Tyler Olivieri HW1

1.) a) is $\mathbb{R}$ a vector space over the field $\mathbb{Q}$?

Recall vector space Axioms (addition on V, scalar multiplication on V s.t

1) commutivity $u+v = v+u \quad \forall v,u \in V$

2) associativity $(u+v)+w = u+(v+w)$ and $(ab)v = a(bv) \quad \forall u,v,w \in V$ and $\forall a,b \in F$

3) additive identity $\exists$ an element $0 \in V$ s.t. $v+0 = v \quad \forall v \in V$

4) additive inverse for every $v \in V$, $\exists w \in V$ s.t. $v+w = 0$

5) multiplicative identity $1v = v \quad \forall v \in V$

6) distributive properties $a(u+v) = au+av$ and $(a+b)u = au+bu$
$\forall a,b \in F$ and $\forall u,v \in V$.

7) addition on V $\Rightarrow$ $u+v \in V$

scalar multiplication on V $\Rightarrow$ $av \in V \quad a \in F \quad v \in V$

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for addition on V, properties 1-4 hold due to $a,b \in \mathbb{R}$, $u,v,w \in \mathbb{R}$
$\mathbb{R}$ elements have commutivity, associativity, additive identity, additive inverse

Recall $\mathbb{Q} \subseteq \mathbb{R}$

for multiplication on V, $a,b \in \mathbb{Q}$, $u,v,w \in \mathbb{R}$

$(ab)v = a(bv)$ because $\mathbb{Q} \subseteq \mathbb{R}$ and property holds $\forall a,b \in \mathbb{R}$, $v \in \mathbb{R}$

$1v = v$ holds for real numbers

again since $\mathbb{Q} \subseteq \mathbb{R}$ $a(u+v) = au+av$ and $(a+b)u = au+bu$
and property holds for real elements, distributivity holds.

all properties are satisfied

To show closure, $\forall$

addition on V: $u+v \in \mathbb{R}$ $\quad \forall\, u, v \in \mathbb{R}$

This holds because sum of any two $\mathbb{R}$ elements is $\mathbb{R}$

multiplication on V: $av \in \mathbb{R}$ $\quad \forall a \in \mathbb{Q}$ $\quad \forall v \in \mathbb{R}$

a real element multiplied by a real element is real

since $\mathbb{Q} \subseteq \mathbb{R}$ all elements of $\mathbb{Q}$ are $\mathbb{R}$

and then $av \in \mathbb{R}$

$\Rightarrow$ $\mathbb{R}$ is a vectorspace over $\mathbb{Q}$

1b) Is Q a vector space over $\mathbb{R}$? Tyler Olivieri

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No for Q to be a vector space over $\mathbb{R}$

vector multiplication on Q properties fail

$av \in Q \quad \forall a \in \mathbb{R} \quad \forall v \in Q$

let a be a real, irrational number.

then av is irrational

and $av \notin Q$

which serves as a counter example.

1c) is set of purely imaginary complex #'s a vector space over $\mathbb{R}$?

All addition properties hold due to addition of purely imaginary numbers

addition on purely imaginary complex $v, w \in$ purely imaginary complex

$v+w \in$ purely imaginary complex $\forall v, w$

$a(u+v) = au + av$ $\quad a, b \in \mathbb{R}$ $\quad u, v \in$ purely imag. complex.

$(a+b)u = au + bu$

multiplication on purely img. complex.

$av \in$ purely img. complex $\quad a \in \mathbb{R}$ $\quad v \in$ purely img. complex

This holds due to how multiplication of complex # properties

$(a+bi)(c+di) = (ac-bd) + (bc+ad)i$

but $b=0$, $c=0$ $\Downarrow$ adi which is purely img.

all properties hold

$\Rightarrow$ purely imaginary complex #'s are a vector space over $\mathbb{R}$

True.

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1d) Is the set of complex #'s of abs. val 1 a vector space over $\mathbb{R}$?

It is not a valid vector space

Assume a valid vector space

let $x, y \in$ complex w/ abs value of 1

a valid vector space would have vector addition ~~on~~ closure

$x+y \in$ complex w/ abs value of 1

but $x+y = 2$ for any $x, y \in$ complex w/ abs 1

$2 \notin$ complex w/ abs 1

$\Rightarrow$ contradiction

1e) is the set of symmetric 3x3 real matrices a vector space over $\mathbb{R}$?

let $A, B, C \in \mathbb{R}^{3\times3}$ and $[A]_{ij}$, $[B]_{ij}$, $[C]_{ij}$ be ij^{th} pos of A, B, C respectively

Matrix addition is element-wise.

(*) $[A]_{ij} = [A]_{ji}$ $[B]_{ij} = [B]_{ji}$ $[C]_{ij} = [C]_{ji}$

commutivity holds due to matrix elements being real $\forall ij$

associativity holds due to " "

additive identity " " "

additive inverse " " "

$[A]_{ij} + [B]_{ij} = [A]_{ji} + [B]_{ji}$ $\Rightarrow$ Symmetric matrix addition is still symmetric matrix of real elements

due to (*) $\Rightarrow$ closure in addition.

multiplicative identity holds, $1A = A$

$$[1]_{ij}[A]_{ij} = [A]_{ij} \quad \forall i,j$$

due to real elements A

distributive holds aggin due to real elements.

closeure of multiplication.

$c \in \mathbb{R}$ $\quad [A]_{ij} = [A]_{ji}$

$cA = c[A]_{ij} = c[A]_{ji} \quad \forall i,j$

$\Rightarrow cA \in \mathbb{R}$ due to real number times real number is real.

thus, real symmetric 3x3 matrices are a vector space over $\mathbb{R}$

1f) Is the set of invertible 3x3 real matrices a vector space over $\mathbb{R}$?

$AB = BA = I \iff A, B$ invertible

Assume it is a valid vector space. Let A, B be invertible 3x3 real matrices

then $A+B$ is real invertible 3x3 matrix

let A be I. I is invertible $II = II = I$

let B be $-I$. $-I$ is invertible $(-I)(-I) = (-I)(-I) = I$

$I + -I = 0$ $\quad$ 0 matrix is not invertible $0 \cdot 0 = 0 \cdot 0 \neq I$

$\Rightarrow$ it is not a valid vector space.

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2a) Set of 2x2 real matrices (under matrix addition and multiplication)

Is it a field? why or why not?

It is a field if each pair of elements $x, y \in F$ yield

$x+y \in F$

$xy \in F$

and 1) Addition is commutative

$x+y = y+x \quad \forall x, y \in F$

2) Addition is associative

$x+(y+z) = (x+y)+z \quad \forall x, y, z \in F$

3) There is a unique element 0 in F s.t. $x+0 = x$ for every x in F

4) To each x in F there corresponds a unique element $(-x)$ in F such that $x+(-x) = 0$.

5) Multiplication is commutative $\forall x, y$ in F

$xy = yx$

6) Multiplication is associative $x(yz) = (xy)z \quad \forall x, y$, and z in F

7) There is a unique non-zero element 1 in F such that $x1 = x$ for every x in F.

8) To each non-zero x in F there corresponds a unique element x^{-1} or $(1/x)$ in F such that $xx^{-1} = 1$

9) Multiplication distributes over addition; that is $x(y+z) = xy + xz$

$\forall x, y$, and z in F

Assume it is a valid field

let $A, B \in \mathbb{R}^{2\times 2}$

then $AB = BA$

but let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix} \neq \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} = BA$$

Contradiction

$\Rightarrow$ it is not a valid field.

2b) The set of irrational real numbers (under addition and multiplication of real numbers) Tyler Olivieri HW 1

Assume the set of irrational real numbers is a field.

Then two elements $x, y \in$ set of irrational real numbers

have the property $xy \in$ set of irrational real numbers.

let $x = \sqrt{2}$ $y = \sqrt{2}$

$xy = \sqrt{2}\sqrt{2} = 2$

2 is rational real number

$2 \notin$ set of irrational real numbers

contradiction.

2c) The set of complex numbers of the form $a+bi$ w/ a, b each rational (under addition and multiplication of real number)

addition of complex numbers $(a+bi)+(c+di) = (a+c) + (b+d)i$

multiplication of complex numbers $(a+bi)(c+di) = (ac-bd) + (bc+ad)i$

We note that $(a+c), (b+d), (ac-bd), (bc+ad)$ are rational numbers because addition/multiplication of rational numbers are rational. (rational numbers are a valid field, $\mathbb{Q}$).

Thus, the set of complex numbers of the form $a+bi$ have elements x, y s.t $x+y \in$ set of complex numbers " " ' '

$xy \in$ set of complex numbers " " "

Similary since rational numbers are a valid field, the complex numbers w/ rational coefficients will have properties (1-9) from 2a) with $(a+b), (b+d), (ac-bd), (bc+ad)$

$\Rightarrow$ valid field.

20) The set $\mathbb{R}^2$ under the usual addition of vectors, and with multiplication defined by $(a,b)\cdot(c,d) = (ac-bd, ad+bc)$

Since $a,b,c,d \in \mathbb{R}$ $(*)$ and $\mathbb{R}$ is a valid field with commutative, associative multiplication and addition

then $(a,b)(c,d) = (c,d)(a,b)$

$(a,b) + ((c,d) + (e,f)) = ((a,b) + (c,d)) + (e,f)$

$(a,b) + (0,0) = (a,b)$

$(a,b) + (-a,-b) = (0,0)$

again due to $a,b,c,d \in \mathbb{R}$ $(*)$ closure also occurs due $a,b,c,d \in \mathbb{R}$

$(a,b)(c,d) = (c,d)(a,b)$ due to $(*)$

$(a,b)((c,d)(e,f)) = ((a,b)(c,d))(e,f)$ due to $(*)$

$(a,b)(1,0) = (a,b)$ for multiplicative identity.

$(a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = \left(\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}, \frac{-ab}{a^2+b^2} + \frac{ab}{a^2+b^2}\right)$

$= (1,0)$ for multiplicative inverse.

and multiplication distributing over addition comes from $(*)$

Also since $(a+c), (b+d), (ac-bd), (ad+bc)$ are $\mathbb{R}$

addition and multiplication is closed

$\Rightarrow$ $\mathbb{R}^2$ is field under normal addition and multiplication as defined in problem

3)

$y''' + f_2(x)y'' + f_1(x)y' + f_0(x)y = c$

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c is fixed real number

$f_i(x)$ $i=1,2,3$ are real valued functions

V_c is set of all 3-times differentiable functions $y(x)$ that are solution to homogenous linear differential equation.

Show V_c is a vector space over field of scalars $\mathbb{R}$

iff $c=0$

Assume $c=0$. Let y_0, y_1 be two solutions to the linear diff eq.

thus (1) $y_0''' + f_2(x)y_0'' + f_1(x)y_0' + f_0(x)y_0 = 0$

(2) $y_1''' + f_2(x)y_1'' + f_1(x)y_1' + f_0(x)y_0 = 0$

$y_0 + y_1 \in V_c$ for v.s. Add eq (1) to eq(2) to get eq (3)

(3) $(y_0+y_1)''' + f_2(x)(y_0+y_1)'' + f_1(x)(y_0+y_1)' + f_0(x)(y_0+y_1) = 0$

$\Rightarrow (y_0+y_1)$ is solution to diff eq

$\Rightarrow (y_0+y_1) \in V_c$

$\alpha y_0 \in V_c$ $\quad \alpha \in \mathbb{R}$ for v.s

$(\alpha y_0)''' + f_2(x)(\alpha y_0)'' + f_1(x)(\alpha y_0)' = 0$

$\alpha(y_0''' + f_2(x)y_0'' + f_1(x)y_0') = 0$

$\Rightarrow \alpha y_0$ is solution

$\Rightarrow \alpha y_0 \in V_c$

for homogenous linear differential equations, $y=0$ is a possible solution

Due to V_c over $\mathbb{R}$

functions will commute, w/ addition and multiplication

have associative properties

have distributive properties.

Due to $\forall$ $0 \in V_c$, additive identity exists

$1 \in V_c$ because $f_0(x)$ returns a real value, since $0 \in \mathbb{R}$,

$f_0(x)$ can always return 0

$\Rightarrow$ multiplicative identity exists.

$\Rightarrow$ V_c is a vector space for satisfying axioms and being closed under addition and multiplication.

$\Rightarrow$ V_c is a vector space iff $c = 0$

4) a) Write down addition and multiplication tables of a field having exactly 3 elements

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let $\mathbb{F}_3$ be the finite field consisting of 3 elements.

$\{0,1,2\} \in \mathbb{F}_3$

(mod 3) addition

	0	1	2
0	0	1	2
1	1	2	0
2	2	3	1

multiplication mod 3

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

4) b) Do the same for a finite field of some other number of elements, greater than 3 and less than 10

let $\mathbb{F}_5$ be the finite field consisting of 5 elements with addition/multiplication mod 5. $\{0,1,2,3,4\} \in \mathbb{F}_5$

Addition mod 5

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Multiplication mod 5

	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Is it a valid field? Yes.

properties (1), (2), (5), (6), (a) from (2a) hold due to elements being real.

(3) additive identity is satisfied. Any element in $F_5 + 0$ is same element.

(4) additive inverse is satisfied

$0+0=0$
$1+4=0$
$2+3=0$
$3+2=0$
$4+1=0$

(7) multiplicative identity is satisfied any element in $F_5 \cdot 1$ is same element in $\mathbb{F}_5$

(8) multiplicative identity is satisfied. $1 \cdot 1 = 1$
$2 \cdot 3 = 1$
$3 \cdot 2 = 1$
$4 \cdot 4 = 1$

$\Rightarrow$ valid field.