Assignment 6

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1 INTRODUCTION

This week we will look at the Finite Volume Method (FVM) where we first will look at its conservation properties, then how control volumes relate to our mesh followed by how we can apply FVM to the Magnetostatic problem. We then look at differences between the finite methods and lastly we look at a validation experiment.

2 THEORY

2.1 Conservation properties of FVM

We will begin by going into detail about the conservation properties of FVM. When we have a large global volume and split it into several smaller local volumes we introduce new common surfaces between the local volumes. This have an effect when we apply the Gauss-Divergence theorem to go from volume integrals to surface integrals as we now integrate over the surfaces. To illustrate this we can define some vector function $\mathbf{f}(\mathbf{u})$ and integrate the divergence of this function over some fixed volume V giving us:

$$\int_V \nabla \cdot \mathbf{f}(\mathbf{u}) dV$$

Converting this to a surface integral we apply Gauss-Divergence theorem giving us:

$$\int_S \mathbf{f}(\mathbf{u}) \cdot \mathbf{n} dS$$

Where $\mathbf{n}$ is the outwards unit normal. However, as discussed and illustrated in Figure 1 by splitting the global volume we have introduced a new surface. Looking at Figure 1 we have V_a and V_b share the common surface S_c, the boundary of V_a is $S_a \cup S_c$ and the boundary of V_b is $S_b \cup S_c$. We also note $V = V_a \cup V_b$ and $S = S_a \cup S_b$. We can thus integrate over the local volumes and get:

$$\begin{aligned}\int_{V_a \cup V_b} \nabla \cdot \mathbf{f}(\mathbf{u}) dV = &\int_{S_a} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n_a} dS + \int_{S_c} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n_c} dS \\ &+ \int_{S_b} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n_b} dS + \int_{S_c} \mathbf{f}(\mathbf{u}) \cdot (-\mathbf{n_c}) dS\end{aligned}$$

We here see the integrals over the common surface cancels out giving us:

$$\int_{V_a \cup V_b} \nabla \cdot \mathbf{f}(\mathbf{u}) dV = \int_{S_a \cup S_b} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n} dS$$

Which is equivalent to:

$$\int_V \nabla \cdot \mathbf{f}(\mathbf{u}) dV = \int_S \mathbf{f}(\mathbf{u}) \cdot \mathbf{n} dS$$

This means conservation is guaranteed on both a local and a global scale.

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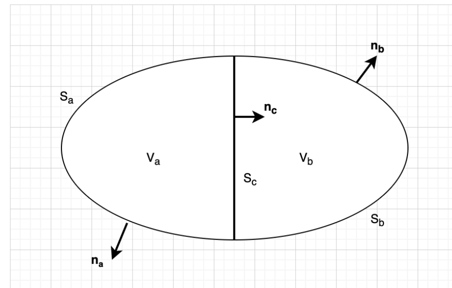


Fig. 1. Illustration of a global volume getting split introducing a new common surface.

2.2 Control volumes

An integral part of FVM is the control volumes. Control volumes are introduced as part of our discretization to simplify our problem such that we look at a local subproblem which is easier to solve. Control volumes can be of any shape as long as we have the geometric information of them, e.g. cell centers, surface centers, neighbouring cells, surface connectivity etc. For this week's assignment we will construct a regular triangle mesh. In this mesh we will define the control volumes 'around' vertices of the mesh. That is, in the interior of the mesh we will construct octagons and squares going between the centers of our triangle mesh. At the boundaries of our mesh we will create polygons. This can be seen in Figure 2. Although there are no formal requirements on the shape of the control volumes we need the edges of the control volumes to be perpendicular to the grid lines in our mesh, otherwise the outwards unit normal of the control volume surface will point in a different direction than the line segments connecting our vertices and we will introduce errors in our later approximations. Because of this we need to tailor the shape of our control volumes such that they match with our choice of mesh.

2.3 FVM applied to Magnetostatic problem

We can now apply FVM to our problem in this week's notebook, namely the Magnetostatic problem which has the PDE $\nabla \cdot \nabla \phi(\mathbf{x}) = \nabla \cdot \mathbf{M}(\mathbf{x})$. The first step of FVM is to decide on a mesh and control volume layout. As we are dealing with a 2D problem in the notebook we will use a regular triangle mesh. For control volumes we will use the centers in the triangles to draw octagons and squares between interior nodes, and simple lines to the boundary. This is illustrated in Figure 2 where black lines is our mesh and red lines constitute our control volumes. The next step is to convert our PDE to a volume integral and then to a surface integral using the *Gauss-Divergence* *theorem.*

$$\nabla \cdot \nabla\phi(\mathbf{x}) = \nabla \cdot \mathbf{M}(\mathbf{x})$$
$$\int_V \nabla \cdot \nabla\phi(\mathbf{x})dV = \int_V \nabla \cdot \mathbf{M}(\mathbf{x})dV$$
$$\int_S \nabla\phi(\mathbf{x}) \cdot \mathbf{n}dS = \int_S \mathbf{M}(\mathbf{x}) \cdot \mathbf{n}dS$$

Where $\mathbf{n}$ is the outward unit normal. These surface integrals means we need to integrate over the edges of our control volumes. We can now exploit that the edges are piecewise continuous, making our integrals *piecewise continuous integrals.*

$$\sum_e \int_{S_e} \nabla\phi(\mathbf{x}) \cdot \mathbf{n_e}dS = \sum_e \int_{S_e} \mathbf{M}(\mathbf{x}) \cdot \mathbf{n_e}dS$$

Where $\mathbf{n_e}$ denotes the outward unit normal for the edges in the control volume. We can now use the midpoint approximation rule to remove the integral. This can be done because the outward unit normal is constant along each S_e part.

$$\sum_e [\nabla\phi(\mathbf{x}) \cdot \mathbf{n_e}]_c l_e = \sum_e [\mathbf{M}(\mathbf{x}) \cdot \mathbf{n_e}]_c l_e$$

Here l_e denote the length of the control volume edge we are looking at. To approximate $[\nabla\phi(\mathbf{x}) \cdot \mathbf{n_e}]_c$ we realise we can rewrite it as a directional derivative. We note because of the way we have defined our control volumes we have n_e points in the same direction as the line segment between two vertices and we can thus make the approximation:

$$\nabla\phi(\mathbf{x}) \cdot \mathbf{n_e} = \frac{\partial\phi(\mathbf{x})}{\partial\mathbf{n_e}} \approx \frac{\phi(\mathbf{x})_j - \phi(\mathbf{x})_i}{l_{ij}}$$

Where $\phi(\mathbf{x})_i$ is the ϕ value at a vertex i in our mesh, $\phi(\mathbf{x})_j$ is the ϕ value at a vertex j in our mesh and l_{ij} is the length of the line segment connecting the two vertices. Substituting this in and cleaning up we end with the discretization:

$$\sum_e \frac{(\phi(\mathbf{x})_j - \phi(\mathbf{x})_i)}{l_{ij}} l_e = \sum_e [\mathbf{M}(\mathbf{x}) \cdot \mathbf{n_e}]_c l_e$$
$$\sum_e \frac{l_e}{l_{ij}}(\phi(\mathbf{x})_j - \phi(\mathbf{x})_i) = \sum_e [\mathbf{M}(\mathbf{x}) \cdot \mathbf{n_e}]_c l_e$$

This approximation does however introduce small error along the boundary. This is illustrated in Figure 3 where we end up estimating the ϕ value only halfway to the boundary (red dot) instead of at the boundary (green dot) because the control volume edge only goes to the boundary. We could instead use a more complicated approximation where we use shape functions like in FEM to interpolate the value to the boundary. This would mean we would have $\nabla\phi(\mathbf{x}) = \sum_\alpha \nabla N_\alpha \hat{\phi}_\alpha = \mathbf{N}\hat{\phi}$ where N_α is a shape function and $\hat{\phi}$ is a set of discrete ϕ values we would then solve for.

2.4 Handling of unit circle

In this week's assignment when we look at the right hand side integral, $\int_V \nabla \cdot \mathbf{M}(\mathbf{x})dV$, we require the integrand, $\nabla \cdot \mathbf{M}(\mathbf{x})$, to be a continuous real valued function. When we consider a control volume completely outside the unit circle $\mathbf{M}(\mathbf{x})$ is per definition $[0, 0]^T$ and thus continuous. When we consider a control volume completely inside the unit circle $\mathbf{M}(\mathbf{x})$ is per definition $[0, -1]^T$ and

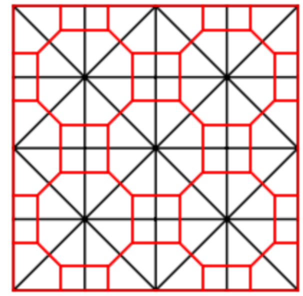


Fig. 2. Our mesh drawn in black and our control volumes drawn in red.

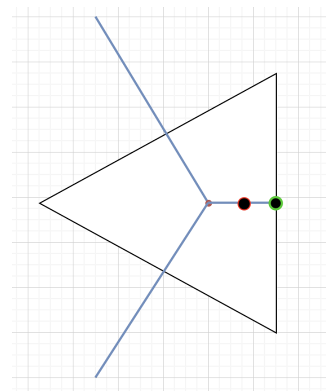


Fig. 3. Small error introduced by not using shape functions for interpolation. Blue lines illustrate edges in control volumes and the black lines illustrate a single boundary triangle in our mesh.

thus also continuous. However, when we have a control volume which is partially inside and partially outside then $\mathbf{M}(\mathbf{x})$ suddenly jumps from one value to another and is thus discontinuous. We can handle this by splitting the integral into two parts, one over the part of the control volume which is outside the unit circle, and one which is inside and then sum the two parts. That is we would have:

$$\int_V \nabla \cdot \mathbf{M}(\mathbf{x})dV = \int_{V_{inside}} \nabla \cdot \mathbf{M}(\mathbf{x})dV + \int_{V_{outside}} \nabla \cdot \mathbf{M}(\mathbf{x})dV$$

This split can either be done on the fly when we realise our control volume is getting split by the unit circle, or we could preemptively design our control volumes such that their boundaries conforms with the unit circle.

2.5 Difference between Finite Methods

FVM resembles some of the same concepts as FEM as in both methods we start with a big system which we then divide into smaller subsystems which we then can solve more easily. However, where FEM uses trial functions and integration by parts to convert from strong form to weak form, FVM takes a more direct approach and uses Gauss-Divergence theorem to get surface integrals. FEM also relies on the approximation from using shape functions to interpolate positions inside the domain, where FVM can make use of shape functions, but can also as seen earlier, make use of the outward unit normal from the control volumes. Comparing to FDM, FVM and FEM are much more easily applied to unstructured meshes as it is not always obvious which nodes to use for FDM in an unstructured mesh. However, FDM can quite easily be formally verified though analysis of the Taylor remainder terms, which is not as obviously done for FVM and FEM.

3 EXPERIMENT 1

In this experiment we would like to perform validation on our solution of ϕ as to see if we get the expected results. First we will investigate how we expect our solution to look like. From [Wikipedia 4/5/2021] we have the illustrations in Figure 4. In Figure 4a and Figure 4b we have a close up view of the electric field around a positive and negative point charge. Where here see the contour lines around the centers are circular and that the electric field lines are perpendicular to these contours. In Figure 4b we again see the two charges, but now opposite each other. We see that the contours drawn gets effected by the other charge, and had there been drawn more contours we would have seen they would get prolonged as they got closer to the middleground. We also see the full size of the electric field lines. This is what we will expect to see from our model.

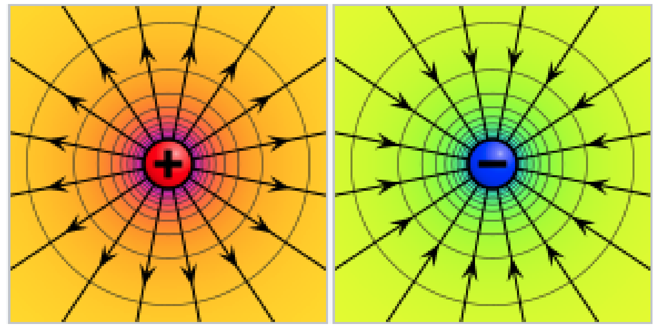
We can now compute our solution for ϕ on different grid sizes and resolutions and visually validate our model. In Figure 5 we see our results of this. We note that the white lines have been manually drawn to give an idea of how the electrical field lines would look like. These have been attempted drawn such that they are perpendicular to the contour lines, as they represent the direction of $\nabla\phi$.

3.1 Discussion of results

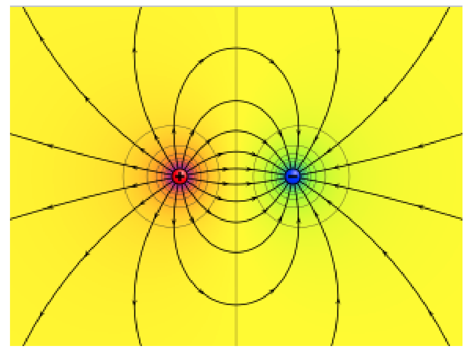
The first thing we note about our results is all contour lines are rather 'square' and not as perfect round as we had expected. This is probably due to error introduced when we handle the unit circle, but can perhaps also partially stem from the error introduced due to the boundary conditions. We do see however, that as we move towards the middle of the poles, the contours correctly get flat and prolonged. We also note on the 6 by 6 grids that the contours are round behind the poles, more so on the finer resolution image than the coarse. When we look at the electrical field lines in white, we see the lines behaves as expected, going in an arc from one pole to the other. In conclusion we do have some errors, probably due to the handling of the unit circle discretization, but in broad lines, our model is in accordance to our expectation.

4 CONCLUSION

In conclusion we have seen due to how the shared surfaces cancel out when we apply Gauss-Divergence theorem, we have both global and local conservation. We have also seen how the shape of our control volumes plays a big role in FVM and how it can introduce errors if we do not get it right. Lastly we conclude there has been introduced errors into our model, and most likely due to how we handle the control volumes which are both inside and outside the unit circle, but in broad lines our model meets our expectations.

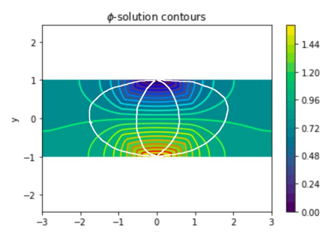


(a) Electric potential of seperate positive and negative point charges.

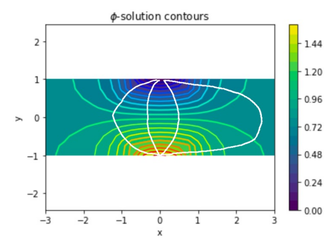


(b) Electric potential near two opposite point charges.

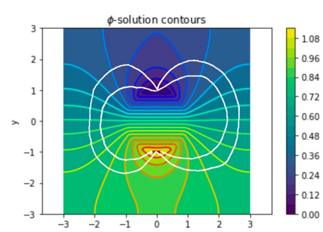
Fig. 4. Illustrations of electric potential.



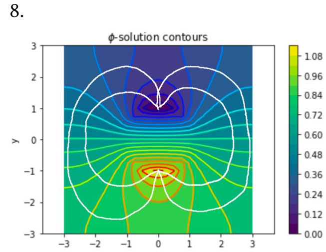
(a) Grid size: 6 by 2, resolution: 72 by 24.



(b) Grid size: 6 by 2, resolution: 24 by 8.



(c) Grid size: 6 by 6, resolution: 72 by 72.



(d) Grid size: 6 by 6, resolution: 24 by 24.

Fig. 5. Results for solving the potential field ϕ for different grid sizes and resolution.

REFERENCES

Wikipedia. *Electric potential.* https://en.wikipedia.org/wiki/Electric_potential, 4/5/2021.