# LOGISTIC REGRESSION

Spring 2023

CS6431 Natural Language Processing

B1:

*Speech and Language Processing (Third Edition draft – Jan2022)*

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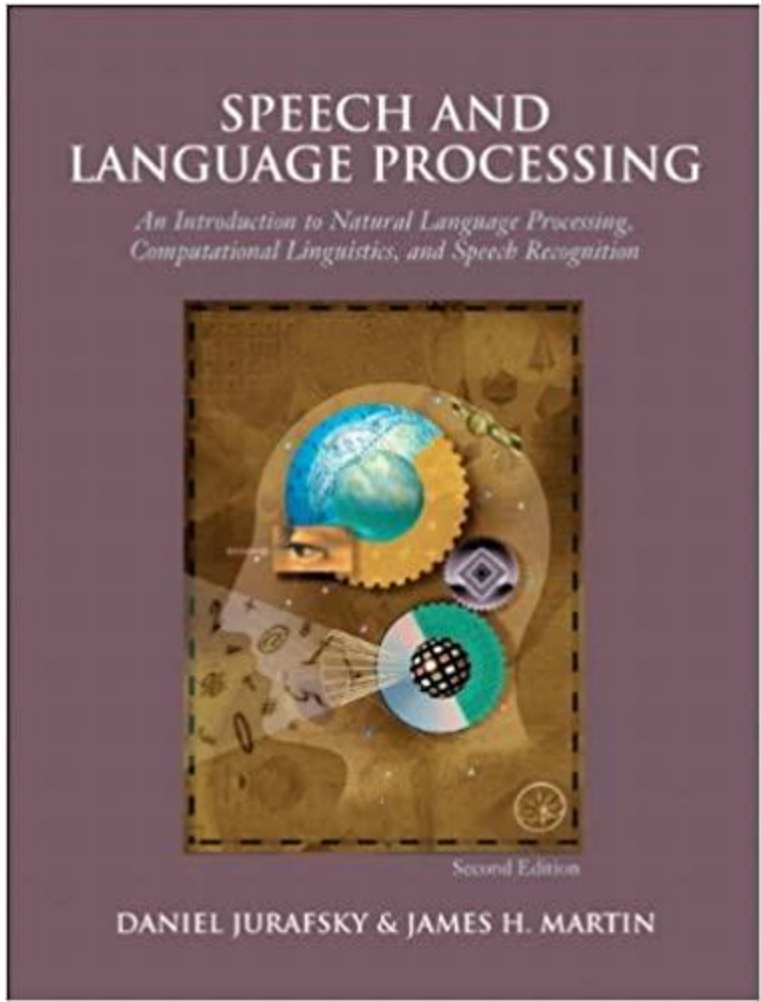


# Credits

1. B1

# Assignment

**Read:**

B1: Chapter 5

**Problems:**

# Generative and Discriminative Classifiers

- Generative: knows how to generate features if it belonged to a particular class
  - E.g. Naïve Bayes

$$\hat{c} = \underset{c \in C}{\operatorname{argmax}} \overbrace{P(d|c)}^{\text{likelihood}} \overbrace{P(c)}^{\text{prior}}$$

- Discriminative model: directly compute $P(c|d)$ by giving more importance to features that are better at discriminating output classes
  - Logistic Regression

# Logistic Regressor

- A single layer neural network with
  - Sigmoid/SoftMax as the activation function
  - Cross-entropy loss function
  - Uses stochastic gradient descent for optimization

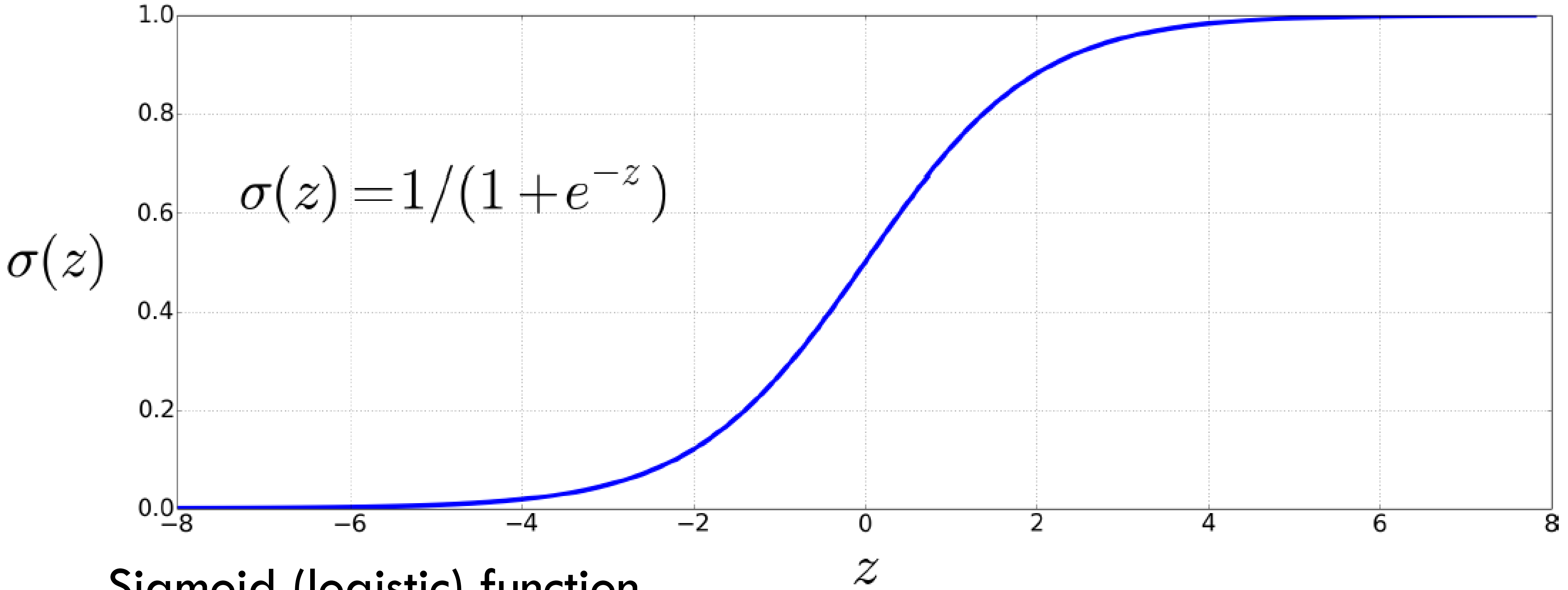
# Sigmoid/SoftMax function

- Inputs, weights, and bias

$$z = \left( \sum_{i=1}^{n} w_i x_i \right) + b$$

- Or $z = \mathbf{w} \cdot \mathbf{x} + b$

if this sum is high , we say y=1
if low y=0;




Sigmoid (logistic) function

- Output (in case of binary classification)

$$\begin{aligned} P(y=1) &= \sigma(\mathbf{w}\cdot\mathbf{x}+b) \\ &= \frac{1}{1+\exp(-(\mathbf{w}\cdot\mathbf{x}+b))} \end{aligned}$$

$$\begin{aligned} P(y=0) &= 1-\sigma(\mathbf{w}\cdot\mathbf{x}+b) \\ &= 1-\frac{1}{1+\exp(-(\mathbf{w}\cdot\mathbf{x}+b))} \\ &= \frac{\exp(-(\mathbf{w}\cdot\mathbf{x}+b))}{1+\exp(-(\mathbf{w}\cdot\mathbf{x}+b))} \end{aligned}$$

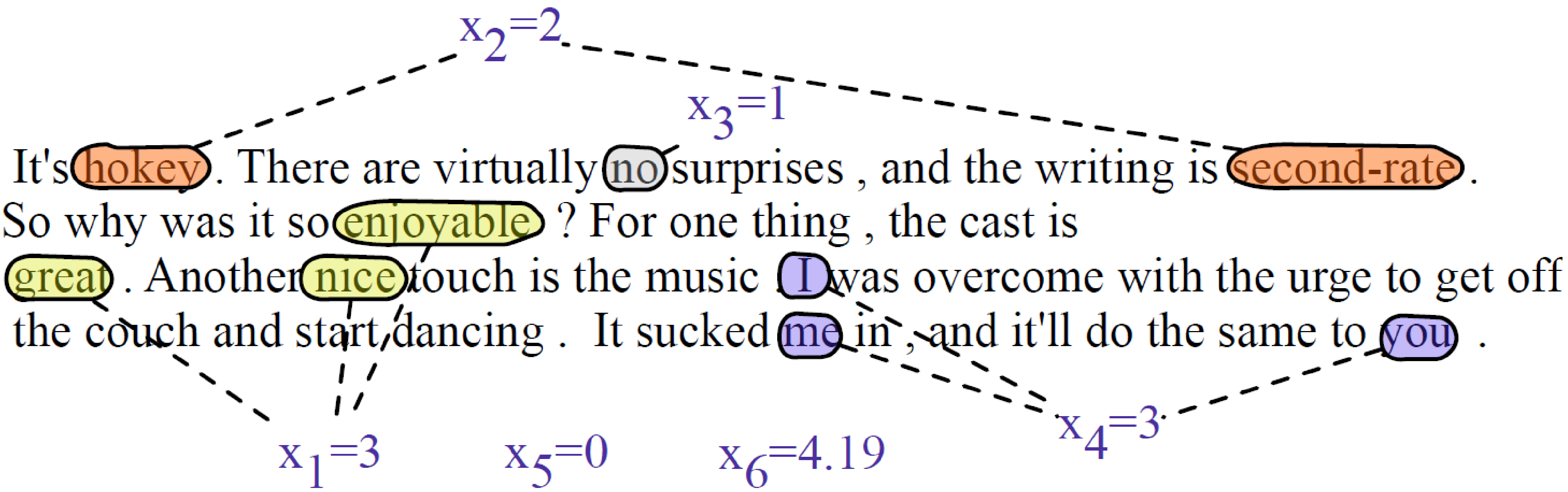
- For sigmoid

$$1-\sigma(x)=\sigma(-x)$$

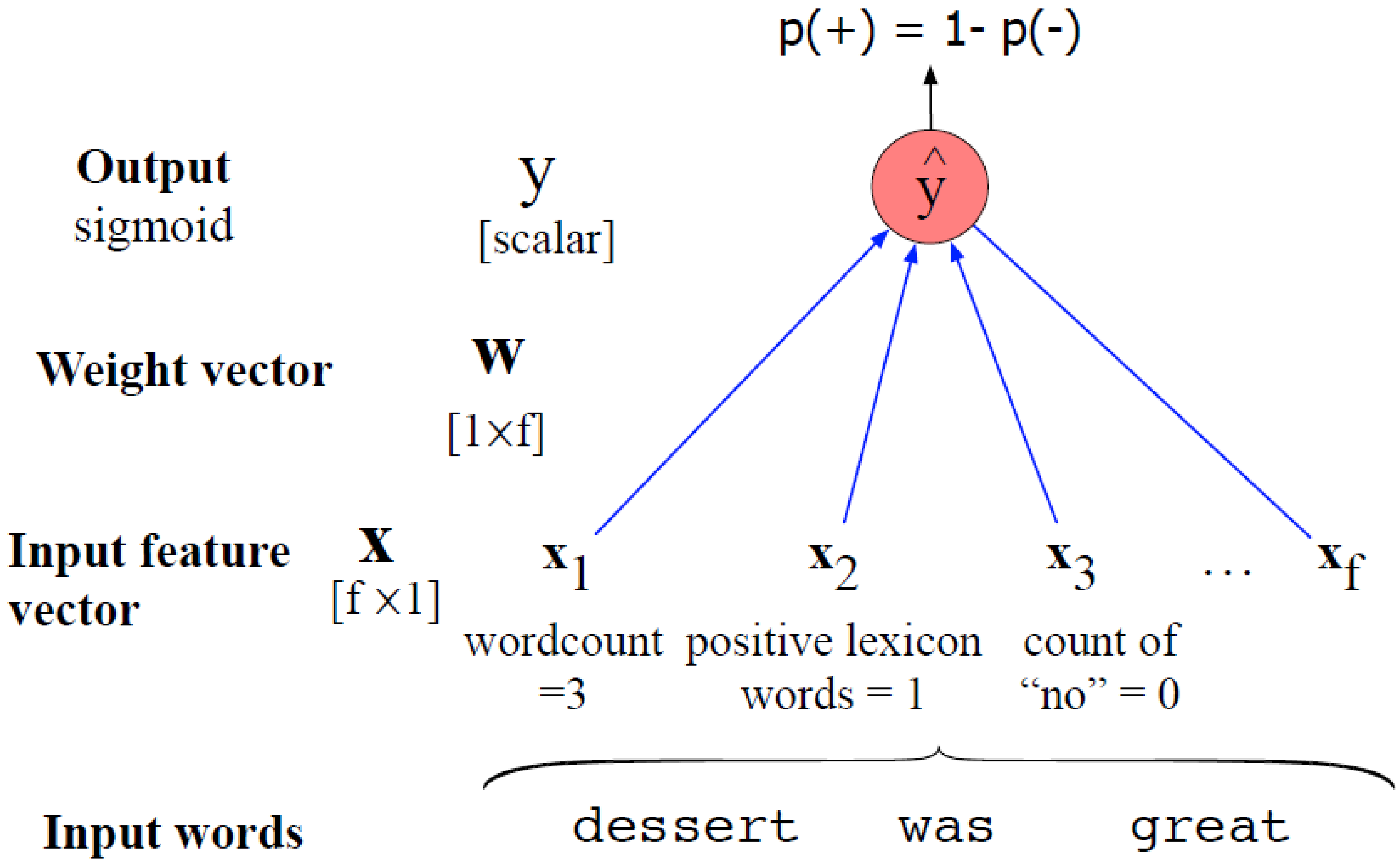
$$P(y=0) \text{ as } \sigma(-(\mathbf{w}\cdot\mathbf{x}+b))$$

$$\text{decision}(x) = \begin{cases} 1 & \text{if } P(y=1|x) > 0.5 \\ 0 & \text{otherwise} \end{cases}$$

decision boundary




| Var | Definition | Value in Fig. |
|---|---|---|
| $x_1$ | count(positive lexicon words $\in$ doc) | 3 |
| $x_2$ | count(negative lexicon words $\in$ doc) | 2 |
| $x_3$ | $\begin{cases} 1 & \text{if “no”} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$ | 1 |
| $x_4$ | count(1st and 2nd pronouns $\in$ doc) | 3 |
| $x_5$ | $\begin{cases} 1 & \text{if “!”} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$ | 0 |
| $x_6$ | log(word count of doc) | $\ln(66) = 4.19$ |

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Let the corresponding six weights be [2.5, -5.0,-1.2, 0.5, 2.0, 0.7] and $b$= 0.1

# Period disambiguation

- What sort of features would you suggest?
- Hand-crafted features
  - Feature interactions
  - Feature templates
- Representation Learning

# Scaling Input Features

- Z-normalization
  - Zero mean
  - Unit variance

$$\mu_i = \frac{1}{m}\sum_{j=1}^{m} x_i^{(j)} \qquad \sigma_i = \sqrt{\frac{1}{m}\sum_{j=1}^{m}\left(\mathbf{x}_i^{(j)} - \mu_i\right)}$$

$$\mathbf{x}_i' = \frac{\mathbf{x}_i - \mu_i}{\sigma_i}$$

$$\sigma^2 = \frac{\Sigma(x_i^j - \mu_i)}{m}$$

- Or, simply normalize as ($\in [-1, +1]$)

$$\mathbf{x}_i' = \frac{\mathbf{x}_i - \min(\mathbf{x}_i)}{\max(\mathbf{x}_i) - \min(\mathbf{x}_i)}$$

# Logistic Regression vs. Naïve Bayes

- Naïve Bayes has a overly strong conditional independence assumption
  - Problem with correlated features
- Logistic Regression is much more robust to correlated features
- Naïve Bayes pluses
  - Works well on small datasets
  - Easy to implement and fast to train (no optimization step)

# Multinomial logistic regression

- Or SoftMax Regression
- Only one among more than two classes can be true
  - Both predicted output $\hat{y}$ and actual output $y$ are of size $k$
    - $\hat{y}_i$ estimates $P(y_i = 1|x)$
- Probabilistically normalized version of sigmoid

For a vector $\mathbf{z}$ of dimensionality $K$, the softmax is defined as:

$$\text{softmax}(\mathbf{z}_i) = \frac{\exp(\mathbf{z}_i)}{\sum_{j=1}^{K} \exp(\mathbf{z}_j)} \quad 1 \le i \le K$$

write on desk

The softmax of an input vector $\mathbf{z} = [\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_K]$ is thus a vector itself:

$$\operatorname{softmax}(\mathbf{z}) = \left[ \frac{\exp(\mathbf{z}_1)}{\sum_{i=1}^{K} \exp(\mathbf{z}_i)}, \frac{\exp(\mathbf{z}_2)}{\sum_{i=1}^{K} \exp(\mathbf{z}_i)}, \ldots, \frac{\exp(\mathbf{z}_K)}{\sum_{i=1}^{K} \exp(\mathbf{z}_i)} \right]$$

- E.g., input:

$$\mathbf{z} = [0.6, 1.1, -1.5, 1.2, 3.2, -1.1]$$

- Output:

$$[0.055, 0.090, 0.006, 0.099, 0.74, 0.010]$$

- For logistic regression

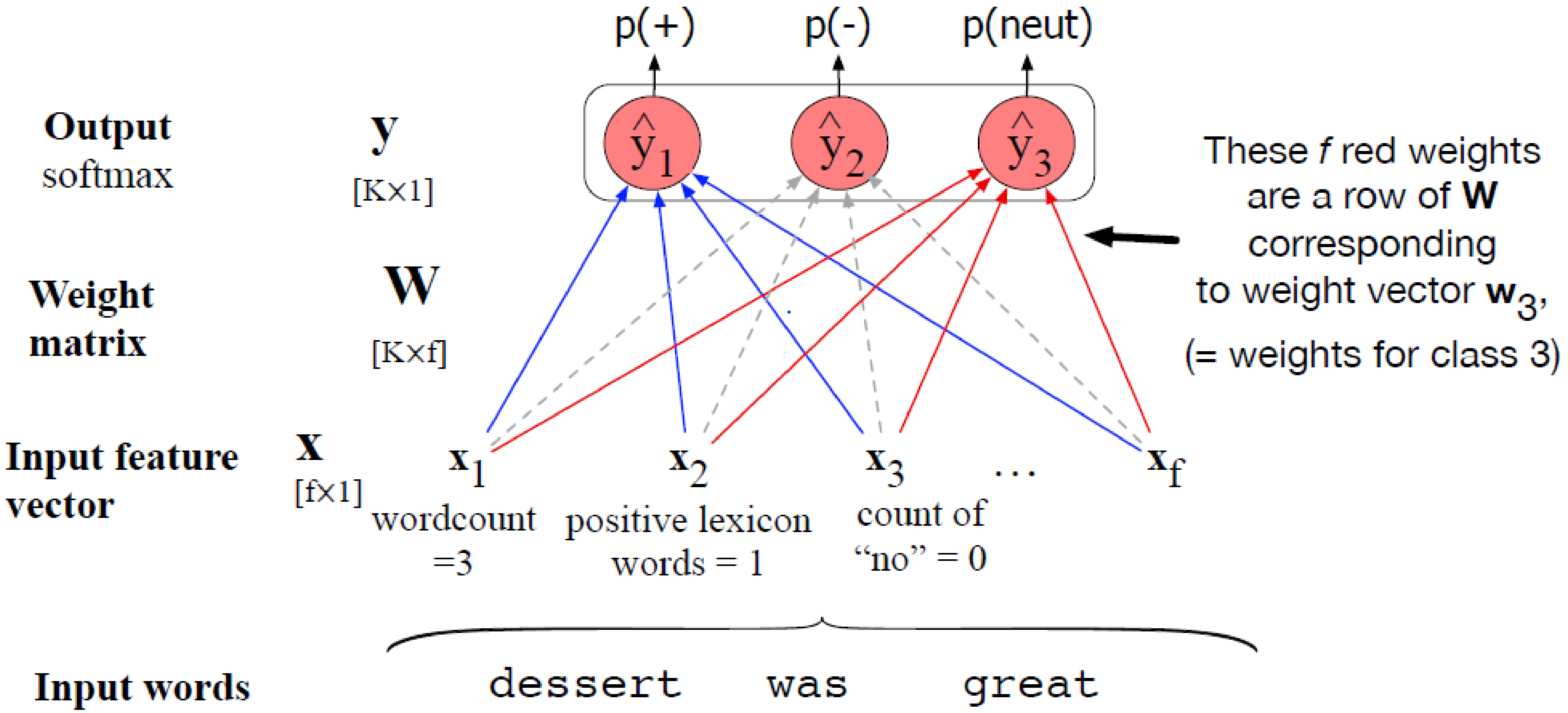
$$p(\mathbf{y}_k = 1|\mathbf{x}) = \frac{\exp(\mathbf{w}_k \cdot \mathbf{x} + b_k)}{\sum_{j=1}^{K} \exp(\mathbf{w}_j \cdot \mathbf{x} + b_j)}$$

Or,

$$\hat{\mathbf{y}} = \mathrm{softmax}(\mathbf{W}\mathbf{x} + \mathbf{b})$$

$z = Wx + b$

# Multinomial Logistic Regression



- In multimodal, a feature can be evidence for or against each individual class.
  - An exclamation mark '!' may indicate positive or negative emotion, but not neutral

| Feature | Definition | $\mathbf{w}_{5,+}$ | $\mathbf{w}_{5,-}$ | $\mathbf{w}_{5,0}$ |
|---|---|---|---|---|
| $f_5(x)$ | $\begin{cases} 1 & \text{if "!"} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$ | 3.5 | 3.1 | $-5.3$ |

# Cross-entropy loss function

$$L(\hat{y}, y) = \text{How much } \hat{y} \text{ differs from the true } y$$

- Conditional maximum likelihood estimation: Choose $w$ and $b$ that maximize the log $p(y|x)$ in the training data given the observations $x$.
- Only two possible outcomes: Bernoulli distribution

$$p(y|x) = \hat{y}^y (1-\hat{y})^{1-y}$$

  - Note: if y=1, $p(y|x) = \hat{y}$, else if y=0, $p(y|x) = (1-\hat{y})$
- Taking log

$$\log p(y|x) = \log\left[\hat{y}^y (1-\hat{y})^{1-y}\right]$$
$$= y\log\hat{y} + (1-y)\log(1-\hat{y})$$

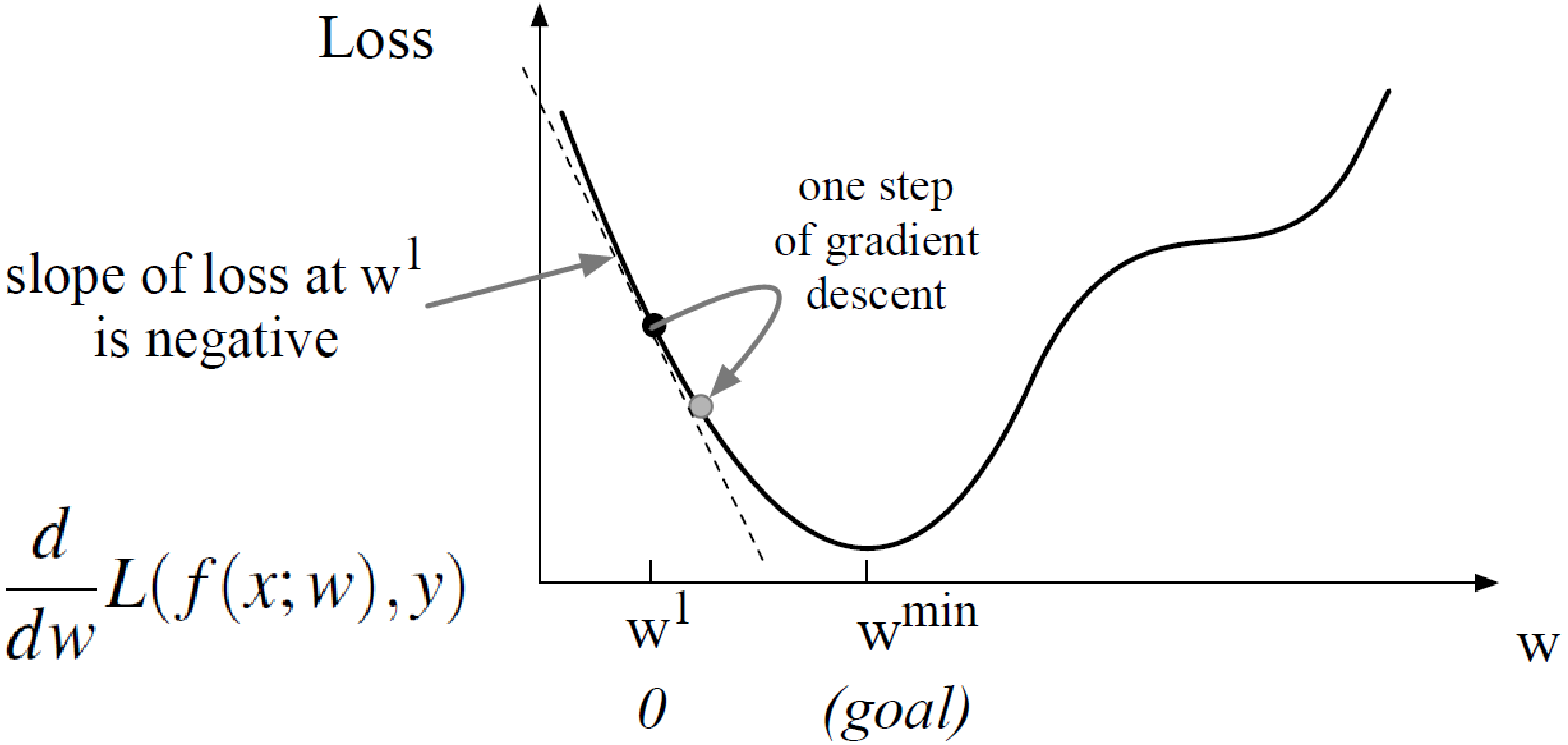
- To make it a loss function

$$L_{\text{CE}}(\hat{y}, y) = -\log p(y|x) = -[y\log\hat{y} + (1-y)\log(1-\hat{y})]$$

$$L_{\text{CE}}(\hat{y}, y) = -[y\log\sigma(\mathbf{w}\cdot\mathbf{x}+b) + (1-y)\log(1-\sigma(\mathbf{w}\cdot\mathbf{x}+b))]$$

# Stochastic Gradient Descent

- Figuring out in which direction the function's slope is rising the most steeply, and moving in the opposite direction
- Logistic regression: convex error function
  - Vs. Neural network: non-convex (multiple local minima)




$$w^{t+1} = w^t - \eta \frac{d}{dw} L(f(x;w), y)$$

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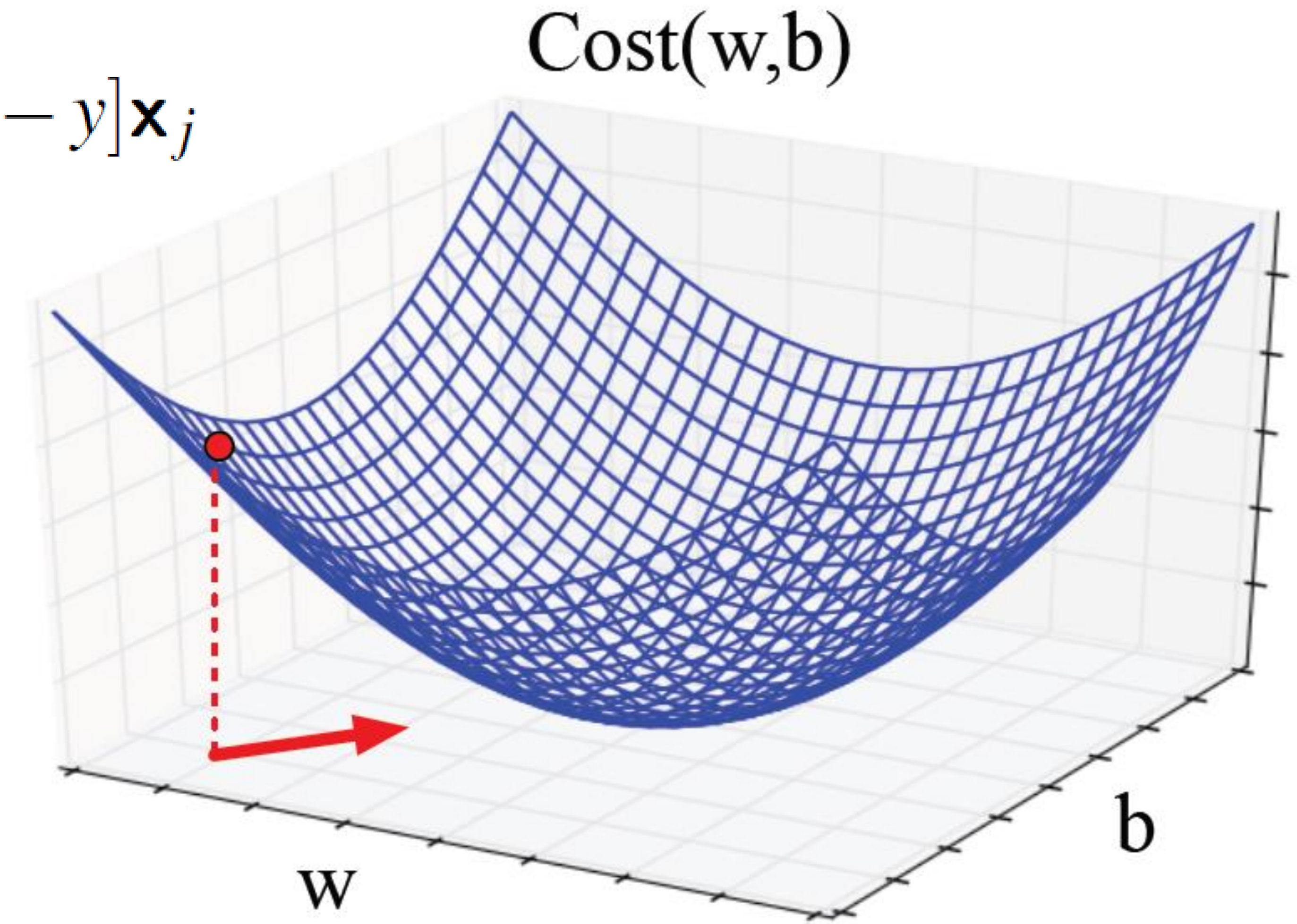
The partial derivative tells the steepest along that dimension

$$\frac{\partial L_{\mathrm{CE}}(\hat{y}, y)}{\partial \mathbf{w}_j} = [\sigma(\mathbf{w} \cdot \mathbf{x} + b) - y]\mathbf{x}_j$$

$$= (\hat{y} - y)\mathbf{x}_j$$



**function** STOCHASTIC GRADIENT DESCENT($L(), f(), x, y$) **returns** $\theta$

\# where: L is the loss function

\#  f is a function parameterized by $\theta$

\#  x is the set of training inputs $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$

\#  y is the set of training outputs (labels) $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$

$\theta \leftarrow 0$

**repeat** til done

For each training tuple $(x^{(i)}, y^{(i)})$ (in random order)

1. Optional (for reporting): # How are we doing on this tuple?

   Compute $\hat{y}^{(i)} = f(x^{(i)}; \theta)$ # What is our estimated output $\hat{y}$?

   Compute the loss $L(\hat{y}^{(i)}, y^{(i)})$ # How far off is $\hat{y}^{(i)}$ from the true output $y^{(i)}$?

2. $g \leftarrow \nabla_\theta L(f(x^{(i)}; \theta), y^{(i)})$ # How should we move $\theta$ to maximize loss?

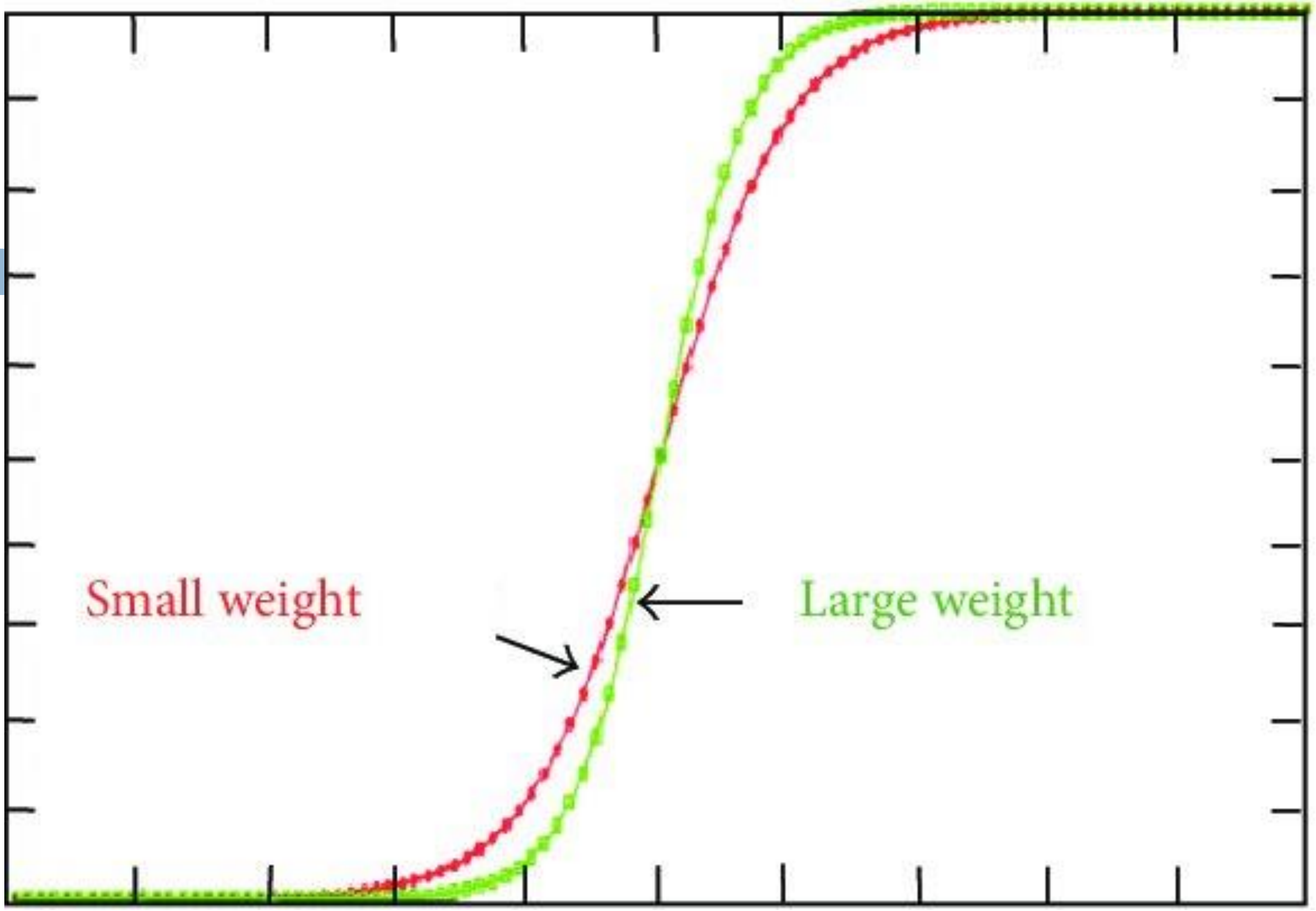
3. $\theta \leftarrow \theta - \eta\, g$ # Go the other way instead

return $\theta$

# Regularization

- Large weights => over generalization (overfitting)

$$S(x) = \frac{1}{1+e^{-x}}$$




- Add a penalty for large weights

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{m} \log P(y^{(i)} | x^{(i)}) - \alpha R(\theta)$$

- L1 Regularization: Linear function of weights

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \left[ \sum_{1=i}^{m} \log P(y^{(i)} | x^{(i)}) \right] - \alpha \sum_{j=1}^{n} |\theta_j|$$

- L2 Regularization: Quadratic function of weight values

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \left[ \sum_{i=1}^{m} \log P(y^{(i)} | x^{(i)}) \right] - \alpha \sum_{j=1}^{n} \theta_j^2$$

# L1 vs. L2

- L1
  - Linear but non-continuous at 0, complex derivative
  - Laplace prior on the weights
  - Prefers a sparse weight matrix with a few large weights
- L2
  - Simple derivative
  - Gaussian prior with zero mean
  - Prefers weight vectors with many small weights