Tyler Oliv... PS6 1) H & K Pg 83 #3

3) Let T be a linear operator on $\mathbb{R}^3$ defined by

$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$

Is T invertible?

$T(v) = 0$ $v = 0$ only sol

$\Leftrightarrow$ T is invertible

(1) $3x_1 = 0$ $\Rightarrow x_1 = 0$

(2) $x_1 - x_2 = 0$

(3) $2x_1 + x_2 + x_3 = 0$

then (2) becomes $0 - x_2 = 0$

$\Rightarrow x_2 = 0$

then (3) becomes $2 0 + 0 + x_3 = 0$

$\Rightarrow x_3 = 0$

T has $\ker(T) = \{0\}$

$\dim(\ker(T)) = 0$

by rk-nullity

$n = rk(T) + \ker(T)$

$\Rightarrow rk(T) = n$

$\Rightarrow T$ is surjective and injective

$\Rightarrow$ invertible

Tyler Olivieri PS6

H&K p 84 5

Let $C^{2\times 2}$ be the complex vector space of 2x2 matrices w/ complex entries.

Let $B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$ and let T be a lin operator on $C^{2\times2}$ defined by $T(A) = BA$

what is the rank(T)?

$$\text{rank}(T) = \dim(\text{Img}(T)) \le \dim(C^{2\times 2}) = 4$$

$$\text{rank}(T) = 4 - \dim(\text{null}(T))$$

$$\text{null}(T) = \{ A \in C^{2\times2} \mid BA = 0 \}$$

we know the form of BA. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$BA: \quad A_{11} \longmapsto A_{11} - 4A_{21}$$
$$A_{21} \longmapsto -A_{11} + 4A_{21}$$
$$A_{12} \longmapsto A_{12} - 4A_{22}$$
$$A_{22} \longmapsto -A_{12} + 4A_{22}$$

$$\Rightarrow D = \begin{bmatrix} 1 & -4 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

$$\text{RREF}(D) = \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$\text{rank}(D) = 2$

$\Rightarrow \text{rank}(T) = 2$

b

$T^2(A) = T(T(A)) = T(BA) = B^2A$

Tyler Olivares PSC #7 p 85 H&K

7) Let V and W be vector spaces over the field F and let U be an isomorphism of V onto W. Prove that $T \mapsto UTU^{-1}$ is an isomorphism of $L(V,V)$ onto $L(W,W)$

STS that the function that sends T to UTU^{-1} is invertible for it to be an isomorphism.

let this function be f

so $f(T) = UTU^{-1}$

if f is invertible $\Leftrightarrow$ f is 1 to 1 (isomorphic)

$$f(cT_1+T_2) = U(cT_1+T_2)U^{-1} = (UcT_1+UT_2)U^{-1}$$

$$= (cUT_1+UT_2)U^{-1} = cUT_1U^{-1} + UT_2U^{-1}$$

$$= cf(T_1) + f(T_2)$$

thus f is linear.

since $U: V \longrightarrow W$

then $U^{-1}: W \longrightarrow V$

since $f(T)$ is linear, we can view UTU^{-1} as composition of linear transformations.

to show f is invertible

$\exists$ some function f^{-1} s.t $f^{-1}f = I$

$ff^{-1} = I$

let $f(T) = UTU^{-1}$

$f^{-1}f = f^{-1}UTU^{-1}$

let $f^{-1} = U^{-1}T^{-1}U^{-1}$

$f^{-1}f = (UT^{-1}U^{-1})(UTU^{-1}) = I$

$ff^{-1} = (UTU^{-1})(UT^{-1}U^{-1}) = I$

$\Rightarrow$ f is invertible $\Leftrightarrow$ f is an isomorphism

from $L(V,V)$ to $L(W,W)$

Tyler Olivieri PS6 # 1a 95 H 8k

1a) Let T be a linear operator on C^2 defined by $T(x_1, x_2) = (x_1, 0)$. Let β be the standard ordered basis for C^2 and let $\beta' = \{\alpha_1, \alpha_2\}$ be the ordered basis defined by $\alpha_1 = (1, i)$ $\alpha_2 = (-i, 2)$

a) What is the matrix of T relative to the pair β, β'

$T : C^2 \rightarrow C^2$

$$T_{matrix} = \begin{bmatrix} T(1,0) & T(0,1) \end{bmatrix}_{\beta'}$$

$T(1,0) = 1$

$T(0,1) = 0$

$$[T(1,0)]_{\beta'} \quad [1]_{\beta'} = 2\alpha_1 - i\alpha_2 = 2(1,i) - i(-i,2)$$

$$= (2, 2i) + (i^2, -2i) = (2, 2i) + (-1, -2i)$$

$$= (1, 0) \checkmark$$

$$[T(0,1)]_{\beta'} = \begin{bmatrix} 0 \end{bmatrix}_{\beta'} = 0\alpha_1 + 0\alpha_2 = 0 + 0 = 0$$

$$T_{matrix} = \begin{bmatrix} 2 & 0 \\ -i & 0 \end{bmatrix}$$

p 96 #5

5) Let T be the linear operator on $\mathbb{R}^3$, the matrix of which in the standard basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$\begin{matrix} -1+2-1 = 0 \\ 0+1-1 = 0 \\ 1+3-4 \end{matrix}$$

Find a basis for the range of T and a basis for the nullspace of T.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$rk(A) = 2$ $\dim(\text{range}(T)) = 2$

$\Rightarrow \dim(\ker(T)) = 1$ by rank-nullity theorem

basis for range of T = $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

We can see col 3 is col 2 − col 1

we have two independent vectors in range(T), which leaves the first 2.

basis for nullspace of T = $\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

$$A\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1+2-1 \\ 0+1-1 \\ 1+3-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Tyler Olivier HBK #8 p97

Let θ be a real #. Prove that the following two fields are similar over the field of complex #'s:

Hint: let T be the lin op on C^2 which is represented by the first matrix in the std. ordered basis. Then find vectors α_1 and α_2 s.t. $T\alpha_1 = e^{i\theta}\alpha_1$, $T\alpha_2 = e^{-i\theta}\alpha_2$, and $\{\alpha_1, \alpha_2\}$ is a basis.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

similar if $\exists$ P that is invertible s.t

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = P \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} P^{-1}$$

$$T\alpha_1 = (i\cos\theta - \sin\theta, i\sin\theta + \cos\theta) \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$= (i\cos\theta + i^2\sin\theta, e^{i\theta}) \qquad T\alpha_1 = e^{i\theta}\alpha_1$$

$$\alpha_1 = (i, 1)$$

$$= (i(\cos\theta + i\sin\theta), e^{i\theta}) \qquad e^{i\theta}(i,1) \neq (ie^{i\theta}, e^{i\theta})$$

$$= (ie^{i\theta}, e^{i\theta})$$

$T\alpha_2' = e^{-i\theta}\alpha_2$ A $\qquad \alpha_2 = (1,-1)$

$$e^{-i\theta}(\alpha_2) = e^{-i\theta}(i,-1)$$

$T\alpha_2$ (

$$= (ie^{-i\theta}, -e^{-i\theta})$$

$$T\alpha_2 = (i\cos\theta + \sin\theta, \; i\sin\theta - \cos\theta)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$T\alpha_2 = (i\cos\theta + \sin\theta, -(\cos\theta - i\sin\theta)))$$

$$T\alpha_2 = (i\cos\theta + \sin\theta, -e^{-i\theta})$$

$$= (i\cos\theta - i^2\sin\theta; -e^{-i\theta})$$

$$= (i(\cos\theta + i\sin\theta), -e^{-i\theta})$$

$$= (ie^{-i\theta}, -e^{-i\theta})$$

α_1, α_2 are independent, thus $\{\alpha_1, \alpha_2\}$ are a basis for C2.

and since we can express $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ as T w.r.t $\{\alpha_1, \alpha_2\}$, the matrices are similar.

Tyler Olivieri 856

2) Suppose that A and B are similar $n \times n$ matrices

$B = C^{-1} A C$ for some invertible $n \times n$ matrix C.

a) Show A^n, B^n are similar

A and B are similar

$$B = C^{-1} A C$$

Assume A^{n-1} and B^{n-1} are similar

$$B^{n-1} = C^{-1} A^{n-1} C$$

Then,

$$B^n = BB^{n-1} = C^{-1} A C C^{-1} A^{n-1} C$$

$$B^n = C^{-1} A C C^{-1} A^{n-1} C$$

$$B^n = C^{-1} A I A^{n-1} C$$

$$B^n = C^{-1} A A^{n-1} C$$

$$B^n = C^{-1} A^n C$$

A^n, B^n are similar by induction

b) Show that if $f(x)$ is a polynomial, then $f(A)$ and $f(B)$ are similar.

A, B are similar

$$B = C^{-1}AC$$

$$f(x) = \sum_{i=0}^{n} a_i x^i$$ by definition of polynomial

$$\Rightarrow f(A) = \sum_{i=0}^{n} a_i A^i$$

$$f(B) = \sum_{i=0}^{n} a_i B^i$$

WTS $f(B) = C^{-1} f(A) C$

$$f(B) = f(C^{-1}AC)$$

$$\sum_{i=0}^{n} a_i B^i = \sum_{i=0}^{n} a_i (C^{-1}AC)^i$$

from part (a) we know $(C^{-1}AC)^i = C^{-1}A^iC$

$$f(B) = \sum_{i=0}^{n} a_i (C^{-1}AC)^i = \sum_{i=0}^{n} a_i C^{-1}A^iC$$

$$= \sum_{i=0}^{n} C^{-1} a_i A^i C$$ scalar will commute w/ matrix

$$= C^{-1}\left(\sum_{i=0}^{n} a_i A^i\right)C$$

$$f(B) = C^{-1} f(A) C$$

A, B similar $\Rightarrow$ $f(B), f(A)$ are similar

2c) Tyler Olivier

Show that if $f(x)$ is a polynomial

s.t. $f(A) = 0$ then $f(B) = 0$

A, B similar

$B = C^{-1}AC$

Since C is invertible

$CBC^{-1} = A$

$f(A) = f(CBC^{-1})$

$0 = f(CBC^{-1})$

from part (b) when $f(x)$ is a polynomial

$f(CBC^{-1}) = Cf(B)C^{-1}$

$0 = Cf(B)C^{-1}$

C is invertible

$C^{-1}0 = C^{-1}Cf(B)C^{-1}$

$0 = If(B)C^{-1}$

$0C = f(B)C^{-1}C$

$0 = f(B)I$

$0 = f(B)$

A, B similar and $f(A) = 0 \Rightarrow f(B) = 0$

3) Let $T: V \to V$ be a linear transformation, where V is a finite dvs. Let T^n denote $T \circ T \circ \ldots \circ T$ with n factors of T and let $r_n = \text{rank}(T_n)$

Prove that $r_{n+1} \leq r_n \quad \forall n$

deduce that the sequence $r_1, r_2, r_3, \ldots$ is eventually constant.

Case 1 from PS5 #2

$\text{kernel}(T) \subset \text{kernel}(T^2)$

$\Rightarrow \dim(\ker(T)) \leq \dim(\ker(T^2))$

$n - rk(T) \leq n - rk(T^2)$

$rk(T^2) \leq rk(T)$

base case $r_2 \leq r_1$

In general, let $x \in \text{kernel}(T^n)$

$T^n(x) = 0$

$T(T^n(x)) = T(0) = 0$

$T^{n+1}(x) = 0$

$x \in \text{kernel}(T^{n+1})$

$\text{kernel}(T^n) \subset \text{kernel}(T^{n+1})$

and $\text{Kernel}(T) \subset \text{Kernel}(T^2) \subset \ldots \text{kernel}(T^n) \subset \text{kernel}(T^{n+1})$

because n was arbitrary above, due to rk-nullity theorem, the same as above

$rk_{n+1} \leq rk_n \leq rk. \leq rk_1$

3) It is a non-increasing sequence bounded from below by 0. and since every rk is constant, the sequence of rk of T^n will converge to 0.

Tyler Olivieri PS6 4)

Let n be a non-negative integer, and let $\alpha_1, \dots, \alpha_{n+1}$ be distinct real numbers. Let P_n be the vector space of real polynomials $f(x)$ of degree $\leq n$. Define $F: P_n \to \mathbb{R}^{n+1}$ by $f \mapsto (f(\alpha_1), \dots, f(\alpha_{n+1}))$

a) Show that F is an isomorphism. hint: $\dim(P_n) = ?$ $\text{kernel}(F) = ?$

$\dim(P_n) = n+1$

basis of $P_n = \{1, x, x^2, \dots, x^n\}$ which has $n+1$ elements. This has been shown to be a basis in H&K and in class.

$\dim(\text{Kernel}(F)) + rk(F) = \dim(P_n)$

$\dim(\text{Kernel}(F)) + rk(F) = n+1$

$\dim(\text{Kernel}(F)) = n+1 - rk(F)$

claim $rk(F) = n+1$

then $\dim(\text{Kernel}(F)) = n+1-(n+1) = 0$

$\dim(\text{Kernel}(F)) = 0 \iff F$ is an isomorphism

Proof of claim

$L_i(f) = f(\alpha_i)$ is a linear functional on P_n. Ex 22 in section 3.5 H&K showed that if α_i are distinct, the linear functionals are independent. Furthermore, they showed that they are a basis for P_n^* and have $\dim = \dim(P_n^*) = n+1$

4b) Tyler Olivieri

Explicitly find $f^{-1}(e_1), \dots, F^{-1}(e_{n+1})$ (where $e_1, \dots, e_{n+1}$ are std. basis vectors in $\mathbb{R}^{n+1}$)

in the case $n=2$ and $a_j = j$ for $j = 1, 2, 3$

hint (where does $(X-a)(X-b)$ vanish?)

$a_1 = 1 \quad a_2 = 2 \quad a_3 = 3$

let $p_1(x) = F^{-1}(e_1) \qquad P_2(x) = F^{-1}(e_2) \qquad P_3(x) = F^{-1}(e_3)$

$F(P_1(x)) = (1, 0, 0)$

$F(p_1(x)) = (p_1(1), p_1(2), p_1(3))$

$p_1(2) = 0 = p_1(3)$

$p_1(1) = 1$

let $p_1(x) = \dfrac{(x-2)(x-3)}{(1-2)(1-3)}$

$p_1(1) = \dfrac{(1-2)(1-3)}{(1-2)(1-3)} = 1$

$p_1(2) = \dfrac{(2-2)(2-3)}{(1-2)(1-3)} = 0$

$p_1(3) = \dfrac{(3-2)(3-3)}{(1-2)(1-3)} = 0$

$F(p_1(x)) = (1, 0, 0) = e_1$ ✓

$F^{-1}(e_1) = \dfrac{(x-2)(x-3)}{(1-2)(1-3)}$

Similarly, let $p_2(x) = \dfrac{(x-1)(x-3)}{(2-1)(2-3)}$

then $p_2(1) = 0 \quad p_2(2) = 1 \quad p_2(3) = 0$

$F(p_2(x)) = (0, 1, 0) = e_2$

$F^{-1}(e_2) = p_2(x)$

$F^{-1}(e_2) = \dfrac{(x-1)(x-3)}{(2-1)(2-3)}$

and let $p_3(x) = \dfrac{(x-1)(x-2)}{(3-1)(3-2)}$

so $p_3(1) = 0 \quad p_3(2) = 0 \quad p_3(3) = 1$

$F(P_3(x)) = (0, 0, 1) = e_3$

$F^{-1}(e_3) = P_3(x)$

$F^{-1}(e_3) = \dfrac{(x-1)(x-2)}{(3-1)(3-2)}$

Tyler Olivieri PS6 # 5)

Let P_n be the v.s of real polynomials $f(x)$ of degree at most n. Define $T: P_2 \to P_2$ by $f(x) \mapsto f(x) - (x-1) f'(x)$ where $f'(x)$ is the derivative of $f(x)$.

a) Show that T is a lin. trans.

T is lin tran if

$T(cV_1(x) + V_2(x)) = cTV_1(x) + TV_2(x)$ for $V_1(x), V_2(x) \in P_2$, $c \in F$

$$T(cV_1(x) + V_2(x)) = cV_1(x) + V_2(x) - (x-1)\left(cV_1(x) + V_2(x)\right)'$$

$$= cV_1(x) + V_2(x) - (x-1)\left[cV_1'(x) + V_2'(x)\right]$$

$$= cV_1(x) - (x-1)cV_1'(x) + V_2(x) - (x-1)V_2'(x)$$

$$= c\left[V_1(x) - (x-1)V_1'(x)\right] + V_2(x) - (x-1)V_2'(x)$$

$$= cTV_1(x) + TV_2(x)$$

$\Rightarrow$ T is a linear transformation

b) Find the kernel of T. (Hint: Diff eq)

$$\ker(T) = \{ v(x) \in P_2 \mid T v(x) = 0 \}$$

$$\ker(T) = \{ v(x) \in P_2 \mid v(x) - (x-1) v'(x) = 0 \}$$

$$v(x) = (x-1) v'(x)$$

$$\frac{v(x)}{(x-1)} = v'(x)$$

$$\frac{1}{(x-1)} = \frac{v'(x)}{v(x)}$$

$$\Rightarrow v(x) = (x-1)$$

$$v'(x) = (1)$$

$$(x-1) = (x-1)(1) = (x-1)$$

$$\ker(T) = (x-1)$$

Tyler Olivieri 4c) PS6

In general, deduce that $F^{-1}(e_1), \dots, F^{-1}(e_{n+1})$ form a basis of P_n. In the special case done in (b), express the polynomial x as a linear combination of these basis elements.

We know $e_1, \dots, e_{n+1}$ form a bais for $\mathbb{R}^{n+1}$.

Since F is an isomorphism, it sends a basis to a basis.

$\sum_{i=1}^{n+1} c_i e_i = 0$ and since $\{e_1, \dots, e_{n+1}\}$ are a basis they are lin ind.

$c_i = 0 \quad \forall i$

$F^{-1}\left(\sum_{i=1}^{n+1} c_i e_i\right) = F^{-1}(0)$

F^{-1} is a lin transformation (proved in H8k) Theorem 7. sec 3.1

$\Rightarrow F^{-1}(0) = 0$

$\sum_{i=1}^{n+1} c_i F^{-1}(e_i) = 0$

this is actually Theorem 8

$\sum_{i=1}^{n+1} c_i P_i(x) = 0$ $\Rightarrow$ $P_i(x)$ are lin ind.

$n+1$ elements

$\Rightarrow \{P_1(x), \dots, P_{n+1}(x)\}$ are a basis.

$$p(x) = c_1 \frac{(x-2)(x-3)}{(1-2)(1-3)} + c_2 \frac{(x-1)(x-3)}{(2-1)(2-3)} + c_3 \frac{(x-1)(x-2)}{(3-1)(3-2)}$$

5) d) Find the matrix T w.r.t basis $\{f_1, f_2, f_3\}$ of P_2

where $f_i = F^{-1}(e_i)$

$$T_{\{f_1,f_2,f_3\}} = \begin{bmatrix} T(f_1) & T(f_2) & T(f_3) \end{bmatrix}$$

$f_1 = F^{-1}(e_1)$

$$T(f_1) = T(F^{-1}(e_1)) = T\left(\frac{(x-2)(x-3)}{(1-2)(1-3)}\right)$$

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)} - (x-1)\, d/dx \left[\frac{(x-2)(x-3)}{(1-2)(1-3)}\right]$$

need to express this in terms of the basis elements $\{f_1, f_2, f_3\}$

$2x-5$

$x^2-3x-2x+6$

$$= f_1 - (x-1)\, d/dx\left[\frac{(x-2)(x-3)}{(1-2)(1-3)}\right]$$

$2x^2-7x+5$ $(+1-1)$

$$= f_1 - \frac{(x-1)(2x-5)}{(1-2)(1-3)}$$

x^2-5x+6 $\quad +x^2-2x-1$

$(x-1)(x+1)$

$$= f_1 - \frac{(x-2)(x-3)}{(1-2)(1-3)} + \frac{x^2-2x-1}{(1-2)(1-3)}$$

$$= \cancel{f_1 - f_1} + \frac{x^2-2x-1}{(1-2)(1-3)}$$

couldn't finish algebra in time :(

Tyler Olivieri

5(c) Find the matrix of T w.r.t the basis $\{1, x, x^2\}$ of P_2.

$$T_{\{1,x,x^2\}} = \left[T(1) \;\; T(x) \;\; T(x^2) \right]$$

$$T(1) = 1 - (x-1)\, d/dx[1]$$

$$= 1 - (x-1)(0)$$

$$= 1 \qquad = 1 + 0x + 0x^2$$

$$T(x) = x - (x-1)\, d/dx[x]$$

$$= x - (x-1)(1)$$

$$= x - (x-1)$$

$$= x - x + 1$$

$$= 1 \qquad = 1 + 0x + 0x^2$$

$$T(x^2) = x^2 - (x-1)\, d/dx[x^2]$$

$$= x^2 - (x-1)(2x)$$

$$= x^2 - 2x^2 + 2x$$

$$= -x^2 + 2x + 0$$

$$T_{\{1,x,x^2\}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$