17 Peeking into Current Research

We have seen quite a few optimization algorithms, must mostly shortest path, or maximal reliability. A path is a sequence edge and a shortest path is the minimal sum of edge weights, whereas the reliability is the maximal product. In both cases, the principle of optimality applies. Also in both cases, the algorithms closely follow some kind of matrix-manipulation.

This observation is more universal. What we have called the "driving equation" is a step toward solving an equation that is often called "Bellman's functional equation" or "Bellman's Objective function". In all cases we considered, we had a situation where the new function value was the result of considering a discrete number of options. If this would have been continuous, a partial differential equation would have resulted, the "Hamilton-Jacobi-Bellman equation", see Wiki for details.

But there is another way to formalize the approach. If you take a matrix multiplication algorithm, but replace the "$\times$" by a Boolean AND and the "$+$" by a Boolean OR has been alluded to several times. But similarly, replace the "$\times$" by a regular $+$ and the "$+$" by the "min" -operation, then you get shortest paths ideas. So a rather new matrix-formalism has been created that does exactly that, and is called the "min-plus" algebra.

Following the same idea, replace the "$\times$" by a regular $\max$-operator and the "$+$" by the "$\times$" -operation, then you get the "max-plus" algebra. Both the "min-plus" and the "max-plus" can be used without thinking about the underlying optimization problems, and you can multiply matrices, find eigenvalues, and so on. Note that matrix-algebra only needs operations that are associate and distributive, which these indeed satisfy. The result is in the eye of the beholder: in the application domain. So what are the eigenvalues of these matrices? Consider the paper by Sennosuke Watanabe and Yoshihide Watanabe, *Min-Plus Algebra and Networks*, RIMS, B47, 041-54, 2014.

The Bellman equation for various discrete problems have been formulated as linear equations over a somewhat strange looking matrix-algebra. But a step in the traditional Bellman-Ford algorithm for instance, has an analogue in the matrix-algebra setting. Consider the shortest path problem from node v_1 to all other nodes in a graph without a negative cycle and let y_i be the shortest distance from v_1 to v_i. The Bellman equations is similar to the all-source-all-destinations as shown in (10), but now for the single source variation. Let $A(i,\ j)$ be the matrix of weights between the nodes, then

$$v_j = \min_{\text{2 options}} \left\{ \min_{\ell,\text{ with } 1\leq \ell \leq n} \{ v_\ell + A(\ell,\ j)\},\quad v_j, \right\} \tag{26}$$

Introduce a new symbol $\textcircled{\scriptsize min}$ and write $\min\{a,b,c\} = a \textcircled{\scriptsize min} b \textcircled{\scriptsize min} c$ and write the equation as

$$\begin{aligned} v_j &= \Big\{ \{ v_1 + A(1,\ j)\} \textcircled{\scriptsize min} \{ v_2 + A(2,\ j)\} \textcircled{\scriptsize min} \{ v_3 + A(3,\ j)\} \textcircled{\scriptsize min} \quad \dots \ \{ v_n + A(n,\ j)\} \quad \textcircled{\scriptsize min} v_j \Big\} && (27)\\ &= \Big\{ \{ v_1 \textcircled{\scriptsize x} A(1,\ j)\} \textcircled{\scriptsize min} \{ v_2 \textcircled{\scriptsize x} A(2,\ j)\} \textcircled{\scriptsize min} \{ v_3 \textcircled{\scriptsize x} A(3,\ j)\} \textcircled{\scriptsize min} \quad \dots \ \{ v_n \textcircled{\scriptsize x} A(n,\ j)\} \quad \textcircled{\scriptsize min} v_j \Big\} && (28)\end{aligned}$$

The last entry can be omitted, since it duplicates the term $\{ v_j \textcircled{\scriptsize x} A(j,\ j)\}$. Also, we wrote $\textcircled{\scriptsize x}$ in stead of the starndard $+$ to indicate that the $\textcircled{\scriptsize x}$ takes precedence over the $\textcircled{\scriptsize min}$. So with the "new" interpretation, this becomes a fairly standard matrix equation

$$\mathbf{v} = \mathbf{v}\,\mathbf{A} + \mathbf{b}, \tag{29}$$

where the vector $\mathbf{b}$ will be explained shortly. This equation is solved easily, by

$$\mathbf{v}\big(\mathbf{I} - \mathbf{A}\,\big) = \mathbf{b}, \tag{30}$$

or

$$\mathbf{v} = \mathbf{b}\,\big(\mathbf{I} - \mathbf{A}\,\big)^{-1} = \mathbf{b}\,\big(\mathbf{I} + \mathbf{A}\ + \mathbf{A}^2 + \mathbf{A}^3 + \dots\dots\big) \tag{31}$$

Now add the initial condition that $v_1 = 0$, so that $\mathbf{b}$ is the vector with 1 on its first component, and all other entries are set to ∞.
It now becomes clear that Dijkstra's algorithm, Bellman's algorithm and others can be interpreted as solving linear equations by iterations, much like the Jacobi iteration. Noting that the Gauss-Seidel algorithm accelerates the Jacobi iteration, its analogue in this new formalism accelerates the Bellman-Ford algorithm. Could matrix-squaring by used? What about Strassen's algorithm? and so on. (There is a small problem: the "$\min$"-operations is replaced by the "$+$", but the "$+$" has an inverse (i.e. for all x, there a number, which we call $(-x)$, such that $x + (-x) = 0$. These things have to do with the mathematical notion of a ring, a semi-ring, and unit-elements w.r.t. operations)

The same paper also argues that if you start with a standard weighted graph (for shortest paths calculations), then the induced matrix in the "min-plus" algebra has an smallest eigenvalue, and this value corresponds to the minimal average edge weight of a cycle in the graph (average here is the sum of the edge weights of a cycle, divided by the number of edges in that cycle).

Now add stochastic entries as edge weights, and you end up with Network calculus. These are all promising new directions for classical CS-problems.