Elliptical Filter Design

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1 Theory behind Elliptical Filter

Why Elliptic filter?
We prefer elliptic approximation to realise filters because, for a given order, the transition band is the smallest. This allows us to achieve the best approximation with the fewest resources. On the contrary we get very less phase control.

1.1 Transfer Function

The transfer function for an elliptic filter (or any type of FIR filter in general) is as follows:

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon_p^2 U_N^2(w)}; w = \frac{\Omega}{\Omega_p}$$

where, N is the order of filter and F_N is given by

$$F_N^2(w) = \left\{ \begin{array}{lr} cd(NuK_1, k_1) & \text{Elliptic} \\ w^2 & \text{Butterworth} \\ C_N(w), & \text{Chebyschev} \end{array} \right\}$$

Here, C_N is Chebyschev function and cd is a Jacobi elliptic function.

k (selectivity parameter) is defined as the ratio of passband to stopband frequency

k_1 (discrimination parameter) is defined as the ratio of ϵ_p and ϵ_s

$$\epsilon_p = \sqrt{(\frac{1}{1-\delta_1})^2 - 1}$$

$$\epsilon_s = \sqrt{(\frac{1}{\delta_2})^2 - 1}$$

For the filter specifications provided to us we have $\delta 1 = \delta 2 = 0.15$

1.2 Elliptic Function

Elliptic Functions are meromorphic and doubly periodic functions extendable to Complex plane. There are total 3 types of elliptic functions, but here we are going to use only the first kind for designing Elliptic filter.

$$\textbf{Elliptic Function of First Kind: } F(\phi,k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}} \tag{1}$$

k is also called the eccentricity or elliptical modulus.(This can be related to arc length finding integral of ellipse)

We define another quantity K(k) as the complete elliptical integral of order 1.

$$K(k) = F(\frac{\pi}{2}, k) \tag{2}$$

Also for further calculation we introduce k' = $\sqrt{1-k^2}$ 'complementary elliptical modulus, with following relation:

$$K(k') = K'(k)$$

For calculations in Matlab,we use same standard naming convention.

1.3 Jacobi Elliptical Function

There are total of 12 function represented as pq(z,k) where p,q $\in$ s,c,d,n and for p = q the value of function becomes trivially 1. Therefore total of 12 functions to study.
For making our elliptical filter we are focused only on cd(z,k) and sn(z,k).
These functions can be generated from theta functions, but here we will use Elliptical First Kind integral. We define z = F(ϕ,k) (from equation 1). We then find inverse of F(ϕ,k) with respect to ϕ and call it *Jacobi Amplitude.*

$$\phi = amp(z,k)$$

$$sn(z,k) = \sin(\phi) = \sin(amp(z,k)) \tag{3}$$

$$cn(z,k) = \cos(\phi) = \cos(amp(z,k)) \tag{4}$$

$$dn(z,k) = \frac{d}{dz}(\phi) = \frac{d}{dz}(amp(z,k)) \tag{5}$$

Equation 3, 4, 5 form the basis of Jacobi Elliptical Functions and rest 9 can formed from these 3 by following relation.

$$pq(z,k) = \frac{1}{qp(z,k)}$$

$$pq(z,k) = \frac{pr(z,k)}{qr(z,k)}; \text{where } r \in \{s,n,c,d\}$$

Here, z is the arbitrary complex vector, and k is the elliptic modulli.
We will be using only the following 2 functions majorly : cd(z,k) and $sn^{-1}(u,k)$

$$cd(z,k) = cde(z/K,k)$$
$$sn(z,k) = sne(z/K,k)$$
$$v = asne(u,k) \text{ means v is solution of : } sn(v \cdot K,k) = u$$

(The above 3 equations are for equivalent matlab functions used)

1.4 Parameters, Poles and Zeroes

We already know how k and k' are evaluated section 2.1 . Now to kind Order:

$$\textbf{Degree Equation : } N = ceil(\frac{K(k)\cdot K'(k_1)}{K'(k)\cdot K(k_1)}) \tag{6}$$

N can be represented as 2l+r, where r $\in$ {0,1}
Now, because there is ceiling function, so there will be some overdesign(meeting more that specified requirements) i.e. $|H(j\Omega)|$ will fall below δ_2 before reaching stopband frequency as N is not exact integer without ceil. So we can change our specs to new value of k such that N comes out to be same exact integer without ceil.

In matlab, this is done by using *ellipdeg(N,*k_1*)*, this returns the value of k which satisfies the degree equation (6) for given value of N and K_1.
Now the value of k has been updated, and this value will be used further in design.
Zeroes: Zeroes of H(jΩ) are poles of $U_N(\Omega)$. Let us first define few things:

$$u_i = \frac{2i-1}{N} \text{ ; where } i \in 1,2,...,l$$
$$\zeta_i = cd(u_i \cdot K(k),k) = cde(u_i,k)$$
$$U_N(\Omega) = (\Omega)^r \Pi_{i=1}^{l}\left[\left(\frac{\Omega^2-\zeta_i^2}{1-\Omega^2k^2\zeta_i^2}\right)\cdot\left(\frac{1-k^2\zeta_i^2}{1-\zeta_i^2}\right)\right] \tag{7}$$

U_N has total 2l poles therefore there are total of 2l zeroes of H(jΩ), which are given by :

$$z_i = \frac{j}{k\zeta_i} \text{ ; where } i \in 1,2,...,l \tag{8}$$

Remaining zeroes are z_i^* i.e. conjugates of z_i

Poles:

$$U_N(\Omega) = \frac{\pm j}{\epsilon_p}$$

There are 2l+r i.e. N poles to $|H(j\Omega)|$
Let v $\in \mathbb{R}$ be a solution to $sn(jv \cdot N \cdot K(k), k) =$ j/ϵ_p, therefore

$$v = \frac{-j}{N \cdot K(k)} sn^{-1}(\frac{j}{\epsilon_p}, k_1) = \frac{-j}{N} asne(\frac{j}{\epsilon_p}, k_1) \tag{9}$$

Poles are given by:

$$p_i = jcd((u_i - jv)K(k), k) = jcde(u_i - jv, k) \text{ ,where } i \in 1, 2, ..., l \tag{10}$$

$$p_0 = jcd((1 - jv)K(k), k) = jcde(1 - jv, k) \tag{11}$$

p_0 is the pole on negative real axis whixh occurs only when N is odd.Remaining poles are given by p_i^* i.e. conjugates of p_i.

$$H(s) = \frac{A}{(s - p_0)^r} \Pi_{i=0}^{l} \left[\frac{(s - z_i)(s - z_i^)}{(s - p_i)(s - p_i^)} \right] \tag{12}$$

A is normalisation factor for making gain=1.

2 IIR Multi-Band pass Filter

2.1 Un-normalized Discrete Time Filter Specifications

Filter Number M = 21
M = 11Q + R
Q = Quotient when M is divided by 11 = 1
R = Remainder when M is divided by 11 = 10
Passband 1 specifications:
B_L(m) = 40 + 5Q = 40 + 5*1 = 45KHz
B_H(m) = 70 + 5Q = 70 + 5 = 75KHz
Passband 2 specifications:
B_L(m) = 170 + 5R = 170 + 5*10 = 220KHz
B_H(m) = 200 + 5R = 200 + 50 = 250KHz
Therefore the specifications of the **Multi-Band pass** Filter are:

- Passband : **45 - 75 KHz** and **220 - 250 KHz**
- Stopband : **0 - 40 KHz**, **80 - 215 KHz** and **255 - 300 KHz** (As **sampling rate** is **600 KHz**)
- Transition band : **5KHz** on either side of the passband and stopband
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Nature : Passbands and Stopbands are **oscillatory**

Sampling Rate = **600 KHz**

To design such a filter, we will cascade two filters, a Bandpass and a Bandstop filter, each of them being **Elliptic** filters. The specifications of these two filters are mentioned below:

3 Bandpass Elliptic Filter

3.1 Un-normalized Discrete Time Filter Specifications

The specifications of this filter are :

- Passband : **45 - 250 KHz**
- Stopband : **0 - 40 KHz** and **255 - 300 KHz**
- Transition band : **5KHz** on either side of passband
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Nature : Passbands and stopbands are **oscillatory**

3.2 Normalized Digital Filter Specifications

Sampling rate = 600Khz
In the normalized frequency axis, sampling rate corresponds to 2π
Therefore, any frequency can be normalized as follows :

$$\omega = \frac{\Omega * 2\pi}{\Omega_s}$$

where Ω_s is the Sampling Rate.

For the normalized discrete filter specifications, the nature and tolerances being the dependent variables remain the same while the passband and stopband frequencies change as per the above transformations.

- Passband : **0.15 - 0.833** π
- Stopband : **0 - 0.133** π and **0.85 - 1** π
- Transition band : **0.0167** π on either side of stopband

3.3 Bilinear Transformation

To convert to analog domain, we use the following bilinear transformation :

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$\Omega_{analog} = \tan(\frac{w}{2})$$

Applying the transformation at Band Edges we get :

ω	Ω
0	0
0.133 π	0.213
0.15 π	0.24
0.833 π	3.732
0.85 π	4.165
π	∞

Therefore, the corresponding specifications are :

- Passband : **0.24** (Ω_{p1}) **- 3.732** (Ω_{p2})
- Transition band : Between the passband and stopband edges
- Stopband : **0 - 0.213**(Ω_{s1}) and **4.165** (Ω_{s2}) **-** ∞

3.4 Frequency Transformation and Relevant Parameters

We need to convert the Band - Pass filter into a Low - Pass analog filter as we are aware of it's frequency response in order to keep equiripple passband and stopband. For that purpose we use the following frequency transformation with two parameters B and Ω_o

$$\Omega_l = \frac{\Omega^2 - \Omega_o^2}{B\Omega}$$

If we follow the convention that the passband edges are mapped to +1 and -1, the parameters, in terms of the passband edges can be obtained by solving two equations and are given by :

$$\Omega_o = \sqrt{\Omega_{p1}\Omega_{p2}} = \sqrt{0.24 * 3.732} = 0.946$$

$$B = \Omega_{p2} - \Omega_{p1} = 3.492$$

Ω	Ω_L
0^+	- ∞
0.213 (Ω_{s1})	-1.142
0.24 (Ω_{p1})	-1
0.946 (Ω_o)	0
3.732 (Ω_{p2})	1
4.165 (Ω_{s2})	1.131
∞	∞

To make the filter as close to ideal as possible we choose the more stringent stopband for the lowpass filter i.e. $\Omega_{sL} = \min(\Omega_{sL1}, -\Omega_{sL2}) = 1.131$. (where Ω_{sL} stands for the stopband for the lowpass filter)

Therefore the analog lowpass filter specifications are as follows:

- Passband Edge : 1 (Ω_{pL})
- Stopband Edge : 1.131 (Ω_{sL})

3.5 Analog Elliptic Lowpass Transfer function

As now we have Ω_{pL} and Ω_{sL}, we can plug them into formula of k and k_1 :

$$\epsilon_p = \sqrt{(\frac{1}{0.85})^2 - 1} = 0.6197$$

$$\epsilon_s = \sqrt{(\frac{1}{0.15})^2 - 1} = 6.5912$$

$$k_1 = \frac{\epsilon_p}{\epsilon_s} = \frac{0.6197}{6.5912} = 0.0940$$

$$k = \frac{\Omega_{pL}}{\Omega_{sL}} = \frac{1}{1.131} = 0.8842$$

$$N = ceil(\frac{K(k) \cdot K'(k_1)}{K'(k) \cdot K(k_1)}) = ceil(3.1708)$$

Thus giving N = 4; $\implies$ l=2 and r=0

Thus rounding off N, we get k=0.9595 and the modified value of Ω_{sL}=1.0422

We now just have to get the transfer function so we use equation 8,9 for getting both the things.

Zeros are: z_1=j/(0.9796*0.9595)=1.0639j
z_1^*=-1.0639j
z_2=j/(0.5891*0.9595)=1.7692j
z_2^*=-1.7692j

Poles are: p_1=-0.0310 + 0.9995i
p_1^*=-0.0310 - 0.9995i
p_2=-0.3536 + 0.7071i
p_2^*=-0.3536 - 0.7071i

As N is even, we also need to account for the DC gain (which changes the normalization factor) which is $1/\sqrt(1+\epsilon^2)$, where ϵ is given by :

$$\epsilon = \sqrt{\frac{1}{(\delta_1 - 1)^2} - 1} \implies \frac{1}{\sqrt{1+\epsilon^2}} = 1 - \delta_1$$

$$H_{analog\_LP}(s) = \frac{0.15s^4 + 0.6392s^2 + 0.5313}{s^4 + 0.7691s^3 + 1.6689s^2 + 0.7459s + 0.6251} \tag{13}$$

Pole-zero Map:

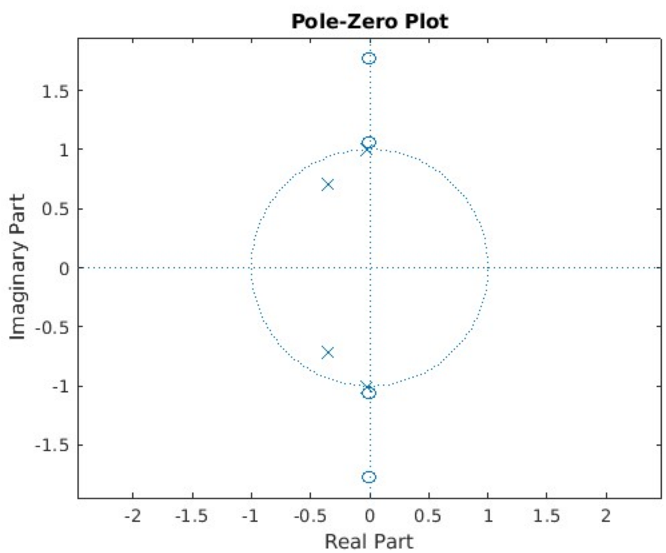


Figure 1: Pole-Zero Plot

3.6 Analog Bandpass Transfer Function

The transformation is given by :

$$s_L = \frac{s^2 + \Omega_o^2}{Bs}$$

Substituting the values of B and Ω_o

$$s_L = \frac{s^2 + 0.8949}{3.492s}$$

Using the above transformation we get we get $H_{analog,BPF}$(s) from $H_{analog,LPF}$(s)

$$H_{analog\_BS}(s) = \frac{0.1500s^8 + 8.3311s^6 + 93.6738s^4 + 6.6722s^2 + 0.0962}{1s^8 + 2.6857s^7 + 23.9299s^6 + 38.9718s^5 + 134.1735s^4 + 34.8765s^3 + 19.1649s^2 + 1.9249s^1 + 0.6414}$$

3.7 Discrete Time Filter Transfer Function

To transform the analog domain transfer function into the discrete domain, we need to make use of the Bilinear Transformation which is given as :

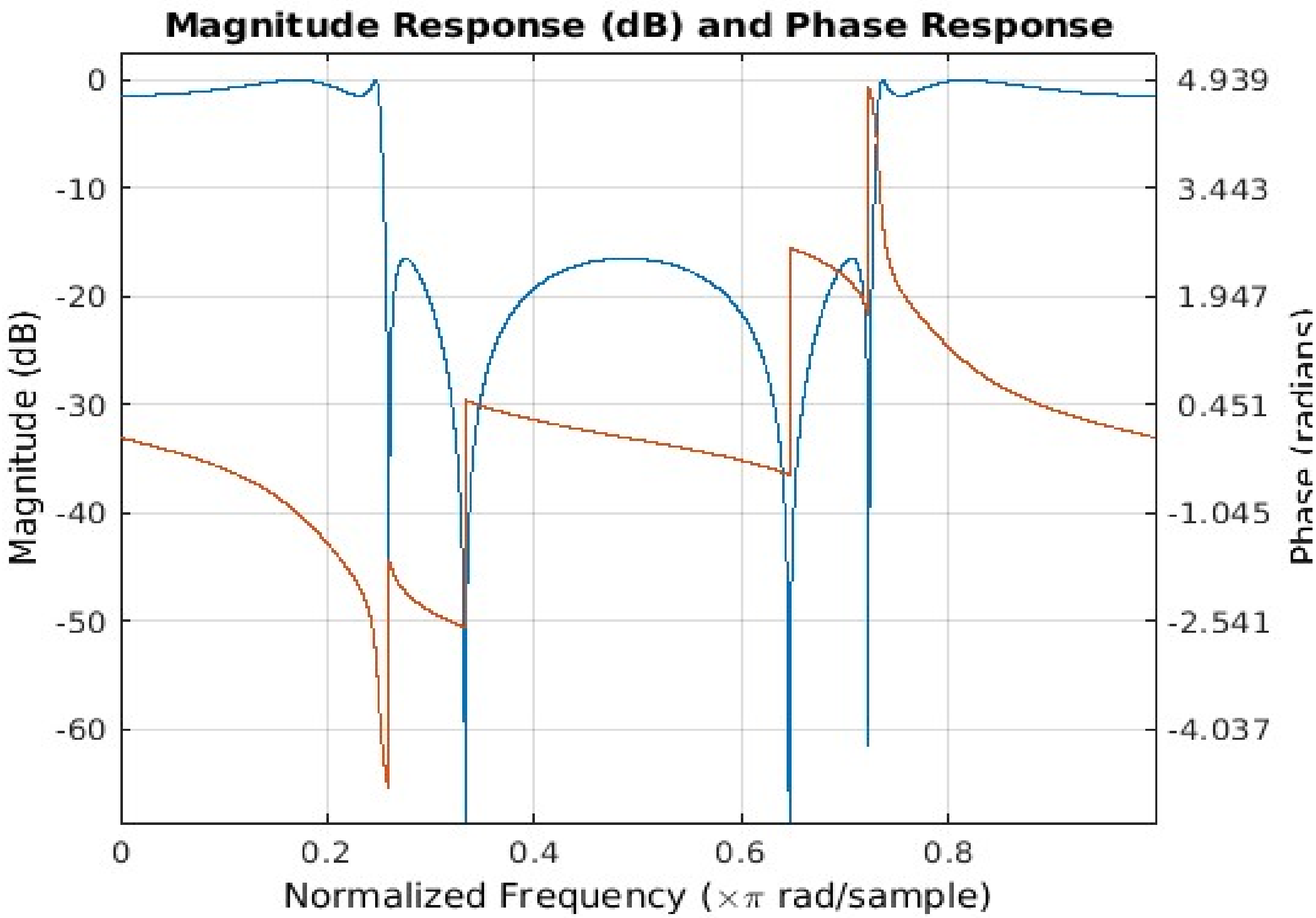
$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

Using above equation we get $H_{discrete,BPF}$(z) from $H_{analog,BPF}$(s)

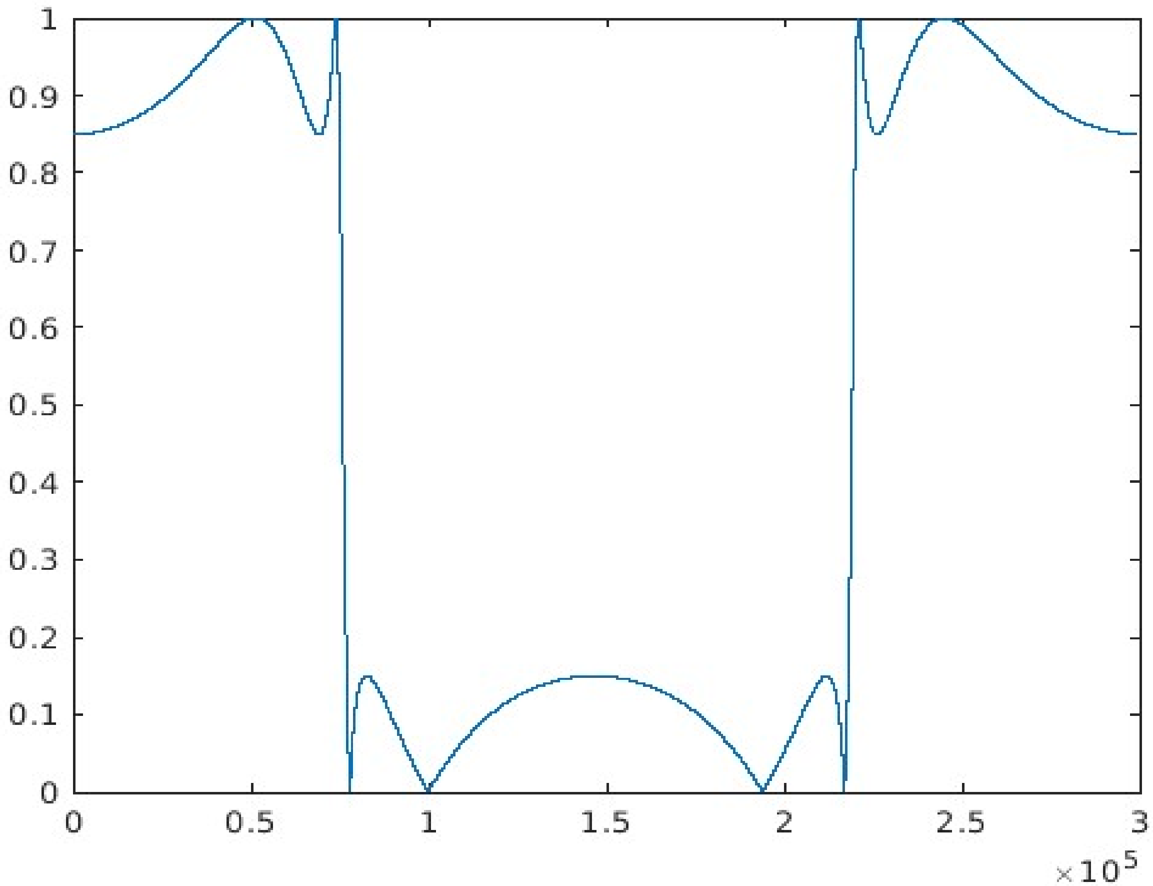
$$H_{discrete\_BS}(z) = \frac{0.4232z^8 - 0.0275z^7 - 1.1959z^6 + 0.0141z^5 + 1.6678z^4 + 0.0141z^3 - 1.1959z^2 - 0.0275z + 0.4232}{1z^8 - 0.1348z^7 - 1.5600z^6 + 0.0501z^5 + 1.9000z^4 - 0.0581z^3 - 0.9139z^2 - 0.0356z + 0.3903}$$

3.8 Matlab Plots

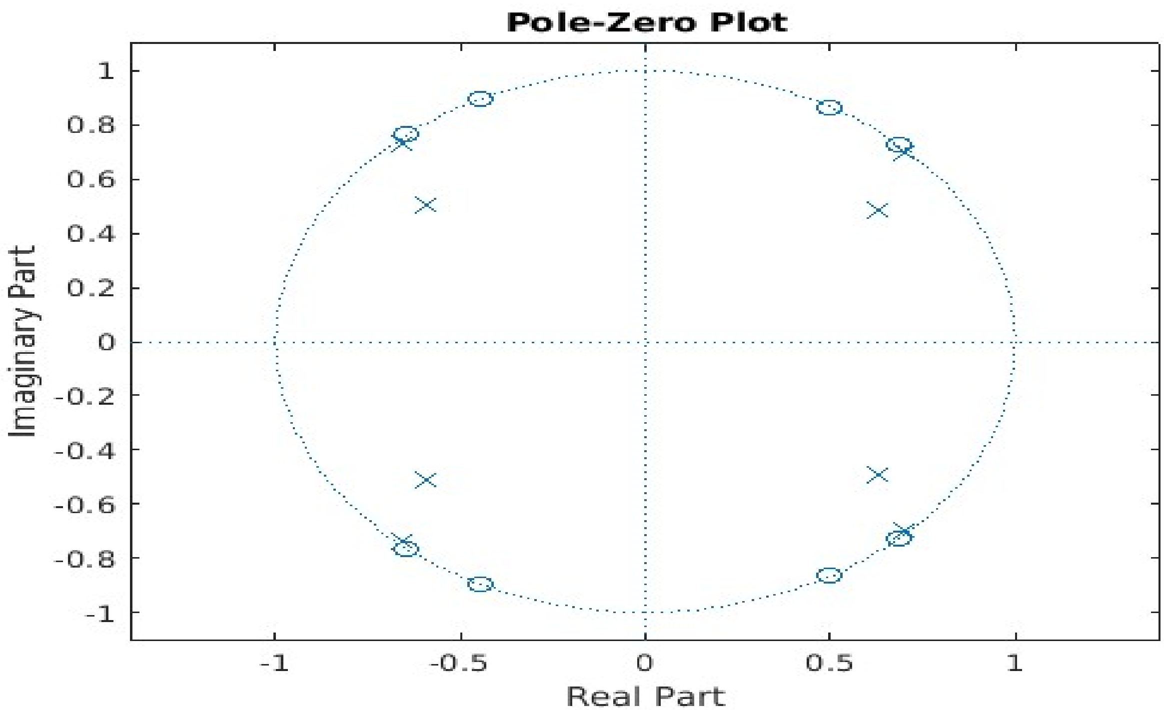
3.8.1 Frequency Response



3.8.2 Magnitude Response



3.8.3 Pole - Zero Plot



As we can see, all poles of above transfer function lie within unit circle. So, the system is stable.

4 Bandstop Elliptic Filter

4.1 Un-normalized Discrete Time Filter Specifications

The specifications of this filter are :

- Stopband : **80 - 215 KHz**
- Passband : **0 - 75 KHz** and **220 - 300 KHz**
- Transition band : **5KHz** on either side of passband
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Nature : Passbands and stopbands are **oscillatory**

4.2 Normalized Digital Filter Specifications

Sampling rate = 600KHz
In the normalized frequency axis, sampling rate corresponds to 2π

Therefore, any frequency can be normalized as follows :

$$\omega = \frac{\Omega * 2\pi}{\Omega_s}$$

where Ω_s is the Sampling Rate.

For the normalized discrete filter specifications, the nature and tolerances being the dependent variables remain the same while the passband and stopband frequencies change as per the above transformations.

- Stopband : **0.267 - 0.717** π
- Passband : **0 - 0.25** π and **0.733 - 1** π
- Transition band : **0.0167** π on either side of stopband

4.3 Bilinear Transformation

To convert to analog domain, we use the following bilinear transformation :

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$\Omega_{analog} = \tan(\frac{w}{2})$$

Applying the transformation at Band Edges we get :

ω	Ω
0	0
0.25 π	0.414
0.267 π	0.445
0.717 π	2.097
0.733 π	2.246
π	∞

Therefore, the corresponding specifications are :

- Stopband : **0.445** (Ω_{s1}) **- 2.097** (Ω_{s2})
- Transition band : Between the passband and stopband edges
- Passband : **0 - 0.414**(Ω_{p1}) and **2.246** (Ω_{p2}) **-** ∞

4.4 Frequency Transformation and Relevant Parameters

We need to convert the Band - Stop filter into a Low - Pass analog filter as we are aware of it's frequency response in order to keep equiripple passband and monotonic stopband. For that purpose we use the following frequency transformation with two parameters B and Ω_o

$$\Omega_l = \frac{B\Omega}{\Omega_o^2 - \Omega^2}$$

If we follow the convention that the passband edges are mapped to +1 and -1, the parameters, in terms of the passband edges can be obtained by solving two equations and are given by :

$$\Omega_o = \sqrt{\Omega_{p1}\Omega_{p2}} = \sqrt{0.414 * 2.246} = 0.964$$

$$B = \Omega_{p2} - \Omega_{p1} = 1.832$$

Ω	Ω_L
0^+	0^+
0.414 (Ω_{p1})	+1
0.445 (Ω_{s1})	1.115
0.946^- (Ω_o)	∞
0.946^+ (Ω_o)	$-\infty$
2.097 (Ω_{s2})	-1.108
2.246 (Ω_{p2})	-1
∞	0^-

To make the filter as close to ideal as possible we choose the more stringent stopband for the lowpass filter i.e. Ω_{sL} = min(Ω_{sL1} , - Ω_{sL2}) = 1.108. (where Ω_{sL} stands for the stopband for the lowpass filter)

Therefore the analog lowpass filter specifications are as follows:

- Passband Edge : 1 (Ω_{pL})
- Stopband Edge : 1.108 (Ω_{sL})

4.5 Analog Elliptic Lowpass Transfer function

As now we have Ω_{pL} and Ω_{sL}, we can plug them into formula of k and k_1 :

$$\epsilon_p = \sqrt{(\frac{1}{0.85})^2 - 1} = 0.6197$$

$$\epsilon_s = \sqrt{(\frac{1}{0.15})^2 - 1} = 6.5912$$

$$k_1 = \frac{\epsilon_p}{\epsilon_s} = \frac{0.6197}{6.5912} = 0.0940$$

$$k = \frac{\Omega_{pL}}{\Omega_{sL}} = \frac{1}{1.108} = 0.9025$$

$$N = ceil\left(\frac{K(k) \cdot K'(k_1)}{K'(k) \cdot K(k_1)}\right) = ceil(3.3094)$$

From this we get N = 4 $\implies$ l=2 and r=0 and thus rounding off N, we get k = 0.9595. Thus we get modified value of Ω_{sL}=1.0422.

We now just have to get the transfer function so we use equation 8,9 for getting both the things.

Zeros are: z_1=j/(0.9796*0.9595)=1.0639j
z_1^*=-1.0639j
z_2=j/(0.5891*0.9595)=1.7692j
z_2^*=-1.7692j

Poles are: p_1=-0.0310 + 0.9995i
p_1^*=-0.0310 - 0.9995i
p_2=-0.3536 + 0.7071i
p_2^*=-0.3536 - 0.7071i

Pole-zero Map:

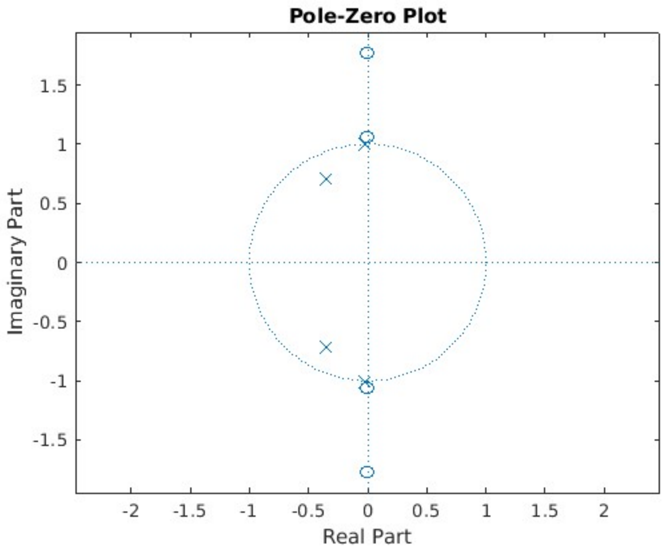


Figure 2: Pole-Zero Plot

As N is even, we also need to account for the DC gain (which changes the normalization factor) which is $1/\sqrt{(1+\epsilon^2)}$, where ϵ is given by :

$$\epsilon = \sqrt{\frac{1}{(\delta_1 - 1)^2} - 1} \implies \frac{1}{\sqrt{1+\epsilon^2}} = 1 - \delta_1$$

$$H_{analog\_LP}(s) = \frac{0.15s^4 + 0.6392s^2 + 0.5313}{s^4 + 0.7691s^3 + 1.6689s^2 + 0.7459s + 0.6251} \tag{14}$$

4.6 Analog Bandstop Transfer Function

The transformation is given by :

$$s_L = \frac{Bs}{s^2 + \Omega_o^2}$$

Substituting the values of B and Ω_o

$$s_L = \frac{1.832s}{s^2 + 0.9293}$$

Using the above transformation we get we get $H_{analog,BS}$(s) from $H_{analog,LP}$(s)

$$H_{analog\_BS}(s) = \frac{0.8500s^8 + 6.5916s^6 + 13.4861s^4 + 5.6924s^2 + 0.6339}{s^8 + 2.1861s^7 + 12.6779s^6 + 13.6600s^5 + 39.8568s^4 + 12.6942s^3 + 10.9486s^2 + 1.7544s + 0.7458}$$

4.7 Discrete Time Filter Transfer Function

To transform the analog domain transfer function into the discrete domain, we need to make use of the Bilinear Transformation which is given as :

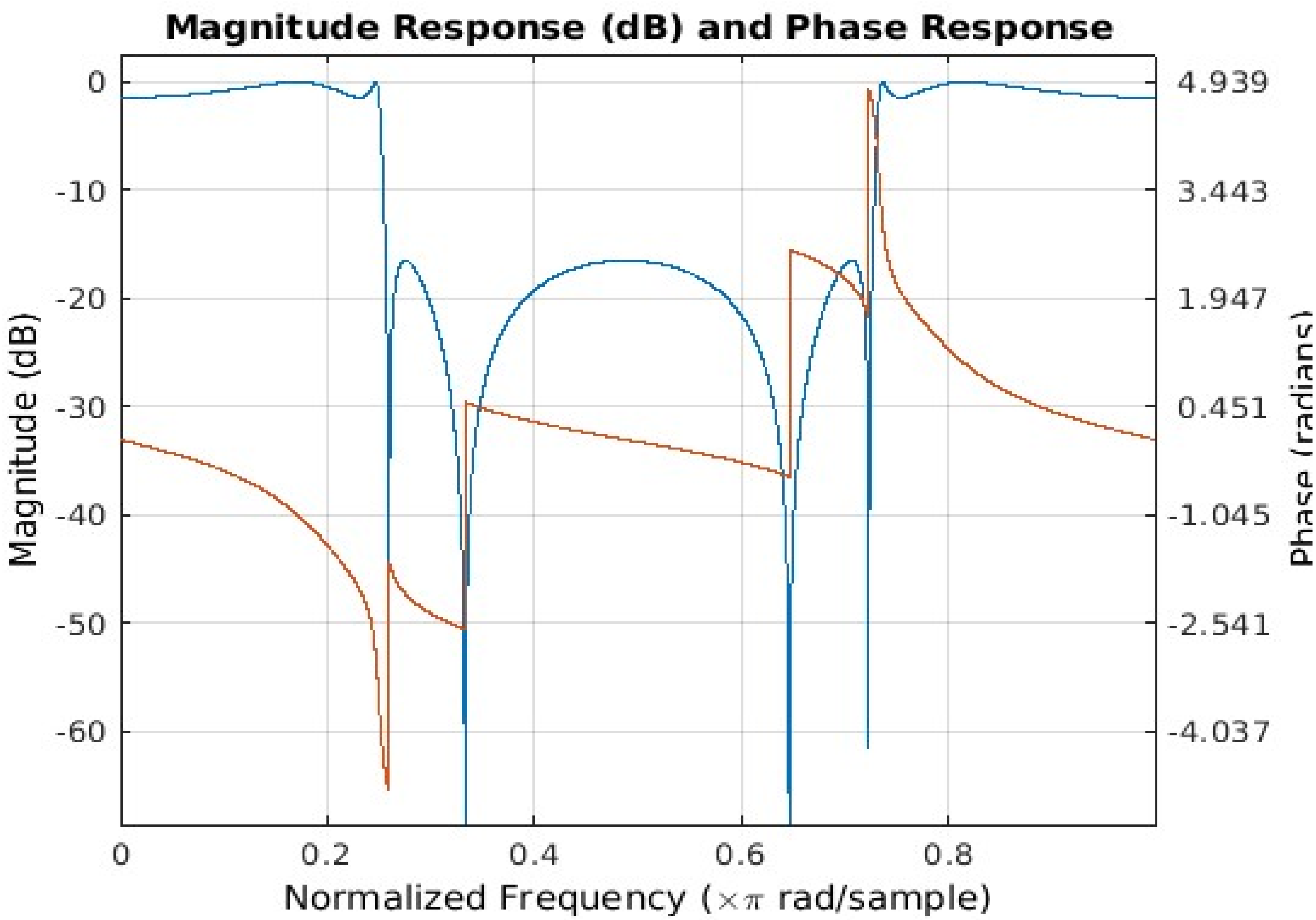
$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

Using above equation we get $H_{discrete,BS}$(z) from $H_{analog,BS}$(s)

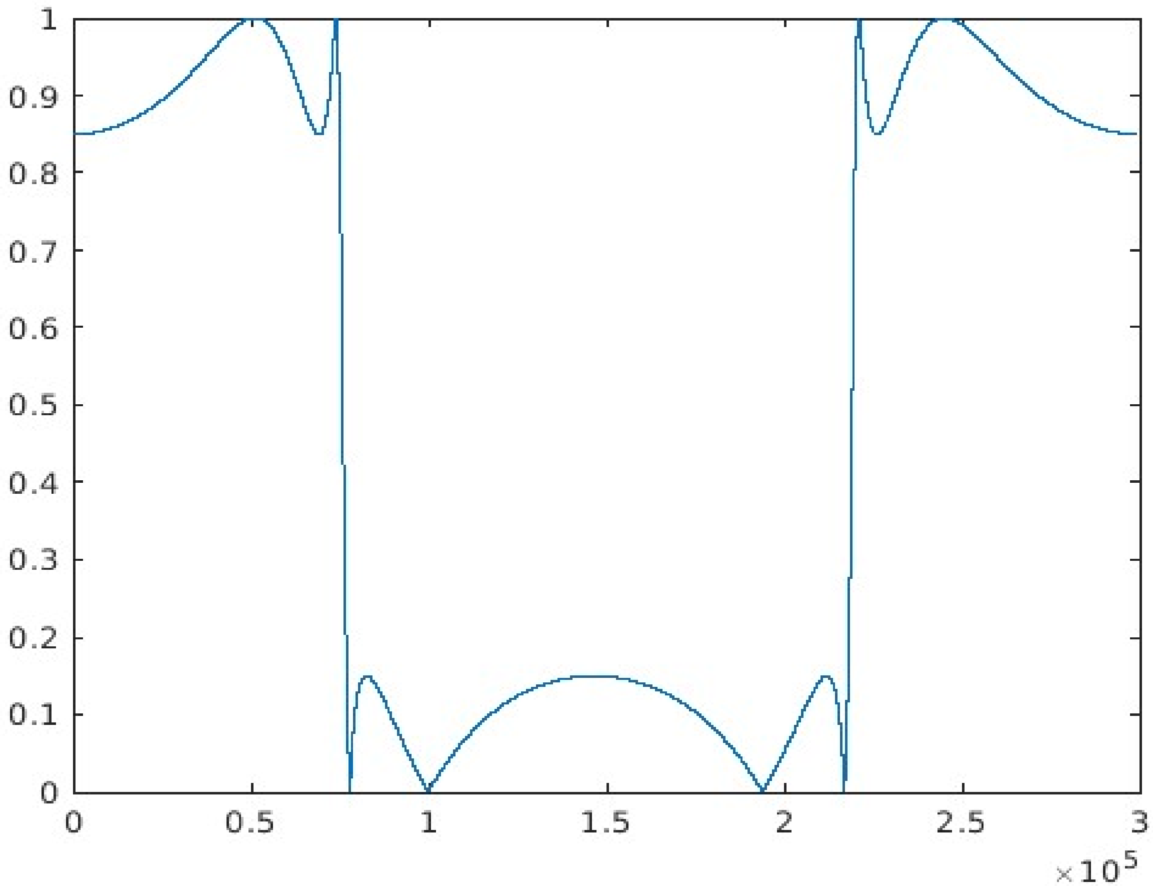
$$H_{discrete,BS}(z) = \frac{0.2853z^8 - 0.0557z^7 + 0.3846z^6 - 0.0890z^5 + 0.6485z^4 - 0.0890z^3 + 0.3846z^2 - 0.0557z + 0.2853}{z^8 - 0.1410z^7 - 0.1422z^6 - 0.0792z^5 + 1.3094z^4 - 0.0740z^3 - 0.1937z^2 - 0.0464z + 0.3657}$$

4.8 Matlab Plots

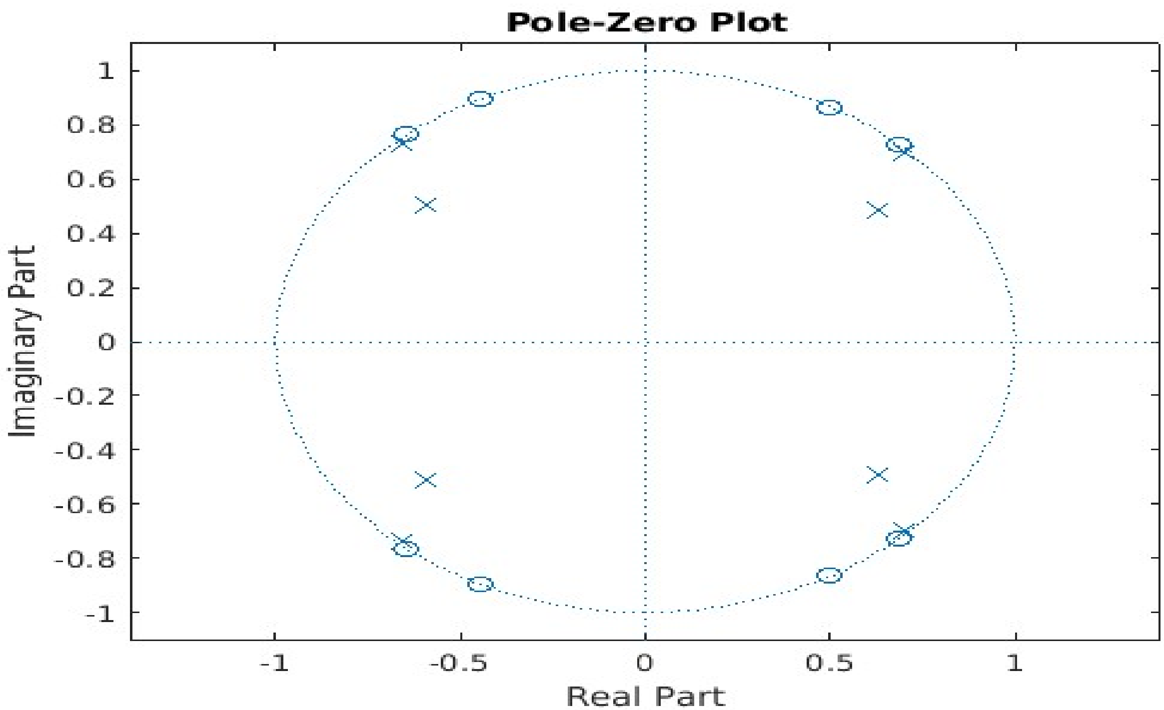
4.8.1 Frequency Response



4.8.2 Magnitude Response



4.8.3 Pole - Zero Plot



As we can see, all poles of above transfer function lie within unit circle. So, the system is stable.

5 Final Results

5.1 Cascaded Transfer Funtion of Multi-Band Filter

Our Final Transfer function has 16 zeroes and 16 poles. The coefficients of the transfer function are as follows:

Degree	Coefficient
z^{16}	0.1207
z^{15}	-0.0314
z^{14}	-0.1769
z^{13}	0.0224
z^{12}	0.2920
z^{11}	-0.0326
z^{10}	-0.3121
z^{9}	-0.0028
z^{8}	0.4037
z^{7}	-0.0028
z^{6}	-0.3121
z^{5}	-0.0326
z^{4}	0.2920
z^{3}	0.0224
z^{2}	-0.1769
z^{1}	-0.0314
z^{0}	0.1207

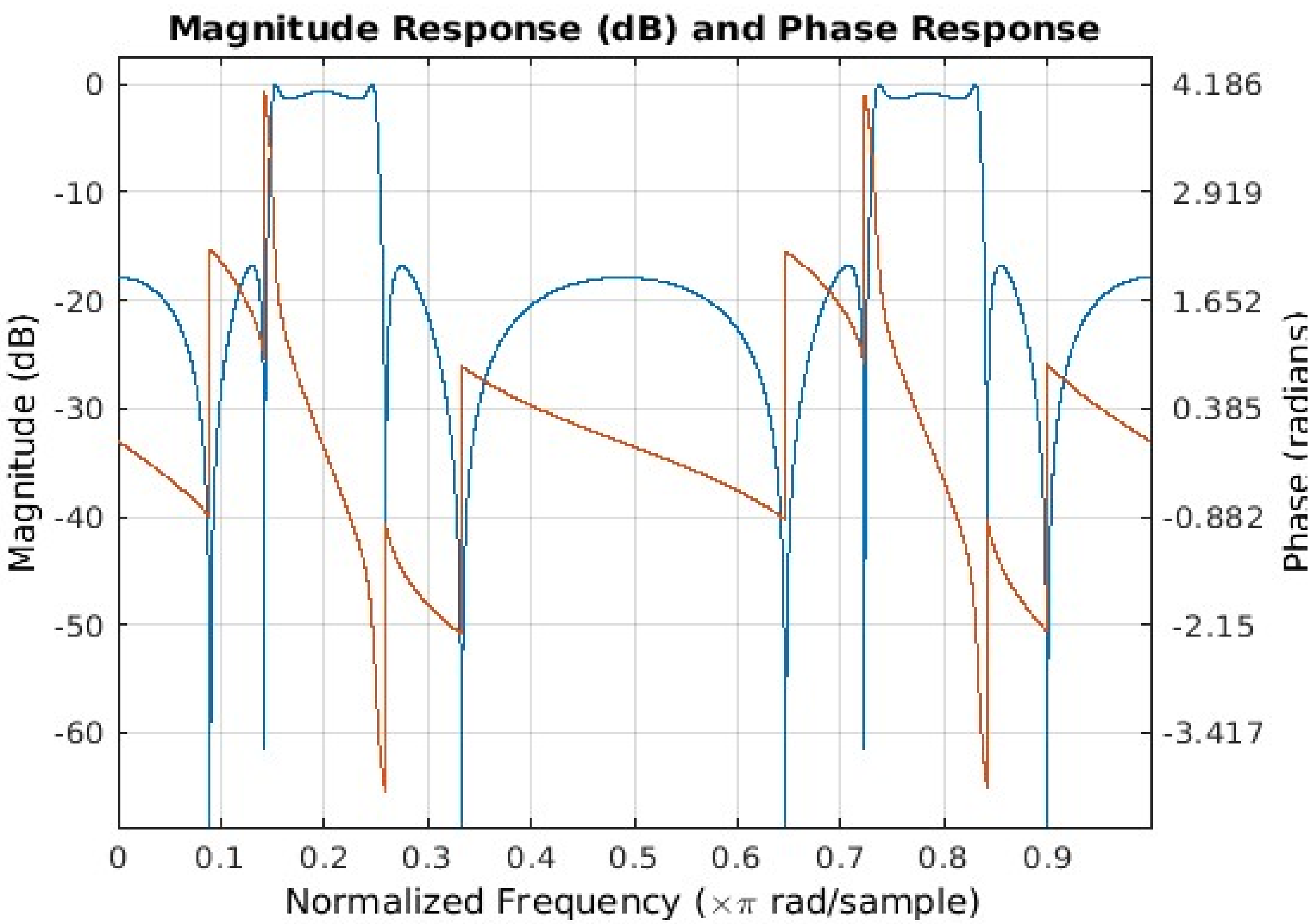
Table 1: Numerator Coefficients

Degree	Coefficient
z^{16}	1
z^{15}	-0.2758
z^{14}	-1.6832
z^{13}	0.2101
z^{12}	3.4348
z^{11}	-0.4601
z^{10}	-3.4062
z^{9}	0.1118
z^{8}	3.6880
z^{7}	-0.1809
z^{6}	-2.1858
z^{5}	-0.0685
z^{4}	1.3882
z^{3}	-0.0008
z^{2}	-0.4082
z^{1}	-0.0311
z^{0}	0.1427

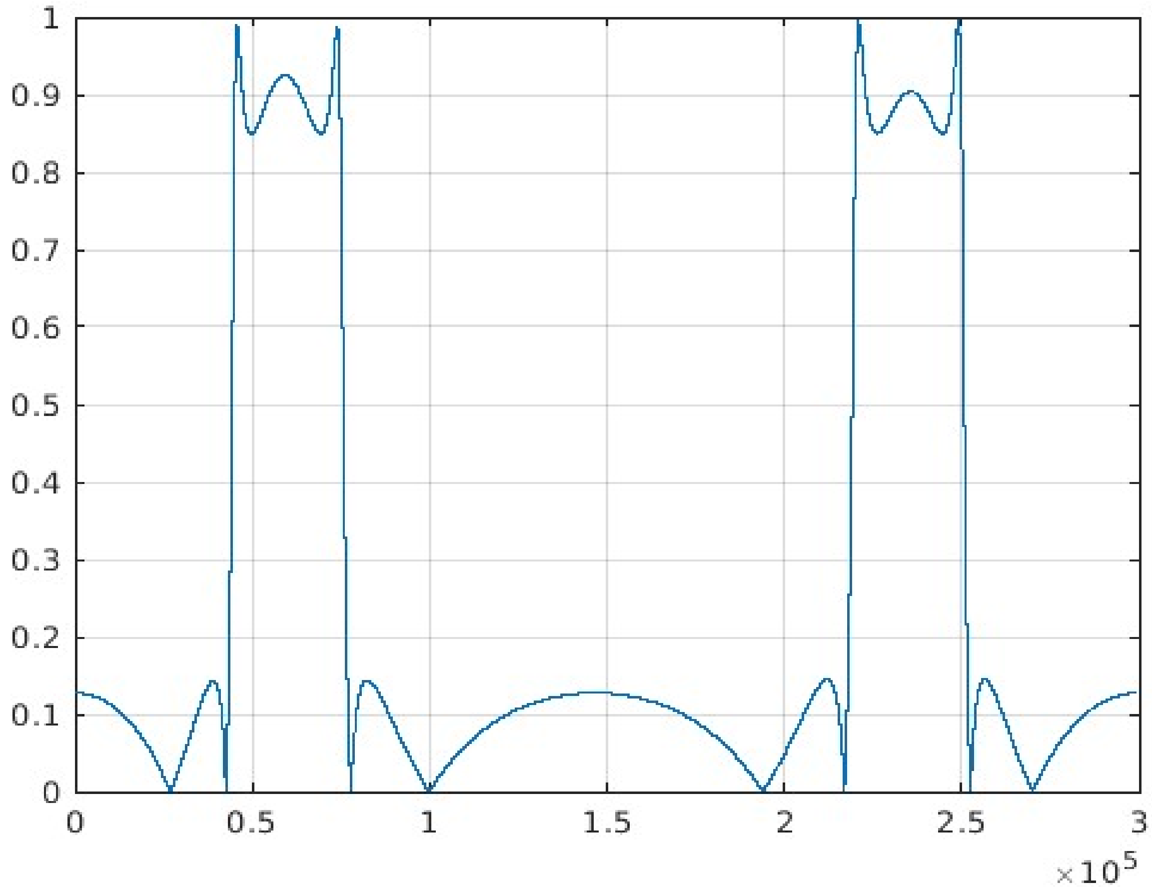
Table 2: Denominator Coefficients

5.2 Matlab Plots

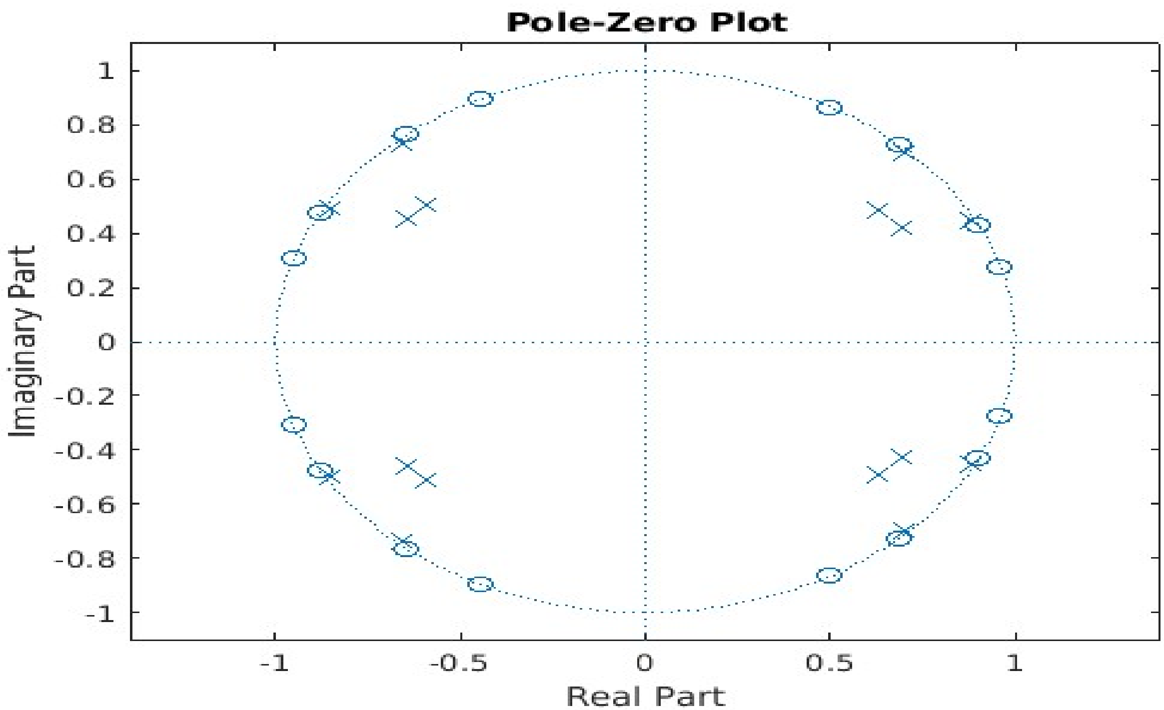
5.2.1 Frequency Response



5.2.2 Magnitude Response



5.2.3 Pole - Zero Plot



As we can see, all poles of above transfer function lie within unit circle. So, the system is stable.

6 Observations on Elliptical Filter Design and comparisons with Butterworth and Chebyschev

- For the same passband and stopband specifications, the order i.e. the number of resources required (which can be seen from the unit sample delays in the signal flow graph or in some sense the degree of the numerator/denominator in Z-transform), is highest for Butterworth and least for Elliptic Filter.
- The phase response becomes more nonlinear as we go from Butterworth to Chebyschev to Elliptic Filter. This is because the Butterworth filter has a maximally flat passband, while the Chebyschev filter has ripples in the passband and the Elliptic filter has ripples in both passband and stopband.

- The Elliptic Filter has the steepest transition band, followed by Chebyschev and then Butterworth. This is because the Elliptic Filter has ripples in both passband and stopband, which allows it to have a steeper transition band.
- The Elliptic Filter has the narrowest transition band, followed by Chebyschev and then Butterworth.

It's important to note that the choice of filter type depends on the specific requirements of the application, such as passband and stopband specifications, transition bandwidth, and phase linearity. Each filter type has its own advantages and trade-offs, and the appropriate filter should be selected accordingly.