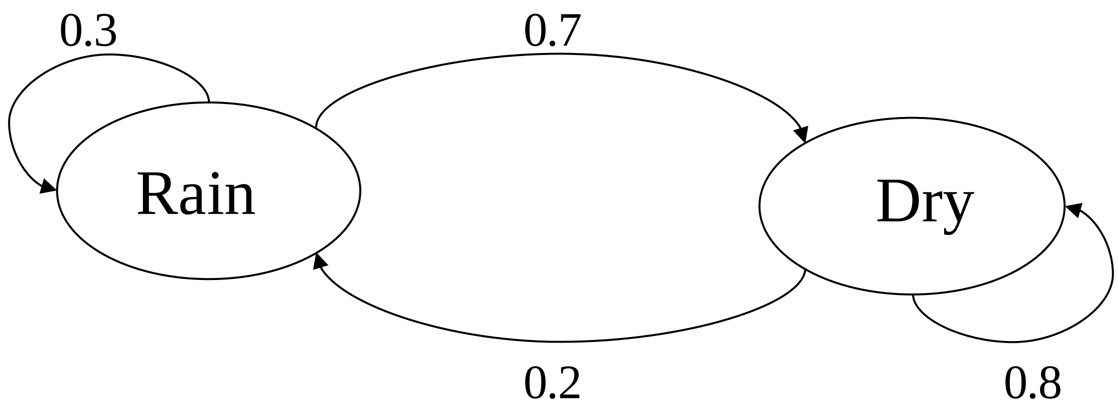
Hidden Markov Models

Example of Markov Model



- Two states : 'Rain' and 'Dry'.
- Transition probabilities: $P(\text{'Rain'}|\text{'Rain'})=0.3$, $P(\text{'Dry'}|\text{'Rain'})=0.7$, $P(\text{'Rain'}|\text{'Dry'})=0.2$, $P(\text{'Dry'}|\text{'Dry'})=0.8$
- Initial probabilities: say $P(\text{'Rain'})=0.4$, $P(\text{'Dry'})=0.6$.

Calculation of sequence probability

- By Markov chain property, probability of state sequence can be found by the formula:

$$P(s_{i1}, s_{i2}, \ldots, s_{ik}) = P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) P(s_{i1}, s_{i2}, \ldots, s_{ik-1})$$

$$= P(s_{ik} \mid s_{ik-1}) P(s_{i1}, s_{i2}, \ldots, s_{ik-1}) = \ldots$$

$$= P(s_{ik} \mid s_{ik-1}) P(s_{ik-1} \mid s_{ik-2}) \ldots P(s_{i2} \mid s_{i1}) P(s_{i1})$$

- Suppose we want to calculate a probability of a sequence of states in our example, {‘Dry’,’Dry’,’Rain’,Rain’}.

P({‘Dry’,’Dry’,’Rain’,Rain’}) =

P(‘Rain’|’Rain’) P(‘Rain’|’Dry’) P(‘Dry’|’Dry’) P(‘Dry’)=

= 0.3*0.2*0.8*0.6

Hidden Markov models.

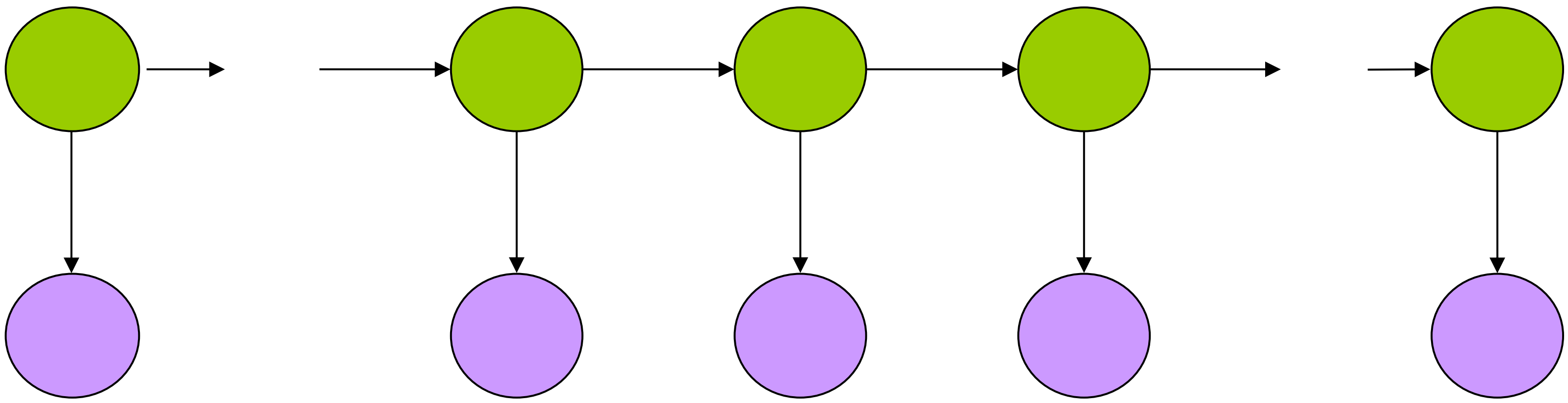
- Set of states: $\{s_1, s_2, \ldots, s_N\}$
- Process moves from one state to another generating a sequence of states : $s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$
- Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

- States are not visible, but each state randomly generates one of M observations (or visible states) $\{v_1, v_2, \ldots, v_M\}$

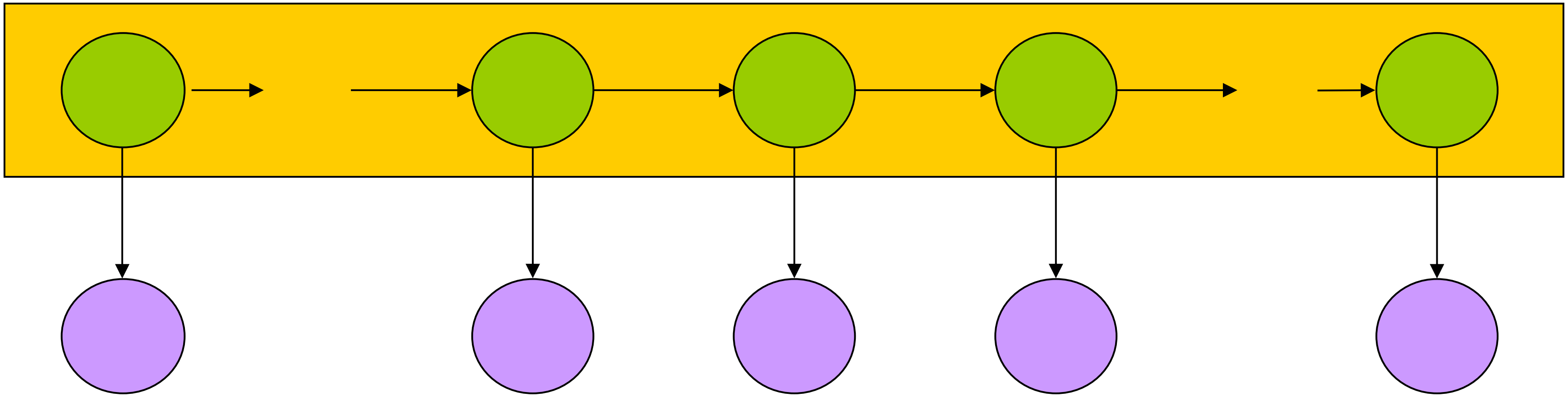
- To define hidden Markov model, the following probabilities have to be specified: matrix of transition probabilities $A=(a_{ij})$, $a_{ij}= P(s_i \mid s_j)$, matrix of observation probabilities $B=(b_i(v_m))$, $b_i(v_m) = P(v_m \mid s_i)$ and a vector of initial probabilities $\pi=(\pi_i)$, $\pi_i = P(s_i)$. Model is represented by $M=(A, B, \pi)$.

What is an HMM?



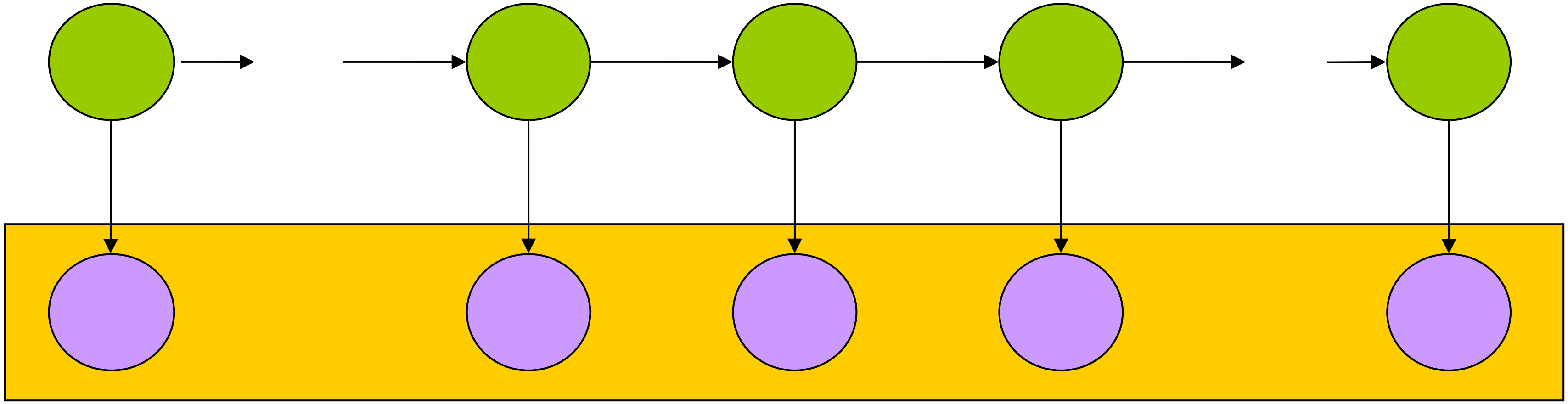
- Graphical Model
- Circles indicate states
- Arrows indicate probabilistic dependencies between states

What is an HMM?



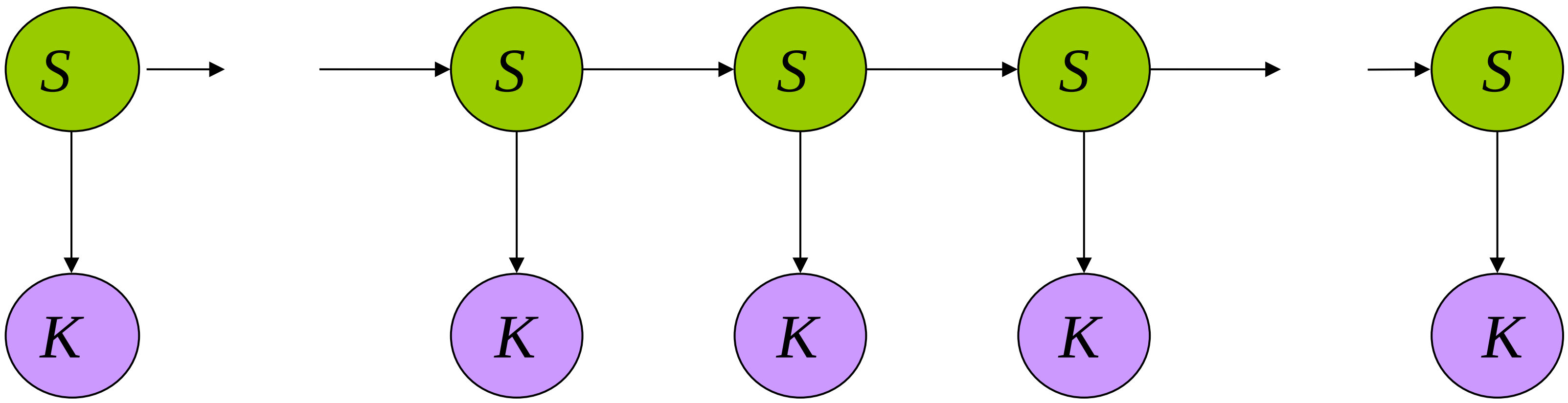
- Green circles are ***hidden states***
- Dependent only on the previous state
- "The past is independent of the future given the present."

What is an HMM?



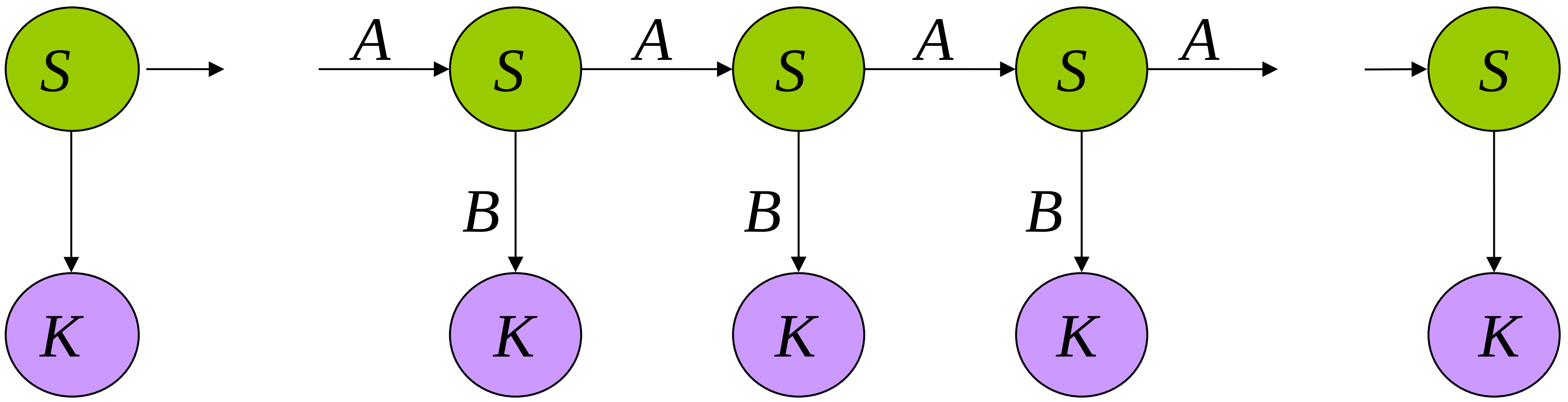
- Purple nodes are ***observed states***
- Dependent only on their corresponding hidden state

HMM Formalism



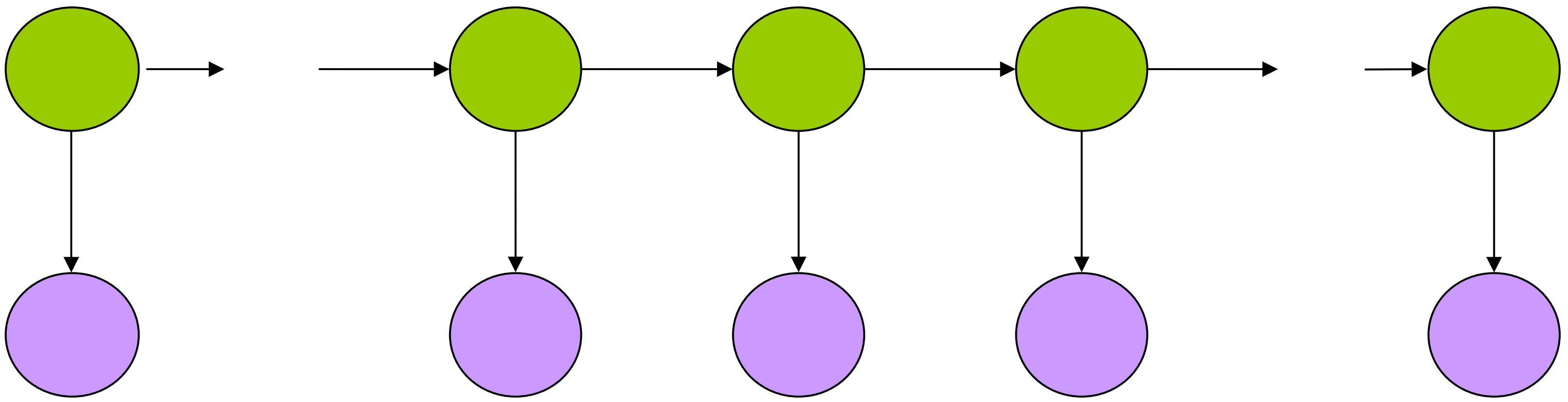
- $\{S, K, \Pi, A, B\}$
- S : $\{s_1 \ldots s_N\}$ are the values for the hidden states
- K : $\{k_1 \ldots k_M\}$ are the values for the observations

HMM Formalism



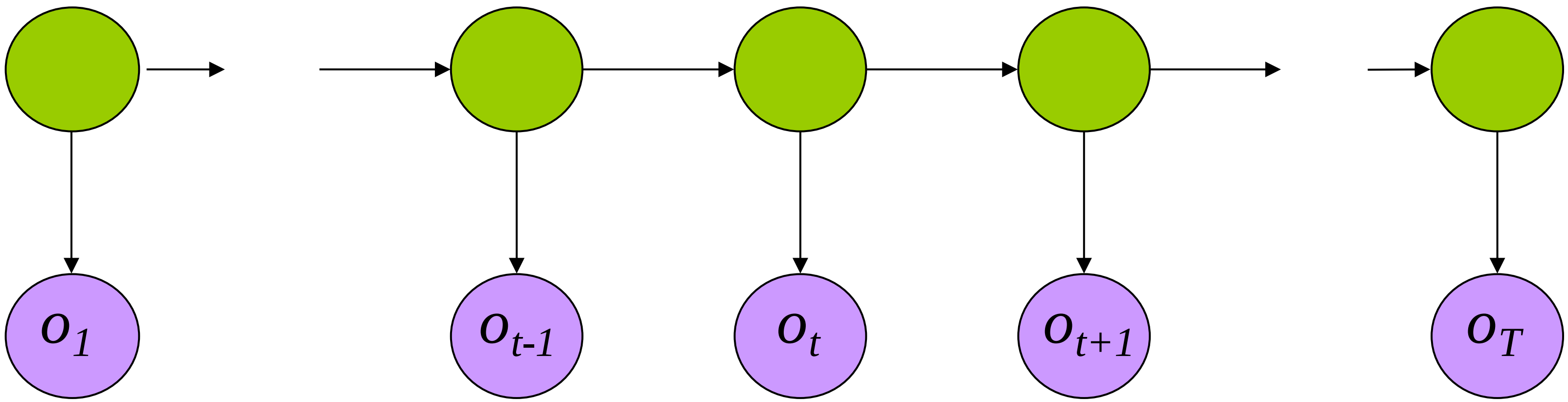
- $\{S, K, \Pi, A, B\}$
- $\Pi = \{\pi_i\}$ are the initial state probabilities
- $A = \{a_{ij}\}$ are the state transition probabilities
- $B = \{b_{ik}\}$ are the observation state probabilities

Inference in an HMM



- Compute the probability of a given observation sequence
- Given an observation sequence, compute the most likely hidden state sequence
- Given an observation sequence and set of possible models, which model most closely fits the data?

Decoding

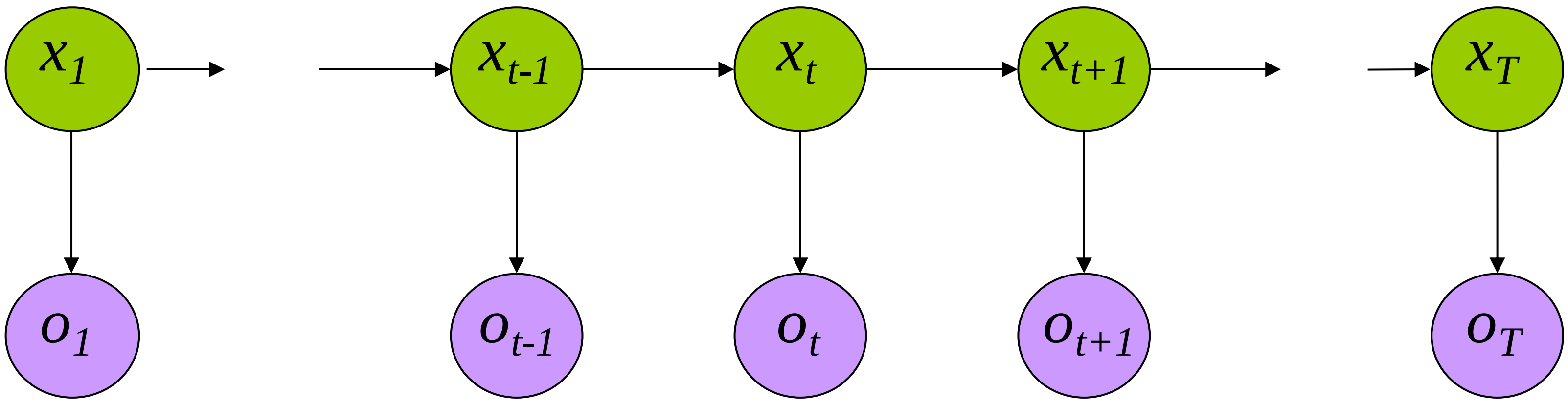


Given an observation sequence and a model, compute the probability of the observation sequence

$$O = (o_1 \ldots o_T),\ \mu = (A, B, \Pi)$$

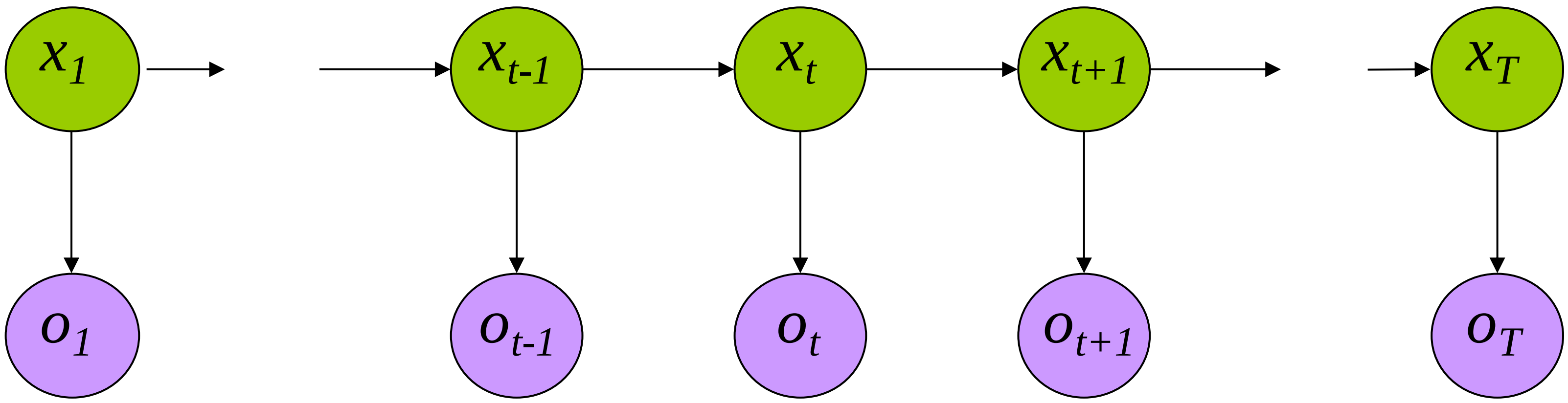
Compute $P(O \mid \mu)$

Decoding



$$P(O \mid X, \mu) = b_{x_1 o_1} b_{x_2 o_2} \dots b_{x_T o_T}$$

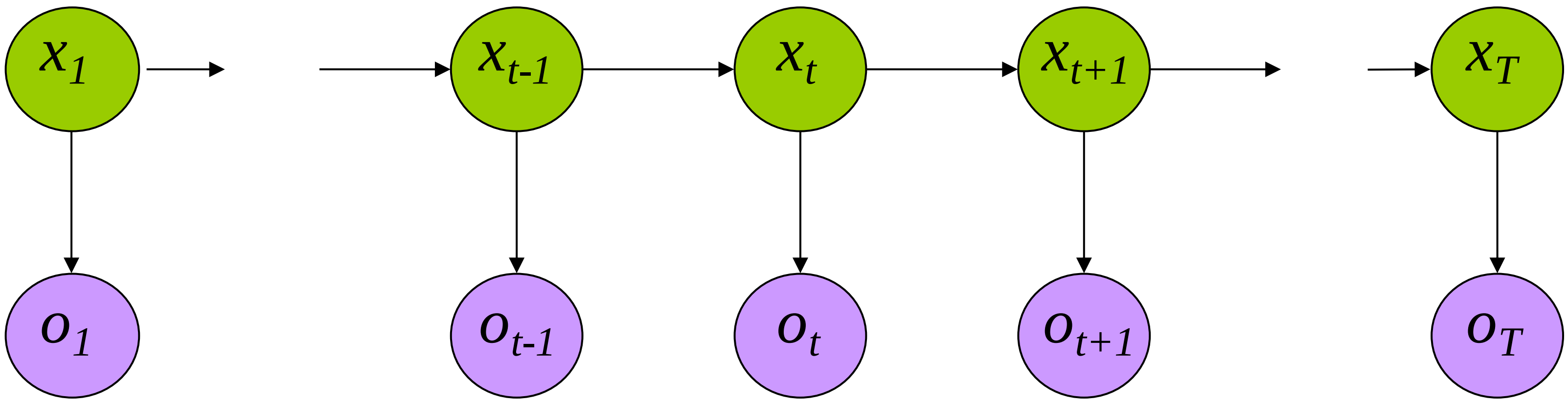
Decoding



$$P(O \mid X, \mu) = b_{x_1 o_1} b_{x_2 o_2} \dots b_{x_T o_T}$$

$$\boxed{P(X \mid \mu) = \pi_{x_1} a_{x_1 x_2} a_{x_2 x_3} \dots a_{x_{T-1} x_T}}$$

Decoding

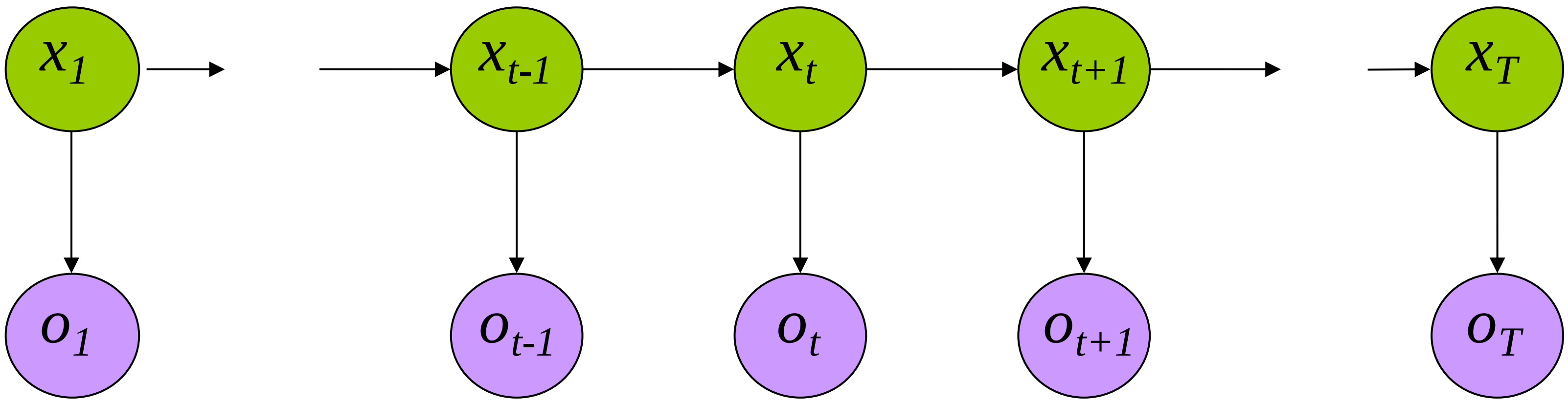


$$P(O \mid X, \mu) = b_{x_1 o_1} b_{x_2 o_2} \ldots b_{x_T o_T}$$

$$P(X \mid \mu) = \pi_{x_1} a_{x_1 x_2} a_{x_2 x_3} \ldots a_{x_{T-1} x_T}$$

$$\boxed{P(O, X \mid \mu) = P(O \mid X, \mu) P(X \mid \mu)}$$

Decoding



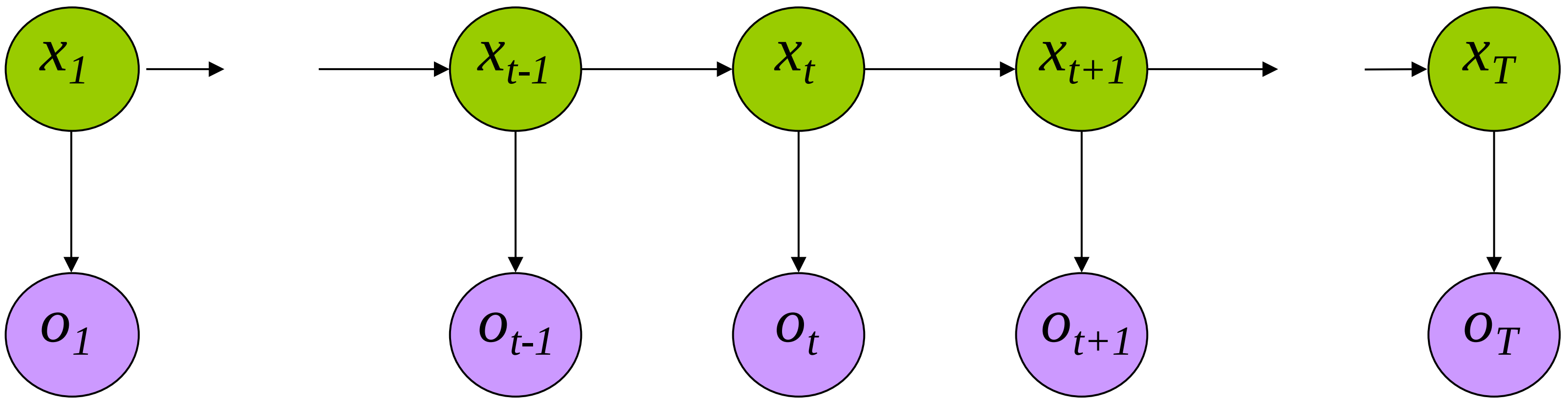
$$P(O \mid X, \mu) = b_{x_1 o_1} b_{x_2 o_2} \ldots b_{x_T o_T}$$

$$P(X \mid \mu) = \pi_{x_1} a_{x_1 x_2} a_{x_2 x_3} \ldots a_{x_{T-1} x_T}$$

$$P(O, X \mid \mu) = P(O \mid X, \mu) P(X \mid \mu)$$

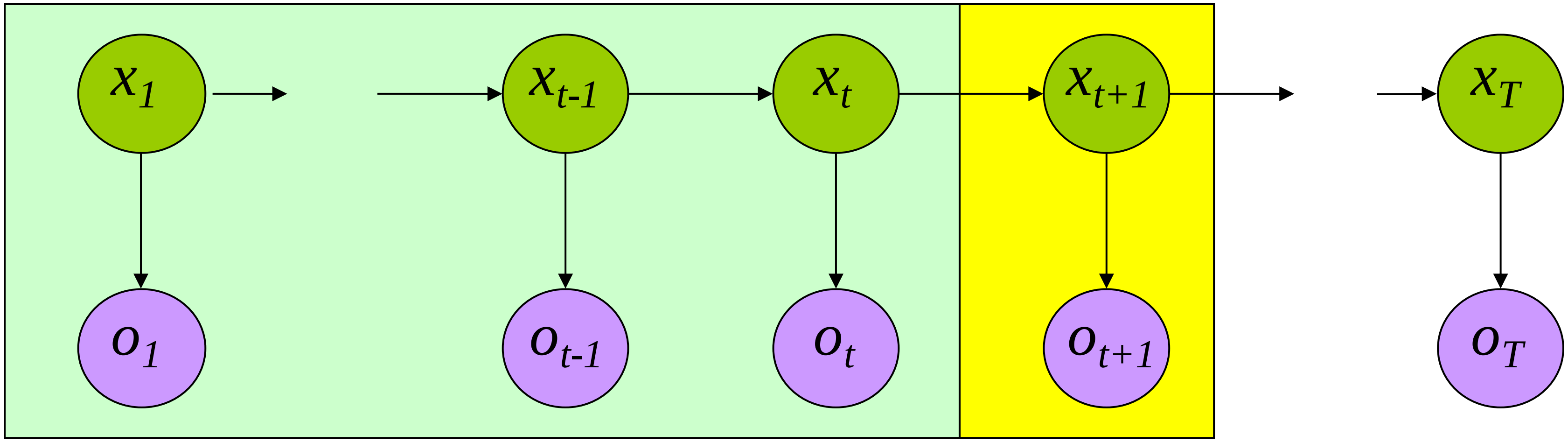
$$\boxed{P(O \mid \mu) = \sum_X P(O \mid X, \mu) P(X \mid \mu)}$$

Decoding



$$P(O \mid \mu) = \sum_{\{x_1 \dots x_T\}} \pi_{x_1} b_{x_1 o_1} \prod_{t=1}^{T-1} a_{x_t x_{t+1}} b_{x_{t+1} o_{t+1}}$$

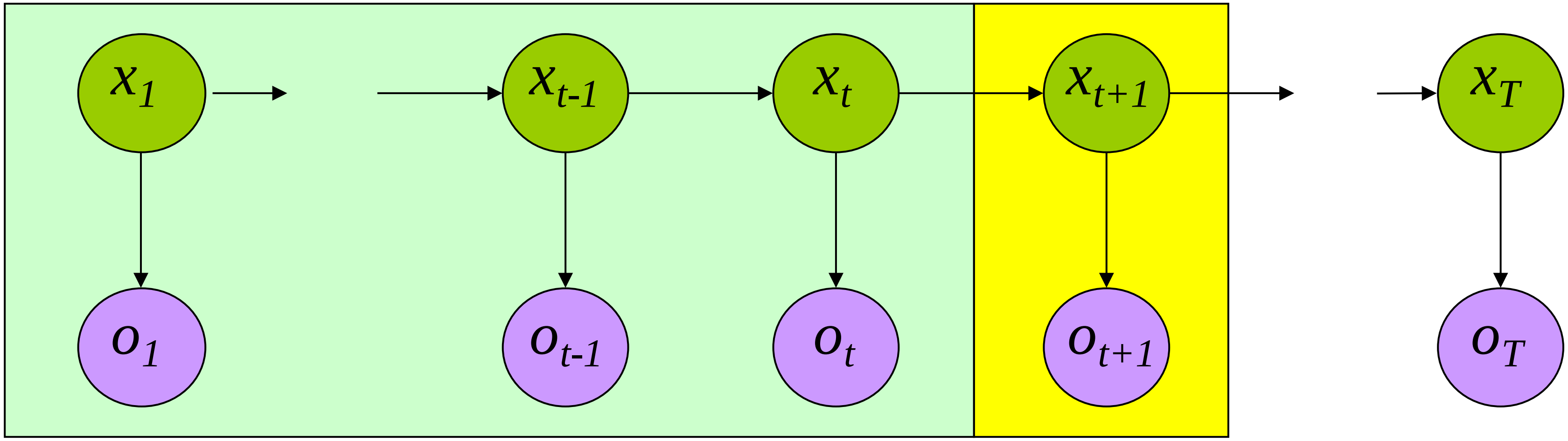
Forward Procedure



- Special structure gives us an efficient solution using *dynamic programming*.
- **Intuition**: Probability of the first t observations is the same for all possible t+1 length state sequences.
- **Define:**

$$\alpha_i(t) = P(o_1 \ldots o_t, x_t = i \mid \mu)$$

Forward Procedure



$\alpha_j(t+1)$

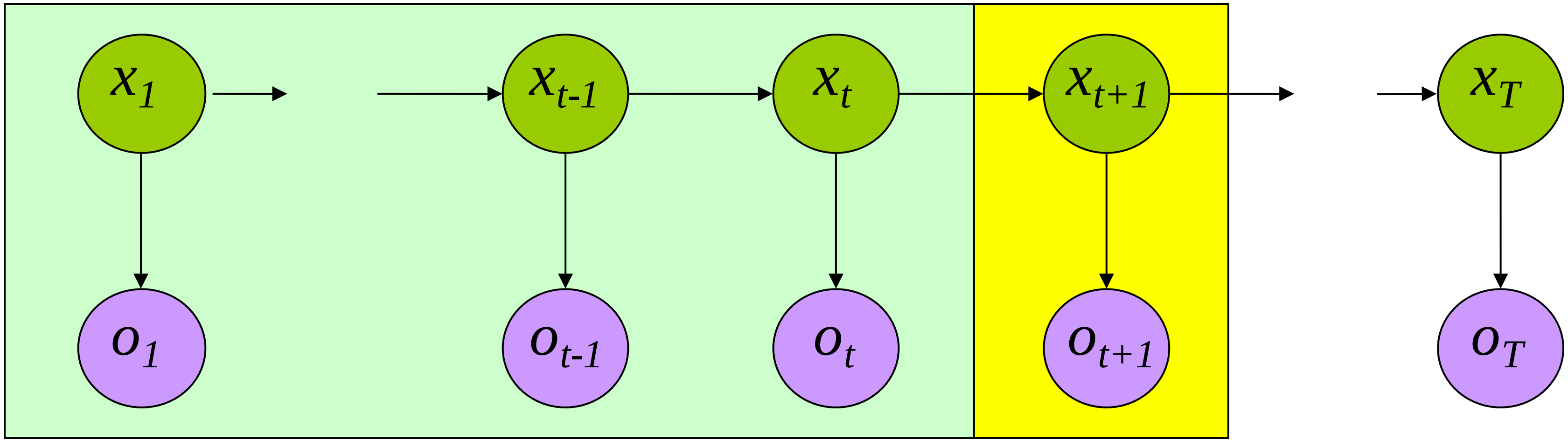
$$= P(o_1 \ldots o_{t+1}, x_{t+1} = j)$$

$$= P(o_1 \ldots o_{t+1} \mid x_{t+1} = j) P(x_{t+1} = j)$$

$$= P(o_1 \ldots o_t \mid x_{t+1} = j) P(o_{t+1} \mid x_{t+1} = j) P(x_{t+1} = j)$$

$$= P(o_1 \ldots o_t, x_{t+1} = j) P(o_{t+1} \mid x_{t+1} = j)$$

Forward Procedure



$$\alpha_j(t+1)$$

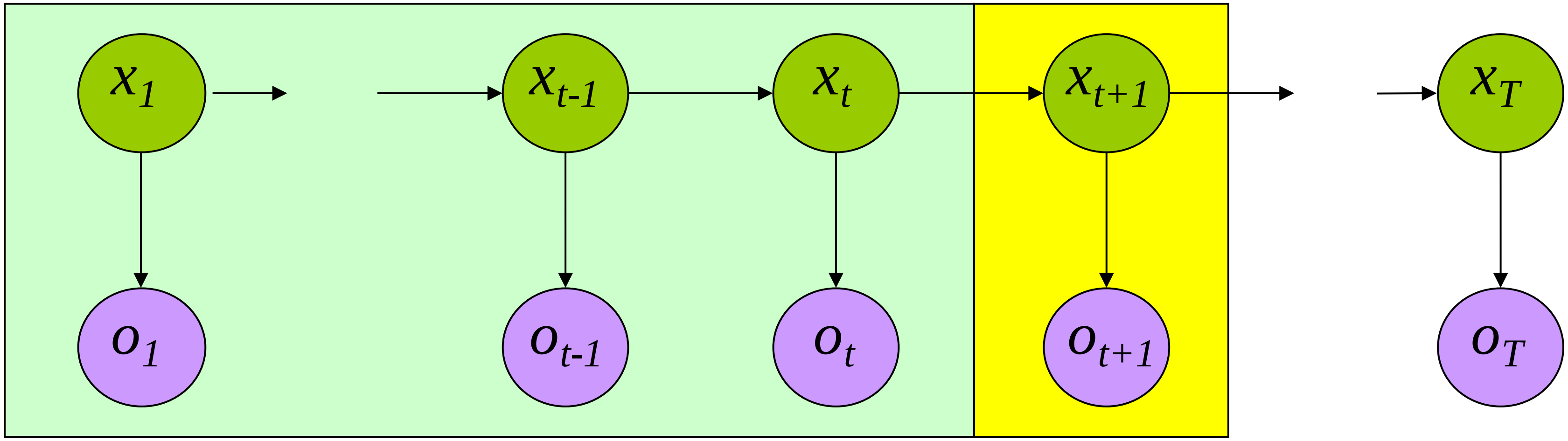
$$= P(o_1 \dots o_{t+1}, x_{t+1} = j)$$

$$\boxed{= P(o_1 \dots o_{t+1} \mid x_{t+1} = j) P(x_{t+1} = j)}$$

$$= P(o_1 \dots o_t \mid x_{t+1} = j) P(o_{t+1} \mid x_{t+1} = j) P(x_{t+1} = j)$$

$$= P(o_1 \dots o_t, x_{t+1} = j) P(o_{t+1} \mid x_{t+1} = j)$$

Forward Procedure



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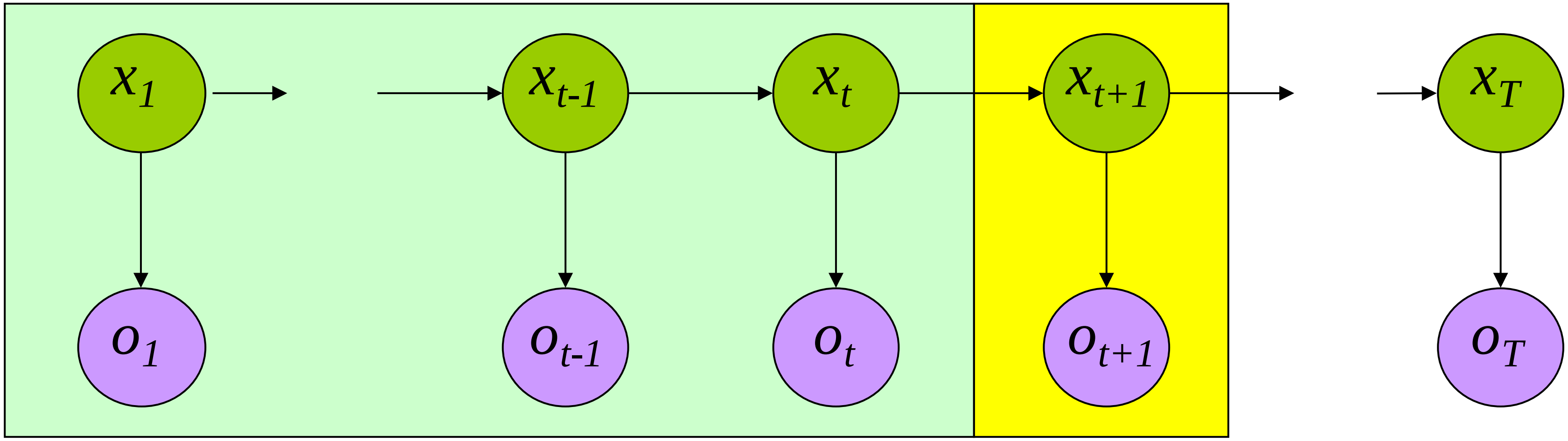
Forward Procedure



$$\alpha_j(t+1)$$

$$= P(o_1 \ldots o_{t+1}, x_{t+1} = j)$$

$$= P(o_1 \ldots o_{t+1} \mid x_{t+1} = j) P(x_{t+1} = j)$$

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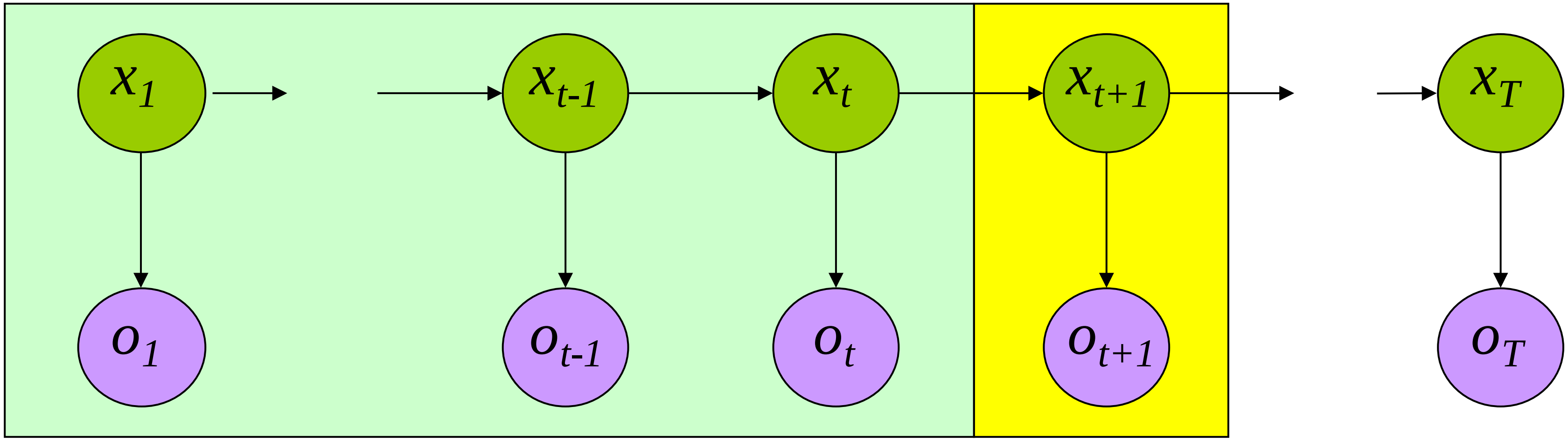
$$\boxed{= P(o_1 \ldots o_t, x_{t+1} = j) P(o_{t+1} \mid x_{t+1} = j)}$$

Forward Procedure



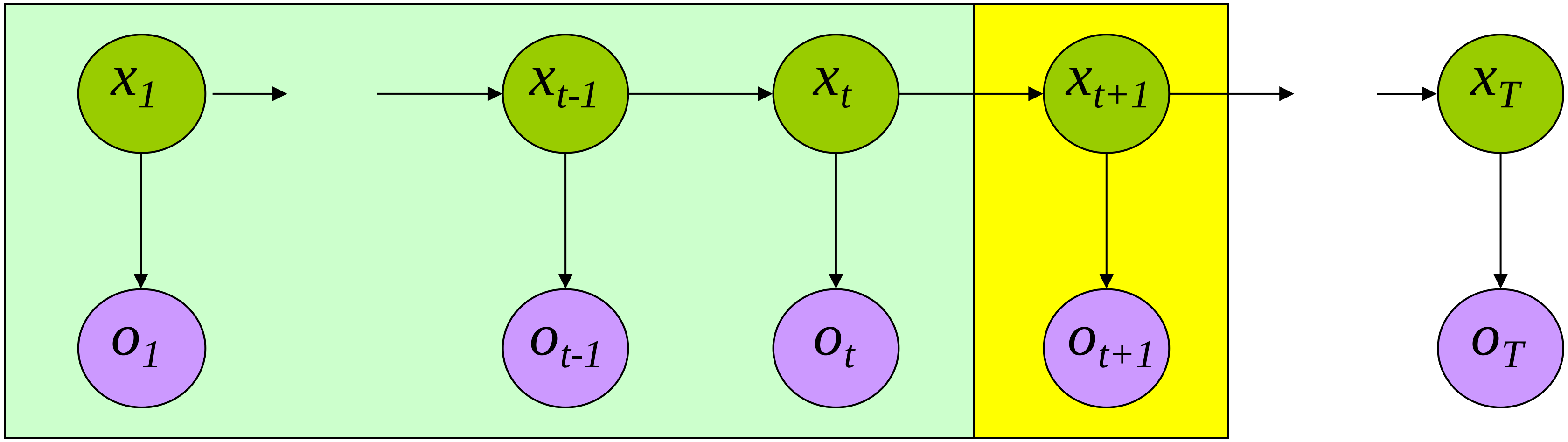
$$= \sum_{i=1...N} P(o_1...o_t, x_t = i, x_{t+1} = j) P(o_{t+1} | x_{t+1} = j)$$

$$= \sum_{i=1...N} P(o_1...o_t, x_{t+1} = j | x_t = i) P(x_t = i) P(o_{t+1} | x_{t+1} = j)$$

$$= \sum_{i=1...N} P(o_1...o_t, x_t = i) P(x_{t+1} = j | x_t = i) P(o_{t+1} | x_{t+1} = j)$$

$$= \sum_{i=1...N} \alpha_i(t) a_{ij} b_{j o_{t+1}}$$

Forward Procedure



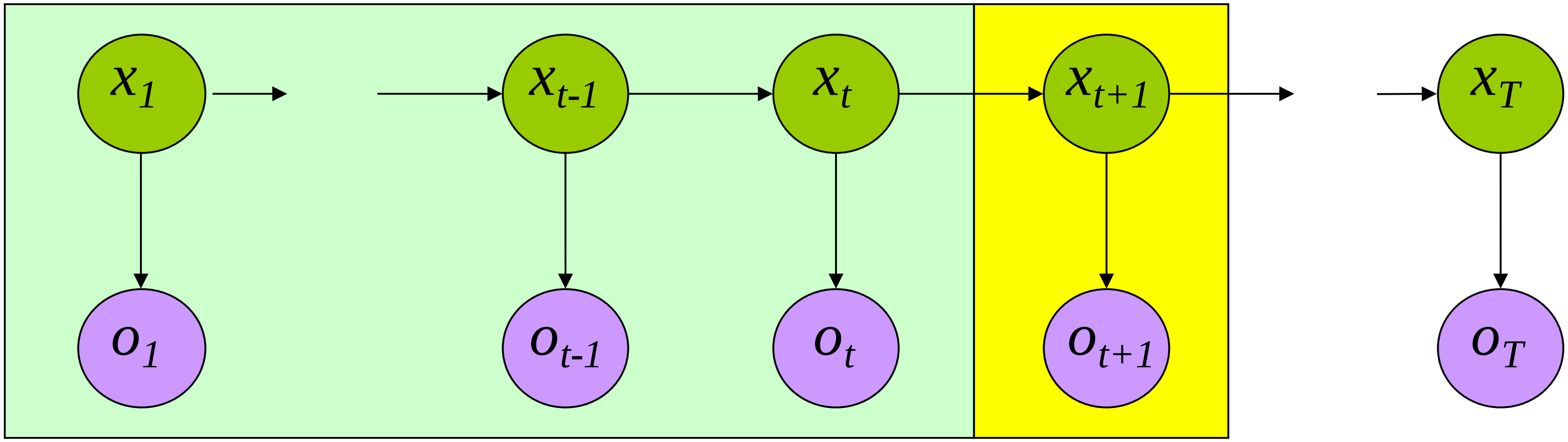
$$= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_t = i, x_{t+1} = j) P(o_{t+1} \mid x_{t+1} = j)$$

$$\boxed{= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_{t+1} = j \mid x_t = i) P(x_t = i) P(o_{t+1} \mid x_{t+1} = j)}$$

$$= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_t = i) P(x_{t+1} = j \mid x_t = i) P(o_{t+1} \mid x_{t+1} = j)$$

$$= \sum_{i=1\ldots N} \alpha_i(t) a_{ij} b_{j o_{t+1}}$$

Forward Procedure



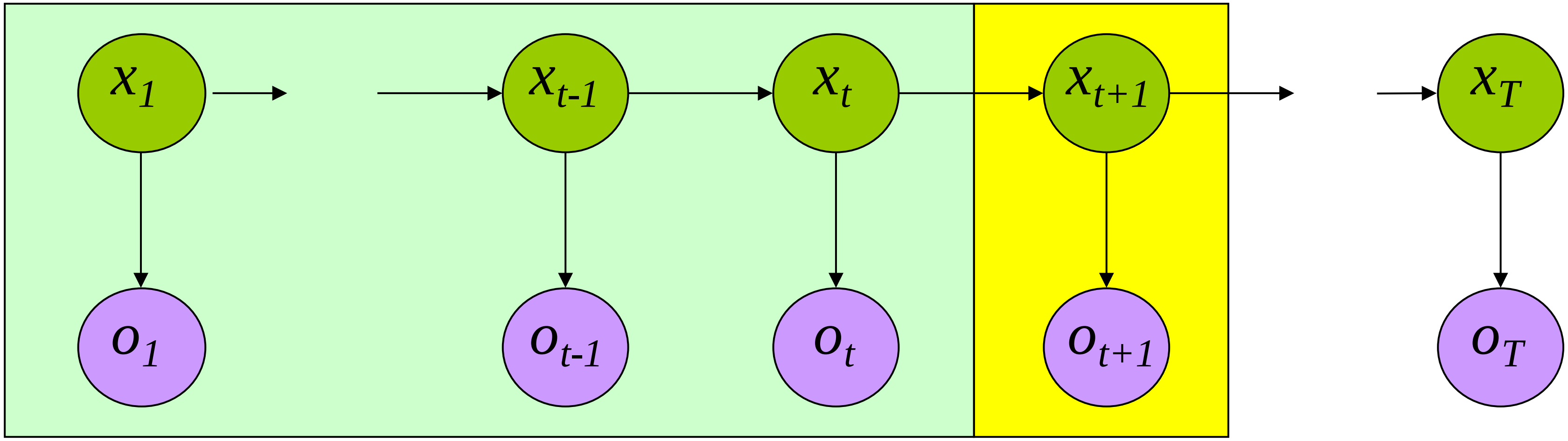
$$= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_t = i, x_{t+1} = j) P(o_{t+1} \mid x_{t+1} = j)$$

$$= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_{t+1} = j \mid x_t = i) P(x_t = i) P(o_{t+1} \mid x_{t+1} = j)$$

$$\boxed{= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_t = i) P(x_{t+1} = j \mid x_t = i) P(o_{t+1} \mid x_{t+1} = j)}$$

$$= \sum_{i=1\ldots N} \alpha_i(t) a_{ij} b_{j o_{t+1}}$$

Forward Procedure



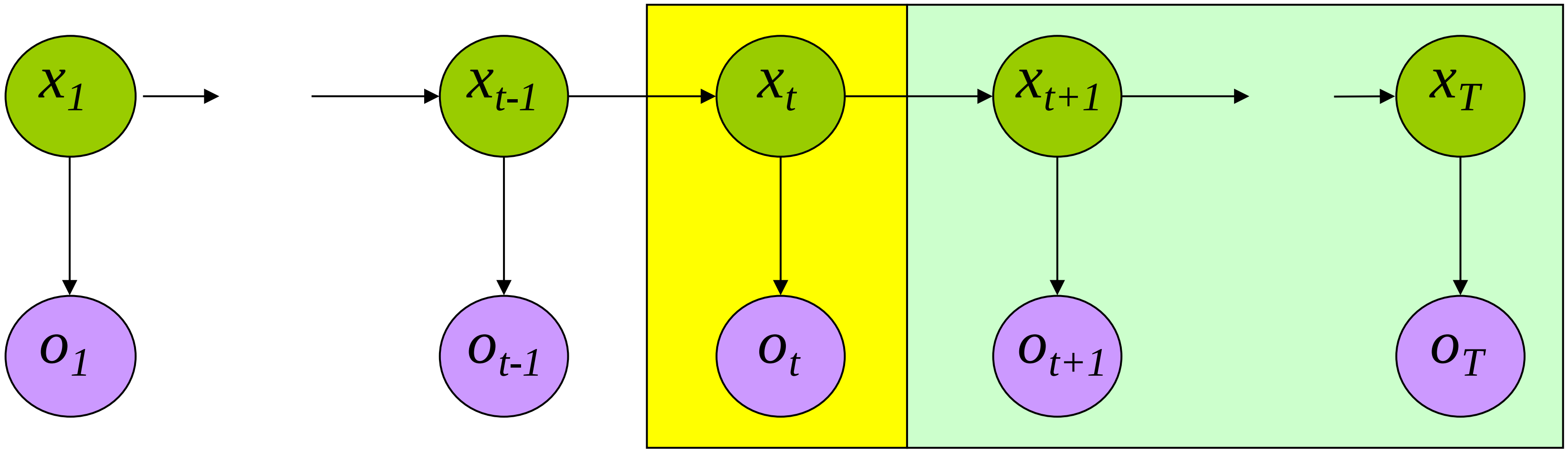
$$= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_t = i, x_{t+1} = j) P(o_{t+1} \mid x_{t+1} = j)$$

$$= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_{t+1} = j \mid x_t = i) P(x_t = i) P(o_{t+1} \mid x_{t+1} = j)$$

$$= \sum_{i=1\ldots N} P(o_1 \ldots o_t, x_t = i) P(x_{t+1} = j \mid x_t = i) P(o_{t+1} \mid x_{t+1} = j)$$

$$= \sum_{i=1\ldots N} \alpha_i(t) a_{ij} b_{j o_{t+1}}$$

Backward Procedure



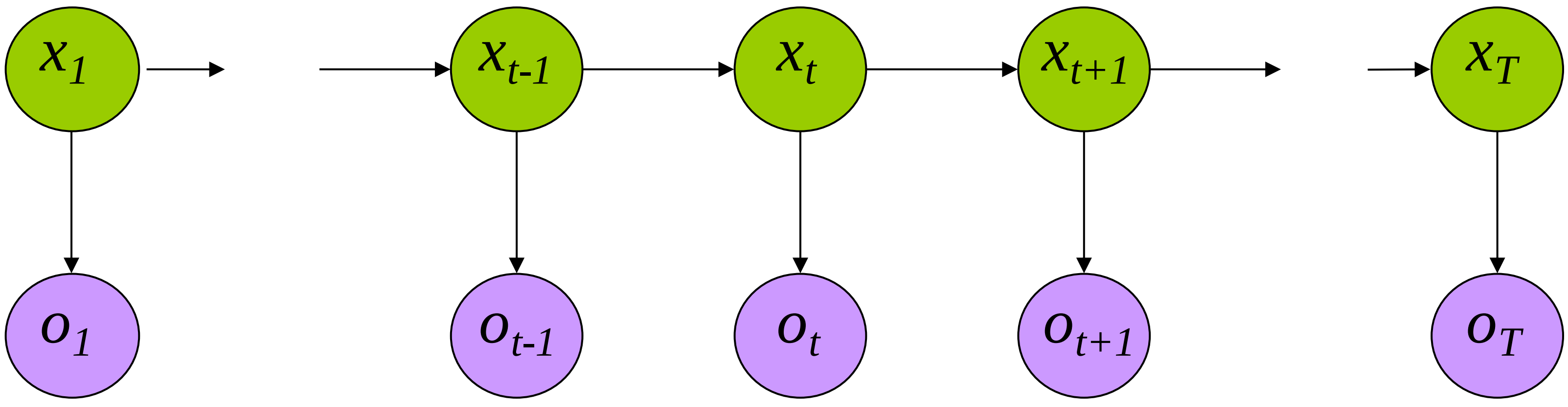
$$\beta_i(T+1) = 1$$

$$\beta_i(t) = P(o_t \ldots o_T \mid x_t = i)$$

$$\beta_i(t) = \sum_{j=1\ldots N} a_{ij} b_{io_t} \beta_j(t+1)$$

Probability of the rest of the states given the first state

Decoding Solution



$$P(O \mid \mu) = \sum_{i=1}^{N} \alpha_i(T)$$

Forward Procedure

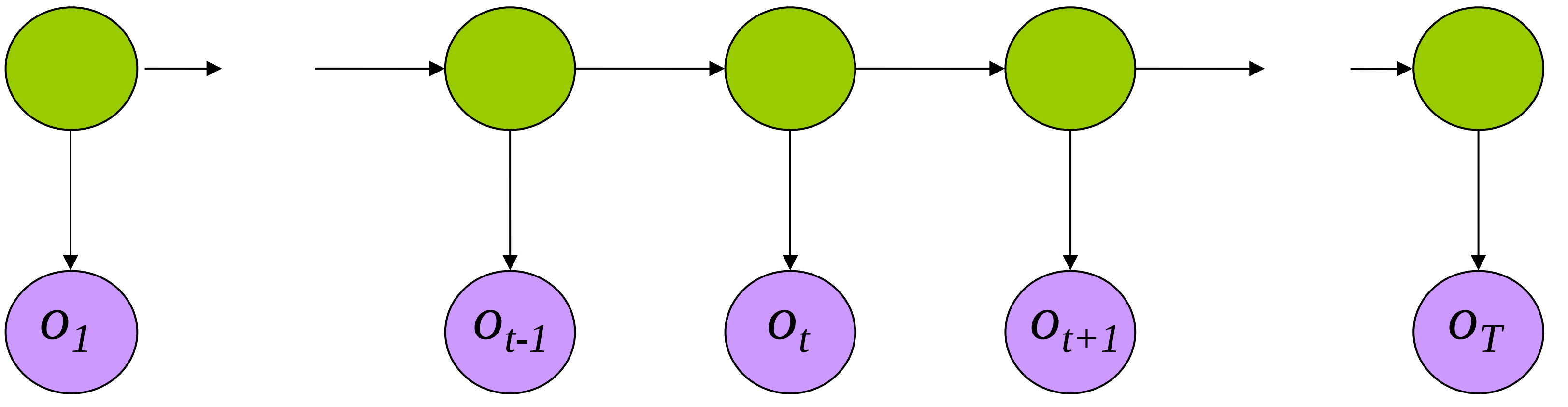
$$P(O \mid \mu) = \sum_{i=1}^{N} \pi_i \beta_i(1)$$

Backward Procedure

$$P(O \mid \mu) = \sum_{i=1}^{N} \alpha_i(t) \beta_i(t)$$

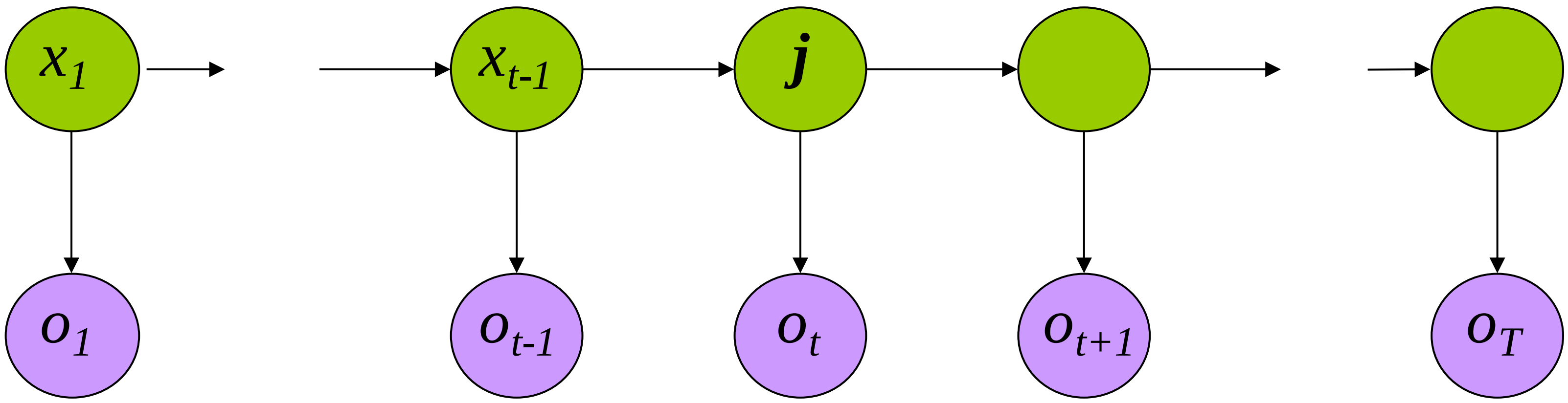
Combination

Best State Sequence



- Find the state sequence that best explains the observations
- **Viterbi** algorithm
- $\arg\max_X P(X \mid O)$

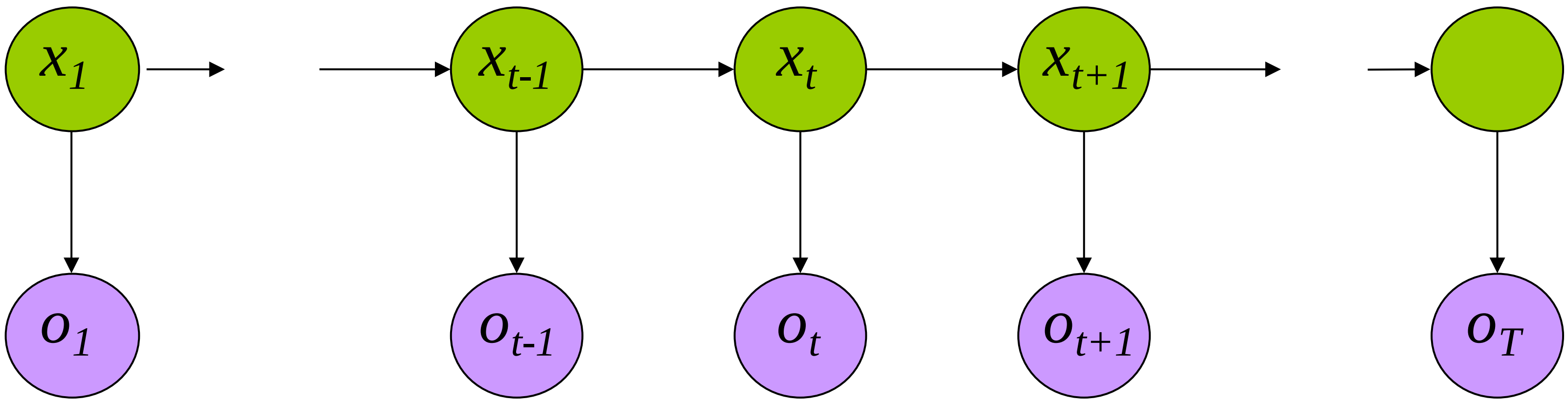
Viterbi Algorithm



$$\delta_j(t) = \max_{x_1 \ldots x_{t-1}} P(x_1 \ldots x_{t-1}, o_1 \ldots o_{t-1}, x_t = j, o_t)$$

The state sequence which maximizes the probability of seeing the observations to time t-1, landing in state j, and seeing the observation at time t

Viterbi Algorithm



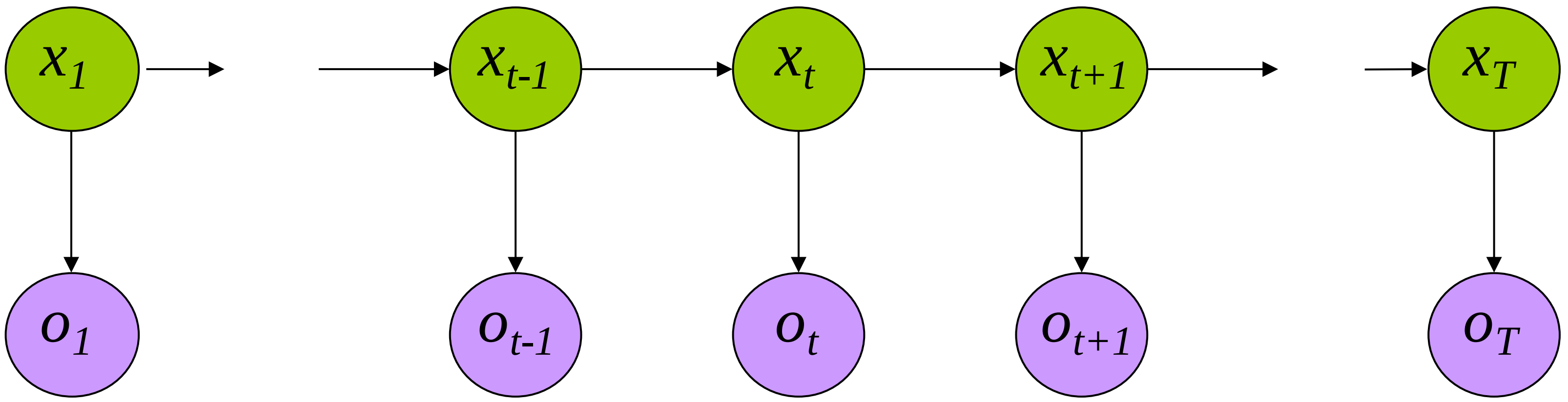
$$\delta_j(t) = \max_{x_1 \ldots x_{t-1}} P(x_1 \ldots x_{t-1}, o_1 \ldots o_{t-1}, x_t = j, o_t)$$

$$\delta_j(t+1) = \max_i \delta_i(t) a_{ij} b_{j o_{t+1}}$$

$$\psi_j(t+1) = \arg\max_i \delta_i(t) a_{ij} b_{j o_{t+1}}$$

Recursive Computation

Viterbi Algorithm



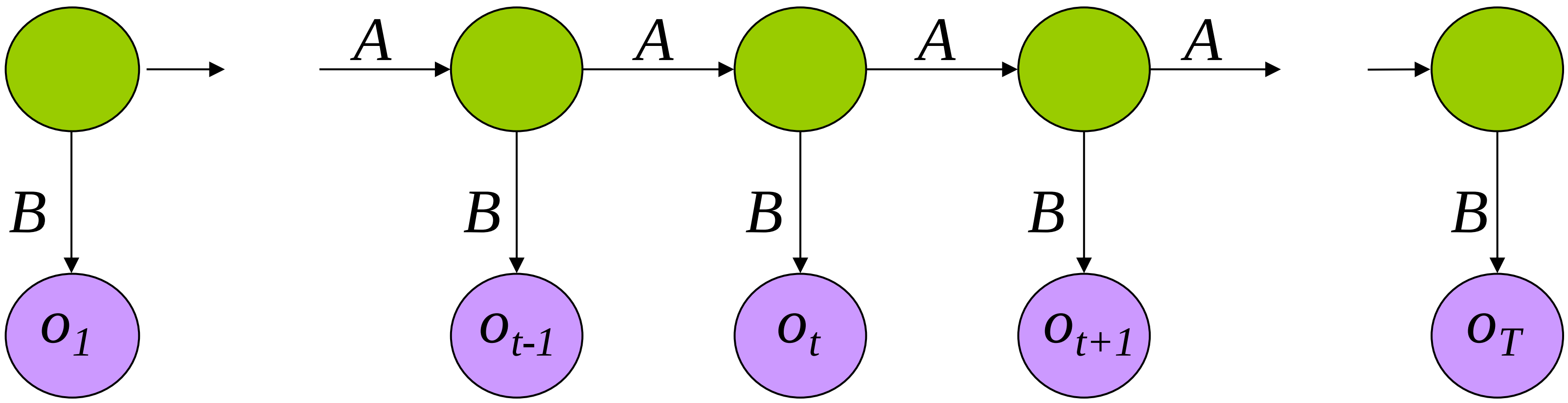
$$\hat{X}_T = \arg\max_i \delta_i(T)$$

$$\hat{X}_t = \psi_{\hat{X}_{t+1}}(t+1)$$

$$P(\hat{X}) = \arg\max_i \delta_i(T)$$

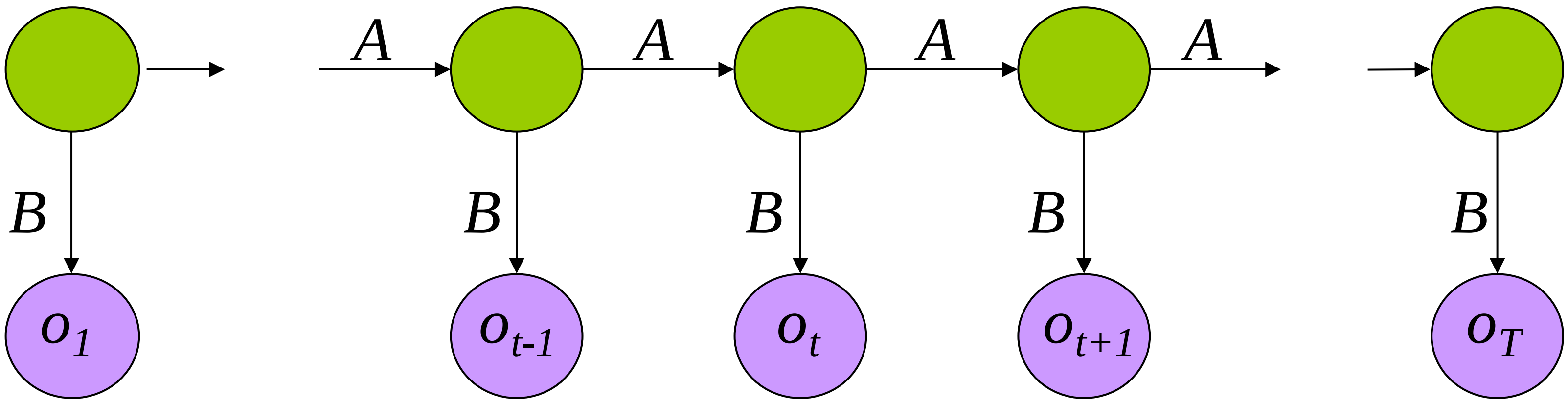
Compute the most likely state sequence by working backwards

Parameter Estimation



- Given an observation sequence, find the model that is most likely to produce that sequence.
- No analytic method
- Given a model and observation sequence, update the model parameters to better fit the observations.

Parameter Estimation



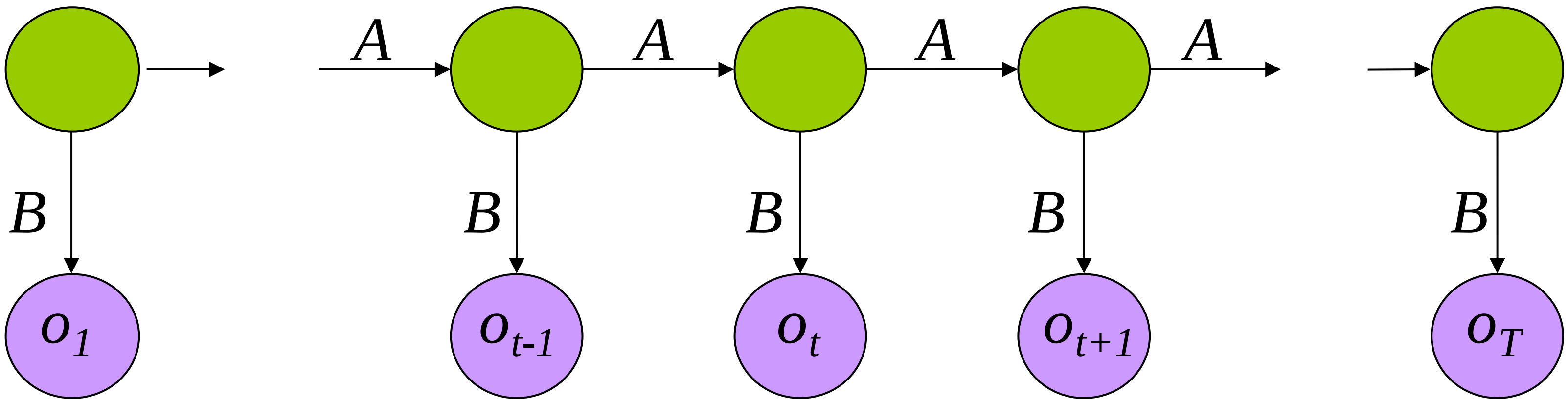
$$p_t(i,j) = \frac{\alpha_i(t) a_{ij} b_{jo_{t+1}} \beta_j(t+1)}{\sum_{m=1\ldots N} \alpha_m(t)\beta_m(t)}$$

Probability of traversing an arc

$$\gamma_i(t) = \sum_{j=1\ldots N} p_t(i,j)$$

Probability of being in state *i*

Parameter Estimation



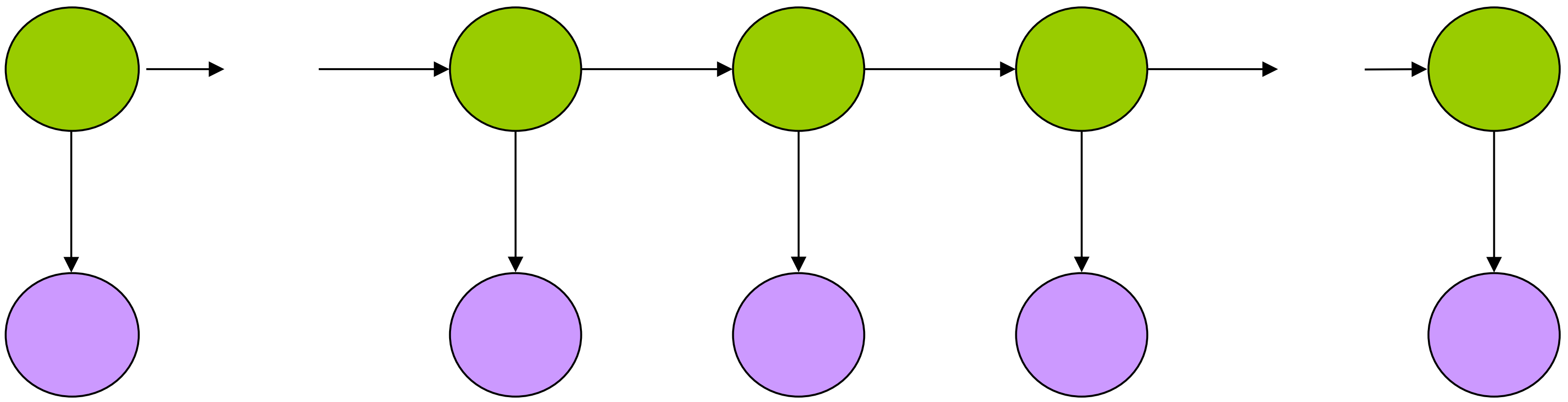
$$\hat{\pi}_i = \gamma_i(1)$$

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T} p_t(i,j)}{\sum_{t=1}^{T} \gamma_i(t)}$$

$$\hat{b}_{ik} = \frac{\sum_{\{t:o_t=k\}} \gamma_t(i)}{\sum_{t=1}^{T} \gamma_i(t)}$$

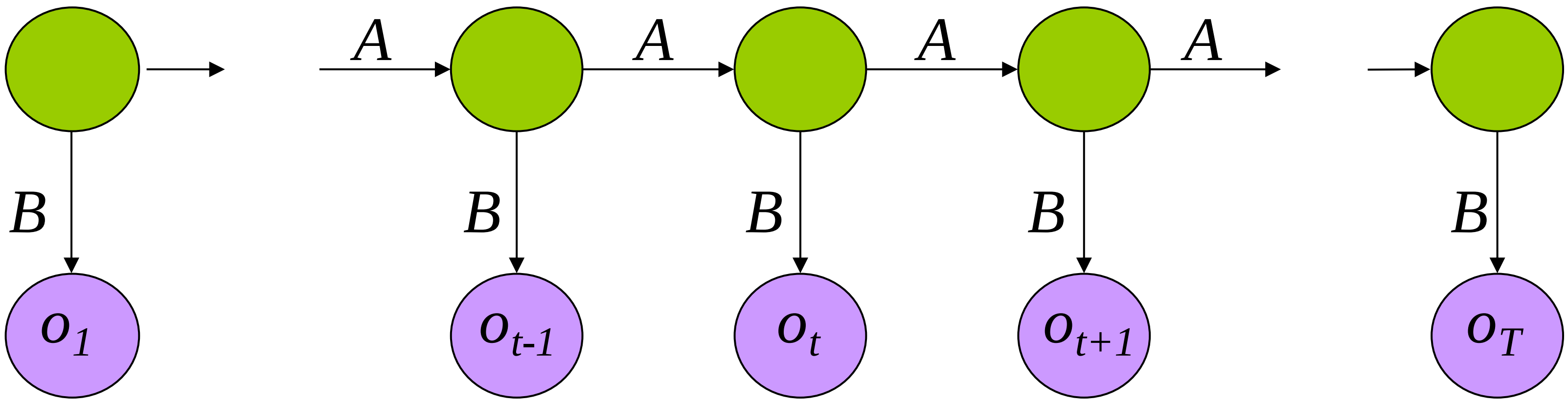
Now we can compute the new estimates of the model parameters.

HMM Applications



- Generating parameters for n-gram models
- Tagging speech
- Speech recognition

The Most Important Thing



We can use the special structure of this model to do a lot of neat math and solve problems that are otherwise not solvable.