What we will learn

1 Null and Alternative Hypotheses

2 Test Statistic

3 *P*-Value

4 Significance Level

5 One-Sample *z* Test

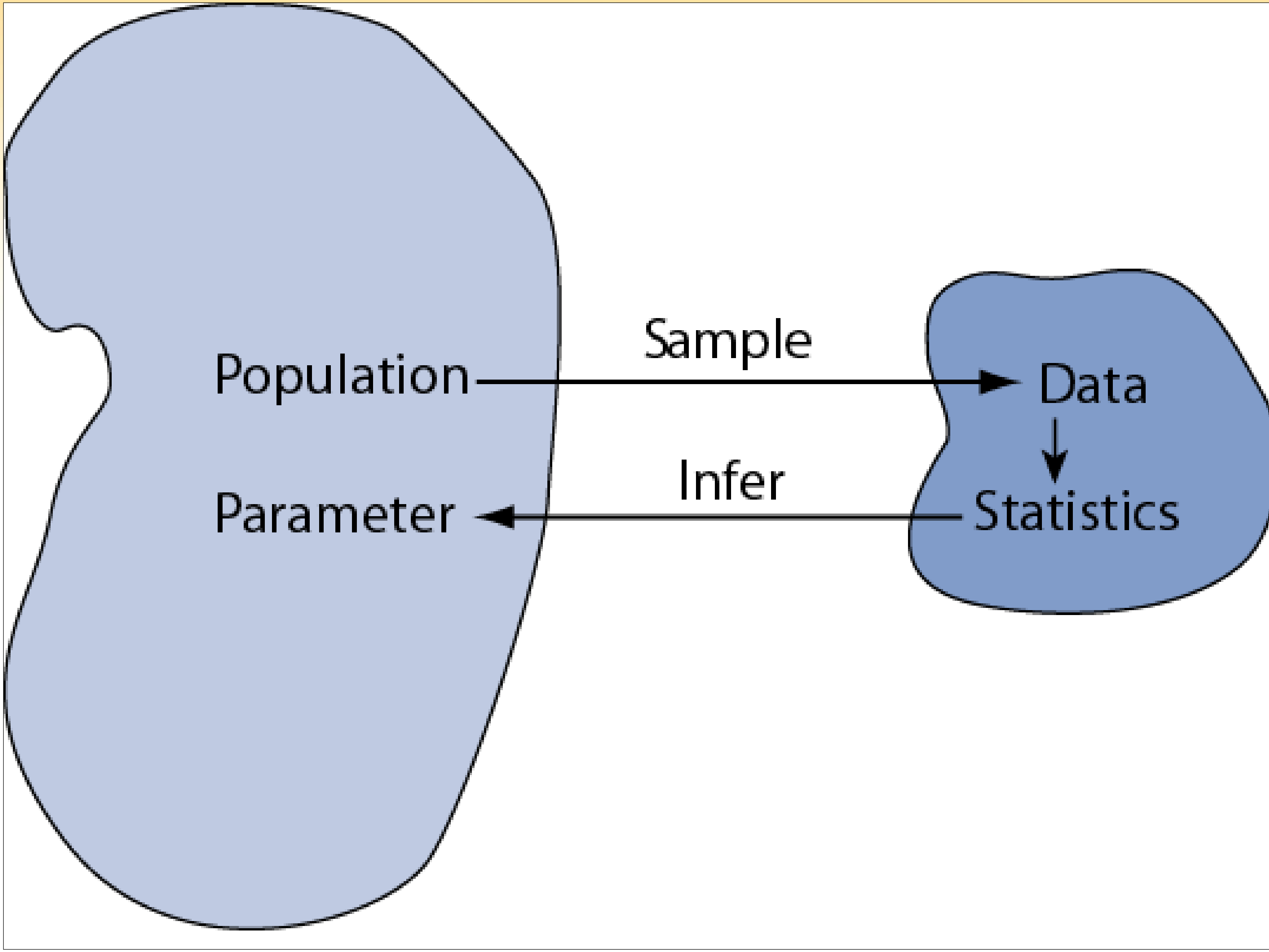
6 Power and Sample Size

Terms Introduce in Prior Chapter

- **Population** ≡ all possible values
- **Sample** ≡ a portion of the population
- **Statistical inference** ≡ generalizing from a sample to a population with calculated degree of certainty
- Two forms of statistical inference
  - **Hypothesis testing**
  - **Estimation**
- **Parameter** ≡ a characteristic of population, e.g., population mean μ
- **Statistic** ≡ calculated from data in the sample, e.g., sample mean ($\bar{x}$)

Distinctions Between Parameters and Statistics (Chapter 8 review)

	Parameters	Statistics
Source	Population	Sample
Notation	Greek (e.g., μ)	Roman (e.g., *xbar*)
Vary	No	Yes
Calculated	No	Yes

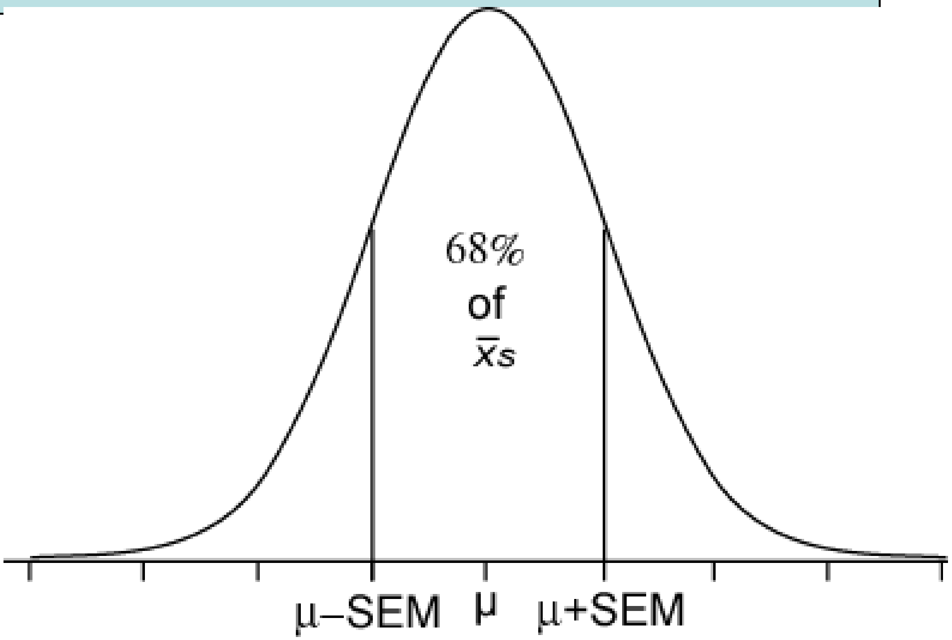

Sample
Population
Data
Infer
Parameter
Statistics

Sampling Distributions of a Mean (Introduced in Ch 8)

The **sampling distributions of a mean (SDM)** describes the behavior of a sampling mean

$$\bar{x} \sim N\left(\mu, SE_{\bar{x}}\right)$$

$$\text{where } SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$



Hypothesis Testing

- Is also called *significance testing*
- Tests a claim about a parameter using evidence (data in a sample
- The technique is introduced by considering a one-sample z test
- The procedure is broken into four steps
- *Each* element of the procedure must be understood

Hypothesis Testing Steps

A. Null and alternative hypotheses
B. Test statistic
C. P-value and interpretation
D. Significance level

Null and Alternative Hypotheses

- Convert the research question to null and alternative hypotheses
- The **null hypothesis (*H_0*)** is a claim of "no difference in the population"
- The **alternative hypothesis (*H_a*)** claims "H_0 is false"
- Collect data and seek evidence against H_0 as a way of bolstering H_a (deduction)

Illustrative Example: "Body Weight"

- **The problem:** In the 1970s, 20–29 year old men in the U.S. had a mean μ body weight of 170 pounds. Standard deviation σ was 40 pounds. We test whether mean body weight in the population now differs.
- **Null hypothesis** $H_{0:}$ μ = 170 ("no difference")
- The **alternative hypothesis** can be either $H_{a:}$ μ > 170 (**one-sided test**) or
  $H_{a:}$ μ ≠ 170 (**two-sided test**)

Test Statistic

This is an example of a one-sample test of a mean when σ is known. Use this statistic to test the problem:

$$z_{stat} = \frac{\bar{x} - \mu_0}{SE_{\bar{x}}}$$

$$\text{where } \mu_0 \equiv \text{ population mean assuming } H_0 \text{ is true}$$

$$\text{and } SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

Illustrative Example: *z* statistic

- For the illustrative example, $\mu_0 = 170$
- We know $\sigma = 40$
- Take an SRS of $n = 64$. Therefore

$$SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{40}{\sqrt{64}} = 5$$

- If we found a sample mean of 173, then

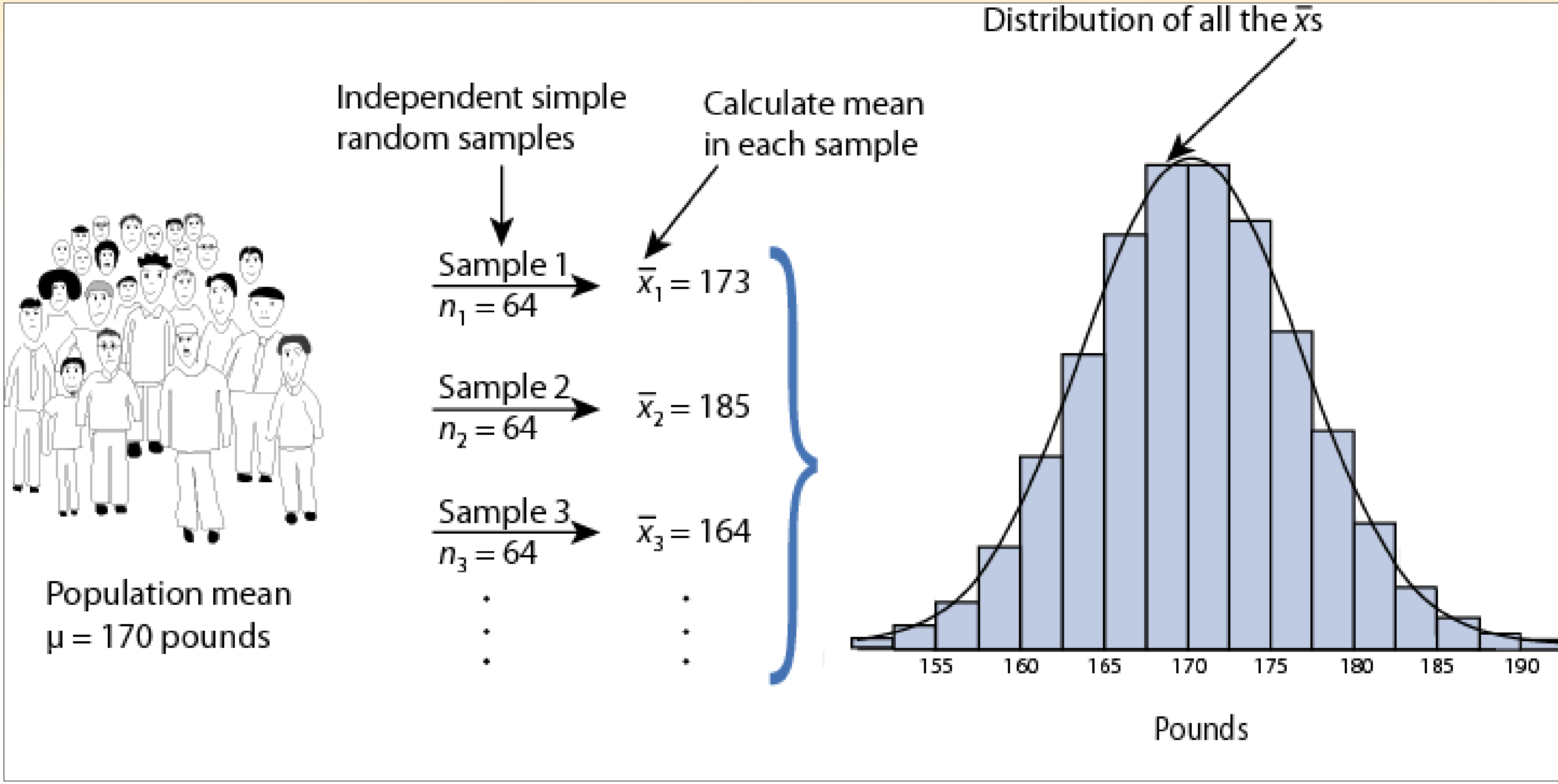
$$z_{\text{stat}} = \frac{\bar{x} - \mu_0}{SE_{\bar{x}}} = \frac{173 - 170}{5} = 0.60$$

Illustrative Example: *z* statistic

If we found a sample mean of 185, then

$$z_{\text{stat}} = \frac{\bar{x} - \mu_0}{SE_{\bar{x}}} = \frac{185 - 170}{5} = 3.00$$

Reasoning Behinμz_{stat}



Sampling distribution of xbar under H_0: μ = 170 for n = 64 ⇒ $\bar{x} \sim N(170, 5)$

P-value

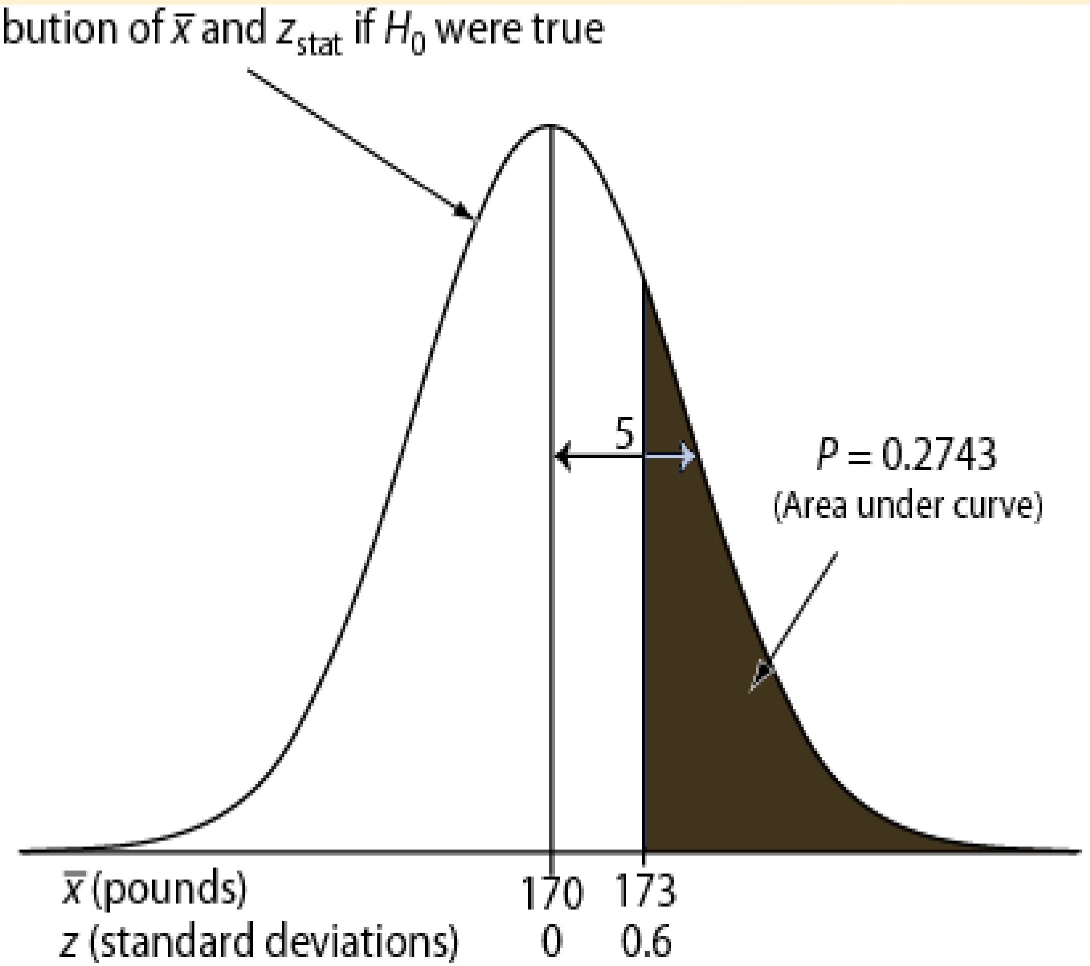
- The *P*-value answer the question: What is the probability of the observed test statistic or one more extreme **when H_0 *is true*?**
- This corresponds to the AUC in the tail of the Standard Normal distribution beyond the $z_{stat.}$
- Convert *z* statistics to *P*-value :

  For H_a: $\mu > \mu_0 \Rightarrow P = \Pr(Z > z_{stat})$ = right-tail beyond z_{stat}

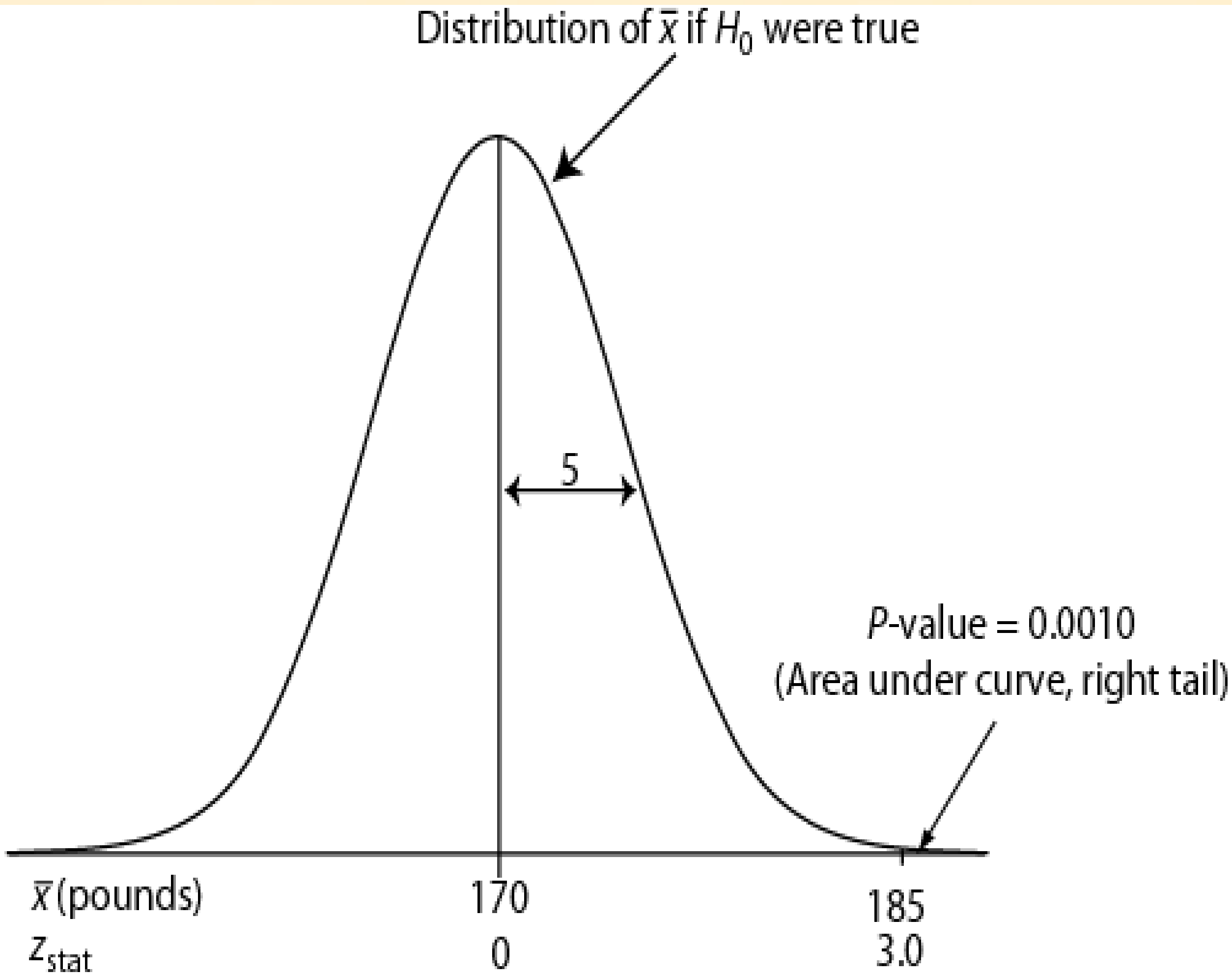
  For H_a: $\mu < \mu_0 \Rightarrow P = \Pr(Z < z_{stat})$ = left tail beyond z_{stat}

  For H_a: $\mu \neq \mu_0 \Rightarrow P = 2 \times$ one-tailed *P*-value
- Use Table B or software to find these probabilities (next two slides).

One-sided P-value for z_{stat} of 0.6

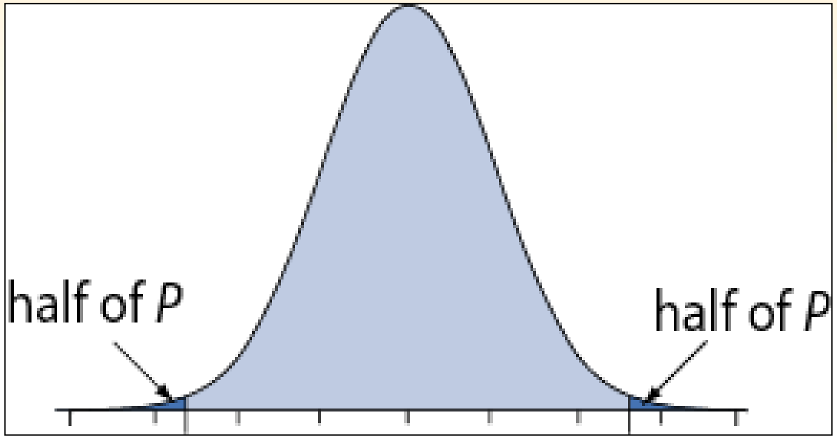


One-sided P-value for z_{stat} of 3.0



Two-Sided P-Value

- One-sided $H_a \Rightarrow$ AUC in tail beyond z_{stat}
- Two-sided $H_a \Rightarrow$ consider potential deviations in both directions $\Rightarrow$ double the one-sided P-value



Examples: If one-sided $P = 0.0010$, then two-sided $P = 2 \times 0.0010 = 0.0020$. If one-sided $P = 0.2743$, then two-sided $P = 2 \times 0.2743 = 0.5486$.

Interpretation

- *P*-value answer the question: What is the probability of the observed test statistic … **when H_0 *is true*?**
- Thus, smaller and smaller *P*-values provide stronger and stronger evidence against H_0
- Small *P*-value $\Rightarrow$ strong evidence

Interpretation

Conventions*

$P > 0.10 \Rightarrow$ non-significant evidence against H_0

$0.05 < P \leq 0.10 \Rightarrow$ marginally significant evidence

$0.01 < P \leq 0.05 \Rightarrow$ significant evidence against H_0

$P \leq 0.01 \Rightarrow$ highly significant evidence against H_0

Examples

$P = .27 \Rightarrow$ non-significant evidence against H_0

$P = .01 \Rightarrow$ highly significant evidence against H_0

*** It is *unwise* to draw firm borders for "significance"**

α-Level (Used in some situations)

- Let α ≡ probability of erroneously rejecting H_0
- Set α threshold (e.g., let α = .10, .05, *or whatever*)
- Reject H_0 when $P \le \alpha$
- Retain H_0 when $P > \alpha$
- Example: Set α = .10. Find $P = 0.27 \Rightarrow$ retain H_0
- Example: Set α = .01. Find $P = .001 \Rightarrow$ reject H_0

(Summary) One-Sample z Test

A. Hypothesis statements

H_0: $\mu = \mu_0$ vs.

H_a: $\mu \neq \mu_0$ (two-sided) or

H_a: $\mu < \mu_0$ (left-sided) or

H_a: $\mu > \mu_0$ (right-sided)

B. Test statistic

$$z_{stat} = \frac{\bar{x} - \mu_0}{SE_{\bar{x}}} \text{ where } SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

C. P-value: convert z_{stat} to P value

D. Significance statement (usually not necessary)

Conditions for z test

- σ known (not from data)
- Population approximately Normal or large sample (central limit theorem)
- SRS (or facsimile)
- Data valid

The Lake Wobegon Example

“where all the children are above average”

- Let X represent Weschler Adult Intelligence scores (WAIS)
- Typically, X ~ N(100, 15)
- Take SRS of n = 9 from Lake Wobegon population
- Data $\Rightarrow$ {116, 128, 125, 119, 89, 99, 105, 116, 118}
- Calculate: x-bar = 112.8
- Does sample mean provide strong evidence that population mean μ > 100?

Example: “Lake Wobegon”

A. Hypotheses:

H_0: μ = 100 versus

H_a: μ > 100 (one-sided)

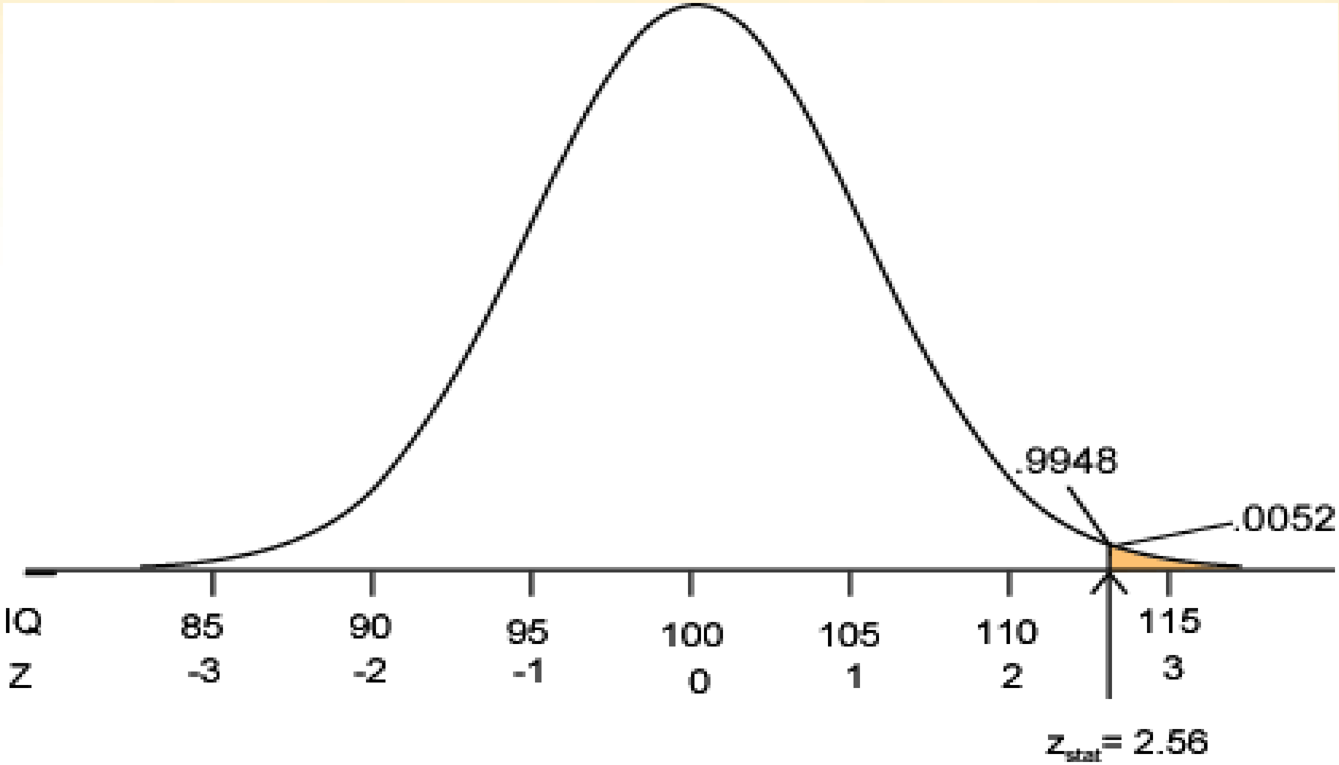
H_a: μ ≠ 100 (two-sided)

B. Test statistic:

$$SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{9}} = 5$$

$$z_{stat} = \frac{\bar{x} - \mu_0}{SE_{\bar{x}}} = \frac{112.8 - 100}{5} = 2.56$$

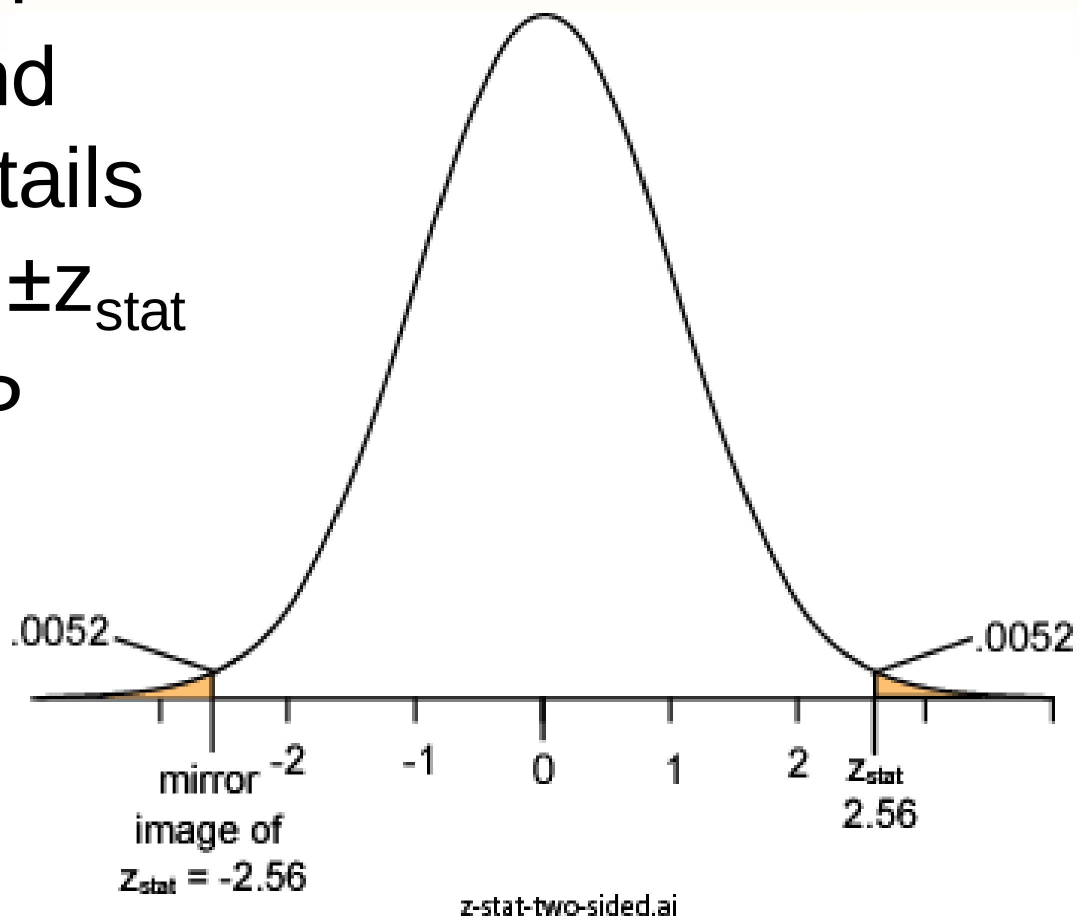
C. *P*-value: $P = \Pr(Z \geq 2.56) = 0.0052$



P =.0052 $\Rightarrow$ it is unlikely the sample came from this null distribution $\Rightarrow$ strong evidence against H_0

Two-Sided *P*-value: Lake Wobegon

- H_a: $\mu \neq 100$
- Considers random deviations "up" and "down" from $\mu_0 \Rightarrow$ tails above and below $\pm z_{stat}$
- Thus, two-sided *P* $= 2 \times 0.0052$ $= 0.0104$



Power and Sample Size

Two types of decision errors:

Type I error = erroneous rejection of true H_0

Type II error = erroneous retention of false H_0

Decision	Truth: H_0 true	Truth: H_0 false
Retain H_0	Correct retention	Type II error
Reject H_0	Type I error	Correct rejection

$\alpha \equiv$ probability of a Type I error

$\beta \equiv$ Probability of a Type II error

Power

- $\beta \equiv$ probability of a Type II error

  $\beta = \Pr(\text{retain } H_0 \mid H_0 \text{ false})$

  (the “|” is read as “given”)

- $1 - \beta$ = “Power” $\equiv$ probability of avoiding a Type II error

  $1 - \beta = \Pr(\text{reject } H_0 \mid H_0 \text{ false})$

Power of a z test

$$1-\beta=\Phi\left(-z_{1-\frac{\alpha}{2}}+\frac{|\mu_0-\mu_a|\sqrt{n}}{\sigma}\right)$$

where

- $\Phi(z)$ represent the cumulative probability of Standard Normal Z
- μ_0 represent the population mean under the null hypothesis
- μ_a represents the population mean under the alternative hypothesis

Calculating Power: Example

A study of n = 16 retains H_0: μ = 170 at α = 0.05 (two-sided); σ is 40. What was the power of test's conditions to identify a population mean of 190?

$$1-\beta=\Phi\left(-z_{1-\frac{\alpha}{2}}+\frac{|\mu_0-\mu_a|\sqrt{n}}{\sigma}\right)$$

$$=\Phi\left(-1.96+\frac{|170-190|\sqrt{16}}{40}\right)$$

$$=\Phi(0.04)$$

$$=0.5160$$

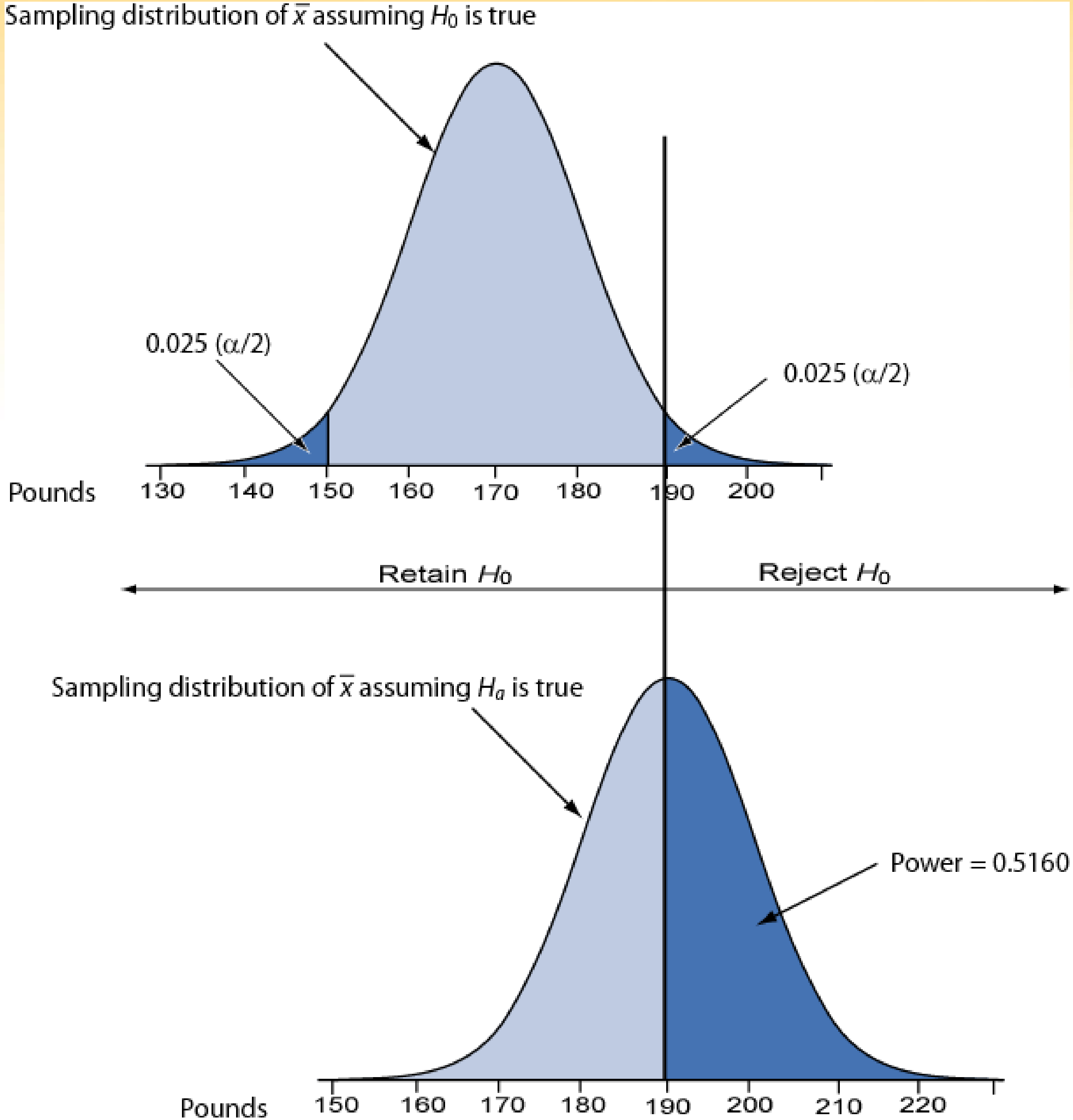
Reasoning Behind Power

- Competing sampling distributions

  Top curve (next page) assumes H_0 is true

  Bottom curve assumes H_a is true

  α is set to 0.05 (two-sided)
- We will reject H_0 when a sample mean exceeds 189.6 (right tail, top curve)
- The probability of getting a value greater than 189.6 on the bottom curve is 0.5160, corresponding to the power of the test



Sampling distribution of $\bar{x}$ assuming H_0 is true
0.025 (α/2)
0.025 (α/2)
Pounds
130 140 150 160 170 180 190 200
Retain H_0
Reject H_0
Sampling distribution of $\bar{x}$ assuming H_a is true
Power = 0.5160
Pounds
150 160 170 180 190 200 210 220

Sample Size Requirements

Sample size for one-sample *z* test:

$$n = \frac{\sigma^2 \left(z_{1-\beta} + z_{1-\frac{\alpha}{2}} \right)^2}{\Delta^2}$$

where

$1 - \beta \equiv$ desired power

$\alpha \equiv$ desired significance level (two-sided)

$\sigma \equiv$ population standard deviation

$\Delta = \mu_0 - \mu_a \equiv$ the **difference worth detecting**

Example: Sample Size Requirement

How large a sample is needed for a one-sample *z* test with 90% power and α = 0.05 (two-tailed) when σ = 40? Let H_0: μ = 170 and H_a: μ = 190 (thus, $\Delta = \mu_0 - \mu_a = 170 - 190 = -20$)

$$n = \frac{\sigma^2 \left(z_{1-\beta} + z_{1-\frac{\alpha}{2}}\right)^2}{\Delta^2} = \frac{40^2 (1.28 + 1.96)^2}{-20^2} = 41.99$$

Round up to 42 to ensure adequate power.

Illustration: conditions for 90% power.