Tyler Olivieri HW2

1a) Suppose that F is a field, and $a, b \in F$. Prove that if $ab = 0$ then either $a = 0$ or $b = 0$.

$ab = 0$

$a^{-1}ab = a^{-1}0$ multiplicative inverse

$1 \cdot b = a^{-1} \cdot 0$ multiplicative identity

$b = a^{-1} \cdot 0$ (proved any number $\cdot\, 0 = 0$ in lab)

$b = 0$

same steps as above, multiply on ~~left~~ right

$ab = 0$

$abb^{-1} = 0b^{-1}$

$a \cdot 1 = 0$

$a = 0$

1b) Suppose that V is a vector space over the field of scalars F and that $c \in F$. Prove $c\mathcal{O} = \mathcal{O}$

$$c\mathcal{O} = \underbrace{c(\mathcal{O} + \mathcal{O})}_{\text{additive identity}} = \underbrace{c\mathcal{O} + c\mathcal{O}}_{\text{distribution property of VS}}$$

$c\mathcal{O} = c\mathcal{O} + c\mathcal{O}$

$-c\mathcal{O} + c\mathcal{O} = -c\mathcal{O} + c\mathcal{O} + c\mathcal{O}$ Additive inverse

$\Rightarrow \mathcal{O} = c\mathcal{O}$

c) Suppose that V is a vector space over the field of scalars F, and that $c \in F$ and $v \in V$. Prove that if $cv = 0$, then either $c = 0$ or $v = 0$

Assume $c \neq 0$, $v \neq 0$

$cv = 0$

$cvv^{-1} = 0v^{-1}$ multiplicative inverse

$c \cdot 1 = 0$ Any vector $\cdot\, 0 = 0$

$c = 0$ multiplicative identity

Contradiction c must $= 0$

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2a) Determine whether W is a subspace of the vector space V over $\mathbb{R}$

$V = \mathbb{R}^2 \quad W = \{ (x,y) \in V \mid xy \geq 0 \}$

vectors (x,y) in W have same sign.

W is not a subspace. It suffices to show a linear combination of vectors in W is not in W.

let $\alpha, \beta \in W$. $\alpha = (1,1) \quad \beta = (-2,-.5)$

$\alpha \in W \quad 1 \cdot 1 = 1 \geq 0$

$\beta \in W \quad -2 \cdot (-.5) = 1 \geq 0$

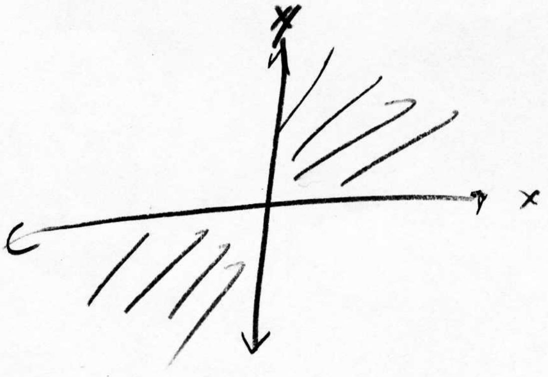
$\gamma = \alpha + \beta = (1,1) + (-2,-.5) = (-1,.5)$

$\gamma \notin W \quad (-1)(.5) = -.5 \leq 0$

thus, by counter example W is not a subspace.

The set of vectors W span quadrants 1 and 3 of the cartesian plane in $\mathbb{R}^2$

//// W



2b) $V = \mathbb{R}^2$ $\quad W = \{ (x,y) \in V \mid x^2 - 2xy + y^2 = 0 \}$

W is a subspace.

W is non-empty (1) $(0,0) \in W$ $\quad (0)^2 - 2(0)(0) + (0)^2 = 0$

$0 = 0$

vectors W have cordinates (x,y) where $x=y$

let $\alpha, \beta \in W$ $c \in \mathbb{R}$ $\alpha = (x_1, x_1)$ $\beta = (y_1, y_1)$ due to cordinates must be equel to be a vector in W.

$c\alpha + \beta \in W$ for W to be a subspace.

it suffices to show that any vector $c\alpha + \beta$ have cordinates that are equal, implying they are in W.

$$c\alpha + \beta = c(x_1, x_1) + (y_1, y_1) = (cx_1, cx_1) + (y_1, y_1)$$

$$= (cx_1 + y_1, cx_1 + y_1)$$

cordinates are equal, and c, x_1, y_1 are arbitrary.

(2) so W is closed under addition and multiplication

$\Rightarrow$ due to (1) and (2) W is a subspace of V.

2c)

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$V = \mathbb{R}^4$, $W = \{(x,y,z,t) \in V \mid x+y+2z=0,\ y+z+2t=0\}$

W is a subspace.

(1) $(0,0,0,0) \in W$ $\quad 0+0+2(0)=0 \quad 0=0 \qquad 0+0+2(0)=0 \quad 0=0$

$x+y+2z=0$ is a plane in $\mathbb{R}^4$ that intersects w/ the origin

$(0,0,0,0)$ $\quad 0+0+2(0)=0$ $\qquad$ A plane is a subspace if it contains the origin, which $x+y+2z=0$ does.

by similar arguments, $y+z+2t=0$ is a plane in $\mathbb{R}^4$ that intersects w/ the origin.

W is set of vectors of intersection of above planes.

Since W is intersection of collection of subspaces of V and contains 0 vector

$\Rightarrow$ W is a subspace

2d) V is set of sol to the diff eq $f''(x)+f'(x)-2f(x)=0$ Tyler Olivieri

W is the set of sol to the diff eq $f'(x)=f(x)$ HW2

additive identity is $f_0 = \emptyset \in W$

$$f_0' = 0$$

$$0 = f_0'(x) = f_0(x) = 0$$

$$f_0' + f_1 = 0 + f_1 = f_1$$

W is closed under addition / multiplication.

$f_1, f_2 \in W$ $\quad \alpha \in F$

$$f_3 = \alpha f_1 + f_2$$

$$f_3' = \alpha f_1' + f_2'$$

$$\alpha f_1'(x) = \alpha f_1(x) \qquad f_2'(x) = f_2(x)$$

add eq.

$$\alpha f_1'(x) + f_2'(x) = \alpha f_1(x) + f_2(x)$$

$$f_3'(x) = f_3(x)$$

$\Rightarrow$ W is subspace of V

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2 e) V is the set of differential real-valued functions on $\mathbb{R}$

c is a fixed real number, and $W = W_c$ is the set of

$f \in V$ s.t $f(1) = 0$ $f'(2) = c$

$W \subseteq V$ trivially and W is subset of differential real-valued functions.

$0 \in W$ additive identity is in W.

let $f_0 = 0$ then $\forall$ $f \in W$ $f + f_0 = f + 0 = f$

because f is differential, real-valued function

W is closed under addition/multiplication under certain restriction on c.

let $f_1, f_2 \in W$ $\alpha \in \mathbb{R}$

$f_3 = \alpha f_1 + f_2$

$f_3(1) = \alpha f_1(1) + f_2(1) = \alpha \cdot 0 + 0 = 0 + 0 = 0$

$f_3' = \alpha f_1' + f_2'$

$f_3'(2) = \alpha f_1'(2) + f_2'(2) = \alpha c + c = (\alpha+1)c$

$(\alpha+1)c = c$ iff $c = 0$

if $c = 0$ W is subspace.

if $c \neq 0$ W is not closed under addition/multiplication.

$(\alpha+1)c \neq c$ and W is not a subspace.

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2.1) V is set of convergent sequences $a_1, a_2, a_3, \ldots$ of real numbers

W is set of convergent sequences $a_1, a_2, a_3, \ldots$ s.t. the series $\sum_{i=1}^{\infty} a_i$ converges.

If V is a vector space, $W \subseteq V$ and W is a subspace if W is closed under addition/multiplication and contains the zero vector.

let A_1 be $a_1, a_2, a_3, \ldots$ s.t. $\sum_{i=1}^{n} a_i = C_A$

B_1 be $a_1, a_2, a_3, \ldots$ s.t. $\sum_{i=1}^{n} b_i = C_B$

$c \in F$

then $cA_1 + B_1$ is also a convergent sequence s.t. the series converges.

The series $cA_1 + B_1$ converges

$$S = cA_1 + B_1 = c\sum_{i=1} a_i + \sum_{i=1} b_i = \sum_{i=1}^{\infty} ca_i + \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} ca_i + b_i$$

$$c\sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i = cC_A + C_B = \sum_{i=1}^{\infty} ca_i + b_i$$

let S be $\overset{s_1}{ca_1+b_1}, \overset{s_2}{ca_2+b_2}, \ldots$ which will converge to $cC_A + C_B$

0 vector $\in W$. 0 vector is sequence of all 0 i.e. $0, 0, 0, \ldots$

0 sequence is additive identity

Furthermore, we know S will be convergent sequence

A series converges if all partial sums converge

$$\lim_{n\to\infty} \sum_{i=1}^{n} s_i = S = \lim_{n\to\infty} \sum_{k=1}^{n-1} s_i + \lim_{n\to\infty} s_n = S + \lim_{n\to\infty} s_n$$

$$S = S + \lim_{n\to\infty} s_n \Rightarrow \lim_{n\to\infty} s_n = 0$$

$\Rightarrow$ W is subspace, it is closed under addition/mult and has 0 vector.

3) Prove that the functions e^x, e^{2x}, e^{3x} are linearly independent in the real vector space V consisting of all real-valued differential functions on $\mathbb{R}$.

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e^x, e^{2x}, e^{3x} are linearly indepent iff

$$ae^x + be^{2x} + ce^{3x} = 0 \quad \forall \text{ with } a,b,c \in \mathbb{R}$$

Differentiate function twice.

$$ae^x + b2e^{2x} + c3e^{3x} = 0$$

$$ae^x + b4e^{2x} + c9e^{3x} = 0$$

The linear combination must hold $\forall x$. let $x=0$

$$a + b + c = 0$$

$$a + 2b + 3c = 0$$

$$a + 4b + 9c = 0$$

The only solution to the system of equations is $a=0, b=0, c=0$

$\Rightarrow e^x, e^{2x}, e^{3x}$ are a linearly independent.

4) Let S be a linearly independent subset of a vector space V. Let $v \in V$ with $v \notin S$, and let $S' = S \cup \{v\}$ Tyler Olivier. HW2

Prove: S' is linearly independent iff v is not in the span of S.

Assume v is in the span of S.

Let $s_1, s_2, \ldots, s_n$ be distinct linearly ind. vectors in S.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be scalars in F

then $\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n = 0$

and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ due to linear independence.

if v is in the span of S,

$$\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n = v$$

(1) $$\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n - v = 0$$

For S' to be linear independent set

$$\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n + \alpha_{n+1} v = 0$$

(2) $$\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha_{n+1} = 0$$

However, we reach a contradiction. If v is in the span of S,

from (1) and (2) $\alpha_{n+1} \neq 0$

$\Rightarrow$ S' is not linearly independent

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5) Let $A = (a_{ij})$ be an $n \times n$ lower triangular matrix i.e. $a_{ij} = 0$ for $i < j$. Suppose that B is an $n \times n$ matrix s.t. $AB = I$

Prove (i) B is also lower triangular

(ii) the diagonal elements of A are all non-zero.

$AB = I$ let $b_1, b_2, \dots, b_n$ be the column vectors of B.

s.t $[b_1 \ b_2 \ \dots \ b_n] = B$

$A[b_1 \ b_2 \ \dots \ b_n] = I$

$[Ab_1 \ Ab_2 \ \dots \ Ab_n] = I$

let e_i be the ith column vector in the standard basis, then.

$I = [e_1 \ e_2 \ \dots \ e_n]$

$[Ab_1 \ Ab_2 \ \dots \ Ab_i \dots Ab_n] = [e_1 \ e_2 \ \dots \ e_i \ \dots \ e_n]$

$Ab_i = e_i$

let $a_1, a_2, \dots, a_n$ be the column vectors of A

$a_1 b_{i1} + a_2 b_{i2} + \dots + a_n b_{in} = e_i$

since e_i is 1 in ith row and 0 elsewhere

for some j, $1 \leq j < i \leq n$, e_i in jth row is 0.

thus $b_{1i}, b_{2i}, \dots, b_{ji}$ must be 0 to ensure e_j is 0.

$a_{j1} b_{1i} + a_{j2} b_{2i} + \dots + a_{jj} b_{ji} + \dots + a_{jn} b_{ni} = 0$

$a_{j(j+1)} = \dots = a_{j(n)} = 0$, A is lower triangular

$a_{j1} b_{1i} + a_{j2} b_{2i} + \dots + a_{jj} b_{ji} = 0$

(continued on back)

$a_{j1} b_{11}$

$$a_{j1} b_{1i} + a_{j2} b_{2i} + \dots + a_{jj} b_{ji} = 0$$

for this to hold for all lower triangular matrices A

$$b_{1i} = b_{2i} = \dots = b_{ji} = 0$$

thus, for any vector b_i, first (or top) j elements must be 0, when $j < i$

$\Rightarrow$ B is lower triangular.

5)

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(ii) the diagonal elements of A are all non-zero.

Since $AB = I$ we know $A^{-1} = B$ and A is invertible.

It suffices to show that if A has a 0 in a diagonal element position, A is not invertible.

It is a proof by contrapositive

invertibility $\rightarrow$ non-zero diagonal elements of A

$\Updownarrow$

zero diagonal element of A $\longrightarrow$ not invertible.

invertibility of A was given as a problem assumption.

let $A_{nn} = 0$. Since A is lower triangular, $(a_{ij} = 0 \quad i<j)$

$$A_{1n} = A_{2n} = \cdots = A_{(n-1)n} = 0$$

thus, A has a column of 0.

A matrix with a column of 0 is not invertible

$$AA^{-1} = A^{-1}A = I$$

but if A has column of 0.

$$\text{"}I\text{"} \quad AA^{-1} = \begin{bmatrix} 1 & & 0 & \\ & 1 & & \cdot \\ 0 & & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

it cannot equal identity.

$\Rightarrow$ not invertible.

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6) Let V be a vector space. Suppose that W_1, W_2 are subspaces of V w/ the property that neither is contained in the other. Prove that the union $W_1 \cup W_2$ is not a subspace of V.

$$W_1 \not\subseteq W_2 \qquad W_2 \not\subseteq W_1$$

let $w_1 \in W_1 \quad w_1 \notin W_2 \qquad w_2 \in W_2 \quad w_2 \notin W_1$

$W_1 \cup W_2$ is not a subspace if $w_1 + w_2 \notin W_1 \cup W_2$ because a subspace is closed under addition and vectors $w_1, w_2 \in W_1 \cup W_2$.

claim $w_1 + w_2 \notin W_1 \cup W_2$. Assume $w_1 + w_2 \in W_1$

$w_1 \in W_1 \Rightarrow (-w_1) \in W_1$ by additive identity property.

Since the subspace W_1 is closed under addition

$$w_1 + w_2 + (-w_1) \in W_1$$

$$w_2 \in W_1$$

Contradiction.

Similarly,

$$w_2 + w_1 + (-w_2) \in W_2$$

$$w_1 \in W_2$$

contradiction

$w_1 + w_2 \notin W_1$, $w_1 + w_2 \notin W_2$ $\Rightarrow w_1 + w_2 \notin W_1 \cup W_2$

$\Rightarrow W_1 \cup W_2$ is not a subspace