Ime in priimek

Vpisna številka

1	
2	
3	
4	
Σ	

Osnove matematične analize: drugi kolokvij

13. januar 2025

Čas pisanja je 90 minut. Dovoljena je uporaba 2 listov A4 formata s formulami. Uporaba kalkulatorja ali drugih pripomočkov ni dovoljena. Vse odgovore dobro utemelji!

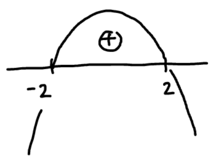
1. naloga (25 točk)

Funkcija f je podana s predpisom $f(x) = x\sqrt{4-x^2}$.

a) (8 točk) Določi definicijsko območje, stacionarne točke in območja naraščanja/padanja.

$4 - x^2 \geq 0$

$(2-x)(2+x) \geq 0$



$\mathcal{D}_f = [-2, 2]$

$$f'(x) = 1 \cdot \sqrt{4-x^2} + x \cdot \frac{1}{2\sqrt{4-x^2}} \cdot (-2x) =$$

$$= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = \frac{4-x^2-x^2}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}} = 0$$

$4 = 2x^2$

$x^2 = 2$

$x = \pm\sqrt{2}$

f': $\ominus$ on $(-2, -\sqrt{2})$, $\oplus$ on $(-\sqrt{2}, \sqrt{2})$, $\ominus$ on $(\sqrt{2}, 2)$

$-2 \quad -\sqrt{2} \quad 0 \quad +\sqrt{2} \quad 2$

f: $\searrow$ min $\nearrow$ max $\searrow$

pada: $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$

narašča: $[-\sqrt{2}, \sqrt{2}]$

b) (7 točk) Na katerih intervalih je funkcija f konveksna in kje konkavna?

$$f'(x) = \frac{4-2x^2}{\sqrt{4-x^2}}$$

$$f''(x) = \frac{-4x\sqrt{4-x^2} - (4-2x^2)\cdot\frac{1}{2\sqrt{4-x^2}}\cdot(-2x)}{4-x^2} = \frac{-4x(4-x^2) + x(4-2x^2)}{(4-x^2)\sqrt{4-x^2}} = \frac{2x^3 - 12x}{(4-x^2)^{3/2}} = \frac{2x(x^2-6)}{(4-x^2)^{3/2}} = 0$$

$\ominus$ $\oplus$ na $\mathcal{D}_f$

$x_1 = 0$ $\quad$ $x_{2,3} = \pm\sqrt{6}$

$\pm\sqrt{6} \notin \mathcal{D}_f$

f'': $\ominus$ on $(-2, 0)$, $\oplus$ on $(0, 2)$

$-2 \quad 0 \quad 2$

f: $\frown$ $\smile$

konkavna: $[-2, 0]$

konveksna: $[0, 2]$

c) (8 točk) S pomočjo prvega in drugega odvoda čim bolj natančno skiciraj graf funkcije f.

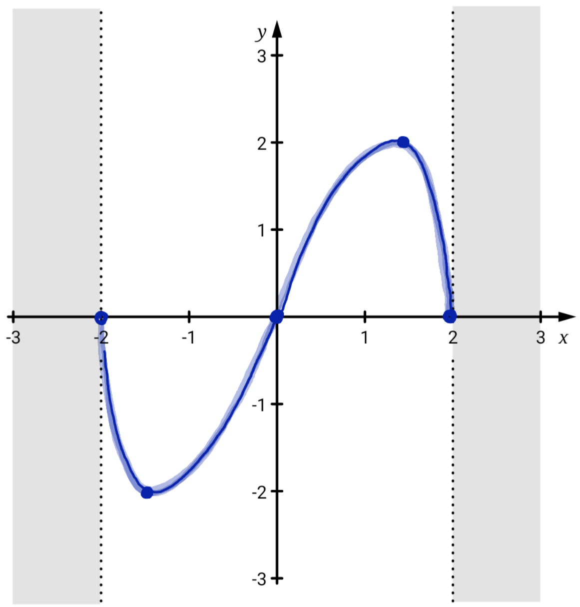
$f(\sqrt{2}) = 2$

$f(-\sqrt{2}) = -2$

$f(0) = 0$

$f(\pm 2) = 0$

liha



2. naloga (25 točk)

Naj bo

$$f(x,y) = x - xy + y.$$

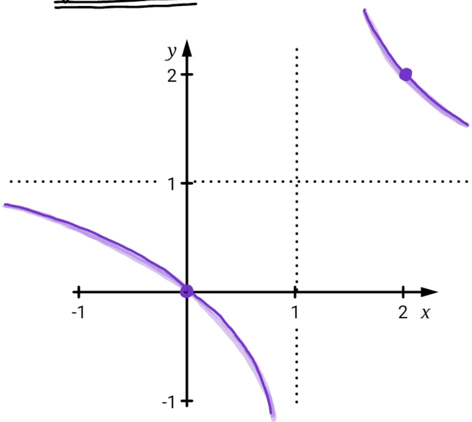
a) (7 točk) Skiciraj nivojnici $f(x,y)=0$ in $f(x,y)=1$.

$x - xy + y = 0$

$y(1-x) = -x$

$x \neq 1$: $\underline{\underline{y = \frac{x}{x-1}}}$

$x = 1$: $0 = -1$ //



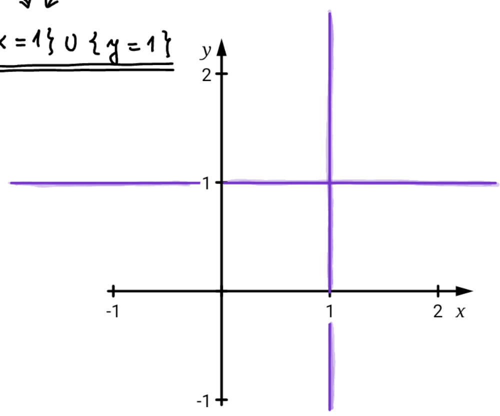
$x - xy + y = 1$

$y(1-x) = 1-x$

$x \neq 1$: $y = 1$

$x = 1$: $0 = 0$ ✓

$\underline{\underline{\{x=1\} \cup \{y=1\}}}$



b) (8 točk) Zapiši enačbo normale na nivojnico $f(x,y)=0$ v točki $(2,2)$.

$y = \frac{x}{x-1}$

$y' = \frac{1\cdot(x-1) - x\cdot 1}{(x-1)^2} = -\frac{1}{(x-1)^2}$

$y'(2) = -1$ ... sm. koef. tangente

$\Rightarrow \underline{\underline{k_n = 1}}$

$y - 2 = 1(x-2)$

$\underline{\underline{y = x}}$ normala v $(2,2)$

c) (8 točk) Določi in klasificiraj stacionarne točke funkcije f.

$f_x = 1 - y = 0$

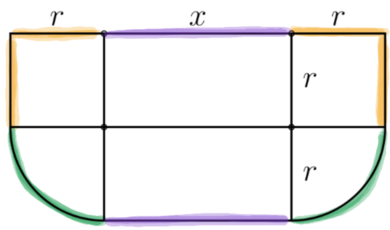
$f_y = -x + 1 = 0$

$x = y = 1 \longrightarrow \underline{\underline{(1,1)}}$ st. točka

$$\left.\begin{array}{l} f_{xx} = 0 \\ f_{xy} = -1 \\ f_{yy} = 0 \end{array}\right\} \det H_f = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = 0 - (-1)^2 = -1 < 0 \Rightarrow \underline{\underline{(1,1) \text{ je sedlo}}}$$

3. naloga (25 točk)

Vrtnar ima na voljo p metrov ograje. Ograditi želi gredico, ki bo sestavljena iz dveh pravokotnikov, dveh kvadratov in dveh četrtin kroga kot prikazuje slika. Pri tem želi porabiti vso ograjo in hkrati poskrbeti, da bo površina gredice največja možna.



a) (5 točk) Izrazi obseg p s stranico x in polmerom r.

$$p = 2x + 4r + 2\cdot\frac{1}{4}\cdot 2\pi r = \underline{\underline{2x+4r+\pi r}}$$

$p = 3x + 10r + \pi r$

če postavi ograjo tudi znotraj vrta

b) (5 točk) Izrazi površino $A(x, r)$ kot funkcijo stranice x in polmera r.

$$A(x,r) = 2xr + 2r^2 + 2\cdot\frac{1}{4}\cdot\pi r^2 = \underline{\underline{2xr + 2r^2 + \frac{1}{2}\pi r^2}}$$

c) (15 točk) Poišči vrednosti x in r, pri katerih bo vrtnar pri danem obsegu p dobil največjo površino gredice.

$$\mathcal{L}(x,r,\lambda) = 2xr + 2r^2 + \frac{1}{2}\pi r^2 - \lambda(2x+4r+\pi r-p)$$

$$\mathcal{L}_x = 2r - 2\lambda = 0 \longrightarrow r = \lambda$$

$$\mathcal{L}_r = 2x + 4r + \pi r - 4\lambda - \pi\lambda = 0$$

$$\mathcal{L}_\lambda = -(2x+4r+\pi r-p) = 0$$

$$2x + 4r + \pi r - 4r - \pi r = 0$$

$$2x = 0$$

$$\underline{\underline{x = 0}}$$

$$4r + \pi r = p$$

$$(4+\pi)r = p$$

$$\underline{\underline{r = \frac{p}{4+\pi}}}$$

$$3x + 10r + \pi r = p$$

$$x = \frac{p}{3} - \frac{\pi+10}{3}r$$

$$A(r) = \frac{2}{3}pr - \frac{2}{3}(\pi+10)r^2 + 2r^2 + \frac{1}{2}\pi r^2$$

$$A'(r) = \frac{2}{3}p - \frac{4}{3}(\pi+10)r + 4r + \pi r = 0 \quad /\cdot 3$$

$$2p - 4(\pi+10)r + 12r + 3\pi r = 0$$

$$2p - \pi r - 28r = 0$$

$$(\pi+28)r = 2p$$

$$\underline{\underline{r = \frac{2p}{\pi+28}}}$$

$$3x = p - (\pi+10)r = p - \frac{(\pi+10)\cdot 2p}{\pi+28} =$$

$$= p\,\frac{\pi+28-2\pi-20}{\pi+28} = p\,\frac{-\pi+8}{\pi+28} \quad /:3$$

$$\underline{\underline{x = \frac{(8-\pi)}{3(28+\pi)}p}}$$

4. naloga (25 točk)

a) (12 točk) Izračunaj

$$* = \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)^2+1}\, dx.$$

$t = \sin x$
$dt = \cos x\, dx$
$x = 0 \to t = 0$
$x = \frac{\pi}{2} \to t = 1$

$$* = \int_0^1 \frac{dt}{t^2+1} = \operatorname{arctg} t \Big|_0^1 = \operatorname{arctg} 1 - \operatorname{arctg} 0 = \underline{\underline{\frac{\pi}{4}}}$$

b) (13 točk) Naj bo

$$g(x) = \frac{2\sqrt{x}}{\sqrt{3\sqrt[3]{1+x}}}.$$

Izračunaj prostornino vrtenine, ki jo dobimo, če graf funkcije f zavrtimo okrog x-osi na intervalu $x \in [0,7]$. *(Namig: Uporabi integriranje po delih.)*

$$g^2(x) = \frac{4x}{3\sqrt[3]{(1+x)^2}} = \frac{4}{3}x\,(1+x)^{-\frac{2}{3}}$$

$$V = \int_0^7 \pi \cdot \frac{4}{3}x\,(1+x)^{-\frac{2}{3}} dx = 4\pi x (1+x)^{\frac{1}{3}} \Big|_0^7 - \int_0^7 4\pi (1+x)^{\frac{1}{3}} dx = 28\pi \cdot \sqrt[3]{8} - 0 - \left(4\pi \frac{(1+x)^{\frac{4}{3}}}{\frac{4}{3}}\right)\Big|_0^7 =$$

$u = \frac{4\pi}{3}x \qquad du = \frac{4\pi}{3}dx$

$dv = (1+x)^{-\frac{2}{3}}dx \qquad v = 3(1+x)^{\frac{1}{3}}$

$$= 56\pi - 3\pi(1+x)^{\frac{4}{3}}\Big|_0^7 = 56\pi - 3\pi \cdot 8^{\frac{4}{3}} + 3\pi =$$

$$= 59\pi - 3 \cdot 2^4 \pi = \underline{\underline{11\pi}}$$