Q1)

a) To prove $k(x,z) = k_1(x,z) + k_2(x,z)$

Given k_1 and k_2 are valid kernals

→ To prove $k(x,z)$ is valid kernal we have to show

① $K(x,z)$ is symmetric
② $k(x,z)$ is positive semidefinate

Symmetric

As $k_1(x,z)$ is valid kernal i.e it is symmetric

$$\Rightarrow k_1(x,z) = k_1(z,x)$$

likewise

$$\Rightarrow k_2(x,z) = k_2(z,x)$$

as both k_1 and k_2 are matrix So, sum of two symmetric matrix is also symmetric.

Hence $k(x,z)$ is symmetric

Positive semidefinate

Let $u \in R^n$ and Gram matrix of k is denoted by G has property

$$G_{i,j} = k(x_i, x_j)$$

$$G = G_1 + G_2$$

$$\Rightarrow \underbrace{\underset{\geq 0}{u G_1 u^T} + \underset{\geq 0}{u G_2 u^T}}_{\text{both } \geq 0} = u G u^T$$

So, sum is also greater than 0

Hence $k(x,z)$ is positive semidefinate.

) To prove $k(x,z) = k_1(x,z)\, k_2(x,z)$

$k(x,z)$ is equivalent to $\langle \phi(\vec{x}), \phi(\vec{z}) \rangle$

$\Downarrow$ proof

let $\vec{x} = (x_1, x_2)$

$\vec{z} = (z_1, z_2)$

$k(x,z) = \langle \vec{x} \cdot \vec{z} \rangle^2$

$k(x,z) = (x_1 z_1 + x_2 z_2)^2$

$= x_1^2 z_1^2 + x_2^2 z_2^2 + 2 x_1 z_1 x_2 z_2$

$= \langle (x_1^2, \sqrt{2}\, x_1 x_2, x_2^2), (z_1^2, \sqrt{2}\, z_1 z_2, z_2^2) \rangle$

$k(x,z) = \langle \phi(\vec{x}) \cdot \phi(\vec{z}) \rangle$

To prove - $k(x,z) = k_1(x,z)\, k_2(x,z)$

$k_1(x,z) = \phi_1(x) \cdot \phi_1(z)$

$k_2(x,z) = \phi_2(x) \cdot \phi_2(z)$

So, $k(x,z) = \phi_1(x) \cdot \phi_1(z) \cdot \phi_2(x) \cdot \phi_2(z)$

$= \underbrace{\phi_1(x) \cdot \phi_2(x)}_{\phi(x)} \cdot \underbrace{\phi_1(z) \cdot \phi_2(z)}_{\phi(z)}$

proof :- let $\phi(x) = (x_1^2, \sqrt{2}\, x_1 x_2, x_2^2)$

$= \phi_1(x) \cdot \phi_2(x)$

$\Rightarrow \phi_1(x) = (x_1^2, \sqrt{2}\, x_1, 1)$

$\phi_2(x) = (1, x_2, x_2^2)$

Hence,

$k(x,z) = \phi(x) \cdot \phi(z)$

c) To prove :-

$$K(x, z) = h(k_1(x, z))$$ where h is polynomial function with positive coeficient

→ polynomial function involves only non-negative powers of x, and one more constrait they have given is coeficients are also positive.

So, lets take a polynomial with positive coeficient

ex :- $x^3 + x^2 + x$

$$= \underbrace{[k_1(x, z)]^3}_{\text{product of valid kernals is also a valid kernal}} + [k_1(x, z)]^2 + [k_1(x, z)] \quad -① $$

product of valid kernals is also a valid kernal So, (b - part as we proved)

eqn ① is transformed to

$$= \underbrace{k_3(x, z) + k_4(x, z) + k_1(x, z)}_{\text{sum of three valid kernals}}$$

sum of three valid kernals is also a valid kernal

So,

$k(x, z) = h(k_1(x, z))$ is a valid statement

Hence proved

d) To prove :-

$$k(x,z) = \exp(k_1(x,z))$$

as we know expansion of exponential function is

$$\exp(x) = \lim_{i \to \infty} (1 \ldots + x^i / \underline{i})$$

So,

$k(x,z)$ = polynomial function of kernals.

As we proved in (c) part that the polynomial function of kernal is also a kernal likewise here also, we end up with a same case So,

$k(x,z)$ is a valid kernal where it is equal to $\exp(k_1(x,z))$

e) To prove :-

$$k(x,z) = \exp\left(-\frac{\|x-z\|_2^2}{\sigma^2}\right)$$

Solving R.H.S

$$= \exp\left(\frac{-\|x\|^2 - \|z\|^2 + 2x^T z}{\sigma^2}\right)$$

$$= \underbrace{\exp\left(\frac{-\|x\|^2}{\sigma^2}\right)}_{g(x)} \underbrace{\exp\left(-\frac{\|z\|^2}{\sigma^2}\right)}_{g(z)} \underbrace{\exp\left(\frac{2x^T z}{\sigma^2}\right)}_{\text{valid kernal as proved in d part}}$$

Now, to prove $g(x) \cdot g(z)$ is also valid as $\phi : x \to H$ (mapping function)

as we know $k(x,y) = \phi(x) \cdot \phi(y)$

$\Rightarrow \underbrace{[g(x) \cdot g(z)]}_{\text{valid kernel}} \cdot \underbrace{\exp(k_1(x,z))}_{\text{valid kernel}}$

Hence product is also a valid kernel

Hence proved