Supplementary Material for
"Probing band topology using modulational instability"

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1 Linear Stability Analysis

Here we consider the linear stability of nonlinear Bloch waves in a genric tight binding lattice described by the nonlinear evolution equation

$$i\partial_t \left|\psi(\boldsymbol{r},t)\right\rangle = (\hat{H}_L + \hat{H}_{NL}) \left|\psi(\boldsymbol{r},t)\right\rangle . \tag{S1}$$

We assume that the nonlinear part of the Hamiltonian $\hat{H}_{NL}$ is a diagonal matrix with real elements dependent only on the local on-site intensity, i.e. $\hat{H}_{NL} = \Gamma \mathrm{diag}[f(|\psi_a|^2), f(|\psi_b|^2), ...]$, where f describes the intensity-dependent nonlinear frequency shift and $a, b, ...$ indexes the sublattice degree of freedom. The linear part of the Hamiltonian $\hat{H}_L$ can be expanded in real space as

$$\hat{H}_L \left|\psi(\boldsymbol{r})\right\rangle = \sum_{\boldsymbol{\delta}} \hat{C}(\boldsymbol{\delta}) \left|\psi(\boldsymbol{r}+\boldsymbol{\delta})\right\rangle , \tag{S2}$$

where summation $\boldsymbol{\delta}$ is over neighbouring unit cells. Transforming to Fourier space, $\left|\psi(\boldsymbol{r})\right\rangle = \sum_{\boldsymbol{k}} \left|\psi(\boldsymbol{k})\right\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, we obtain the Bloch Hamiltonian

$$\hat{H}(\boldsymbol{k}) \left|\psi(\boldsymbol{k})\right\rangle = \left(\sum_{\boldsymbol{\delta}} \hat{C}(\boldsymbol{\delta}) e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}} \right) \left|\psi(\boldsymbol{k})\right\rangle . \tag{S3}$$

Note that under the Fourier transform $\hat{H}_L^* \left|\psi(\boldsymbol{r})\right\rangle \to \hat{H}^*(-\boldsymbol{k}) \left|\psi(\boldsymbol{k})\right\rangle$.

To perform the linear stability analysis we consider small perturbations about some nonlinear steady state $\left|\phi(\boldsymbol{r})\right\rangle$ with energy E, i.e. $\left|\psi(\boldsymbol{r},t)\right\rangle = (\left|\phi(\boldsymbol{r})\right\rangle + \left|p(\boldsymbol{r},t)\right\rangle) e^{-iEt}$. First, by Taylor expansion of the diagonal nonlinear term and neglecting terms quadratic in the perturbation, we obtain a linearised evolution equation for the perturbation,

$$(i\partial_t + E) \left|p(\boldsymbol{r})\right\rangle = \hat{H}_L \left|p(\boldsymbol{r})\right\rangle + \Gamma \sum_{j=a,b,...} \left[(f(|\phi_j|^2) + f'(|\phi_j|^2)|\phi_j|^2) p_j(\boldsymbol{r}) + f'(|\phi_j|^2)\phi_j^2 p_j^*(\boldsymbol{r}) \right] \left|j\right\rangle , \tag{S4}$$

The solution to this set of coupled first order linear differential equations can be expanded in terms of exponential functions as $\left|p(\boldsymbol{r},t)\right\rangle = \left|u(\boldsymbol{r})\right\rangle e^{-i\lambda t} + \left|v^*(\boldsymbol{r})\right\rangle e^{i\lambda^* t}$. We collect terms with the same time dependence to obtain the eigenvalue problem

$$\lambda \left|u(\boldsymbol{r})\right\rangle = (\hat{H}_L - E) \left|u(\boldsymbol{r})\right\rangle + \Gamma \sum_{j=a,b,...} \left[(f(|\phi_j|^2) + f'(|\phi_j|^2)|\phi_j|^2) u_j(\boldsymbol{r}) + f'(|\phi_j|^2)\phi_j^2 v_j(\boldsymbol{r}) \right] \left|j\right\rangle , \tag{S5}$$

$$\lambda \left|v(\boldsymbol{r})\right\rangle = -(\hat{H}_L^* - E) \left|v(\boldsymbol{r})\right\rangle - \Gamma \sum_{j=a,b,...} \left[(f(|\phi_j|^2) + f'(|\phi_j|^2)|\phi_j|^2) v_j(\boldsymbol{r}) + f'(|\phi_j|^2)\phi_j^{*2} u_j(\boldsymbol{r}) \right] \left|j\right\rangle . \tag{S6}$$

Now we assume that steady state is a nonlinear Bloch wave such that $\left|\phi(\boldsymbol{r})\right\rangle = \left|\phi\right\rangle e^{i\boldsymbol{k_0}\cdot\boldsymbol{r}}$. Fourier transforming the above equations, there is coupling between perturbation fields $\left|u(\boldsymbol{k})\right\rangle$ and $\left|v(\boldsymbol{k}-2\boldsymbol{k}_0)\right\rangle$. We obtain the coupled equations

$$\lambda \left|u(\boldsymbol{k}+\boldsymbol{k}_0)\right\rangle = (\hat{H}(\boldsymbol{k}_0+\boldsymbol{k}) - E) \left|u\right\rangle + \Gamma \sum_{j=a,b,...} \left[(f(|\phi_j|^2) + f'(|\phi_j|^2)|\phi_j|^2]) u_j + f'(|\phi_j|^2)\phi_j^2 v_j \right] \left|j\right\rangle , \tag{S7}$$

$$\lambda \left|v(\boldsymbol{k}-\boldsymbol{k}_0)\right\rangle = -(\hat{H}^*(\boldsymbol{k}_0-\boldsymbol{k}) - E) \left|v\right\rangle - \Gamma \sum_{j=a,b,..} \left[(f(|\phi_j|^2) + f'(|\phi_j|^2)|\phi_j|^2]) u_j + f'(|\phi_j|^2)\phi_j^{*2} u_j \right] \left|j\right\rangle . \tag{S8}$$

This eigenvalue problem has a built-in particle hole symmetry: eigenvalues λ must occur in complex conjugate pairs. Real λ correspond to stable perturbation modes, purely imaginary λ result in exponential instabilities, and complex λ correspond to oscillatory instabilities.

In two band tight binding models the Bloch Hamiltonian can be parameterized using the Pauli matrices as $\hat{H}(k) = \boldsymbol{d}(k) \cdot \hat{\boldsymbol{\sigma}}$, where $\boldsymbol{d}(k)$ is a real 3 component vector. We obtain the explicit matrix form of the above linear stability equations,

$$\lambda \begin{pmatrix} |u\rangle \\ |v\rangle \end{pmatrix} = \begin{pmatrix} \boldsymbol{d}(\boldsymbol{k}_0+\boldsymbol{k}) \cdot \hat{\boldsymbol{\sigma}} - E + \Gamma \sum_j (f_j + f_j' |\phi_j|^2) |j\rangle\langle j| & \Gamma \sum_j f_j' \phi_j^2 |j\rangle\langle j| \\ -\Gamma \sum_j f_j' \phi_j^{*2} |j\rangle\langle j| & -\boldsymbol{d}(\boldsymbol{k}_0-\boldsymbol{k}) \cdot \hat{\boldsymbol{\sigma}}^* + E - \Gamma \sum_j (f_j + f_j'|\phi_j|^2) |j\rangle\langle j| \end{pmatrix} \begin{pmatrix} |u\rangle \\ |v\rangle \end{pmatrix}, \tag{S9}$$

where $f_j = f(|\phi_j|^2)$ and $f_j' = f'(|\phi_j|^2)$.

For the case of a nonlinear Bloch wave with intensity I_0 localized to the a sublattice analyzed in the main text, we have $|\phi\rangle = (\sqrt{I_0}, 0)$, $d_{x,y}(\boldsymbol{k}_0) = 0$, $d_z(\boldsymbol{k}_0) = \Delta - 4J_2$, and $E = d_z(\boldsymbol{k}_0) + \Gamma f(I_0)$. The $\boldsymbol{k} = 0$ eigenvalue problem takes the simple form

$$\lambda \begin{pmatrix} u_a \\ u_b \\ v_a \\ v_b \end{pmatrix} = \begin{pmatrix} \Gamma f' I_0 & 0 & \Gamma f' I_0 & 0 \\ 0 & -2d_z(\boldsymbol{k}_0) - \Gamma f & 0 & 0 \\ -\Gamma f' I_0 & 0 & -\Gamma f' I_0 & 0 \\ 0 & 0 & 0 & 2d_z(\boldsymbol{k}_0) + \Gamma f(I_0) \end{pmatrix} \begin{pmatrix} u_a \\ u_b \\ v_a \\ v_b \end{pmatrix}, \tag{S10}$$

yielding $\lambda = 0, 0, \pm[2d_z(\boldsymbol{k}_0) + \Gamma f(I_0)]$ and a fourfold degeneracy when $\Gamma f(I_0)/2 = -d_z(\boldsymbol{k}_0)$, i.e. when the nonlinear energy shift on the a sublattice is sufficient to close the band gap. Complex instability eigenvalues emerge beyond this threshold intensity. Note that this transition is independent of $f'(I_0)$, i.e. the precise form of the nonlinear response function.

2 Nonlinear Dirac model

In this section, we examine linear stability of the high-symmetry nonlinear Bloch wave with the in-plane wave vector $\mathbf{k}_0 = [\pi, 0]$. In the vicinity of $\mathbf{k}_0$, $\mathbf{k} = \mathbf{k}_0 + \mathbf{p}$, the series expansion in the Bloch Hamiltonian $\hat{H}_L(\mathbf{k})$ at $|\mathbf{p}| \ll 1$ leads to the Dirac-like Hamiltonian

$$\hat{H}_D = -J_1\sqrt{2}\left(-p_x\hat{\sigma}_y + p_y\hat{\sigma}_x\right) + \left(\Delta - 4J_2 + J_2\left[p_x^2 + p_y^2\right]\right)\hat{\sigma}_z. \tag{S1}$$

The corresponding evolution equations including the local Kerr nonlinearity $f(I) = I$ can be formulated in the real space in terms of spatial derivatives by substituting $p_{x,y} = -i\partial_{x,y}$:

$$i\partial_t \boldsymbol{\psi} = \begin{pmatrix} \Delta - 4J_2 - J_2\left[\partial_x^2 + \partial_y^2\right] + \Gamma|\psi_1^2| & J_1\sqrt{2}\left(i\partial_y - \partial_x\right) \\ J_1\sqrt{2}\left(i\partial_y + \partial_x\right) & -\Delta + 4J_2 + J_2\left[\partial_x^2 + \partial_y^2\right] + \Gamma|\psi_2^2| \end{pmatrix} \boldsymbol{\psi}. \tag{S2}$$

2.1 Nonlinear dispersion of bulk modes

We search for the solution of (S2) in the form of weakly nonlinear Bloch waves:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{-iEt + ip_x x + ip_y y}. \tag{S3}$$

Plugging the spinor (S3) into Eq.(S2) results in the system of equations for the amplitudes A and B

$$\begin{cases} \left(-E + \Delta - 4J_2 + J_2\left[p_x^2 + p_y^2\right] + \Gamma|A^2|\right)A - J_1\sqrt{2}\left(p_y + ip_x\right)B = 0 \\ \left(E + \Delta - 4J_2 + J_2\left[p_x^2 + p_y^2\right] - \Gamma|B^2|\right)B + J_1\sqrt{2}\left(p_y - ip_x\right)A = 0 \end{cases} \tag{S4}$$

Denoting the total wave intensity $|A|^2 + |B|^2 = I_0$, we first find the solutions for the lower and upper bands at the zero wave vector $\mathbf{p} = 0$:

$$A^{(0)} = 0, \quad |B^{(0)}|^2 = I_0, \quad E_2^{(0)} = -\Delta + 4J_2 + \Gamma|B^{(0)}|^2 = -\Delta + 4J_2 + \Gamma I_0, \tag{S5}$$

$$B^{(0)} = 0, \quad |A^{(0)}|^2 = I_0, \quad E_1^{(0)} = \Delta - 4J_2 + \Gamma|A^{(0)}|^2 = \Delta - 4J_2 + \Gamma I_0. \tag{S6}$$

At the intensities above the critical value $I_0 \geq \pm 2\frac{(\Delta - 4J_2)}{\Gamma}$, we get the additional doubly degenerate solution

$$|A^{(0)}|^2 = \frac{I_0}{2} - \frac{\Delta - 4J_2}{\Gamma}, \quad |B^{(0)}|^2 = \frac{I_0}{2} + \frac{\Delta - 4J_2}{\Gamma}, \quad E_3^{(0)} = \frac{\Gamma I_0}{2} \tag{S7}$$

with the eigenvectors:

$$\begin{pmatrix} A^{(0)} \\ B^{(0)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi}\sqrt{I_0 - \dfrac{2\left(\Delta - 4J_2\right)}{\Gamma}} \\ \pm\sqrt{I_0 + \dfrac{2\left(\Delta - 4J_2\right)}{\Gamma}} \end{pmatrix}, \tag{S8}$$

where φ is an arbitrary phase depending on which direction we approach the degeneracy point $\mathbf{p}=0$.

To find the dispersion in the neighborhood of the point $p_x=p_y=0$, we employ the perturbation theory. Treating p_x and p_y as small perturbations, we expand all quantities to the first order $E=E^{(0)}+E^{(1)}+...;\ A=A^{(0)}+A^{(1)}+...;\ B=B^{(0)}+B^{(1)}+...$. We obtain a cross-like solution describing the nonlinear Dirac cone:

$$E=E_3^{(0)}+E^{(1)}=\frac{\Gamma I_0}{2}\pm\frac{\sqrt{2}J_1\sqrt{p_x^2+p_y^2}}{\sqrt{1-\frac{4(\Delta-4J_2)^2}{I_0^2\Gamma^2}}}. \tag{S9}$$

Thus, at $\Gamma^2 I_0^2>4\left(\Delta-4J_2\right)^2$ one of the dispersion curves develops a loop.

Next, we derive the exact implicit expression for the nonlinear dispersion $E(p_x,p_y)$. To simplify our derivations, we set $p_x=0$ and rewrite the system in the form:

$$\begin{pmatrix} E_n-M_n & J_1\sqrt{2}p_y \\ J_1\sqrt{2}p_y & E_n+M_n\end{pmatrix}\begin{pmatrix} A \\ B\end{pmatrix}=0\,, \tag{S10}$$

denoting $E_n=E-\Gamma I_0/2$, $M_n=\Delta-4J_2+\dfrac{\Gamma I_0\left(\Delta-4J_2+J_2p_y^2\right)}{2(E-\Gamma I_0)}+J_2p_y^2$. The nonlinear dispersion is then given by

$$E_n^2=2J_1^2p_y^2+M_n^2(p_y^2). \tag{S11}$$

The eigenvectors are

$$|A|^2=\frac{I_0}{2}+\frac{\left(\Delta-4J_2\right)I_0}{2(-\Gamma I_0+E)}+\frac{J_2p_y^2I_0}{2(-\Gamma I_0+E)}, \tag{S12}$$

$$|B|^2=\frac{I_0}{2}-\frac{\left(\Delta-4J_2\right)I_0}{2(-\Gamma I_0+E)}-\frac{J_2p_y^2I_0}{2(-\Gamma I_0+E)}. \tag{S13}$$

The implicit relation (S11) can be posed as

$$\left((E-\Gamma I_0/2)^2-2J_1^2p_y^2\right)\left(E-\Gamma I_0\right)^2=\left(\Delta-4J_2+J_2p_y^2\right)^2\left(E-\frac{\Gamma I_0}{2}\right)^2. \tag{S14}$$

Note, this dispersion relation supports the existence of 2 more loops in addition to the loop at the point $p_y=0$, described above. This bifurcation occurs at $d_z(\mathbf{p})=0$:

$$p_y^{BHZ}=\pm\sqrt{\frac{4J_2-\Delta}{J_2}}, \tag{S15}$$

in the non-trivial phase only, $|\Delta|<4J_2$. The energies at the point p_y^{BHZ} are:

$$E^0_{3BHZ}=\Gamma I_0, \tag{S16}$$

$$E^0_{2,1BHZ}=\Gamma I_0/2\pm\sqrt{\frac{2J_1^2(4J_2-\Delta)}{J_2}}. \tag{S17}$$

Specifically, the energy E^0_{3BHZ} correspond to two additional cross points, BHZ crosses, which appear only in the nontrivial case with the eigenvectors

$$\begin{pmatrix} A^{(0)}_{BHZ} \\ B^{(0)}_{BHZ}\end{pmatrix}=\begin{pmatrix} e^{i\phi}\sqrt{\dfrac{I_0}{2}+\sqrt{\dfrac{I_0^2}{4}-2J_1^2(p_y^{BHZ})^2/\Gamma^2}} \\ \pm\sqrt{\dfrac{I_0}{2}-\sqrt{\dfrac{I_0^2}{4}-2J_1^2(p_y^{BHZ})^2/\Gamma^2}}\end{pmatrix}. \tag{S18}$$

The additional crosses appear at the intensities higher $\Gamma I_0/2=\pm\sqrt{\dfrac{2J_1^2(4J_2-\Delta)}{J_2}}$ (the sign is chosen depending on sign of the nonlinearity Γ), which is defined by the degeneracy of the cross point and one of the bands at $p_y=p_y^{BHZ}$

As we can see in Fig. 2 of the main text, the next dispersion plane bifurcation (d) $\to$ (e) happens when $E_1^0=E^0_{2BHZ}$ or $E_2^0=E^0_{1BHZ}$ for different signs of Γ (according to Eqs. (S5), (S16)), that corresponds to

$$\Gamma=\frac{2}{I_0}\left(\mp\left(\Delta-4J_2\right)\pm\sqrt{\frac{2J_1^2(4J_2-\Delta)}{J_2}}\right). \tag{S19}$$

2.2 Modulation instability

To examine linear stability of the nonlinear Bloch modes, we introduce small complex-valued perturbations to the amplitudes: $A = A_0 + \delta a$, $B = B_0 + \delta b$ and look for the solution in the form:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A_0 + \delta a \\ B_0 + \delta b \end{pmatrix} e^{-iEt + ip_x x + ip_y y}. \tag{S20}$$

The equations for deviations $\delta a,\ \delta b$ can be recast as

$$i\frac{\partial}{\partial t}\begin{pmatrix} \delta a \\ \delta b \\ \delta a^* \\ \delta b^* \end{pmatrix} = \hat{L}\begin{pmatrix} \delta a \\ \delta b \\ \delta a^* \\ \delta b^* \end{pmatrix}, \tag{S21}$$

where operator $\hat{L}$ is the 4×4 matrix

$$\hat{L} = \begin{pmatrix} \hat{H}_D(\partial_x,\partial_y) + H_D(p_x,p_y) - E\hat{I} + 2\Gamma\begin{pmatrix} |A_0|^2 & 0 \\ 0 & |B_0|^2 \end{pmatrix} & \Gamma\begin{pmatrix} A_0^2 & 0 \\ 0 & B_0^2 \end{pmatrix} \\ -\Gamma\begin{pmatrix} A_0^{*2} & 0 \\ 0 & B_0^{*2} \end{pmatrix} & -\hat{H}_D^*(\partial_x,\partial_y) - H_D^*(p_x,p_y) + E\hat{I} - 2\Gamma\begin{pmatrix} |A_0|^2 & 0 \\ 0 & |B_0|^2 \end{pmatrix} \end{pmatrix}, \tag{S22}$$

where

$$H_D(p_x,p_y) = \begin{pmatrix} J_2\left(p_x^2+p_y^2\right) & -J_1\sqrt{2}\left(p_y+ip_x\right) \\ -J_1\sqrt{2}\left(p_y-ip_x\right) & -J_2\left(p_x^2+p_y^2\right) \end{pmatrix}, \tag{S23}$$

$$\hat{H}_D(\partial_x,\partial_y) = \begin{pmatrix} \Delta - 4J_2 - J_2\left(\partial_x^2+\partial_y^2\right) & J_1\sqrt{2}\left(i\partial_y-\partial_x\right) \\ J_1\sqrt{2}\left(i\partial_y+\partial_x\right) & -\Delta + 4J_2J_2\left(\partial_x^2+\partial_y^2\right) \end{pmatrix}. \tag{S24}$$

To study modulational instability, we take $[\delta a;\delta a^*;\delta b;\delta b^*] = [C_1;C_2;C_3;C_4]e^{-i\lambda t+i\kappa_x x+i\kappa_y y} = \boldsymbol{C}e^{-i\lambda t+i\kappa_x x+i\kappa_y y}$ and set $\kappa_x = 0$ to simplify further considerations. Eq. (S21) leads to the system of equations for amplitudes $\boldsymbol{C}$ $:\left(\hat{L}-\lambda\hat{I}\right)\boldsymbol{C} = 0$. The positive imaginary part of λ, found from $\mathbf{det}\left(\hat{L}-\lambda\hat{I}\right) = 0$, indicates instability.

For the cross point at $p_x = p_y = 0$, existing at the intensities $\Gamma I_0 > \pm 2\left(\Delta - 4J_2\right)$ at the energy $E^{(0)} = \frac{\Gamma I_0}{2}$, with the amplitudes $|A_0|^2 = \dfrac{I_0}{2} - \dfrac{\Delta-4J_2}{\Gamma}$, $|B_0|^2 = \dfrac{I_0}{2} + \dfrac{\Delta-4J_2}{\Gamma}$, we find the energy detuning λ along the straight lines $I_0\Gamma + C = -2(\Delta - 4J_2)$ in the parameter plane (Γ,Δ):

$$\lambda = \pm\sqrt{\pm\frac{\sqrt{e^{-2i\varphi}\kappa_y^2\left(C^2\left(-1+e^{2i\varphi}\right)^2 J_1^2 + 2C\left(-1+e^{2i\varphi}\right)^2\Gamma I_0 J_1^2\right) + 2\Gamma^2 I_0^2 J_2^2\kappa_y^4}}{\sqrt{2}} + CJ_2\kappa_y^2 + \Gamma I_0 J_2\kappa_y^2 + 2J_1^2\kappa_y^2 + J_2^2\kappa_y^4} \tag{S25}$$

We analyse Eq. (S25) for $C = 0$ at the line $I_0\Gamma = -2(\Delta - 4J_2)$, which is the negatively inclined existence boundary of the cross solution:

$$\lambda_{1,2} = \pm\sqrt{J_2^2\kappa_y^4 + 2\kappa_y^2 J_1^2}, \tag{S26}$$

$$\lambda_{3,4} = \pm\sqrt{-4(\Delta-4J_2)J_2\kappa_y^2 + J_2^2\kappa_y^4 + 2J_1^2\kappa_y^2}. \tag{S27}$$

The imaginary part $\mathrm{Im}(\lambda_{1,2})$ is zero for all values of the wave number κ_y, therefore, $\lambda_{1,2}$ do not show any instability. The area of the stability can be determined from $\lambda_{3,4}$: it is a purely real quantity for $\Gamma > -\dfrac{J_1^2}{I_0 J_2}$ or equivalently $2J_1^2 \geq 4\left(\Delta - 4J_2\right)J_2$. In the nontrivial case, since $J_2\left(\Delta - 4J_2\right) < 0$, we conclude that $2J_1^2 \geq 4\left(\Delta - 4J_2\right)J_2$ for any J_2, Δ. Therefore, the cross point is stable. But in the trivial case the area of parameters J_2, J_1 exists, for which $\mathrm{Im}(\lambda_{3,4}) > 0$, and the cross point becomes unstable. The boundary value of the detuning in the trivial phase is

$$\Delta_c = 4J_2 + \frac{J_1^2}{2J_2}. \tag{S28}$$

Note, for the given intensity, $-\Gamma I_0/2 = \Delta - 4J_2$, the upper branch and the point of the cross are degenerate. Hence, the line of stability $I_0\Gamma = -2(\Delta - 4J_2)$ appears in Fig. 1 in the man text for the upper branch at $\Delta < \Delta_c$. For $\Delta > \Delta_c$, we analytically obtain the maximum growth rate $\max_{\kappa_y} \operatorname{Im} \lambda_{3,4}$ at the wavenumber κ_y^{max}:

$$\max_{\kappa_y} \operatorname{Im} \lambda_{3,4} = \frac{|J_1^2 + \Gamma I_0 J_2|}{|J_2|}; \tag{S29}$$

$$\kappa_y^{max} = \pm \sqrt{\frac{|\Gamma I_0 J_2 + J_1^2|}{J_2^2}}. \tag{S30}$$

Equation (S25) at the other boundary of the existence of the cross solution ($C = -2\Gamma I_0$) takes the form:

$$\lambda_{1,2,3,4} = \pm \sqrt{\pm \kappa_y^2 \Gamma I_0 J_2 - \Gamma I_0 J_2 \kappa_y^2 + 2J_1^2 \kappa_y^2 + J_2^2 \kappa_y^4}, \tag{S31}$$

from which we obtain the area of stability $\Gamma < \dfrac{J_1^2}{I_0 J_2}$.

Let us consider Eq. (S25) for the case $\varphi = \pi n, n \in \mathbb{Z}$:

$$\lambda_{1,2} = \pm \sqrt{J_2^2 \kappa_y^4 + C J_2 \kappa_y^2 + 2J_1^2 \kappa_y^2}, \tag{S32}$$

$$\lambda_{3,4} = \pm \sqrt{J_2^2 \kappa_y^4 + C J_2 \kappa_y^2 + 2J_2 \Gamma I_0 \kappa_y^2 + 2J_1^2 \kappa_y^2}. \tag{S33}$$

The boundaries of the cross stability are located on lines with $C = -\dfrac{2J_1^2}{J_2} - 2\Gamma I_0$ and $C = -\dfrac{2J_1^2}{J_2}$. These are the straight lines $I_0\Gamma = 2(\Delta - 4J_2 - \frac{J_1^2}{J_2})$ ~~and~~ $\cancel{I_0\Gamma = -2(\Delta - 4J_2 - \frac{J_1^2}{J_2})}$.

The color maps of the maximum increment value $\max_\lambda(\operatorname{Im}(\lambda)$ in the parameter space for the cross solution $E = \Gamma I_0/2$ are plotted in Fig. 2.2 by using Eq. (S25).

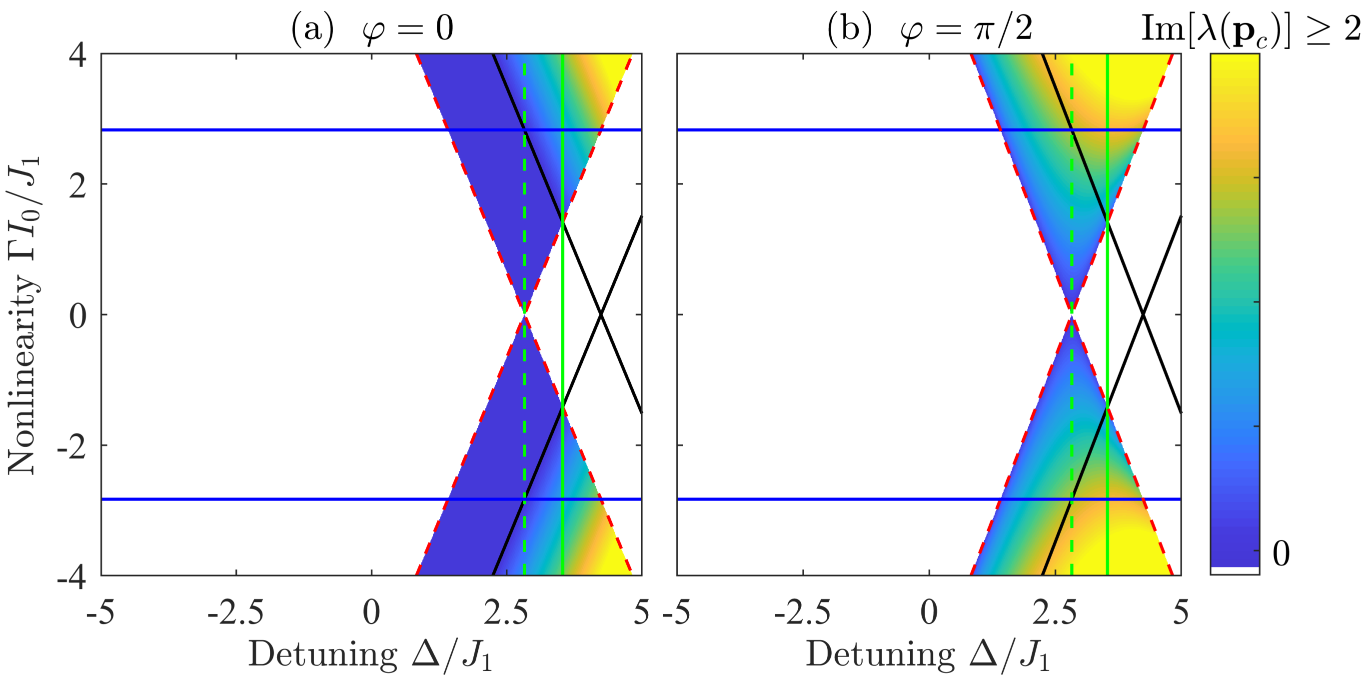

Figure S1: The maximum increment value $\max_\lambda(\operatorname{Im}(\lambda)$ color-coded in the plane of parameters Δ/J_1, $\Gamma I_0/J_1$ for the cross point $E = \Gamma I_0/2$ at $p = 0$. Parameters are $J_2 = J_1/\sqrt{2}$, (a) $\varphi = 0$, (b) $\varphi = \pi/2$. Solid dashed lines $\Gamma = \pm \dfrac{2}{I_0}(4J_2 - \Delta)$ highlight the boundaries of the existence of the cross solution. On these lines, the cross point is stable in the nontrivial domain, $|\Delta| < 2\sqrt{2}$, whose upper boundary is marked with the green dashed line. In the trivial domain, the cross point is unstable at detunings larger $\Delta = J_1^2/2J_2 + 4J_2$ marked with a solid green line. The boundaries of the cross point stability for $\varphi = 0$ are black straight lines $I_0\Gamma = \pm 2(\Delta - 4J_2 - J_1^2/J_2)$. At $\varphi = 0$, the intersection point of the black lines with the boundary of the trivial phase at $\Gamma = \pm \frac{2J_1^2}{J_2 I_0}$ (straight blue lines) defines the intensity, for which, by changing Δ, we can distinguish the trivial phase from the non-trivial one observing a transition from stability to instability.