FACULTY OF SCIENCE
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Bachelor's project

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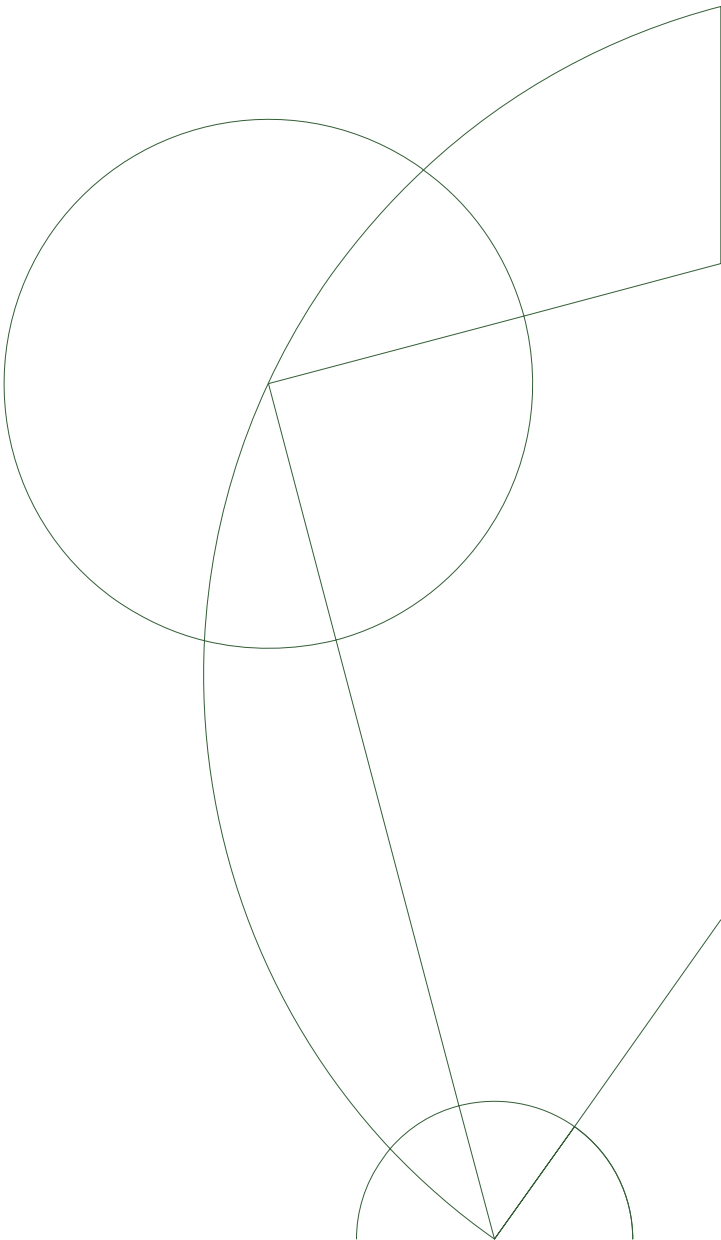
Parallel Bit-Counting

for Approximate Similarity Searching



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Abstract

This project regards an analysis of recent advances in solving the Approximate Jaccard Similarity Search Problem, specifically in regards to how one can achieve sublinear query time using parallel bit counting as presented by Knudsen[11]. This contains a theoretical analysis of both runtime and correctness of the bit-counting algorithm as well as an empirical comparison to existing methods. The findings include a theoretical run time advantage to using parallel bit counting compared to a simple, linear time algorithm, but empirical results show that this advantage is hard to achieve in practice. Furthermore, reflections are made on how the results might scale on more specialized hardware.

Contents

1 Introduction

The *Approximate Similarity Search Problem* regards efficiently finding a set A from a corpus $\mathcal{F}$ that is approximately[1] similar to a query set Q in regards to the *Jaccard Similarity* metric $J(A,Q) = \frac{|A \cap Q|}{|A \cup Q|}$[7][11]. Practical applications includes searching through large corpi of high-dimensional text documents like plagiarism-detection or website duplication checking among others[9]. The main bottleneck in this problem is the *curse of dimensionality*. Any trivial algorithm can solve this problem in $O(n|Q| \max_{A \in \mathcal{F}} |A|)$ time, but algorithms that query in linear time to the dimensionality of the corpus scale poorly when working with high-dimensional datasets. Text documents are especially bad in this regard since they often are encoded using w-*shingles* (w contigous words) which Li, Shrivastava, Moore, *et al.* [6] shows easily can reach a dimensionality upwards of $d = 2^{83}$ using just 5-shingles.

The classic solution to this problem is the MinHash algorithm presented by Broder [1] to perform website duplication checking for the AltaVista search engine. It preprocesses the data once using hashing to perform effective querying in $O(n^\rho \log(n)|Q|)$ time, a significant improvement independent of the dimensionality of the corpus. Many improvements have since been presented to both improve processing time, query time and space efficiency. Notable mentions includes (but are not limited to) the use of *tensoring*[4], *b-bit minwise hashing*[5], *fast similarity sketching*[7]. Simple applications of these techniques leads to efficient querying with a constant error probability. If one wishes to achieve an even better error probability such as $\varepsilon = o(1)$, it is standard practice within the field to use $O(\log_2(1/\varepsilon))$ independent data structures and return the best result, resulting in a query time of $O(\frac{1}{\epsilon}(n^\rho + |Q|))$[2]. Recent advances by Knudsen [11] show that it is possible to achieve an even better query time by sampling these data structures from one large sketch. The similarity between these sub-sketches and a query set needs to be evaluated efficiently when querying, which requires efficient computation of the cardinality of a bit-string. To do this, Knudsen [11] presents a general parallel bit-counting algorithm that computes the cardinality of a list of bit-strings in sub-linear time amortized due to word-parallism. This brings the query time down to $O((\frac{n \log_2 w}{w})^\rho \log(1/\varepsilon) + |Q|)$.

The main focus of this project is to analyse, prove, implement and evaluate this parallel bit-counting technique. The analysis will be based on the original paper [11], but with some modifications to resolve some of the issues with the original algorithm. This will also include a pseudo-code implementation of the algorithm since the original paper only describes it through recurrences. This leads to a proof of correctness that slightly alters from the one presented in the paper, and a runtime analysis that does indeed show the sub-linear run time as claimed.

This theoretical analysis will be backed up by a real-life implementation that can be benchmarked to help show this sub-linear run time in practice. At last, reflections on the results and methods will be made to back up eventual conclusions.

[1] The exact definition of approximate will be explained later.

[2] ρ is a parameter between 0 and 1 that decides how accurate the approximation will be. Its exact definition will be introduced later.

2 Theory

The Similarity Search Problem has many variations and uses. The general idea is to find some data point from a set of datapoints that is the most similar to some query point according to some similarity metric. The LSH framework, one of the first big breakthroughs in this field, regards points in a d-dimensional space and uses the Manhattan distance between them as a similarity metric - two points that are close together are regarded as more similar[3]. In this project, we will regard the similarity between sets and use the Jaccard similarity $J(A, B) = \frac{|A \cap B|}{|A \cup B|}$ as our similarity metric. The Jaccard similarity produces a number between 0 and 1, where a larger number indicates more similarity between the two sets.
This part will first define the problem to be considered in a concise way - then describe the existing solutions in general to the context of the parallel bit counting algorithm, described in the next section to set the context for the rest of the report.

2.1 Problem Definition

The formal definition of the *Jaccard Similarity Search Problem* is as follows: Given a family $\mathcal{F}$ of n sets from some universe U and a query set Q, preprocess $\mathcal{F}$ to efficiently find the set $S \in \mathcal{F}$ such that $S = \max_{A \in \mathcal{F}} J(A, Q)$.
In many practical applications, it may be enough to only consider the *Approximate Jaccard Similarity Search Problem* (as defined by Knudsen [11]): Given a family $\mathcal{F}$ from a universe U, a query set Q and two variables $0 < j_2 < j_1 < 1$, preprocess $\mathcal{F}$ such that one can efficiently find some set $A \in \mathcal{F}$ such that $J(A, Q) \geq j_2$ if there exists some set $B \in \mathcal{F}$ where $J(B, Q) \geq j_1$. Note that it is possible for $A = B$. The intuition to this is that we can dial in how precise we expect our algorithm to be able to solve this problem by fine-tuning j_2 and j_1. We no longer expect the algorithm to return the *best* solution, just one that is "good enough" (if it exists). Algorithms that solve this problem should also be able to return "no answer", if no set $B \in \mathcal{F}$ upholds $J(B, Q) \geq j_1$, which is entirely possible depending on the parameters.
Most existing methods to solve this problem depend on the relationship between j_1 and j_2. This relationship is often described as $\rho = \frac{\log_2(1/j_1)}{\log_2(1/j_2)}$. Note that ρ shrinks when j_1 approaches 1.

2.2 Trivial Solution

The most obvious solution is to compare each coordinate in each set $A \in \mathcal{F}$ with each coordinate in Q to calculate the size of the intersection $|A \cap U|$ and union $|A \cup U|$. This solution will always give the correct answer, and query in $O(d^2 n)$ time trivially, but suffers from the *curse of dimensionality*. As the amount of dimensions in the data set doubles, the runtime quadruples! When working with high-dimensional datasets like often seen in text processing, this can have huge consequences. Li, Shrivastava, Moore, *et al.* [6] has shown that it is easy to reach a dimensionality upwards of $d = 2^{83}$ when using *5-shingles* (5 contigous words) of the 10,000 most common English words. A simple improvement is to make a hash table for D of size d with no collisions and look up every entry in A to find any matches. This could reduce the run time to $O(d + dn) = O(dn)$ per query.
These algorithms are both guaranteed to return the set $S = \max_{A \in \mathcal{F}} J(Q, A)$ on every

query. The next algorithm can query much faster than this by relaxing this condition: if we allow preprocessing the data and relax the requirement of finding an exact match by considering the *approximate similarity search problem*, we can create sketches of the data set before querying. This sketching strategy can drastically reduce the query time.

2.3 Locality Sensitive Hashing Framework

One of the first big contributions to this problem is the Locality Sensitive Hashing Framework by Indyk and Motwani [2]. This framework aims to solve the Approximate Similarity Search problem for any kind of similarity space, including Jaccard similarity, using some appropriate (s_1, s_2, r_1, r_2)-sensitive family of hash functions that excibit the following properties:

Definition 1. *A family of hash functions $\mathcal{H} = \{h : X \to U\}$ is (s_1, s_2, r_1, r_2)-sensitive for some set X and some universe U with a similarity function $S : X \to X \to [0; 1]$ if these conditions apply for any $x, y \in X$:*

- *If $S(x, y) \geq s_1$, then $Pr[h(x) = h(y)] \geq r_1$*
- *If $S(x, y) \leq s_2$, then $Pr[h(x) = h(y)] \leq r_2$*

First, we define a family of functions $\mathcal{G} = \{g : X \to U^k\}$ such that $g(p) = \langle h_0(p), \ldots, h_{k-1}(p)\rangle$ for $h_i \in \mathcal{H}, k \in \mathbb{Z}^+$, where k is to be determined later. We can now regard $g(p)$ as the unique identifier of a "bucket", where we can store the value p. We do this l times with l different functions $g_0, \ldots, g_{l-1}$ for each data point $p \in X$. If a collision happens, we store only one of the colliding points at random.
To query, calculate $g_0(q), \ldots, g_{l-1}(q)$ and look up the points in each corresponding bucket. This results in points $P = \{p_0, \ldots, p_{t-1}\}$ where t is the amount of points found. For each $p_i \in P$, if $S(p_i, q) \geq s_2$, then return it. If no points fulfills this constraint, return false.
Part of implementing this framework for a specific similarity metric (e.g. Jaccard Similarity) requires choosing a suitable value for k and l. Finding the right parameters can in this case result in getting a constant 1-sided error probability, as we will show later.

2.4 MinHash

The MinHash algorithm is one of the classic solutions to the *approximate similarity search problem*, introduced by Broder [1] for the AltaVista search engine.
It is based on the following theorem:

Theorem 1. *Let g be a hash function from $V \cup W \to [M]$ for some $M \in \mathbb{Z}^+$ where $W \cup V \subseteq U$. Let $H(W) = \min_{w \in W} g(w)$. Then*

$$Pr[H(W) = H(V)] = \frac{|V \cap W|}{|V \cup W|} = J(V, W)$$

Proof. If you sample a random element $x \in V \cup W$, then you can find the probability that $x \in V \cap W$ like so:

$$Pr[x \in V \cap W] = \frac{|V \cap W|}{|V \cup W|} = J(V, W)$$

When we calculate $H(W)$ and $H(V)$, the smallest value of the two will be $H(W \cup V)$. Since we use hash functions, this is essentially equivalent of sampling a random value from $W \cup V$. If this value is in the intersection $W \cap V$, then $H(W)$ and $H(V)$ will have picked the same element. This means that the probability of $H(W) = H(V)$ is the same as the probability of picking an element of the intersection from the union, which is $\frac{|V \cap W|}{|V \cup W|} = J(V, W)$ □

The MinHash algorithm works like so: First, preprocess the data by hashing each coordinate using k different hash functions and saving the smallest value for each hash function as a sketch $S(A) = \langle H_0(A), \ldots, H_{k-1}(A) \rangle$. To query a set Q, calculate its sketch $S(Q)$ and then calculate

$$d(S(A), S(Q)) = \frac{1}{k} \sum_{i \in [k]} [H_i(A) = H_i(Q)]$$

The expected value of this will then be the Jaccard Similarity:

$$\mathbb{E}[d(S(A), S(Q))] = \mathbb{E}[\frac{1}{k} \sum_{i \in [k]} [H_i(A) = H_i(Q)]]$$

$$= \frac{1}{k} \sum_{i \in [k]} \mathbb{E}[[H_i(A) = H_i(Q)]]$$

$$= \frac{1}{k} \sum_{i \in [k]} J(A, Q)$$

$$= J(A, Q)$$

Since each hash function is independent, we can also use a Chernoff probability bound: Let us call each indicator variable $[H_i(A) = H_i(Q)]$ for X_i and let $X = \sum_{i \in [k]} X_i$ and $\mu = \mathbb{E}[X]$. Then for any $\delta > 0$

$$Pr[X > (1 + \delta)\mu] < \left(\frac{e^{\delta}}{(1 + \delta)^{(1+\delta)}} \right)^{\mu}$$

$$Pr[X < (1 - \delta)\mu] < \left(\frac{e^{-\delta}}{(1 - \delta)^{(1-\delta)}} \right)^{\mu}$$

The more hash functions we choose, the larger k and X will be, the larger μ will be and the stricter the bound will be.

2.4.1 Implementation with the LSH framework

For any hash function h picked randomly from $\mathcal{H}$, we know that $Pr[h(A) = h(B)] = J(A, B)$ as shown earlier. This must mean that h is (j_1, j_2, j_1, j_2)-sensitive. If we then consider a function g picked randomly containing $h_0, \ldots, h_{k-1}$ hash functions. We can then derive that $Pr[g(a) = g(b)] = J(A, B)^k$. This means that g is (j_1, j_2, j_1^k, j_2^k)-sensitive. By choosing $k = \lceil \frac{\log_2(n)}{\log_2(1/j_2)} \rceil$, we can derive that g now is $(j_1, j_2, O(1/n^{\rho}), 1/n)$-sensitive where $\rho = \frac{\log_2 1/j_1}{\log_2 1/j_2}$. This has the consequence that the error probability now is dependendant on $1/n$, which can be converted to a constant error probability by repeating the experiment multiple times. By keeping $l = \lceil n^{\rho} \rceil$ different data structures,

we reduce the error probability to $O(1)$. This means that the run time needed to create the required sketches is $O(l \cdot k \cdot d) = O(n^\rho \log_2(n)|Q|)$ per data point. The query time is now $O(l \cdot k \cdot |Q|) = O(n^\rho \log_2(n)|Q|)$ since we need to calculate a min-hash value of $|Q|$ for each of the l sketches which takes $O(k \cdot |Q|)$ time per sketch. This process will lead to l different candidate points, from which a true positive can be linearily searched in $O(l \cdot |Q|)$ time.

2.5 Later Improvements

Many improvements have since been discovered. Dahlgaard, Knudsen, and Thorup [7] have introduced an improved sketch, which is much faster to create and preserves the same properties as the MinHash sketch, while reducing the query time to $O(k \cdot l + |Q|) = O(n^\rho \log_2(n) + |Q|)$. Part of this improvement is due to a special algorithm that can eliminate "bad matches" by estimating if the Jaccard similarity between the query set and candidate set is above some threshold within a certain probability. Here, a "bad match" is defined as a candidate set A where $J(A, Q) < j_2$. This reduces the filtering step to only take $O(l + |Q|)$ time, since the bad matches can be eliminated in $O(l)$ time and the amount of matches remaining is $O(1)$. Christiani [8] has since improved this query time to $O(n^\rho + |Q|)$.

Until now, the data structures presented error with a constant probability $O(1)$. If one wants to reach a better error probability ε (could for example be $\varepsilon = O(1/n)$), it is common practice within the field to use $O(\log_2(1/\varepsilon))$ independent data structures. This will usually result in a query time of $O((n^\rho + |Q|) \log_2(1/\varepsilon))$.

Knudsen [11] has shown a method of achieving an improved $O(n^\rho \log_2(1/\varepsilon) + |Q|)$ query time instead, which is very beneficial when working with highly dimensional datasets and a small ε. This is done by creating the M different sketches of Q nescessary for querying from one big sketch of size $M \cdot \log_2^3(n)$, with each sketch segment fed into each LSH datastructure.

Knudsen [11] has further improved the query time to $O((\frac{n \log_2(d)}{d})^\rho \log_2(1/\varepsilon) + |Q|)$ (where d is the word-size) by reducing k such that $h_i \in \mathcal{H}$ becomes $(j_1, j_2, O((n/b)^\rho), n/b)$-sensitive where $b = d/\log_2(d)$ and then efficiently filtering through the b expected "bad" matches.

Understanding why this filtering is efficient requires understanding the b-bit minwise hashing trick presented by Li, Shrivastava, Moore, *et al.* [6]: By only storing the b lowest bits of each hash value, one can pack $O(d/b)$ hash values per word. Knudsen [11] uses 1-bit minwise hashing to create a sketch of size $O(\log_2(d))$ per set $A \in \mathcal{F}$. While 1-bit hashing greatly increases the chance of collision per hash value, it allows us to use many more hash values per sketch, which is suprisingly effective at negating this [6]. This allows us to store $O(\frac{d}{\log_2(d)})$ sketches per word. When querying, instead of comparing our sketches of Q with each of the $O(n^\rho)$ sketches per data structure one at a time, we can compare it to $O(\frac{d}{\log_2(d)})$ different sketches per operation.

By calculating the amount of bits per sketch for A and Q that are equal, one can estimate the Jaccard Similarity to a degree which is good enough to filter a candidate match.

This calculation can be done by bitwise XNOR'ing a word packed with sketches of a set $A \in \mathcal{F}$ with a word packed with sketches of Q. Then, one can calculate the amount of bits set. Every bit set will then indicate that Q and A hashed to the same value, which means that we can filter each match based on whether or not the total amount

of bits set is over some threshold.
It is this calculation on how to count these bits efficiently that the rest of this project will focus on.

3 Parallel Bit Counting

In the same article where Knudsen [11] presents the improvements for algorithms with an $o(\varepsilon)$ error probability, a parallel algorithm to compute the cardinality of a bit-string is also presented. It is only described by recurrences, and omits any implementation details. I will try to describe the same algorithm in a much simpler fashion and prove its correctness and run time while doing it. This means that this section is my own interpretation of the algorithm, proofs and analysis described in [11]. A naive algorithm to perform bit-counting could be described like in algorithm 1.

Algorithm 1 A naive linear time algorithm

```
1: function CARDINALITY(w, d)        ▷ w is the input word, d is the word-size
2:     x ← 0
3:     i ← 0
4:     while i ≤ d do
5:         x ← x + (w ≫ i) ∧ 1
6:         i ← i + 1
7:     end while
8:     return x
9: end function
```

This algorithm trivially runs in linear time to the dimensionality[3] d of the input word w. For n words of size d, this algorithm runs in $O(nd)$ time. Knudsen [11] presents two improvements to this: The first will improve the run time to $O(n \log(d))$ time by utilizing a divide-and-conquer technique and the second to $O(n + \log(d))$ time by introducing parallelism.

3.1 Divide-and-Conquer

To explain how divide-and-conquer methods can be used to improve run-time, we must first introduce a bit mask from [11] defined like so:

$$m_{i,j} = \underbrace{0 \ldots 0}_{2^j - 2^i} \underbrace{1 \ldots 1}_{2^i} \ldots \underbrace{0 \ldots 0}_{2^j - 2^i} \underbrace{1 \ldots 1}_{2^i}$$

where $j > i$ and j and i are non-negative integers. The notation m_i is a shorthand for $m_{i,i+1}$ and indicates a mask with an equal amount of 1's and 0's. By computing $w \wedge m_{i,j}$ for some word w and some integers i, j, we can replace every other sub-string of size 2^i in the bit-string with 0's. This will be described as *isolating* segments of size 2^i.
We will use this in the following operation:

$$T(w, m, k) = (w \gg k) \wedge m \tag{1}$$

This operation isolates the segments indicated by the bit-mask starting from the kth position of the word w. This is useful for partitioning a bit-string, since any word now can be described as

$$w = T(w, m_i, 0) + (T(w, m_i, 2^i) \ll 2^i)$$

[3]For the rest of this project, it is assumed that d is a power of 2.

The algorithm to calculate the cardinality can then be described like in algorithm 2

Algorithm 2 A divide-and-conquer approach

```
1: function CARDINALITY(w, d)          ▷ w is the input word, d is the word-size
2:     i ← 0
3:     while i ≤ log_2(d) do
4:         w ← T(w, m_i, 0) + T(w, m_i, 2^i)
5:         i ← i + 1
6:     end while
7:     return w
8: end function
```

The simplest way to gain intuition of this algorithm is to prove it. A loop-invariant is presented like so:

Invariant 1. *At the ith iteration of the algorithm, each bitstring-segment of size 2^i of the word $w^{(i)}$ will contain the cardinality of the corresponding segment of the word $w^{(0)}$. Furthermore, the operation can be done in constant time.*

Proof. When $i = 0$, then each bit-string of size $2^0 = 1$ in $w^{(0)} = w$ will be 1 if the bit is 1 and 0 if the bit is 0. This proves our base case.
If at step i the word $w^{(i)}$ contains $\frac{d}{2^i}$ segments of size 2^i, then invariant 1 will be upheld if we combine each segment with its neighbor to form a segment of size 2^{i+1}. Since the cardinality of a combined bitstring is equal to the cardinality of its parts, we can divide all of the segments into pairs of size 2^{i+1} and add them together to form the new pair, which is done by the operation $T(w, m_i, 0) + T(w, m_i, 2^i)$ in constant time if the masks are pre-computed. This operation works because $T(w, m_i, 0)$ isolates every other segment of size 2^i starting from the first segment and $T(w, m_i, 2^i)$ isolates every other segment of size 2^i starting from the second segment.

□

When the algorithm terminates after $\log_2(d)$ iterations, the segments will have size $2^{\log_2(d)} = d$ and thus span the entire original word, which means that we have the bit-count of the original word.

3.2 Parallelism

To introduce parallelism into the algorithm, we must first realize the following: When two segments of 2^i bits gets combined, they will not need all of the 2^{i+1} bits to represent their sum. It is actually such that the bits used at iteration i is exactly $i + 1$.

Invariant 2. *At the ith iteration of the algorithm, the amount of bits set in a given segment of a word is at most $i + 1$.*

Proof. We will prove this by induction.
At $i = 0$, the size of the segments are $2^0 = 1$, which is equal to $i + 1$.
If it is true at iteration i, then at iteration $i + 1$ the algorithm will add two words of size $i + 1$ which creates a word of size $i + 2$, which fulfills the loop invariant. □

Now, we will introduce a function $l(i)$, which produces the smallest number such that $2^{l(i)} \geq i + 2$. If we use the bit mask $m_{l(i),i+1}$ instead of m_i, we will get the same result. This also means that we can pack $2^{i-l(i)}$ words into one by utilizing the empty space in each segment.

Algorithm 3 A parallel divide-and-conquer algorithm

1: **function** COMPUTE(S, d) ▷ S is the input set, d is the word-size
2: $n \leftarrow S.length$
3: **for** $i \in [\log_2(d)]$ **do**
4: $k' \leftarrow \{\}$
5: $t \leftarrow \{\}$
6: **for** $w \in S$ **do**
7: $k' \leftarrow k' \cup \{T(w, m_i, 0) + T(w, m_i, 2^i)\}$
8: **end for**
9: **if** $i < 2$ **then**
10: $S \leftarrow k'$
11: **else**
12: **if** $l(i) = l(i+1)$ **then**
13: **for** $j \in [\lceil S.length/2 \rceil]$ **do**
14: $t \leftarrow t \cup \{k'_{2j} + (k'_{2j+1} \ll 2^i)\}$
15: **end for**
16: **else**
17: **for** $j \in k'$ **do**
18: $t \leftarrow t \cup \{T(k'_j, m_{l(i)}, 0) + (T(k'_j, m_{l(i)}, 2^{l(i)}) \ll 2^i)\}$
19: **end for**
20: **end if**
21: $S \leftarrow t$
22: **end if**
23: **end for**
24: $S' \leftarrow \{\}$
25: **for** j in $[S.length]$ **do** ▷ For every word in the final set
26: **for** k in $[2^{l(\log_2(d))}]$ **do** ▷ For every original word embedded in S_j
27: $S' \leftarrow S' \cup \{T(S_j, m_{l(\log_2(d)),\log_2(d)+1}, k \cdot 2^{l(\log_2(d))})\}$
28: **end for**
29: **end for**
30: **return** S'
31: **end function**

This algorithm is quite a bit more complex than the first one, and to prove it we need some more terminology.
When we pack the words $t_0, \ldots, t_{2^{i-l(i)}-1}$ into a word t, we say that t is i-packed. This means one can extract all $2^{i-l(i)}$ different words by performing the operation:

$$t_j = T(t, m_{l(i),i+1}, j \cdot 2^{l(i)}) \tag{2}$$

for all $j \in [2^{i-l(i)} - 1]$. The point of the algorithm is to ensure that after every step of the algorithm that every word in S is $(i + 1)$ packed. When the first loop of the algorithm terminates at $i = \log_2(d) - 1$, then every word will be $\log_2(d)$-packed, which

means that it fits $2^{\log_2(d)-l(\log_2(d))}$ words. When working with 64-bit words, this will result in $2^{6-3} = 8$ words per packed word.

Invariant 3. *At the end of the ith iteration of the first loop of the algorithm, every word in S will be $(i+1)$-packed for any $i \geq 1$*

Proof. First, the first two iterations of the loop are run, such that $i = 1$. During the first two iterations of the loop, no words are combined since it just uses the naive algorithm, which upholds the invariant because any 2-packed words embeds $2^0 = 1$ of the original words.
For any iteration where $i \geq 2$, one out of two things can happen. Either, $l(i+1) = l(i)$ (we do not need more bits to describe each segment), or $l(i+1) = l(i) + 1$ (we need twice the amount of bits to describe each segment).
In the first case, we will combine the words using bit-shifting. Since every word in k' uses the same amount of bits to represent each segment even though we just combined segments when creating k', there must be an empty space of at least size $l(i+1)$ in every segment. We can fill out that empty space by adding another word bit-shifted by a factor of 2. Now, the amount of words embedded is equal to the sum of both of its parts (i.e. has doubled). This upholds the loop invariant since any $(i+1)$-packed word must have twice the amount of words embedded than a i-packed word if $l(i+1) = l(i)$ by definition.
In the second case, $l(i+1) = l(i) + 1$. In this case, we need to relocate our segments that actually use $2^{l(i)}$ bits so they fit into the segments of size 2^i bits that were created when creating k'. The same amount of words have been embedded into each words as before, but as $l(i+1) = l(i) + 1$ a $(i+1)$-packed word should only embed the same amount of words as a i-packed word, which upholds the invariant. Furthermore, since we relocated the segments, we can now still extract each original word by performing the operation in equation 2.
The algorithm will terminate when $i = \log_2(d) - 1$, from which each word will embed $2^{\log_2(d)-l(\log_2(d))}$ of the original words. □

3.3 Discrepansies

In the original article, combining the sections of a word (like we do in algorithm 2 and on line 7 and 14 in algorithm 3) is done with the following operation:

$$t^{(i+1)} = T(T(t^{(i)}, m_i, 0) + T(t^{(i)}, m_i, 2^i), m_{i+1}, 0)$$

Here, the algorithm is described as a recurrence, where $t^{(i)}$ is the word t after i iterations. This definition is troublesome since the outermost call to T causes information loss. A counter-example to this is just the word $t^{(0)} = 0\texttt{b}1111$. We can see that this word contains 4 bits set. Using this operation, we can see that

$$t^{(1)} = T(T(0\texttt{b}1111, 0\texttt{b}0101, 0) + T(0\texttt{b}1111, 0\texttt{b}0101, 2), 0\texttt{b}0011, 0) = 0\texttt{b}0010$$

The outermost application of the bit mask means that we lose information about the two most significant bits of t. If we were to calculate $t^{(2)}$ and thus let the algorithm terminate, we would only have counted half of all the bits!
This is simply solved by removing the outermost call to T, since no information will be lost and invariant 1 will apply.

Another issue with the original article is that the order of the cardinalities in the final set S' is not consistent with the order of the original set $S^{(0)}$. When the algorithm shifts the sections in each word in line 17-19 of the algorithm, every other section gets shifted 2^i bits to the left. When the algorithm has terminated, the cardinality of $S_i^{(0)}$ might be contained in S'_j where $i \neq j$. The original paper by Knudsen [11] does not mention this, and it is not said whether this affects the criteria of correctness, but for any practical usage, this does seem like it could be an issue. This is only a problem when $\log_2(d) \leq 7$, since the affected branch is only run when $l(i+1) = l(i) + 1$ and $i > 2$ (it is also run when $i = 2$, but causes no problem since no words have been packed yet, which means that they will not get shifted). In the case where $d = 128$, the following formula calculates an index j from i such that $|S_i| = S'_j$.

$$j(i) = \begin{cases} (i \gg 1) + ((i \gg 3) \ll 2) & \text{if } i\&1 = 0 \\ (i \gg 1) + 4 + ((i \gg 3) \ll 2) & \text{otherwise} \end{cases}$$

Proof. When $i = 6$, every word in S contains $2^{6-3} = 8$ words. Before lines 17-19 are run, each word from the original set is described as one bitstring segment of $2^{l(6)} = 8$ bits each. Lines 17-19 then takes every other segment and shifts it $2^6 = 64$ bits to the left. Since this is the last iteration of the algorithm, the for loop in lines 25-29 gets run immediately after, which extracts each segment into S' in the order of the least significant first for each word.
This means that every other word from the original set will be shifted 4 indices later in S', since 64 bits can contain $64/8 = 8$ words and half of them get shifted as well. The calculation presented to find $j(i)$ has two branches: one branch if i is even and one if i is uneven. If i is even, $j(i)$ can be found by calculating $(i \gg 1) + ((i \gg 3) \ll 2)$ (as is done using bit-shift calculation). If i is odd, then this calculation is just offset by 4. This works, because the shifting makes such that S' contains series of the first 4 even indices of the original set, then the first 4 odd indices, then the next 4 even indices and so on. The $(i \gg 3)$ calculates the index of which of these series that i is found in, and by shifting it 2 bits to the left, we effectively multiply this index by 4 to find which series that our number is in. Then we calculate which of these 4 indices in the series that correspond to our number by calculating $(i \gg 1)$. If i is odd, we will have to add 4, since we are interested in the odd indices. □

If we were to use a word size that is large enough such that lines 17-19 get run again, then we would have to find a more complex computation to find the correct index. This will only happen when $d \geq 16384$, so we can safely assume that this will not be relevant in the near future unless new computers get invented with such large word sizes.

3.4 Runtime Analysis

We can now move on to showing the run time of the algorithm. First, we can try to find the run time of the first for-loop. At each iteration i, we will describe the amount of elements in $S^{(i)}$ as $n^{(i)}$ with $n = n^{(0)}$. We can then describe $n^{(i)}$ like so:

$$n^{(i)} = \lceil \frac{n}{2^{i-l(i)}} \rceil \tag{3}$$

which means that the first for-loop in total takes

$$\sum_{i=2}^{\log_2(d)} \lceil \frac{n}{2^{i-l(i)}} \rceil$$

Since the last loop iterates over every element of the original set, it must take $O(n)$ time. We can now derive the total run time:

$$O(n) + \sum_{i=2}^{\log_2(d)} \lceil \frac{n}{2^{i-l(i)}} \rceil$$

$$= O(n) + \sum_{i=2}^{\log_2(d)} \lceil n2^{l(i)-i} \rceil$$

$$\leq O(n) + \sum_{i=2}^{\log_2(d)} (n2^{l(i)-i} + 1)$$

$$\leq O(n + \log(d)) + n \sum_{i=2}^{\log_2(d)} 2^{l(i)-i}$$

Now we use the fact that $l(i)$ is always the smallest number that satisfies $2^{l(i)} \geq i + 2$. Since it is the smallest number, then $2^{l(i)} < 2(i + 2)$, which can be proved by contradiction: $2^{l(i)} \geq 2(i + 2) \implies 2^{l(i)-1} \geq i + 2$, which shows that there is a number that is 1 smaller than $l(i)$ that upholds the bound $2^{l(i)} \geq i + 2$. Therefore $2^{l(i)} < 2(i + 2)$. We can now use this fact:

$$\leq O(n + \log(d)) + n \sum_{i=0}^{\infty} \frac{2(i+2)}{2^i} = O(n + \log(d))$$

The run time of the algorithm can therefore be bound to $O(n + \log(d))$.

4 Methods

For the rest of the project, the main focus will be on showing the practical feasability of the parallel bit-counting algorithm compared to simpler algorithms. Furthermore, a comparison will be made to the `popcnt` CPU instruction that is implemented on most x64-86 CPUs and has the same purpose [16].

4.1 Benchmarking

To be able to quantify whether one algorithm is faster than the other, one can try to implement it in practice, run it on a specified data set and measure how long it takes for the algorithm to terminate. Benchmarking this way is common in the industry, but it can be hard to be able to generalise from results, due to the many possible sources of error such as:

- Results can vary from machine to machine due to factors like difference in instruction sets, processor infrastructure, cache hierachy, compiler compatibility, memory capacity etc.
- The input dataset can be biased towards one specific kind of architecture or implementation.
- Caches become more efficient when ”warmed up” e.g. when the computer has recently fetched from the same memory addresses. Two identical benchmarks might produce different results based on whether or not the caches are filled.
- A process that benchmarks a program will have to share the CPU cores with other processes running on the computer. This can produce a high variance between identical benchmarks on the same machine.
- Different compilers can produce more og less optimized code, which might produce varying results on different compilation targets.

The scope of this project disallows running benchmarks on a wide varity of computers, which presents some uncertancy in the validity of the results. Some of the other sources of error are mitigable.

- The benchmark should be run multiple times with different pseudo-randomly generated input to mitigate bias in the input data.
- The benchmark should be run multiple times per input data set, until a significant mean runtime can be calculated.
- The benchmark should perform a few ”dry runs” before each run to warm up the caches.

This applications of this software lends itself to large-scale machine learning applications, search engines and data mining. Therefore, the most relevant results would come from running the benchmarks on a machine with access to a GPU, and for the algorithm to be written with parallelism[4] in mind. This has not been considered within the scope of this project however, but instead lends itself as an open research problem.

[4]In this case, parallelism is meant in a GPU context unlike word parallelism as mentioned previously

5 Implementation

5.1 Technology

The implementation to be measured upon has been written in Rust, a memory safe, fast, compiled programming language[14] that has great tooling and typing for writing, testing and benchmarking algorithms.
It uses the `rustc` compiler with a `LLVM` compilation back-end. It does not use a garbage collector but relies on compile time borrow checking, which leads to a language that is comparable to or faster than C[10].
The binary to be benchmarked is compiled and run on a laptop workstation as described in appendix A. The code for the implementation is available in appendix B.

5.2 Design

One of the main design challenges of the implementation is to support multiple word sizes simultaneously to give an accurate estimate on how the run time scales with the word size. This has the consequence of making it harder to hard-code values like $m_{i,j}$ and $l(i)$, since a theoretical implementation with a very large word size would have to compute these values at run-time.
This implementation uses hard-coded values however, since any real-life implementation with a reasonable word-size would do so as well.
Another challenge is code optimization. Rust has an `#[inline(always)]` macro that forces function inlining, which reduces the time needed to handle the stack. A nice feature about the algorithm is that it does not use multiplication, which usually is a very slow operation. It does however operate on very large lists of numbers, which need to be handled efficiently. For this, the heap-allocated `Vec` data structure was used, as it functions as a dynamic array and contains many features to help with optimizations. For instance, the `Vec::with_capacity(n)` constructor specifies an initial capacity of the dynamic array such that unnescessary allocations are avoided, and the Vec::truncate() allows constant time shortening of arrays[14].

5.3 Correctness

It is luckily easy to verify the correctness of the implementation of the algorithm since it is a deterministic algorithm. That means that it is possible to calculate any expected output by hand, which both is useful for validating and for debugging.
To be able to show that the implementation works on a wide array of inputs, we can generate a large list of random numbers to help cover eventual edge cases. We can then compare the results to some reference implementation - in this case the built-in `count_ones()` method.
It is much harder to verify whether the code works with multiple different word-sizes. This is due to the fact that the else branch on line 16 of algorithm 3 only gets hit once when working with $d = 64$ (when $i = 2$) and twice when $d = 128$. Since the applications of this algorithm really benefits when using large word sizes, it is very relevant that this gets tested.

5.4 Benchmarking

To benchmark the algorithm, two reference implementations were created to be used as a baseline models alongside the CPU instruction. These implementations were based on algorithm 1 and 2. A list of 2^{24} random numbers were to be generated and used as input for each of the algorithms.
The Rust library `Criterion.rs`[12] was used to perform the benchmarking. Criterion.rs was very useful for multiple reasons.

- First of all, Criterion.rs integrates into standard Rust tooling easily and allows benchmarking on binary code built for production (e.g. with full optimizations and no debugging symbols)
- Secondly, Criterion.rs provides a compiler "opaque" functions that makes sure that repeated calls to the same function do not get optimized away.
- Criterion.rs handles warming up caches by running the algorithms multiple times before beginning to measure.
- Lastly, Criterion.rs does not record the run time of a single operation - instead it runs multiple samples each consisting of a number of iterations. The run time is the calculated as the runtime per sample divided by the amount of iterations.

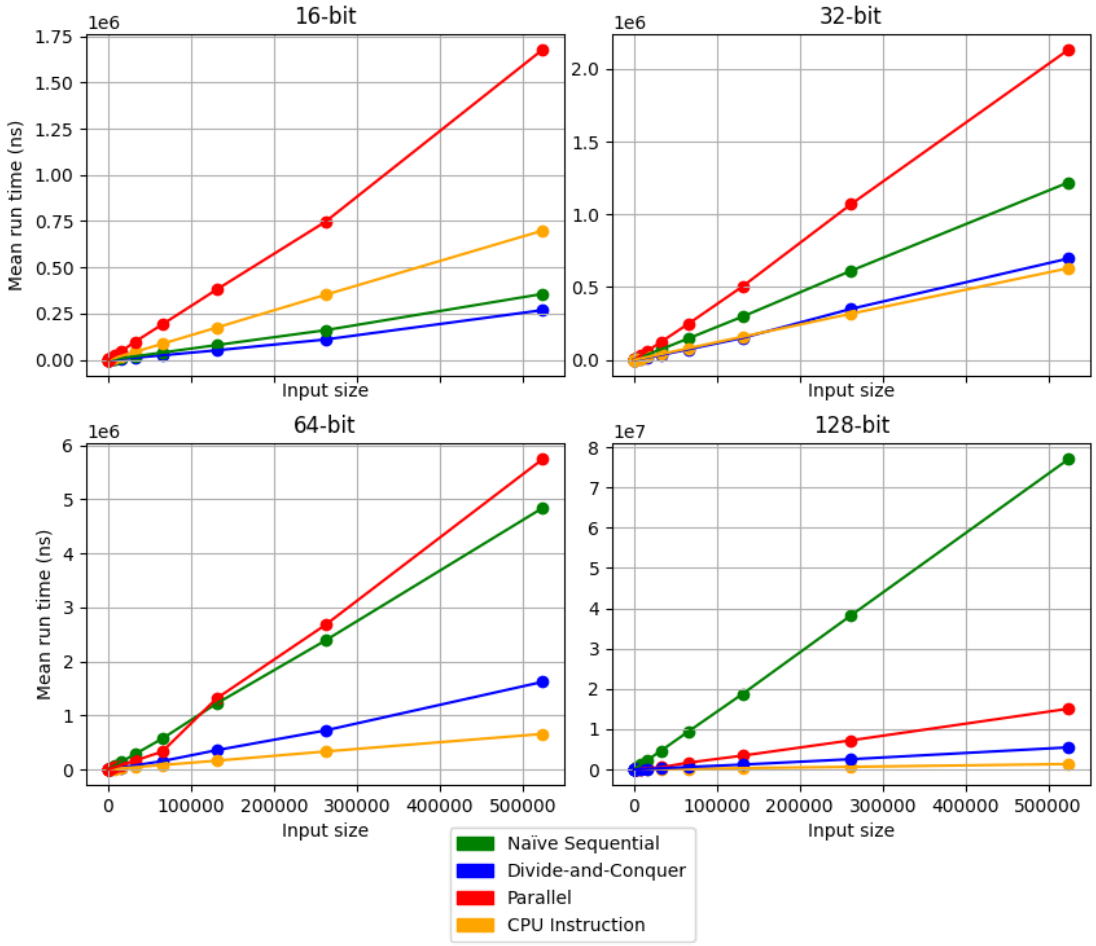

Figure 1: Run time of each algorithm on different word-lengths

6 Results

The benchmarks were run on different input sizes that were all powers of two - in this case $\{2^4, 2^5, \ldots 2^{19}\}$ and with word sizes $\{16, 32, 64, 128\}$. The results can be seen in figure 1. It can be seen that the parallel algorithm is significantly slower for small word sizes, and only becomes more efficient than the naïve sequential algorithm when $d = 128$. This must mean that the hidden factors hidden in the big-O notation of the parallel algorithm's run-time outweigh the d factor of the naïve algorithm. This makes sense, since the parallel algorithm is much more complex and only starts benefitting from parallelism after the first three iterations. Interestingly enough, when $d = 64$, the parallel algorithm only seems to be slower when the input size is larger than 2^{16}. This can be due to the input data being able to fit inside 128 KiB Li1 data cache on my machine (see appendix A for machine specifications).
When run on 128-bit size words, the naïve sequential algorithm performs much worse than than both the parallel and the divide-and-conquer algorithm. This shows that when we increase the word size, algorithms that have a linear dependency on d suffer much more than algorithms that have a logarithmic dependency, which is as expected. It is also seen that the divide and conquer algorithm significantly outperforms both the parallel and the naïve algorithm at all word sizes. This is might be due to the fact

that it is much simpler than the parallel algorithm and is asymptotically more optimal than the naïve sequential algorithm. The final thing to note is that the `popcnt` CPU instruction is the ultimately fastest option out of the algorithms. This shows that these algorithms might only be useful on hardware that does not support this instruction or on very large wordsizes. Even then, the divide-and-conquer algorithm seems to be sufficiently efficient for many uses or even superior to the parallel algorithm despite its suboptimal theoretical run time.

7 Discussion

Even though the results indicate a clear advantage to simpler algorithms to perform bit counting, the parallel bit counting algorithm should not be completely dismissed. The experiment does not tell anything about how the algorithm performs on very large word size (e.g. 256 to 384 bit words) as seen on some GPU units[13]. Since the applications of the similarity search problem usually apply to very large data sets, it is often appropriate to perform highly efficient parallel calculations when possible. Parallel programming comes with many new challenges that were not considered in this project such as minimizing read/write operations to the main memory of the computer, the memory capacity of the graphics card and the size of the register files. This makes an actual implementation of the algorithm 3 non-trivial, especially since most programming interfaces for interacting with a GPU relies on intricate low-level systems programming. A language like Futhark might ease this however[15].
Another thing that this experiment does not take into account is the index calculation time required to actually look up into the list of cardinalities. This was not included, since the correctness criteria of the algorithm may or may not take into account whether or not the cardinalities should have the same index in the final list as the input data. A general formula for this was not found, so the calculation time cannot be shown to be asymptotically positive in regards to the word size (although one might expect it to be), which makes any results unconclusive to real life applications anyways.
The implementation that this experiment relied upon was optimized as per the best of my abilities, but futher optimizations might be possible. A nice feature of the Rust programming language is that the asymptotic run time very often is documented in the documentation, which was very useful when working with heap-allocated vectors of numbers. The implementation was also written to be as generic as possible in regards to word size such that the experiment would be performed as unbiased as possible. This was possible due to Rust's generics, and while Rust design philosphy relies upon zero-cost abstractions[14] (like C++), this might cause some optimization issues. Further optimizations might also be possible by manually analyzing the produced assembly code, which was regarded as out-of-scope for this project.
In general, while parallel bit counting theoretically is more efficient than its sequential and instruction-level counterparts, it has yet to be shown to be the case in practice. While further research might research how it performs on specialized hardware, the algorithm behind this will probably first be useful when larger word-sizes become more easily available and computable, if ever.

8 Conclusion

An in-depth analysis of existing solutions to the Approximate Similarity Search Problem described how many existing solutions utilize sketching to achieve a low query time within a constant error probability. New research by [11] shows how to achieve an even better error probability by segmenting a large sketch into multiple sections and regarding them as semi-independant experiments. An important part of this optimization relies in being able to quantify the results of each experiment very efficiently - in this case by counting the cardinality of a bit-string, which [11] presents a presumably very efficient algorithm to do.
An analysis of the algorithm shows its correctness with a few modifications alongside a theoretical run time which matches [11]. Benchmarks performed on an actual application tells otherwise however, showing that the word parallelism used to achieve a low amortized cost actually has a too high complexity to match simpler algorithms in practice when used on standard word-sizes. Further research might focus on benchmarking the algorithm on more specialized hardware, although rewritting the implementation seems to be non-trivial.

Mathematical Notation

Although sets by definition are not ordered, it is commonly seen in the litterature to index into a set (see [11] for an example). Therefore, sets are regarded as ordered for this project.
Expressions within square brackets are indicator variables denoted using Iverson notation:

$$[X = Y] \in \{0, 1\}$$

$$0 \leq Pr[[X = Y] = 1] \leq 1$$

Integers within square brackets denotes the integer interval between 0 and the number.

$$[n] = [0 \ldots (n-1)] = \{0, 1, \ldots, n-1\}$$

Numbers in a different base than 10 will be indicated with a subscript.

$$11010110_2 = 214$$

Bit-shifting is done using C-style notation. Therefore

$$0101_2 \ll 1 = 1010_2$$

$$1010_2 \gg 1 = 0101_2$$

Logical operations between integers are meant as bit-wise.

$$\text{AND:} \quad 0101_2 \wedge 0011_2 = 0001_2$$

$$\text{XNOR:} \quad 0101_2 \odot 0011_2 = 1001_2$$

For this project, it is often useful to reference the same word after different iterations of an algorithm. To do this, the notation $x^{(i)}$ means a word after i iterations of an algorithm.

Appendices

A Machine Specifications

Model	Lenovo Thinkpad E580
Operating System	Arch Linux kernel 5.15.5-arch1-1
Word size	64bit
Processor model	Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz
ISA	x86_64
Memory Capacity	32 GB
Caches	128 KiB L1d, 128 KiB L1i, 1MiB L2, 8 MiB L3

B Code

The code for the implementation part can be found on in this Github repository: `https://github.com/Snailed/parallel-bit-counting`.

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