Tyler Olivieri ~~PS~~ PS8 #12 H&K p 105-107

Let V be a fdvs over F and let W be a subspace of V. If f is a linear functional on W, prove that there is a linear functional g on V s.t. $g(\alpha) = f(\alpha)$ for each α in the subspace W.

let $\{\gamma_1, \dots, \gamma_k\}$ be a basis for W

$\{\gamma_1, \dots, \gamma_k, \beta_1, \dots, \beta_n\}$ be a basis for V.

with dual basis' $\{\gamma_1^*, \dots, \gamma_k^*\}$

$\{\gamma_1^*, \dots, \gamma_k^*, \beta_1^*, \dots, \beta_n^*\}$

(Δ) $f = \sum_{i=1}^{k} f(\gamma_i)\gamma_i^*$

$g = \sum_{i=1}^{k} g(\gamma_i)\gamma_i^* + \sum_{i=1}^{n} g(\beta_i)\beta_i^*$

if $\alpha \in W$ if $f \in W^*$

$\alpha = \sum_{i=1}^{k} a_i\gamma_i$ $f(\alpha) = \sum_{i=1}^{k} a_i f(\gamma_i)$

now for vectors $v \in V$ functional $g \in V^*$

$v = \sum_{i=1}^{k} v_i\gamma_i + \sum_{i=1}^{n} b_i\beta_i$ $g(v) = \sum_{i=1}^{k} v_i g(\gamma_i) + \sum_{i=1}^{n} b_i g(\beta_i)$

(see (Δ))

since f, g share some dual basis elements we can state $g(\alpha) = f(\alpha)$ letting $g(\beta_1) = g(\beta_2) = \dots = g(\beta_n) = 0$ and $f(\gamma_i) = g(\gamma_i)$ for vectors $\alpha \in W \subseteq V$. For vectors $v \notin W$ but $v \in V$ where $g(\gamma_i) = 0$ we can define $g(v) = \sum_{i=1} b_i g(\beta_i)$

thus, there exists a functional defined on V that has the property $g(\alpha) = f(\alpha)$ for every α in W.

b)

let $f \in (W_1 \cap W_2)^0$

$\Rightarrow f(w) = 0 \quad \forall w \in W_1 \cap W_2 \quad (*)$

Since V is a fdvs, W_1 and W_2 are fdvs as they are subspaces of V

we have a finite element basis of $W_1 \cap W_2$

$\{\alpha_1, \dots, \alpha_k\}$

for $W_1 \quad \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m\}$

$W_2 \quad \{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_n\}$

$V \quad \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n, z_1, \dots, z_j\}$

s.t for $v \in V$

$$v = \sum_i a_i \alpha_i + \sum_i b_i \beta_i + \sum_i y_i \gamma_i + \sum_i d_i z_i$$

$$f(v) = \sum_i a_i f(\alpha_i) + \sum_i b_i f(\beta_i) + \sum_i y_i f(\gamma_i) + \sum_i d_i f(z_i)$$

since $f \in (W_1 \cap W_2)^0$ then $a_i = 0 \quad (*)$

$$f(v) = \sum_i b_i f(\beta_i) + \sum_i y_i f(\gamma_i) + \sum_i d_i f(z_i)$$

let $f(v) = f_1(v) + f_2(v)$

where $f_1(v) = \sum_i b_i f(\beta_i)$

$$f_2(v) = \sum_i y_i f(\gamma_i) + \sum_i d_i f(z_i)$$

if $v \in W_1 \quad \sum_i a_i \alpha_i + \sum_i b_i \beta_i$ then $f_2(v) = 0$

as has $y_i = 0 \quad d_i = 0$

$\Rightarrow f_2 \in W_1^0$

if $v \in W_2$ $\quad v = \sum_i a_i \alpha_i + \sum_i y_i \gamma_i$

then $f_1(v) = 0$ as v has $b_i = 0$

$f_1 \in W_2^\circ$

thus $f \in W_1^\circ + W_2^\circ$

$(W_1 \cap W_2)^\circ \subseteq W_1^\circ + W_2^\circ$

$\Rightarrow (W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$

Tyler Olivieri PS8 #1 H&K Chpt 3 p105-107 #11

Let W_1 and W_2 be subspaces of a finite-dimensional vector space V

a) Prove that $(W_1+W_2)^\circ = W_1^\circ \cap W_2^\circ$

let $f_1 \in (W_1+W_2)^\circ$

$w \in W_1+W_2$

$f_1(w) = 0$ $\qquad (W_1+W_2)^\circ = \{f \in (W_1+W_2)^* \mid f(w_1+w_2)=0,\ w_1 \in W_1,\ w_2 \in W_2\}$

$f(w_1+w_2) = 0 \quad \forall w_1 \in W_1 \quad \forall w_2 \in W_2$

thus, let $w_1 = 0$

$f(w_2)=0 \Rightarrow f \in W_2^\circ$

$f(w_1+w_2)=0$ $\qquad \Rightarrow f \in W_1^\circ \cap W_2^\circ$

and let $w_2 = 0$ $\qquad \Rightarrow (W_1+W_2)^\circ \subseteq W_1^\circ \cap W_2^\circ$

$f(w_1) = 0 \Rightarrow f \in W_1^\circ$

let $f_1 \in W_1^\circ \cap W_2^\circ$ $\quad f_1 \in W_1^\circ \wedge f_1 \in W_2^\circ$

$\exists$

$f_1(w_1)=0$ for $w_1 \in W_1$

$f_1(w_2)=0$ $\quad w_2 \in W_2$

$f_1(w_1)+f_1(w_2) = 0+0$

$f_1(w_1+w_2) = 0$

$f_1 \subset (W_1+w_2)^\circ$

$\Rightarrow W_1^\circ \cap W_2^\circ \subseteq (W_1+W_2)^\circ$

$\Rightarrow W_1^\circ \cap W_2^\circ = (W_1+W_2)^\circ$

11 b) Prove that $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$

let $f \in (W_1 \cap W_2)^\circ$ $\hookrightarrow f(w_1) + f(w_2) = 0$ $\forall w_1 \in W_1$ $\forall w_2 \in W_2$

$\Rightarrow f(w) = 0 \quad \forall w \in W_1 \cap W_2$

let $w = w_1 + w_2$ $w_1 \in W_1$ $w_2 \in W_2$ $w \in W_1 \cap W_2$

W_1 can have a basis for $W_1 \cap W_2$ $\{\alpha_1, \dots, \alpha_K\}$

a basis for W_1 $\{\alpha_1, \dots, \alpha_K, \beta_1, \dots, \beta_n\}$

a basis for W_2 $\{\alpha_1, \dots, \alpha_K, c_1, \dots, c_m\}$

a basis for V $\{\alpha_1, \dots, \alpha_K, \beta_1, \dots, \beta_n, c_1, \dots, c_m, d_1, \dots, d_j\}$

s.t for $v \in V$

$v = \sum a\alpha + \sum b\beta + \sum \gamma c + \sum z d$

continued on next page / ordering got weird here because I was stuck on this part.

let $f \in W_1^\circ + W_2^\circ \Rightarrow f = f_1(w_1) + f_2(w_2)$

(*) where $f_1(w_1) = 0$ $f_1 \in W_1^\circ$ $\forall w_1 \in W_1$

$f_2(w_2) = 0$ $f_2 \in W_2^\circ$ $\forall w_2 \in W_2$

let $w \in W_1 \cap W_2 \Rightarrow w \in W_1$, $w \in W_2$

$f(w) = (f_1 + f_2)(w)$

$= f_1(w) + f_2(w)$

$= 0 + 0$ (from (*))

$= 0$

$f \in (W_1 \cap W_2)^\circ$

$W_1^\circ + W_2^\circ \subseteq (W_1 \cap W_2)^\circ$

Tyler Olivieri P58 #17 p105-107 HWK

Let W be the space of $n \times n$ matrices over the field F, and let W_0 be the subspace spanned by the matrices C of the form $C = AB - BA$. Prove that W_0 is exactly the subspace of matrices which have trace zero. (Hint: what is the dimension of the space of matrices of trace zero? Use the matrix 'units', i.e. matrices with exactly one non-zero entry, to construct enough linearly independent matrices of the form $AB - BA$).

$C \in W_0$

(*) $\text{trace}(C) = \text{trace}(AB - BA) = \text{trace}(AB) - \text{trace}(BA)$

$= \text{trace}(AB) - \text{trace}(AB) = 0$

$C_1, C_2 \in W_0$ $\quad a \in F$

C_1, C_2 have 0 trace see (*)

$aC_1 + C_2$

$\text{trace}(aC_1 + C_2) = \text{trace}(aC_1) + \text{trace}(C_2)$

$= a\,\text{trace}\,C_1 + \text{trace}\,C_2$

$= a0 + 0$

$= 0$ $\Rightarrow aC_1 + C_2 \in W_0$

thus $W_0 \subseteq$ subspace of matrices with trace 0.

to show that

W_0 = subspace of matrices w/ trace 0.

WTS $\dim(W_0)$ = dim (subspace of matrices w/ trace 0).

What is dim (subspace of matrices w/ trace 0?)

trace: $\mathbb{R}^{n \times n} \to \mathbb{R}$

As mentioned in H&K is actually a linear functional Ex 19 3.5 (linear functional is still linear trans)

thus, using rk-nullity

$\dim(\mathbb{R}^{n \times n}) = \dim \operatorname{Im}(\text{trace}) + \dim \ker(\text{trace})$

trace is onto, thus $\dim \operatorname{Im}(\text{trace}) = 1$

$\dim(\mathbb{R}^{n \times n}) = n^2$

$n^2 = 1 + \dim \ker(\text{trace})$

$n^2 - 1 = \dim \ker(\text{trace})$

where $\ker(\text{trace}) = \{ A \in \mathbb{R}^{n \times n} \mid \text{trace}(A) = 0 \}$

$\Rightarrow$ dim (subspace of matrices w/ trace 0) $= n^2 - 1$

WTS $\dim(W_0) = n^2 - 1$

basis of W_0? $n^2 - n$ unit matrices off diagonal

they do not affect trace. need $n-1$ more to complete basis.

let β represent the basis for W_0

$\beta = \{ \{\beta_d\}, \{\beta_{od}\} \}$ d - diagonal, od - off diagonal

by construction, 0 in diagonal.

$\text{trace}(\{\beta_d\} + \{\beta_{od}\}) = 0$ $\quad \text{trace}(\{\beta_d\}) + \underbrace{\text{trace}(\{\beta_{od}\})}_{= 0} = 0$

we want to construct β_d s.t $\text{trace}(\{\beta_d\}) = 0$

let elements in β_d have a 1 in ii position and a -1 in some other position jj $i \neq j$, thus trace of these elements are 0.

These elements are also lin ind. because they have a 1 in ii position and 0 elsewhere

Now if it can be shown that these matrices are basis elements of W_0, then $\dim(W_0) = n^2 - 1$ and the proof will conclude.

Recall $\beta_{od_{ij}}$ is off diagonal basis el $i \neq j$

β_d is diagonal basis element w/ 1 in ii, -1 in jj pos.

$\text{trace}(2\beta_{od}) = 0$

$$2\beta_{od_{ij}} = \beta_{d_{ij}}\beta_{od_{ij}} - \beta_{od_{ij}}\beta_{d_{ij}}$$

is basis element for W_0

$\Rightarrow \beta_{od_{ij}}$ is basis element for W_0

$$\beta_{d_{ij}} = \beta_{od_{ij}}\beta_{od_{ji}} - \beta_{od_{ji}}\beta_{od_{ij}}$$

is basis element for W_0

$\Rightarrow \beta_{d_{ij}}$ is basis element for W_0

thus $\beta = \{\beta_{od}, \beta_d\}$ is a basis for W_0 with $n^2 - 1$ elements

$\Rightarrow \dim(W_0) = n^2 - 1$

$\Rightarrow W_0 =$ subspace of $n \times n$ matrices with 0 trace.

Ex illustration of basis in 2x2

$$2\begin{bmatrix}0&0\\1&0\end{bmatrix} = \begin{bmatrix}0&0\\2&0\end{bmatrix} = \begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix} - \begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}-1&0\\0&1\end{bmatrix}$$

$$= \begin{bmatrix}0&0\\-1&0\end{bmatrix} - \begin{bmatrix}0&0\\-1&0\end{bmatrix} = \begin{bmatrix}0&0\\2&0\end{bmatrix}$$

$$\begin{bmatrix}-1&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} - \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

$$= \begin{bmatrix}0&0\\0&1\end{bmatrix} - \begin{bmatrix}1&0\\0&0\end{bmatrix} = \begin{bmatrix}-1&0\\0&1\end{bmatrix}$$

Tyler Olivier P58 p 115 - 116 #1 H8K

Let F be a field and let f be the linear functional on F^2 defined by $f(x_1, x_2) = ax_1 + bx_2$. For each of the following linear operators T, let $g = T^*f$ and find $g(x_1, x_2)$

a) $T(x_1, x_2) = (x_1, 0)$ What is T^*?

$$g(x_1, x_2) = (T^*f)(x_1, x_2) = f(T(x_1, x_2))$$

$$= f((x_1, 0)) = ax_1$$

b) $T(x_1, x_2) = (-x_2, x_1)$

$$g(x_1, x_2) = (T^*f)(x_1, x_2) = f(T(x_1, x_2))$$

$$= f(-x_2, x_1) = -ax_2 + bx_1$$

c) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$

$$g(x_1, x_2) = (T^*f)(x_1, x_2) = f(T(x_1, x_2))$$

$$= f(x_1 - x_2, x_1 + x_2) = a(x_1 - x_2) + b(x_1 + x_2)$$

$$= ax_1 + bx_1 - ax_2 + bx_2$$

$$= (a+b)x_1 + (-a+b)x_2$$

Tyler Olivieri PS8 # 7 H&K p116

Let V be a fdvs over F. Show that $T \to T^*$ is an isomorphism of $L(V,V)$ onto $L(V^*, V^*)$

Since V is a fdvs.

H&K explains the dimension of $L(V,V)$ is $(\dim V)(\dim V)$

we also know $\dim V = \dim V^*$

thus $\dim(L(V^*, V^*)) = \dim(V^*)\dim(V^*) = \dim(V)\dim(V)$

$\Rightarrow \dim(L(V,V)) = \dim(L(V^*,V^*))$

$\Rightarrow$ since $L(V,V)$ and $L(V^*,V^*)$ are fdvs themselves,

$L(V,V) \xrightarrow{\sim} L(V^*,V^*)$

$T \xrightarrow{\sim} T^*$ is also an isomorphism

as foreach T there is a unique T^* Theorem 21 H&K

and $L(V,V) \xrightarrow{\sim} [T]_\beta$ theorem 12 H&K

thus
$$\begin{array}{ccc} L(V,V) & \xrightarrow{\sim} & [T]_\beta \\ \downarrow \wr & & \downarrow \wr \\ L(V^*,V^*) & \longrightarrow & [T^*]_{\beta^*} \end{array}$$

this must also be an isomorphism.

$\Rightarrow$ $T \to T^*$ is an isomorphism of $L(V,V)$ onto $L(V^*,V^*)$

Tyler Olivieri 258 Chpt 4 P 122-123 #1a H8K

Let F be a subfield of the complex numbers and let A be the following 2x2 matrix over F

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Compute $f(A)$

a) $f = x^2 - x + 2$

$$A^2 = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$$

$$f(A) = A^2 - A + 2I$$

$$= \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ -4 & 7 \end{bmatrix}$$

Tyler Olivieri PS8 # 4 H & K p 122-123

4) If f and g are independent polynomials over a field F and h is a non-zero polynomial over F, show that fh and gh are independent.

f and g are independent polynomials over a field F

$\Rightarrow c_1 f + c_2 g = 0 \qquad c_1, c_2 \in F \qquad c_1 = 0 \quad c_2 = 0$

$\Rightarrow f \neq 0 \quad g \neq 0$ (if they were 0, they couldn't be independent)

$f \neq g$

$h \neq 0$

$(c_1 f + c_2 g) h = 0h$

$c_1 fh + c_2 gh = 0$

$h \neq 0 \Rightarrow fh \neq 0 \quad gh \neq 0$ Theorem 1 a) H & K Chpt 4.

$\Rightarrow c_1 = 0 \qquad c_2 = 0$

$\Rightarrow fh$ and gh are independent

---

or if scalars did exist s.t. $c_1 \neq 0$, $c_2 \neq 0$

$c_1 fh = -c_2 gh$

by corollary 2

$c_1 f = -c_2 g$

which implies $c_1 f + c_2 g = 0$ w/ $c_1 \neq 0$, $c_2 \neq 0$

this contradicts independence,

thus $c_1 = 0$, $c_2 = 0$ in $c_1 fh + c_2 gh = 0$ and fh is independent of gh.

Tyler Olivieri PS8 #6 H&K Chpt 4 p 122-123

Let S be a set of non-zero polynomials over a field F
If no two elements of S have the same degree, show that
S is an independent set in $F[x]$.

S is an indepent set if $c \in F$

(*) $c_1 S_1(x) + c_2 S_2(x) + \ldots + c_d S_d(x) = 0 \quad c_1 = c_2 = \ldots = c_d = 0$

Since all polynomials in S have different degrees,
consider the polynomial in S_d with maximum degree s.t.
no other polynomial in S has a degree $\geq d$
if this poynomial S_d has a non-zero coefficient in the
d^{th} slot. If it was 0, it would not have degree d.
Also since S_d has largest degree in S no other
polynomial in S has coefficents in slot $K > d$.
Since no two polynomials in S have the same degree.
S_d has the unique non-zero coefficient in d^{th} slot.

thus in the d^{th} slot we have the following result.

assuming polynomials are ordered by degree then $c_1 0 + c_2 0 + \ldots + c_d S_d = 0$

$c_d S_d = 0$

$S_d \neq 0$

$\Rightarrow c_d = 0$

The same argument can be made for every degree less than d now that we know $c_d = 0$ and with the knowledge that no two polynomials have same degree

$$c_{d-1} s_{d-1} = 0$$
$$s_{d-1} \neq 0$$
$$c_{d-1} = 0$$

and this will terminate once we go over all polynomials with degree less than d. It will eventually terminate as degree ≥ 0

thus $c_1 = c_2 = \dots = c_d = 0$

and the polynomials are lin independent.

Tyler Olivieri PS8 # 7 H&K Chpt 4 p123

If a and b are elements of a field F and $a \neq 0$ show that the polynomials 1, $ax+b$, $(ax+b)^2$, $(ax+b)^3, \ldots,$ form a basis of $F[x]$.

if $b=0$ we have the polynomials

$\{1, ax, ax^2, \ldots, ax^n, \ldots\}$ which is clearly a basis as established previously in class.

if $b \neq 0$ we have to establish the polynomials are linearly independent and span the space of $F[x]$

We note that from theorem H&K

$\deg(fg) = \deg f + \deg g$ f, g are non-zero

degrees of polynomials	polynomials
0	1
1	$ax+b$
$1+1=2$	$(ax+b)^2 = (ax+b)(ax+b)$
$1+1+1=3$	$(ax+b)^3 = (ax+b)(ax+b)(ax+b)$
$\vdots$	$\vdots$

Note that every polynomial in the set will have a different degree. From the previous problem, we can see that this is a linearly independent set.

Now WTS it spans $F[x]$ and then the proof can be concluded.

It spans all scalar polynomials $\in F[x]$ $c \in F$

$$c1 = c$$

$$c_1 1 + c_2(ax+b) = c_2 ax + c_2 b + c_1$$

let $c_2 = 1/a \quad c_1 = -b/a$

$$1/a \, a x + b 1/a \, b - b/a$$

$$= x$$

thus c_1, c_2 can be chosen s.t $c_1 1 + c_2(ax+b) = x$

scaling c_1, c_2 would give a scaled x

$\Rightarrow \{1, ax+b\}$ span polynomials of degree 1.

The same argument can be made for $\{1, ax+b, (ax+b)^2\}$ for polynomials of degree 2.

Thus, $\{1, ax+b, (ax+b)^2, \dots\}$ span $F[x]$

$\Rightarrow \{1, ax+b, (ax+b)^2, \dots\}$ is a basis for $F[x]$

Tyler Olivieri PS8 # 2

a) Show that if $T: V \to W$ is a surjective linear transformation of finite dimensial v.s with kernel N, then $\dim(V/N)$

$\dim(V/N) = \dim(V) - \dim(N)$ [hint consider dimension W]

T is surjective $\iff \operatorname{Im}(T) = W$

$\dim(\operatorname{Im}(T)) = \dim(W)$

by rank-nullity theorem $\dim V = \dim \operatorname{Im}(T) + \dim \ker(T)$

$\dim V = \dim(W) + \dim N$

$(*)$ $\dim V - \dim N = \dim W$

STS $\dim(W) = \dim(V/N)$

By the last theorem in appendix A4 in H&K

W is isomorphic to V/N

if two fdvs are isomorphic, they are the same dimension. take a basis for one space and transform them into the other, the linearity of the transform preserves independence and thus dimension

$\Rightarrow \dim(W) = \dim(V/N)$ $(**)$

$\Rightarrow \dim(V) - \dim(N) = \dim(W) = \dim(V/N)$ $(*)$ $(**)$

b) Illustrate this with an example $V=\mathbb{R}^3$, $W=\mathbb{R}$

$T(x,y,z) = x+y+z$

$\dim(V) = 3 \qquad \dim(N) = \dim(V) - \dim(\mathrm{Im}(T))$ (rk-nullity)

Clearly T is surjective $\Rightarrow \dim(\mathrm{Im}(T)) = \dim(W) = 1$

$\dim(N) = 3-1 = 2$

$\dim(V/N) = \dim(V) - \dim(N) = 3-2 = 1$

V/N is vector space of cosets of N

for $v=(a,b,c) \in V$, $v+N \in V/N \qquad N = \{v \in V \mid a+b+c=0\}$

V/N is line perpendicular to plane $a+b+c=0$

$v+W$ describes that every vector $v \in V$ (a,b,c)

Claim: basis of V/W is $(1,1,1)+W$ and since V/N is 1-dimensional

$V/N = \{v+W \in V/N \mid v = c(1,1,1)\}$

$(1,1,1)$ is linearly independent and is the appropriate elements for a 1-d space.

It is also in V/N because it is not in W.

To see this consider a basis of N $\{(1,-1,0),(0,1,-1)\}$

$(1,1,1)$ is lin independent of

Consider the quotient map $Q: V \to V/W \qquad Q(\alpha) = \alpha + N$

the nullspace of this transformation is N, thus

since $(1,1,1)$ is not in N, it is in $\mathrm{Range}(Q)$, Q is linear

and therefore in V/W.

V/N is the space of translated planes $x+y+z=0$ along the vectors $c(1,1,1)$

Tyler Olivier: PS8 #3

Let $T: V \to W$ and $S: W \to Z$ be linear transformations of fdvs. Consider the transpose transformations, $T^*: W^* \to V^*$ and $S^*: Z^* \to W^*$

a) show that $T^* = 0$ iff $T = 0$

Assume $T^* = 0$

let g be nonzero in W^* $\alpha \neq 0 \in V$

$T^* g(\alpha) = g T(\alpha)$

$0 g(\alpha) = g T(\alpha)$

$0 = g(T(\alpha))$

g is non zero

$g(0) = 0$

$\Rightarrow T(\alpha) = 0$

$\alpha \neq 0$

$\Rightarrow T = 0$

Assume $T = 0$ Assume $g(\alpha) \neq 0$

$T^* g(\alpha) = g T(\alpha)$

$T^* g(\alpha) = g(0) = 0$

$T^* g(\alpha) = 0$

$\Rightarrow T^* = 0$

$T = 0 \Leftrightarrow T^* = 0$

b) Show that $(S \circ T)^* = T^* \circ S^*$ and deduce that if $S \circ T = 0$ then $T^* \circ S^* = 0$

$(S \circ T)^* : Z^* \rightarrow V^*$

let $z \in Z^*$

$(S \circ T)^*(z) = z \circ (S \circ T) = (z \circ S) \circ T$

$= T^* \circ (z \circ S) = (T^* \circ z) \circ S$

$= S^* \circ (T^* \circ z) = (S^* \circ T^*)(z)$

$\Rightarrow (S \circ T)^* = S^* \circ T^*$

if $S \circ T = 0$

$(S \circ T)^* (z) = z \circ (S \circ T) = z(0) = 0$

because z is a linear functional in Z^*

Tyler Olivier

3c) Show that if T is surjective then T^* is injective.

[hint: What is the kernel of T^*] $\quad T: V \to W$

WTS $\ker T^* = 0 \iff T^*$ injective $\quad T^*: W^* \to W^*$

or $\dim \ker T^* = 0$

what is the kernel of T^*?

All functionals $g \in W^*$ s.t $g(T\alpha) = 0$ for every α in V.

$\dim(W^*) = \dim(W)$

$\dim(W^*) = \dim \mathrm{Im}(T^*) + \dim \ker(T^*)$

$\dim \mathrm{Im}(T^*) = \mathrm{rk}\, T^*$

$\dim \mathrm{Im}(T^*) = \mathrm{rk}\, T^* = \mathrm{rk}\, T = \dim \mathrm{Im}(T)$ (Theorem 22 H&K 3.7)

$\dim \mathrm{Im}(T^*) = \dim \mathrm{Im}(T)$

$\Rightarrow \dim(W) = \dim \mathrm{Im}(T) + \dim \ker(T^*)$

T is surjective $\Rightarrow \mathrm{Im}(T) = W$

$\Rightarrow \dim \mathrm{Im}(T) = \dim(W)$

$\dim(W) = \dim(W) + \dim \ker(T^*)$

$\Rightarrow \dim \ker T^* = 0$

$\Rightarrow T^*$ is injective

Tyler Olivieri

3d) Show that, if T is injective then T^* is surjective

[hint: Pick a basis β of V, and show that $T(\beta) \subset W$ extends to a basis of W]

$T: V \to W$

$T^*: W^* \to V^*$

T is injective $\Leftrightarrow$ dim ker$(T) = 0$

or ker$(T) = 0$

The annihilator of the null space of T is the set of functionals $g(v) = 0 \quad \forall v \in \ker(T)$

since $\ker(T) = 0$ $v = 0$

$g(0) = 0$

this is true for every functional $g \in V^*$

annihator of the null space of T is V^*

H&K theorem 22 (ii)

the range of T^* is the annihalator of the nullspace of T

thus the range of T^* is V^*

$\Rightarrow$ T^* is surjective

3e). Conclude that T is an isomorphism iff T^* is an isomorphism

Assume T is an isomorphism

$\Rightarrow T$ is bijective $\Rightarrow T$ is injective and surjective

$\Rightarrow T$ is injective $\Rightarrow T^*$ is surjective $\Rightarrow$

T is surjective $\Rightarrow T^*$ is injective

$\Rightarrow T^*$ is bijective

$\Rightarrow T^*$ is an isomorphism

Assume T^* is an isomorphism

$\Rightarrow T^*$ is bijective $\Rightarrow T^*$ is injective and surjective

T^* is surjective $\Rightarrow$ $\mathrm{Im}(T^*) = V^*$

$\mathrm{rk}(T^*) = \dim(V^*) = \dim(V)$

$\mathrm{rk}(T^*) = \mathrm{rk}(T)$

$\mathrm{rk}(T) = \dim(V)$

$\Rightarrow \ker(T) = 0$ by rk-nullity

$\Rightarrow T$ is injective

T^* is injective $\Rightarrow$ $\dim \ker T^* = 0$

$\mathrm{rk}\, T^* = \dim W^*$ by rk nullity

$\mathrm{rk}\, T = \dim W$

$\Rightarrow T$ is surjective

$\Rightarrow T$ is bijection $\Rightarrow T$ is an isomorphism

T is an isomorphism $\Leftrightarrow T^*$ is an isomorphism.

Tyler Olivieri PS8 #4

(+, scalar mult, ·)

not commutative
distribute over
associative +

Let A be an algebra over a field F

a) Show that $0 \cdot a = 0$ for all $a \in A$
where $0 \in A$ is the additive identity in A

additive identity ↓ ↓ distribution

$$a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$$

$$a \cdot 0 = a \cdot 0 + a \cdot 0$$

additive inverse

$$a \cdot 0 + (-a \cdot 0) = a \cdot 0 + a \cdot 0 + (-a \cdot 0)$$

$$0 = a \cdot 0$$

b) Suppose that A has a multiplicative identity $1 \in A$
Prove that $(-1) \cdot a = -a$ $\forall a \in A$ where $-a \in$ denotes the additive inverse of $a \in A$.

↓ from part a ↓ additive inverse ↓ distributive

$$0 = 0 \cdot a = (1 + -1) \cdot a = 1 \cdot a + -1 \cdot a$$

multiplicative identity

$$= a + -1 \cdot a$$

$$0 = a + -1 \cdot a$$

additive inverse

$$+(-a) = +(-a) + a + -1 \cdot a$$

$$-a = 0 + -1 \cdot a$$

additive identy

$$-a = -1 \cdot a$$

Tyler Olivieri PS8 KFS

a) Let F be a field, let $f(x) \in F[x]$ and let A be an $n \times n$ matrix over F. suppose that $f(x) = f_1(x) f_2(x)$ in $F[x]$. Prove that $f(A) = f_1(A) f_2(A)$ as matrices.

$$f(x) = f_1(x)\, f_2(x) \qquad \deg(f) = \deg(f+g)$$

$$f(x) = \sum_{i=1}^{d_1+d_2} d_i x^i = \sum_{i=1}^{d_1} f_{1i} x^i \sum_{i=1}^{d_2} f_{2i} x^i$$

$$= \sum_{k=0}^{d_1+d_2} \sum_{i=0}^{k} f_{1i} f_{2k-i} x^k$$

$$= f_{10} x^0 f_{20} x^0 + f_{10} x^0 f_{21} x^1 + f_{11} x^1 f_{20} x^0 + f_{10} x^0 f_{22} x^2 + f_{11} x^1 f_{21} x^1 + f_{12} x^2 f_{20} x^0 + \ldots$$

$$= \underbrace{f_{10} f_{20} x^0}_{f_0} + \overbrace{f_{10} f_{21} x + f_{11} f_{20} x}^{x_1} + f_{10} f_{22} x^2 + f_{11} f_{21} x^2 + f_{12} f_{20} x^2 + \ldots$$

$$= f_0 x^0 + f_1 x^1 + f_2 x^2 + \ldots + f_{d_1+d_2} x^{d_1+d_2}$$

$$= \sum_{i=1}^{d_1+d_2} f_i x^i = f(x)$$

$$f(A) = \sum_{i=1}^{d_1+d_2} f_i A^i = \sum_{i=1}^{d_1} f_{1i} x^i \sum_{i=1}^{d_2} f_{2i} A^i$$

(*) proof can break down here

$$= f_{10} A^0 f_{20} A^0 + f_{10} A^0 f_{21} A^1 + f_{11} A^1 f_{20} A^0 + f_{10} A^0 f_{22} A^2 + f_{11} A^1 f_{21} A^1 + f_{12} A^2 f_{20} A^0 + \dots$$

$$= f_{10} f_{20} A^0 + (f_{10} f_{21} + f_{11} f_{20}) A^1 + (f_{10} f_{22} + f_{11} f_{21} + f_{12} f_{20}) A^2 + \dots$$

$$= \sum_{i=1}^{d_1+d_2} f_i A^i = f(A)$$

in general, the matrices will commute with the scalars.

the matrices can commute with themselves,

so the factorization will hold. As mentioned in c, polynomial functions of multiple matrices may not factorize due to matrices not commuting with other matrices in general.

5b) Let $f(x,y) \in F[x,y]$, the algebra of polynomials in x and y with coefficients in F. Show that $f(A,B)$ is not necessarily equal to the matrix $f_1(A,B) f_2(A,B)$

hint

let $f(x,y) = x^2 - y^2$ and pick two 2x2 matrices

$$f(x,y) = (x+y)(x-y) \qquad f_1(x,y) = (x+y) \qquad f_2(x,y) = (x-y)$$

$$f(x,y) = f_1(x,y) f_2(x,y)$$

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$f(A,B) = A^2 - B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$f_1(A,B) f_2(A,B) = (A+B)(A-B) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

thus, $f(A,B) \neq f_1(A,B) f_2(A,B)$ in general.

$$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

thus

5c) The proof for (a) breaks d[illegible] for the case in (b)

when ~~the~~ I assumed matrices would commute.

which is true for the same matrix $A \cdot A = A \cdot A$

but $A \cdot B \neq B \cdot A$ in general