Tyler Olkowicz HW3

1) a) Express $u = (7,-1,5)$ as a linear combination of $v = (3,-1,2)$ and $w = (1,1,1)$

$u = 2v + w$

$(7,-1,5) = 2(3,-1,2) + (1,1,1)$

$(7,-1,5) = (6,-2,4) + (1,1,1)$

$(7,-1,5) = (7,-1,5)$

b) Show that $z = (0,0,1)$ is not a linear combination of v and w.

It will STS z, v, w are linearly independent. Then z is not a linear combination of v and w.

let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$\alpha_1 v + \alpha_2 w + \alpha_3 z = 0$

$\alpha_1(3,-1,2) + \alpha_2(1,1,1) + \alpha_3(0,0,1) = (0,0,0)$

$\alpha_1 3 + \alpha_2 1 + \alpha_3 0 = 0 \Rightarrow 3\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = \frac{-\alpha_2}{3}$

$\alpha_1 -1 + \alpha_2 1 + \alpha_3 0 = 0 \Rightarrow -\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = -\alpha_2$

$\alpha_1 2 + \alpha_2 1 + \alpha_3 1 = 0 \Rightarrow 2\alpha_1 + \alpha_2 + \alpha_3 = 0$

The eq: $\alpha_1 = -\frac{\alpha_2}{3}$ $\alpha_1 = -\alpha_2$ can only be satisfied if $\alpha_1 = \alpha_2 = 0$

w/ $\alpha_1 = \alpha_2 = 0$, $2\alpha_1 + \alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_3 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow \{w, v, z\}$ is a lin. indep. set

1) c) Can z be expressed as a linear combination of the vectors u, v, w?

No z cannot be expressed as a linear combination of u, v, w.

$\{u, v, w\}$ are a linearly dependent set (part a)

$\{v, w, z\}$ are a linearly indepenent set (part b)

STS z is not in span of $\{u, v, w\}$ / $\{v, u, w, z\}$ is lin ind set

Assume z can be expressed as a linear combination of $\{u, v, w\}$ s.t $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and either $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, $\alpha_4 \neq 0$

$\alpha_1 u + \alpha_2 v + \alpha_3 w = \alpha_4 z$ (linear dependent set)

since $\{u, v, w\}$ are a linearly dependent set $\exists$ a linear combination of $\{v, w\}$ to express u (part a)

$\beta_1, \beta_2 \in \mathbb{R}$ either $\beta_1 \neq 0$, $\beta_2 \neq 0$

$\beta_1 v + \beta_2 w = u$

$\alpha_1 (\beta_1 v + \beta_2 w) + \alpha_2 v + \alpha_3 w = \alpha_4 z$

$\alpha_1 \beta_1 v + \alpha_1 \beta_2 w + \alpha_2 v + \alpha_3 w = \alpha_4 z$

$(\alpha_1 \beta_1 + \alpha_2) v + (\alpha_1 \beta_2 + \alpha_3) w = \alpha_4 z \Rightarrow (\alpha_1 \beta_1 + \alpha_2) v + (\alpha_1 \beta_2 + \alpha_3) w - \alpha_4 z =$

but since $\{v, w, z\}$ are lin. ind. we reach a contradiction.

$(\alpha_1 \beta_1 + \alpha_2) = 0$ $(\alpha_1 \beta_2 + \alpha_3) = 0$ $\alpha_4 = 0$

This linear combination does not exist. z is not in span of $\{v, w\}$

z cannot be expressed as a linear combination of u, v, w.

Tyler Olivier HW3

1) d) Can the vector $(\pi, e, \sqrt{2})$ be expressed as a lin combination of the vectors v, w, z?

We are concerned with vectors in $\mathbb{R}^3$.

We know $\mathbb{R}^3$ is a vector space

$\mathbb{R}^3$ has dimension 3. Thus $\mathbb{R}^3$ has a basis of ≤ 3 elements

A basis is a set of vectors that are linearly independent and span a vector space by definition

proved in lecture $\{v, w, z\}$ are 3 linearly independent elements in a 3 dimensional vector space $\mathbb{R}^3 \Rightarrow \{v, w, z\}$ basis of $\mathbb{R}^3 \Rightarrow \{v, w, z\}$ span $\mathbb{R}^3$

Also, in a finite dim V.S, every non-empty lin indep set is part of basis. # elements in basis $\leq \dim(V.S)$

$(\pi, e, \sqrt{2})$ can be expressed as a linear combination of the basis $\{v, w, z\}$ as long as $(\pi, e, \sqrt{2}) \in \mathbb{R}^3$

$(\pi, e, \sqrt{2}) \in \mathbb{R}^3 \Rightarrow$ can be expressed as a linear combination of $\{v, w, z\}$

2a) Tyler Olivieri HW3

$(\alpha,1,0)$, $(1,\alpha,1)$, $(0,1,\alpha)$ Find all real numbers α s.t. vectors are lin independent in $\mathbb{R}_3$

let $\beta_1, \beta_2, \beta_3 \in F$

$$\beta_1(\alpha,1,0) + \beta_2(1,\alpha,1) + \beta_3(0,1,\alpha) = 0$$

and if $\beta_1 = \beta_2 = \beta_3 = 0$, vectors are lin independent.

$$(\beta_1\alpha, \beta_1, 0) + (\beta_2, \beta_2\alpha, \beta_2) + (0, \beta_3, \beta_3\alpha) = (0,0,0)$$

$$\beta_1\alpha + \beta_2 = 0 \Rightarrow -\beta_1\alpha = \beta_2 \Rightarrow \beta_1 = \frac{-\beta_2}{\alpha} \overset{(1)}{\Rightarrow} \alpha = \frac{-\beta_1}{\beta_2}$$

$$(3)\ \beta_1 + \beta_2\alpha + \beta_3 = 0$$

$$\beta_2 + \beta_3\alpha = 0 \Rightarrow \beta_2 = -\beta_3\alpha \Rightarrow \beta_3 = -\beta_2/\alpha \overset{(2)}{\Rightarrow} \alpha = \frac{-\beta_2}{\beta_3}$$

substitute (1) and (2) into (3)

$$\frac{-\beta_2}{\alpha} + \beta_2\alpha - \frac{\beta_2}{\alpha} = 0$$

$$\beta_2\left(-\frac{1}{a} + \alpha - 1/\alpha\right) = 0$$

$$\beta_2(-2/\alpha + \alpha) = 0$$

if $\left(-\frac{2}{\alpha}+\alpha\right) \neq 0 \Rightarrow \beta_2 = 0$

if $(-2/\alpha + \alpha) \neq 0$, $-\beta_3\alpha = \beta_2 = 0$

$-\beta_3\alpha = 0 \Rightarrow \beta_3 = 0$ if $\alpha \neq 0$

if $(-2/\alpha + \alpha) \neq 0 \Rightarrow -\beta_1\alpha = \beta_2 = 0$

$-\beta_1\alpha = 0 \Rightarrow \beta_1 = 0$ if $\alpha \neq 0$

vectors are independent if $(-2/\alpha + \alpha) \neq 0$ and $\alpha \neq 0$

2b) Does the answer change if instead you work over the rational vector space Q^3 or C^3 (and allow α to be in Q or C respectively).

I don't think it changes. $\beta_1, \beta_2, \beta_3$ are still zero (vectors are lin indep.) with 2 conditions on α.

$$\left(-2/\alpha + \alpha\right) \neq 0 \qquad \alpha \neq 0.$$

3) Tyler Olivier HW3

Show that if a vector space V is not finite dimensional, then there ~~exists~~ is an infinite linearly independent subset $\{v_1, v_2, v_3, \ldots\}$ of V. Does V have a finite spanning set?

V is not finite dimensional $\Rightarrow$ V is infinite dimensional

finite linearly ind subset $\Rightarrow$ finite dimensional (contrapositive)

infinite dimensional $\overset{\text{by definition}}{\Rightarrow}$ infinite elements in basis

basis by definition spans a vector space

Assume V is infinite dimensional $\overset{\text{by definition}}{\Rightarrow}$ V has infinite basis elements

basis is lin ind $\overset{\text{(by def)}}{\Rightarrow}$ V has infinite element lin. indep subset

$\dim(V) = \infty$. let $v_1 \in V$ $v_2 \notin \text{span}(v_1)$ $v_2 \in V$.

$\text{span}(v_1) \neq V$ because only a basis spans a vector space
V has ∞ elements, $|\{v_1\}| < \infty$. Since $v_2 \notin \text{span}(v_1)$ $\{v_1, v_2\}$ are lin ind subset

cardinality, or # elements in set

let $v_3 \in V$ $v_3 \notin \text{span}(\{v_1, v_2\})$. By same argument as above $\{v_1, v_2, v_3\}$ are lin ind subset. since cardinality of $\{v_1, v_2\} < \infty$ $\exists$ another vector in V that is lin indep.

This "terminates" when cardinality of lin indep subset is ∞.

4.1) Tyler Olivier HW 3

let $v_0, v_1, \ldots, v_n$ be vectors in v.s V s.t. $\{v_1, \ldots, v_n\}$ is a basis of V, and such that v_0 is not in the span of $v_1, \ldots, v_{n-1}$. Prove that $\{v_0, v_1, \ldots, v_{n-1}\}$ is a basis of V.

$\{v_0, v_1, \ldots, v_{n-1}\}$ is a basis of V $\Leftrightarrow$

need to show

$\{v_0, v_1, \ldots, v_{n-1}\}$ span V.

$\{v_0, v_1, \ldots, v_{n-1}\}$ are lin independent

Since $\{v_1, \ldots, v_n\}$ are a basis they are lin indep.

every subset of a lin independent set is lin independent

$\Rightarrow \{v_1, \ldots, v_{n-1}\}$ are lin independent

v_0 is not in span of $\{v_1, \ldots, v_{n-1}\}$

$\Rightarrow \{v_0, v_1, \ldots, v_{n-1}\}$ are lin ind. PS2 #4

$\dim(V) = n$ because $\{v_1, \ldots, v_n\}$ has n elements

$\{v_0, v_1, \ldots, v_{n-1}\}$ has n elements, w/ $\dim(V) = n$ [because $\{v_1, \ldots, v_n\}$ has n elements basis], is subset of V,

and is lin indep.

$\Rightarrow \{v_0, v_1, \ldots, v_{n-1}\}$ basis of V

Suppose $\{v_0, v_1, \ldots, v_{n-1}\}$ doesn't span V, then $\exists\ v_{n+1} \in V$ that is not in the span of $\{v_0, v_1, \ldots, v_{n-1}\}$ s.t. $\{v_0, v_1, \ldots, v_{n-1}, v_{n+1}\}$ form a lin. indep. set

We arrive at a contradiction.

Theorem 4, section 2.3, H & K states that if $\{V_1, \dots, V_n\}$ spans V [which it does by def of basis]

then there is no lin independent set of vectors with more than $|\{V_1, \dots, V_n\}|$ ($\dim(V)$) elements.

Thus since $\{V_0, \dots, V_{n-1}\}$ and $\{V_1, \dots, V_n\}$ have the same number of elements and are both lin independent, there is no vector $V_{n+1} \in V$ s.t. $V_{n+1} \notin \operatorname{span}\{V_0, \dots, V_{n-1}\}$

$\Rightarrow \{V_0, \dots, V_{n-1}\}$ spans V

$\Rightarrow \{V_0, \dots, V_{n-1}\}$ is a basis for V

5) Tyler Olivieri HW 3

Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^4$. Suppose that the set $S = \{v_1, v_2\}$ is lin independent, and that the set $S' = \{v_3, v_4\}$ is also lin independent. Suppose also that $span(S) \cap span(S') = 0$. Prove that $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$.

show $B = \{v_1, v_2, v_3, v_4\}$ is lin independent.

$\Rightarrow B$ is basis

B spans $\mathbb{R}^4$.

First we expand S to B and show it is lin independent.

let $v_3 \neq 0$ $v_3 \notin span(S)$ b/c $v_3 \in span(S')$. This vector exists because the only vector in both span(s) and span(s') is 0 vector.

$v_3 \neq 0$, $span(S) \cap span(S') = 0$, $v_3 \notin span(S) \Rightarrow v_3 \in span(S')$

By PS2 #4 $S'' = \{v_1, v_2, v_3\}$ is a lin independent set because $S = \{v_1, v_2\}$ are lin independent. $v_3 \notin span(S)$

By a similar argument $S''' = \{v_1, v_2, v_4\}$ is a lin independent set. $S'''' = \{v_2, v_3, v_4\}$, $S''''' = \{v_1, v_3, v_4\}$ are lin independent set again by the same argument.

$\Rightarrow B = \{v_1, v_2, v_3, v_4\}$ is a lin independent set. All subsets $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, S, S', S'', S''', S'''', S'''''$ are lin independent.

Suppose B does not span $\mathbb{R}^4$.

then $\exists\ V_5 \notin span(B)$ and $V_5 \in \mathbb{R}^4$.

By PS2 #4 $B = \{V_1, V_2, V_3, V_4, V_5\}$ is lin independent.

But by same theorem used in problem 4, we reach number of elements in $B \geq dim(V)$

a contradiction.

B is lin independent

$\Rightarrow$ B spans $\mathbb{R}_4$

$\Rightarrow$ B is a basis for $\mathbb{R}_4$