Tyler Olivieri PS7 H8K p 105

1) In $\mathbb{R}^3$, let $\alpha_1 = (1,0,1)$ $\alpha_2 = (0,1,-2)$, $\alpha_3 = (-1,-1,0)$

a) If f is a linear functional on $\mathbb{R}^3$ s.t.

$$f(\alpha_1) = 1 \qquad f(\alpha_2) = -1 \qquad f(\alpha_3) = 3$$

$\alpha = (a,b,c)$ find $f(\alpha)$

$$1x_1 + 0x_2 + 1x_3 = 1$$

$$0x_1 + 1x_2 - 2x_3 = -1$$

$$-1x_1 - 1x_2 + 0x_3 = 3$$

$$\Rightarrow \quad x_1 = 4 \quad x_2 = -7 \quad x_3 = -3$$

$$f(\alpha) = 4a - 7b - 3c$$

b) Describe explicitly the linear functional f on $\mathbb{R}^3$ s.t

$$f(\alpha_1) = f(\alpha_2) = 0 \quad \text{but} \quad f(\alpha_3) \neq 0$$

$$x_1 + x_3 = 0 \quad \Rightarrow \quad x_3 = -x_1$$

$$x_2 - 2x_3 = 0 \quad \Rightarrow \quad x_2 = -2x_1$$

$$-x_1 - x_2 \neq 0 \quad \Rightarrow \quad x_1 \neq 0$$

$$\Rightarrow x_1 \neq 0 \quad x_2 = 2x_1 \quad x_3 = -x_1$$

$$\Rightarrow f(a) = 1x_1 - 2x_2 - x_3$$

$$\Rightarrow x_1 \neq 0 \quad x_2 = -2x_1 \quad x_3 = -x_1$$

c) Let f be any linear functional s.t. $f(\alpha_1) = f(\alpha_2) = 0$

and $f(\alpha_3) \neq 0$

If $\alpha = (2, 3, -1)$ show that $f(\alpha) \neq 0$

$$f(\alpha) = 1(2) - 2(3) - 1(-1) = 2 - 6 + 1 = 3 - 6 = -3 \neq 0$$

from functional found in (b)

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2) Let $\beta = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for C^3 defined by

$\alpha_1 = (1,0,-1)$ $\alpha_2 = (1,1,1)$ $\alpha_3 = (2,2,0)$

Find the dual basis of β

dual basis $f_i(\alpha_j) = \delta_{ij}$

$f_1(x)$ is solution to

$$x_1 - x_3 = 1$$

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 2x_2 = 0$$

$$\Rightarrow x_1 = 1 \quad x_2 = -1 \quad x_3 = 0$$

$$f_1(a) = a_1 - a_2$$

$f_2(x)$ is sol. to

$$x_1 - x_3 = 0$$

$$x_1 + x_2 + x_3 = 1$$

$$2x_1 + 2x_2 = 0$$

$$\Rightarrow x_1 = 1 \quad x_2 = -1 \quad x_3 = 1$$

$$f_2(a) = a_1 - a_2 + a_3$$

$f_3(x)$ is sol to

$$x_1 - x_2 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 2x_2 = 1$$

$$\Rightarrow x_1 = -\tfrac{1}{2} \quad x_2 = 1 \quad x_3 = -\tfrac{1}{2}$$

$$f_3(a) = -\tfrac{1}{2}a_1 + a_2 - \tfrac{1}{2}a_3$$

$\{f_1, f_2, f_3\}$ is the dual basis

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3) If A and B are $n \times n$ matrices over F,
Show that $\text{trace}(AB) = \text{trace}(BA)$.

$\text{trace}(A) = \sum_i [A]_{ii}$

ii entry of AB / dot row col

$\text{trace}(AB) = \sum_i [AB]_{ii} = \sum_i \vec{a}_i \cdot \vec{b}_i$

dot product is commutative

$= \sum_i \vec{b}_i \vec{a}_i = \sum_i [BA]_{ii} = \text{trace}(BA)$

Now show that similar matrices have the same trace.

$B = C^{-1}AC$ B, A similar

$\text{trace}(B) = \text{trace}(C^{-1}AC)$

$= \text{trace}(C^{-1}(AC))$ from above

$= \text{trace}((AC)C^{-1})$

$= \text{trace}(AI)$

$\text{trace}(B) = \text{trace}(A)$

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#4 p106

Let V be the v.s of all polynomial functions p from R into R which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)\,dx \quad f_2(p) = \int_0^2 p(x)\,dx \quad f_3(p) = \int_0^{-1} p(x)\,dx$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is the dual

find p_1, p_2, p_3 s.t.

$$f_i(p_j) = \delta_{ij}$$

$$f_1(p_1) = \int_0^1 p_1(x)\,dx = 1 \qquad f_1(p_2) = \int_0^1 p_2(x)\,dx = 0$$

$f_1(p_3)$

Tyler Ollivieri H&X p 111 #5

If A and B are $n \times n$ complex matrices,

show that $AB - BA = I$ is impossible.

look at trace of $(AB - BA)$ and trace I

$$\text{trace}(AB - BA) = \sum_i [AB - BA]_{ii} = \sum_i [AB]_{ii} - [BA]_{ii}$$

$$= \text{trace}(AB) - \text{trace}(BA) = \text{trace}(AB) - \text{trace}(AB) = 0$$

from problem 3.

but if I is $n \times n$ trace $I = n$

therefore, if we assume equality

$$AB - BA = I$$

we arrive at a contradiction, because the diagonals of the LHS and RHS are different, therefore, not equal.

1) p111

Let n be a positive integer and F a field. Let W be the set of all vectors $(x_1, x_2, \ldots, x_n)$ in F^n s.t

$$x_1 + x_2 + \cdots + x_n = 0 \quad (*)$$

a) Prove that W^0 consists of all linear functionals f of the form $f(x_1, x_2, \ldots, x_n) = c \sum_{j=1}^{n} x_i$

W^0 is by definition the set of lin functionals s.t. $f(\alpha) = 0$ for every $\alpha \in W$.

STS $f(x_1, x_2, \ldots, x_n) = c \sum_{j=1}^{n} x_i = 0$ for every $x \in W$

let $x = (x_1, \ldots, x_n) \in W$ $(*)$

$f(x) = f(x_1, \ldots, x_n) = c \sum_{j=1}^{n} x_i = c(0) = 0$

thus $f(x) = c \sum_{j=1}^{n} x_i$ is the anhilator of W,

W^0

it is also linear

$$f(ax_1 + x_2) = c \sum_{j=1}^{n} ax_{j1} + x_{j2} = ca \sum_{j=1}^{n} x_{j1} + \sum_{j=1}^{n} x_{j2}$$

$$= ac \sum_{j=1}^{n} x_{j1} + \sum_{j=1}^{n} x_{j2}$$

$$= af(x_1) + f(x_2)$$

Tyler Olivieri 857 H & K p111 #1 b

Show that the dual space W^* of W can be

'naturally' identified with the linear functionals.

$$f(x_1, x_2, \cdots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

on F^n which satisfy $c_1 + \cdots + c_n = 0$

we can identify elements in W^* with a linear functional

$f(x_1, \cdots, x_n) = c_1 x_1 + \cdots c_n x_n$ where $\sum c_i = 0$

because $W \longrightarrow W^*$ is an isomorphism.

since we have this isomorphism, the dual space

can be identified from W.

W^* is described through the dual basis which are

linear functionals.

Tyler Olivieri PS1

#2) Let m, n be positive integers and consider vectors $v_1, \dots, v_m \in \mathbb{R}^n$

Show that these vectors are linearly independent as vectors in $\mathbb{R}^n$ over the field of scalars $\mathbb{R}$ iff they are linearly independent as vectors in $\mathbb{C}^n$ over the field of scalars $\mathbb{C}$.

WTS that $\sum_{i=1}^{m} c_i v_i = 0$ with $c_i = 0 \ \forall i$ $c_i \in \mathbb{R}$

$$\Updownarrow$$

$\sum_{i=1}^{m} c_i v_i = 0$ with $c_i = 0 \ \forall i$ $c_i \in \mathbb{C}$

Assume $v_1, \dots, v_m \in \mathbb{R}^n$ are linearly independent in $\mathbb{C}^n$ over $\mathbb{C}$

then $\sum_{i=1}^{m} c_i v_i = 0$ with $c_i = 0 \ \forall i$ $c_i = a + bi \in \mathbb{C}$

if $c_i = 0$ both $a = b = 0$

then, c_i can be viewed as $0 \in \mathbb{R}$ because there is no imaginary component.

Since $v_i \in \mathbb{R}$, $c_i = 0 \in \mathbb{R}$

$\sum_{i=1}^{m} c_i v_i = 0$ is linearly independent in $\mathbb{R}$

$\therefore v_1, \dots, v_m \in \mathbb{R}^n$ lin ind $\in \mathbb{C}^n$ over $\mathbb{C}$ $\Rightarrow$ $v_1, \dots, v_m \in \mathbb{R}^n$ lin ind $\in \mathbb{R}^n$ over $\mathbb{R}$

In $\mathbb{C}$, there is more freedom to find a non-zero solution to for c_i

$\sum_{i=1}^{m} c_i v_i = 0$

if a solution cannot be found except $c_i = 0$, further restricted the set of possible c_i's will certainly not give a solution

Now Assume $v_1, \dots, v_m \in \mathbb{R}^n$ are vectors in $\mathbb{R}^n$ over $\mathbb{R}$.
and are lin independent

$$\sum_{i=1}^{m} c_i v_i = 0 \qquad c_i = 0 \quad \forall i \quad c_i \in \mathbb{R}$$

hint you can tell if rows are lin independent if # rows = # pivots because row rank = col rank or no zero row

Consider the matrix $\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$

for any matrix with linear independent rows can be row reduced to a matrix R with the first non-zero entry of each row being 1 and each column containing the leading non-zero entry of some row has all its other entries 0. In other words, a lin ind set of vectors in $\mathbb{R}^n$ is equivalent to considering $\{e_i\}$ for some set of pos. integer i

Consider extending the field to $\mathbb{C}$. The test for lin independence stays the same.

$$\sum_{i=1}^{m} c_i v_i = 0 = 0 + 0i \quad c_i \in \mathbb{C}$$

$$\sum_{i=1}^{m} (a + bi) v_i = 0 + 0i$$

$$\sum_{i=1}^{m} a v_i + i b v_i = 0 + 0i$$

$$\sum_{i=1}^{m} a v_i + i \sum_{i=1}^{m} b v_i = 0 + 0i$$

but $a, b \in \mathbb{R}$, thus we know from the vectors being lin ind in $\mathbb{R}^n$, $\sum_{i=1}^{m} a v_i = 0$ $\sum_{i=1}^{m} b v_i = 0$ with $a_i = b_i = 0$

$$\Rightarrow \sum_{i=1}^{m} (a + bi) v = 0 \quad \text{with} \quad (a + bi) = 0$$

$\Rightarrow$ vectors are lin independent in $\mathbb{C}^n$ over $\mathbb{C}$

Tyler Olivieri PS7 #3

Let V be a v.s, Let W be a subspace of V.

Let S be a subset of V

a) if S is a linearly independent subset of V, must $S \cap W$ be a linearly independent subset of W?

Yes. The intersection of $S \cap W$ will contain either all of S, if $S \subset W$ or a subset of S.

In the first case, when $S \subset W$ and the intersection $S \cap W = S$ then the intersection is certainly linearly independent because S is assumed to be a linearly independent subset. and contained in W.

In the other scenario, $S \cap W$ will be always it will be a subset of S, which will still be linearly independent.

Any subset of a linearly independent set is linearly independent. H&K Section 2.3.

b) If S spans V, must $S \cap W$ span W?

Yes If S Spans V,

$W \subset V = span(S)$ because W is subspace of V.

$W = span(W) \subset span(S) = V$

$span(W) \subset span(S \cap W)$ due to

$span(W) = W$ of
$span(W) \subset span(S)$
restricting S to W
is at least as big
as $span(W)$.

So $S \cap W$ must span W

if S spans V and $W \subset V$

Tyler Olivien PS7

3 c)

If S is a basis of V, must $S \cap W$ be a basis of W.

If S is a basis of V, S is lin independent and S spans V

$S \cap W$ will be a bais of W iff $S \cap W$ is lin independent set in W. $S \cap W$ spans W.

from part a, if S is a linearly independent subset of V, $S \cap W$ will be a linearly independent subset of W.

Since S is a bais of V, S is a linearly independent subset of V. Therefore, $S \cap W$ is a linearly indepent subset of W.

from part b, if S spans V, $S \cap W$ spans W.

Since S is a basis, it spans V, so $S \cap W$ spans W.

Now $S \cap W$ is a linearly independent subset of W

$S \cap W$ spans W

$\Rightarrow S \cap W$ is a basis of W.

d) If $ann(W) \subset ann(S)$, must $S \subset W$?

$ann(W) \subset ann(S)$

$\Rightarrow ann\, W = ann\, W \cap ann\, S$

$= ann(W \cup S)$ not sure if equality holds here

$\subset ann(span(W \cup S))$

$= ann(W+S)$ ← subspace

Since W is a subspace of V. (assuming V is finite dimensional)

$\dim ann(W) = \dim V - \dim W$

is $W+S$ a subspace? i think so

$\dim(ann(W+S)) = \dim V - \dim(W+S)$

$\Rightarrow \dim(W) \geq \dim(W+S)$

but $W \subset W+S$

$\therefore W = W+S$

$W+S \supset S$

$\Rightarrow S \subset W$

not quite sure about this one?

Tyler Olivieri PS7 #4

If X, Y are subspaces of a v.s V, write $V = X \oplus Y$ if every element $v \in V$ can be written in exactly one way as

$v = x + y$ with $x \in X$ and $y \in Y$.

a) If $V = \mathbb{R}^3$ and X is the x-axis, find a subspace $Y \subset V$ s.t. $V = X \oplus Y$. Find the dimensions of $V, X, Y, V^*, \text{ann}(X), \text{ann}(Y)$. What relationships do you notice among these dimensions?

clearly, $\dim(V) = 3$ ($\mathbb{R}^3$ has dim 3)

$X = \{ v \in V \mid v = (a, 0, 0), a \in \mathbb{R} \}$

$\Rightarrow \dim(X) = 1$

then if $V = X \oplus Y$, $v = x + y$ for $x \in X$ $y \in Y$

$Y = \{ v \in V \mid v = (0, b, c), b, c \in \mathbb{R} \}$

$\Rightarrow \dim(Y) = 2$

$\cancel{V} = X + Y = \{ x + y ; (a, 0, 0) + (0, b, c) \} = \{ v ; (a, b, c) \}$

furthermore, $0 \in V$ $x \in X$, $y \in Y$

$x + y = 0$

$(a, 0, 0) + (0, b, c) = 0$

$\Rightarrow a = b = c = 0$

$x = 0, y = 0$ (or $X \cap Y = \{0\}$)

$\Rightarrow V = X \oplus Y$

$\dim(V^*) = \dim(V) = 3$

dim(ann

$\dim(\text{ann}(X)) = ?$

Since X is a subspace of V

$\dim V = \dim(X) + \dim(\text{ann}(X))$

$3 = 1 + \dim(\text{ann}(X))$

$\dim(\text{ann}(X)) = 2$

Similarly for Y.

$\dim V = \dim(Y) + \dim(\text{ann}(Y))$

$3 = 2 + \dim(\text{ann}(Y))$

$\dim(\text{ann}(Y)) = 1$

$\dim(X) < \dim(Y)$ but $\dim(\text{ann}(Y)) > \dim(\text{ann}(X))$

$\dim(X) + \dim(Y) = \dim(V)$

4b) Let V be any finite dim v.s with subspaces X, Y.
Show that $V = X \oplus Y$ iff the following two conditions hold:

$X + Y = V$ and $X \cap Y = 0$

Assume $V = X \oplus Y$ i.e. every element $v \in V$ is written in exactly one way $x + y$ for $x \in X$ $y \in Y$

clearly, if $V = X \oplus Y$, $V = X + Y$ because $X \oplus Y$ is a more stringent condition.

$V = X + Y$ is the set of all possible sums $x + y$.

If $V = X \oplus Y$ holds, there is at least one sum, and therefore $V = X + Y$. $\boxed{V = X \oplus Y \Rightarrow V = X + Y}$

Now, if $V = X \oplus Y$, $X \cap Y = 0$. This comes from the fact that for ~~x every~~ every vector $v \in V$ $v = x + y$ in exactly one way.

Assume $X \cap Y \neq 0$ then $\exists$ $x_1, a \in X$ $y_1, a \in Y$ $x, y \subset V$

$x_1 + y_1 = a$

$x_1 + y_1 - a = 0$

$(x_1 - a) \in X$ $(y_1 - a) \in Y$

$(x_1 - a) + y_1 = 0$

$(y_1 - a) + x_1 = 0$

$\Rightarrow V \neq X \oplus Y$ because $0 \in V$ is not described in exactly one way

$X \cap Y \neq 0 \Rightarrow V \neq X \oplus Y$ by contrapositive

$V = X \oplus Y \Rightarrow X \cap Y = 0$

so $V = X \oplus Y \Rightarrow V = X + Y$ and $X \cap Y = 0$

Now assume $X+Y=V$ and $X\cap Y=0$ WTS $V=X\oplus Y$

clearly for every $v\in V$, there exists $x\in X$, $y\in Y$ $v=x+y$ in exactly one way
s.t $v=x+y$ from $V=X+Y$ but the sum is not gaurenteed to be unique.

consider having ~~a~~ vector $v\in V$

$$v=x_1+y_1 \qquad v=x_2+y_2 \qquad x_1,x_2\in X \quad y_1,y_2\in Y.$$

then

$$x_1+y_1=x_2+y_2$$

$$x_1=x_2+y_2-y_1$$

$$y_1=x_2+y_2-x_1$$

this shows that vectors x_1 is a linear combination of vectors in X and Y.

similarly, for y_1

$$\Rightarrow X\cap Y\neq\{0\}$$

and we arrive at a contradiction to our assumption.

therefore, $v=x+y$ in exactly one way from

$X+Y=V$ $X\cap Y=0$

$X+Y=V$ and $X\cap Y=0 \Rightarrow V=X\oplus Y$

and finally

$$V=X\oplus Y \Leftrightarrow X+Y=V \text{ and } X\cap Y=0$$

b) Explain what $\varphi_{V,B}$ does to each basis vector of V. Tyler Olivieri PS

and show that $\varphi_{V,B} : V \to V^{**}$ is an isomorphism.

Also show that $\varphi_{V,B}$ is the same as the isomorphism $ev : V \to V^{**}$

given by $v \to ev_v$ where $ev_v(f) = f(v)$ for $f \in V^*$.

(Hint: Show $\varphi_{V,B}(v_i) = ev_{v_i}$ for all i). Then deduce that $\varphi_{V,B}$

does not depend on the choice of basis β (and in that sense

is "natural")

$\varphi_{V,B}$ takes basis vectors in V and evaluates the vector on

the corresponding basis vector in V^*, giving the ith cordinate

of a vector.

The proof that $\varphi_{V,B} : V \to V^{**}$ is an isomorphism is given

in H&K Theorem 17 in 3.6

They show $\varphi_{V,B}$ is linear.

let $\gamma_{v,\beta} = c\alpha + \beta$, for each f in V^*

$$\varphi_{\gamma,\beta}(f) = f(\gamma) = f(c\alpha + \beta) = f(c\alpha) + f(\beta) = cf(\alpha) + f(\beta)$$

$$= c\varphi_{\alpha,\beta}(f) + \varphi_{\beta,\beta}(f)$$

so $\varphi_{V,B} : V \to V^{**}$ is a linear transformation.

and $\varphi_{V,B} = 0$ iff $v = 0$ $\Leftrightarrow$ non-singular

furthermore, $\dim V^{**} = \dim V^* = \dim V$ $\Leftrightarrow$ invertible transformation

$\Leftrightarrow$ isomorphism

Tyler Olivieri P57

4c) Let V be a fdvs w/ subspaces X, Y s.t.

$V = X \oplus Y$

i) Show that if A is a basis of X and B is a basis of Y, then $A \cup B$ is a basis of V.

Since $V = X \oplus Y$, $X \cap Y = 0$ (proved in 4b)

therefore, X and Y are disjoint subspaces.

furthermore, $\forall x \in X$ $x \notin \text{span}(Y)$

$\forall y \in Y$ $y \notin \text{span}(X)$

the basis A, B are linearly independent sets, and due to the disjoint property of the subspaces,

$A \notin \text{span } B$, $B \notin \text{span } A$

$\Rightarrow$ $A \cup B$ is a linearly independent set.

furthermore since $V = X \oplus Y$ $V = x + y$ for $x \in X$, $y \in Y$

$x \in \text{span}(A)$, $x \in \text{span}(A \cup B)$

$y \in \text{span}(B)$, $y \in \text{span}(A \cup B)$

$x + y \in \text{span}(A \cup B)$

$\Rightarrow$ $A \cup B$ spans V

$\Rightarrow$ $A \cup B$ is a basis for V.

ii) Prove that the numerical relationships you noticed in part a hold.

Prove $\dim V = \dim X + \dim Y$ if $V = X \oplus Y$

We know $\dim(V) = \#$ elements in bais.

Since we know the basis for V is the union of the basis of X and Y., then

$\dim V = \dim X + \dim Y$.

as the length of the union of the disjoint bases will be the length of the first basis + length of the second basis.

4c iii) Show that $V^* = ann(X) \oplus ann(Y)$ Tyler Olivieri PS7

$\Uparrow$

STS $V^* = ann(X) + ann(Y)$ and $ann(X) \cap ann(Y) = 0$

we know there is a unique dual basis A^*, B^* for X, Y.

A unique dual basis $(A \cup B)^*$ for V

we know $\dim V = \dim V^*$

$\dim X + \dim(ann(X)) = \dim V$ } H & K

$\dim Y + \dim(ann(Y)) = \dim V$

$\dim X + \dim Y = \dim V$. part (4c ii)

$\Rightarrow \dim(ann(X)) = \dim Y$

$\Rightarrow \dim(ann(Y)) = \dim(X)$

$\Rightarrow \dim(ann(X)) + \dim(ann(Y)) = \dim V$

$\Rightarrow \dim(ann(X)) + \dim(ann(Y)) = \dim V^*$ (1)

Since $ann(X), ann(Y) \in V^*$ we have that

basis $(A \cup B)^*$ is concatenation of basis A^* and B^*

$\Rightarrow (A \cup B)^* = A^* \cup B^*$

and $f_v = f_x + f_y$ for $f_v \in V^*$, $f_x \in ann(X)$, $f_y \in ann(Y)$

$\Rightarrow V^* = ann(X) + ann(Y)$

but also $\dim V^* = \dim(ann(X) + ann(Y)) = \dim(ann(X)) + \dim(ann(Y)) - \dim(ann(X) \cap ann(Y))$

and from (1)

$\Rightarrow \dim(ann(X) \cap ann(Y)) = 0$

$\Rightarrow ann(X) \cap ann(Y) = 0$

$\Rightarrow V^* = ann(X) \oplus ann(Y)$

Tyler Olivieri PS7

5) For any fd vs V with basis $\beta = \{v_1, \dots, v_n\}$ and corresponding dual basis $\beta^* = \{f_1, \dots, f_n\}$ of V^*, define $\phi_{V,\beta} : V \to V^*$ by $\sum_{i=1}^{n} a_i v_i \longmapsto \sum_{i=1}^{n} a_i f_i$.

In particular, since V^* is finite dimensional with basis β^*, we can also consider the map $\phi_{V^*,\beta^*} : V^* \to V^{**}$.

Let $\varphi_{V,\beta} = \phi_{V^*,\beta^*} \circ \phi_{V,\beta} : V \to V^{**}$

a) Show that $\phi_{V,\beta} : V \to V^*$ is an isomorphism, but it depends on choice of basis β. [Hint: For the second part, choose two different bases β, β' of some vector space V; e.g. take V to be the one-dimensional space $\mathbb{R}$. Then compare the two maps $\phi_{V,\beta}$ and $\phi_{V,\beta'}$, and verify they are not the same.]

Two vector spaces are isomorphic if there is an invertible linear map from one vector space onto the other one. the map $\phi_{V,\beta}$ is clearly linear (sending linear combination of basis vectors to a linear combination of basis vectors), where there is a unique linear map between the basis vectors since $\phi_{V,\beta}$ sends $\sum_{i=1}^{n} a_i v_i \longmapsto \sum_{i=1}^{n} a_i f_i$. It is clearly surjective because $\{f_1, \dots, f_n\}$ span V^* and it is injective because $\{f_1, \dots, f_n\}$ are linearly independent. Thus $\phi_{V,\beta}$ is bijective $\Longrightarrow$ $\phi_{V,\beta}$ is an isomorphism.

To show it depends on the choice of basis we consider the one-dimensional v.s $\mathbb{R}$. with two different basis for $V = \mathbb{R}$

$\beta = \{1\}$ $\beta' = \{2\}$

Clearly, $\beta \neq \beta'$

$\phi_{V,\beta} : V \to V^*$ $a_i \beta_1 \mapsto a_i f_1$

$f_1(\beta_1) = 1$

$f_1(1) = 1$

$f_1(a) = a$

$\phi_{V,\beta'} : V \to V^*$ $a_i \beta_1' \mapsto a_i f_1'$

$f_1'(\beta_1') = 1$

$f_1'(2) = 1$

$f_1'(a) = 1/2 a$

clearly $f_1(a) \neq f_1'(a)$

So we have two different isomorphisms from two different choice of Basis β, β'

$\psi_{v_i,\beta}(f) = f(v_i)$ $\qquad ev_{v_i}(f_i) = f(v_i)$ $\qquad$ function of vector, not basis?

$$\psi_{V,\beta}(a) = \psi_{V,\beta}\left(\sum_{i=1}^{n} a_i v_i\right) = \phi_{V^*,\beta^*} \circ \phi_{V,\beta} : V \to V^{**}$$

$$= \phi_{V^*,\beta^*}\left(\phi_{V,\beta}\left(\sum_{i=1}^{n} a_i v_i\right)\right)$$

$$= \phi_{V^*,\beta^*}\left(\sum_{i=1}^{n} a_i f_i\right) = \sum_{i=1}^{n} a_i \phi_{V^*,\beta^*}(f_i)$$

$$= \sum_{i=1}^{n} a_i L_i(f_j)$$

$\llcorner$ evalutes to 1 or 0.

$$= (a_1, a_2, \dots, a_n) = a$$

which doesn't depend on choice of basis

for a different choice of basis β' we induce $ev : V \to V^{**}$

same as previous case

$$ev_a(a) = ev\left(\sum_{i=1}^{n} a_i v_i'\right) = \phi_{V^{*\prime},\beta^{*\prime}} \circ \phi_{V,\beta'} : V \to V^{**}$$

$$= \phi_{V^{*\prime},\beta^{*\prime}}\left(\sum_{i=1}^{n} a_i f_i'\right)$$

$$= \sum_{i=1}^{n} a_i L_i'(f_j') = (a_1, a_2, \dots, a_n) = a$$