University of Colorado

The Kuramoto Model

A presentation in partial satisfaction
of the requirements for the degree of
MSc in Applied Mathematics

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2008 April 24

The Kuramoto Model

Motivation: Phenomenon of collective synchronization – a large number of independent oscillators spontaneously synchronize to a common frequency and phase.

Examples:

- Pacemaker cells in the heart

- Metabolic cycles of yeast cells

- Crickets chirp in unison

- Synchronously flashing fireflies

The fireflies which are found in Siam are very common and are remarkable for the intensity of their light. One sees these insects fly separately from one tree to another and in all directions, but more often they are assembled by thousands on a great tree on the bank of the river. It is a magnificent spectacle to see spring out at one time from all the branches of this tree as it were, thousands of great electric sparks, because these fireflies emit not a continuous light but one interrupted by the effect of a sort of respiration. It is difficult to explain how this emission of light is simultaneous for several thousands of individuals.

Bishiop Jean Baptiste Pallegoix (1854)

The bushes literally swarm with fireflies, which flash out their intermittent light almost contemporaneously; the effect being that for an instant the exact outline of all the bushes stands prominently forward, as if lit up with electric sparks, and next moment all is jetty dark – darker from the momentary illumination that preceded. These flashes succeed one another every three or four seconds for about ten minutes, when an interval of similar duration takes place; as if to allow the insects to regain their electric or phosphoric vigor.
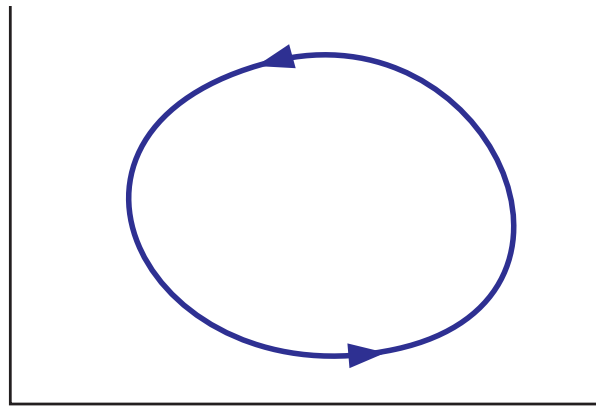
<div align="right">John Cameron (1865)</div>

Quotations from:

- John Bonner Buck. Synchronous Rhythmic Flashing of Fireflies. *The Quarterly Review of Biology*, **13**(3):301–314, September 1938.

Construct a Model of Coupled Oscillators

Stable limit-cycle oscillators:

Limit cycle in phase space

Two timescales:

- Oscillators relax to their limit cycles on one timescale

- Oscillators interact with one another on a second timescale

Assume first timescale $\ll$ second timescale.

This is what we mean by weak coupling.

So oscillators are always close to their limit cycles and we can describe an oscillator's state by a *phase θ*.

Let the oscillator's *natural frequency* be ω.

If we have N oscillators with no coupling, then

$$\dot{\theta}_i = \omega_i \qquad i = 1 \ldots N$$

Kuramoto showed that for a very general system of coupled oscillators we can write

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i)$$

where Γ_{ij} is the *interaction function*.

Kuramoto chose a specific interaction function

$$\Gamma_{ij} = \frac{K}{N} \sin(\theta_j - \theta_i)$$

where K is the *coupling parameter*.

So the *Kuramoto model* is

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \qquad (1)$$

This is easy to model and solve in Matlab (2 lines!).

Matlab code:
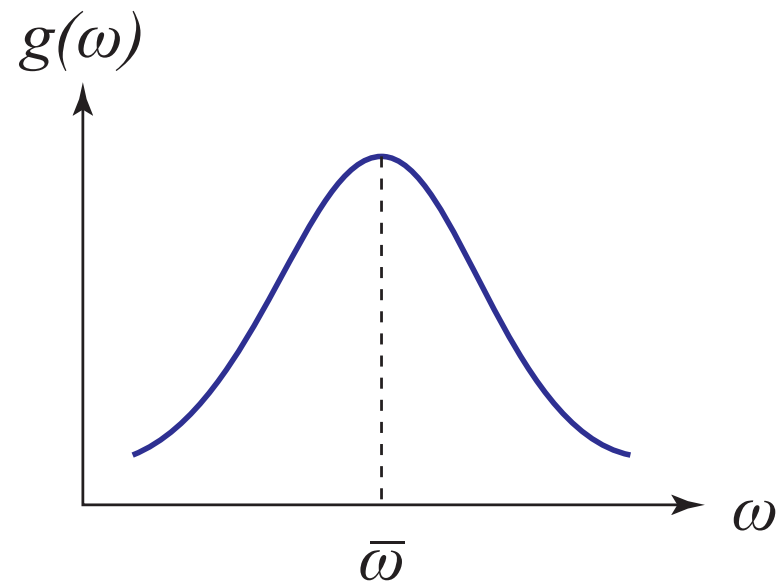
Define Kuramoto model dynamics:

```
dPdt = @(P,W,c) W+c*sum(sin(meshgrid(P)-meshgrid(P)'),2);
```

Solve the system:

```
[T P] = ode45(@(t,p)dPdt(p,W,c),[0 tmax],P0);
```

Distribution of Frequencies

Assume the natural frequencies are distributed according to some probability density $g(\omega)$, where g is *symmetric* and *unimodal* about some mean frequency $\bar{\omega}$

Redefine θ_i by $\theta_i + \bar{\omega}t$ and equation (1) becomes
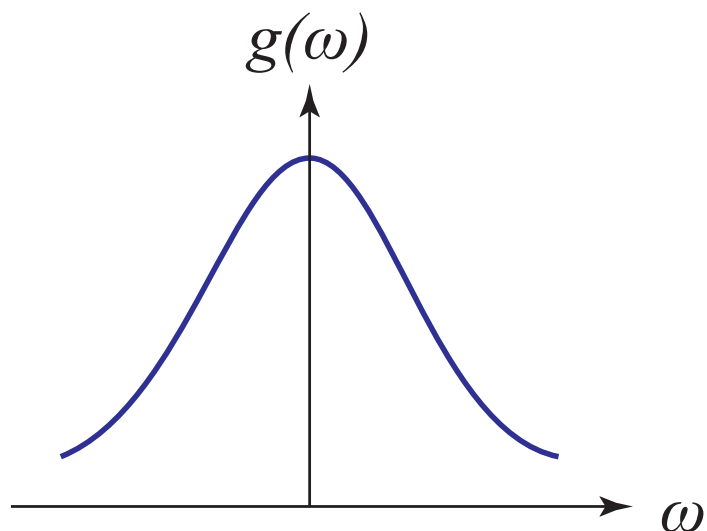
$$\dot{\theta}_i + \bar{\omega} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j + \bar{\omega}t - \theta_i - \bar{\omega}t)$$

$$\dot{\theta}_i = (\omega_i - \bar{\omega}) = +\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

Redefine ω_i by $\omega_i - \bar{\omega}$ and we have equation (1) back again

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

The distribution function g becomes symmetric about zero

symmetric: $g(\omega) = g(-\omega)$

unimodal: nowhere increasing on the interval $[0, \infty)$
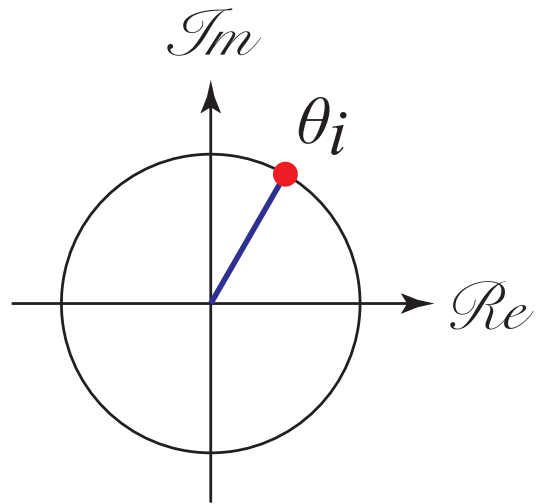
The Order Parameter

We can easily simulate a collection of N oscillators using Matlab.

We find that oscillators become synchronized for certain values of the coupling parameter K.

To get a quantitative idea of what is happening we need a measure of how synchronized the system is.

Imagine the phases as points on the unit circle

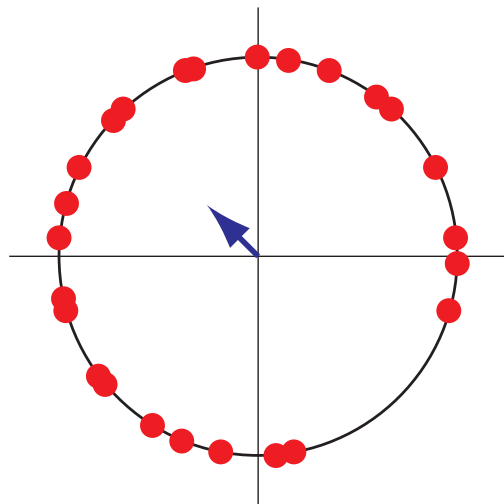Represent the point as a complex number $e^{i\theta_i}$ and take the average over all oscillators

$$R = \frac{1}{N} \sum_{j=1}^{\infty} e^{i\theta_j}$$

This defines the complex *order parameter R.*

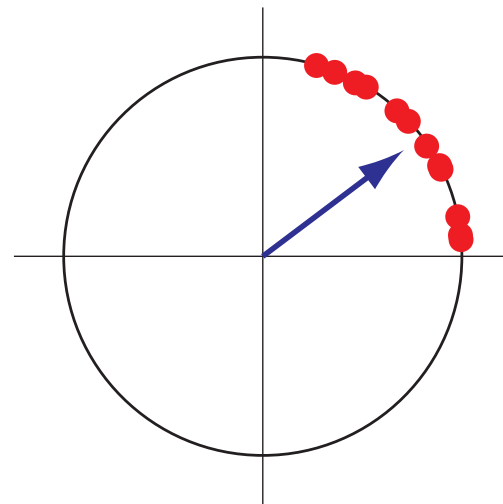Write R as $re^{i\phi}$

r is a measure of the *phase coherence*

ϕ is the *average phase*

scattered phases

r small

no synchronization

clumped phases

r large

synchronized

We can rewrite equation (1) in terms of R as follows

$$R = re^{i\phi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}$$

Multiply by $e^{-i\theta_i}$

$$re^{i(\phi-\theta_i)} = \frac{1}{N} \sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}$$

Take the imaginary part

$$r\sin(\phi - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

So equation (1) becomes

$$\dot{\theta}_i = \omega_i + Kr\sin(\phi - \theta_i)$$

It looks like we have uncoupled the equations!

But we haven't, of course.

The equations are still coupled through the values of r and ϕ.

The values r and ϕ are "averages" of the state of the system.

So we see that an oscillator interacts with the others via an "average" of the current state of the system.

The Kuramoto model is a *mean field* model.

We see that:

- The phase θ_i of an oscillator is pulled towards the mean phase ϕ (steady state: $\phi = \theta_i$)

- The strength of the coupling is proportional to the coherence r (positive feedback)

Time Evolution of the System

We can solve the system using Matlab.

What we find is

- For weak coupling the system evolves into a disordered (incoherent) state.

- For strong coupling the system becomes orderly but not all oscillators synchronize.

Weak Coupling

For small values of K, and starting in any state, the system evolves into a disordered (incoherent) state.

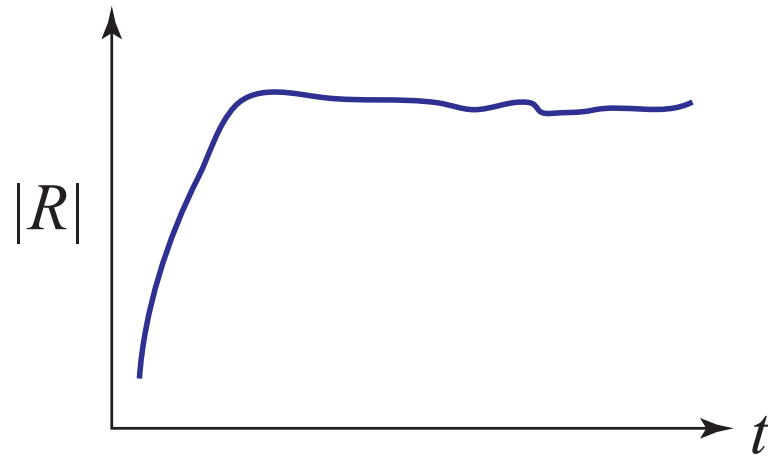Magnitude of order parameter vs. time
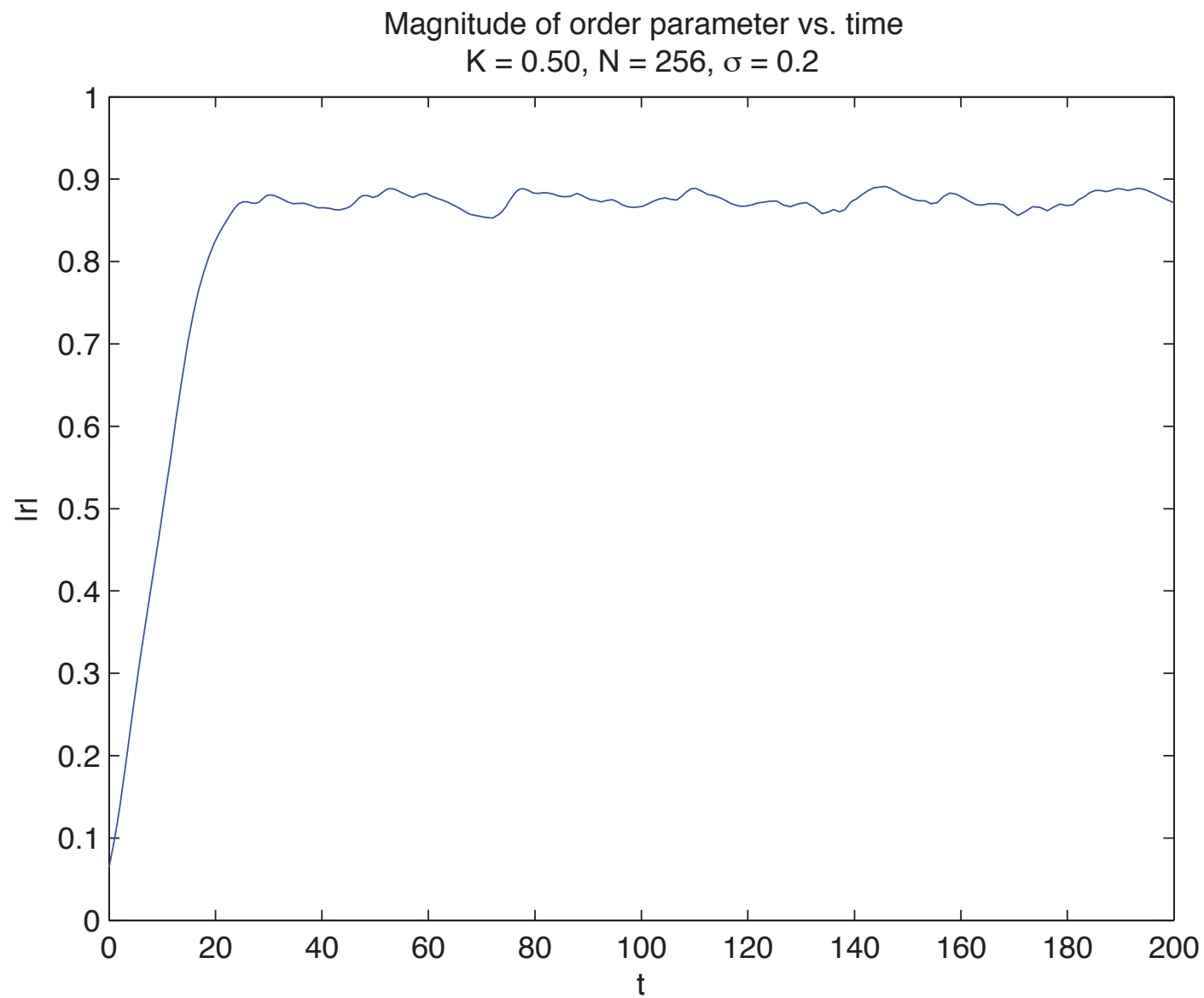K = 0.10, N = 256, σ = 0.2
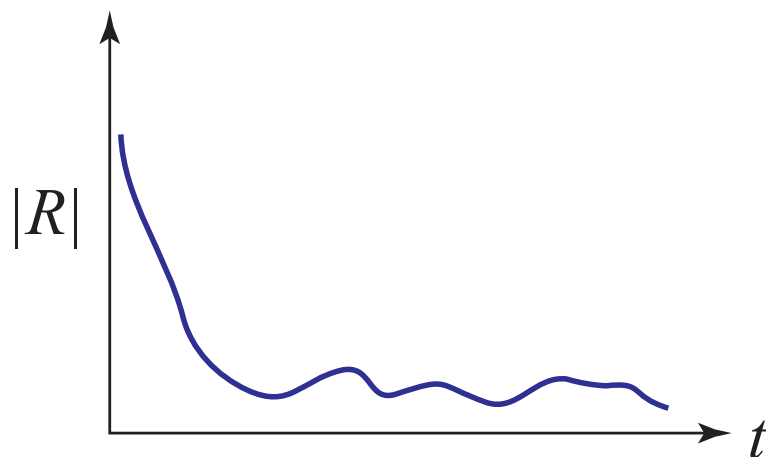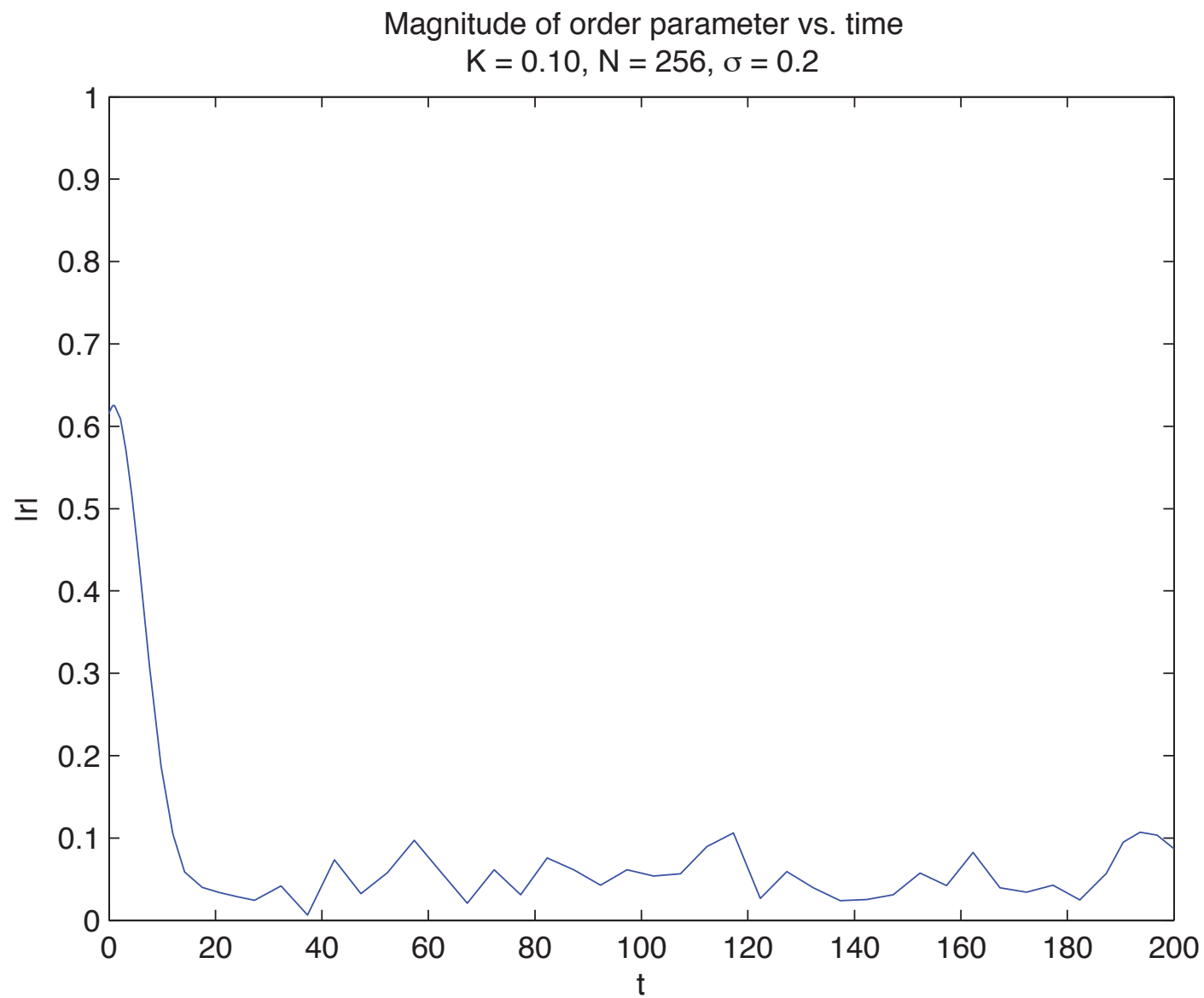
Strong Coupling

For large values of K the system becomes orderly but not all oscillators synchronize.

The system levels off with a value of $r = r_\infty < 1$.
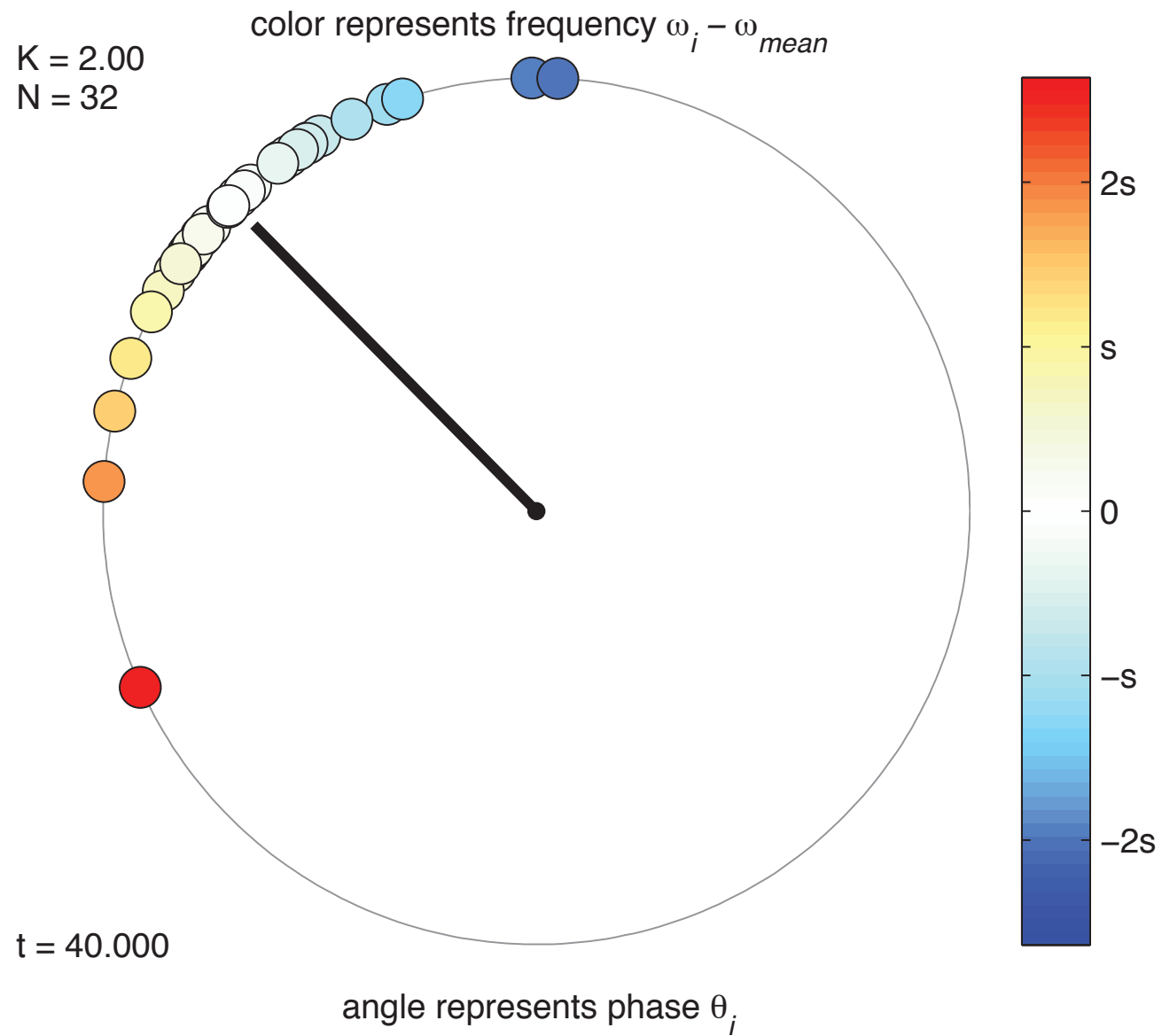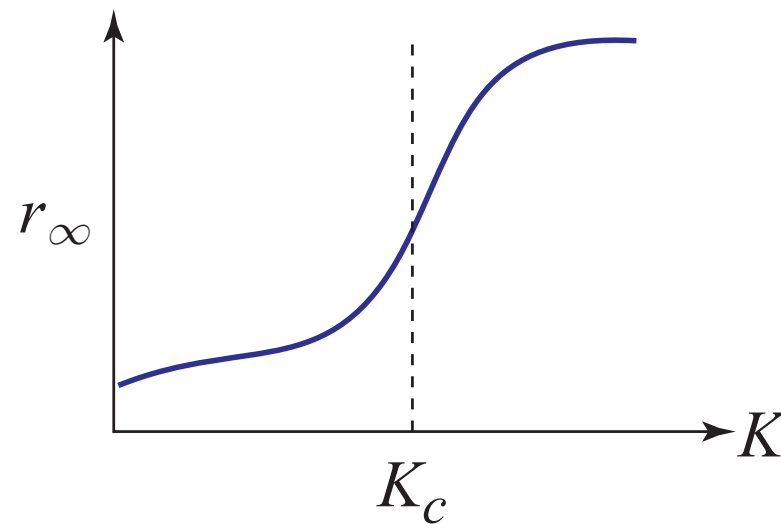
The system is only partially synchronized.

Magnitude of order parameter vs. time
K = 0.50, N = 256, σ = 0.2

In the partially synchronized case the oscillators are in two groups.

The first group has natural frequencies ω_i within 2σ of $\bar{\omega}$. These have synchronized at $\bar{\omega}$ and rotate with the average phase ϕ.

The second group has ω_i outside 2σ of $\bar{\omega}$. These run near their natural frquencies and drift relative to the cluster of synchronized oscillators.

color represents frequency $\omega_i - \omega_{mean}$

K = 2.00
N = 32
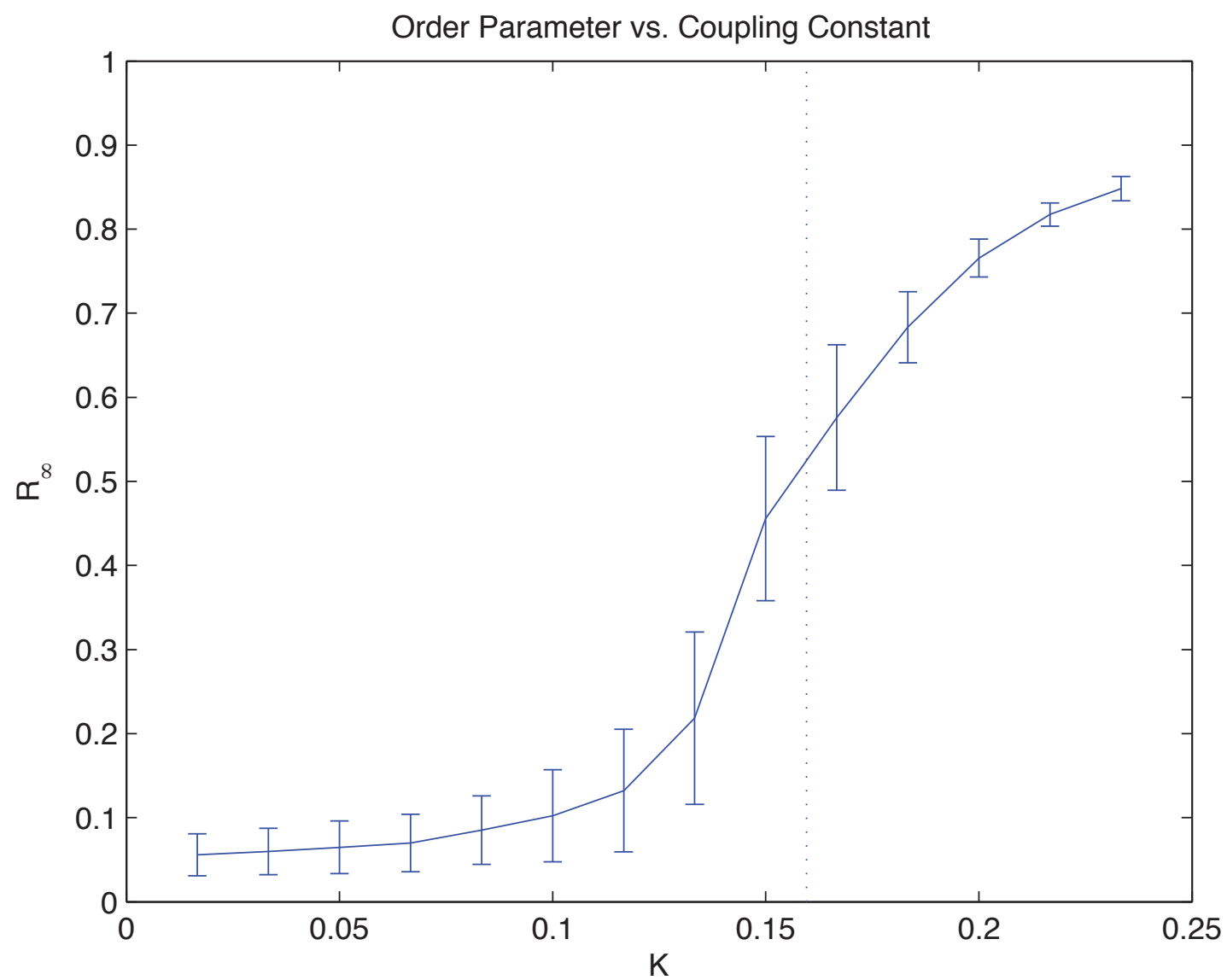
t = 40.000

angle represents phase θ_i
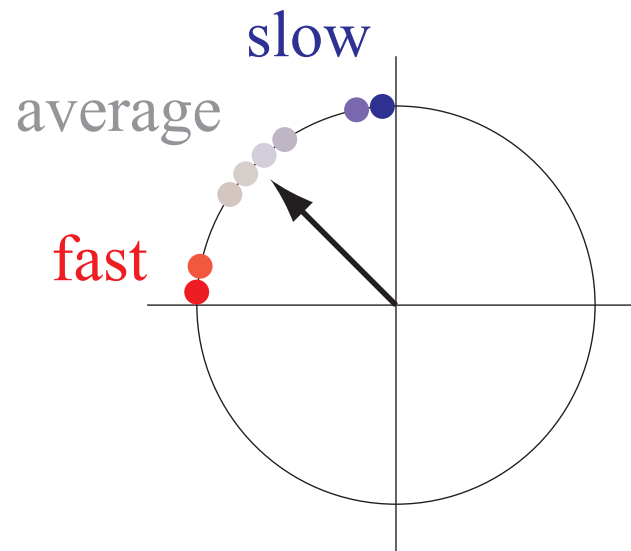
We can plot the value of r_∞ as a function of K.

For values of K less than some critical value K_c the system stays disordered.

But for $K > K_c$ the system becomes ordered.

This looks like a *phase transition*.

Order Parameter vs. Coupling Constant

Kuramoto's Analysis

Kuramoto was able to get formulas for K_c and r_∞.

$$K_c = \frac{2}{\pi g(0)}$$

This is an exact formula for any g satisfying the conditions above.

For the special case of

$$g(\omega) = \frac{\gamma}{\pi \left(\gamma^2 + \omega^2\right)}$$

Kuramoto was able to get

$$r_\infty = \sqrt{1 - \frac{K_c}{K}}$$

Stability

Kuramoto and Nishikawa tried to prove that the system was stable.

They published two papers in 1987.

Neither turned out to be correct.

Strogatz and Mirollo (1991) and Crawford (1995) proved that the incoherent state is stable.

But as of 2000, nobody has been able to prove that the ordered state is stable.

- Yoshiki Kuramoto and Ikuko Nishikawa. Statistical Macrodynamics of Large Dynamical Systems: Case of a Phase Transition in Oscillator Communities. *Journal of Statistical Physics*, **29**:569–606, 1987.

- Yoshiki Kuramoto and Ikuko Nishikawa. Cooperative Dynamics in Complex Physical Systems: Proceedings of the Second Yukawa International Symposium, Kyoto, Japan, August 24–27, 1988. Hajime Takayama, editor.

- Steven H. Strogatz and Renato E. Mirollo. Stability of Incoherence in a Population of Coupled Oscillators. *Journal of Statistical Physics*, **63**:613–635, 1991.

- John D. Crawford. Amplitude Expansions for Instabilities in Populations of Globally Coupled Oscillators. *Journal of Statistical Physics*, **74**:1047–1084, 1994.

- John D. Crawford. Scaling and Singularities in the Entrainment of Globally Coupled Oscillators. *Physical Review Letters*, **74**:4341–4344, 1995.

- John D. Crawford and K. T. R. Davies. Synchronization of Globally Coupled Phase Oscillators: Singularities and Scaling for General Couplings. *Physica D: Nonlinear Phenomena*, **125**:1–46, 1999.