Tyler Olivieri H&K P510 P 187 #1

1) Find characteristic polynomial of T and U and characteristic values T and U

Find basis for corresponding space of characteristic vectors.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Lin transformation T w.r.t std orded basis.

Characteristic polynomial: $\det(A - cI) = (1-c)(0-c) = (-c)(1-c)$
$= c^2 - c = c(c-1)$

characteristic values $c = 0, c = 1$

$c=1$ - characteristic vector is any vector $\{(a,0)\}$
So basis is $(1,0)$

$c=0$ - charactristic vector is $\{(0,a)\}$ so $(0,1)$ forms basis

For U, the results are the same because the characteristic polynomial factors into the reals.

$A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

Characteristic polynomial: $\det(A - cI) = (2-c)(1-c) - (3)(-1)$
$= (2 - 2c - c + c^2) + 3 = c^2 - 3c + 5$

Characteristic values quadratic formula, $c_1 = \frac{3 + \sqrt{11}\, i}{2}$ $c_2 = \frac{3 - \sqrt{11}\, i}{2}$

Thus for T, there are no characteristic values, they are complex, and T is over $\mathbb{R}$ (and no characteristic vectors) for U, solve $(A - c_1 I)X = 0$

$(A - c_2 I)X = 0$

for characteristic vectors.

$$\begin{bmatrix} \frac{-1+\sqrt{11}i}{2} & 3 \\ -1 & \frac{1+\sqrt{11}i}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$X_1 = \left(\frac{1+\sqrt{11}i}{2}, 1 \right)^T$$

$$\begin{bmatrix} \frac{-1+\sqrt{11}i}{2} & 3 \\ -1 & \frac{1-\sqrt{11}i}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$X_2 = \left(\frac{1-\sqrt{11}i}{2}, 1 \right)^T$$

Which form the basis for the characteristic vector space respectively.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det(A - cI) = (1-c)(1-c) - 1 = (1 - c - c + c^2) - 1$$

$$= c^2 - 2c = c(c-2)$$ Characteristic polynomial

Characteristic values are $c_1 = 0$ $c_2 = 2$

The corresponding characteristic vectors

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad x_1 = (1, -1)^T \quad \text{for } c_1$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad x = (1, 1)^T \quad \text{for } c_2$$

They will be the same for T and u as we have real characteristic values.

Tyler Olivieri H&K P169 #3 PS10

3) Let A be an $n \times n$ triangular matrix over the field F. Prove that the characteristic values of A are diagonal entries of A.

characteristic values of A are values c s.t. $\det(A - cI) = 0$

Thm 1 sect 6.2 H&K
Definition that follows

A is upper triangular

$$A = \begin{bmatrix} a_{11} & & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & 0 & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \qquad A - cI = \begin{bmatrix} a_{11}-c & & \cdots & a_{1n} \\ & a_{22}-c & \cdots & a_{2n} \\ & 0 & \ddots & \vdots \\ & & & a_{nn}-c \end{bmatrix}$$

$$\det(A - cI) = \sum_{\sigma} (\operatorname{sgn} \sigma)(A - cI)(1, \sigma_1) \cdots (A - cI)(n, \sigma_n)$$

Prop All terms will be 0 except $(A-cI)(1,1)(A-cI)(2,2)\cdots(A-cI)(n,n)$ in the product because every other term would have a 0

then $\det(A) = (a_{11} - c)(a_{22} - c) \cdots (a_{nn} - c)$

and

$\det(A) = 0$ when $a_{11} = c, a_{22} = c, \ldots, a_{nn} = c$

$\Rightarrow$ characterstic values are the diagonal of upper triangular matrix

because $\det(A) = \det(A^T)$, we see this holds for a general triangular matrix.

Proof of prop

It is easily seen that the identity permutation gives a non-zero term.

Consider any transposition (a permutation of exchanging two elements). Then generally we have this permutation contributing $(A-cI)(1,\sigma_1)\cdots(A-cI)(1,\sigma_n)$ contributing to the determinant. two indices i, j where $i<j$ a transposition exchanging i and j leaves the elements $(A-cI)(i,j)$, $(A-cI)(j,i)$ in the determinant product.

Consider two facts after the transposition,

1) in upper triangular matrix $(A-cI)(i,j)=0$ if $i>j$
2) element $(A-cI)(j,i)$ has $j>i$ and thus is zero.

Since permutations are symmetric group under composition, it can be seen that any permutation except the identity is zero. Any transposition gives a 0 term in the det, and any permutation is a composition of transpositions.

Tyler Olivieri H&K #5 P 189 P5/0

5) Let $A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$

is A similar over $\mathbb{R}$ to a diagonal matrix?

" " " $\mathbb{C}$ " "

A is similar to diagonal matrix if it is diagonalizable, which is when A has a basis of characteristic vectors.

if there are n distinct

$A - cI = \begin{bmatrix} 6-c & -3 & -2 \\ 4 & -1-c & -2 \\ 10 & -5 & -3-c \end{bmatrix}$

$\det(A - cI) = -22c + (-c-3)(-c-1)\cdot(-c+6) - 16$

$\det(A - cI) = 0$ when $c = 2$ $\pm i$.

over $\mathbb{R}$ $\pm i$ are not characteristic values ($\pm i \notin \mathbb{R}$)

$(A - 2I)X = 0$ when $[X] = [(1, 0, 2)]$

this cannot be a basis for $\mathbb{R}^3$ as there is only one characteristic vector, thus A is not diagonalizable over $\mathbb{R}$.

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over $\mathbb{C}$, we

over $\mathbb{C}$, A has characteristic values $2, \pm i$

$(A - iI)[x] = 0$ when $x = [(3/5 + i/5,\ 3/5 + i/5,\ 1)]$

$(A - (-i)I)[x] = 0$ when $x = [(3/5 - i/5,\ 3/5 - i/5,\ 1)]$

$(A - 2I)[x] = 0$ when $x = [(1, 0, 2)]$

thus, A has a basis of charactristic vectors and can be diagonalized.

let $P = \begin{bmatrix} 1 & 3/5 - i/5 & 3/5 + i/5 \\ 0 & 3/5 - i/5 & 3/5 + i/5 \\ 2 & 1 & 1 \end{bmatrix}$

then $D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{bmatrix}$

Tyler Olivieri PS10 H&k P 189 #c

6) Let T be a linear operator on $\mathbb{R}^4$ which is represented by in the std ordered basis by

$$T_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

under what conditions is T diagonalizable?

T is diagonalizable if it has a basis of characteristic vectors for $\mathbb{R}^4$.

$\det(T_A - cI) = 0$

$\det(T_A - cI) = c^4$ (determinant of triangular matrix is product of diagonal)

$\Rightarrow c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$ are eigenvalues of T_A.

T is diagonalizable iff the eigenvectors corresponding to the eigenvalues are linearly independent. In other words find 4 lin ind solutions to $T_A X = 0$ or dim nullspace $(T) = 4$

the $0_{4\times 4}$ matrix is the only matrix with nullspace of dim 4.

$\Rightarrow a = 0, b = 0, c = 0$ for diagonalizable T.

Tyler Olivieri HW 8 PS10 P 187

Let A and B be $n \times n$ matrices over F.

Prove that if $(I-AB)^{-1}$ is invertible then $(I-BA)$ is invertible and $(I-BA)^{-1} = I + B(I-AB)^{-1}A$

if $(I-AB)$ is invertible $\quad (I-AB)(I-AB)^{-1} = I$

$(I-AB)^{-1}(I-AB) = I$

$(I-BA)(I-BA)^{-1} = I$

$(I-BA)(I+B(I-AB)^{-1}A)$

$= I + B(I-AB)^{-1}A - BA - BAB(I-AB)^{-1}A$

$= I + B(\cancel{I-AB})^{-1} - \cancel{I} - AB(I-AB)^{-1})A$

$= I + B(-I + (I-AB^{-1}) - AB(I-AB)^{-1})A$

$= I + B(-I + (I-AB)(I-AB)^{-1})A$

$= I + B(-I + I)A$

$= I + B(0)A$

$(I-BA)(I+B(I-AB)^{-1}A) = I$

$\Rightarrow (I+B(I-AB)^{-1}A) = (I-BA)^{-1}$

$\Rightarrow$ if $(I-AB)^{-1}$ is invertible, $(I-BA)$ is invertible and its inverse is $(I+B(I-AB)^{-1}A)$

Tyler Olivieri PS10 Hfk p 189 #9

Use the result of Ex 8 to prove that if A and B are $n \times n$ matrices over F, then AB and BA have precicely the same characteristic values in F.

If 0 is a characteristic value of AB

$\Rightarrow AB$ is singular

$\Rightarrow \det(AB) = 0$

$\Rightarrow \det(A)\det(B) = 0$

$\det(B)\det(A) = 0$

$\det(BA) = 0$

$\Rightarrow BA$ is singular

$\Rightarrow BA$ has characteristic value of 0.

Suppose $c \neq 0$ and is characteristic value of AB

$\det(cI - AB) = 0$

$\det(c(I - \frac{1}{c}AB)) = 0$

$c^n \det(I - \frac{1}{c}AB) = 0$

$\Rightarrow c^n \det(I - \frac{1}{c}BA) = 0$ from problem 8.

$(I - AB)$ invertible

$\Rightarrow (I - BA)$ invertible.

$\Rightarrow \det(Ic - BA) = 0$

$\Rightarrow c$ is characteristic value of BA.

Tyler Olivieri PS10 #2

Let $c_1, \dots, c_k \in F$ be distinct elements in a field F.

a) Show that there is no non-zero vector $(z_0, \dots, z_{k-1}) \in F^k$ s.t. $\sum_{j=0}^{k-1} z_j c_i^j = 0$ for all $i = 1, \dots, k$

By the fundamental theorem of algebra, a polynomial with degree $k-1$ can have at most $k-1$ distinct roots.

Thus, for a non-zero vector $(z_0, z_1, \dots, z_{k-1})$, the polynomial $\sum_{j=0}^{k-1} z_j c_i^j = 0$ for at most $k-1$ values of $c_i \in F$.

However, we have k distinct elements of $c \in F$, thus, for $\sum_{j=0}^{k-1} z_j c_i^j = 0$ to hold for all k distinct c, z must be the zero vector, as only $k-1$ values of $c_i \in F$ can make the polynomial zero.

b) Let $a_1, \dots, a_k \in F$

show that A is invertible unless $a_i = 0$

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ a_1 c_1 & a_2 c_2 & \cdots & a_k c_k \\ a_1 c_1^2 & a_2 c_2^2 & \cdots & a_k c_k^2 \\ \vdots & \vdots & & \vdots \\ a_1 c_1^{k-1} & a_2 c_2^{k-1} & \cdots & a_k c_k^{k-1} \end{pmatrix}$$

consider $A^T x$ where $x = [(z_0, \dots, z_{k-1})^T]$

then we have $\sum_{j=0}^{k-1} a_i c_i^j z_j = a_i \sum_{j=0}^{k-1} z_j c_i^j$

from part a $\sum_{j=0}^{k-1} z_j c_i^j \neq 0$ for $x \neq 0$

$\Rightarrow a_i \sum_{j=0}^{k-1} z_j c_i^j \neq 0$ unless $a_i = 0$ or $x = 0$.

therefore the only vector x to give solution

$A^T x = 0$ is $x = 0$

$\Rightarrow A^T$ is invertible. if $a_i \neq 0$ $\forall i$

$\Rightarrow A$ is invertible.

Tyler Olivieri PS10 # 3

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ take $(a,b) \mapsto (17a - 30b, 9a - 16b)$

a) Find the matrix of T wrt std basis e_1, e_2

$$T_A = \begin{bmatrix} 17 & -30 \\ 9 & -16 \end{bmatrix}$$

b) find the matrix of T wrt basis $f_1 = (1,1)$, $f_2 = (1,-1)$

$Tf_1 = (-13, -7) = -10 f_1 - 3f_2$

$Tf_2 = (47, 25) = 36 f_1 + 11 f_2$

$$T_B = \begin{bmatrix} -10 & 36 \\ -3 & 11 \end{bmatrix}$$

c) Find a basis of $\mathbb{R}^2$ for which the matrix of T is the diagonal matrix. $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$

characteristic values, c, of T $\det(T - cI) = 0$

characteristic values of T_A $\det(T_A - cI) = 0$

$(17-c)(-16-c) - 9(-30) = 0$

$(17-c)(-16-c) - 270 = 0$

$(c-2)(c+1) = 0$

$c = 2$ $c = -1$ are characteristic values.

$(T_A - (-1)I)\, X = 0 \qquad X = (5,3)^T$

$(T_A - 2I)\, X = 0 \qquad X = (2,1)^T$

thus let $P = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$D = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

d) there is no basis of $\mathbb{R}^2$ which T has matrix

$T_2 \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. determinant of T is invariant to choice of basis, or similar matrices have the same det.

$\det(T_A) = -2 \qquad \det(D) = -2 \qquad \det(T_2) = 12$

thus $[T]_{[\beta]} \neq \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ or it cannot be diagonalized into that form with any choice of basis

Tyler Olivieri PS10 #4

Let P_2 be the real vector space of polynomials in $\mathbb{R}[x]$ of degree at most 2. Let $T: P_2 \to P_2$ be the lin. trans. given by $T(f) = g$ where $g(x) = (x+1) f'(x)$

Find the eigenvalues and eigenvectors of T.

1) Find the matrix of T relative to the basis $\beta \; \{1, x, x^2\}$

$$T_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$T(1) = (x+1)(0) = 0 = 0(1) + 0(x) + 0(x^2)$

$T(x) = (x+1)(1) = x+1 = 1(1) + 1(x) + 0(x^2)$

$T(x^2) = (x+1)(2x) = 2x^2 + 2x = 0(1) + 2(x) + 2(x^2)$

eigenvalues c $\det(T_\beta - cI) = 0$

since T_β is triangular, $\det(T_\beta - cI)$ is product of diagonal

$\det(T_\beta - cI) = c(1-c)(2-c)$ eigenvalues $c_1 = 0$ $c_2 = 1$ $c_3 = 2$

eigenvector for $c_1 = 0$

$v_1 = (1, 0, 0)^T$

$c_2 = 1$

$v_2 = (1, 1, 0)^T$

$c_3 = 2$ $v_3 = (1, 2, 1)^T$

ii) Instead use seperation of variables to solve the differential eq $(x+1)\,dy/dx = cy$ where c is a constant,

$$\frac{(x+1)}{dx} = \frac{cy}{dy}$$

$$\frac{dx}{(x+1)} = \frac{dy}{cy}$$

$$\int \frac{dx}{(x+1)} = \int \frac{dy}{cy}$$

$Ty = (x+1)\,dy/dx$

$Ty = cy$

thus, solution y is an eigenvector with corresponding eigenvalue c.

$$\log(x+1) + k_1 = c\log(y) + k_2$$

$k_3 = k_1 - k_2$

$$\log(x+1) + k_3 = c\log(y)$$

$$\exp\left(\frac{\log(x+1) + k_3}{c}\right) = y$$

eigenvector

eigenvalue c

$$c = \frac{\log(x+1) + k_3}{\log(cy)}$$

Tyler Olivieri PS10 #5

a) Show that if $T: V \to V$ is a linear transformation and $v \in V$ is an eigenvector with eigenvalue c, then v is also an eigenvector for T^K, with eigenvalue c^K.

v is an eigenvector for T. w/ corresponding eigenvalue c.

base case

$Tv = cv$

assume v is an eigen vector for T^{k-1}, with eigenvalue c^{k-1}

$T^{k-1}v = c^{k-1}v$

show $T^K v = c^K v$ then v is an eigenvector of T with corresponding eigenvalue T.

$$T(T^{k-1}v) = T(c^{k-1}v)$$

$$T^k v = c^{k-1}T(v) = c^{k-1}cv = c^k v$$

$$T^k v = c^k v$$

Thus, proof is concluded by induction.

b) use this to find the eigenvalues and eigenvectors of A^{253},

where $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix}$

$\det(A - cI) = 0$ gives eigenvals of A.

$= -c(-c-1)(-c+1)$ det is product of diagonal for triangular matrix.

$c_1 = 0$, $c_2 = 1$, $c_3 = -1$ are eigenvalues.

eigenvalues of A^{253} are $c_1 = 0^{253} = 0$ $c_2 = 1^{253} = 1$

$c_3 = (-1)^{253} = -1$

eigenvectors of A are

$AV_1 = 0$ $V_1 = (-2, 1, 0)^T$ $c_1 = 0$

$(A - I)V_2 = 0$ $V_2 = (1, 0, 0)^T$ $c_2 = 1$

$(A + I)V_3 = 0$ $V_3 = (-7, 4, 2)^T$ $c_3 = -1$

eigenvectors of A^{253} are V_1, V_2, V_3 as listed above