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Linear Algebra And its Applications

(1) Find the equation of the parabola $y = A + Bx + Cx^2$ that passes through 3 points $(1,1)$, $(2,-1)$ and $(3,1)$ using guassian elimination.

Sol:-

Eq'n of parabola $\Rightarrow$ $y = A + Bx + Cx^2$

$(1,1) \Rightarrow$ $1 = A + B + C$

$(2,-1) \Rightarrow$ $-1\cancel{2} = A + 2B + 4C$

$(3,1) \Rightarrow$ $1 = A + 3B + 9C$

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & \cancel{3}4 \\ 1 & 3 & 9 \end{bmatrix}} \underset{X}{\begin{bmatrix} A \\ B \\ C \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

$$[A \ \ b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ $\quad R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$2C = 4 \Rightarrow C = 2$ //

$B + 3C = -2$

$B + 6 = -2$

$B = -8$ //

$A + B + C = 1$

$A - 8 + 2 = 1$

$A = 7$ //

$\therefore$ The equation of the parabola :- $y = 2x^2 - 8x + 7$ //

(2) Find LU decomposition of the matrix :-

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

Sol:-

$R_2 \to R_2 - 2R_1$ $\qquad l_{21} = 2$

$R_3 \to R_3 + 5R_1$ $\qquad l_{31} = -5$

$R_4 \to R_4 - 5R_1$ $\qquad l_{41} = 5$

$$A \sim \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$ $\qquad l_{32} = -2$

$R_4 \to R_4 + 2R_2$ $\qquad l_{42} = -2$

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$$\sim \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$ $\qquad$ $l_{43} = 3$

$$\sim \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} = U$$ //

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix}$$ //

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

(3) Let $T: R^3 \to R^3$ defined by
$T(x,y,z) = (x+2y-z,\ y+3,\ x+y-2z)$

(i) Find the matrix T relative to the standard basis of R^3

(ii) Find the basis for 4 fundamental subspaces of T

(iii) Find the eigen values and eigen vectors of T.

(iv) Decompose $T = QR$

Sol:- (i)

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y-z \\ y+z \\ x+y-2z \end{bmatrix}$$

Standard basis of R^3 : $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$x+2y-z \Rightarrow x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-z)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y+z \Rightarrow \overset{x}{0}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \overset{y}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \overset{z}{1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x+y-2z \Rightarrow \overset{x}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \overset{y}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \overset{z}{(-2)}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} //$$

(ii) For $C(A)$ or $C(T)$:- $AX = b$

$$[A\ b] = [T\ b] = \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 1 & -2 & b_3 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$[T\ b] \sim \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & -1 & -1 & b_3 - b_1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{bmatrix}$$

Basis for $C(T)$: $\{(1,0,1), (2,1,1)\}$ //

Basis for $C(A^T)$ or $C(T^T)$: $\{(1,2,-1), (0,1,1)\}$ //

For $N(A)$ or $N(T)$:-

$$T \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

convert to Row-reduced form :-

$R_1 \to R_1 - 2R_2$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Consider the non-pivot / free vectors: $(-3, 1, 0)$

$$\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ Basis of $N(A)$: $\{(3, -1, 1)\}$ //

$C(A) \perp N(A^T)$

$b_3 - b_1 + b_2 = 0 \Rightarrow -b_1 + b_2 + b_3 = 0$

$\Downarrow$

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Basis of $N(A^T)$: $\{(-1, 1, 1)\}$ //

(iii)

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$
$$|T - \lambda I| = 0$$

$$T - \lambda I = \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 1-d & 2 & -1 \\ 0 & 1-d & 1 \\ 1 & 1 & -2-d \end{vmatrix} = 0$$

$$(1-d)\left[(1-d)(-2-d) - 1\right] - 2\left[-1\right] - 1\left[-(1-d)\right] = 0$$

$$(1-d)\left[-2 - d + 2d + d^2 - 1\right] + 2 + 1 - d = 0$$

$$(1-d)(-3 + d + d^2) + 3 - d = 0$$

$$-\cancel{3} + \cancel{d} + \cancel{d^2} + 3d - \cancel{d^2} - d^3 + \cancel{3} - \cancel{d} = 0$$

$$d^3 - 3d = 0$$

$$d(d^2 - 3) = 0$$

$$d = 0 \quad , \quad d^2 - 3 = 0$$

$$d = \pm\sqrt{3}$$

Eigen values: $d_1 = 0$, $d_2 = +\sqrt{3}$, $d_3 = -\sqrt{3}$ //

$AX = dx$

For $d_1 = 0$:- $AX = 0$ or $Tx = 0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$R_1 \to R_1 - 2R_2$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \quad //$$

For $\lambda = +\sqrt{3}$

$$Ax = \sqrt{3}x$$
$$Ax - \sqrt{3}x = 0$$

$$A - \lambda I = \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{(1-\sqrt{3})} R_1$$

$$\sim \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & -\frac{(1+\sqrt{3})}{(1-\sqrt{3})} & \frac{-(2+\sqrt{3})(1-\sqrt{3})+1}{(1-\sqrt{3})} \end{bmatrix}$$

$1 - \frac{1}{(1-\sqrt{3})} \times 2$

$1 - \frac{2}{(1-\sqrt{3})}$

$\frac{(1-\sqrt{3})-2}{1-\sqrt{3}}$

$R_3 \rightarrow R_3 + \frac{(1+\sqrt{3})}{(1-\sqrt{3})^2} R_2$

$\frac{-1-\sqrt{3}}{}$

$-\frac{(1+\sqrt{3})}{(1-\sqrt{3})}$

$$\sim \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 0 & \frac{(-(2+\sqrt{3})(1-\sqrt{3})+1)(1-\sqrt{3})+(1+\sqrt{3})}{(1-\sqrt{3})^2} \end{bmatrix}$$

$-(2+\sqrt{3}) + \frac{1}{(1-\sqrt{3})}$

$\frac{-(2+\sqrt{3})(1-\sqrt{3})+1}{(1-\sqrt{3})}$

$+\frac{(1+\sqrt{3})}{(1-\sqrt{3})}$

$= 2 \times (1-\sqrt{3})$

$\frac{(1+\sqrt{3})}{(1-\sqrt{3})^2}$

$$(-(2-2\sqrt{3}+\sqrt{3}-3)+1)(1-\sqrt{3})+(1+\sqrt{3})$$

$$= (-2+2\sqrt{3}-\sqrt{3}+3+1)(1-\sqrt{3}) + (1+\sqrt{3})$$

$\frac{-(2+\sqrt{3})(1-\sqrt{3})+1+}{(1-\sqrt{3})} \quad \frac{(1+\sqrt{3}}{(1-\sqrt{3}}$

$$= (2+\sqrt{3})(1-\sqrt{3}) + (1+\sqrt{3})$$

$$= (2 \cancel{-2\sqrt{3}} + \cancel{\sqrt{3}} - 3) + 1 + \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$$

$\frac{}{(1-\sqrt{3})^2}$

$$\sim \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 0 & \cancel{\sqrt{3}/(1-\sqrt{3})^2}\ 0 \end{bmatrix}$$

$\cancel{R_3 \rightarrow R_3 \times \frac{(1-\sqrt{3})^2}{\sqrt{3}}}$

$$\cancel{\sim \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\cancel{R_2 \rightarrow R_2 - R_3}$

$$\sim \begin{bmatrix} (1-\sqrt{3}) & 2 & -1 \\ 0 & (1-\sqrt{3}) & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 / 1-\sqrt{3}$

$R_2 \to R_2 / 1-\sqrt{3}$

$$\sim \begin{bmatrix} 1 & 2/(1-\sqrt{3}) & -1/(1-\sqrt{3}) \\ 0 & 1 & 1/(1-\sqrt{3}) \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - \frac{2}{(1-\sqrt{3})} R_2$$

$$\begin{array}{cccc} & x & y & z \\ \sim & \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \\ 0 \end{array} & \left.\begin{array}{c} \sqrt{3}/(1-\sqrt{3}) \\ 1/1-\sqrt{3} \\ 0 \end{array}\right] \end{array}$$

$$V_2 = \begin{bmatrix} -\sqrt{3}/(1-\sqrt{3}) \\ -1/(1-\sqrt{3}) \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} (\sqrt{3}+3)/2 \\ (1+\sqrt{3})/2 \\ 1 \end{bmatrix} //$$

For $\lambda = -\sqrt{3}$

$Ax = -\sqrt{3} \cdot x$

$Ax + \sqrt{3}x = 0$

$$A - \lambda I = \begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{(1+\sqrt{3})} R_1$$

$$\frac{-1}{(1-\sqrt{3})} - \frac{2 \cdot 1}{(1-\sqrt{3})(1-\sqrt{3})}$$

$$\frac{-1}{(1-\sqrt{3})} - \frac{2}{(1-\sqrt{3})^2}$$

$$\frac{-(1-\sqrt{3})-2}{(1-\sqrt{3})^2}$$

$$-1+\sqrt{3}-2$$

$$\frac{\sqrt{3}-3}{(1-\sqrt{3})^2}$$

$$\sqrt{3}(1-\sqrt{3})$$

$$\frac{\sqrt{3}}{(1-\sqrt{3})}$$

$$-\frac{\sqrt{3} \times (1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$1-3$$

$$-2$$

$$\frac{\sqrt{3}(1+\sqrt{3})}{2}$$

$$\frac{\sqrt{3}+3}{2}$$

$$\frac{+1(1+\sqrt{3})}{\cancel{(1-\sqrt{3})}(1+}$$

$$\frac{(-2+\sqrt{3})(1+\sqrt{3})+1}{(1+\sqrt{3})}$$

$$1 - \frac{2}{(1+\sqrt{3})}$$

$$\frac{1+\sqrt{3}-2}{1+\sqrt{3}}$$

$$\frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$-2+\sqrt{3}+\frac{1}{(1+\sqrt{3})}$$

$$\sim \begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & \frac{\sqrt{3}-1}{1+\sqrt{3}} & \frac{(-2+\sqrt{3})(1+\sqrt{3})+1}{1+\sqrt{3}} \end{bmatrix}$$

$$R_3 \to R_3 - \frac{\sqrt{3}-1}{(1+\sqrt{3})^2} R_2$$

$$\sim \begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 / 1+\sqrt{3}$

$R_2 \to R_2 / 1+\sqrt{3}$

$$\sim \begin{bmatrix} 1 & 2/1+\sqrt{3} & -1/1+\sqrt{3} \\ 0 & 1 & 1/1+\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - \frac{2}{(1+\sqrt{3})} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-\sqrt{3}-3}{(1+\sqrt{3})^2} \\ 0 & 1 & 1/1+\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} \frac{\sqrt{3}+3}{(1+\sqrt{3})^2} \\ -1/1+\sqrt{3} \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \\ 1 \end{bmatrix} //$$

$\frac{-1}{(1+\sqrt{3})} \frac{(1-\sqrt{3})}{(1-\sqrt{3})}$

$\frac{(1-\sqrt{3})}{2}$

$\frac{\sqrt{3}-1}{1+\sqrt{3}} = \frac{2(1-\sqrt{3})}{}$

$-2 - 2\sqrt{3} + \sqrt{3} + 3 + 1$

$\frac{(2-\sqrt{3})}{(1+\sqrt{3})}$

$-\frac{(\sqrt{3}-1)}{(1+\sqrt{3})^2} 1$

$\frac{(2-\sqrt{3})(1+\sqrt{3}) - (\sqrt{3}-1)}{(1+\sqrt{3})^2}$

$2 + \cancel{2\sqrt{3}} - \cancel{\sqrt{3}} - 3 - \cancel{\sqrt{3}} + 1$

0

$\frac{-1}{(1+\sqrt{3})} - \frac{2}{(1+\sqrt{3})} \cdot \frac{1}{1+\sqrt{3}}$

$\frac{-(1+\sqrt{3}) - 2}{(1+\sqrt{3})^2}$

$-1 - \sqrt{3} - 2$

$-\sqrt{3} - 3$

$-(3-\sqrt{3})$

$-\cancel{\sqrt{3}}(-\sqrt{3}$

$\sqrt{3} - 3$

$\frac{\sqrt{3}(\cancel{\sqrt{3}}1-\sqrt{3})}{(1+\sqrt{3})^2}$

$\sqrt{3}(1+\sqrt{3})$

$\frac{\sqrt{3}}{(1+\sqrt{3})}$

$\frac{\sqrt{3}}{(1+\sqrt{3})} \frac{(1-\sqrt{3})}{(1-\sqrt{3})}$

$\frac{\sqrt{3}-3}{-2}$

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Eigen values: $\lambda_1 = 0$, $\lambda_2 = \sqrt{3}$, $\lambda_3 = -\sqrt{3}$

Eigen vectors: $v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} \frac{3+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \\ 1 \end{bmatrix}$

T = QR

$$T = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \end{array}$$

Gram Schmidth Process :-

$$q_1 = \frac{a}{\|a\|} = \frac{(1,0,1)}{\sqrt{1^2+1^2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$B = b - (q_1^T b) q_1$$

$$= (2,1,1) - (1/\sqrt{2} \;\; 0 \;\; 1/\sqrt{2}) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= (2,1,1) - \frac{3}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$B = \begin{bmatrix} 1/\cancel{\sqrt{2}} \\ 1 \\ -1/\cancel{\sqrt{2}} \end{bmatrix}$$

$$q_2 = \frac{B}{\|B\|} = \frac{(1/\cancel{2} \;\; 1 \;\; -1/\cancel{2})}{\sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + \left(-\frac{1}{2}\right)^2}} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$C = c - (q_1^T c) q_1 - (q_2^T c) q_2$$

$Ax = R$

$x = A^T B$

$$C = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \left(\frac{-3}{\sqrt{2}}\right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \left(\frac{3}{\sqrt{6}}\right) \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ $\Rightarrow$ dependent on the other two vectors or no independent component

$$\underset{3\times 3}{T} = \underset{3\times 2}{Q}\ \underset{2\times 3}{R}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$R = Q^T T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \end{bmatrix}$$

$$T = QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \end{bmatrix}$$

//

(4) Find a best fit line $y = C + Dx$ for the following data

x	-4	1	2	3
y	4	6	10	8

Sol:-

$$x = \begin{bmatrix} C \\ D \end{bmatrix}$$

$$Ax = b$$

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

Least square method:-

normal equation: $A^T A \hat{x} = A^T b$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \hat{x} = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{116}\begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$= \frac{1}{116}\begin{bmatrix} 772 \\ 80 \end{bmatrix} = \begin{bmatrix} 193/29 \\ 20/29 \end{bmatrix}$$

Best fit line : $y = C + Dx = \frac{193}{29} + \frac{20}{29}x$ //

(5) Find the projection matrices P and Q onto the plane $x_1 + x_2 + 3x_3 + 4x_5 = 0$ and its orthogonal complement respectively.

$Ax = b$

$$\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$Ax = 0$

$A = \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}$

$P = A(A^T A)^{-1} A^T$

$N(A)$ is : $\{(-1,1,0,0,0), (-3,0,1,0,0), (0,0,0,1,0), (-4,0,0,0,1)\}$

$$N(A) = A = \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 26/27 & -3/27 & 0 & -4/27 \\ -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 1 & 0 \\ -4/27 & -12/27 & 0 & 11/27 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 26/27 & -1/27 & -3/27 & 0 & -4/27 \\ -1/27 & 26/27 & -3/27 & 0 & -4/27 \\ -3/27 & -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 0 & 1 & 0 \\ -4/27 & -4/27 & -12/27 & 0 & 11/27 \end{bmatrix}$$

$C(A^T)$ is the orthogonal complement of $N(A)$

$$C(A^T) = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix}$$

$P = I - Q$

$Q = I - P$

$$\Rightarrow \quad Q = \begin{bmatrix} 1/27 & +1/27 & 3/27 & 0 & 4/27 \\ 1/27 & 1/27 & 3/27 & 0 & 4/27 \\ 3/27 & 3/27 & 9/27 & 0 & 12/27 \\ 0 & 0 & 0 & 0 & 0 \\ 4/27 & 4/27 & 12/27 & 0 & 16/27 \end{bmatrix}$$

(6) For which range of number's 'a' the matrix A is positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

which 3x3 matrix (symmetric) B produces this function $f = x^T A x$?

where $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

Sol:- $$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

$2 - \frac{2 \cdot 2}{a}$

$2 - \frac{4}{a}$

$\frac{2a-4}{a}$

$R_2 \to R_2 - \frac{2}{a} R_1$

$R_3 \to R_3 - \frac{2}{a} R_1$ $\Rightarrow a \neq 0$ and $a > 0$

$$A = \begin{bmatrix} a & 2 & 2 \\ 0 & \frac{a^2-4}{a} & \frac{2a-4}{a} \\ 0 & \frac{2a-4}{a} & \frac{a^2-4}{a} \end{bmatrix}$$

$\frac{a^2-4}{a} > 0 \Rightarrow a^2 > 4$

$a > 2$

$R_3 \to R_3 - \left(\frac{2a-4}{a^2-4}\right) R_2$

$$A = \begin{bmatrix} a & 2 & 2 \\ 0 & \frac{a^2-4}{a} & \frac{2a-4}{a} \\ 0 & 0 & \frac{a^2-4}{a} - \frac{(2a-4)^2}{(a^2-4)a} \end{bmatrix}$$

$$\frac{a^2-4}{a} - \frac{(2a-4)^2}{(a^2-4)a} > 0$$

$$\frac{(a^2-4)(a^2-4)-(2a-4)^2}{(a^2-4)a} > 0$$

$$(a^2-4)(a^2-4)-(2a-4)^2 > 0$$

$$a^4 + \cancel{16} - 8a^2 - 4a^2 - \cancel{16} + 16a > 0$$

$$a^4 - 12a^2 + 16a > 0$$

$$a^3 - 12a + 16 > 0$$

$$a > -4 \quad , \quad a \neq 2 \; ; \quad a > 2$$

Range of a :- $\boxed{a > 2}$

$$(x_1 \;\; x_2 \;\; x_3)\, A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} = f$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

$$B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \; //$$

(7) Find the SVD decomposition of A:

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3\times 2}$$

Sol:-

$$\underset{3\times3}{A} = U \underset{3\times2}{\Sigma} \underset{2\times2}{V^T}$$

$$A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

$$\lambda^2 - 90\lambda + 0 = 0$$

$$\lambda(\lambda - 90) = 0$$

Eigen values :- $\lambda_1 = 90 \quad \lambda_2 = 0$

Eigenvectors for $\lambda_1 = 90$

$$A^TA - \lambda_1 I = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$= \begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{-9}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \qquad X_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

vector $V_1 = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$

Eigen vectors for $\lambda_2 = 0$

$$A^T A - \lambda_2 I = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

$R_2 \to R_2 + \frac{1}{3} R_1$

$$= \begin{bmatrix} 81 & -27 \\ 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 / 81$

$$\sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1 \end{bmatrix}$$

$$\text{vector } V_2 = \begin{bmatrix} -1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$$

$\sigma_1 = \sqrt{90} \qquad \sigma_2 = \sqrt{0}$

$$\underset{3\times 2}{\Sigma} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

Eigen Values : $\lambda^3 - 90\lambda^2 + 0 = 0$

$\lambda^2(\lambda - 90) = 0$

$\lambda_1 = 90 \quad \lambda_2 = 0 \quad \lambda_3 = 0$

$$AA^T - \lambda_1 I = \begin{bmatrix} -80 & -20 & -20 \\ -20 & -50 & 40 \\ -20 & 40 & -50 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{4}R_1$

$R_3 \to R_3 - \frac{1}{4}R_1$

$$\sim \begin{bmatrix} -80 & -20 & -20 \\ 0 & -45 & 45 \\ 0 & 45 & -45 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$R_1 \to R_1 / -80$

$R_2 \to R_2 / -45$

$$\sim \begin{bmatrix} 1 & 1/4 & 1/4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - \frac{1}{4}R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$$

$$V_1 = \frac{X_1}{\|X\|} = \frac{(-1/2, 1, 1)}{\sqrt{\left(\frac{-1}{2}\right)^2 + 1^2 + 1^2}} = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

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$$AA^T - \lambda_2 I = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 2R_1$

$R_3 \rightarrow R_3 + 2R_1$

$$= \begin{bmatrix} 10 & -20 & -20 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1/10$

$x = 2y + 2z$

$$= \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -1/2 \\ -5/4 \\ 1 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \qquad V_3 = \begin{bmatrix} -2/\sqrt{45} \\ -5/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix}$$

$$U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 0 & -5/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 0 & -5/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$$