MA-203

* Books :-

1. Advanced Engineering Mathematics
   [E. Kreyszig]

2. Differential Equations
   [G.F. Simmons]

3. Special Function for Scientists & Engineers [W.W. Bell]

(1) Integral Transform : An integral transform 'maps' an eq'n from its original domain into another domain. Manipulating and sol'n the eq'n in the translate domain can be much easier than manipulation & sol'n in the original domain.

$$T\{f(x), s\} = \int_a^b f(x) \underbrace{K(x,s)}_{\text{Kernel}} dx$$

$$K(x,s) = \begin{cases} 0 & x < 0 \\ e^{-sx} & x \geq 0 \end{cases}$$

$(a,b) \to (0,\infty)$

$$L\{f(t), s\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad -(1)$$

Provided (1) exist

$s \to$ parameter

$Re(s) > 0$

• Laplace transform :- let $f(x)$ is a function for $x \geq 0$, then Laplace transform of $f(x)$ is defined by,

$$F(s) = L\{f(t), s\} = \int_0^\infty e^{-st} f(t)\, dt \quad \text{—(1)}$$

$Re(s) > 0$

provided (1) exist

Table of Laplace Transforms

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t^n	$\frac{n!}{s^{n+1}}$ $\left[\frac{\Gamma(n+1)}{s^{n+1}}\right]$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$t\sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
$\frac{\sin(bt)-bt\cos(bt)}{2b^3}$	$\frac{1}{(s^2+b^2)^2}$
$\frac{t\sin(bt)}{2b}$	$\frac{s}{(s^2+b^2)^2}$
$\theta_a(t)$	$\frac{e^{-as}}{s}$
$\theta_a(t)\, f(t-a)$	$e^{-as} F(s)$

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt \qquad \int_0^\infty e^{-st}\, t^n\, dt$$

$L\{t^n ; s\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$; $\Gamma(n+1) = n\,\Gamma(n)$

(1) $f(t) = 1 \quad \forall \quad t \in [0, \infty)$

$$F(s) = \int_0^\infty e^{-st}\, dt = \lim_{h \to \infty} \int_0^h e^{-st}\, dt$$

$$= \lim_{h \to \infty} \left[-\frac{e^{-st}}{s} \right]_0^h = -\lim_{h \to \infty} \frac{e^{-sh}}{s} + \lim_{h \to \infty} \frac{1}{s}$$

For $s > 0 \Rightarrow sh > 0$

$$\therefore \lim_{h \to \infty} \frac{e^{-sh}}{s} = 0 \qquad \therefore F(s) = \frac{1}{s}$$

(2) $f(t) = e^{at} \quad \forall \quad t \geq 0$

$$F(s) = \int_0^\infty e^{-(s-a)t}\, dt = \lim_{h \to \infty} \int_0^h e^{-(s-a)t}\, dt$$

$$= \lim_{h \to \infty} \left[-\frac{e^{-(s-a)t}}{s-a} \right]_0^h = -\lim_{h \to \infty} \frac{e^{(s-a)h}}{s-a} + \lim_{h \to \infty} \frac{1}{s-a}$$

For $s > a \Rightarrow (s-a)h > 0$

$$\therefore \lim_{h \to \infty} \frac{e^{(s-a)h}}{s-a} = 0 \quad \therefore F(s) = \frac{1}{s-a}$$

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

And, $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

$$\cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

* Basic idea : A function $f(t)$ has L.T. if does not grow fast. , i.e,

$$|f(t)| < M e^{\gamma t}$$

Function of Exponential order

A function $f(x)$ is said to exponential order of γ as,

$$\left[\lim_{x \to \infty} e^{-\gamma t} f(t) = \text{finite} \right]^{*}$$

Or a given x_0, $\exists$ a real no. $M>0$ s.t.

$$|e^{-\gamma x} f(x)| < M \qquad \forall \; x \geq x_0$$

$$|f(x)| < M e^{\gamma x}$$

Q. Prove that x^2 is exp. order

2

$$\lim_{x\to\infty} e^{-2x} x^2$$

Eg $f(x) = e^{x^3}$

$$\lim_{x\to\infty} e^{-\gamma x} \cdot e^{x^3} = \lim_{x\to\infty} \frac{e^{x^3}}{e^{\gamma x}} \to \infty$$

(diverges)

* Existence of L.T. :

1

Let $f(x)$ is defined and piecewise continous on every finite interval $0 \leq x \leq x_0$ and is exp. order γ as $x \to \infty$. Then,

2

$$L\{f(x); s\} \text{ exist } \forall s > \gamma$$

[Sufficient but not necessary]

Proof :- To Prove,

$$\int_0^\infty e^{-sx} f(x)\,dx \to \text{exist} \quad x > \gamma$$

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^{x_0} f(t) e^{-st}\,dt \Big] + \left| \int_{x_0}^{\infty} f(t) e^{-st}\,dt \right|$$

finite

$$\le \int_0^\infty e^{-st} |f(t)|\,dt$$

$\therefore$ It exist

$$\le \int_0^\infty e^{-st} M e^{\gamma t}\,dt$$

$$= \frac{M}{s-\gamma}$$

Note :-

(1) $L\{x^n; s\}$ exist for $-1 < x < 0$

(2) $L\{x^{-1/2}; s\}$ exist ; when $\frac{1}{\sqrt{x}}$ is not conti.

$$L\{x^{-1/2}\} = \int_0^\infty e^{-sx} \frac{1}{\sqrt{x}} dx$$

$sx = t$

$$\therefore \frac{1}{\sqrt{s}} \int_0^\infty e^{-t} t^{-1/2} dt = \sqrt{\frac{\pi}{s}}$$

* Properties of Laplace :-

① Linearity Prop :- Let $F_i(s)$ are L.T. of $f_i(x)$ $\forall$ $i = 1$ to n. and c_i are constants. Then

$$L\{c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)\}$$

$$= c_1 L\{f_1(x)\} + c_2 L\{f_2(x)\} + \dots$$

$$c_n L\{f_n(x)\}$$

② Change of scale prop. : Let,

$$L\{f(x); s\} = F(s)$$

Then, $L\{f(ax);s\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

② First Shifting Prop (Translation prop) of LT :-

If $L\{f(x);s\} = F(s)$ then

$$L\{e^{-ax} f(x)\} = F(s+a) \quad *$$

$$\left[\text{③} \quad L\left[e^{at} \underbrace{f(t)}_{F(s)}\right] = F(s-a) \right]$$

④ Second Shifting Prop. of L.T. :-

If $L\{f(x);s\} = F(s)$

and $g(x) = \begin{cases} f(x-a) & x > a \\ 0 & x < a \end{cases}$

Then, $\left[L\{g(x),s\} = e^{-as} F(s) \right] *$

* Heaviside Unit Step function :-

It is defined by,

$$H(x-a) = \begin{cases} 1 & x > a \\ 0 & x < a \end{cases}$$

$$L\{H(x-a)\} = \frac{e^{-as}}{s} \quad ; \quad L\{H(x)\} = \frac{1}{s}$$

Eg. $\sin 5x \cos 3x = \frac{1}{2}[\sin 8x + \sin 2x]$

$$L\{\sin 5x \cos 3x] = \frac{1}{2}\left[\frac{8}{s^2+64} + \frac{2}{s^2+4}\right]$$

Eg. $f(x) = e^{-x}[3\sinh 2x - 5\cosh 2x]$

$$L\left[3\sinh 2x - 5\cosh 2x\right] = \frac{6}{s^2-4} - \frac{5s}{s^2-4}$$

$$f(x) \longrightarrow f'(x) \qquad f(x) \rightarrow x^n f(x)$$

$$\rightarrow \int f(x)\,dx \qquad \rightarrow \frac{f(x)}{x^n}$$

<u>eg</u>

$$f(t) = \begin{cases} 0 & 0<t<1 \\ t & 1<t<2 \\ 0 & t>2 \end{cases}$$

<u>eg</u>

$$f(x) = \left(1+xe^{-x}\right)^3$$

<u>eg</u>

$$f(x) = e^{-2x}\cos^2 x$$

* L.T. of Multiplication and division by t.

(1) If $L\{f(x);s\} = F(s)$ *

Then, $\boxed{L\{x^n f(x);s\} = (-1)^n \frac{d^n F(s)}{ds^n}}$

*

Base Case → Mathematical Ind. → $n=k+1$

*

(2) If $L\{f(x);s\} = F(s)$

Then, $L\left\{\frac{f(x)}{x};s\right\} = \int_s^{\infty} F(u)\,du$

Provided $\lim_{x\to 0} \frac{f(x)}{x}$ exist.

* $G(x) = \frac{f(x)}{x} \Rightarrow xG(x) = f(x)$

Take Laplace both side ; Use (1) property * eg $L\left\{\frac{\sin x}{x}\right\} = \frac{\pi}{2} - \tan^{-1} s$

$= \tan^{-1}(1/s)$

$\int_0^\infty \frac{\sin x}{x}\, dx = ?$ $\quad = \pi/2$

$\int_0^\infty e^{-sx} \frac{\sin x}{x} = \tan^{-1}\left(\frac{1}{s}\right)$

(4) $L\left[t^n \underbrace{f(t)}_{F(s)}\right] = (-1)^n \frac{d^n F(s)}{ds^n}$

$$\left[\star \lim_{s \to \infty} F(s) = 0\right]$$

$L\left\{\frac{f(x)}{x^2}; s\right\} = \int_s^\infty \int_{u_1}^\infty F(u)\, du$

* L.T. of Derivatives and Integrals :-

Let, $L\{f(x);s\} = F(s)$

$L\{f'(x);s\} = sF(s) - f(0)$

(1) $\vdots$

$$L\{f^n(x);s\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$$

*

$$\left[L[f^n(t)] = S^n L[f(t)] - S^{n-1} f(0) - S^{n-2} f'(0) - \dots - f^{n-1}(0) \right]$$

*

Proof by

Mathematical Induction $n \geq 1$

② $L\left\{\int_0^x f(u)\,du\right\} = \frac{F(s)}{s}$

$$L\left\{\int_0^x \int_0^{u_2} \int_0^{u_1} \cdots \int_0^{u} f(u)\,du\,du_1\,du_2\right\} = \frac{F(s)}{s^n}$$

eg

$$L\left\{\frac{\cos at - \cos bt}{t};s\right\}$$

$$L\{\cos at - \cos bt\} = \frac{s^2}{s^2+a^2} - \frac{s^2}{s^2+b^2}$$

$$= \frac{s^2(b^2-a^2)}{(s^2+a^2)(s^2+b^2)}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \frac{u b^2}{\cancel{(a^2+a^2)}\,(u^2+b^2)} - \int_s^\infty \frac{u a^2}{\cancel{(a^2+b^2)}\,(u^2+a^2)}$$

$$\int_s^\infty \frac{u\,du}{u^2+a^2} - \int_s^\infty \frac{u\,du}{u^2+b^2}$$

$$= \frac{1}{2}\int_{s^2+a^2}^\infty \frac{dt}{t} - \frac{1}{2}\int_{s^2+b^2}^\infty \frac{dt}{t}$$

$$= \frac{1}{2}\ln\left|\frac{s^2+b^2}{s^2+a^2}\right| = \ln\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

Gamma function:

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

$$\Gamma n+1 = n\,\Gamma n$$

* Γn is not defined for -ve integer

$$\Gamma n+1 = n! \quad \text{when } n \text{ is integer}$$

$$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

Q) $$\int_0^\infty \frac{e^{-x}\sin x}{x}\,dx$$

$$L\{\sin x\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin x}{x}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du = \frac{\pi}{2} - \tan^{-1} s$$

$$= \tan^{-1}(1/s)$$

$$L\left\{\frac{e^{-x}\sin x}{x}\right\} = \tan^{-1}\left(1/s+1\right)$$

$$\int_0^\infty e^{-st}\,\frac{e^{-x}\sin x}{x}\,dx = \tan^{-1}\left(\frac{1}{s+1}\right)$$

$[s=0]$

$$= \int_0^\infty \frac{e^{-x}\sin x}{x}\,dx = \tan^{-1}(1) = \frac{\pi}{4}$$

Eg: Find the Laplace Transform of $\sin\sqrt{t}$, i.e., $L\{\sin\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s}$

Also find LT $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$.

Sol: We know that,

$$\sin(\sqrt{t}) = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\cdots \infty$$

$$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\}$$

$\frac{1}{2}\Gamma\frac{1}{2}$ $\qquad$ $\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}$ $\qquad$ $\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}$ ---- ∞

$$= \frac{\Gamma\frac{3}{2}}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma\frac{5}{2}}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma\frac{7}{2}}{s^{7/2}} - \cdots\cdots \infty$$

And, $\Gamma(n+1) = n\Gamma(n)$, $\Gamma\frac{1}{2} = \sqrt{\pi}$

$$\frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{2^3 s^{5/2}} + \frac{\sqrt{\pi}}{2^6 s^{7/2}} - \cdots \infty$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \left[\underset{\downarrow}{1} - \underset{\downarrow}{\frac{1}{2^2 s}} \cdot \frac{1}{1!} + \underset{\downarrow}{\frac{1}{2^4 s^2}} \cdot \frac{1}{2!} \cdots \infty \right]$$

$$\quad 4^0 \qquad 4^1 \qquad 4^2$$

$$e^{-1/4s} \longrightarrow$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

$$(\sin\sqrt{t})' = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{1}{4}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

LT of periodic function :-

<u>Def</u> :- A function $f(x)$ is set to be periodic with period α, if $f(x+n\alpha) = f(x)\ \forall x$, α is +ve Constant. The least value of α is Called period of $f(x)$.

Let, $f(x)$ be a periodic function with period α, i.e, $f(x+n\alpha) = f(x)\ \forall x$, $n = 1, 2, 3, \ldots$

Then,
$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-s\alpha}} \int_0^{\alpha} e^{-st} f(t)\,dt}$$

<u>eg</u>

$$f(x) = \begin{cases} \sin x & ; \ 0 < x < \pi \\ 0 & ; \ \pi < x < 2\pi \end{cases}$$

$T = 2\pi$

<u>Ans</u>: $L\{f(x)\} = \dfrac{1}{1-e^{-2\pi s}} \displaystyle\int_0^{\pi} e^{-st} (\sin t)\,dt$

Initial Value Theorem :-

Let $f(x)$ be Continuous $\forall\ x \geq 0$ and Exp. order exist. Also Suppose that $f'(x)$ is Class of A then,

$$\left[\lim_{x \to 0} f(x) = \lim_{s \to \infty} sF(s)\right] *$$

Final Value Theorem :-

Let $f(x)$ is Continuous $\forall\ x \geq 0$ & Exp. Order. Also Suppose $f'(x)$ belongs to Class of A then,

$$\left[\lim_{x \to \infty} f(x) = \lim_{s \to 0} sF(s)\right] *$$

(Laplace exists)

$\Rightarrow F(s) = \dfrac{3}{s+2}$

eg $f(x) = 3e^{-2x}$

FVT ✗

IVT ✓

Beta Function :-

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$B(m,n) = B(n,m)$$

$$B(m,n) = \frac{\Gamma m \; \Gamma n}{\Gamma m+n}$$

$L\{B(m,n)\} =$

$\Gamma m \; \Gamma n / s^{m+n}$

Bessel Function :-

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r! \; \Gamma n+r+1}$$

$$= 1 - \frac{x^2}{2^2 \cdot (1!)^2} + \frac{x^4}{2^4 \cdot (2!)^2} - \cdots\cdots$$

$$L\{J_0 x\} = \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2}$$

$$\boxed{L\{J_0(ax)\} = \frac{1}{\sqrt{a^2+s^2}}}$$

Eg

$$L\{x J_0(ax)\} = \frac{s}{(s^2+a^2)^{3/2}}$$

$$\int_0^\infty J_0(x)\,dx = 1$$

$$\Rightarrow \int_0^\infty e^{-st} J_0(x)\,dx = \frac{1}{\sqrt{s^2+1}} \quad [s=0]$$

$$\int_0^\infty x e^{-3x} J_0(4x)\,dx = 3/125$$

$$\Rightarrow \int_0^\infty x J_0(ax)\, e^{-sx}\,dx = \frac{s}{(s^2+a^2)^{3/2}} \quad \begin{bmatrix} s=3 \\ a=4 \end{bmatrix}$$

Eg

$$L\{J_1(x)\} = 1 - \frac{s}{(s^2+1)^{1/2}}$$

Few properties of Bessel function :-

1. $$J_{-n}(x) = (-1)^n J_n(x)$$

2. $$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

3. $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

More functions :-

$$S_i(x) = \int_0^x \frac{\sin t}{t}\,dt \qquad C_i(x) = \int_x^\infty \frac{\cos t}{t}\,dt$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$L\{S_i(x)\} = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$$

Prove this $\Longleftarrow$ $$L\{C_i(x)\} = \frac{\log(p^2+1)}{2p}$$

• Proof of $L\{C_i(x)\} = \dfrac{\log(p^2+1)}{2p}$

Let, $f(t) = \displaystyle\int_x^\infty \frac{\cos t}{t}\,dt \Rightarrow f'(t) = -\frac{\cos x}{x}$

$L\{x f'(t)\} = -L\{\cos x\}$ $\left[f(0) = \lim_{x\to 0} \frac{\cos x}{x} = 0\right]$

$(-1)\dfrac{d}{ds}[sF(s) - \cancel{f(0)}^{\,0}] = \dfrac{-x}{1+x^2}$ $*$

$sF(s) = \dfrac{1}{2}\displaystyle\int_0^s \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln|s^2+1| + c$

$\left[\lim_{s\to\infty} \ln\sqrt{s^2+1} = 0\right]$

$sF(s) = \dfrac{1}{2}\ln|s^2+1| + C$

$\lim_{s\to\infty} sF(s) = \lim_{x\to 0} f(x) = 0 \Rightarrow C = 0$

$\therefore F(s) = \dfrac{1}{2s}\ln(s^2+1)$

Error Function:

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt$$

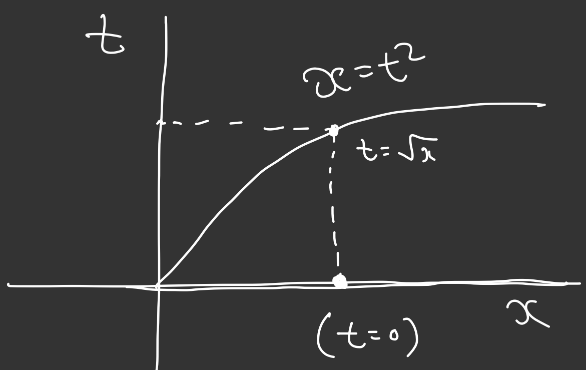
$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$L\{\operatorname{erf}(\sqrt{x})\} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-sx}\left\{\int_0^{\sqrt{x}} e^{-t^2}\,dt\right\}dx$$

$t = 0$, $t = \sqrt{x}$

$x = 0$, $x = \infty$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \int_{t^2}^\infty e^{-sx}\,dx\,dt$$

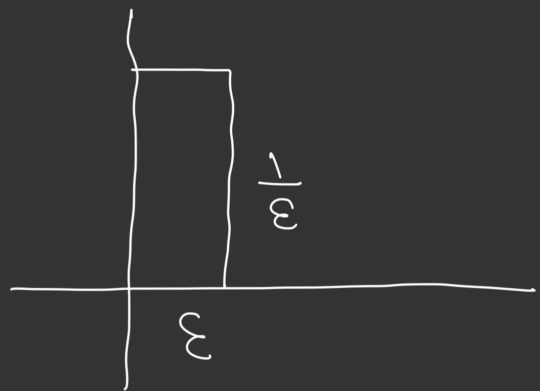


$$\boxed{L\{\operatorname{erf}(\sqrt{x})\} = \frac{1}{s\sqrt{s+1}}} \ast$$

eg $$L\left\{\operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}\left(1 - e^{-a\sqrt{s}}\right)$$

Unit impulsive function (Dirac - Delta function)

$$F_\varepsilon(x) = \begin{cases} 1/\varepsilon & 0 \le x \le \varepsilon \\ 0 & \text{Otherwise} \end{cases}$$



$$\delta(x) = \lim_{\varepsilon \to 0} F_\varepsilon(x)$$

Properties of dirac - delta :-

$$\int_0^\infty \delta(x)\,dx = 1$$

\# $L\{f(t)\cdot\delta(t-a)\}$

$= e^{-as} f(a)$ *

$$\int_0^\infty \delta(x-a)\,G(x) = G(a)$$

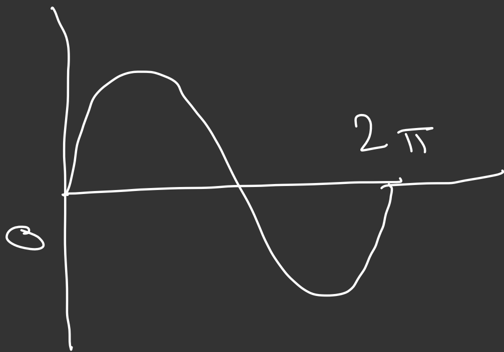
$$L\{\delta(x)\} = 1$$

$$\left[\int_0^\varepsilon \frac{1}{\varepsilon} e^{-sx}\,dx + \int_\varepsilon^\infty 0\cdot e^{-sx}\,dx\right]$$

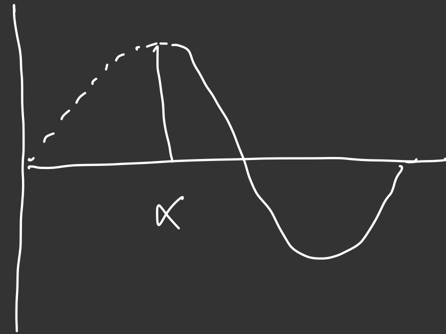
$$L\{\delta(x-a)\} = e^{-sa}$$

\# Unit Step function :-

$$H(x-a) = \begin{cases} 1 & x > a \\ 0 & x < a \end{cases}$$



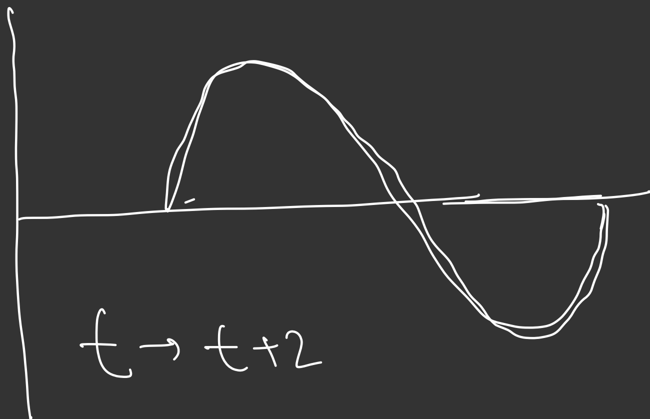
$f(t) = 5\sin t$



$f(t) = 5\sin t \, H(t-2)$

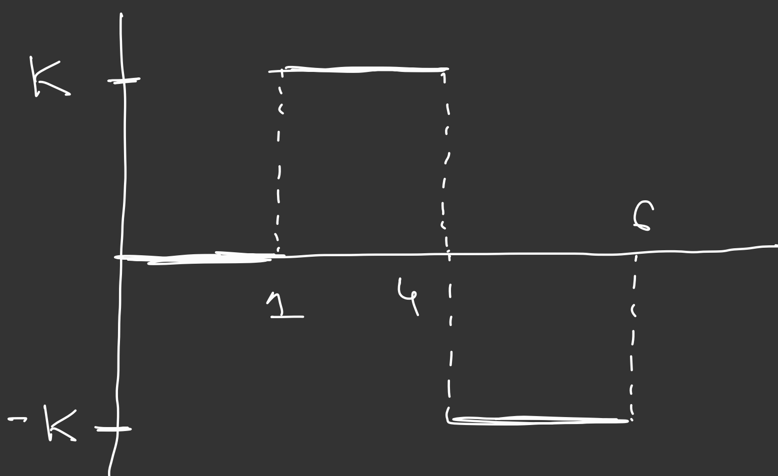
$* \; L\{f(t-a)H(t-a)\}$

$= e^{-as} F(s)$



$t \to t+2$

eg



$$L\{f(t)\} = K\left[H(t-1) - H(t-4)\right] +$$

$$(-K)\left[H(t-4) - H(t-6)\right]$$

<u>eg</u>

$$f(t) = \begin{cases} 2 & 0 < t < 1 \\ \frac{1}{2}t^2 & 1 < t < \frac{1}{2}\pi \\ \cos t & t > \frac{1}{2}\pi \end{cases}$$

<u>Sol'n</u>: $f(t) = 2 + \left(\frac{t^2}{2} - 2\right)H(t-1) +$

$$\left(\cos t - \frac{t^2}{2}\right) H\left(t - \frac{\pi}{2}\right)$$

$$f(t-1) = \frac{t^2}{2} - 2$$

$$f(t) = \frac{(t+1)^2}{2} - 2 = \frac{t^2}{4} + \frac{t}{2} - \frac{3}{4}$$

$$F(s) = \frac{1}{2s^3} + \frac{1}{2s^2} - \frac{3}{4s}$$

$$f\left(t-\frac{\pi}{2}\right) = \cos t - \frac{t^2}{2}$$

$$f(t) = \cos\left(t+\frac{\pi}{2}\right) - \frac{\left(t+\frac{\pi}{2}\right)^2}{2}$$

$$f(t) = -\sin t - \left[\frac{t^2}{2} + \frac{\pi^2}{8} + \frac{\pi t}{2}\right]$$

$$F(s_2) = \frac{-1}{s^2+1} - \frac{1}{s^3} - \frac{\pi^2}{8s} - \frac{\pi}{2s^2}$$

$$\therefore F(s) = \frac{2}{s} + e^{-s}\left(\frac{1}{2s^3} + \frac{1}{2s^2} - \frac{3}{4s}\right)$$

$$+ e^{-\frac{\pi}{2}s}\left(\frac{-1}{s^2+1} - \frac{1}{s^3} - \frac{\pi^2}{8s} - \frac{\pi}{2s^2}\right)$$

To Prove: $\operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right) = \frac{1}{s}\left(1 - e^{-a\sqrt{s}}\right)$

$$\operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{a/2\sqrt{t}} e^{-x^2}\, dx$$

$$L\left\{\operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-st} \int_0^{a/2\sqrt{t}} e^{-x^2}\, dx\, dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} \int_{a^2/4x^2}^{\infty} e^{-st}\, dt\, dx \qquad \text{Let, } \alpha = \frac{a^2 s}{4}$$

$$= \frac{2}{\sqrt{\pi}\, s} \int_0^{\infty} e^{-x^2} \cdot e^{-\frac{sa^2}{4x^2}}\, dx = \frac{2}{\pi\sqrt{s}} \int_0^{\infty} e^{-\left(x^2 + \frac{\alpha}{x^2}\right)}\, dx$$

$$= \frac{2}{\pi\sqrt{s}} \left[\int_0^{\infty} e^{-\left(x + \frac{\alpha}{x}\right)^2 + 2\alpha}\, dx \right]$$

Example 3.2.7

If $f(t) = erf\left(\frac{a}{2\sqrt{t}}\right)$, then

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}(1 - e^{-a\sqrt{s}}), \tag{3.2.23}$$

where $erf(t)$ is the *error function* defined by (2.5.13).

To prove (3.2.23), we begin with the definition (3.2.5) so that

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \int_0^\infty e^{-st}\left[\frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-x^2}dx\right] dt,$$

which is, by putting $x = \frac{a}{2\sqrt{t}}$ or $t = \frac{a^2}{4x^2}$ and interchanging the order of integration,

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}dx \int_0^{a^2/4x^2} e^{-st}dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\frac{1}{s}\left\{1 - \exp\left(-\frac{a^2 s}{4x^2}\right)\right\}dx$$

$$= \frac{1}{s}\cdot\frac{2}{\sqrt{\pi}}\left[\int_0^\infty e^{-x^2}dx - \int_0^\infty \exp\left\{-\left(x^2 + \frac{sa^2}{4x^2}\right)\right\}dx\right],$$

where the integral

$$\int_0^\infty \exp\left\{-\left(x^2+\frac{\alpha^2}{x^2}\right)\right\}dx = \frac{1}{2}\left[\int_0^\infty\left(1-\frac{\alpha}{x^2}\right)\exp\left[-\left(x+\frac{\alpha}{x}\right)^2+2\alpha\right]\right.$$

$$\left.+\int_0^\infty\left(1+\frac{\alpha}{x^2}\right)\exp\left[-\left(x-\frac{\alpha}{x}\right)^2-2\alpha\right]\right]dx,$$

which is, by putting $y = \left(x \pm \frac{\alpha}{x}\right)$, $dy = \left(1 \mp \frac{\alpha}{x^2}\right)dx$, and observing that the first integral vanishes,

$$= \frac{1}{2}e^{-2\alpha}\int_{-\infty}^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}e^{-2\alpha}, \qquad \alpha = \frac{a\sqrt{s}}{2}.$$

Consequently,

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}e^{-a\sqrt{s}}\right] = \frac{1}{s}[1 - e^{-a\sqrt{s}}].$$

We use (3.2.23) to find the Laplace transform of the complementary error function defined by (2.10.14) and obtain

$$\mathscr{L}\left\{erfc\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}e^{-a\sqrt{s}}. \tag{3.2.24}$$

The proof of this result follows from $erfc(x) = 1 - erf(x)$ and $\mathscr{L}\{1\} = \frac{1}{s}$.

$\square$

Ex $\int_0^\infty \frac{1-\cos t}{t^2}\,dt = \pi/2$ Prove it.

$L\{1-\cos t;p\} = \frac{1}{p} - \frac{p}{1+p^2}$ ← u is dummy variable in integral.

$L\{\frac{1-\cos t}{t};p\} = \int_p^\infty \left[\frac{1}{u} - \frac{u}{1+u^2}\right]du$ provided $\lim_{t\to 0}\frac{1-\cos t}{t}$ exist. (it is exist. use L-Hospital Rule)

$= \left[\log u - \frac{1}{2}\log(1+u^2)\right]_p^\infty$

$= \left[\log\frac{u}{\sqrt{1+u^2}}\right]_p^\infty = \left[\log\frac{1}{\sqrt{\frac{1}{u^2}+1}}\right]_p^\infty$ (Taken u^2 common in denominator)

$= \log 1 - \log\frac{1}{\sqrt{\frac{1}{p^2}+1}} = 0 - \log\frac{p}{\sqrt{1+p^2}}$

Again

$L\{\frac{1-\cos t}{t^2};p\} = -\int_p^\infty \log\frac{u}{\sqrt{1+u^2}}\,du$ provided $\lim_{t\to 0}\frac{1-\cos t}{t^2}$ = exist

(use concept of $\int\log x$ integral)

$= -\left[u\log\frac{u}{\sqrt{1+u^2}} - \int\frac{\sqrt{u^2+1}}{\cancel{u}}\cdot\left\{\frac{1}{\sqrt{u^2+1}} - \frac{u\cdot 2u}{\cancel{2}(u^2+1)^{3/2}}\right\}\cdot\cancel{u}\,du\right]$

$= -\left[u\log\frac{u}{\sqrt{1+u^2}} - \int\frac{\sqrt{u^2+1}}{\sqrt{u^2+1}}\left\{1 - \frac{u^2}{(1+u^2)}\right\}du\right]$

$= -\left[u\log\frac{u}{\sqrt{1+u^2}} - \int\frac{1}{u^2+1}\,du\right]$

$= -\left[u\log\frac{u}{\sqrt{1+u^2}} - \tan^{-1}u\right]_p^\infty$

$L\{\frac{1-\cos t}{t^2};p\} = -0 + \tan^{-1}\infty + p\log\frac{p}{\sqrt{1+p^2}} - \tan^{-1}p$

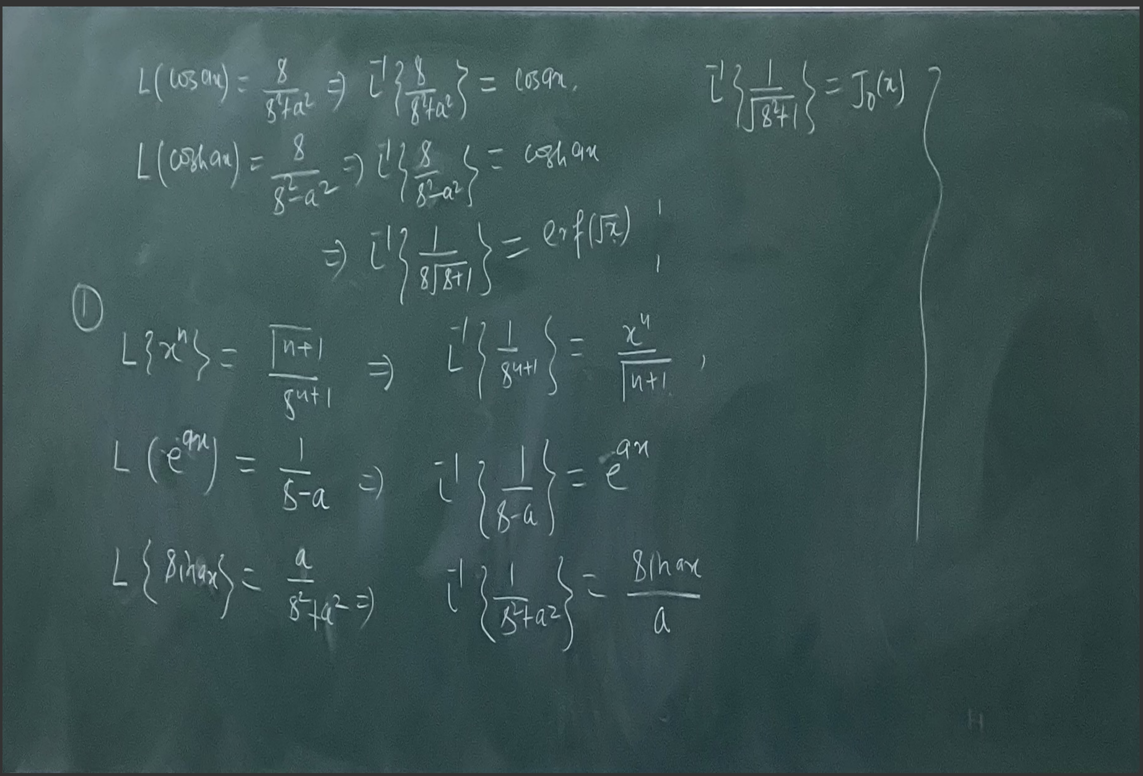
$\int_0^\infty e^{-pt}\left(\frac{1-\cos t}{t^2}\right)dt = \cancel{\tan^{-1}\infty}\ \pi/2 + p\log\frac{p}{\sqrt{1+p^2}} - \tan^{-1}p$

put, p=0, we get

$\int_0^\infty\left(\frac{1-\cos t}{t^2}\right)dt = \pi/2 + 0\cdot(\ ) - \tan^{-1}0 = \pi/2$ Ans.

Laplace Inverse :-

$$L\{f(x)\} = F(s) \implies f(x) = L^{-1}\{F(s)\}$$



Eg $L^{-1}\left\{ \frac{1}{s^4} - \frac{3p}{s^2+16} + \frac{5}{s^2+4} \right\}$

Eg $L^{-1}\left\{ \frac{1}{s} e^{-1/s} \right\} = J_0 (2\sqrt{x})$ *

Change of Scale :

If, $L^{-1}\{F(s)\} = f(x)$

Then, $L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{x}{a}\right)$

Proof :- $F(as) = \int_0^\infty e^{-ast} f(x)\,dx$

$$= \frac{1}{a} \int_0^\infty e^{-st} f(t/a)\,dt$$

$$= \frac{1}{a} L\{f(x/a)\}$$

1st Shifting property | Translation property :-

If, $L^{-1}\{f(x)\} = F(s)$

Then, $L^{-1}\{F(s-a)\} = e^{ax} f(x)$

Inverse LT of derivative :-

If $L^{-1}\{F(s)\} = f(x)$

Then, $L^{-1}\left\{\frac{d^n F(s)}{d s^n}\right\} = (-1)^n x^n f(x)$

Inverse LT of integral :-

If, $L^{-1}\{F(s)\} = f(x)$

Then, $L^{-1}\left\{\int_s^\infty F(u)\,du\right\} = \frac{f(x)}{x}$

Given, $\lim_{x \to 0} \frac{f(x)}{x}$ exist

Multiplication of power s :

$L\{f'(x)\} = sF(s) - f(0)$

$f'(x) = L^{-1}\{sF(s)\} - L^{-1}\{f(0)\}$

$= L^{-1}\{sF(s)\}$ when, $f(0) = 0$

$= L^{-1}\{sF(s)\} - \delta(x) f(0)$ When, $f(0) \neq 0$

Inverse LT of division by s.

Let, $L^{-1}[F(s)] = f(x)$

Then, $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^x f(t)\,dt$

2^{nd} Shifting :-

$$L\{g(x)\} = e^{-as} F(s)$$

or $L\{f(x-a)H(x-a)\} = e^{-as} F(s)$

If, $L^{-1}[F(s)] = f(x)$. Then,

$$f(x-a)H(x-a) = g(x) = L^{-1}\{e^{-as}F(s)\}$$

Eg $\frac{1}{(p+3/2)^{7/2}}$

A: $\left(e^{-3/2 x} \frac{x^{5/2}}{\Gamma 7/2}\right)$

Eg $\frac{1}{\sqrt{2p+3}}$

A: $\left(\frac{1}{\sqrt{2}} e^{-3/2 x} \frac{x^{1/2}}{\Gamma 3/2}\right)$

Eg $\frac{p}{(p+1)^5} \Rightarrow \left[\frac{p+1}{(p+1)^5} - \frac{1}{(p+1)^5}\right]$

eg $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Convolution of Two Functions:

Let $f(x)$ and $g(x)$ are two functions. Then Convolution of f and g is given by,

$$f * g = \int_0^x f(x-u)\, g(u)\, du$$

$$= \int_0^x g(x-u)\, f(u)\, du$$

Convolution Theorem of L.T. :-

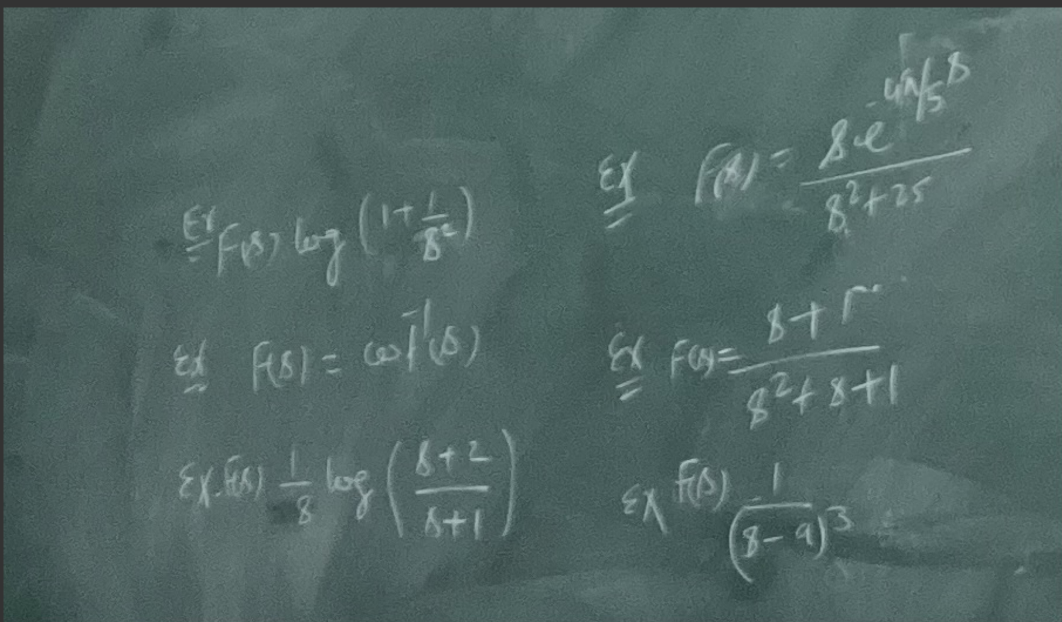
Let $f(x)$ and $g(x)$ be two function of Class A and

$$L^{-1}\{F(s)\} = f(x) \quad \& \quad L^{-1}\{G(s)\} = g(x)$$

Then, $L^{-1}\{F(s)\cdot G(s)\} = f(x) * g(x)$

$$\int_0^x \sin(x-t)\cos t\,dt = \sin x * \cos x$$

Eg $L^{-1}\left\{\dfrac{1}{s^3\sqrt{s+1}}\right\}$



* For $\tan^{-1}$, $\cot^{-1}$ $\Rightarrow$ Differentiate

* For log funct. $\Rightarrow$ IVT, FVT

$\Rightarrow$ Also differentiate

(1) $F(s) = \log\left(1+\frac{1}{s^2}\right)$

$$F'(s) = \frac{2s}{(s^2+1)} - \frac{2}{s}$$

$$-x f(x) = 2\cos x - 2$$

$$f(x) = \frac{1-\cos x}{x}$$

(2) $F'(s) = -1/1+s^2$

$$\Rightarrow f(x) = \frac{\sin x}{x}$$

(3) Expand

(4)
(5) 1st Shifting Property
(6)

Application of L.T. to Solve the Diff. Eq. :-

$$(D^2+9)\,y = \cos 2x \quad \text{if} \quad y(0)=1, \; y\left(\frac{\pi}{2}\right)=-1$$

- Take L.T. $\longrightarrow$ Simplify

$$L\{y(x)\} = \bar{y}(s) = h(s)$$

$$y(x) = L^{-1}\{h(s)\}$$

Eg. $\dfrac{d^2y}{dx^2} + 9y = \cos 2x$

$$\Rightarrow s^2\bar{y}(s) - s\,y(0) - y'(0) + 9\bar{y}(s) = \frac{s}{s^2+4}$$

Boundary Condition: $y(0)=1$

$y\left(\frac{\pi}{2}\right)=-1$

$$\Rightarrow [s^2+9]\bar{y}(s) - s - \underbrace{y'(0)}_{\Downarrow \; A \; (\text{let})} = \frac{s}{s^2+4}$$

$$\bar{y}(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$\Downarrow$$

$$y(x) = L^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\} + L^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$+ L^{-1}\left\{\frac{A}{s^2+9}\right\}$$

$$= \int_0^x \sin 2u \cos(3x-u)\, du + \cos 3x + A \sin 3x$$

At, $y = \frac{\pi}{2}$, $A =$ ______

Ans: $y(x) = \frac{4}{5}\cos 2x + \frac{4}{5}\sin 3x + \frac{1}{5}\cos 2x$

Q) $$\frac{d^4y}{dx^4} - y = 1$$

$y(0) = y'(0) = y''(0) = y'''(0) = 0$

$$L\left\{\frac{d^4y}{dx^4}\right\} = s^4\bar{y}(s) - s^3\cancel{y(0)}^{0} - \cancel{s^2y'(0)}^{0} - \cancel{sy''(0)}^{0} - \cancel{y'''(0)}^{0}$$

$$s^4\bar{y}(s) - \bar{y}(s) = 1/s$$

$$\bar{y}(s) = \frac{1}{s(s^4-1)} = \frac{1}{s(s^2+1)(s^2-1)}$$

$$L^{-1}\left\{\frac{1}{s^4-1}\right\} = \frac{1}{2}\left[\sinh x - \sin x\right]$$

$$L^{-1}\left\{\frac{1}{s(s^4-1)}\right\} = \int_0^x \frac{1}{u^4-1}\,du$$

Eg $\quad xy'' - (x-1)y' - y = 0$

$y(0) = 5$ as $y \to 0$ as $x \to \infty$

$$L[xy''] - L[(x-1)y'] - \bar{y}(s) = 0$$

$$-\frac{d}{ds}[L(y'')] + \frac{d}{ds}[L(y')] + \underline{L[y']}$$

$$\underline{-\bar{y}(s) = 0}$$

$$-\frac{d}{ds}\left[s^2\bar{y}(s) - sy(0) - y'(0)\right] + \frac{d}{ds}\left[s\bar{y}(s) - y(0)\right]$$

$$= \underline{-2s\bar{y}(s)} - \underline{s^2\bar{y}'(s)} - 5 - 0 + \cancel{\bar{y}(s)} + \underline{s\bar{y}'(s)} + \underline{s\bar{y}(s)} - 5 - \cancel{\bar{y}(s)} = 0$$

$$= -s\bar{y}(s) + (s - s^2)\bar{y}'(s) - 10 = 0$$

$$= s(s-1)\bar{y}'(s) - s\bar{y}(s) - 10 = 0$$

$$= \bar{y}'(s) + \left(\frac{-s}{s(s-1)}\right) - \frac{10}{s(s-1)} = 0$$

$\downarrow$ Using IF

$\bar{y}(s) =$ ______

$\downarrow$ Inverse

$y(x) =$ ______ ✓

Fourier Transformation :-

$$T\{f(x)\} = \int_a^b K(s,x)\, f(x)\,dx$$

$$K(x,t) = \frac{1}{\sqrt{2\pi}} e^{-ist} \quad \& \quad (a,b) \to (-\infty,\infty)$$

$$F\{f(x),s\} = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$f(x) = F^{-1}\{F(s)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\,ds$$

eg Find F.T. : $e^{-a|x|}$ $\forall$ $a>0$

$$F\{e^{-a|x|}\}$$

F.T of $f(x)$

$$\mathcal{F}\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx \qquad \Bigg| \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$$

and Inverse Transformation is given by

$$\mathcal{F}^{-1}\{F(s)\} = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} F(s)\,ds \qquad \Bigg| \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} F(s)\,ds$$

Provided integral exists.

Sufficient Cond'n for FT :-

let $f(x)$ is absolutely integrable on $(-\infty, +\infty)$

$\left[\int_{-\infty}^{\infty} |f(x)| < \infty\right]$ or Convergence of

integral $\int_{-\infty}^{\infty} e^{isx} f(x)\,dx$ follows $f(x)$ is

Square integral. $\left[\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\right]$

* $f(x)$ Should be piece-wise Continuous.

These are called Complex F.T.

eg $$F\left(e^{-ax^2}\right) = F(s) \quad [\text{For}, a>0]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx - ax^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(is - ax)x} dx$$

$$\left[\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \quad \text{or} \quad \int_{0}^{\infty} e^{-t^2} dt = \sqrt{\pi}/2 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left[x^2 - \frac{isx}{a}\right]} dx$$

$$\left[\left(x - \frac{is}{2a}\right)^2 = x^2 - \frac{is}{a} - \frac{s^2}{4a^2} \left(+\frac{s^2}{4a^2}\right) = x^2 - \frac{is}{a} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left[x - \frac{is}{2a}\right]^2 - \frac{s^2}{4a}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a} \cdot \frac{1}{\sqrt{a}} \cdot \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2a}} e^{-s^2/4a}$$

Properties of F.T. :-

① Linearity prop. :-

$$F\left(c_1 f_1(x) + c_2 f_2(x)\right) = c_1 F_1(s) + c_2 F_2(s)$$

② Change of Scale prop. :- *

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \quad [\text{let, } ax = t]$$

(3) 1^{st} Shifting prop. :- (*)

$$F\{e^{iax} f(x)\} = F(s+a)$$

[For the given defined Fourier Transform]

(4) 2^{nd} Shifting prop. :- (*)

$$F\{f(x-a)\} = e^{isa} F(s)$$

[For the given defined Fourier Transform]

(5) Modulation Prop of F.T. :-

If, $F\{f(x)\} = F(s)$

Then find, $F\{f(x) \cos ax ; s\} =$

$$\frac{F(s+a)}{2} + \frac{F(s-a)}{2}$$

(6) Duality Prop. : $F\{f(x)\} = F(s)$

Then, $F\{F(x)\} = f(-s)$

eg $F\left\{\frac{\sin ax}{x}\right\}, \quad a > 0$

$$F\left\{\frac{\sin ax}{x}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \frac{\sin ax}{x} \, dx$$

$e^{isx} = \cos sx + i \sin sx$

$$\Rightarrow F\left\{\frac{\sin ax}{x}\right\} = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \frac{\cos sx \sin ax}{x} \, dx + \cancel{i \int_{-\infty}^{\infty} \frac{\sin sx \sin ax}{x} \, dx}^{\,0} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\sin(s+a)x - \sin(s-a)x}{x}$$

By Using next quest.

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty \frac{\sin(s+a)x}{x} - \int_0^\infty \frac{\sin(s-a)x}{x} \right]$$

$$= \sqrt{\pi/2} \qquad |s| < a$$

$$0 \qquad |s| > a$$

Eg Find FT of $f(x) = 1 \quad |x| < a$

$= 0 \quad |x| > a$

Also find: (i) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(sa)\cos(sx)}{s}\, ds$

(ii) $\displaystyle\int_0^\infty \frac{\sin s}{s}\, ds$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\, is} \left(e^{isa} - e^{-isa} \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s} = H(x-a)$$

$$f(x) = F^{-1}\{F(s)\} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \frac{\sin sa}{s}\, ds$$

$$\begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx \sin sa}{s}\, ds - 0$$

$$\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s}\, ds = \begin{cases} \pi & |x| < a \\ 0 & |x| > a \end{cases}$$

For $x = 0$, $a = 1$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \Rightarrow \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

eg

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

$$\int_{0}^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} \, dx = -\frac{3\pi}{16}$$

Proof of Duality Prop. :

Proof of the Duality Prop

$$F\{F(x)\} = f(-s)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds$$

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(x)\, dx$$

$$\Rightarrow f(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} F(x)\, dx$$

$$\boxed{f(-s) = F\{F(x)\}}$$

Proof of composition $\quad F\{f(x)\} = F(s)$

$$\int_{-\infty}^{\infty} F(s)\, g(s)\, e^{isx}\, dx = \int_{-\infty}^{\infty} f(\xi)\, G(\xi - x)\, d\xi$$

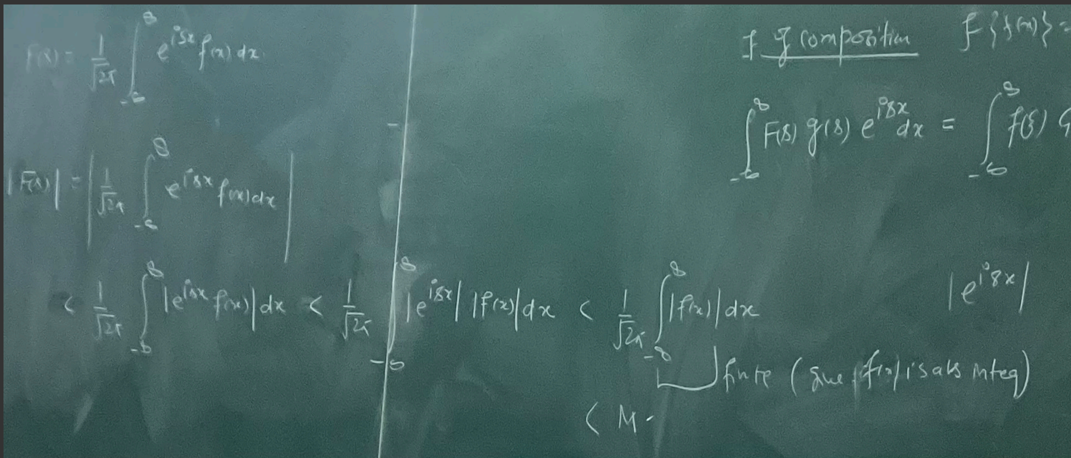
$$G(s) = F\{g(x)\}$$

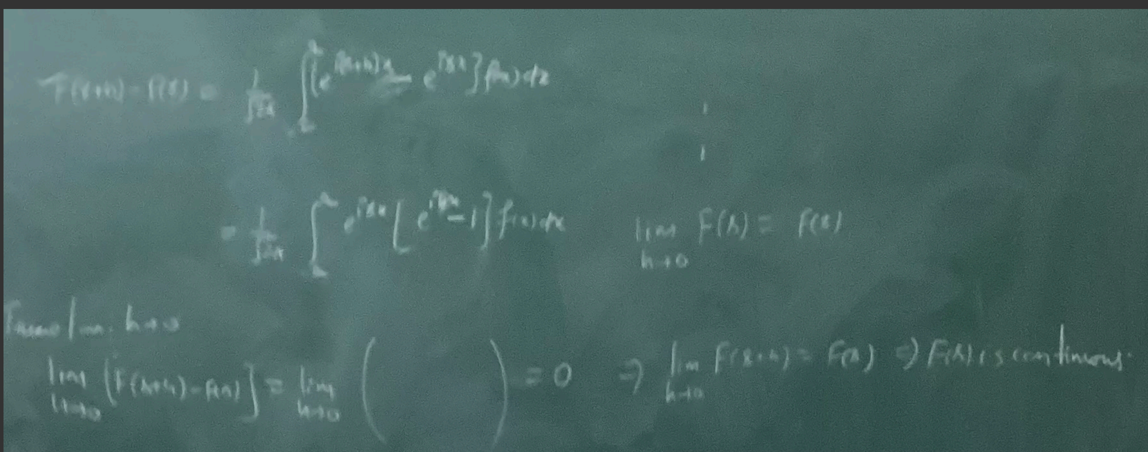
Theorem: If $f(x)$ is piecewise Continuous & diff. and absolutely integer

Then (i) $F(s)$ is bounded b/w $(-\infty, \infty)$

(ii) $F(s)$ is Continuous b/w $(-\infty, \infty)$

Proof: (i) & (ii)





If $f(x)$ is continuous and diff. and $f(x) \to 0$ as $|x| \to \infty$ Then,

$$F\{f'(x)\} = (-is)\, F\{f(x)\}$$

$$= -is\, F(s)$$

$$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\left[f(x) e^{isx} \right]_{-\infty}^{\infty}}_{=0} - is \int_{-\infty}^{\infty} f(x) e^{isx} \right]$$

$$\Rightarrow -\frac{is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx = -is\, F\{f(x)\} = -is\, F(s)$$

Note: If $f(x)$ is Conti and n times diff. and $|f^k(x)| \to 0$ as $|x| \to \infty$ for $k = 1, 2, \ldots\ldots n-1$. Then, Fourier Transf. of n^{th} derivative is.

$$F\{f^n(x)\} = (-is)^n F(s)$$

Defination of Convolution prop. :-

The Convolution of two integrable function $f(x)$ and $g(x)$ denoted by $f * g$.

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi$$

- Convolution Theorem for Fourier Trans. :-

$\Rightarrow$ If $F\{f(x)\} = F(s)$ and $F\{g(x)\} = G(s)$

then $F\{f * g\} = F(s)\, G(s)$

$$f * g = F^{-1}\{F(s)\, G(s)\}$$

Proof : $F\{f * g\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi)\, d\xi \int_{-\infty}^{\infty} e^{is(t+\xi)} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\xi} g(\xi)\, d\xi \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$$

$$= F\{g(x)\} \cdot F\{f(x)\}$$

- Property: If $F\{f(x)\} = F(s)$

and, $F\{g(x)\} = G(s)$

Then, $$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)} = \int_{-\infty}^{\infty} F(s)\,\overline{G(s)}\,ds$$

[This is called Parseval's identity]

eg $$f(x) = \left(1 - \frac{|x|}{a}\right) H\left(1 - \frac{|x|}{a}\right)$$

$$* \quad F_c\{e^{-ax}; s\} = \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right)$$

$$* \quad F_c\{u(t-x); s\} = \sqrt{\frac{2}{\pi}} \frac{\sin st}{s}$$

$$* \quad F\{e^{-ax^2}; s\} = \frac{e^{-s^2/4a}}{\sqrt{2a}}$$

Riemann - Lebesgue lemma:

If $F(s) = \mathcal{F}\{f(x)\}$ then

$$\lim_{|s|\to\infty} |F(s)| = 0$$

Proof: $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx \quad \text{--- } 0$$

[Given]

$$e^{isx} = -e^{isx + i\pi} \quad \text{--- } 1$$

Using 1 in 0 we get,

$$F(s) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx+i\pi} f(x)\,dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\left(x+\frac{\pi}{s}\right)} f(x)\,dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f\left(x-\frac{\pi}{s}\right)dx \quad \text{--- } 2$$

Add ① & ② we get,

$$\overline{F}(s) = \frac{1}{2\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{isx} \left[f(x) - f\left(x - \frac{\pi}{s}\right) \right] dx \right]$$

$$|F(s)| = \left| \frac{1}{2\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{isx} \left[f(x) - f\left(x - \frac{\pi}{s}\right) \right] dx \right] \right.$$

$$\leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{isx}| \left| f(x) - f\left(x - \frac{\pi}{s}\right) \right| dx$$

$$\leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{s}\right) \right| dx \quad \text{—} ④$$

$$\lim_{|s| \to \infty} |F(s)| \leq \lim_{|s| \to \infty} \frac{1}{2\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \left[f(x) - f\left(x - \frac{\pi}{s}\right) \right] dx \right.$$

$$\lim_{|s|\to\infty} |F(s)| \leq 0 \quad \& \quad |F(s)| \geq 0$$

[Modulus]

$$\Rightarrow \lim_{|s|\to\infty} |F(s)| = 0$$

For, $H(a-|x|) = H(-|x|+a)$

$= 1 \qquad |x| < a$

$= 0 \qquad |x| > a$

Characteristic Function:

eg Find F.T. of Characteristic funct. :-

$$\chi_{[-a,a]} = H(a-|x|) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

$$F(s) = F\{\chi_{[-a,a]}\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right)$$

Good Functions :-

A good function $g(x)$ is a function in $C^{\infty}(\mathbb{R})$ [All derivatives are Conti], that decays sufficiently & rapidly that $g(x)$ and all its derivative decay to zero faster than $|x|^{-N}$ as $x \to \infty$.

- Unit function: $I(x)$

$$\int_{-\infty}^{\infty} I(x)\, g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx$$

- Heaviside Unit Step funct : $H(x)$

$$\int_{-\infty}^{\infty} H(x)\, g(x) = \int_{0}^{\infty} g(x)\, dx$$

• Signum funct :-

$$\int_{-\infty}^{\infty} sgn(x)\, g(x)\, dx = \int_{0}^{\infty} g(x)\,dx - \int_{-\infty}^{0} g(x)\,dx$$

$$\left[Sgn(x) = 2H(x) - I(x) \right]$$

Dirac-Delta Funct :-

$$\delta(x) = 0 \quad \text{if} \quad x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

Let, $\delta_n = \sqrt{\frac{n}{\pi}}\, e^{-nx^2}$, $n = 1, 2, 3, \ldots\ldots$

$$\left[\lim_{n \to \infty} \delta_n(x) = \delta(x) \right] *$$

$$\left[\begin{array}{l} \displaystyle\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0) \\ \displaystyle\int_{-\infty}^{\infty} \delta(x-a)\,\phi(x)\,dx = \phi(a) \end{array}\right] *$$

Find the F.T. of $\delta(x)$.

$$F(\delta(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\,\delta(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}$$

$*$

I.F.T. of $1/\sqrt{2\pi}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx}\,ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx}\,ds$$

$$= \delta(x) \quad *$$

eg Find F.T. of Const. funct. c

$$F\{c;s\} = c\sqrt{2\pi}\,\delta(x)$$

$$F\{c\,e^{-\frac{x^2}{4n}};s\} = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(-isx + x^2/4n)}\,dx$$

$$= \sqrt{2\pi}\cdot c\cdot\sqrt{\frac{n}{\pi}}\; e^{-ns^2}$$

For $\lim_{n\to\infty}$, $F\{c\} = \sqrt{2\pi}\cdot c\cdot \delta(x)$

eg Find F.T. of $e^{-ax}H(x)$

Ans: $\dfrac{1}{\sqrt{2\pi}(is-a)}$

Fourier Cosine Transform:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx \longrightarrow \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(sx)\, f(x)\,dx$$

Sine and Cosine transformation is defined as,

$$F\{f(x)\} = F_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(sx)\, f(x)\,dx$$

$$F\{f(x)\} = F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(sx)\, f(x)\,dx$$

And Corresponding inverse transforms are,

$$F^{-1}\{F_c(s)\} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(sx)\, F_c(s)\,ds$$

$$F^{-1}\{F_s(s)\} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(sx)\, F_s(s)\,ds$$

Properties:

1. If, $F_c\{f(x)\} = F_c(s)$ &

$F_s\{f(x)\} = F_s(s)$

$$F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

$$F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right)$$

2. $F_c\{f'(x)\} = \underline{s F_s(s) - \sqrt{\frac{2}{\pi}}\, f(0)}$

[Using by parts]

$$F_s\{f'(x)\} = \underline{-s F_c(s)}$$

For double derivative,

$$F_c\{f''(x)\} = -s^2 F_c(s) - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$F_s\{f''(x)\} = -s^2 F_s(s) + \sqrt{\frac{2}{\pi}}\, s\, f(0)$$

Q. Find F.C.T & F.S.T of x^{m-1} ☆

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{m-1} \cos(sx)\, dx$$

$$\Gamma m = \int_0^\infty e^{-x} x^{m-1}\, dx$$

$x = its$

$dx = is\,dt$

$$= \int_0^\infty e^{-ist} (ist)^{m-1}\, is\, dt$$

$$= (is)^m \cdot \int_0^\infty e^{-its} t^{m-1}\, dt$$

$$\Rightarrow \frac{\Gamma m}{s^m} = i^m \int_0^\infty e^{-ist} t^{m-1}\, dt$$

$$= e^{im\frac{\pi}{2}} \int_0^\infty e^{-ist} t^{m-1}\, dt$$

$$\int_0^\infty e^{-ist}\, t^{m-1}\, dt = \frac{e^{-im\frac{\pi}{2}}\,\Gamma m}{s^m}$$

$$\int_0^\infty \left[\cos st - i\sin(st)\right] t^{m-1}\, dt = \frac{\Gamma m}{s^m}\left\{\cos m\frac{\pi}{2} - i\sin\frac{m\pi}{2}\right\}$$

Comparing,

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(st)\, t^{m-1}\, dt = \frac{\Gamma m}{s^m} \cos\left(\frac{m\pi}{2}\right) \sqrt{\frac{2}{\pi}}$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \sin(st)\, t^{m-1}\, dt = \frac{\Gamma m}{s^m} \sin\left(\frac{m\pi}{2}\right) \sqrt{\frac{2}{\pi}}$$

Eg Find the F.C.T of e^{-x^2}.

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos(sx)\, dx$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2} \sin(sx)\, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{(-2x e^{-x^2})}_{d(e^{-x^2})} \sin(sx)\, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\left[e^{-x^2} \cancel{\sin(sx)} \right]_0^\infty - s \int e^{-x^2} \cos(sx)\, dx \right]$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[-s \int e^{-x^2} \cos(sx)\, dx \right] = -\frac{1}{2} s I$$

$$\Rightarrow I = C e^{-s^2/4}$$

Now, $I\Big|_{s=0} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x^2} dx = \frac{1}{\sqrt{2}}$

At $s = 0$, $I = \frac{1}{\sqrt{2}} \Rightarrow C = \frac{1}{\sqrt{2}}$

$$\boxed{I = \frac{1}{\sqrt{2}} e^{-s^2/4}}$$

eg Find F.C.T of $\frac{1}{1+x^2}$. & Also FST

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\cos(sx)}{1+x^2} dx = I(let)$$

$$\frac{dI}{ds} = -x \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin(sx)}{1+x^2} dx$$

$$\frac{d^2 I}{ds^2} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x^2 \cos(sx)}{1+x^2}\, dx$$

$$\frac{d^2 I}{ds^2} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(x^2+1-1)\cos(sx)}{1+x^2}\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \left[\cancel{\int_0^\infty \cos(sx)\, dx}^{\,0} - \int_0^\infty \frac{\cos(sx)}{1+x^2}\, dx \right]$$

$$= I \Rightarrow \frac{d^2 I}{ds^2} - I = 0$$

$$\Downarrow$$

$$I = Ae^{s} + Be^{-s} \Rightarrow \frac{dI}{ds} = Ae^{s} - Be^{-s}$$

At $s = 0$, $I = \sqrt{\frac{\pi}{2}}$

$$\therefore A + B = \sqrt{\frac{\pi}{2}}$$

$s \to \infty$, $I = 0$ $\therefore A = 0$

$A = 0$ & $B = \sqrt{\frac{\pi}{2}}$

$\therefore I = \sqrt{\frac{\pi}{2}} e^{-s}$

Eg. Find F.T. of $f(x)$.

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

$$\int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}$$

Using Parseval Identity, $\int_{-\infty}^{\infty} F(s) \overline{G(s)} = \int_{-\infty}^{\infty} f(x) \overline{g(x)}$

$f(x) = g(x)$, $\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$

eg Solve the $f(x)$ the integral eq.

$$\int_0^{\infty} f(x) \cos(sx) \, dx = 1 - s \qquad 0 \le s \le 1$$

$$= 0 \qquad s > 1$$

Prove that: $\int_0^{\infty} \frac{\sin^2 t}{t^2} \, dt = \frac{\pi}{2}$

$$F_c(s) = (1-s)\sqrt{\frac{2}{\pi}} \qquad 0 \le s \le 1$$

$$= 0 \qquad s > 1$$

$$f(x) = F^{-1}\{F_c(s); x\}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^1 (1-s) \cos(sx)\, ds$$

$$= \frac{2}{\pi} \left[\int_0^1 \cos(sx)\, ds - \int_0^1 s \cos(sx)\, ds \right]$$

$$= \frac{2}{\pi} \left[\frac{1 - \cos x}{x^2} \right]$$

By putting in initial integral and $s = 0$.

$$\Rightarrow \quad \frac{2}{\pi} \int_0^\infty \frac{\sin^2(x/2)}{x^2} = f(x)$$

Eg $F_s\{\operatorname{erfc}(ax)\}$

$$\operatorname{erfc}(ax) = \frac{2}{\sqrt{\pi}} \int_{ax}^{\infty} e^{-t^2}\,dt$$

$$F_s\{\operatorname{erfc}(ax)\} = \frac{2}{\sqrt{\pi}} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(sx) \int_{ax}^{\infty} e^{-t^2}\,dt$$

$$= \frac{2\sqrt{2}}{\pi} \int_0^{\infty} \sin(sx) \int_{ax}^{\infty} e^{-t^2}\,dt$$

$\downarrow$ Change of Variable

Convolution of Fourier Sine & Cosine:

If $F_c\{f(x)\} = F_c(s)$ &

$F_c\{g(x)\} = G_c(s)$

$$F_c^{-1}\{F_c(s)\,G_c(s)\} =$$

$$\frac{1}{2\pi}\int_0^\infty f(\xi)\left[g(x+\xi)+g(x-\xi)\right]d\xi$$

eg

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad \text{[Heat eq.]}$$

$$\frac{\partial^2 u}{\partial t^2} = K\frac{\partial^2 u}{\partial x^2} \quad \text{[Wave eq.]}$$

N/B: BC at the lower end

$u(0,t) \longrightarrow$ FS transform

$u_x(0,t) \longrightarrow$ FC transform

$$F\{u(x,t)\} = \bar{u}(s,t)$$

$$F\left\{\frac{\partial u}{\partial x}\right\} = (-is)\,\bar{u}(s,t)$$

$$F\left\{\frac{\partial^2 u}{\partial x^2}\right\} = (-is)^2\,\bar{u}(s,t)$$

$$F\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\bar{u}(s,t)$$

eg Consider a initial boundary value prob. in heat eq. in $0 < x < \infty$ without source.

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$$

$$u(x,0) = 0 \qquad 0 < x < \infty$$

$$u(0,t) = f(t), \quad u(x,t) \to 0 \text{ as } x \to \infty$$

Take Sine F.T. of eq ①, we get

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin(sx)\, dx = K \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin(sx)\, dx$$

$$\Rightarrow \frac{d\bar{u}(s,t)}{dt} = K \sqrt{\frac{2}{\pi}} \left[\cancel{\left[\frac{\partial u}{\partial x} \sin(sx) \right]_0^\infty}^{\,0} - s \int_0^\infty \frac{\partial u}{\partial x} \cos(sx)\, dx \right]$$

$$= K \sqrt{\frac{2}{\pi}} \left[-s \left[\left[u \cos(sx) \right]_0^\infty + s \int_0^\infty u \sin(sx)\, dx \right] \right]$$

$$= K \sqrt{\frac{2}{\pi}} \left[-s \left\{ -f(t) + s \int_0^\infty u \sin(sx)\, dx \right] \right]$$

$$= Ks \sqrt{\frac{2}{\pi}} f(t) - Ks^2 \bar{u}(s,t)$$

$$\frac{d\bar{u}}{dt} + k s^2 \bar{u} = k s \sqrt{\frac{2}{\pi}} f(t)$$

$\downarrow$ Using linear diff. eq.

$$\bar{u}(s,t) = \left[\qquad \right]$$

$\downarrow$ inverse Cosine FT

$$u(x,t)$$

eg

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

[For Steady State heat eq.. laplace]

$$u_x(0,t) = -\mu$$

$$u(x,t) \to 0 \quad \text{at } x \to \infty \qquad u(x,0) = 0$$

A:

$$\frac{d\bar{u}}{dt} + s^2 \bar{u} = 0$$

use the method of F.T to determine the displacement $u(x,t)$ of infinite string given that is initially at rest and initial displacement is $f(x)$, $-\infty < x < \infty$

Show that the sol^n is $u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct) \right)$

$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t}$

$\frac{d\bar{u}}{dt} =$

Solution: Using Wave eq^n,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty$$

Given.

$$u(x,0) = f(x)$$

$$u_t(x,0) = 0$$

Taking F.T of eq^n,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \frac{\partial^2 u}{\partial t^2} dx = \frac{1 \cdot c^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx$$

$$\frac{d^2 \bar{u}}{dt^2} = c^2 \left[\frac{1}{\sqrt{2\pi}} \left[\left[\cancel{\frac{\partial u}{\partial x} e^{isx}} \right]_{-\infty}^{+\infty} - is \int_{-\infty}^{\infty} e^{isx} \frac{\partial u}{\partial x} \right] \right.$$

$$= \frac{c^2}{\sqrt{2\pi}} \left[-is \left[\left[u e^{isx} \right]_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} e^{isx} u(x,t)\, dx \right] \right]$$

$$= -c^2 s^2 \bar{u}$$

$$\Rightarrow \frac{d^2 \bar{u}}{dt^2} + c^2 s^2 \bar{u} = 0$$

$$\Rightarrow \bar{u}(s,t) = A \cos(cst) + B \sin(cst)$$

$$\Rightarrow \underbrace{\bar{u}_t(st)}_{0} = -Acs \sin(cst) + Bcs \cos(cst)$$

$$\Longrightarrow B = 0$$

$$\bar{u}(s,t) = A \cos(cst)$$

$$\bar{u}(s,0) = \bar{f}(s) = A$$

$$\bar{u}(s,t) = \bar{f}(s) \cos(cst)$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{-isx} \underline{\cos(cst)}\, ds$$

$$\left(\frac{e^{icst} + e^{-icst}}{2} \right)$$

$$= \frac{1}{2} \left[f(x+ct) + f(x-ct) \right]$$

eg Laplace eq. : $u_{xx} + u_{yy} = 0$

$$0 < x, y < \infty$$

B.C : $u(0,y) = a$ $\qquad u(x,0) = 0$

$\nabla u \to 0$ as $\sqrt{x^2+y^2} \to \infty$

Taking Sine Transform,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin(sx)\, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial y^2} \sin(sx)\, dx = 0$$

$$\frac{d^2 \bar{u}(s,y)}{dy^2} + \sqrt{\frac{2}{\pi}} \left[\cancel{\left[\frac{\partial u}{\partial x} \sin(sx) \right]_0^\infty}^{\,0} - s \int_0^\infty \frac{\partial u}{\partial x} \cos(sx) \right] = 0$$

$$\frac{d^2 \bar{u}(s,y)}{dy^2} + \sqrt{\frac{2}{\pi}} \left[-s \left[\left[u \cos(sx) \right]_0^\infty + s \int_0^\infty u \sin(sx) \right] \right]$$

$$\frac{d^2\bar{u}(s,y)}{dy^2} + \sqrt{\frac{2}{\pi}}\left[-s\left[-a + s\int_0^\infty u\sin(sx)\right]\right]$$

$$\frac{d^2\bar{u}(s,y)}{dy^2} + \sqrt{\frac{2}{\pi}}\left[as - s^2\int_0^\infty u\sin(sx)\right]$$

$$\frac{d^2\bar{u}(s,y)}{dy^2} + \sqrt{\frac{2}{\pi}}\,as - s^2\bar{u}(s,y) = 0$$

$$\frac{d^2\bar{u}(s,y)}{dy^2} - s^2\bar{u}(s,y) = -\sqrt{\frac{2}{\pi}}\,as$$

$$\bar{u}(s,y) = C.I + P.I$$

$$= c_1e^{sy} + c_2e^{-sy} + \sqrt{\frac{2}{\pi}}\,\frac{a}{s}$$

At ∞, $\bar{u}(s,y) \to 0 \implies c_1 = 0$

$$\bar{u}(s,0) = 0 \implies C_2 + \sqrt{\frac{2}{\pi}}\,\frac{a}{s} = 0$$

$$\implies C_2 = -\sqrt{\frac{2}{\pi}}\,\frac{a}{s}$$

$$\bar{u}(s,y) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s}\left(1 - e^{-sy}\right)$$

$$u(x,y) = \frac{2a}{\pi} \int_0^\infty \left(\frac{1 - e^{-sy}}{s}\right) \sin sx \, ds$$

Ans : $\frac{2a}{\pi} \tan^{-1}\left(\frac{y}{x}\right)$

Calculation of Integral :

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} \qquad a>0, \; b>0$$

If F.T. of $f(x) = e^{-a|x|}$ is $F(s)$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

Using Parseval Identity.

$$\int_{-\infty}^{\infty} F(s)\, G^*(s) = \int_{-\infty}^{\infty} f(x)\, g^*(x)$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{ab}{(s^2+a^2)(s^2+b^2)} = \int_{-\infty}^{\infty} e^{-(a+b)|x|}\,dx$$

$$= 2\int_{0}^{\infty} e^{-(a+b)x}\,dx$$

$$= -\frac{[0-1]}{(a+b)} \Rightarrow \int_{-\infty}^{\infty}\frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{(a+b)}$$

eg $a>0,\ b>0$ Show:

$$\int_{0}^{\infty}\frac{x^2\,dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)}$$

$$F_s\left(e^{-a|x|}\right) = \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}$$

Using Parseval Identity,

$$\int_{-\infty}^{\infty} F(s)\, G^{*}(s) = \int_{-\infty}^{\infty} f(x)\, g(x)$$

$$\Rightarrow \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+b^2)}\, ds = \int_{-\infty}^{\infty} e^{-(a+b)|x|}\, dx$$

$$\Rightarrow \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = -\frac{[0-1]}{(a+b)}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2(a+b)}$$

eg

$$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^4} = \frac{\pi}{(2a)^5}$$

$$f(x) = \frac{1}{2(x^2+a^2)}$$

$$\overline{F}\{f(x)\} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2a} \cdot e^{-a|s|}$$

Take Fourier

$$\longleftarrow \quad f'(x) = \frac{x}{(x^2+a^2)^2}$$

and Use $\overline{F}\{f'(x)\}$

Use Parseval Identity

Method of Variable Separation :-

$$A(x)\frac{\partial^2 u}{\partial x^2} + B(x)\frac{\partial^2 u}{\partial x \partial y} + C(x)\frac{\partial^2 u}{\partial y^2}$$

$$+ f(u_x, u_y, u) = 0$$

$B^2 - 4AC > 0 \rightarrow$ Hyperbolic eq.

$B^2 - 4AC < 0 \rightarrow$ Elliptic

$B^2 - 4AC = 0 \rightarrow$ Parabolic

Heat eq.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$A = 1,\ B = 0,\ C = 0$

$B^2 - 4AC = 0$

$\rightarrow$ Parabolic

Lap. eq.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\downarrow$

Elliptic

Wave eq.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$B = 0,\ A = 1,$

$C = -1$

$B^2 - 4AC > 0$

$\rightarrow$ Hyperbolic

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

With Condition: $u(0,t) = 0$

$u(l,t) = 0$

$u(x,0) = f(x)$

$0 < x < l$

$$u(x,t) = X(x)\,T(t)$$

$$X(x)\frac{dT}{dt} = T(t)\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{\frac{dT}{dt}}{T(t)} = \frac{d^2X}{dx^2}\Big/ X = \lambda^2 \text{ (say)}$$

$$\therefore \frac{dT}{dt} - \lambda^2 T(t) = 0 \quad \& \quad \frac{d^2X}{dx^2} - \lambda^2 X = 0$$

Case-I: For $\lambda^2 = 0$,

$$X = Ax + B$$

$$u(0,t) = 0 = X(0)\,T(t)$$

$$u(l,t) = 0 = X(l)\,T(t)$$

For non-trivial, $X(0) = 0$

$X(l) = 0$

$\therefore X(x) = 0 \longrightarrow$ Not Possible

Case-II: $\lambda^2 > 0$

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0$$

$$\Downarrow$$

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

$\hookrightarrow$ B.C.: $X(0) = A + B = 0$

$X(l) = Ae^{\lambda l} + Be^{-\lambda l} = 0$

Exp. is never 0

$\therefore A = B = 0$

$\Rightarrow$ Not Possible

Case - III : $\lambda^2 < 0 \longrightarrow -\mu^2 > 0$

$$\frac{d^2 X}{dx^2} + \mu^2 X = 0$$

$$\Downarrow$$

$$X(x) = A \cos \mu x + B \sin \mu x$$

$$X(0) = A = 0$$

$$X(l) = A \cos \mu l + B \sin \mu l = 0$$

$\therefore \sin \mu l = 0$ ($B \neq 0$ for a non-trivial sol.)

$$\mu l = n\pi$$

$$\mu = \frac{n\pi}{l} \implies \mu^2 = \frac{n^2 \pi^2}{l^2}$$

$$\therefore X(x) = B \sin \left(\frac{n \pi x}{l} \right)$$

$$\frac{dT}{dt} + \mu^2 T = 0$$

$$\Downarrow$$

$$T = A e^{-\mu^2 T}$$

$$= A e^{-\frac{n^2\pi^2}{l^2} t}$$

$$u(x,t) = B \sin\left(\frac{n\pi x}{l}\right) \cdot A e^{-\left(\frac{n^2\pi^2}{l^2} t\right)}$$

$$= C \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2}{l^2} t\right)}$$

$$u_n(x,t) = C_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2}{l^2} t\right)}$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum C_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2 t}{l^2}\right)}$$

$$u(x,t) = \sum f(x)\, e^{-\left(\frac{n^2\pi^2 t}{l^2}\right)}$$

$$C_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$u(x,0) = \sin\left(\frac{m\pi x}{l}\right) f(x) =$$

$$\left(c_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{2\pi x}{l}\right) + \dots\right) \sin \frac{m\pi x}{l}$$

$$\int_0^l f(x) \sin\left(\frac{m\pi x}{l}\right) = C_n \frac{l}{2}$$

What if ??

$u(0,t) = 2$ $\qquad$ $u(l,t) = 5$

$V(x,t) = u(x,t) + Ax + B$

$V(x,t) = 0$ At $x = 0$ & $x = l$

$\therefore$ $B = -2$ & $A = -3/l$

And solve for $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial V}{\partial t}$

Eg $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0$ $\qquad$ $u_t(x,0) = 0$

(B.C.)

$u(l,t) = 0$ $\qquad$ $u(x,0) = f(x)$

Use the method of V.S.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0$$

$$0 \leq x \leq 1$$

$u(0,t) = 2$, $u(1,t) = 3$, $u(x,0) = x(1-x)$

Define,

$V(x,t) = u(x,t) + Ax + B$

$V(0,t) = 0$

$\therefore \; u(0,t) + B = 0 \implies B = -2$

$V(1,t) = 0$

$\therefore \; u(1,t) + A + B = 0 \implies A = -1$

$V(x,t) = u(x,t) - (x+2)$

$u(x,t) = V(x,t) + x + 2$

$$\frac{\partial u}{\partial t} \to \frac{\partial v}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} \to \frac{\partial^2 v}{\partial x^2}$$

$$\therefore \quad c^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

$$V(x,0) = u(x,0) - (x+2)$$

$$= -(x^2+2)$$

eg Find the sol. of 1-D wave eq.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{———} \quad (1)$$

$u(0,t) = 0$

$u(l,t) = 0$

$u(x,0) = f(x)$

$u_t(x,0) = 0$

$u(x,t) = X(x)\,T(t)$ [By separating variables]

put in (1), $u(x,t)$ is sol. of eq. (1)

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= X(x)\frac{d^2T}{dt^2} \\ \frac{\partial^2 u}{\partial x^2} &= T(t)\frac{d^2X}{dx^2}\end{aligned}\right\} \quad (2)$$

(2) in (1) we get,

$$\frac{1}{c^2}\cdot\frac{1}{T(t)}\cdot\frac{d^2T}{dt^2} = \frac{1}{X(x)}\cdot\frac{d^2X}{dx^2} = \lambda^2$$

(say)

$$\frac{d^2X}{dx^2} - \lambda^2 X(x) = 0$$

&

$$\frac{d^2T}{dt^2} - c^2\lambda^2 T(t) = 0$$

Case 1 : $\lambda^2 = 0$

$$\frac{d^2X}{dx^2} = 0 \implies X(x) = Ax + B$$

B.C. :- $u(0,t) = X(0)T(t) = 0$

$$\implies X(0) = 0$$

$u(l,t) = X(l)T(t) = 0$

$$\implies X(l) = 0$$

$T(t) \neq 0$ [For Non-Trivial Sol.]

We get, $X(x) = 0$

We get, $u(x,t) \longrightarrow$ Trivial Sol.

Case 2 : $\lambda^2 > 0$

$$\frac{d^2X}{dx^2} - \lambda^2 X(x) = 0 \quad \& \quad \frac{d^2T}{dt^2} - \lambda^2 T(t) = 0$$

$$\therefore \quad X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

$$X(0) = A + B = 0 \longrightarrow A = -B$$

$$X(l) = Ae^{\lambda l} + Be^{-\lambda l} = 0$$

$$\therefore \quad A = B = 0$$

$$u(x,t) = 0 \qquad \text{(which is Trivial Sol.)}$$

<u>Case 3</u>: $\lambda^2 < 0 \implies \mu^2 > 0$

$(\lambda^2 = -\mu^2)$

$$\frac{d^2X}{dx^2} + \mu^2 X(x) = 0$$

$$X(x) = A\cos\mu x + B\sin\mu x$$

$$X(0) = 0 \implies A = 0$$

$$X(l) = 0 \implies \sin\mu l = 0 \qquad [B \neq 0]$$

$$\mu l = n\pi \implies \mu = \frac{n\pi}{l}$$

$$\therefore X(x) = B\sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{d^2T}{dt^2} + \mu^2 c^2 T(t) = 0$$

$$T(t) = C\cos(\mu ct) + D\sin(\mu ct)$$

$$\Rightarrow T'(t) = \mu c\left[-C\sin(\mu ct) + D\cos(\mu ct)\right]$$

$$\Rightarrow T'(t) = 0 \Rightarrow D = 0$$

$$T(t) = C\cos(\mu ct)$$

$$u(x,t) = B\sin\left(\frac{n\pi x}{l}\right) c\cos\left(\frac{n\pi ct}{l}\right)$$

$$= A\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi c t}{l}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n u_n$$

$$u(x,t) = \sum A \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi c t}{l}\right)$$

$$\left[= A_1 \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c t}{l}\right) + \text{-------}\right] \times$$

$$\sin \frac{\pi x}{l}$$

$$u(x,0) = \int_0^l \left[A_1 \sin\left(\frac{\pi x}{l}\right) + A_2 \sin\left(\frac{2\pi x}{l}\right) \cdots\right]$$

$$\sin \frac{\pi x}{l} \qquad \times \sin \frac{\pi x}{l}$$

$$\int_0^l f(x) \sin\frac{\pi x}{l} = A_1 \int_0^l \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi x}{l}\right)$$

$$A_1 = \frac{\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)}{\int_0^l \sin\frac{\pi x}{l}\sin\frac{\pi x}{l}} \longrightarrow (l/2)$$

Also, $f(x) = u(x,0) = \sum A_n \sin\frac{n\pi x}{l}$ —— (B)

$$\int f(x)\sin\left(\frac{m\pi x}{l}\right)dx = A_n \int_0^l \sin\left(\frac{m\pi x}{l}\right)\cdot\sin\left(\frac{n\pi x}{l}\right)$$

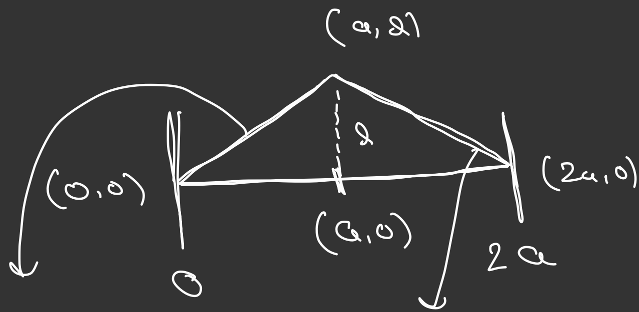
$$= A_n\,(l/2)$$

$$A_n = \frac{\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)}{(l/2)}$$

Eg. Find the eq. of the vibrating string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

If the string is plucked at middle point by giving an initial displacement 'l' from the mean position.



$u(0,t) = 0$

$u(2a,t) = 0$

$u_t(x,0) = 0$

$u(x,0)$

$y = \frac{l}{a}x$ $\qquad$ $y = l - \frac{l}{a}(x-a)$

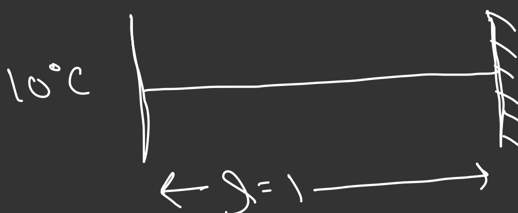
$$= \frac{l}{a}x \qquad 0 \le x \le a$$

$$= l - \frac{l}{a}(x-a) \qquad a \le x \le 2a$$

Eg Find the temp. distributions $u(x,t)$ in a uniform bar of unit length whose one end is kept at temp. $10°C$ & other end is insulated. Further given that

$$u(x,0) = 1-x, \quad 0<x<1$$

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$$



$$\left.\begin{aligned} u(0,t) &= 10° \\ u_x(1,t) &= 0 \end{aligned}\right] \Longrightarrow V(x,t) = u(x,t) + Ax + B$$

$$u(x,0) = 1-x, \quad 0<x<1$$

eg $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(x,0) = 0$ $\qquad$ $u(b,y) = 0$

$u(x,b) = 0$ $\qquad$ $u(a,y) = f(y)$

$$u(x,y) = X(x)\,Y(y)$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \quad (\text{say})$$

$\lambda^2 > 0, \quad \lambda = 0, \quad \lambda^2 < 0$

① $\lambda^2 > 0, \quad \dfrac{d^2X}{dx^2} - \lambda^2 X = 0$

$$\Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

$$\frac{d^2Y}{dy^2} + \lambda^2 y = 0 \Rightarrow Y(y) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$$

$u(x,0) = X(x)Y(0) = 0 \implies Y(0) = 0$

$u(x,b) = X(x)Y(b) = 0 \implies Y(b) = 0$

$y(0) = 0 = C_1 \implies C_1 = 0$

$y(b) = 0 \implies \sin \lambda b = 0 \quad (C_2 \neq 0)$

$$\lambda = \frac{n\pi}{b}$$

(For other 2 cases, we have trivial sol.)

Dirichlet Problem for a Circle:

eg Find the sol. of 2-D polar form of Laplace eq:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad r < a$$

$u(a,\theta) = f(\theta)$

$u(r,\theta) = R(r)\,H(\theta)$

From eq ① we get,

$$H(\theta)\frac{d^2R}{dr^2} + \frac{H(\theta)}{r}\frac{dR}{dr} + \frac{R(r)}{r^2}\frac{d^2H}{d\theta^2} = 0$$

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{Hr^2}\frac{d^2H}{d\theta^2} = 0$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + \frac{1}{H}\frac{d^2H}{d\theta^2} = 0$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} = -\frac{1}{H}\frac{d^2H}{d\theta^2} = \lambda^2$$

$$\Rightarrow r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 R = 0$$

$$\& \quad \frac{d^2H}{d\theta^2} + \lambda^2 H = 0$$

$\lambda^2 > 0, \quad \lambda^2 < 0, \quad \lambda^2 = 0$

For, $\log r = z \implies r = e^z$

(1) $\lambda^2 = 0$

$$\frac{d^2\Theta}{d\theta^2} = 0 \implies \Theta = A\theta + B$$

$[f(\theta + 2\pi) = f(\pi]$ Not Periodic!

(2) $\lambda^2 > 0$

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - \lambda^2 R = 0$$

$$[D^2 - \lambda^2] R = 0$$

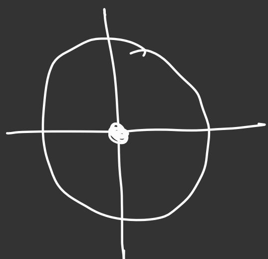
$$R = C_1 e^{\lambda z} + C_2 e^{-\lambda z}$$

$$= C_1 r^\lambda + C_2 \left(\frac{1}{r}\right)^\lambda$$

$$H(\theta) = C_3 \cos(\lambda\theta) + C_4 \sin(\lambda\theta)$$

$$R(r)\,H(\theta) = \left(C_1 r^{\lambda} + \cancel{C_2\left(\frac{1}{r}\right)^{\lambda}}\right)$$

$$\left(C_3 \cos\lambda\theta + C_4 \sin\lambda\theta\right)$$



(At $r = 0$, it should be finite)

$$u(r,\theta) = r^{\lambda}\left(A\cos\lambda\theta + B\sin\lambda\theta\right)$$

$$u(r,\theta) = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{\lambda}\left[A_n \cos\lambda\theta + B_n \sin\lambda\theta\right] + a_0/2$$

$$A_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos(\lambda\theta)$$

$$B_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin(\lambda\theta)$$

- Tools:-

$$x^2\frac{d^2y}{dx} + x\frac{dy}{dx} + y = 0$$

$\hookrightarrow a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0$

$\downarrow$

$\log x = z$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x\frac{dy}{dx} = Dy$$

$\therefore \left[D(D-1) + D + 1\right]y = 0 \Rightarrow \left[D^2+1\right]y = 0$

$\Rightarrow \quad y = C_1\cos(z) + C_2(\cos z)$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$\frac{d^2y}{dx^2} + \left(\frac{-2x}{1-x^2}\right)\frac{dy}{dx} + \left(\frac{n(n+1)}{1-x^2}\right)y = 0$$

$P(x)$ & $Q(x)$??

→ $P(x)$ & $Q(x)$ are defined then Sol.

→ $P(x)$ & $Q(x)$ not defined then Sol.

We Consider a 2^{nd} order diffe. eq.,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

If $P(x)$ & $Q(x)$ are well defined for some points, these points are called Ordinary point.

If not defined, then these points are called Singular point. (Either $P(x)$ or $Q(x)$ or both)

Eg Find Singular & Ordinary pnt of given diff. eq. :

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$\Rightarrow$

$$\frac{d^2y}{dx^2} + \left(\frac{-2x}{1-x^2}\right)\frac{dy}{dx} + \left(\frac{n(n+1)}{1-x^2}\right)y = 0$$

Ordinary point : $\mathbb{R} - \{-1, 1\}$

Sing point : $x = \pm 1$

Legendre diffg. eq. :

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

OR $$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + n(n+1)y = 0$$

Ordinary pnt. → Power Series method

Singular pnt. → Frobenius method

Power Series :

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$$

Or $= \sum a_n x^n$ about $x = 0$

$$= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

Or $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ power series

about $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} (x-x_0)^n$$

$$a_n = \frac{f^n(x_0)}{n!}$$

~~eg~~ $y' = y$ about $x = 0$

Let, $y = a_0 + a_1 x + a_2 x^2 + \cdots$

is a sol. of ①

$y = \sum a_n x^n$ is a sol. of diff. eq.

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$\Rightarrow$ $a_0 = a_1$

$\Rightarrow$ $a_1 = 2a_2$,

$$a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$a_2 = 3a_3$

$$a_3 = \frac{a_0}{6}$$

$$a_4 = \frac{a_0}{24}$$

$$y = a_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right]$$

$$y = a_0 e^x$$

Eg $y'' + y = 0$ about $x = 0$

Eg Find Sol. about $x = 0$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$\Rightarrow$ $x = 0$ is an ordinary point of given diff. eq.

let, $y = \sum_{0}^{\infty} a_n x^n$ is a Sol. of diff. eq.

$$y' = \sum_{1}^{\infty} n a_n x^{n-1} = \sum_{0}^{\infty} a_{n+1}(n+1)x^n$$

$$y'' = \sum_{2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{0}^{\infty} a_{n+2}(n+1)(n+2)x^n$$

$$x^2 y'' = \sum a_{n+2} (n+2)(n+1) x^{n+2}$$

$$= \sum a_n \, n(n-1) x^n$$

$$x y' = \sum a_n \, n \, x^n$$

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) = 0$$

If $P(x)$ and $Q(x)$ are well defined (Analytic), then we get sol. by power series method.

$$y = f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad x_0 \text{ is}$$

an ordinary point.

Eg Find Sol. about $x=0$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad -(1)$$

Let, $y = \sum a_n x^n$ is Sol. of eq (1)

$$(1-x^2)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$+ p(p+1)\sum a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^n$$

$$- \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} p(p+1)a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} p(p+1)a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2a_n n + p(p+1)a_n\right] x^n = 0$$

$$\Rightarrow a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2a_n + p(p+1)a_n = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{n(n-1) + 2n - p(p+1)}{(n+2)(n+1)} a_n}$$

$$\Rightarrow y = (\_\_\_\_)a_0 + (\_\_\_\_)a_1$$

eg $$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$$

Find Sol. about $x=0$

Sol $x=0$ is an ordinary pnt. then eq (1) has power series Sol.

let, $y = \sum_{n=0}^{\infty} a_n x^n$ is a Sol. of eq (1)

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum a_n n x^{n-1}$$

$$+ x^2 \sum a_n x^n + \sum 2a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n +$$

$$\sum_{n=0}^{\infty} a_n x^{n+2} + 2\sum_{n=0}^{\infty} 2a_n x^n = 0$$

$\downarrow$ Conversion

$$\Rightarrow \sum_{n=0}^{\infty} a_n (n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n +$$

$$\sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow \left[2a_2 + 6a_3 x\right] + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

$$+ a_1 x + \sum_{n=2}^{\infty} a_n n x^n + \sum_{n=2} a_{n-2} x^n +$$

$$(2a_0 + 2a_1 x) + \sum_{n=2}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow (2a_2 + 2a_0) + (6a_3 + 3a_1)x +$$

$$\sum_{n=2}^{\infty} \left[a_{n+2}(n+2)(n+1) + n a_n + a_{n-2} + 2a_n\right] x^n = 0$$

$$\Rightarrow 2a_2 + 2a_0 = 0 \quad \& \quad 6a_3 + 3a_1 = 0$$

$$\& \quad a_{n+2}(n+2)(n+1) + n a_n + a_{n-2} + 2a_n = 0$$

To Solve Power Series :-

1. Let, $y = \Sigma$

2. Put it in eq. ⟶

   Convert all into x^n.

[Note: leave indices, i.e, n as it is]

3. Collect the extra term and make all series to a single starting n

4. Using extra term & series, calculate y.

$$a_{n+2} = \frac{(n+2)a_n + a_{n-2}}{(n+2)(n+1)} \qquad n \geq 2$$

$$\therefore \; y = a_0 \left(\_\_\_\_ \right) + a_1 \left(\_\_\_\_ \right)$$

Solution of diff. eq. at singular point. :

Eg $$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y = 0$$

$x = 0$ is a singular point.

Regular — Irregular

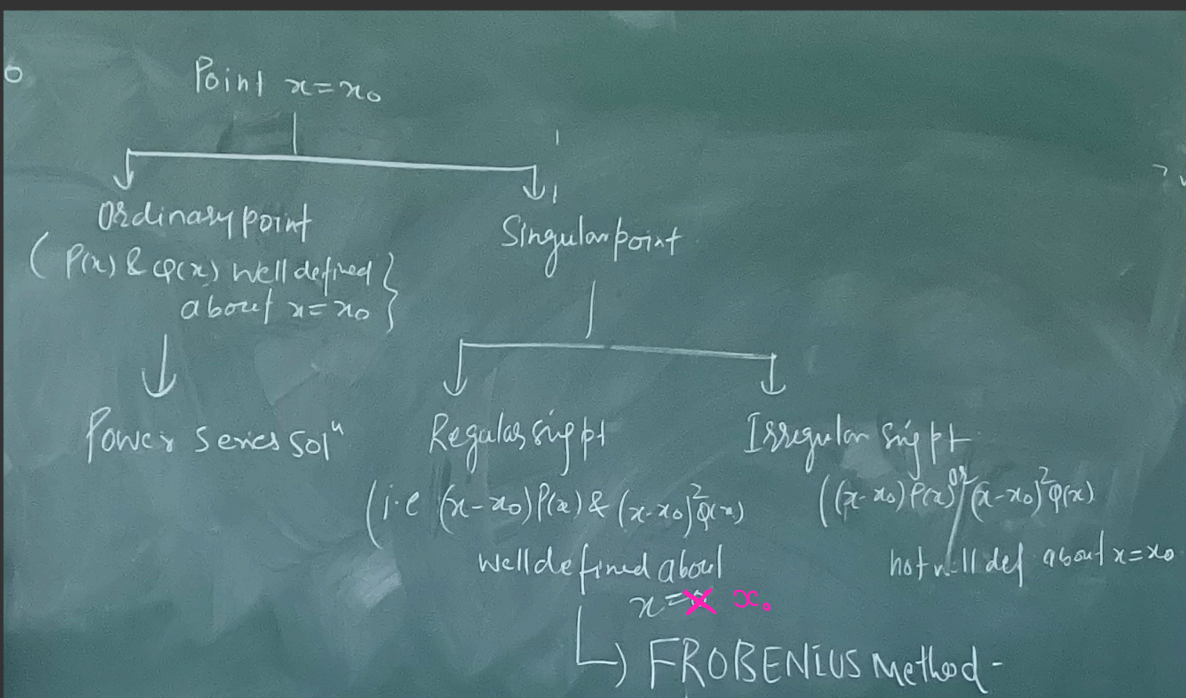
Regular : $x = x_0$ is a sing. pnt.

$$\lim_{x\to x_0} (x-x_0)P(x) \ \& \ \lim_{x\to x_0} (x-x_0)^2 Q(x) \Rightarrow \text{Finite}$$

For this Consider,

$$y = x^r \sum a_n x^n = \sum a_n x^{n+r}, \quad a_0 \neq 0$$

$$y = \sum a_n (x-x_0)^{n+r}, \quad a_0 \neq 0$$



To Solve Forbenius :-

① Establish the Series

[Don't Care abt indicies]

② Combine all x^{n+r} terms & using

$n=0$ in that, get indicial eq.

Case 1: $r_2 - r_1 \neq N$

$\Rightarrow$ Convert all series to x^{n+r}

We get, $y = a_0 [f(r)]$

Using r_1 & r_2, $y = Au + Bv$

For terms like $a_1, a_3, \ldots$, get to the common terms of x.

~~Eg~~ Find Sol. of differential eq. about $x=0$,

$$2x^2y'' + x(2x+1)y' - y = 0$$

$$y'' + \frac{x(2x+1)}{2x^2}y' - \frac{y}{2x^2} = 0$$

$\lim_{x\to 0} xP(x)$ & $\lim_{x\to 0} x^2Q(x)$ = finite

$\therefore$ let, $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$

is a solution of diff. equation.

$$y' = \sum a_n (n+r) x^{n+r-1}$$

$$y'' = \sum a_n (n+r)(n+r-1) x^{n+r-2}$$

$$\Rightarrow$$

$$2x^2 \sum a_n (n+r)(n+r-1) x^{n+r-2} +$$

$$2x^2 \sum a_n (n+r) x^{n+r-1} +$$

$$x \sum a_n (n+r) x^{n+r-1} - \sum a_n x^{n+r} = 0$$

$$\Rightarrow \sum 2a_n (n+r)(n+r-1) x^{n+r} +$$

$$\sum 2a_n (n+r) x^{n+r+1} + \sum a_n (n+r) x^{n+r}$$

$$- \sum a_n x^{n+r} = 0$$

$$\Rightarrow 2a_0 r(r-1) + a_0 r - a_0 = 0$$

$$\Rightarrow \left[2r(r-1) + (r-1)\right] a_0 = 0$$

$$\Rightarrow a_0 \neq 0 \text{ so, } 2r^2 - r - 1 = 0$$

(Indicial eq.)

$$r = 1, -\frac{1}{2}$$

r_1 & r_2 (i) If $r_1 - r_2 \neq N$, N is an integer

(ii) If $r_1 - r_2 = N$

(iii) If $r_1 - r_2 = 0$

$$\sum \left[2a_n(n+r)(n+r-1) + a_n(n+r) - a_n\right] x^{n+r}$$

$$+ \sum \left[2a_{n-1}(n+r-1)\right] x^{n+r} = 0$$

$$\Rightarrow a_n\left[2(n+r)(n+r-1)+(n+r)-1\right] + 2a_{n-1}(n+r-1)=0$$

$$\Rightarrow a_n = \frac{-2(n+r-1)}{\underbrace{\left[2(n+r)(n+r-1)+(n+r)-1\right]}_{\text{Reducing to (by } (n+r)=t)}}$$

$$a_n = \frac{-2(n+r-1)}{\left[2(n+r)(n+r-1)+(n+r)-1\right]}\, a_{n-1}$$

$$= \frac{-2(n+r-1)}{(n+r-1)\left(n+r+\frac{1}{2}\right)}\, a_{n-1}, \quad n\geq 1$$

$$a_1 = \frac{-2r}{r(r+\frac{1}{2})} a_0 = \frac{-2}{(r+\frac{1}{2})} a_0$$

$$a_2 = f(a_0, r) \; \ldots\ldots \; a_3 = f(a_0, r)$$

$$\therefore \; y = \underline{a_0} \left[\frac{f(r)}{\qquad} \right]$$

For $\underline{r_1}$ & $\underline{r_2}$ individually.

$$\therefore \; \left[y = A u + B v \right] \; ?$$

Frobenius Method :-

$$y = \sum a_n x^{n+r}, \quad a_0 \neq 0$$

be a sol. of diffr. eq. about regular singular point $x = 0$.

- Indicial eq : lowest power coeff is zero

↓

Real roots

① Roots are dist. & diff. by not an integer

② Roots are equal

③ Roots are distinct & diff by an integer.

(Bessel Function)

eg

$$x^2 y'' + x y' + (x^2 - m^2) y = 0$$

Find sol. about $x = 0$. (regular point.)

According to Frobenius Method,

$$y = \sum a_n x^{n+r}, \quad a_0 \neq 0$$

$$y' = \sum a_n (n+r) x^{n+r-1}$$

$$y'' = \sum a_n (n+r)(n+r-1) x^{n+r-2}$$

$$x^2 \sum a_n (n+r)(n+r-1) x^{n+r-2} +$$

$$+ x \sum a_n (n+r) x^{n+r-1} +$$

$$x^2 \sum a_n x^{n+r} - \sum m^2 a_n x^{n+r} = 0$$

$$\Rightarrow \sum a_n \left[(n+r)(n+r-1) + (n+r) - m^2 \right]$$

$$+ \sum a_n x^{n+r+2} = 0$$

For Indicial eq., Take $n = 0$

$\therefore \underset{(\neq 0)}{\underline{a_0}} \left[r(r-1) + r - m^2 \right] = 0$

$r = \pm m$ (Roots of indicial eq.)

Case: If diff. of roots are not an integer (i.e, $r_1 - r_2 \neq N, N \in \mathbb{Z}$)

$$\left[(n+r)(n+r-1) + (n+r) - m^2 \right] a_n + a_{n-2} = 0$$

Take $(n+r) = p$

$$\therefore \ p(p-1) + p - m^2$$

$$\therefore \ (n+r+m)(n+r-m) a_n + a_{n-2} = 0$$

$$\Rightarrow \quad a_n = \frac{-a_{n-2}}{(n+r+m)(n+r-m)} \qquad n \geq 2$$

Consider,

$$a_n\left[(n+r)(n+r-1)+(n+r)-m^2\right]$$

$$a_1\left\{\underbrace{(r+1)(r)+(r+1)-m^2}_{\neq 0}\right] = 0 \quad \Rightarrow a_1 = 0$$

$$a_2 = \frac{-a_0}{(r+m+2)(r-m+2)}$$

$a_3 = 0$ - - - - —

For $r = m$,

$$a_2 = \frac{-a_0}{2(2m+2)} \; ; \; a_4 = \frac{-a_2}{(2m+4)}$$

$$= \frac{a_0}{8(2m+2)(2m+4)}$$

$$a_1 = a_3 = a_5 = \dots = 0$$

$$y = x^m \left[a_0 - \frac{a_0}{4(m+1)} x^2 + \frac{1}{32(m+1)(m+2)} x^4 + \dots \right]$$

$$= a_0 x^m \left[1 - \frac{1}{4(m+1)} x^2 + \frac{1}{32(m+1)(m+2)} x^4 + \dots \right]$$

Let, $a_0 = \dfrac{1}{2^m \Gamma(m+1)}$

$$y = \frac{x^m}{2^m \Gamma(m+1)} \left[1 - \frac{1}{4(m+1)} x^2 + \frac{1}{32(m+1)(m+2)} x^4 + \cdots \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^r \left(x/2\right)^{m+2n}}{n! \, \Gamma(m+n+1)} = J_m(x)$$

Second Sol, $r = -m$,

$$y_2 = J_{-m}(x)$$

Final Sol. : $A\,J_m(x) + B\,J_{-m}(x)$

is a sol. of eq (i) when roots of Indicial eq. $2m$ is not an integer.

Case - II : If roots of Indicial eq. are equal, i.e., $r_1 = r_2 = r$

- We need two sol. y_1 at $r = r_1 = Au$

$$y_2 = \left.\frac{\partial y}{\partial r}\right|_{r=r_1} = Bv$$

$$y = Au + Bv$$

Eg $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$ about $x = 0$

roots of Indicial eq. : 0 , 0

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)y = 0$$

$y = \sum a_n x^{n+r}$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0, \text{ about } x = 0$$

$$\sum a_n (n+r)(n+r-1)x^{n+r-1} + \sum a_n (n+r) x^{n+r-1} + \sum a_n x^{n+r+1} = 0$$

$$\sum a_n \left[(n+r)(n+r-1) + (n+r)\right] x^{n+r-1} + \sum a_n x^{n+r+1}$$

$$r^2 - \cancel{r} + \cancel{r} = 0$$

$$r^2 = 0 \Rightarrow r = 0, 0$$

$$\left[(n+r)(n+r-1) + (n+r)\right] a_n + a_{n-2} = 0 \quad \text{Recursion relation.}$$

$$p(p-1) + p$$

$$(p^2 - \cancel{p} + \cancel{p})\, a_n + a_{n-2}$$

$$\Rightarrow \quad a_n = -\frac{a_{n-2}}{(n+r)^2}$$

$$y = a_0 x^r \left[1 - \frac{1}{(r+2)^2} x^2 + \frac{1}{(r+2)^2 (r+4)^2} x^4 \right.$$

$$\left. \cdots\cdots \right]$$

$r = 0,$

$$y = a_0 \left[1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 \cdot 4^2} x^4 + \cdots \right]$$

$$= A u$$

$$\left. \frac{\partial y}{\partial r} \right|_{r=0} = a_0 x^r \log x \left[\text{——} \right]$$

$$+ a_0 x^r \left\{ \frac{2x^2}{(r+2)^3} - \left[\text{——} \right] \right]$$

$$= B v$$

③ Roots of Indicial eq. differ by an integer.

(a) Roots of indicial eq. differ by an integer, making a coeff of y ∞.

Roots: r_1, r_2 $(r_2 > r_1)$

Coeff. of y becomes ∞, when $r = r_2$

We modify the form of y by replacing a_0 by $d_0(r - r_1)$

then the 2 sol. can be obtained by putting $r = r_1$ in the modified form y & $\frac{\partial y}{\partial r}$.

The sol. obtained by $r = r_1$ is rejected bcz it only gives a numerical multiple of obtained sol at $r = r_2$

~~Q~~ $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y = 0$

$$y = \sum a_n x^{n+r}, \quad a_0 \neq 0$$

$$\sum a_n (n+r)(n+r-1) x^{n+r} +$$

$$\sum a_n (n+r) x^{n+r} + \sum a_n x^{n+r+2}$$

$$- \sum a_n x^{n+r} = 0$$

$$\sum a_n \left[(n+r)(n+r-1) + (n+r) - 1 \right]$$

$$+ \sum a_n x^{n+r+2} = 0$$

At $r = 0$

Indicial eq. : $r = \pm 1$

Now, $a_0 \neq 0$,

$$a_n\left[(n+r)(n+r-1) + (n+r) - 1\right] + a_{n-2} = 0$$

$$\Rightarrow \quad a_n = \frac{-a_{n-2}}{(n+r+1)(n+r-1)}$$

$$a_1\left[(r+1)(r) + (r+1) - 1\right] = 0 \Rightarrow a_1 = 0$$

$$a_2 = \frac{-a_0}{(r+3)(r+1)}$$

$$a_4 = a_0 / (r+1)(r+3)^2(r+2)$$

$$a_0 = d_0 (r+1)$$

$$y = a_0 x^r \left[1 - \frac{1}{(r+3)(r+1)} x^2 + \frac{1}{(r+1)(r+3)^2(r+5)} x^4 + \cdots \right]$$

$$y = d_0 x^r \left[(r+1) - \frac{1}{(r+3)} x^2 + \frac{1}{(r+3)^2(r+5)} x^4 - \cdots \right]$$

put $r = -1$,

(b) The roots of indicial eq. differ by an integer making coeff indeterminate.

If this type of possibility & $r_1 > r_2$.
If one of the coeff becomes indeterminant the complete sol. is given by putting $r = r_2$ in y which contains 2 arbitrary coeff.

eg $x^2 \frac{d^2y}{dx^2} + (x+x^2)\frac{dy}{dx} + (x-9)y = 0$

$y = \sum a_n (n+r)(n+r-1) x^{n+r} +$

$\sum a_n (n+r) x^{n+r} + \sum a_n (n+r) x^{n+r+1}$

$+ \sum a_n x^{n+r+1} - \sum 9 a_n x^{n+r} = 0$

$$\Rightarrow \sum \left[a_n(n+r)+1\right] x^{n+r+1} +$$

$$\sum a_n \left[(n+r)(n+r-1)+(n+r)-9\right] x^{n+r} = 0$$

Indicial eq : $r^2 - 9 = 0$, $r = \pm 3$

$$a_{n-1}\left[n-1+r+1\right] + a_n (r+n+3)(n+r-3) = 0$$

$$a_n = \frac{-a_{n-1}(n+r)}{(n+r+3)(n+r-3)}$$

$$a_1 = \frac{-a_0(r+1)}{(r+4)(r-2)} \qquad a_2 = \frac{-a_1(r+2)}{(r+3)(r-1)}$$

$$a_6 = [\text{———}] \rightarrow \frac{0}{0} \text{ form}$$

$$a_7 \; \times \; \times$$

$$\therefore y = a_0 [\text{———}] + a_6 [-]$$

$$\qquad\qquad\qquad Au \qquad\qquad\qquad Bv$$

Special Function:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)\, y = 0$$

– Bessel

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m(m+1)\, y = 0$$

– Legendre

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)\, y = 0$$ – Legendre

Legendre Function:

For, $y \to \infty$, $y = \sum a_n x^{r-n}$

$x = \infty \to$ reg. singular pnt

$\downarrow$

$t = \frac{1}{x} \longrightarrow$ diff $t = 0$ — ord / sing

$$y_1 = a x^n \left[1 - \frac{n(n-1)}{2(2n-1)} x^{-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x^{-4} + \cdots \right] = A P_n$$

$$P_n = \boxed{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}} \left\{ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \cdots \right\}$$

$$y_2 = b x^{-n-1} \left[1 + \frac{(n+1)(n+2)}{2(2n+3)} x^{-2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 (2n+3)(2n+5)} x^{-4} + \cdots \right] = B Q_n$$

Generating Function for Legendre polynomial:

$$\left(1-2hx+h^2\right)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

$$|h|<1$$

$$\left(1-h(2x-h)\right)^{-1/2} =$$

$$1+\frac{1}{2}h(2x-h)+\frac{1}{2}\cdot\frac{3}{2}h^2(2x-h)^2$$

$$\cdots\quad\cdots$$

$$+\frac{1\cdot 3\cdot 5\cdots(2n-5)}{2\cdot 4\cdot 6\cdots(2n-4)}h^{n-2}(2x-h)^{n-2}$$

$$+\frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}h^{n-1}(2x-h)^{n-1}$$

$$+ \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} h^n (2x-h)^n$$

Coeff of h^n

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} h^n (2x-h)^n$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!} x^n$$

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)} h^{n-1} (2x-h)^{n-1} =$$

$$- \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!} \frac{n}{2} \frac{(n-1)}{(2n-1)} x^{n-2}$$

Coeff of $h^n \longrightarrow P_n(x)$

$$(1-2hx+h^2)^{-1/2} = \sum h_n P_n(x)$$

$$= P_0(x) + h P_1(x) + h^2 P_2(x) + \cdots$$

$P_n(1) = 1$ & $P_n(-1) = (-1)^n$

$$P_n(x) = (-1)^n P_n(x) \quad \text{——— (1)}$$

$$(1-h)^{-1} = \sum h_n^n P_n(1)$$

$$1 + h + h^2 + \cdots = P_0(1) + P_1(1) + h^2 P_2(1) + \cdots$$

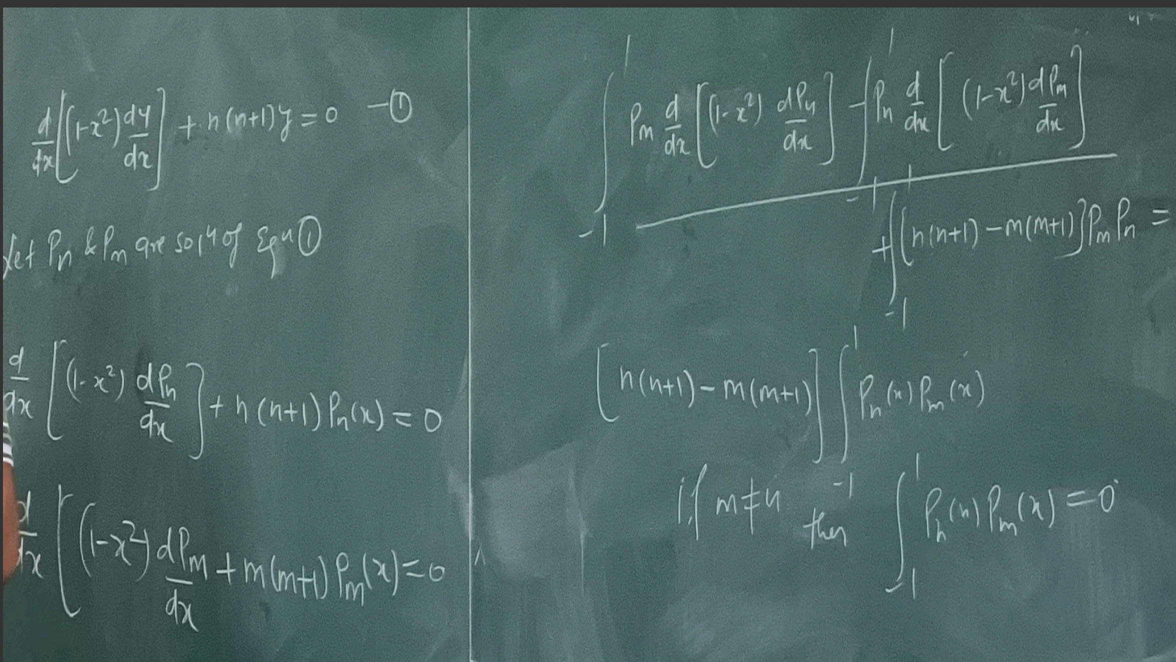
$$P_n(1) = 1$$

Orthogonal Prof. of Legendre Polynomial

Prove that :

$$\int_{-1}^{1} P_n(x)\, P_m(x) = 0 \quad \text{if } m \neq n$$

$$= \frac{2}{2n+1} \quad \text{if } m = n$$



Case 2 :- Using generating function,

$$\left[1-2xh+h^2\right]^{-1/2} = \sum h^n P_n(x)$$

$$\left[1-2xh+h^2\right]^{-1} = \left[\sum h^n P_n(x)\right]$$

$$\left[1-2xh+h^2\right]^{-1/2}$$

$$= \sum h^n P_n(x) \cdot \sum h^m P_m(x)$$

$$= \sum h^{2n} P_n^2(x) + \sum h^{m+n} P_n(x) P_m(x)$$

$$\int_{-1}^{1} \frac{1}{1-2xh+h^2}\, dx = \sum h^{2n} \int_{-1}^{1} P_n^2(x)\, dx + \sum h^{m+n} \cancel{\int_{-1}^{1} P_n(x) P_m(x)} \rightarrow 0$$

(By Orthogonality Property)

$$\int_{-1}^{1} \frac{1}{1-2xh+h^2}\, dx = \sum h^{2n} \int_{-1}^{1} P_n^2(x)\, dx$$

$$\downarrow$$

Coeff. of h^{2n}

$$= \frac{-1}{2h} \log\left(1-2xh+h^2\right) \Big|_{-1}^{1}$$

$$= \frac{1}{2h}\left(\log(1+h)^2 - \log(1-h)^2 \right)$$

$$= \frac{\log(1+h) - \log(1-h)}{h}$$

$$= \frac{\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots\cdots\right) + \left(h + \frac{h^2}{2} + \frac{h^3}{3} - \cdots\right)}{h}$$

$$= \frac{2h + \frac{2h^3}{3} + \cdots}{h}$$

$$= 2\left(1 + \frac{h^2}{3} + \cdots\right)$$

$$\downarrow$$

Coeff. of $2n$ = $1/2n+1$

$$\int_{-1}^{1} P_n^2(x)\,dx = 2/2n+1$$

Rodrigues Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{(5x^3 - 3x)}{2}$$

$P_n(x) \to$ odd when $n \to$ odd

$P_n(x) \to$ even when $n \to$ even

Proof: $y = (x^2-1)^n$

$$\frac{dy}{dx} = n(x^2-1)^{n-1} \cdot 2x$$

$$(x^2-1)\frac{dy}{dx} = n(x^2-1)^n 2x$$

$$= 2nyx$$

$$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 2ny + 2nx\frac{dy}{dx}$$

$$(x^2-1)y_2 + 2xy_1 = 2ny + 2nxy_1 \quad \rightarrow (3)$$

$$D^n(uv) = (D^n u)v + {}^nC_1 (D^{n-1}u)(Dv) + {}^nC_2 (D^{n-2}u)(D^2 v) + \dots D^n v$$

Diff eq (3) n times,

$$\left[y_{n+2} + n\,y_{n+1}(2x) + {}^nC_2\, y_n\, 2\right]$$

$$+ 2y_{n+1} + 2\,{}^nC_1\, y_n =$$

$$2n\,y_n + 2n\left[y_{n+1} + {}^nC_1\, y_n\right]$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+1}} -$$

$$n(n+1)\frac{d^n y}{dx^n} = 0$$

Let, $\dfrac{d^n y}{dx^n} = Z$

$$(x^2-1)\frac{d^2 Z}{dx^2} + 2x\frac{dZ}{dx} - n(n+1)Z = 0$$

Or

$$(1-x^2)\frac{d^2 Z}{dx^2} - 2x\frac{dZ}{dx} + n(n+1)Z = 0$$

Let, $Z = CP_n \Rightarrow \dfrac{d^n y}{dx^n} = CP_n$

$$\Rightarrow CP_n(1) = \left.\frac{d^n y}{dx^n}\right|_{x=1}$$

$$\Rightarrow C = \left.\frac{d^n y}{dx^n}\right|_{x=1}$$

$$y = (x^2-1)^n = (x-1)^n (x+1)^n$$

$$\frac{d^n y}{dx^n} = \frac{d^n}{dx^n}\left[(x-1)^n (x+1)^n\right]\Bigg|_{x=1}$$

$$= 2^n n!$$

$$2^n n!\, P_n(x) = Z = \frac{d^n y}{dx^n}$$

$$= \frac{d^n}{dx^n}(x^2-1)^n$$

$$\Rightarrow P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

Obs $P_n(x)$ is even funct. as n is even
$P_n(x)$ is odd funct. as n is odd.

Fourier Leg. Series (expansion)

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad \text{—} (1)$$

Multiply eq (1) with $P_n(x)$ & integrate in domain $[-1, 1]$

$$\Rightarrow \int_{-1}^{1} P_m(x) f(x)\, dx = \sum_{n=0}^{\infty} \int_{-1}^{1} a_n P_n(x) P_m(x)\, dx$$

By Orthogonality Property L.P. :-

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \int_{-1}^{1} a_n P_n \cdot P_n\, dx + 0$$

$$\Rightarrow \int_{-1}^{1} P_m(x) f(x)\, dx = \int_{-1}^{1} \cancel{a_0 P_0(x) P_m(x)}^{\,0} +$$

$$\int_{-1}^{1} \cancel{a_1 P_1(x) P_m(x)}^{\,0} + \dots \int_{-1}^{1} a_n P_n(x) P_n(x)$$

$(m = n)$

$$+ \int_{-1}^{1} \cancel{a_{n+1} P_{n+1}(x) P_m}^{\,0}$$

$$\Rightarrow a_n = \frac{(2n+1)}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

Eg Find the 1^{st} three terms in the expansion of the following in terms of Leg. polynomial

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$$

$$\Rightarrow f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$$

[By Fourier Leg. Series]

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

Where, $a_n = \left(n+\frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx$

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(x) P_0(x)\, dx = \frac{1}{2} \int_{-1}^{1} f(x)\, dx$$

$$= \frac{1}{4}$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} f(x) P_1(x)\, dx = \frac{3}{2} \int_{0}^{1} x^2\, dx$$

$$= 1/2$$

$$a_2 = \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)\,dx = \frac{5}{16}$$

$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x)$$

Recurrence Relation of leg. poly. :-

1. $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

2. $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

3. $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

4. $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$

5. $(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

6. $(1-x^2)P_n'(x) = (n+1)\left[xP_n(x) - P_{n+1}(x)\right]$

Proof:

(1) $(1-2xh+h^2)^{-1/2} = \sum h^n P_n(x)$

$\downarrow$

diff. w. r. t to h

$$-\frac{1}{2}(1-2xh+h^2)^{-1/2}(-2x+2h) = \sum n h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2) \sum n h^{n-1} P_n(x)$$

$$\Rightarrow (x-h) \sum h^n P_n(x) =$$

$$(1-2xh+h^2) \sum n h^{n-1} P_n(x)$$

Considering Coeff. of h,

$$\Rightarrow x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2xn P_n(x) + (n-1) P_{n-1}(x)$$

Rearranging we get ① relation,

② $(1-2xh+h^2)^{-1/2} = \sum h^n P_n(x)$

↓

diff. w.r.t to x,

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2}(-2h) = \sum h^n P_n'(x) \quad \text{— } ①$$

diff. w.r.t to h

$$-\frac{1}{2}(1-2xh+h^2)^{-1/2}(-2x+2h) = \sum n h^{n-1} P_n(x)$$

$$(x-h)\sum h^n P_n(x) =$$

$$(1-2xh+h^2)\sum n h^{n-1} P_n(x) \quad \text{— } ②$$

Using ① & ② & rearranging we get,

② recurrence relation.

③ [diff w.r.t to x & use ② relation]

④ [diff ① relation & use ② relation]

Q. $P_n(x)$ is even funct. as n is even

$P_n(x)$ is odd funct. as n is odd

$$P_n(x) = (-1)^n P_n(x)$$

$$\Downarrow$$

$$(1-2xh+h^2)^{-1/2} = \sum h^n P_n(x)$$

$$(1+2xh+h^2)^{-1/2} = \sum h^n P_n(-x)$$

$\downarrow h \to -h$

$$(1+2(-h)(x)+(-h)^2)^{-1/2} = \sum (-h)^n P_n(-x)$$

$$\sum h^n P_n(x)$$

Equating we get, $P_n(-x) = (-1)^n P_n(x)$

If n is even, $n = 2m$

$$P_{2m}(-x) = (-1)^{2m} P_{2m}(x)$$

$$\Rightarrow P_{2m}(-x) = P_{2m}(x)$$

Similarly, $P_{2m+1}(-x) = -P_{2m+1}(x)$

Eg $$\int_{-1}^{1} (x^2-1) P_{n+1}(x) P_n'(x)\, dx$$

Applying (5) recurrence relation,

$$-\int_{-1}^{1} n\left(P_{n-1}(x) - xP_n(x)\right) P_{n+1}(x)\, dx =$$

$$-n \int_{-1}^{1} P_{n-1}(x) P_{n+1}(x) + \int x P_n P_{n+1}(x)$$

0

$$= \frac{1}{2n+1} \int_{-1}^{1} \left(P_{n+1} \left((n+1) P_{n+1}(x) + n P_{n-1}(x) P(x) \right) \right] dx$$

$$= \frac{1}{2n+1} \cdot \frac{2(n+1)}{2n+2+1}$$

eg $$\int_{-1}^{1} x^2 P_{n+1}(x) P_{n-1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+2)(2n+3)}$$

eg $$P_n'(1) = \frac{1}{2} n(n+1)$$

$$P_n'(-1) = (-1)^{n-1} \frac{1}{2} n(n+1)$$

$$\left[\begin{array}{l} \text{By, } (1-x^2) \dfrac{d^2 P_n(x)}{dx^2} - 2x \dfrac{d P_n(x)}{dx} + n(n+1) P_n(x) = 0 \\[2em] \text{at } x=1, \quad P_n'(1) = \dfrac{1}{2} n(n+1) P_n(1) \end{array} \right]$$

eg

$$\int_{-1}^{1} P_n(x)\,dx = 0 \quad \text{When } n \neq 0$$

$$\& \quad \int_{-1}^{1} P_0(x)\,dx = 2$$

Bessel Function :-

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

$x=0$ is regular singular point.

→ Sol. by Forbenius method.

When $2n$ is not an integer.

$y = AJ_n(x) + BJ_{-n}(x)$ When n is not integer

When n is an integer,

$y = AJ_n(x) + BY_n(x)$

$$J_n(x) = \sum \frac{(-1)^r \, (x/2)^{n+2r}}{r! \, \Gamma(n+r+1)}$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \, (x/2)^{-n+2r}}{r! \, \Gamma(-n+r+1)}$$

$$= \sum_{r=n}^{\infty} \frac{(-1)^r \, (x/2)^{-n+2r}}{r! \, (r-n)!}$$

put $r - n = s$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{s+n} \, (x/2)^{-n+2s+2n}}{(s+n)! \; s!}$$

$$= (-1)^n \sum \frac{(-1)^s \, (x/2)^{n+2s}}{\Gamma(n+s+1) \; s!}$$

$= (-1)^n J_n(x)$

Recurrence Relation for Bessel funct. :-

1. $\dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

2. $\dfrac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Diff. 1,

$$n x^{n-1} J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x)$$

$$\Rightarrow J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x) \quad \text{—(3)}$$

Diff. 2,

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \quad \text{—(4)}$$

Add ③ & ④, $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ — ⑤

Subtract ③ & ④, $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ — ⑥

Prove that

(i) $\lim_{x \to 0} \frac{J_n(x)}{x^n} = \frac{1}{2^n \sqrt{n+1}}$

(ii) $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - - - - - -$

Generating Function of Bessel Function :-

$0 < |z| < \infty$

$$e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

$$e^{\frac{xz}{2}} \cdot e^{\frac{-x}{2z}} = \sum \frac{(xz/2)^n}{n!} \sum \frac{(-x/2z)^n}{n!}$$

$$= \sum (\quad) z^n$$

$$\left[\text{As, } e^x = \sum x^n / n! \right]$$

~~Eg~~ Prove that: $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\frac{1}{2}+2r}}{r! \, \Gamma(r+3/2)}$$

$$= \frac{\left(\frac{x}{2}\right)^{1/2}}{\Gamma(3/2)} - \frac{\left(\frac{x}{2}\right)^{5/2}}{\Gamma(5/2)} + \frac{\left(\frac{x}{2}\right)^{9/2}}{2! \, \Gamma(7/2)} - \cdots$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3/2}} \cdot \frac{1}{\sqrt{x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

Orthogonal Property of Bessel Funct. :-

<u>Prove</u> : $\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$

if $\alpha \neq \beta$

$$\int_0^1 x \left[J_n(\alpha x) \right]^2 dx = \frac{1}{2} \left[J_n'(\alpha) \right]^2 \text{ if}$$

$\alpha = \beta$

Where α & β are zeros of $J_n(x)$, i.e,

$J_n(\alpha) = J_n(\beta) = 0$

<u>Proof</u> : $J_n(x)$ is sol. of Bessel diffg. eq.,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

$J_n(\alpha x)$ is a sol. of :

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\alpha^2 x^2 - n^2)y = 0 \quad \text{— (1)}$$

$u = J_n(\alpha x)$ & $v = J_n(\beta x)$

$$\therefore \quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} \; (\beta^2 x^2 - n^2)y = 0 \quad \text{— (2)}$$

Multiply eq (1) $\times \frac{v}{x}$ & (2) $\times \frac{u}{x}$ &

Subtract,

$$v x \frac{d^2u}{dx^2} - u x \frac{d^2v}{dx} + v\frac{du}{dx} - u\frac{dv}{dx}$$

$$+ \left(\alpha^2x^2 - \cancel{n^2} - \beta^2x^2 + \cancel{n^2}\right)\frac{uv}{x} = 0$$

$$\Rightarrow \frac{d}{dx}\left[x\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\right] + (\alpha^2-\beta^2)xuv = 0$$

And integrate it from 0 to 1,

$$x\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\Big|_0^1 + \int_0^1 (\alpha^2-\beta^2)xuv = 0$$

$$\downarrow$$

$$v\frac{du}{dx} - u\frac{dv}{dx} + \underbrace{(\alpha^2-\beta^2)\int_0^1 xuv\,dx}$$ $= 0$

$$\left[\alpha J_n(\beta)J_n'(\alpha) - \beta J_n(\alpha)J_n'(\beta)\right] + (\quad)$$

$= 0$

$$(\alpha^2-\beta^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$$

if $\alpha \neq \beta$, $\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$

For, $\alpha = \beta$,

$\beta = \alpha + \varepsilon$ When $\varepsilon \to 0$

$$(\alpha^2-\beta^2)\int_0^1 x J_n(\alpha x) J_n(\beta x) =$$

$$-x\Big[\alpha J_n(\beta x) J_n'(\alpha x) - \beta J_n(\alpha x) J_n'(\beta x)\Big]_0^1$$

$$-x\Big[\alpha J_n\big((\alpha+\varepsilon)x\big) J_n'(\alpha x) -$$

$$(\alpha+\varepsilon) J_n(\alpha x)\, \cancel{J_n'}\big((\alpha+\varepsilon)x\big)\Big]_0^1$$

$$= -\left[\alpha J_n(\alpha+\varepsilon) J_n'(\alpha)\right]$$

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n((\alpha+\varepsilon)x)\, dx =$$

$$\frac{1}{\alpha^2-(\alpha+\varepsilon)^2}\left[\alpha J_n(\alpha) + \varepsilon \alpha J_n'(\alpha) + \varepsilon^2(\quad) + \cdots\right]$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$-2\alpha\varepsilon + \varepsilon^2$$

(Taylor Series)

$$\Rightarrow \int_0^1 x\left(J_n(\alpha x)\right)^2 = \frac{1}{2}\left(J_n'(\alpha)\right)^2$$

Bessel Fourier Series :-

$$f(x) = \sum_{i=1}^{n} a_n J_n(\alpha_i x)$$

$$\int_0^1 x \cdot f(x) J_n(\alpha_j x)\, dx = \sum a_n \int_0^1 x J_n(\alpha_i x) J_n(\alpha_j x)\, dx$$

$$= \frac{a_n}{2} \left(J_n'(\alpha_i)\right)^2 \quad [\text{For } \alpha_i = \alpha_j]$$

$$\Rightarrow a_n = \frac{2}{(J_n'(\alpha_i))^2} \int_0^1 x f(x) J_n(\alpha_j x)\, dx$$

Fourier Bessel Coefficient

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r! \, \Gamma(n+r+1)}$$

$$J_0(0) = ?? = 1$$

$$J_n(0) = 0 \quad \text{if } n \neq 0$$

$$= 1 \quad \text{if } n = 0$$

Proof of recurrence relation :-

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r! \, \Gamma(n+r+1)}$$

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) \, x^{n+2r-1}}{2^{n+2r} \, \Gamma(n+r+1) \, r!}$$

$$= n \sum_{r=0}^{\infty} \frac{(-1)^r \, x^{n+2r-1}}{2^{n+2r} \, \Gamma(n+r+1)} \cdot \frac{1}{r!} \quad +$$

$$\sum_{r=0}^{\infty} \frac{2(-1)^r \, r \, x^{n+2r-1}}{2^{n+2r} \, \Gamma(n+r+1) \, r!}$$

let,
$r-1 = s$
(Series manipulation)

$$= \frac{n}{x} J_n(x) - J_{n+1}(x)$$

Generating Function:-

$$e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x) \quad \text{—} (1)$$

$$z = e^{i\phi}$$

$$e^{\frac{x}{2}\left(e^{i\phi} - e^{-i\phi}\right)} = e^{(xi \sin\phi)}$$

$$= \cos(x \sin\phi) + i \sin(x \sin\phi) \quad (LHS)$$

RHS,

$$= J_0(x) + Z J_1(x) + \frac{1}{Z} J_{-1}(x) +$$

$$Z^2 J_2(x) + \frac{1}{Z^2} J_{-2}(x) + \cdots$$

$$= J_0(x) + \left(Z - \frac{1}{Z}\right) J_1(x) +$$

$$\left(Z^2 + \frac{1}{Z^2}\right) J_2(x) + \left(Z^3 - \frac{1}{Z^3}\right) J_3(x) + \cdots$$

$$\left(Z = e^{i\phi}\right)$$

$$= J_0(x) + 2i \sin\phi \, J_1(x) + 2\cos(2\phi) J_2(x)$$

$$+ 2i \sin(3\phi) J_3(x) + \cdots$$

Comparing real & Imaginary part,

$$\cos(x\sin\phi) = J_0(x) + 2\cos(2\phi) J_2(x) + 2\cos(4\phi) J_4(x) + \cdots$$

$$\sin(x\sin\phi) = 2\sin\phi\, J_1(x) + 2\sin(3\phi) J_3(x) + \cdots$$

Now, Prove that:

$$J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) = 1$$

(By Putting $\phi = 0$ in $\cos(x\sin\phi)$ expansion)

$$\& \quad x = 2\sum_{n=0}^{\infty} (2n+1) J_{2n+1}(x)$$

Eg (a) $\frac{d}{dx}\{J_n^2(x) + J_{n+1}^2(x)\} =$

$$2\left[\frac{n}{x} J_n^2(x) - \frac{(n+1)}{x} J_{n+1}^2(x)\right]$$

(b) $\frac{d}{dx}\{x J_n(x) J_{n+1}(x)\} =$

$$x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

(c) $= 1 = J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots)$

(d) $x = 2J_0J_1 + 6J_1J_2 + \cdots$

$$2(2n+1)J_nJ_{n+1} + \cdots$$

Also prove that,

$$|J_0(x)| \le 1, \quad |J_n(x)| \le 2^{-1/2} \quad (n \ge 1)$$

Solution :

(a) Diff we get,

$$2J_n J_n'(x) + 2J_{n+1}'(x) J_{n+1}(x)$$

Using ⑤ recurrence relation we get, RHS

(b) Diff we get,

$$J_n(x) J_{n+1}(x) + x\left[J_n'(x) J_{n+1}(x) + J_{n+1}'(x) J_n(x)\right]$$

Using recurrence we get, RHS

(c) Using 1st Problem.

$$\frac{d}{dx}\left[J_0^2(x)+J_1^2(x)\right] = 2\left(0-\frac{1}{x}J_1^2(x)\right)$$

$$\frac{d}{dx}\left[J_1^2(x)+J_2^2(x)\right] = 2\left(\frac{1}{x}J_1^2(x)-\frac{2}{x}J_2^2(x)\right)$$

$$\frac{d}{dx}\left[J_2^2(x)+J_3^2(x)\right] = 2\left(\frac{2}{x}J_2^2(x)-\frac{3}{x}J_3^2(x)\right)$$

$$\vdots$$

Adding all these we get,

$$\frac{d}{dx}\left[J_0^2(x)+2\left(J_1^2(x)+J_2^2(x)+\dots\right)\right] = 0$$

$$\Rightarrow J_0^2(x)+2\left(J_1^2(x)+J_2^2(x)+\dots\right) = C$$

Using, $J_0(0) = 1$ & $J_n(0) = 0$ if $n \neq 0$

$\therefore C = 1,$

$$\Rightarrow J_0^2(x) + 2\left(J_1^2(x) + J_2^2(x) + \cdots\right) = 1$$

$$|J_0(x)| \leq 1 \Rightarrow |J_0(x)| \leq 1$$

Also, $|J_n(x)| \leq 2^{-1/2}$

(d) Using (b) Problem we get by same Procedure.

$$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}} \quad a > 0$$

$$x \int_0^1 y^{n+1} J(xy)\,dy = J_{n+1}(x)$$

eg $\frac{d}{dx}\left[\frac{J_{-n}(x)}{J_n(x)}\right] = \frac{-2\sin(n\pi)}{\pi x J_n^2(x)}$

eg let y be a polynomial solution of diff. eq. :-

$$(1-x^2)y'' - 2xy' + 6y = 0$$

If $y(1) = 2$ then find the value of

$$\int_{-1}^{1} y^2\,dx$$

eg let $y : [-1,1] \to R$ with $y(1) = 1$, satisfying diff eq. :-

$$(1-x^2)y'' - 2x\frac{dy}{dx} + 6y = 0 \quad |x| < 1$$

Find : $\int_{-1}^{1} y(x)\,[x+x^2]\,dx$

Sturm - Liouville Problem :-

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = S(x)$$

$$a_1 y(a) + a_2 y'(a) = C_1$$

$$b_1 y(b) + b_2 y'(b) = C_2$$

If, $S(x) = 0$ & $C_1 = C_2 = 0 \Rightarrow$ Homo.

$\int y_m y_n = 0$ if $m \neq n$

Orthogonal

$\int y_m^2 = 1$ if $m = n$

Orthonormal

$$\int y_m y_n = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

A diff. eq. of the form

$$\frac{d}{dx}\left[b(x) \frac{dy}{dx} \right] + \left[q(x) + \lambda r(x) \right] y = 0$$

With B.C. :-

$$a_1 y(a) + a_2 y'(a) = 0$$
$$b_1 y(b) + b_2 y'(b) = 0$$

Where, $a_1, a_2, b_1, b_2 \rightarrow$ Const.

(a_1, a_2 & b_1, b_2 not together 0)

And $p(x)$, $q(x)$ & $r(x)$ are diff. in domain $[a, b]$. λ is Unknown parameter independent of x. λ defines the SLP problem & is an important parameter.

A 2^{nd} order diff. eq. of the form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + [P_2(x) + \lambda P_3(x)]y = 0$$

$$, \quad P_0(x) \neq 0$$

$$\Downarrow$$

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [Q(x) + \lambda r(x)]y = 0$$

if

$$p(x) = \int \frac{p_1(x)}{p_0(x)}$$

$$q(x) = \frac{p_2(x)}{p_0(x)} \cdot p(x)$$

$$r(x) = \frac{p_3(x)}{p_0(x)} \cdot p(x)$$

<u>Prop 1</u> :- The eigen values of SLP are real

<u>Proof</u>: Let,

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\,y = 0 \quad \text{--- (1)}$$

And, $\quad a_1 y(a) + a_2 y'(a) = 0$

$\quad b_2 y(b) + b_2 y'(b) = 0 \quad$ is a SLP

$p(x), q(x)$ & $r(x)$ → all are define & real.

Take Conjugate of Eq ①,

$$\frac{d}{dx}\left[p(x)\frac{d\bar{y}}{dx}\right] + \left[q(x) + \bar{\lambda}\, r(x)\right]\bar{y} = 0 \quad \text{— ②}$$

Multiply Eq ① × $\bar{y}$ & Eq ② × y

& Subtract,

$$\bar{y}\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - y\frac{d}{dx}\left[p(x)\frac{d\bar{y}}{dx}\right]$$

$$+ q(x)\cancel{y\bar{y}} + \lambda r(x) y\bar{y} - q(x)\cancel{y\bar{y}}$$

$$- \bar{\lambda}\, r(x) y\bar{y} = 0$$

$$\Rightarrow \bar{y}\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - y\frac{d}{dx}\left[p(x)\frac{d\bar{y}}{dx}\right]$$

$$= (\bar{\lambda} - \lambda)\, r(x)\, y\bar{y}$$

$$\Rightarrow \bar{y}\left[\frac{dp}{dx}\frac{dy}{dx} + p(x)\frac{d^2y}{dx^2}\right] -$$

$$y\left[\frac{dp}{dx}\frac{d\bar{y}}{dx} + p(x)\frac{d^2\bar{y}}{dx^2}\right] =$$

$$(\bar{\lambda}-\lambda)\, r(x)\, |y|^2$$

$$\Rightarrow p(x)\left[\bar{y}\frac{d^2y}{dx^2} - y\frac{d^2\bar{y}}{dx^2}\right] +$$

$$\frac{dp}{dx}\left[\bar{y}\frac{dy}{dx} - y\frac{d\bar{y}}{dx}\right] = (\bar{\lambda}-\lambda)\, r(x)\, |y|^2$$

$$\Rightarrow \frac{d}{dx}\left[p(x)\left[y\bar{y}' - \bar{y}y'\right]\right] = (\bar{\lambda}-\lambda)\, r(x)\, |y|^2$$

$$\Rightarrow p(x)\left\{y\bar{y}' - \bar{y}y'\right\}\Big|_a^b = (\bar{\lambda}-\lambda)\int_a^b r(x)\,|y|^2$$

$$\Rightarrow p(b)\left[y(b)\,\bar{y}'(b) - \bar{y}(b)\,y'(b)\right] -$$

$$p(a)\left[y(a)\,\bar{y}'(a) - \bar{y}(a)\,y'(a)\right]$$

Case 1 : $p(a) = p(b) = 0$

$$\lambda = \bar{\lambda}$$

Case 2 : $p(a) = 0 \;\;\&\;\; p(b) \neq 0$

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

$\Downarrow$ Conjugate

$$a_1 \bar{y}(a) + a_2 \bar{y}'(a) = 0$$

$$b_1 \bar{y}(b) + b_2 \bar{y}'(b) = 0$$

Suppose $b_1 \neq 0$ (as a_1 & $a_2 = 0$ & b_1 & $b_2 = 0$ together ✗)

Eliminate b_2,

$$b_1 (\bar{y} y' - y \bar{y}') = 0$$

<u>Case 3</u>: $p(a) \neq 0$ & $p(b) = 0$

Same as <u>Case 2</u>

<u>Case 4</u>: $p(a) \neq 0$ & $p(b) \neq 0$

Solve simultaneously

<u>Prop 2</u>: For every eigenvalue of SLP, there corresponds only one linearly indep. eigen vectors (eigen function)

, i.e, eigenvalues of SLP are simple.

Wronskein

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

y_1 & y_2 are L.I. $W(x) \neq 0 \; \forall x$

y_1 & y_2 are L.D. $W(x) = 0 \; \forall x$

<u>Proof</u> :

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right]y = 0 \quad \text{— (1)}$$

With $a_1 y(a) + a_2 y'(a) = 0$

$b_1 y(b) + b_2 y'(b) = 0$

let, y_1 & y_2 are two sol. of SLP

With λ eigenvalue.

$$\frac{d}{dx}\left[p(x)\frac{dy_1}{dx}\right] + \left[q(x) + \lambda\, r(x)\right] y_1 = 0 \quad \text{--- (2)}$$

$$\frac{d}{dx}\left[p(x)\frac{dy_2}{dx}\right] + \left[q(x) + \lambda\, r(x)\right] y_2 = 0 \quad \text{--- (3)}$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \quad \text{--- (4)}$$

$$W'(x) = y_1 y_2'' - y_2 y'' \quad \text{--- (5)}$$

$$W(a) = W(b) = 0 \quad \text{--- (6)} \qquad (\text{How ??})$$

Multiply eq (2) $\times\, y_2$ & eq (3) $\times\, y_1$ & Subtract,

$$\Rightarrow y_2 \frac{d}{dx}\left[p(x)\frac{dy_1}{dx}\right] - y_1 \frac{d}{dx}\left[p(x)\frac{dy_2}{dx}\right] = 0 \quad \text{--- (7)}$$

$$\Rightarrow \quad p\left[y_1'' y_2 - y_2'' y_1\right] +$$

$$p'(x)\left[y_1' y_2 - y_1 y_2'\right] = 0$$

$$\Rightarrow \quad d\left[p(x)\, W(x)\right] = 0$$

$$\Rightarrow \quad p(x)\, W(x) = C$$

$\Rightarrow$ Applying B.C. :-

$$p(a)\, W(a) = C$$

$$p(b)\, W(b) = C \qquad C = 0$$

$$W(a) = W(b) = 0 \qquad [\text{Based on B.C.}]$$