Tyler Olivieri PS5 4

1) ~~For each~~ For each give an ex or explain why no such example exists

a) A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ whose kernel is 1-D.

This exists.

let $T: (x,y) \longmapsto (x,0)$ $Im(T)$ is x-axis or line

T is clearly linear.

by rank nullity theorem

$v_1, v_2 \in \mathbb{R}^2$ $c \in F$

$\dim(Ker(T)) + \dim(Im(T)) = n = 2$

$T(cv_1 + v_2) =$

$\dim(Ker(T)) + 1 = 2$

$T((cv_{11}+v_{21}, cv_{12}+v_{22}))$

$\dim(Ker(T)) = 1$

$= T((cv_{11}+v_{21})$

b) $T: \mathbb{R}^4 \to \mathbb{R}^3$ whose kernel is trivial

$= T(c(v_{11},0) + (v_{21},0)$

by rank-nullity theorem

$= cT(v_1) + T(v_2)$

$\dim(Ker(T)) + \dim(Im(T)) = 4 = n$

$\Rightarrow \dim(Im(T)) \leq 3$

$\dim(Ker(T)) + 3 = 4$

$\dim(Ker(T)) = 1$

$\dim(Ker(T)) \neq 0$

false

c) A linear transformation

$T: \mathbb{R}^3 \to \mathbb{R}^2$ taking $(1,2,3)$ to $(4,5)$

let $T: (x,y,z) \mapsto (x+z, x+y+z)$

T is clearly linear.

let $a, b \in \mathbb{R}^3$ $c \in F$

$T(ca+b) = T((ca_1+b_1, ca_2+b_2, ca_3+b_3))$

$= (ca_1+b_1+ca_3+b_3, ca_1+b_1+ca_2+b_2+ca_3+b_3)$

$= c(a_1+a_3, a_1+a_2+a_3) + (b_1+b_3, b_1+b_2+b_3)$

$= cT(a) + T(b)$

2) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation

a) Show that the kernel of T is contained in the kernel of $T^2 = T \circ T$. Can the two kernels ever be unequal?

WTS $\ker(T) \subset \ker(T \circ T)$

$$\ker(T) = \{x \in \mathbb{R}^n \mid T(x) = 0\}$$

TWTS $\forall x \in \mathbb{R}^n$ s.t. $T(x) = 0 \Rightarrow (T \circ T)(x) = 0$

let $x \in \ker(T)$ then $\ker(T) \subseteq \ker(T \circ T)$

then $T(x) = 0$

Apply T to both sides

$T(T(x)) = T(0)$ Any linear transformation $T(0) = 0$

$(T \circ T)(x) = 0$

Can the two kernels ever be unequal? Yes.

Suppose $n=3$, $T: \mathbb{R}^3 \to \mathbb{R}^3$

let dim kernel $(T) \geq 1$, so that it is non-trivial.

dim range $(T) = 2$

kernel of T is all x s.t. $T(x) = 0$

kernel of T^2 is all vectors x s.t. $T(T(x)) = 0$

kernel of T^2 includes kernel of T from above, but now includes the range(T) as well. (union of subspaces)

$\Rightarrow$ kernel$(T) \neq$ kernel(T^2) is this case.

T^2 sends all vectors to 0.

so entire space $\mathbb{R}^3$, where T does not necessarily do this.

b) Show that image $(T) \subset$ kernel $(T) \iff T^2 = 0$

Assume image$(T) \subset$ kernel(T) $(*)$

let $b \neq 0 \in \mathbb{R}^n$ $\forall x \in$ image(T) $\Rightarrow x \in$ kernel (T) due to $(*)$

let $x \in$ image(T)

definition $T(b) = x$ $T(x) = 0$

$T(T(x)) = T(0)$

Apply T $T(T(b)) = T(x)$

$(T \circ T)(x) = 0$

$x \in$ kernel(T) $(T \circ T)(b) = T(x)$

$(T \circ T)(b) = 0$

$T^2 b = 0$

$b \neq 0$

Image $(T) \subset$ kernel $(T) \Rightarrow T^2 = 0$

Assume $T^2 = 0$

let $a \in \mathbb{R}^n$ $a \neq 0$

$b \in$ Image (T) $b \neq 0$ if non-empty

if Image (T) is empty, it will always be contained in the kernel, because kernel(T) has at least 0 vector.

$T(a) = b$

$T(T(a)) = T(b)$

$T^2 a = T(b)$

$0 a = T(b)$

$0 = T(b)$

$b \in$ kernel(T)

$T^2 = 0 \Rightarrow$ Image $(T) \subset$ kernel (T)

Image $(T) \subset$ kernel $(T) \iff T^2 = 0$

Tyler Olivieri PS5 # 3

Let $T: V \to W$ and $S: W \to X$ be linear transformations of v.s. What inequalities (i.e. $\leq$) can you find relating these quantities: rank(T), rank(S), rank(SoT), dim(V), dim(W), dim(X)?

[Last inequality is the best]

can I rank(SoT) to all other inequalities.

(1) $\text{rank}(T) \leq \dim(W)$ $\qquad$ $\text{rank}(S) \leq \dim(W)$

$\text{rank}(T) \leq \dim(V)$ $\qquad$ $\text{rank}(S) \leq \dim(X)$

same argument as T

because

(1) $\text{rank}(T) = \dim(\text{range}(T))$ by definition.

range(T) is a subspace of W

$\text{rank}(T) = \dim(\text{range}(T)) \leq \dim(W)$

$\Rightarrow \text{rank}(T) \leq \dim(W)$

(2) $\text{rank}(T) = \dim(\text{range}(T))$

$\dim(V) = n$

$\dim(\text{range}(T)) + \dim(\text{kernel}(T)) = n$ rank-nullity theorem

$\text{rank}(T) + \dim(\text{kernel}(T)) = n = \dim(V)$

$\text{rank}(T) = \dim(V) - \dim(\text{kernel}(T))$

$\text{rank}(T) \leq \dim(V)$

SoT is lin transform by Theorem 6 section 3.2 H&K

$\text{rank}(S \circ T) = \dim(\text{range}(S \circ T))$

$\dim(\text{range}(S \circ T)) + \dim(\text{kernel}(S \circ T)) = \dim(V) = n$

$\text{rank}(S \circ T) + \dim(\text{kernel}(S \circ T)) = \dim(V)$

$\text{rank}(S \circ T) = \dim(V) - \dim(\text{kernel}(S \circ T))$

$\text{rank}(S \circ T) \leq \dim(V)$

$T: V \to W \quad S: W \to X$

let $\hat{T}'$ be a linear transformation from a subspace of V, kernel$(S \circ T)$ to W. [it has been established that kernel$(S \circ T) \subseteq T$

$T' : \text{kernel}(S \circ T) \to W$

$T(v) = T'(v)$ for $v \in \text{kernel}(S \circ T)$

T' is a linear transformation, applying the rank-nullity theorem

$\dim(\text{range}(T')) + \dim(\text{kernel}(T')) = \dim(\text{kernel}(S \circ T))$

Prop $\text{Image}(T') \subseteq \text{kernel}(S)$

Proof $v \in \text{kernel}(S \circ T)$

$(S \circ T)(v) = 0$

$T(v) \in \text{kernel}(S)$

$T'(v) \subseteq T(v)$

$T'(v) \subseteq \text{kernel}(S)$

by construction of T', $\text{Image}(T') \subseteq T'(v)$

$\Rightarrow \text{Image}(T') \subseteq \text{kernel}(S)$

corollary (*) $\dim(\text{Image}(T')) \leq \dim \text{kernel}(S)$

Prop

$\text{kernel}(T') \subseteq \text{kernel}(T)$

Proof $T'(v) = T(v)$ by construction of T'

(*) $\dim \text{kernel}(T') \leq \dim \text{kernel}(T)$

$\Rightarrow \dim(\text{kernel}(S \circ T)) \leq \dim(\text{Image}(T')) + \dim(\text{kernel}(T'))$

$\dim(\text{kernel}(S \circ T)) \overset{(*)}{\leq} \dim(\text{kernel}(S)) + \dim \text{kernel}(T)$

(1) $\dim \ker(S \circ T) \leq \dim \ker(S) + \dim \text{kernel}(T)$ Proved previously

Apply rank - nullity theorem

(2) $\dim \ker(S \circ T) = n - rk(S \circ T)$

$\dim \ker(S) = n - rk(S)$

$\dim \ker(T) = n - rk(T)$

(1) and (2)

$n - rk(S \circ T) \leq n - rk(S) + n - rk(T)$

rearrange

$\cancel{n} - n - n + rk(S) + rk(T) \leq rk(S \circ T)$

$(*)$ $rk(S) + rk(T) - n \leq rk(S \circ T)$ this is good

Prop $rk(S \circ T) \leq rk(T)$

Proof

$\text{Kernel}(T) \subseteq \text{kernel}(S \circ T) \Rightarrow \dim(\ker(T)) \leq \dim(\ker(S \circ T))$

$\hookrightarrow T(x) = 0$

$S(T(V)) = 0$

$S(0) = 0$

Apply rk - null theorem

$n - rk(T) \leq n - rk(S \circ T)$

$(**)$ $rk(S \circ T) \leq rk(T)$

<u>Prop</u> $rk(S \circ T) \leq rk(S)$

<u>Proof</u>

$\text{Image}(S \circ T) \subseteq \text{Image}(S)$

$\text{Image}(S) = \{ x \in X : S(w) = x \text{ for } w \in W \}$

$\text{Image}(S \circ T) = \{ x \in X : S(w) = x \text{ for } w = T(v) \text{ for } v \in V \}$

since $\text{Image}(T) \subseteq W$

there are possibly less vectors in the domain of $S(w) = x$ when $w = \text{image}(T)$ instead of all vectors $w \in W$.

$\Rightarrow \text{Image}(S \circ T) \subseteq \text{Image}(S)$

$\dim(\text{Image}(S \circ T)) \leq \dim(\text{Image}(S))$

$(\ast\ast\ast)$ $rk(S \circ T) \leq rk(S)$

$\Rightarrow (\ast\ast) \quad (\ast\ast\ast)$

$\Rightarrow rk(S \circ T) \leq \min\{rk(S), rk(T)\}$

finally w/ $\ast$ and above

$rk(S) + rk(T) - n \leq rk(S \circ T) \leq \min\{rk(S), rk(T)\}$

↑ good inequality :)

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#4) Let A be a 2×3 matrix.
Show that there cannot be any 3×2 matrix s.t. $BA = I_{3\times3}$

let T_A be a linear transformation

$$T_A : F^3 \to F^2$$

$\dim(Im(T_A)) + \dim(ker(T_A)) = 3$

$\dim(Im(T_A)) \leq 2$

$\dim(Im(T_A)) + \dim(ker(T_A)) = 3$

$3 - \dim(Im(T_A)) = \dim(ker(T_A))$

$\Rightarrow \dim(ker(T_A)) \geq 1 \quad (*)$

let T_B be a linear transformation

$$T_B : F^2 \to F^3$$

T_{BA} lin. trans.

$$T_{BA} : F^3 \longrightarrow F^3$$

$$T_{BA} = T_B \circ T_A$$

$$T_{BA} : F^3 \to F^2 \to F^3$$

$Ker(T_A)$ is subspace of $\mathbb{R}^3$

$(*)$ $\dim(ker(T_A)) \geq 1$

$\Rightarrow rk(T_A) \leq 2$ from rank nullity theorem.

$rk(T_{BA}) \leq \min\{rk(T_A), rk(T_B)\}$

$rk(T_{BA}) \leq \min\{2, 2\}$ from last inequality problem 3

$rk(T_{BA}) \leq 2$

But $rk(I_{3\times3}) = 3$

$\Rightarrow BA \neq I_{3\times3}$

Show that depending on A, it may be possible to find a 3×2 matrix B s.t $AB = I_{2\times 2}$

A is 2×3

B is 3×2

example $(x,y) \overset{B}{\mapsto} (x,y,0) \overset{A}{\mapsto} (x,y)$

$T_A \circ T_B = T_{AB} : F^2 \longrightarrow F^3 \longrightarrow F^2$

let A have full row rank | $B = A^*(AA^*)^{-1}$
$= A^T(AA^T)^{-1}$

$AB = I_{2\times 2}$

If it didn't, after applying A, the vector it sent to F^2 would not be 2 dimensional

due to $\text{rank}(A) = \dim(\text{Range}(A))$
$\text{rank}(A) = \min(\text{row rank}(A), \text{col rank}(A))$
so $\exists$ vectors $x \in F^2$ that would not equal $ABx \neq x$

$$rk(A)+rk(B)-n \le rk(AB) \le \min\{rk(A), rk(B)\} \quad \text{from problem 3}$$

$$2+2-2 \le rk(AB) \le \min\{2,2\}$$

$$2 \le rk(AB) \le 2$$

$\max\{rk(A)\} = 2$

$\max\{rk(B)\} = 2$

Because rank = row rank
= col rank

and this can be seen from dimensions of the matrices.

$rk(AB) = 2$ (if A, B are rk 2)
$rk(I_{2\times 2}) = 2$
$\Rightarrow AB = I_{2\times 2}$ exists

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5) Let $W \subset \mathbb{R}^3$ be the subspace given by $x+y+z=0$. Find a basis of W and extend it to a basis of $\mathbb{R}^3$. Find another basis of $\mathbb{R}^3$ that does not contain any vector in W.

$\vec{\text{coordinates}}$

basis is lin indp, spans space. $x+y+z=0$ is a plane in $\mathbb{R}^3$

possible basis vector $(1,-1,0)$, $(1,1,-2)$

$x+y+z=0$

lin independent extension $(1,1,1)$

This was done using cross product and knowing the normal vector of $x+y+z=0$ is $(1,1,1)$

Basis for $\mathbb{R}^3$ containing basis vectors for W.

$$\{(1,-1,0),\ (1,1,-2),\ (1,1,1)\}$$

Another basis of $\mathbb{R}^3$ that does not contain any vector in W, is the standard basis. $\mathbb{R}^3$ obviously contains W (so it is in the span of the below vectors)

$(1,0,0)$, $(0,1,0)$, $(0,0,1)$

but $x+y+z = 1 = 1 = 1 \neq 0$

for the standard basis vectors