FUNKCIJE VEČ SPR.

$f: D_f \subseteq \mathbb{R}^n \to \mathbb{R}$

$(x_1, x_2, \dots, x_n) \mapsto u = f(x_1, x_2, \dots, x_n)$

nivojnice C D_f, $f(x_1, x_2, \dots, x_n) = C$

limite: f-ja v (a,b) zvezna, če:

$L = \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

* $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ *

ODVOD

$f'(x_0) = \frac{\Delta f}{\Delta x} = \frac{f(x_0+h)-f(x_0)}{h}$

→ grafično: tangenta na $f(x)$ v dani točki

→ rel. Δ vrednosti $f(x_0)$ ob majhni Δx_0

→ hitrost spr. funkcije f v x_0

·) $f'_-(x_0) = \lim_{h\nearrow 0} \frac{f(x_0+h)-f(x_0)}{h}$

= "levi odvod = leva limita $\frac{\Delta f}{\Delta x}$"

·) $f'_+(x_0) = \lim_{h\searrow 0} \frac{f(x_0+h)-f(x_0)}{h}$ diferenčni kvocient

= "desni odvod = desna limita $\frac{\Delta f}{\Delta x}$"

ODVODI $\left(f'(x) = \frac{df}{dx} \right)$

$(f \pm g)' = f' \pm g'$

$(c \cdot f)' = c \cdot f'$

$(f \cdot g)' = f' \cdot g + f \cdot g'$

$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$ $(g \neq 0)$

$(g \circ f)' = g'(f(x)) \cdot f'(x)$

$(f^{-1})(f(x)) = \frac{1}{f'(x)}$

$c' = 0$

$x' = 1$

$(x^n)' = n \cdot x^{n-1}$

$(x^{\frac{m}{n}})' = \frac{m}{n} x^{\frac{m}{n}-1}$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, $x \neq 0$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, $x > 0$

$(\sqrt[n]{x})' = \frac{1}{n\sqrt[n]{x^{n-1}}}$, $x > 0$

$(e^x)' = e^x$

$(e^{kx})' = ke^{kx}$

$(a^x)' = a^x \ln a$, $a > 0$

$(x^x)' = x^x(1+\ln x)$, $x > 0$

$(\ln x)' = \frac{1}{x}$, $x > 0$

$(\log_a x)' = \frac{1}{x \ln a}$, $a, x > 0$

$(\sin x)' = \cos x$

$(\sin(ax))' = a\cos(ax)$

$(\cos x)' = -\sin x$

$(\cos(ax))' = -a \cdot \sin(ax)$

$\left.\begin{array}{l} (\tan x)' = \frac{1}{\cos^2 x}, \ x \neq \frac{\pi}{2}+k\pi \\ (\mathrm{ctan}\, x)' = -\frac{1}{\sin^2 x}, \ x \neq k\pi \end{array}\right\} k \in \mathbb{Z}$

$\left.\begin{array}{l} (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \\ (\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \end{array}\right\} |x| < 1$

$(\arctan x)' = \frac{1}{1+x^2}$

$(\mathrm{arcctan})' = -\frac{1}{1+x^2}$

$\left(\ln(x+\sqrt{x^2 \pm a^2})\right)' = \frac{1}{\sqrt{x^2 \pm a^2}}$

UPORABE ODVODA

LINEARNA APROX

$L(x) = f(x_0) + f'(x_0)(x-x_0)$

x_0 … približek za x

$f(x_0+h) \doteq f(x_0) + f'(x_0) \cdot h$

"približek, ko je h majhen"

NARAŠČANJE IN PADANJE

·) narašča (strogo): $f'(x) \geq 0$

·) pada (strogo): $f'(x) \leq 0$

STAC. TOČKE IN EKSTREMI

↳ $f'(x) = 0$

Lok. min.: $f(x) \geq f(x_0)$ za $\forall x \in (x_0-\delta, x_0+\delta)$

$f''(x_0) > 0$

Lok. max.: $f(x) \leq f(x_0)$ za $\forall x \in (x_0-\delta, x_0+\delta)$

$f''(x_0) < 0$

$\boxed{f^{(n)} = (f^{(n-1)})'(x)}$

1) n = liho → v x_0 ni lok. ekstrema

2) n = sodo: i) $f^{(n)}(x_0) > 0$ → lok. min.

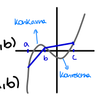
ii) $f^{(n)}(x_0) < 0$ → lok. max.

globalni ekstrem: pogoj velja za $\forall x \in D_f$

KONVEKSNOST / KONKAVNOST

·) konveksna: $f''(x) \geq 0$ za $\forall x \in (a,b)$

·) konkavna: $f''(x) \leq 0$ za $\forall x \in (a,b)$



RAČUNANJE LIMIT

Velja: $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

ALI $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$

$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

TAYLORJEVI POLINOMI

$f(x) \approx \underline{f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}$

$R_n(f(x)) = f(x) - \underline{T_n(x)}$

↳ "napaka"

·) $e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + R_n(e^x)$, $R_n(e^x) = \frac{e^c}{(n+1)!} \cdot x^{n+1}$, $c \in [x, x_0]$

·) $\sin x = x - \frac{x^3}{3!} + \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R_{2k+1}(\sin x)$

·) $\cos x = 1 - \frac{x^2}{2} + \dots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(\cos x)$

·) $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + R_n\left(\frac{1}{1-x}\right)$

TAYLORJEVA VRSTA

$f(x) \to f(x_0) + f'(x_0) \cdot (x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$

ČE $R_n(f(x)) \xrightarrow[n\to\infty]{} 0 \Rightarrow$ Taylorjeva vrsta konvergira in je enaka $f(x)$

TAYLORJEV POLINOM $f(x_0, y_0)$ 2. ST.

$T_2(x,y) = f(x_0,y_0) + \left(f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\right)$

$+ \frac{1}{2}\left(f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right)$

ODVOD F-JE VEČ SPR.

$f_x(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$

$f_y(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$

·) $\mathrm{grad} f(x_0,y_0) = \nabla f(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0))$

·) $f_{x_i}(a_1,\dots,a_n) = \frac{\partial f}{\partial x_i}(a_1,\dots,a_n) =$

$= \lim_{h\to 0} \frac{f(a_1,\dots,a_i+h,\dots,a_n) - f(a_1,\dots,a_i,\dots,a_n)}{h}$

·) $\mathrm{grad} f(a_1,\dots,a_n) = (f_{x_1}(a_1,\dots,a_n), \dots, f_{x_n}(a_1,\dots,a_n))$

·) $g(t) = f(x(t), y(t))$ ⟶ VERIŽNO PRAVILO

·) $g'(t) = f_x(x(t),y(t)) \cdot x'(t) + f_y(x(t),y(t)) \cdot y'(t)$

·) $g_{t_i}(t) = \sum_{j=1}^{n} f_{x_j}(x_1(t),\dots,x_n(t))(x_j)'_{t_i}(t)$

SMERNI ODVOD F-JE VEČ SPR.

$x(t) = x_0 + t \cdot e_1$, $y(t) = y_0 + t \cdot e_2$, $t \in \mathbb{R}$

↳ enačba premice skozi (x_0,y_0) s smernim vektorjem

$g(t) = f(x(t),y(t)) = f(x_0+t\cdot e_1, y_0+t\cdot e_2)$

$g'(t) = f_x(x(t),y(t)) \cdot e_1 + f_y(x(t),y(t)) \cdot e_2 =$

$= \mathrm{grad}\, f(x(t),y(t)) \cdot \vec{e} =: f_{\vec{e}}(x(t),y(t))$

⟶ premik iz t v smeri $\vec{e}$

$f_{\vec{e}}(x_0,y_0) = \mathrm{grad} f(x_0,y_0) \cdot \vec{e} = |\mathrm{grad} f(x_0,y_0)| \cdot \cos\varphi$

↳ $\vec{e}$ kaže v smeri $\mathrm{grad} f(x_0,y_0) \Rightarrow$ max. smerni odvod

UPORABA ODVODA F-JE VEČ SPR.

LINEARNA APROX.

$f(x,y) \doteq L(x,y) = f(x_0,y_0) + df(x_0,y_0)$

·) $df(x_0,y_0) \doteq \Delta f = f(x,y) - f(x_0,y_0) =$

$= f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

NARAŠČANJE / PADANJE

·) narašča (strogo): $f_{\vec{e}}(x_0,y_0) \geq 0$

·) pada (strogo): $f_{\vec{e}}(x_0,y_0) \leq 0$

STAC. TOČKE IN EKSTREMI

stac. točka

1) $\mathrm{grad} f = 0$ "vsi parc. odvodi so 0"

2) $f_{\vec{e}}(x_0,y_0) = 0$ "vsi smerni vektorji so 0"

3) $df(x_0,y_0) = 0$ "totalni diferencial je 0"

ekstremi

HESSEJEVA MATRIKA

$H_f(x_0,y_0) = \begin{bmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{xy}(x_0,y_0) & f_{yy}(x_0,y_0) \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$

$D(x_0,y_0) := A \cdot C - B^2$

1) lok. min.: $D(x_0,y_0) > 0 \wedge A > 0$

2) lok. max.: $D(x_0,y_0) > 0 \wedge A < 0$

3) sedlo: $D(x_0,y_0) < 0$

4) ???: $D(x_0,y_0) = 0$

VEZANI EKSTREMI

$\mathrm{grad} f(x_0,y_0) = \lambda\, \mathrm{grad}\, g(x_0,y_0)$

$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

- $L_x(x_0,y_0,\lambda_0) = f_x(x_0,y_0) - \lambda_0 g_x(x_0,y_0) = 0$
- $L_y(x_0,y_0,\lambda_0) = f_y(x_0,y_0) - \lambda_0 g_y(x_0,y_0) = 0$
- $L_\lambda(x_0,y_0,\lambda_0) = -g(x_0,y_0) = 0$

INTEGRALI

$F'(x) = f(x) \wedge \int f(x)dx = F(x) + C$

$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$

$\int k \cdot f(x)dx = k\int f(x)dx$

* vpeljava nove spremenljivke npr.: $t = 2x+1$, $dt = 2dx$, $dx = \frac{dt}{2}$

* per partes: $\int u\,dv = u \cdot v - \int v \cdot du$

$\int_a^a f(x)dx = 0$

$\int_b^a f(x)dx = -\int_a^b f(x)dx$

$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) = P$

$\int k\,dx = k \cdot x + C$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\int \frac{1}{x} dx = \ln|x| + C$

$\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$

$\int e^x dx = e^x + C$

$\int e^{nx} dx = \frac{e^{nx}}{n} + C$

$\int x \cdot e^x dx = e^x(x-1) + C$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$\int \ln x\, dx = x \cdot \ln x - x + C$

$\int \log_a x\, dx = x \cdot \log_a x - \frac{x}{\ln a} + C$

$\int \sin x\, dx = -\cos x + C$

$\int \sin(nx)dx = -\frac{\cos(nx)}{n} + C$

$\int \cos x\, dx = \sin x + C$

$\int \cos(nx)dx = \frac{\sin(nx)}{n} + C$

$\int \tan x\, dx = -\ln|\cos x| + C$

$\int \mathrm{ctan}\, x\, dx = \ln|\sin x| + C$

THROWBACK

FUNKCIJE

inj. – vsak el. se slika v drugo sliko
surj. – slike po celi y-osi oz. $B = Z_f$
} oboje hkrati: bijektivna

soda – $f(x) = f(-x)$ "simetrija na y-os", liha – $f(-x) = -f(x)$ "simetrija na (0,0)"

monotonost (stroga) – "na danem $[a,b]$ funkcija samo pada / narašča" (strogo)

STOŽNICE

krog: $(x-p)^2 + (y-q)^2 = r^2$, $S(p,q)$

$x^2 + y^2 + ax + by + c = 0$

elipsa: $\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$, $S(p,q)$

$a > b$: $e^2 = a^2 - b^2$, $\varepsilon = \frac{e}{a}$, $b > a$: $e^2 = b^2 - a^2$, $\varepsilon = \frac{e}{b}$

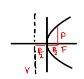
hiperbola: $\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = \pm 1$

a-re, b-im | en. = 1, $e^2 = a^2 + b^2$, $\varepsilon = \frac{e}{a}$

a-im, b-re | en. = −1, $e^2 = a^2 + b^2$, $\varepsilon = \frac{e}{b}$

} asimptoti: $y = \pm \frac{b}{a} x$

parabola: $(y-b)^2 = 2p(x-a)$



$F = \left(\frac{p}{2}+a, 0+b\right)$, $x = -\frac{p}{2} + a$

1. *Periodičnost*

$$\sin(x+2\pi)=\sin x \qquad \cos(x+2\pi)=\cos x$$
$$\tan(x+\pi)=\tan x \qquad \cot(x+\pi)=\cot x$$

2. *Sodost, lihost*

$$\sin(-x)=-\sin x \qquad \cos(-x)=\cos x$$
$$\tan(-x)=-\tan x \qquad \cot(-x)=-\cot x$$

3. *Zveze med kotnimi funkcijami*

$$\sin^2 x+\cos^2 x=1$$
$$1+\tan^2 x=\tfrac{1}{\cos^2 x} \qquad 1+\cot^2 x=\tfrac{1}{\sin^2 x}$$

4. *Prehodi med koti*

$$\sin(\tfrac{\pi}{2}+x)=\cos x \qquad \sin(\pi+x)=-\sin x$$
$$\cos(\tfrac{\pi}{2}+x)=-\sin x \qquad \cos(\pi+x)=-\cos x$$
$$\tan(\tfrac{\pi}{2}+x)=-\cot x$$

5. *Adicijski izreki*

$$\sin(x+y)=\sin x\cos y+\cos x\sin y \qquad \sin(x-y)=\sin x\cos y-\cos x\sin y$$
$$\cos(x+y)=\cos x\cos y-\sin x\sin y \qquad \cos(x-y)=\cos x\cos y+\sin x\sin y$$
$$\tan(x+y)=\tfrac{\tan x+\tan y}{1-\tan x\tan y} \qquad \tan(x-y)=\tfrac{\tan x-\tan y}{1+\tan x\tan y}$$

6. *Dvojni in polovični koti*

$$\sin(2x)=2\sin x\cos x \qquad \sin^2 x=\tfrac{1-\cos 2x}{2}$$
$$\cos(2x)=\cos^2 x-\sin^2 x \qquad \cos^2 x=\tfrac{1+\cos 2x}{2}$$
$$\tan(2x)=\tfrac{2\tan x}{1-\tan^2 x} \qquad \tan x=\tfrac{1-\cos 2x}{\sin 2x}$$

7. *Produkti in vsote*

$$\sin x+\sin y=2\sin\tfrac{x+y}{2}\cos\tfrac{x-y}{2} \qquad \sin x\sin y=\tfrac{1}{2}\left(\cos(x-y)-\cos(x+y)\right)$$
$$\cos x+\cos y=2\cos\tfrac{x+y}{2}\cos\tfrac{x-y}{2} \qquad \cos x\cos y=\tfrac{1}{2}\left(\cos(x-y)+\cos(x+y)\right)$$
$$\cos x-\cos y=-2\sin\tfrac{x+y}{2}\sin\tfrac{x-y}{2} \qquad \sin x\cos y=\tfrac{1}{2}\left(\sin(x-y)+\sin(x+y)\right)$$