Tyler Olivieri HW 3 #1

Suppose $X_1, \dots, X_n$ are an iid random sample of size n w/ sample mean $\hat{\bar{X}} = 12$ and sample variance $s^2 = 5$

a) Let $n = 5$ and suppose samples are drawn from a normal distribution w/ unknown mean μ and known variance $\sigma^2 = 9$

Let the null hypothesis be $H_0 : \mu = 10$; $H_a : \mu \neq 10$

Calculate the relevant test statistic value and p-value.

Want to use z-test. The relevant test statistic

is $z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{12 - 10}{3/\sqrt{5}} = \frac{2\sqrt{5}}{3}$

p-value will be smallest significance level at which the null hypothesis would be rejected.

$p = \Pr\{\text{event more contradictory than observed} \mid H_0 \text{ is true}\}$

$= \Pr(Z > z \cup Z < -z \mid H_0 \text{ is true}\}$

Due to symmetry under null hypothesis

$= 2\Pr(Z > |z| \mid H_0 \text{ is true})$

$= 2(1 - \Pr\{Z \leq |z| \mid H_0 \text{ is true}\})$

$= 2(1 - \Phi(|z|))$

$= 2\left(1 - \Phi\left(\frac{2\sqrt{5}}{3}\right)\right) = 2(1 - \Phi(1.5)) = .13 = \text{p-value}$

b) Using the acceptance region of this test, construct a 95% confidence interval.

let $\alpha = .05$ for 95% confidence

$$1-\alpha = \Pr\left(-z \le \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le z\right)$$

from Normal table

$$.95 = \Pr\left(-1.96 \le \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le 1.96\right)$$

$$.95 = \Pr\left(-1.96(\sigma/\sqrt{n}) - \bar{x} \le -\mu_0 \le 1.96(\sigma/\sqrt{n}) - \bar{x}\right)$$

$$.95 = \Pr\left(1.96(\sigma/\sqrt{n}) + \bar{x} \ge \mu_0 \ge -1.96(\sigma/\sqrt{n}) + \bar{x}\right)$$

$$.95 = \Pr\left(\bar{x} - 1.96(\sigma/\sqrt{n}) \le \mu_0 \le \bar{x} + 1.96(\sigma/\sqrt{n})\right)$$

substitute $\bar{x}$, σ, and n from known values

$$.95 = \Pr\left(12 - 1.96(3/\sqrt{5}) \le 12 \le 12 + 1.96(3/\sqrt{5})\right)$$

$\Rightarrow$ 95% confidence interval for μ_0

$$\left[12 - 1.96(3/\sqrt{5}),\ 12 + 1.96(3/\sqrt{5})\right]$$

$$[9.37,\ 14.63]$$

(a) Determine decision rule for $\alpha = .05$

$$.05 = \Pr\{\text{type I error}\}$$

$$= \Pr\{H_0 \text{ is rejected} \mid H_0 \text{ is true}\}$$

$$= \Pr\{Z < -Z_+ \cup Z > Z_+ \mid H_0 \text{ is true}\}$$

$$= 2\Pr\{Z < -Z_+ \mid H_0 \text{ is true}\}$$

$$.05 = 2\Phi(-Z_+)$$

$$\frac{.05}{2} = \Phi(-Z_+)$$

$$\Phi^{-1}(.05/2) = -Z_+$$

$$-\Phi^{-1}\left(\frac{.05}{2}\right) = Z_+ = 1.96$$

Accept H_0 if $-Z_+ < Z < Z_+$ $\quad -1.96 < Z < 1.96$

reject otherwise.

Tyler Olivieri HW3

1c) Now let $n=5$ again but suppose the samples are drawn from an unspecified distribution w/ unknown mean μ and unknown variance σ^2. Let the null hypothesis be $H_0: \mu = 10$ and the alternate hypothesis be $H_a: \mu > 10$.

Calculate the relevant test statistic value and p-value.

Determine the decision rule for $\alpha = .05$.

The relevant test statistic will be t, because we do not know the distribution and n is small

$$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{12 - 10}{\sqrt{5}/\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{5}} = 2$$

one sided

$$p\text{-value} = \text{Prob}\{ t > T \mid H_0 \text{ is true} \}$$

$$= 1 - \text{Prob}\{ t \le T \mid H_0 \text{ is true} \}$$

$$= 1 - t_{n-1}(T) = 1 - t_4(T) = 1 - t_4(2)$$

CDF of t-dist with 4 deg of freedom

$$= .06$$

$\alpha = P_r\{\text{Type 1 error}\}$

should be $t_{\alpha, n-1}$

$.05 = P_r(t \geq T_\alpha \mid H_0 \text{ is true})$

but I am getting $t^{-1}_{\alpha, n-1}$

$.05 = t_4(t \geq T_a)$

$.05 = 1 - t_4(T_a)$

$1 - .05 = t_4(T_a)$

critical value $= T_a = t_4^{-1}(1 - .05) = t_4^{-1}(.95) = 2.13$

Tyler Olivieri HW 2 2)

Suppose that $Y_1, \dots, Y_n$ are an iid random sample of size n drawn from a Poisson distribution w/ unknown parameter λ. Using $\sum_{i=1}^{n} Y_i$ as the test statistic, find the critical value and rejection region at level α for the test.

let $T = \sum_{i=1}^{n} Y_i$

$$\begin{cases} H_0: \lambda = \lambda_0 \\ H_a: \lambda = \lambda^* \end{cases} \quad \text{where } \lambda^* > \lambda_0$$

let η be the critical value

$\alpha = \Pr\{\text{Type I error occurs}\}$

$\alpha = \Pr\{\text{reject } H_0 \mid H_0 \text{ is true}\}$

reject H_0 when $T > \eta$

$\alpha = \Pr\{T > \eta \mid H_0 \text{ is true}\}$

$\alpha = 1 - \Pr\{T \leq \eta \mid H_0 \text{ is true}\}$

H_0 is true $\lambda = \lambda_0$

Since $T \sim \text{poisson}(\lambda)$, given $\Pr\{T \leq \lambda \mid H_0 \text{ is true}\}$ is cdf of poisson with λ_0 where $\mathbb{E}[Y_i] = \lambda$, $Var(Y_i) = \lambda$

However, as $n \to \infty$ $\frac{T - n\lambda_0}{\sqrt{\lambda/n}} \xrightarrow{d} N(0,1)$

piazza note allows assumption of n large

than we can say $\frac{T - n\lambda_0}{\sqrt{\frac{\lambda_0}{n}}}$ is approximately distributed gaussian conditioned under the null hypothesis being true.

let $$z = \frac{T - n\lambda_0}{\sqrt{\frac{\lambda_0}{n}}}$$

reject when $z > \ell$ where ℓ is a threshold

$$\frac{T - n\lambda_0}{\sqrt{\frac{\lambda_0}{n}}} > \ell$$

$$T > \ell\sqrt{\frac{\lambda_0}{n}} + n\lambda_0 \qquad (*)$$

$$\alpha = \Pr\{z > \ell \mid H_0 \text{ is true}\}$$

$$\alpha = 1 - \Pr\{z \leq \ell \mid H_0 \text{ is true}\}$$

$z \sim N(0,1)$ given H_0 is true

$$\alpha = 1 - \Phi(\ell)$$

$$1 - \alpha = \Phi(\ell)$$

$$\ell = \Phi^{-1}(1-\alpha)$$

we can translate this threshold to use the statistic T instead of z using $(*)$

The rejection region is any $z > \ell$ so (ℓ, ∞)

The rejection region for T is $T > \ell\sqrt{\frac{\lambda_0}{n}} + n\lambda_0$

$$\left(\ell\sqrt{\frac{\lambda_0}{n}} + n\lambda_0, \infty\right)$$

Tyler Olivieri HW3 #3)

Consider a random sample of size $n = 100$ w/ sample proportion $\hat{p} = .2$ from a population w/ a true unknown proportion p.

a) For the test $H_0: p = .25$ versus $H_a: p < .25$, calculate the relevant test statistic value and p-value. Determine the decision rule for $\alpha = .05$ and $\alpha = .01$

Test statistic: We can use the z-test here because we have large n. From CLT $\hat{p}$ calculation will be distributed approx Normal.

Variance of prop $= \frac{p(1-p)}{n}$

$$\text{test } z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

$$= \frac{.2 - .25}{\sqrt{\frac{.25(1-.25)}{100}}} = \frac{-.05}{.043} = -1.163$$

$\alpha = Pr\{\text{Type I error}\} = Pr\{H_0 \text{ is rejected} \mid H_0 \text{ is true}\}$

$\alpha = Pr\{z < \eta \mid H_0 \text{ is true}\}$ η is threshold.

This is just one-sided gaussian. $z \sim N(0,1)$ when H_0 is true.

$\alpha = \Phi(\eta)$

$\eta = \Phi^{-1}(\alpha)$

when $\alpha = .05$

$$2 = \Phi^{-1}(.05) = -1.65$$

p value $= \Pr\{z < Z \mid H_0 \text{ is true}\}$

$$= \Phi(-1.163)$$

$$= .12$$

when $\alpha = .01$

$$2 = \Phi^{-1}(.01) = -2.33$$

accept H_0 when $Z > 2$

reject H_0 when $Z < 2$

b)

For the test $H_0: p = .25$ versus $H_a: p \neq .25$

Calculate the relevant test statistic value and p-value.

Determine the decision rule for $\alpha = .05$ and $\alpha = .01$

The test statistic does not change

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} = -1.163$$

p value $= 2\Pr\{Z < Z \mid H_0 \text{ is true}\}$

$$= 2\Phi(-1.163) = .12$$

$\alpha = \Pr\{\text{Type I error}\} = \Pr\{H_0 \text{ is rejected} \mid H_0 \text{ is true}\}$

$$\alpha = 2(1 - \Phi(2)) \Rightarrow 2 = \pm\Phi^{-1}(1 - \alpha/2)$$

for $\alpha = .05$

$$2 = \Phi^{-1}(1 - .05/2), -\Phi^{-1}(1 - .05/2)$$

$$\alpha/2 = 1 - \Phi(2)$$

$$2 = 1.96, -1.96$$

$$1 - \alpha/2 = \Phi(2)$$

accept if $-2 < Z < 2$ reject o.w

for $\alpha = .025$

$$z = \Phi^{-1}(1 - .01/2) \quad , \quad -\Phi^{-1}(1 - .01/2) \qquad = 2.518 \ , \ -2.58$$

accept H_0 when $-z < Z < z$ $\quad -2.58 < Z < 2.58$

reject H_0 otherwise.