MATHEMATICS PROJECT

FINDING THE IMAGE OF A LINE WITH RESPECT TO A GIVEN PLANE

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STANDARD: XII
ROLL NO.: 15

1 CONTENTS

2 INTRODUCTION

A plane is a flat, two-dimensional surface that extends infinitely far. A plane is the two-dimensional analog of a point (zero dimensions), a line (one dimension), and three-dimensional space. A plane in three-dimensional space has the equation

$Ax + By + Cz + D = 0$,

where at least one of the numbers a, b, and c must be non-zero. A plane in 3D coordinate space is determined by a point and a vector that is perpendicular to the plane.

An equation of the plane containing the point (x_0, y_0, z_0) with normal vector N= <A, B, C> is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$.

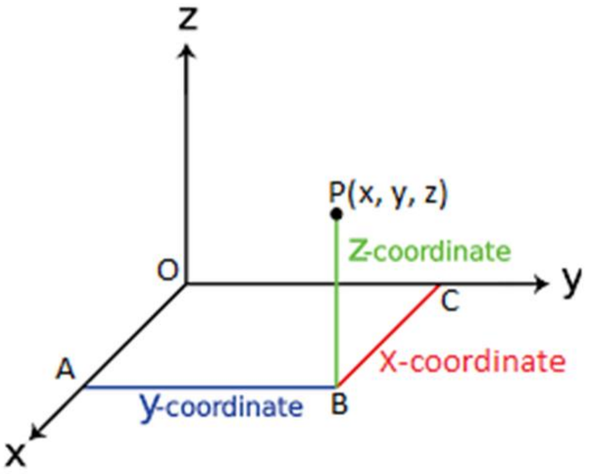
Intercept form of equation of the plane having intercepts a, b, c on the x, y, z axes respectively is $x/a + y/b + z/c = 1$.

3 COORDINATE SYSTEM FOR THREE DIMENSIONAL SPACE COORDINATES OF A POINT IN THREE DIMENSIONAL SPACE

In three-dimensional space, the Cartesian coordinate system is based on three mutually perpendicular coordinate axes: the x-axis, the y-axis, and the z-axis.

3.1 THREE DIMENSIONAL COORDINATE SYSTEM

Let there be a point P in space. If we drop a perpendicular PB on the XY plane and then from point B, we drop perpendiculars BA and BC on the x-axis and y-axis respectively. Assuming the length of the perpendiculars BC, BA and PB as x, y and z respectively. These lengths x, y and z are known as the co-ordinates of the point P in three-dimensional space. It must be noted that while giving the coordinates of a point, we always write them in order such that the co-ordinate of x-axis comes first, followed by the co-ordinate of the y-axis and the z-axis. Thus for each point in space there exist an ordered 3-tuple of real numbers for its representation.



4 METHOD OF LOCATING DIFFERENT POINTS IN THREE DIMENSIONAL SPACE

Graphing in xyz-space can be difficult because, unlike graphing in the xy-plane, depth perception is required. To simplify plotting of points, one can make use of projections onto the coordinate planes. The projection of a point (x, y, z) onto the xy-plane is obtained by connecting the point to the xy-plane by a line segment that is perpendicular to the plane, and computing the intersection of the line segment with the plane.

4.1 HOW TO PLOT THE POINTS IN THREE-DIMENSIONAL PLANE?

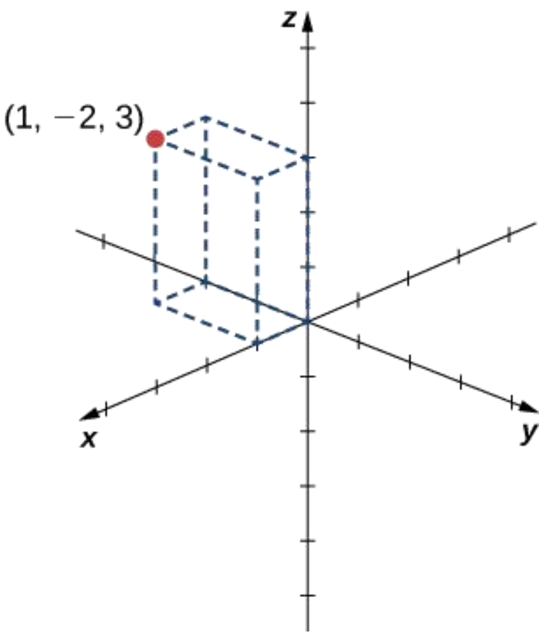
- Locate the point “x” on the X-axis.
- From the point x, moving parallel to the Y-axis, locate the point “y”.
- Similarly, from the determined point, moving parallel to the Z-axis, locate the point “z”.
- This is the final coordinate point in the three-dimensional plane, which we are looking for.

Example:

Sketch the point (1, −2, 3) in three-dimensional space.

Solution

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction.



5 DISTANCE BETWEEN POINTS

Derived from the Pythagorean Theorem, the distance formula is used to find the distance between two points in the plane. The Pythagorean Theorem, $a_2 + b_2 = c_2$ is based on a right triangle where a and b are the lengths of the legs adjacent to the right angle, and c is the length of the hypotenuse.

5.1 SECTION FORMULA

5.1.1 INTERNALLY

The coordinates of the point P(x, y, z) dividing the line segment joining the points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) in the ratio m:n internally are given by:

($mx_2 + nx_1$/ m + n, $my_2 + ny_1$/ m + n, $mz_2 + nz_1$/ m + n)

5.1.2 EXTERNALLY

If the given point P divides the line segment joining the points A(x1, y1, z1) and B(x2, y2, z2) externally in the ratio m:n, then the coordinates of P are given by replacing n with –n as:

$[(mx_2 – nx_1)/ (m – n), (my_2 – ny_1)/ (m−n), (mz_2 – nz_1)/ (m − n)]$

5.2 MID POINT FORMULA

The midpoint m of the segment joining x_1, y_1, z_1 and x_2, y_2, z_2 are $x_1 + x_2$ over 2, $y_1 + y_2$ over 2 and $z_1 + z_2$ over 2 and that's the midpoint formula.

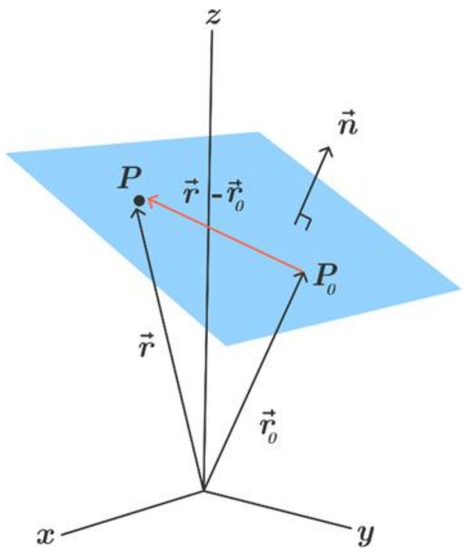
$M = [(x_1 + x_2)/2, (y_1 + y_2)/2, (z_1 + z_2)/2]$

6 EQUATION OF COORDINATE PLANE

If we know the normal vector of a plane and a point passing through the plane, the equation of the plane is established by A(x − x 1) + B(y – y1) + C(z – z1) = 0

The general equation of a plane in the Cartesian coordinate system is represented by the linear equation,

Ax + By + Cz + D=0.



6.1 COORDINATES OF ANY POINT ON XY PLANE, YZ PLANE AND XZ PLANE

- X coordinate of XY plane is x.
- Y coordinate of XY plane is y.
- Z coordinate of XY plane is 0.
- X coordinate of YZ plane is 0.
- Y coordinate of YZ plane is y.
- Z coordinate of YZ plane is z.
- X coordinate of XZ plane is x.
- Y coordinate of XZ plane is 0.
- Z coordinate of XZ plane is z.

7 EQUATION OF PLANE

7.1 PARALLEL TO COORDINATE PLANES

In $Ax + By + Cz + D = 0$,

- If A=B=0, the plane is parallel to the xy-plane;
- If B=C=0, the plane is parallel to the yz-plane;
- If A=C=0, the plane is parallel to the xz-plane.

The equation of a plane which is parallel to each of the xy-, yz-, and zx- planes and going through a point P = (A, B, C) is determined as follows:

1) The equation of the plane which is parallel to the xy -plane is $z = C$.

2) The equation of the plane which is parallel to the yz -plane is $x = A$.

3) The equation of the plane which is parallel to the zx -plane is $y = B$.

7.2 PARALLEL TO COORDINATE AXES

In $Ax + By + Cz + D = 0$,

- If A=0, the plane is parallel to the x-axis;
- If B=0, the plane is parallel to the y-axis;
- If C=0, the plane is parallel to the z-axis;
- If D=0, the plane passes through the origin.

The equation of a plane which is parallel to x-, y-, and z- axis and going through a point P = (A, B, C) is determined as follows:

1) Equation of plane parallel to X-axis is $By + Cz + D = 0$

2) Equation of plane parallel to Y-axis is $Ax + Cz + D = 0$

3) Equation of plane parallel to Z-axis is $Ax + By + D = 0$

4) Equation of the plane passing through the origin is $Ax + By + Cz = 0$.

8 EQUATION OF LINES PARALLEL TO COORDINATE AXES

- The equations of a line parallel to X-axis are $y = B$, $z = C$.
- The equations of a line parallel to Y-axis are $z = C$, $x = A$.
- The equations of a line parallel to Z-axis are $x = A$, $y = B$.

9 DIRECTION COSINES

Direction cosines of a vector are the cosines of the angles between the vector and the three coordinate axes. Equivalently, they are the contributions of each component of the basis to a unit vector in that direction.

Example:

Question

Find the direction cosines of the vector i^ + 2j^ + 3k^.

Answer

Let r = i + 2j + 3k be the given vector.

The magnitude of vector r is $|r| = \sqrt{(1^2 + 2^2 + 3^2)} = \sqrt{14}$

Let the direction cosines of vector r be cos α, cos β, cos γ

We have $\cos\alpha = r.i/|r| = 1/\sqrt{14}$

We have $\cos\beta = r.j/|r| = 2/\sqrt{14}$

We have $\cos\gamma = r.k/|r| = 3/\sqrt{14}$

Therefore, the direction cosines of given vector are $1/\sqrt{14}$, $2/\sqrt{14}$, $3/\sqrt{14}$.

10 DIRECTION RATIOS

Any number proportional to the direction cosine is known as the direction ratio of a line. These direction numbers are represented by a, b and c.

Example:

Question

The direction ratios of the line joining the points (4, 3, −5) and (−2, 1, −8) are?

Answer

Given points P(4, 3, −5) Q(−2, 1, −8)

Direction ratios of PQ = Position vector of P − Position vector of Q

=4−(−2), 3−1, −5−(−8)

=6,2,3

11 ANGLE BETWEEN TWO LINES

If one of the line is parallel to y- axis, then the angle between two straight lines is given by tan θ = ±1/m where 'm' is the slope of the other straight line. If the two lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the formula becomes $\tan \theta = |(a_1b_2 - b_1a_2)/(a_1a_2 + b_1b_2)|$

Two lines are parallel if their slopes are equal and they have different y-intercepts.

If two non-vertical lines in the same plane intersect at a right angle, then they are said to be perpendicular. Horizontal and vertical lines are perpendicular to each other i.e. the axes of the coordinate plane. The slopes of two perpendicular lines are negative reciprocals.

Example:

Question

If A (-2, 1), B (2, 3) and C (-2, -4) are three points, fine the angle between the straight lines AB and BC.

Answer

Let the slope of the line AB and BC are m_1 and m_2 respectively.

Then,

$m_1 = (3 – 1)/(2 + 2)$

$m_2 = (-4 – 3)/(-2 – 2)$

Let θ be the angle between AB and BC. Then,

$\tan \theta = (m_2 – m_1)/(1 + m_1m_2)$

$\Rightarrow \theta = \tan^{-1} (2/3)$, which is the required angle.

12 STRAIGHT LINE IN SPACE

Any straight line in space can be written in two forms – Vector

equation, Cartesian form.

12.1 VECTOR FORM

Vector equation of a line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is given as, $r = a + t\,\vec{b}$ where t is a scalar.

Here a is the point lying on the line and b is the direction of line. Vector equation of a line passing through two points: If a line is passing through two points a and $\vec{b}$ then its vector equation is written as, $r = a + t\,(\vec{b} - a)$

12.2 CARTESIAN FORM

Cartesian equations of a line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $x-x_1/\, x2-x1 = y-y_1/\, y2-y1 = z-z_1/\, z_2-z_1$.

This is also called symmetric equations of line.

Example:

Question

The Cartesian(symmetric) equations of a line are $6x_2 = 3y + 1 = 2z - 2$. Find the vector equation of the line.

Answer

In symmetric form of line, coefficients of x, y and z must be unity. So to get proper form of symmetric equations, we make the coefficients of x, y and z as unity.

$6x-2 = 3y+1 = 2z - 2$

$6(x- 2/6) = 3(y+1/3) = 2(z-1)$

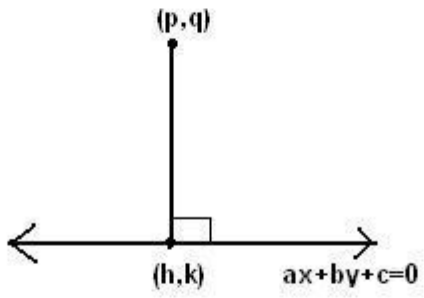
Dividing all sides by 6, $(x - 1/3)/1 = (y + 1/3)/2 = z - 1/3$

This shows line is passing through point $(1/3, -1/3, 1)$ and has direction vector as $(1,2,3)$.

So vector form of this line is given as, $r = (1/3, -1/3, 1) + t(1,2,3)$

13 FINDING FOOT OF PERPENDICULAR, LENGTH OF PERPENDICULAR AND IMAGE OF A POINT IN LINE

Let ax +by +c =0 be a straight line. If a perpendicular line is drawn from any point on the plane to this straight line, then the point of intersection of the given straight line and its perpendicular is called the foot of the corresponding perpendicular.



Example:

Question

Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point P(5,4,2) to the line r = −i^ + 3j^ + k^ + μ(2i^+3j^−k^). Also find the image of P is this line.

Answer

Any point on the line can be written in parametric form as (2λ−1, 3λ+3, −λ+1)

Assuming this as the foot of perpendicular from (5,4,2), we can equate the dot product of this vector and the line direction to zero.

∴ ((2λ − 1 − 5)i^+(3λ + 3 − 4)j^+(−λ + 1 − 2)k^).(2i^ + 3j^ − k^) = 0

⇒(2λ−6) × 2 + (3λ−1) × 3 +(−λ−1) × (−1) = 0

⇒4λ – 12 + 9λ – 3 + λ + 1 = 0

⇒14λ – 14 = 0

⇒λ = 1

The coordinates of the point are thus (1,6,0)

The length of the perpendicular can be found out by (5−1)2 + (4−6)2 + (2−0)2 = 16 + 4 + 4 = 24

The foot of perpendicular would be the midpoint of P and the image of P in the line.

∴(5i^+4j^+2k^) + (xi^+yj^+zk^) = 2×(i^+6j^)

⇒x = 2−5 = −3, y = 12−4 = 8, z = 0−2 = −2

The image of point P is thus (−3, 8, −2)

14 IMAGE OF A LINE WITH RESPECT TO A PLANE

Example:

Question

Find the equation of the image of the line (x – 1)/9 = (y – 2)/–1 = (z + 3)/-3 in the plane 3x – 3y + 10z – 26 = 0

Answer

Given line is (x – 1)/9 = (y – 2)/–1 = (z + 3)/–3 .... (i)

and the plane is 3x – 3y + 10z = 26 ... (ii)

The direction ratios of the line are 9, –1, – 3 and the direction ratios of the normal to the given plane are 3, –3 and 10.

Clearly line (i) is parallel to the plane (ii).

Let Q be the image of the point P(1, 2, –3).

Consider $Q = (\alpha, \beta, \gamma)$

Now,

$$\frac{\alpha - 1}{3} = \frac{\beta - 2}{-3} = \frac{\gamma + 3}{10} = -2\left(\frac{3 - 6 - 30 - 26}{9 + 9 + 100}\right)$$

$$\frac{\alpha - 1}{3} = \frac{\beta - 2}{-3} = \frac{\gamma + 3}{10} = 1$$

$$\alpha = 4,\ \beta = -1,\ \gamma = 7$$

Hence, the required image of the line w.r.t. the given plane is

$$\frac{x - 4}{9} = \frac{y + 1}{-1} = \frac{z - 7}{-3}$$

Question

Find the image of the line (x - 1)/3 = (y - 3)/1 = (z - 4)/-5 in the plane 2x - y + z + 3 = 0

Answer

3(2) + 1 (-1) + (-5) (1) = 6 - 1 - 5 = 0

Therefore, line is parallel to the plane.

Now finding the image of point (1, 3, 4) in the plane. Let the image be (x1, y1, z1). Therefore,

$$\frac{x_1-1}{2}=\frac{y_1-3}{-1}=\frac{z_1-4}{1}$$
$$=\frac{-2(2(1)-(3)+(4)+3)}{2^2+1^2+1^2}=\frac{-2\times\cancel{6}}{\cancel{6}}$$

Therefore,

$x_1=-3, y_1=5, z_1=2$

Therefore, required image of the line is

(x + 3)/3 = (y - 5)/1 = (z - 2)/-5

15 CONCLUSION

1. The reflection of a line must be a line.
2. A line can be defined by any two points on it.
3. If we can find the reflection of one point, the same method can be used to reflect two or more points.
4. The line joining a point and its image (reflection) will be normal to the mirror (the given plane).
5. To find the normal to a plane we can take the cross product of any two vectors in that plane.
6. A vector in a plane can be constructed using any two points in the plane.

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