Tyler Olivieri PS9

Chp 4 p 127 #5 H&K

let n be a positive integer and F a field. Suppose A is an $n \times n$ matrix over F and P is an invertible matrix over F. If f is any polynomial over F,

prove that $f(P^{-1}AP) = P^{-1}f(A)P$

$$f(P^{-1}AP) = \sum_{i=1}^{n} \alpha_i (P^{-1}AP)^i$$

claim $(P^{-1}AP)^i = P^{-1}A^iP$ (which I proved in a previous homework, but I am unsure which problem.

$i=0$ $(P^{-1}AP)^0 = I$ $P^{-1}A^0P = P^{-1}IP = P^{-1}P = I$

$i=1$ $(P^{-1}AP)^1 = P^{-1}AP = P^{-1}A^1P$

$i=2$ $(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P$

assume $i = n-1$ holds $(P^{-1}AP)^{n-1} = P^{-1}A^{n-1}P$

$i=n$ $(P^{-1}AP)^n = P^{-1}AP(P^{-1}AP)^{n-1} = P^{-1}APP^{-1}A^{n-1}P$

$= P^{-1}AIA^{n-1}P = P^{-1}A^nP$

now $f(P^{-1}AP) = \sum_i \alpha_i (P^{-1}AP)^i = \sum_i \alpha_i P^{-1} A^i P$

scalar will commute with matrix

$= \sum_i P^{-1} \alpha_i A^i P \quad = P^{-1}\left(\sum_i \alpha_i A^i\right) P$

$= P^{-1} f(A) P$

Tyler Olivieri 959 Hwk p 148-150 # 5

Let A be a 2x2 matrix over a field F, and

Suppose that $A^2 = 0$. Show for each scalar c that

$\det(cI - A) = c^2$

for a 2x2 matrix A

$$cI - A = \begin{bmatrix} c - a_{11} & a_{12} \\ a_{21} & c - a_{22} \end{bmatrix}$$

$A^2 = 0 \iff A$ is nilpotent

Claim

2x2 nilpotent matrices have the form $\begin{bmatrix} x & b \\ a & -x \end{bmatrix}$ where $ab = -x^2$

$$\begin{bmatrix} x & b \\ a & -x \end{bmatrix}\begin{bmatrix} x & b \\ a & -x \end{bmatrix} = \begin{bmatrix} x^2 + ab & xa - ax \\ bx - xb & ab + x^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

form $\begin{bmatrix} x & b \\ a & -x \end{bmatrix} \Rightarrow$ nilpotent matrix $\exists k$ s.t. $\text{matrix}^k = 0$

nilpotent matrix $A \Rightarrow$ A is not invertible $\Rightarrow \det(A) = 0$

Assume inverse exists, A^{-1}

$AA^{-1} = I$

$AAA^{-1} = AI$

$A^2A^{-1} = A$

$0A^{-1} = A$

$0 = A$

Contradiction.

0 is not invertible

Claim trace(A) = 0 Then it is clear that the matrix will have form $\begin{bmatrix} x & b \\ a & -x \end{bmatrix}$

as trace $= x + -x = 0$

and $\det(\text{form}) = x(-x) - (ab) = -x^2 - ab = 0$

$-x^2 = ab$

proof of claim $A^2 = 0 \Rightarrow \text{trace}(A) = 0$

$\underline{A^2 = 0}$ $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$a^2+bc = 0$ and $cb+d^2 = 0$ and $c(a+d) = 0$ (1)

$(a+d)b = 0$ (2)

Assume $a+d \neq 0$

then (1), (2) imply $c = 0$ and $b = 0$

as in any field $ab = 0 \Rightarrow a = 0$ or $b = 0$

then $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $A^2 = \begin{bmatrix} a^2 & 0 \\ \cancel{b} & d^2 \end{bmatrix} = 0$

$\Rightarrow a^2 = 0 \Rightarrow a = 0$

$\Rightarrow d^2 = 0 \Rightarrow d = 0$

$\Rightarrow a+d = 0+0 = 0$

and we arrive at a contradiction, thus $a+d = 0$

$\Rightarrow \text{trace}(A) = 0$

then $A = \begin{bmatrix} x & b \\ a & -x \end{bmatrix}$ $cI - A = \begin{bmatrix} c-x & b \\ a & c+x \end{bmatrix}$ $ab = -x^2$
$\Rightarrow x^2 = -ab$

$\det(cI-A) = (c-x)(c+x) - ab = c^2 - x^2 - ab = c^2 - x^2 + x^2$

$= c^2$

Tyler Olivieri P59 P 155-156 #2 H8K

Prove that the determinant of the Vandermonde matrix

let $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is $(b-a)(c-a)(c-b)$

by cofactor expansion

$$\det(A) = 1\begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} - a\begin{vmatrix} 1 & b^2 \\ 1 & c^2 \end{vmatrix} + a^2\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix}$$

$$= bc^2 - cb^2 - a(c^2 - b^2) + a^2(c-b)$$

$$= -ac^2 + ab^2 + a^2c - a^2b$$

claim $(b-a)(c-a)(c-b) = -ac^2 + ab^2 + a^2c - a^2b$

$(b-a)(c-a) = bc - ba - ac + a^2$

$(b-a)(c-a)(c-b) = (bc - ba - ac + a^2)(c-b)$

$= bc^2 - bac - \underline{ac^2} + \underline{a^2c} - bcb + \underline{bab} + acb \underline{+a^2b}$

by commutation in mult and addition of the field F

$= -ac^2 + ab^2 + a^2c - ac^2 + \cancel{bc^2} - b^2c - \cancel{bac} + bac$

$= -ac^2 + ab^2 + a^2c - ac^2$

thus $\det(A) = (b-a)(c-a)(c-b)$

Tyler Olivieri PS9 P 162-163 #1 Hlk

1) find inverse of $A = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}$ using

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A)$$

$\det(A) = 72$

$$\operatorname{Adj}(A) = \begin{bmatrix} -3 & -18 & 6 \\ -5 & -6 & -14 \\ 9 & -18 & -18 \end{bmatrix}^T = \begin{bmatrix} -3 & -5 & 9 \\ -18 & -6 & -18 \\ 6 & -14 & -18 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3/72 & -5/72 & 6/72 \\ -18/72 & -6/72 & -18/72 \\ 6/72 & -14/72 & -18/72 \end{bmatrix}$$

Tyler Olivieri PS9 P 162-163 # 2a) H&K

2a) Use Cramers rule to solve

$$\begin{cases} x+y+z=11 \\ 2x-6y-z=0 \\ 3x+4y+2z=0 \end{cases}$$

let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}$

$\det(A) = 11$

let $A_x = \begin{bmatrix} 11 & 1 & 1 \\ 0 & -6 & -1 \\ 0 & 4 & 2 \end{bmatrix}$ $A_y = \begin{bmatrix} 1 & 11 & 1 \\ 2 & 0 & -1 \\ 3 & 0 & 2 \end{bmatrix}$

$A_z = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{bmatrix}$

$\det(A_x) = -88$ $\det(A_y) = -77$ $\det(A_z) = 286$

$$x = \frac{\det(A_x)}{\det(A)} = 11/-88 = -1/8$$

$$y = \frac{\det(A_y)}{\det(A)} = -1/7$$

$$z = \frac{\det(A_z)}{\det(A)} = 11/286$$

Tyler Olivieri P57 H 8k p 162-163 #3

A $n \times n$ matrix A over F is skew-symmetric if $A^t = -A$. If A is a skew-symmetric $n \times n$ matrix with complex entries and n is odd, prove that $\det A = 0$.

$$\det(A) = \sum_\sigma (\operatorname{sgn} \sigma)\, A(1,\sigma 1) \cdots A(n,\sigma n)$$

$$\det(-A) = \sum_\sigma (\operatorname{sgn} \sigma)\, (-A(1,\sigma_1)) \cdots (-A(n,\sigma_n))$$

$$= \sum_\sigma (\operatorname{sgn} \sigma)\, (-1)^n A(1,\sigma_1) \cdots A(n,\sigma n)$$

$$= (-1)^n \sum_\sigma \operatorname{sgn} \sigma\, A(1,\sigma_1) \cdots A(n,\sigma n)$$

$$= (-1)^n \det(A)$$

thus $\det(A) = -\det(-A)$ if n is odd

but $\det(A^T) = \det(A)$

$\det(A^T) = -\det(-A)$ ← must be 0!

thus $\det(A^T) = 0$

$\det(A) = 0$

Tyler Olivien PS9 #4 H&K p. 162-163

An $n \times n$ matrix A over F is called orthogonal if $AA^T = I$

if A is orthogonal, show that $\det A = \pm 1$

$$\det(AA^T) = \det(I) = 1$$

$$\det(A)\det(A^T) = 1$$

$$\det(A)\det(A) = 1$$

$$\det(A)^2 = 1$$

$$\det(A) = \pm\sqrt{1} = \pm 1$$

Tyler Oliviera PS9 #2

Let $a(x), b(x) \in F[x]$ where F is a field. Let J be the set of all polynomials in $F[x]$ that are of the form $a(x)f(x) + b(x)g(x)$ with $f(x), g(x) \in F[x]$.

Let $p(x)$ be the monic polynomial in J of the smallest degree.

a) Show that $p(x)$ divides every polynomial in J, and in particular divides $a(x)$ and $b(x)$.

$p(x) \in J$: $p(x) = a(x)f(x) + b(x)g(x)$

J is an ideal. $J \subseteq F[x]$ when $h \in F(x)$, $d \in J$

$dh \in J$

$$dh(x) = [a(x)f(x) + b(x)g(x)]h(x)$$

$$(dh)(x) = a(x)f(x)h(x) + b(x)g(x)h(x)$$

$$= a(x)(fh)(x) + b(x)(gh)(x)$$

$(fh)(x) \in F[x]$ $(gh)(x) \in F[x]$

due to $F[x]$ being an algebra

$\Rightarrow (dh) \in J$

$\Rightarrow J$ is an ideal.

let $d \in J$, $p \in J$

$d = pq + r$ where $q \in F[x]$ $\quad r = 0$ or $r < \deg p$

by the division algorithm

if p is in J, $pq \in J$ by definition of ideal

and $d - pq = r \in J$

$$\underbrace{d}_{\in J} - \underbrace{pq}_{\in J}$$

$$r = a(x)f(x) + b(x)g(x) - (a(x)t(x) + b(x)m(x))$$

$$r = a(x)(f(x) - t(x)) + b(x)(g(x) - m(x))$$

$f(x) - t(x) \in F[x]$ $\quad g(x) - m(x) \in F(x)$

$\Rightarrow r \in J$

$\Rightarrow r = 0$ because if $r < \deg p(x)$. we contradict the assumption of $p(x)$ being the smallest degree in J.

$\Rightarrow d = pq$ where $d \in J$ $\quad p \in J$ $\quad q \in F(x]$

$\Rightarrow p(x)$ divides all polynomials in J.

If $p(x)$ divides all polynomials in J, $p(x)$ divides $a(x)$ and $b(x)$.. This is because $a(x), b(x) \in J$.

~~let~~ since $d(x) = a(x)f(x) + b(x)g(x)$

clearly $d \in J$. let $f(x) = 1$ $\quad g(x) = 0$

$f(x), g(x) \in F[x]$

$d(x) = a(x) \in J$

let $g(x) = 1$ $\quad f(x) = 0$ $\quad g, f \in E[x]$

$d(x) = b(x) \in J$

$\Rightarrow p$ divides all polynomials in J. $\Rightarrow a, b$ are in J. p divides a and b.

Tyler Olivier! PS9 #2 b

Show that every polynomial $q(x) \in F[x]$ that divides both $a(x)$ and $b(x)$ must also divide $p(x)$, and conclude that $p(x)$ has highest degree among all polynomials that divide both $a(x)$ and $b(x)$

$q(x)$ divides both $a(x)$ and $b(x)$ for $h(x) \in F[x]$, $k(x) \in F[x]$

$a(x) = q(x)h(x)$ $\quad b(x) = q(x)k(x)$ $\quad f(x) \in F[x]$, $g(x) \in F[x]$

$p(x) \in J$ then $p(x) = a(x)f(x) + b(x)g(x)$

$p(x) = q(x)h(x)f(x) + q(x)k(x)g(x)$

$p(x) = q(x)[h(x)f(x) + k(x)g(x)]$

$h(x)f(x) + k(x)g(x) \in F[x]$ due to $F[x]$ being an algebra.

thus $q(x)$ divides $p(x)$.

$p(x)$ has highest degree among all polynomials that divide both $a(x)$ and $b(x)$ because $\deg(p(x)) > \deg q(x)$

$\deg fg = \deg f + \deg g$.

$\deg p(x) = \deg q + \deg[hf + kg]$

either $p = q$ or $\deg p > \deg q$

Tyler Olivieri PS9 #5

let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & c \end{bmatrix} \in M_n(F)$ where $c \in F$ is a scalar

a) using row reduction, determine for which $c \in f$ there is an inverse for A and find A^{-1} for each such c.

row2 = row1 - row 2

row 3 = row 2 - row 3

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1-c \end{bmatrix}$$

row 2 = row 2 - row 3

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1-(1-c) \\ 0 & 0 & 1-c \end{bmatrix}$$

row 1 = row 1 - $k$row3

$k \in F$

$$\begin{bmatrix} 1 & 0 & 1-k(1-c) \\ 0 & 0 & 1-k(k-c) \\ 0 & 0 & 1-c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1-k(1-c) \\ 0 & 1 & 1-k(1-c) \\ 0 & 0 & 1-c \end{bmatrix}$$

A is invertible when it is row equivalent to I.

$c=0$ $\quad$ $1-c=1$

$1-0=1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c=0 \Rightarrow A$ is invertible

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

As long as $c \neq 1$, A is invertible by appropriate choice of k in the row reduction.

$c=1$ gives 0 row and thus A is not invertible.

inverse of A for a general c.

$$A^{-1} = \begin{bmatrix} \frac{-c}{-c+1} & \frac{1}{-c+1} & \frac{1}{-c+1} \\ \frac{c}{c-1} & \frac{-c}{c-1} & \frac{-1}{c-1} \\ \frac{1}{-c+1} & \frac{-1}{-c+1} & \frac{-1}{-c+1} \end{bmatrix}$$

Tyler Olivieri PS9 # 3b

Compute the determinant of A.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & c \end{pmatrix}$$ by cofactor expansion

$$\det(A) = 1\begin{vmatrix} -1 & 0 \\ 1 & c \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & c \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= (-1)(c) + 1 = 1 - c$$

3 c) Using part (b) determine for which $c \in F$ there is an inverse for A and compute A^{-1} using the formula for inverses in terms of determinants. Does this agree with part (a)

$\det(A) \neq 0 \iff A$ is invertible

$\det(A) = 1 - c$ $\quad \det(A) = 0$ when $c = 1$

$\det(A)$ is invertible for $c \neq 1$

This agrees with the answers in part (a)

Tyler Oliviari PS9 #4

Let A be a 3×2 matrix and let B be a 2×3 matrix. Find $\det(AB)$. Connect this to PS5 #4
what are possible values of $\det(BA)$?

$A^+ A = I_n$

$AB \in F^{3\times3}$

let $A' = \left[A \mid 0 \right]$ s.t. $A' \in F^{3\times3}$

$B' = \begin{bmatrix} B \\ \hline 0 \end{bmatrix}$ s.t. $B' \in F^{3\times3}$

$A'B' \in F^{3\times3}$

furthermore $AB = A'B'$ thus $\det(AB) = \det(A'B')$

$A'B' = AB + 0 \cdot 0 = AB$

$\det(A'B') = \det(A')\det(B')$

$\det B' = 0$ because it has a zero row

$\det(AB) = \det(A'B') = \det(A')\det(B') = \det(A')0 = 0$

this is related to PS5 #4 $AB \neq I_{3\times3}$

$\det(AB) = 0 \iff AB$ not invertible. $\iff AB \neq I_{3\times3}$

$\det(BA)$ can be any scalar as $BA = I_{2\times2}$ by appropriate choice of B and A by PS5 #4. It is also possible $\det(BA) = 0$ for choice of ~~BA~~ B and A.

Tyler Olivieri p59 #5

Consider the system of eq. $AX = B$ where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -3 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

a) Solve by row reduction $(A|B)$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -3 & 2 & 0 & 1 \\ 2 & 0 & 1 & 3 \end{array}\right)$$

$$\downarrow \; R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{ccc|c} -3 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 0 & 1 & 3 \end{array}\right)$$

$$\downarrow \; R_2 = R_2 + 1/3 R_1 \qquad R_3 = R_3 + 2/3 R_1$$

$$\left(\begin{array}{ccc|c} -3 & 2 & 0 & 1 \\ 0 & 5/3 & 1 & 7/3 \\ 0 & 4/3 & 1 & 11/3 \end{array}\right)$$

$$\downarrow \; R_3 = 5 \cdot R_3 \qquad R_2 = R_2 - 1 \cdot R_3$$

$$\left(\begin{array}{ccc|c} -3 & 2 & 0 & 1 \\ 0 & 5/3 & 0 & -20/3 \\ 0 & 0 & 1 & 9 \end{array}\right)$$

$$\downarrow \; R_2 = 3/5 R_2 \qquad R_1 = R_1 - 2 \cdot R_2$$

$$\left(\begin{array}{ccc|c} -3 & 0 & 0 & 9 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 9 \end{array}\right)$$

$\downarrow \quad R_1 = -\frac{1}{3} R_1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 9 \end{array}\right)$$

$$\Rightarrow \quad x = -3$$
$$y = -4$$
$$z = 9$$

b) find A^{-1} by writing $X = A^{-1}B$

$$A^{-1} = \frac{Adj(A)}{det(A)}$$

$$det(A) = 1$$

$$Adj(A) = \begin{bmatrix} 2 & -1 & -2 \\ 3 & -1 & -3 \\ -4 & 2 & 5 \end{bmatrix} = A^{-1}$$

$$X = A^{-1}b = \begin{pmatrix} 2 & -1 & -2 \\ 3 & -1 & -3 \\ -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{bmatrix} -3 \\ -4 \\ 9 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

c) Solve by Cramers rule

$\det(A) = 1$ from (b)

let $A_x = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ $A_y = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

$A_z = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$\det(A_x) = -3 \Rightarrow x = \dfrac{\det(A_x)}{\det(A)} = -3$

$\det(A_y) = -4 \Rightarrow y = -4$

$\det(A_z) = 9 \Rightarrow z = 9$