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§1 Lagrangian Mechanics

Let us start from defining the **action**:

$$S\left[\boldsymbol{q}\left(t\right)\right]=\int_{0}^{T}L\left(\boldsymbol{q},\dot{\boldsymbol{q}},t\right)\,dt,$$

the **Eular-Lagrange equation** is derived from $\delta S=0$, with an additional restriction $\delta\boldsymbol{q}\left(0\right)=\delta\boldsymbol{q}\left(T\right)=0$:

$$\begin{aligned}\delta\int_{0}^{T}L\,dt&=\int_{0}^{T}\left(\nabla_{\boldsymbol{q}}L\cdot\delta\boldsymbol{q}+\nabla_{\dot{\boldsymbol{q}}}L\cdot\delta\dot{\boldsymbol{q}}\right)\,dt\\&=\int_{0}^{T}\left(\nabla_{\boldsymbol{q}}L-\frac{d}{dt}\nabla_{\dot{\boldsymbol{q}}}L\right)\cdot\delta\boldsymbol{q}\,dt=0.\end{aligned}$$

Since the choice of $\delta\boldsymbol{q}$ is arbitrary, we obtain the Eular-Lagrange equation (we place the highest order derivative at the beginning):

$$\frac{d}{dt}\boldsymbol{\nabla}_{\dot{\boldsymbol{q}}}L-\boldsymbol{\nabla}_{\boldsymbol{q}}L=\boldsymbol{0}.\tag{1}$$

But what is the **Lagrangian** function L?

†a Free Particle

First, there are some symmetry about the Lagrangian:

1) Space translation: $L\left(\boldsymbol{q},\dot{\boldsymbol{q}},t\right)=L\left(\dot{\boldsymbol{q}}\right)$.
2) Rotation: $L\left(\dot{\boldsymbol{q}}\right)=L\left(\left|\dot{\boldsymbol{q}}\right|^{2}\right)$.

In this sense:

$$2\frac{d}{dt}L'\left(\left|\dot{\boldsymbol{q}}\right|^{2}\right)\dot{\boldsymbol{q}}.$$

Compare to the Newtonian Mechanics:

$$m\ddot{\boldsymbol{q}}=0,$$

we make $L'\left(\left|\dot{\boldsymbol{q}}\right|^{2}\right)$ as a constant $m/2$:

$$L_{\text{free}}=T=\frac{1}{2}m\left|\dot{\boldsymbol{q}}\right|^{2}.\tag{2}$$

†b Conservative Force

In the case of conservative force:

$$m\ddot{\boldsymbol{r}}=\boldsymbol{F}=-\boldsymbol{\nabla}V\left(\boldsymbol{q}\right).$$

Then we have

$$\frac{d}{dt}\boldsymbol{\nabla}_{\dot{\boldsymbol{q}}}T-\boldsymbol{\nabla}_{\boldsymbol{q}}T+\boldsymbol{\nabla}V\left(\boldsymbol{q}\right)=\frac{d}{dt}\boldsymbol{\nabla}_{\dot{\boldsymbol{q}}}\left(T-V\right)-\boldsymbol{\nabla}_{\boldsymbol{q}}\left(T-V\right)=0.$$

So we define

$$L_{\text{conserve}}=T-V=\frac{1}{2}m\left|\dot{\boldsymbol{q}}\right|^{2}-V.\tag{3}$$

†c Constraint Force

Under the following conditions, we can use the **generalized coordinate**:

1) There is some holonomic constraint $f\left(\boldsymbol{q},t\right)$.
2) The constraint force satisfies $\boldsymbol{F}\cdot\delta\boldsymbol{q}=0$.

In Newtonian Mechanics:

$$m\ddot{\boldsymbol{q}}=\boldsymbol{F}_{\text{conserve}}+\boldsymbol{F}_{\text{constraint}}.$$

But in the variation of the action:

$$\begin{aligned}\delta S&=\int_{0}^{T}\left(m\ddot{\boldsymbol{q}}-\boldsymbol{F}_{\text{conserve}}-\boldsymbol{F}_{\text{constraint}}\right)\cdot\delta\boldsymbol{q}\,dt\\&=\int_{0}^{T}\left(m\ddot{\boldsymbol{q}}-\boldsymbol{F}_{\text{conserve}}\right)\cdot\delta\boldsymbol{q}\,dt=0\end{aligned}$$

Chapter 1

Introduction to Statistics

§1 Measure Theory

†a Measurable Space

A **measurable space** is a set Ω with a **σ-algebra** $\mathcal{F} \subseteq 2^{\Omega}$, s.t.

1) $\Omega \in \mathcal{F}$.
2) $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$.
3) $S \subseteq \mathcal{F}, S \preceq \aleph_0 \implies \bigcup S \in \mathcal{F}$.

The **σ-algebra generated by** $S \subseteq 2^{\Omega}$, denoted by

$$\sigma \langle S \rangle ,$$

is defined to be the intersection of all σ-algebra containing S*.

§1.1 *Proposition:*

Let $(\Omega, \mathcal{F})$ be a measurable space:

1) The definition is equivalent to replace (3) by:

$$S \subseteq \mathcal{F}, S \preceq \aleph_0 \implies \bigcap S \in \mathcal{F}.$$

2) $\emptyset \in \mathcal{F}$.
3) $A, B \in \mathcal{F} \implies A \setminus B \in \mathcal{F}$.

An **algebra** is a collection $\mathcal{F} \subseteq 2^{\Omega}$ s.t.

1) $\Omega \in \mathcal{F}$.
2) $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$.
3) $S \subseteq \mathcal{F}, S \prec \aleph_0 \implies \bigcup S \in \mathcal{F}$.

§1.2 *Proposition:*

Let Ω be a set, and let $\mathcal{F} \subseteq 2^{\Omega}$ be an algebra:

1) The definition is equivalent to replace (3) by:

$$S \subseteq \mathcal{F}, S \prec \aleph_0 \implies \bigcap S \in \mathcal{F}.$$

2) $\emptyset \in \mathcal{F}$.
3) $A, B \in \mathcal{F} \implies A \setminus B \in \mathcal{F}$.

§1.3 *Lemma:*

Let $\mathcal{F} \subseteq 2^{\Omega}$. Then $\mathcal{F}$ is a σ-algebra iff. $\mathcal{F}$ is an algebra and satisfies

$$\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \text{ increasing}^{\dagger} \implies \bigcup \{A_n\}_{n\in\mathbb{N}} \in \mathcal{F}.$$

A **π-system** is a collection $C \subseteq 2^{\Omega}$ s.t.

$$S \subseteq C, S \prec \aleph_0 \implies \bigcap S \in C.$$

A **λ-system** is a collection $\mathcal{L} \subseteq 2^{\Omega}$ s.t.

1) $\Omega \in \mathcal{L}$.
2) $A, B \in \mathcal{L}, A \subseteq B \implies B \setminus A \in \mathcal{L}$.
3) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ increasing, then $\bigcup \{A_n\}_{n\in\mathbb{N}} \in \mathcal{L}$.

The **λ-system generated by** $S \subseteq 2^{\Omega}$, denoted by

$$\lambda \langle S \rangle ,$$

is defined to be the intersection of all λ-system containing S.

§1.4 *Lemma: π-λ Theorem*

A collection $\mathcal{F} \subseteq 2^{\Omega}$ is a π-system and a λ-system iff. $\mathcal{F}$ is a σ-algebra.

§1.4.1 *Corollary:*

If C is a π-system, then $\lambda \langle C \rangle = \sigma \langle C \rangle$.

Proof:

Every σ-algebra is a λ-system, we have $\lambda \langle C \rangle \subseteq \sigma \langle C \rangle$.

It suffices to show $\lambda \langle C \rangle$ is a σ-algebra, i.e. $\lambda \langle C \rangle \supseteq \sigma \langle C \rangle$.

Let

$$\lambda_1 (C) = \{A \in \lambda \langle C \rangle \,|\, \forall B \in C : A \cap B \in \lambda \langle C \rangle\} ,$$

and let

$$\lambda_2 (C) = \{A \in \lambda \langle C \rangle \,|\, \forall B \in \lambda \langle C \rangle : A \cap B \in \lambda \langle C \rangle\} ,$$

then $\lambda_1 (C)$ and $\lambda_2 (C)$ are λ-systems containing C, hence $\lambda \langle C \rangle \subseteq \lambda_2 \langle C \rangle \subseteq \lambda_1 \langle C \rangle \subseteq \lambda \langle C \rangle$. Therefore, $\lambda \langle C \rangle = \lambda_2 \langle C \rangle$, $\lambda \langle C \rangle$ is a π-system, by Lem.§1.4, $\lambda \langle C \rangle$ is a σ-algebra. □

The **Borel σ-algebra** of a t.s. (Ω, τ) is defined to be $\sigma \langle \tau \rangle$.

†b Measure and Extension Theorem

A **semialgebra** is a collection $C \subseteq 2^{\Omega}$ s.t.

1) $S \subseteq C, S \prec \aleph_0 \implies \bigcap_{\Omega} S \in C$.
2) $\forall A \in C, \exists S \subseteq C, 0 \prec S \prec \aleph_0$ pairwise disjoint‡ s.t. $\Omega \setminus A = \bigcup S$.

A **measure** of a semialgebra $C \subseteq 2^{\Omega}$ is a function $\mu : C \to [0, +\infty]$ s.t.

1) $\mu (\emptyset) = 0$.
2) If $\{A_n\}_{n\in\mathbb{N}} \subseteq C$ pairwise disjoint, and if $\bigcup \{A_n\}_{n\in\mathbb{N}} \in C$, then

$$\mu \left(\bigcup \{A_n\}_{n\in\mathbb{N}} \right) = \sum_{i=0}^{\infty} \mu (A_i) .$$

*Notice 2^{Ω} is a σ-algebra containing S, the definition is well-defined.

†Which means: for every $n \in \mathbb{N}$, $A_n \subseteq A_{n+1}$.

‡Which means: for every $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j = \emptyset$.

§1.5 *Proposition:*

Let μ be a measure on a semialgebra C:

1) If $A, B \in C$, then $A \subseteq B \implies \mu(A) \leq \mu(B)$.
2) If $\{A_n\}_{n\in\mathbb{N}} \subseteq C$, and if $\bigcup\{A_n\}_{n\in\mathbb{N}} \in C$, then
$$\mu\left(\bigcup\{A_n\}_{n\in\mathbb{N}}\right) \leq \sum_{i=0}^{\infty}\mu(A_i).$$
3) If $\{A_n\}_{n\in\mathbb{N}} \subseteq C$ increasing, and if $\bigcup\{A_n\}_{n\in\mathbb{N}} \in C$, then
$$\mu\left(\bigcup\{A_n\}_{n\in\mathbb{N}}\right) = \lim_{n\to\infty}\mu(A_n).$$
4) If $\{A_n\}_{n=0}^k \subseteq C$ with $\forall 0 \leq n \leq k : \mu(A_n) < \infty$, then
$$\mu\left(\bigcup\{A_n\}_{n=0}^k\right) = \sum_{i=0}^{k}\mu(A_i) - \sum_{0\leq i<j\leq k}\mu(A_i\cap A_j) + \ldots + (-1)^k\mu(A_0\cap\ldots\cap A_k).$$

The **algebra generated by** a semialgebra $C \subseteq 2^{\Omega}$, denoted by
$$\mathcal{A}\langle C\rangle,$$
is defined to be the intersection of all algebra containing C.

§1.6 *Lemma:*

1) An algebra is a semialgebra.
2) Let C be a semialgebra, then
$$\mathcal{A}\langle C\rangle = \left\{A \in 2^{\Omega} | \exists\{B_i\}_{i=0}^k \subseteq C \text{ pairwise disjoint} : A = \bigcup\{B_i\}_{i=0}^k\right\}.$$
3) $\sigma\langle C\rangle = \sigma\langle\mathcal{A}\langle C\rangle\rangle$.

The **measure induced by** μ to the algebra $\mathcal{A}\langle C\rangle$, denoted by $\bar{\mu}$, is defined by $\bar{\mu} : \mathcal{A}\langle C\rangle \to [0,\infty]$:
$$\bar{\mu}(A) = \sum_{i=0}^{k}\mu(B_i),$$
where $\{B_i\}_{i=0}^k$ pairwise disjoint, and $A = \bigcup\{B_i\}_{i=0}^k$.

§1.7 *Theorem:*

Let μ be a measure on a semialgebra C. Then:

1) $\bar{\mu}$ is well-defined.
2) $\bar{\mu}|_C = \mu$.
3) $\bar{\mu}$ is a measure on $\mathcal{A}\langle C\rangle$.

Proof:

3) Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be pairwise disjoint, then for each $n \in \mathbb{N}$, we have
$$A_n = \bigcup\{B_j^n | j \in \mathbb{N} : 0 \leq j \leq k_n \in \mathbb{N}\},$$
where $\left\{B_j^n | j \in \mathbb{N} : 0 \leq j \leq k_n \in \mathbb{N}\right\} \subseteq C$ pairwise disjoint. Similarly,
$$\bigcup\{A_n\}_{n\in\mathbb{N}} = \bigcup\{B_j | j \in \mathbb{N} : 0 \leq j \leq k \in \mathbb{N}\},$$
where $\left\{B_j | j \in \mathbb{N} : 0 \leq j \leq k \in \mathbb{N}\right\} \subseteq C$ pairwise disjoint. Therefore,
$$B_i = B_i \cap \bigcup\{A_n\}_{n\in\mathbb{N}} = \bigcup\{A_n \cap B_i\}_{n\in\mathbb{N}} = \bigcup\{B_j^n \cap B_i\}_{n\in\mathbb{N}, 0\leq j\leq k_n}.$$
and
$$\mu(B_i) = \sum_{n=0}^{\infty}\sum_{j=0}^{k_n}\mu\left(B_j^n \cap B_i\right),$$
which implies
$$\bar{\mu}\left(\bigcup\{A_n\}_{n\in\mathbb{N}}\right) = \sum_{i=0}^{k}\sum_{n=0}^{\infty}\sum_{j=0}^{k_n}\mu\left(B_j^n\cap B_i\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{k}\sum_{j=0}^{k_n}\mu\left(B_j^n \cap B_i\right)\right).$$
While
$$A_n = A_n \cap \bigcup\{B_i\}_{0\leq i\leq k} = \bigcup\{B_j^n \cap B_i\}_{0\leq i\leq k, 0\leq j\leq k_n},$$
i.e.
$$\bar{\mu}(A_n) = \sum_{i=0}^{k}\sum_{j=0}^{k_n}\mu\left(B_j^n \cap B_i\right).$$
Together,
$$\bar{\mu}\left(\bigcup\{A_n\}_{n\in\mathbb{N}}\right) = \sum_{n=0}^{\infty}\bar{\mu}(A_n).$$
□

The **outer measure induced by** μ, denoted by μ^*, is defined by $\mu^* : 2^{\Omega} \to [0,\infty]$:
$$\mu^*(A) = \inf\left\{\sum_{n=0}^{\infty}\mu(A_n) \,|\, \{A_n\}_{n\in\mathbb{N}} \subseteq C : A \subseteq \bigcup\{A_n\}_{n\in\mathbb{N}}\right\}.$$
A set $A \subseteq \Omega$ is μ^***-measurable** iff.
$$\forall E \subseteq \Omega : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap (\Omega \setminus A)).$$
The **Carathéodory σ-algebra**, denoted by
$$\mathcal{M}_{\mu^*},$$
is the collection of all μ^*-measurable sets.

§1.8 *Theorem:* Carathéodory Extension Theorem

Let μ be a measure on a semialgebra C:

1) $\mu^*|_{\mathcal{A}\langle C\rangle} = \bar{\mu}$.
2) If $\{A_n\}_{n\in\mathbb{N}} \subseteq 2^{\Omega}$, then
$$\mu^*\left(\bigcup\{A_n\}_{n\in\mathbb{N}}\right) \leq \sum_{i=0}^{\infty}\mu^*(A_i).$$
3) $C \subseteq \mathcal{M}_{\mu^*}$.
4) $\mathcal{M}_{\mu^*}$ is a σ-algebra.
5) $\mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure.
6) $A \in 2^{\Omega}, \mu^*(A) = 0 \implies 2^A \subseteq \mathcal{M}_{\mu^*}$.

Proof:

1) It suffices to show $\bar{\mu}(A) \leq \mu^*(A)$.

For all $\varepsilon > 0$, there exists a collection $\{A_n\}_{n\in\mathbb{N}} \subseteq C$ s.t.
$$A \subseteq \bigcup\{A_n\}_{n\in\mathbb{N}}, \qquad \sum_{n=0}^{\infty}\mu(A_n) \leq \mu^*(A) + \varepsilon.$$
Let $\{B_i\}_{i=0}^k$ pairwise disjoint s.t. $A = \bigcup\{B_i\}_{i=0}^k$, then
$$A = A \cap \bigcup\{A_n\}_{n\in\mathbb{N}} = \bigcup\{B_i \cap A_n\}_{n\in\mathbb{N}, 0\leq i\leq k}.$$
Therefore,
$$\bar{\mu}(A) \leq \sum_{n=0}^{\infty}\sum_{i=0}^{k}\mu(B_i \cap A_n) = \sum_{n=0}^{\infty}\mu(A_n) \leq \mu^*(A) + \varepsilon.$$
Let $\varepsilon \to 0$, we get $\bar{\mu}(A) \leq \mu^*(A)$.

3) We only show $\forall E \subseteq \Omega, \forall A \in C : \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap (\Omega \setminus A))$.

For all $\varepsilon > 0$, there exists a collection $\{A_n\}_{n\in\mathbb{N}} \subseteq C$ s.t.
$$E \subseteq \bigcup\{A_n\}_{n\in\mathbb{N}}, \qquad \sum_{n=0}^{\infty}\mu(A_n) \leq \mu^*(E) + \varepsilon.$$

Let $\{B_i\}_{i=0}^{k} \subseteq C$ pairwise disjoint s.t. $\Omega \setminus A = \bigcup \{B_i\}_{i=0}^{k}$. Then

$$E \cap A \subseteq \bigcup \{A_n \cap A\}_{n\in\mathbb{N}}, \qquad E \cap (\Omega \setminus A) \subseteq \bigcup \{A_n \cap B_i\}_{n\in\mathbb{N},0\le i\le k}.$$

Hence

$$\begin{aligned}\mu^*(E \cap A) + \mu^*(E \cap (\Omega \setminus A)) &\le \sum_{n=0}^{\infty}\left(\mu(A_n \cap A) + \sum_{i=0}^{k}\mu(A_n \cap B_i)\right)\\ &= \sum_{n=0}^{\infty}\mu(A_n) \le \mu^*(E) + \varepsilon\end{aligned}.$$

Let $\varepsilon \to 0$, we get $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap (\Omega \setminus A))$.

4) We first show that $\mathcal{M}_{\mu^*}$ is an algebra:

For convenience, we write $A^c = \Omega \setminus A$:

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A_1) + \mu^*\left(E \cap A_1^c\right)\\ &= \mu^*(E \cap A_1 \cap A_2) + \mu^*\left(E \cap A_1 \cap A_2^c\right)\\ &\quad + \mu^*\left(E \cap A_1^c \cap A_2\right) + \mu^*\left(E \cap A_1^c \cap A_2^c\right)\end{aligned}.$$

Notice that $A_1 \cup A_2 = (A_1 \cap A_2) \cup \left(A_1^c \cap A_2\right) \cup \left(A_1 \cap A_2^c\right)$, and $(A_1 \cap A_2)^c = A_1^c \cap A_2^c$. We have

$$\mu^*(E) \ge \mu^*(E \cap (A_1 \cup A_2)) + \mu^*\left(E \cap (A_1 \cup A_2)^c\right).$$

Now, we show that if $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}_{\mu^*}$ increasing, then $\bigcup \{A_n\}_{n\in\mathbb{N}} \in \mathcal{M}_{\mu^*}$.

Let $\{B_n\}_{n\in\mathbb{N}}$ be defined by

- $B_0 = A_0$,
- $B_{n+1} = A_{n+1} \setminus A_n$.

Then $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}_{\mu^*}$ pairwise disjoint, with $\bigcup \{A_n\}_{n\in\mathbb{N}} = \bigcup \{B_n\}_{n\in\mathbb{N}}$.

For $n \ge 1$:

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A_n) + \mu^*\left(E \cap A_n^c\right)\\ &= \mu^*(E \cap A_n \cap B_n) + \mu^*\left(E \cap A_n \cap B_n^c\right) + \mu^*\left(E \cap A_n^c\right)\\ &= \mu^*(E \cap B_n) + \mu^*(E \cap A_{n-1}) + \mu^*\left(E \cap A_n^c\right)\\ &= \sum_{i=0}^{n}\mu^*(E \cap B_i) + \mu^*\left(E \cap A_n^c\right)\\ &\ge \sum_{i=0}^{n}\mu^*(E \cap B_i) + \mu^*\left(E \cap \left(\bigcup \{A_n\}_{n\in\mathbb{N}}\right)^c\right)\end{aligned}.$$

Let $n \to \infty$, we have

$$\mu^*(E) \ge \mu^*\left(E \cap \bigcup \{A_n\}_{n\in\mathbb{N}}\right) + \mu^*\left(E \cap \left(\bigcup \{A_n\}_{n\in\mathbb{N}}\right)^c\right).$$

5) Let $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}_{\mu^*}$ pairwise disjoint. Let $\left\{A_j\right\}_{j\in\mathbb{N}}$ be defined by $A_j = \bigcup \{B_n\}_{n=j}^{\infty}$:

$$\begin{aligned}\mu^*(A_0) &= \mu^*(A_0 \cap B_0) + \mu^*\left(A_0 \cap B_0^c\right)\\ &= \mu^*(B_0) + \mu^*(A_1)\\ &= \mu^*(B_0) + \mu^*(B_1) + \mu^*(A_2)\\ &= \sum_{i=0}^{n}\mu^*(B_i) + \mu^*(A_{n+1})\\ &\ge \sum_{i=0}^{n}\mu^*(B_i)\end{aligned}.$$

Let $n \to \infty$, we have

$$\mu^*\left(\bigcup \{B_n\}_{n\in\mathbb{N}}\right) \ge \sum_{i=0}^{\infty}\mu^*(B_i).$$

6) Let $B \in 2^A$, then $\mu^* E \cap B = \mu^*(B) = \mu^* A = 0$, and $\mu^* \ge \mu^*(E \cap B^c)$.

□

†c Measure space

A **measure space** (meas.s.) is a measurable space $(\Omega, \mathcal{F})$ with a measure on $\mathcal{F}^*$. The **μ-null subsets** of a m.s., denoted by $\mathcal{N}_\mu$, is defined by

$$\mathcal{N}_\mu = \{A \subseteq \Omega | \exists B \supseteq A : \mu(B) = 0\}.$$

A m.s. is **complete** iff. $\mathcal{N}_\mu \subseteq \mathcal{F}$.

§1.9 *Theorem: First Littlewood Approximation Principle*

Let μ be a measure on a semialgebra C. If $A \in \mathcal{M}_{\mu^*}$, and if $\mu^*(A) < \infty$. Then for each $\varepsilon > 0$, there exists $\{B_n\}_{n=0}^{k} \subseteq C$ pairwise disjoint, s.t.

$$\mu^*\left(A \Delta \bigcup \{B_n\}_{n=0}^{k}\right) < \varepsilon,$$

where $A_1 \Delta A_2$ denote the **symmetric difference** of A_1 and A_2, defined by $A_1 \Delta A_2 = (A_1 \cap (\Omega \setminus A_2)) \cup ((\Omega \setminus A_1) \cap A_2)$.

Proof:

□

§1.9.1 *Corollary:*

Let μ be a measure on a semialgebra C. Then:

1) $\mathcal{M}_{\mu^*} = \left\{A \Delta B | A \in \sigma\langle C\rangle, B \in \mathcal{N}_{\mu^*}\right\}$.
2) $\mathcal{M}_{\mu^*} = \left\{A \cup B | A \in \sigma\langle C\rangle, B \in \mathcal{N}_{\mu^*}\right\}$.

Proof:

□

§1.9.2 *Corollary:*

Let $(\Omega, \mathcal{F}, \mu)$ be a meas.s, and let $\tilde{\mathcal{F}} = \left\{A \cup B | A \in \mathcal{F}, B \in \mathcal{N}_\mu\right\}$.

1) $\tilde{\mathcal{F}}$ is a σ-algebra.
2) Let $\tilde{\mu} : \tilde{\mathcal{F}} \to [0, \infty]$ defined by

$$\forall A \in \mathcal{F}, \forall B \in \mathcal{N}_\mu : \tilde{\mu}(A \cup B) = \mu(A)$$

is well-defined measure on $\tilde{\mathcal{F}}$.

3) $\left(\Omega, \tilde{\mathcal{F}}, \tilde{\mu}\right)$ is a complete meas.s., and $\tilde{\mu}|_{\mathcal{F}} = \mu$.

Proof:

□

Let μ be a measure on a semialgebra C:

1) μ is **finite** iff. $\mu(\Omega) < \infty$.
2) μ is **infinite** iff. $\mu(\Omega) = \infty$.
3) μ is **σ-finite** iff. $\exists \{A_n\}_{n\in\mathbb{N}} \subseteq C : \Omega = \bigcup \{A_n\}_{n\in\mathbb{N}}, \forall n \in \mathbb{N} : \mu(A_n) < \infty$.

§1.10 *Theorem:*

Let μ be a σ-finite measure on a semialgebra C, and let ν be a measure on $(\Omega, \sigma\langle C\rangle)$ s.t. $\nu|_C = \mu$. Then $\mu^*|_{\sigma\langle C\rangle} = \nu$.

Proof:

□

For the real space $\mathbb{R}^k$, the **standard σ-algebra**

[*] Or a measure up to the restriction on $\mathcal{F}$

§2 Integration Theory

Bibliography