theta + B\sin\theta$
(u^{hom})

$\rightsquigarrow u^{part}: \quad u^{part}(\theta) = \frac{GM}{L'^2}$

$$\underbrace{\hspace{10cm}}$$

$$u(\theta) = A\cos\theta + B\sin\theta + \frac{GM}{L'^2}$$

$$\frac{1}{r} = \frac{GM}{L'^2}\left(1 + A'\cos\theta + B'\sin\theta\right)$$

$$= \frac{GM}{L'^2}\left(1 + e\cos(\theta - \varphi)\right)$$

$$\left(e^2 = A'^2 + B'^2\right)$$

$e = 0$ cirkel
$e < 1$ ~~??~~ ellips
$e = 1$ parabool
$e > 1$