Lecture 1

Discrete objects $\xrightarrow{\text{examples}}$: $\mathbb{N}$, $\mathbb{Q}$, collection of Nations, etc.

Def^n (set): collection of "distinct" objects from a fixed universe

$$\underset{\substack{\text{an element}\\ \text{of set } S}}{x} \in \underset{\text{a set}}{S} \subseteq \underset{\text{universe}}{\Omega}$$

Def^n (multiset): Elements come with multiplicity

Example: $\Omega = \mathbb{N}$

$$\underset{\text{a set}}{S} = \{1, 2, 3, 10, 4\} \subseteq \Omega$$

$$\underset{\text{a multiset}}{T} = \{1, 1, 1, 2, 3, 4, 5, 4\}$$

Def^n (frequency): No. of times a particular element occurs in a multiset

Set Operations:

$$A \cup B := \{x \in \Omega : x \in A \text{ or } x \in B\}$$

$$A \cap B := \{x \in \Omega : x \in A \text{ and } x \in B\}$$

$\overline{A} := \{x \in \Omega : x \notin A\}$

properties:

① $A \cup (B \cup C) = (A \cup B) \cup C$

② $A \cup B = B \cup A$ → "commutativity"

$A \cap B = B \cap A$

③ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

④ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Defn (product of sets):

$$A \times B := \{(x, y) : x \in A, y \in B\}$$

Defn (power set): for a given set S, power set of S, denoted 2^S, is defined as

$$2^S := \{\text{subsets of } S\}$$

(i.e, power set of S is the collection of all subsets of S)

$$\boxed{\textbf{NOTATION}: \quad \{1, 2, 3, \ldots, n\} =: [n]}$$

Let $S = \{1, 2, 3, \ldots, n\} = [n]$

and let $\sigma \in 2^{[n]}$

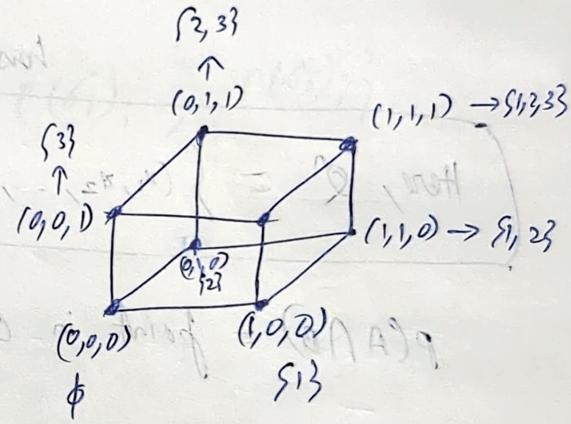
$$p(\sigma) = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$$

'point' (not probability)

" boolean - cube "

Example: $n = 3$

$[3]$



Now, let $\sigma_1, \sigma_2 \in 2^{[n]}$

Suppose $\sigma = \sigma_1 \wedge \sigma_2$

$p(\sigma_1)$, $p(\sigma_2)$

$x, y \in \mathbb{R}^n$

$$x \odot y = (x_1 y_1, x_2 y_2, \ldots, x_n y_n) \in \mathbb{R}^n$$

Suppose $\sigma = \sigma_1 \cup \sigma_2$

$$p(\sigma_1 \cup \sigma_2) = x \oplus y = \left(\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\}\right)$$

"Domination" : - A point "dominates" another if it is coordinatewise bigger (check!)

ie, $(x_1, x_2, \ldots x_n)$ dominates $(y_1, \ldots, y_n)$ if $x_i \geq y_i \quad \forall i = 1, \ldots, n$

Defn: $p(A \cup B)$ = point in Q^n that dominates both $p(A)$ and $p(B)$ and has the smallest no. of 1's

Here, $Q^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \{0,1\}\}$

$p(A \cap B)$ = point in Q^n that

NOTE:

$A \setminus B = \{x \in A : x \notin B\}$

$A \Delta B = (A \setminus B) \cup (B \setminus A)$

NOTATION: $|A|$ = size of A (ie, no. of distinct elements of set A)

$$|\sigma_1 \cap \sigma_2| = ?$$

$x, y \in \mathbb{R}^n$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

So, $\boxed{|\sigma_1 \cap \sigma_2| = \langle p(\sigma_1), p(\sigma_2) \rangle}$

Now, $|\sigma_1 \cup \sigma_2| = ?$

$$|\sigma_1 \cup \sigma_2| = \max \{ p(\sigma_1)_i , p(\sigma_2)_i \}$$

Lecture 2

Recall: $A \times B := \{(x, y) : x \in A,\ y \in B\}$

"ordered pairs"

~~Special case (when B=A)~~

$\{x, y\} \to$ unordered pair

In general, $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) : x_i \in A_i\ \forall i \in [n]\}$

Special case (when $B = A$)

Def^n (relation):

$S \subseteq A \times A$ is a relation on the set A

If $(x, y) \in S$, then we say $x \sim y$

x is related to y

Examples: $(\mathbb{N}, \leq)$

$(\mathbb{R}, <)$

Def^{n}s: ① Reflexive relation: $\forall x \in A,\ x \sim x$

② Symmetric relation:

$x, y \in A$ s.t $x \sim y \Rightarrow y \sim x$

③ Transitive relation:

$x \sim y,\ y \sim z \Rightarrow x \sim z$

Q^n: Construct 8 different examples of relation

$(\_\,,\,\_\,,\,\_)$ → reflexive, symmetric, transitive

0/1

Def^n (Equivalence relation): $\sim$ is an equivalence relation if it is reflexive, symmetric and transitive

Def^n (class):

Let $\sim$ be a relation. Then:

$$[\![a]\!] := \{b \in A : a \sim b\} \longrightarrow \text{class of } a$$

NOTE: If $\sim$ is an equivalence relation, then $[\![a]\!]$ is called the "equivalence class of a"

Def^n (partition): $A \leftarrow$ ground set / base set

Let $A_1, \dots, A_k, \dots$ be collection of non-empty subsets of A s.t $A_i \cap A_j = \phi, \ \forall i,j$

and $A = \bigcup_{i=1}^{k} A_i$

Q^n: $A = [n] = \{1, 2, 3, \dots, n\}$

How many k sized partitions of A are there?

(Hint: Start with $k = 2$)

Theorem 9: Let $\sim$ be an equivalence relation. Then, $\sim$ induces a partition on A, where each set in the partition is an equivalence class w.r.t $\sim$

Theorem 10: Every partition of A gives an equivalence relation on A

pf (10): Let $\{A_\beta\}_{\beta \in I}$ be a partition of A

Lets define the following relation:

$\forall (x, y) \in A \times A$, $x \sim y$ if $\exists \beta \in I$ s.t $x \in A_\beta$ and $y \in A_\beta$

Reflexive: $x \sim x$ ✓

Symmetric: $x \sim y \Rightarrow y \sim x$ ✓

don't need partition for proof

Transitive: $(x \sim y, y \sim z \Rightarrow x \sim z)$

Here, we need partitioning for proof

Check!

---

pf (Thm. 9): $[a], [b], \ldots$

As long as $\sim$ is reflexive, $\{[a]\}_{a \in A}$ will satisfy $\bigcup_{a \in A} [a] = A$ ✓

We just need pairwise disjointness

Suppose $[a] \cap [b] \neq \phi$

Thus, $\exists y \in [a] \cap [b]$

Now, $a \sim y$ and $b \sim y$ (equivalently, $y \sim b$ "symmetry")

$\Rightarrow a \sim b$ (due to transitivity)

Take any $z \in [\![b]\!]$

$a \sim b$, $b \sim z$ $\Rightarrow a \sim z$

$\Rightarrow z \in [\![a]\!]$

~~Propositional Logic~~

~~p, q → two statements~~

~~$p \vee q$~~

Defⁿ (Function): A function from a set A to a set B is a relation b/w A and B s.t each element of A is related to exactly one element of B

Qⁿ: How many relations are there b/w A and B?

Qⁿ: How many functions are there from A to B?

Solⁿ: $|B|^{|A|} = 2^{|A| \log_2 |B|}$

Solⁿ: $2^{|A|\,|B|}$

Qⁿ: If $B = A$, how many equivalence relations do we have on A?

A	$f(x) \in B$
x_1	$f(x_1)$
$\vdots$	$\vdots$
x_n	$f(x_n)$

→ "Truth table" of f

Def^n (Boolean function):

$f : \{0,1\}^n \longrightarrow \{0,1\}$ → can also be $\{-1,+1\}$ and other things as well (with 2 elements)

n-variable Boolean function

Def^n (Types of functions): Let $f : D \to R$ be a function

① **Injective (1-1):**
$\forall y \in R, \ |f^{-1}(y)| \leq 1$

② **Surjective (onto):**
$\forall y \in R, \ f^{-1}(y) \neq \phi$

③ **Bijective (1-1, onto):**
Injective and Surjective

Propositional ~~Proportional~~ Logic

$p \vee q$

p	q	$p \vee q$	$p \wedge q$	$\neg q / \bar{q}$
T	F	T	F	T
F	T	T	F	F
F	F	F	F	T
T	T	T	T	F

$\vee$: OR , $\wedge$: AND , $\neg$: NOT

$$(p \Rightarrow q) = f(p, q)$$

$$p \Leftrightarrow q$$

properties :

① $p \vee (q \vee r) = (p \vee q) \vee r$
$p \wedge (q \wedge r) = (p \wedge q) \wedge r$ } associativity

② $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ } distributivity

③ $\neg (p \vee q) = (\neg p) \wedge (\neg q)$
$\neg (p \wedge q) = (\neg p) \vee (\neg q)$ } De Morgan

Set theoretic formulation :

Ω, A, B, C

$1_A : \Omega \to \{0,1\}$ s.t

$$1_A := \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{o.w} \end{cases}$$

$1_A \, 1_B = 1_{A \cap B}$

$(p \Rightarrow q)$

$\neg p \vee q$

If p is true, then q is true

Lecture 3

$\exists x \quad P(x)$

$\forall x \quad P(x)$

$$\neg\left(\exists x \; P(x)\right) = \forall x \; \neg P(x)$$

$$\neg\left(\forall x \; P(x)\right) = \exists x \; \neg P(x)$$

Induction

To prove : $P(n) \quad \forall n \in \mathbb{N}$

Base case $(n=1)$

Induction hypothesis : Assume that the formula is true for some $n \geq 1$

Induction step : we show that the formula is true for $n+1$

Mathematical Induction follows from the "well ordering" axiom for natural numbers

"Every non-empty subset of $\mathbb{N}$ has a least element"

Lets write all of the above formally :

PTO →

Assume $0 \in \mathbb{N}$

Thm 1 (Mathematical Induction):

Let $S \subseteq \mathbb{N}$ s.t

i) $0 \in S$ and

ii) If $m \in S$, then $(m+1) \in S$

Then, $S = \mathbb{N}$

Pf: Suppose to the contrary, let $S \subsetneq \mathbb{N}$

~~Then, $\exists x \in \mathbb{N} \setminus S$~~

Consider $\bar{S} := \mathbb{N} \setminus S$

(clearly, $\bar{S} \neq \phi$)

Since $\bar{S} \neq \phi$, by WOP, it has a least element, say m

Note that $(m-1) \in \mathbb{N}$ and $(m-1) \in S$

But then, by ii), $(m-1)+1 \in S$

i.e. $m \in S$

$(\Rightarrow \Leftarrow)$

$\therefore S = \mathbb{N}$ ✓

Partial Ordering

a relation on set A

$(A, \sim)$ $\qquad S \subseteq A \times A$

$a \sim b$ if $(a, b) \in S$

$\sim$ is called a "partial ordering" over A if :

① $\sim$ is reflexive (i.e, $a \sim a$, $\forall a \in A$)

② $a \sim b$, $b \sim c \Rightarrow a \sim c$

③ $a \sim b$, $b \sim a \Rightarrow a = b$

NOTE : partial ordering is usually denoted by $\leq$ (instead of $\sim$)

i.e,
① $a \leq a$
② $a \leq b$, $b \leq c \Rightarrow a \leq c$
③ $a \leq b$, $b \leq a \Rightarrow a = b$

Examples (of partial orderings)

$\underbrace{(\mathbb{N}, \leq), \ (\mathbb{R}, \leq)}_{\text{less equal to}}$, $\underbrace{(2^{\mathbb{N}}, \subseteq)}_{\text{but not a total ordering}}$

Defn (Total ordering) :

$(A, \leq)$ is a "total ordering" if it is a partial ordering and $\forall a, b \in A$, either $a \leq b$ or $b \leq a$

Def^n (predecessor) :

Let $(A, \leq)$

We say $a \lhd b$ if :

① $a < b$

② $\nexists\, c \notin \{a, b\} \in A$ s.t $a \leq c \leq b$.

Lemma: Let $(A, \leq)$ be a partial ordering with $|A| < \infty$. If $a < b$, then $\exists\, c_1, \ldots, c_t \in A$ with $t \in \mathbb{N}$ s.t

$a \lhd c_1 \lhd c_2 \lhd \cdots \lhd c_t \lhd b$

Pf: ~~Suppose $\exists c$ s.t $a \leq c \lhd b$~~

~~If $c = a$, we are done~~

~~If not, recursively apply the above logic~~

Apply induction on t

(Exercise!)

Defⁿ (chain) :

Let $(A, \leq)$ be a partial ordering.

A chain in $(A, \leq)$ is of the following form :

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$$

where $a_i \in A$, $\forall i \in [n]$

Defⁿ (anti-chain) :

~~A~~ Let $(A, \leq)$ be a partial ordering

An antichain is a subset $S \subseteq A$ where no two distinct elements from S are related.

$\{b_1, \ldots, b_l\}$ $b_i \nleq b_j$ & $b_j \nleq b_i$

Check : "Dilworth's Theorem"

Example : $1, 2, \ldots, n^2+1 = [n^2+1]$

Consider any sequence $a_1, a_2, \ldots, a_{n^2+1}$ s.t

① $a_i \in [n^2+1]$

② $a_i \neq a_j$, $\forall i \neq j$

Does $\exists$ a monotone subsequence of length $(n+1)$?

(i.e, $b_{i_1} < b_{i_2} < \cdots < b_{i_{n+1}}$ or $b_{i_1} > b_{i_2} > \cdots > b_{i_{n+1}}$)

Sol^n: Yes

$a_1 \;,\; a_2 \;,\; a_3 \;,\; a_i \;,\; \dots \;,\; a_{n^2+1}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$(1, a_1) \quad (2, a_2) \quad (i, a_i) \quad (n^2+1, a_{n^2+1})$

$A = \{ (i, a_i) : i \in [n^2+1] \}$

$(i, a_i) \leq (j, a_j)$
if $i \leq j$ and $a_i \leq a_j$

Take a min. disjoint chain decomposition of A

If $\exists$ a chain in the decomposition of length $\geq n+1$, we are done

By Dilworth's theorem, $\exists$ an antichain

$(j_1, a_{j_1}), \dots, (j_{n+1}, a_{j_{n+1}})$

where $j_1 < j_2 < \dots < j_{n+1}$

But we know that $(j_1, a_{j_1}) \not\leq (j_2, a_{j_2}) \not\leq \dots$

Thus, we can have a decreasing subsequence as required

$\square$

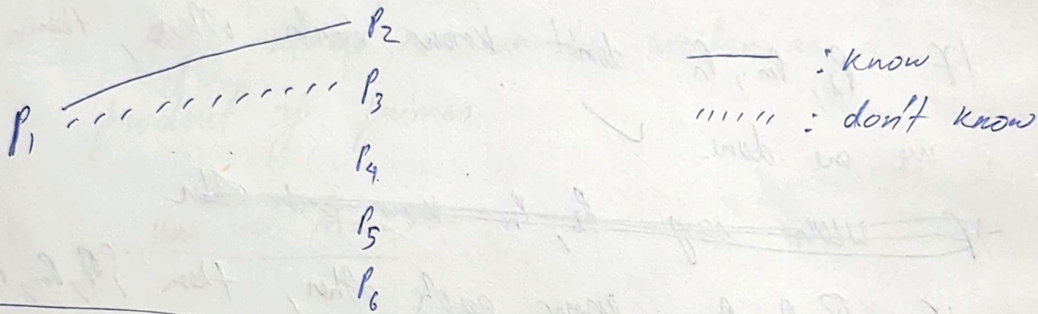
Check: "Hall's theorem"

Lecture 4

Proof by case analysis

<u>Example:</u> Out of 6 people, either there are 3 people who know each other or 3 people who don't know each other

<u>sol^n</u>: $P_1, P_2, P_3, P_4, P_5, P_6$

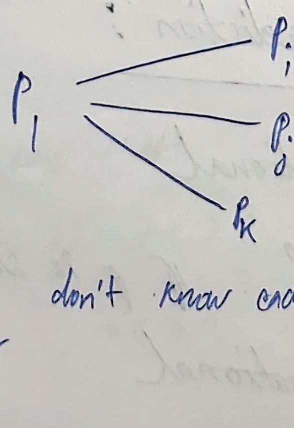


Consider two cases:

<u>case I:</u> ($\exists$ at least three — edges b/w P_1 and rest)

<u>case II</u>: (atmost two — edges b/w P_1 and rest)

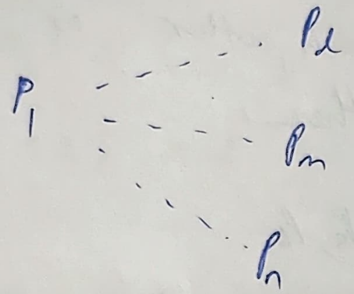
<u>case I:</u>



If P_i, P_j, P_k don't know each other, then we are done ✓

Suppose, WLOG, P_j and P_k know each other. Then, $\{P_1, P_j, P_k\}$ know each other and we are done. ✓

Case II :



If P_l, P_m, P_n don't know each other, then we are done ✓

~~If WLOG say P_l, P_n know each other~~

If P_l, P_m, P_n know each other, then $\{P_l, P_m, P_n\}$ know each other and we are done ✓

If, say, P_l and P_m don't know each other, then, $\{P_1, P_l, P_m\}$ don't know each other and we are done ✓

Proof by contradiction :

Lemma : $\sqrt{2}$ is irrational

Lemma: Let $p \in \mathbb{R}^{\geq 0}$. If p is irrational then $\sqrt{p}$ is also irrational

(Equivalent statement ("contrapositive"): $a \Rightarrow b \Leftrightarrow \neg b \Rightarrow \neg a$)

$\sqrt{p}$ rational $\Rightarrow$ p rational

Disprove by counter-example :

. . . . .

Strong Induction :

Example : Any natural no. can be written as a product of primes.

Pf : Base case ($n=2$) : ✓

Suppose $2, 3, \ldots, n$ can be written as a product of primes.

complete the proof

Asymptotics :

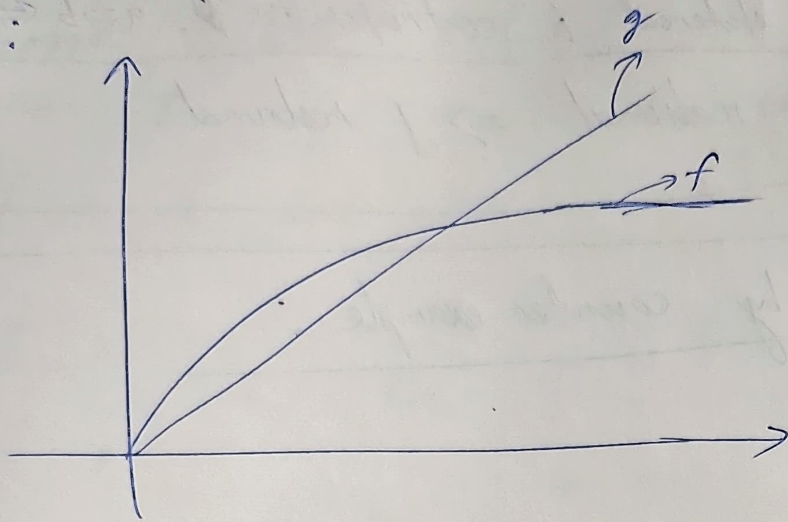
Let $f, g : \mathbb{N} \to \mathbb{N}$ ~~$f(i) \geq f(j)$, $\forall i \geq j$~~

Big-O : We say that

$f(n) = O(g(n))$ if $\exists c > 0$ and $n_0 \in \mathbb{N}$ s.t

$f(n) \leq c\, g(n)$, $\forall n \geq n_0$

Example :



ALT defⁿ (big-O) : $f(n) = O(g(n))$ if $\exists\, c > 0$

s.t $\lim_{n \to \infty} \frac{f(n)}{g(n)} \leq c$

Defⁿ (big-omega) : We say that $f(n) = \Omega(g(n))$

if $\exists\, c > 0$ and $n_0 \in \mathbb{N}$ s.t

$$f(n) \geq c\, g(n) \quad , \forall n \geq n_0$$

Defⁿ (theta): We say $f(n) = \Theta(g(n))$ if

$f = O(g)$ and $f = \Omega(g)$

Example: $f(n) = n!$, $f'(n) = \log(n!)$

$g(n) = n^n$, $g'(n) = \log(n^n)$

Relations b/w f, g, f', g' ? Exercise

Def^n (small-o): We say $f = o(g)$ if $\forall \varepsilon > 0$

$\exists n_0 \in \mathbb{N}$ s.t $g(n) \geq \varepsilon f(n)$, $\forall n \geq n_0$

~~Def^n (small omega):~~

~~We say~~

$$\boxed{\begin{array}{l}\text{NOTE: } f = o(g) \Rightarrow f = O(g) \\ \text{but converse may not be true}\end{array}}$$

Q^n: In the previous example, is $f = o(g)$?

Back to Lecture 3 :

A partial ordering $(P, \leq)$ is called a "poset" if:

① $a \leq a$, $\forall a \in P$

② $a \leq b$, $b \leq c \Rightarrow a \leq c$

③ $a \leq b$, $b \leq a \Rightarrow a = b$

chain : $a_1, \ldots, a_t$

$a_1 \leq a_2 \leq \ldots \leq a_t$

anti-chain : $a_1, \ldots, a_t$

$a_i \not\leq a_j \quad \forall i \neq j$

chain decomposition:

$C_1, C_2, \ldots, C_l$ (here, "length of decomposition" $= l$)

Dilworth's theorem :

Let $(P, \leq)$ be a poset, m = length of the smallest disjoint chain decomposition of $(P, \leq)$ and M = length of the longest antichain in P

finite

Then, $m = M$

~~Pf~~ : $m < M$ is not possible (check!)

So, $m \geq M$

We'll use induction on $|P|$ to prove this

Let C be a maximal chain in P

Now, consider $P \setminus C$

Note that $|P \setminus C| < |P|$

~~Suppose length of maximal antichain in $P \setminus C$ is atmost $M-1$~~

~~By induction or~~

case I (length of maximal antichain in $P \setminus C$ is atmost $M-1$):

Then, from induction hypothesis, $\exists$ a disjoint chain decomposition of $P \setminus C$ of length atmost $M-1$

To this decomposition, add C to get the bound for P

case II ($\exists$ an antichain $\{a_1, a_2, \ldots, a_M\}$ in $P \setminus C$):

$$S^- = \{x \in P : \exists i \in [M] \text{ with } x \leq a_i\}$$

$$S^+ = \{x \in P : \exists i \in [M] \text{ with } x \geq a_i\}$$

note that $a_i \in S^-$ and $a_i \in S^+$, $\forall i$

and thus, $S^- \cap S^+ \neq \phi$

Consider a chain decomposition of S^-:

$$S^- : S_1^-, S_2^-, \ldots, S_M^-$$

$$\downarrow$$

$$a_1$$

Claim: $a_1, a_2, \ldots, a_M$ are maximal elements of $S_1^-, S_2^-, \ldots, S_M^-$

Suppose $x \in S_1^-$ and $x > a_1$

Since $x \in S^-$, $\exists j$ s.t. $x \leq a_j$.

($\Rightarrow \Leftarrow$)

Similarly, $a_1, a_2, \ldots, a_m$ are the minimal elements of $S_1^+, S_2^+, \ldots, S_m^+$

Combining both, we are done ✓

$\square$

- -

$(2^{[n]}, \subseteq) \to \subseteq$

$A_1, \ldots, A_t$ are subsets of $[n]$

s.t $A_i \not\subseteq A_j$, $\forall i \neq j$

Equivalent Qn: What is the length of the maximal antichain in the poset?

Lecture 5

Pigeonhole principle (PHP):

generalized PHP:

Buckets $\longleftrightarrow n$

balls $\longleftarrow kn+r$; $1 \le r \le k-1$

Example: (Recall lecture 3)

$$\{a_1, a_2, \ldots, a_{n^2+1}\} \quad \{1, 2, \ldots, n^2+1\}$$

some permutation of

all distinct

Let $\pi(a_i) :=$ length of the longest increasing subsequence starting at a_i

NOTE: $1 \le \pi(a_j) \le n$ $\forall\, 1 \le j \le n^2+1$

Consider $\{\pi(a_1), \pi(a_2), \ldots, \pi(a_{n^2+1})\}$

By PHP, we know that $\exists\, i \in [n]$ s.t. i appears atleast $(n+1)$ times in the following sequence:

$$\pi(a_1), \pi(a_2), \ldots, \pi(a_{n^2+1})$$

$$\pi(a_1) = i \qquad \pi(a_2) = i$$

$$a_1 > a_2$$

Example: $\{A_1, A_2, \dots, A_m\}$, where

① $A_i \subseteq [n]$, $\forall i$ and

② $A_i \not\subseteq A_j$, $\forall i \neq j$

How large can m be w.r.t n?

(Equivalent Q^n: How large can an antichain be in the poset $(2^{[n]}, \subseteq)$?)

Theorem 2: Let $(\Omega, \leq)$ be a poset, where Ω is a finite set. If $(\Omega, \leq)$ does not contain a chain of length $m+1$, then Ω can be partitioned into atmost m antichains

Pf: Let M be all the maximal elements from Ω

Now, consider $\Omega \setminus M$
$\qquad \qquad \quad \tilde{\Omega}$ ''

Note that $\tilde{\Omega}$ does not contain any chain of length m

Now, apply induction on m (Exercise!)

Theorem (Sperner's theorem):

$$m \leq \binom{n}{\lfloor n/2 \rfloor}$$

pf: Direct consequence of "LYM(B)-inequality"

Theorem (LYM(B) - inequality):

$\{A_1, \dots, A_m\}$ be a collection satisfying the conditions of Sperner's theorem.

Let $\forall k \in [n]$, α_k be the no. of sets of size k in F.

Then, $\sum_{k \in [n]} \frac{\alpha_k}{\binom{n}{k}} \leq 1$

Remark: Using the fact that $\sum \alpha_k = m$ and $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$, we can prove Sperner's thm. from LYM(B) - inequality.

pf (LYM inequality): ① What can we say about maximal chains in $(2^{[n]}, \subseteq)$?

$$\phi \subseteq \{1\} \subseteq \{1,2\} \subseteq \dots \subseteq [n]$$

How many maximal chains are there in $(2^{[n]}, \subseteq)$?

$n!$

) ② $A \subseteq [n]$ with $|A| = k$

How many maximal chains can A be part of?

$k!\,(n-k)!$

$$\phi \subseteq \cdots \subseteq A \subseteq \cdots \subseteq [n]$$
$$A = \{1, 2, \dots, k\} \quad \text{WLOG}$$

$$F = \{A_1, \dots, A_m\}$$

antichain

$$\pi(A_i) = \{ C : C \text{ is a maximal chain and } A_i \in C \}$$

Observe that $\pi(A_i) \cap \pi(A_j) = \phi, \ \forall i \neq j$

Thus, $\displaystyle\sum_{i \in [m]} |\pi(A_i)| \leq n!$ —— (*)

Also, $|\pi(A_i)| = k!\,(n-k)!$, where $k = |A_i|$.

$$\sum_{i \in [m]} |\pi(A_i)| = \sum_{k \in [n]} \sum_{\substack{|A_i| = k, \\ A_i \in F}} k!\,(n-k)!$$

$$= \sum_{k \in [n]} \alpha_k \, k!\,(n-k)!$$

Now, we can use $\circledast$ to get the ③ required inequality

$\square$

---

① $a_1, a_2, \ldots, a_n$

$a_{i_1}, a_{i_2}, \ldots, a_{i_n} \longrightarrow n!$

Suppose k_i ~~[illegible]~~ objects of type i

't' types of objects

Let $\sum_{i=1}^{t} k_i = n$

$$\frac{n!}{|k_1|! \cdots |k_t|!}$$

$a_1, \ldots, a_n \longrightarrow n$ distinct objects

ways of selecting k objects $= \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

How many ways of selecting k objects ?

Exercise

(2) No. of non-negative sol^{n}s to

$$\sum_{i=1}^{K} x_i = n \quad ?$$

○ [] ○ [] ○ [] . . . – – []

no. of ways of arranging $(K-1)$ []'s and n balls $= \binom{n+K-1}{n}$

(3) <u>Property :</u> $\binom{n-1}{K-1} + \binom{n-1}{K} = \binom{n}{K}$

pf : $S = \{1, 2, \ldots, n\}$

No. of ways of choosing K element sets out of S :

i) <u>method 1 (direct) :</u> $\binom{n}{K}$

ii) <u>method 2 :</u> $\binom{n-1}{K-1} + \binom{n-1}{K}$

no. of sets containing 1

no. of sets not containing 1

□

④ $$(1+x)^n = \sum_{i\geq 0} \binom{n}{i} x^i$$

COROLLARIES: ⓘ $$\sum_{i\geq 0} \binom{n}{i} = 2^n$$

no. of possible subsets of $\{1,\dots,n\}$

Recall (before)

$\phi : 2^{[n]} \to \{0,1\}^n$

bijection

use this to prove ⓘ, ⓘⓘ

ⓘⓘ $$\sum_{i \text{ even}} \binom{n}{i} = 2^{n-1} = \sum_{i \text{ odd}} \binom{n}{i}$$

⑤ $$(x+y)^n = \sum_{i\geq 0} \binom{n}{i} x^i y^{n-i}$$

⑥ $$(x_1 + x_2 + \cdots + x_t)^n = \sum_{\substack{k_i \geq 0 \\ \sum_{i=1}^{t} k_i = n}} \binom{n}{k_1\, k_2 \cdots k_t} x_1^{k_1} x_2^{k_2} \cdots x_t^{k_t}$$

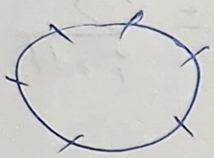
, where $$\binom{n}{k_1\, k_2 \cdots k_t} = \frac{n!}{k_1!\, k_2! \cdots k_t!}$$

$$\binom{n}{k_1\, k_2 \cdots k_t} = \sum_{i=1}^{t} \binom{n-1}{k_1 \quad k_i - 1 \quad k_i}$$

⑦ $\sum_{i \geq 0} \binom{n}{i}^2 = \binom{2n}{n}$

Pf: HINT: $\sum_{i \geq 0} \binom{n}{i}^2 = \sum_{i \geq 0} \binom{n}{i}\binom{n}{n-i}$

no. of ways of choosing n elements out of $2n$ elements

⑧  n elements

no. of ways $\longrightarrow$ $(n-1)!$

Theorem (Erdos-Ko-Rado's theorem):

$\{A_1, A_2, \ldots, A_m\}$, where $A_i \in [n]$, $\forall i$

s.t $A_i \cap A_j \neq \phi$, $\forall i, j$ and

$|A_i| \leq k \leq n/2$

Then, $m \leq \binom{n-1}{k-1}$

Lecture 6

Asymptotic Notations

$\boxed{F_n := \text{no. of primes} \leq n}$

$\{a_n\}_{n \in \mathbb{N}}$ $\{b_n\}_{n \in \mathbb{N}}$

$a_n \sim b_n$ if $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$

$\boxed{\underline{\text{NOTE}}: \quad F_n \sim \left\lceil \frac{n}{\log n} \right\rceil}$

Some estimates:

<u>FACT 1</u>: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$H_n = \Theta(\log n)$

Let $S_k = \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}$

$$\boxed{\frac{\log_2 n}{2} \leq H(n) \leq \log_2 n + 1}$$

FACT 2 : $1+x \le e^x$

FACT 3 (Stirling) :

$$e\left(\frac{n}{e}\right)^n \le n! \le en\left(\frac{n}{e}\right)^n$$

FACT 4 : $\binom{n}{k} \le \left(\frac{en}{k}\right)^k$

1) $(n-1)! \le e\left(\frac{n}{e}\right)^n \le n!$

2) $n! = en\left(\frac{n}{e}\right)^n (1+o(1))$

FACT 5 : $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{k} \le \left(\frac{en}{k}\right)^k$

pf: HINT: $\sum_{0\le i\le k}\binom{n}{i} x^i \le (1+x)^n$

Q1) Let $m \le n$, $f: \{1,2,..,n\} \to \{1,2..,m\}$

where f is surjective

No. of possible such f = ?

Q2) How many ways can we partition $\{1,2,\dots,n\}$ into m subsets?

unordered

Q3) $\pi : \{1,2,\dots,n\} \longrightarrow \{1,2,\dots,n\}$ be an injective map

(i.e, π is a "permutation")

Let $D(n) = \{\pi : \pi(i) \neq i, \ \forall i \in \{1,2,\dots,n\}\}$

$$|D(n)| = ?$$

Q4) Let $m \leq n$. How many non-decreasing functions are there from $\{1,2,\dots,m\}$ to $\{1,2,\dots,n\}$?

(Here, "non-decreasing" $\rightarrow$ $f(i) \geq f(j), \ \forall i \geq j$)

Principle of Inclusion and Exclusion (PIE):

Let $A_1, A_2, \dots, A_n$ be subsets of Ω

want: $|A_1 \cup A_2 \cup \dots \cup A_n| = ?$

we know $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Theorem : Let $A_1, A_2, \ldots, A_n$ be subsets of a finite universe Ω. Then,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Pf (using induction on n):

Base case ($n=2$) : $\boxed{\text{check!}}$

$$|\underbrace{A_1 \cup A_2 \cup \cdots \cup A_n}_{B} \cup A_{n+1}|$$

Exercise

ALT pf:

Let $a \in A_1 \cup A_2 \cup \cdots \cup A_n$

WLOG assume that $a \in A_1 \cap A_2 \cap \cdots \cap A_j$ and $a \notin A_k$, $\forall k \geq j+1$

$$1 - \left[j - \binom{j}{2} + \binom{j}{3} - \binom{j}{4} + \cdots \right] = 0$$

$\boxed{\text{check!}}$

ALT pf:

Let $f_i : \Omega \longrightarrow \{0,1\}$ s.t

$$f_i(a) = \begin{cases} 1, & \text{if } a \in A \\ 0, & \text{o.w} \end{cases}$$

where $1 \le i \le n$

Suppose $F : \Omega \longrightarrow \{0,1\}$ s.t

$$F(x) := \prod_{i=1}^{n} (1 - f_i(x))$$

Then, $$F(a) = \begin{cases} 0, & \text{if } a \in \bigcup_{i=1}^{n} A_i \\ 1, & \text{o.w} \end{cases}$$

$$\Rightarrow \sum_{a \in \bigcup_{i=1}^{n} A_i} F(a) = 0$$

Look at $\prod_{i \in S \subseteq [n]} f_i(a)$

$\boxed{\text{Exercise!}}$

Solutions to Q's 1>, 2>, 3> :

Q 1> $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$, $m \leq n$

f is surjective (onto)

We'll use PIE to solve this

Let $A_i = \{ f : f^{-1}(i) = \phi \}$

$f \in A_1 \cup A_2 \cup \dots \cup A_m \longrightarrow$ not surjective

No. of such f = total no. of f^ns from $\{1, \dots, n\}$ to $\{1, \dots, m\}$

$- |A_1 \cup A_2 \cup \dots \cup A_m| \longrightarrow$ no. of non surjective f's

$= m^n - |A_1 \cup A_2 \cup \dots \cup A_m|$

NOTE: $|A_1 \cap A_2| = (m-2)^n$

$|A_{j_1} \cap A_{j_2} \dots \cap A_{j_k}| = (m-k)^n$

Complete the solution Exercise

Q 2) $\{1, 2, \ldots, n\} \xrightarrow{\text{partitioning}} m$ subsets

method 1 (recursive):

$\{1, 2, \ldots, n\} \longrightarrow \{1\}, \{2, 3, \ldots, n\}$

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}$$

Q 3) "Dearrangement"

$\pi : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ be a 1-1 map

Let $A_i := \{\pi : \pi(i) = i\}$

$\pi \in A_1 \cup A_2 \cup \ldots \cup A_n$

$$|D(n)| = n! - |A_1 \cup A_2 \cup \ldots \cup A_n|$$

~~NOTE:~~ $\cancel{(n-2)!}$

$\cancel{|A_{j_1} \cup A_{j_2} \cup \ldots \cup}$

NOTE: $|A_1 \cap A_2| = (n-2)!$

$|A_{j_1} \cap A_{j_2} \cap \ldots \cap A_{j_k}| = (n-k)!$

Complete the soln Exercise

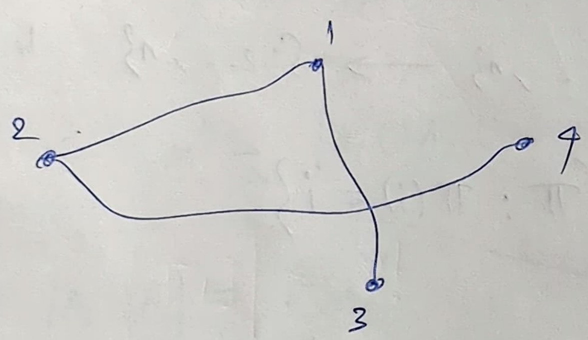
Remark: $\Delta(n) \sim \dfrac{n!}{e}$

Graph Theory

$G = (V, E)$

$E \subseteq \binom{V}{2}$

Example: $V = \{1, 2, 3, 4\}$



Defⁿ (Graph Homomorphism): Suppose we have two graphs $G = (V, E)$ and $G' = (V', E')$

A map $f : V \longrightarrow V'$ is called a homomorphism if $\{u, v\} \in E \Rightarrow \{f(u), f(v)\} \in E'$

Def^n (injective homomorphism) : 1-1 homomorphism

NOTE: injective homomorphism is also called "embedding"

Def^n (surjective homomorphism) : onto homomorphism

Def^n (Isomorphism) : ~~both injective and surjective homomorphism~~

Def^n (Isomorphism) : Let $G = (V, E)$ and $G' = (V', E')$ be two graphs

$f : V \longrightarrow V'$ is called an isomorphism if

f is injective, surjective and

$$\{u, v\} \in E \iff \{f(u), f(v)\} \in E'$$

Remark : f can be an injective and surjective homomorphism but NOT an isomorphism (the 3rd condition is also required)

Def^n (Automorphism) :

Lecture 7

Let $F(n) :=$ { all graphs whose vertex set is $\{1,2,3,\dots,n\}$ }

NOTE : $|F(n)| = 2^{\binom{n}{2}}$

Suppose $G \sim G'$ if G and G' are isomorphic

Theorem : $\sim$ as defined above is an equivalence relation on $F(n)$

~~Pf~~ : Exercise

- - - - - - - - - -

$$[G] \in F(n)/\sim$$

"equivalence class containing G"

Note that $\big|[G]\big| \leq n!$

Observation : $\big|F(n)/\sim\big| \geq \dfrac{2^{\binom{n}{2}}}{n!}$

- - - - - - - - - -

NOTATIONS : for a graph G,

$V(G) = \{$vertices of $G\}$

$E(G) = \{$edges of $G\}$

Defⁿ (subgraph): Let G, G' be two graphs.

We say that G' is a subgraph of G if:

$$V(G') \subseteq V(G) \quad \text{and} \quad E(G') \subseteq E(G)$$

Defⁿ (induced subgraph): for two graphs G, G',

G' is called an "induced subgraph" of G if:

$$V(G') \subseteq V(G) \quad \text{and} \quad E(G') = E(G) \cap \binom{V(G')}{2}$$

~~Examples~~

NOTATION: $S \subseteq V(G)$

$G[S] :=$ induced subgraph with vertex set S

Defⁿ (Independent Set): $G = (V, E)$

$S \subseteq V$ is called an "independent set" if

$G[S]$ has no edges

Qⁿ: Given G, what is the size of the largest independent set in G?

(this is an "NP-complete" problem)

Def^n (vertex cover):

$G = (V, E)$ and $S \subseteq V$

We say S is a "vertex cover"

if $e \cap S \neq \phi$, $\forall e \in E$

(i.e, atleast one end point of any edge of G is in S)

Lemma: Let G be a graph and $S \subseteq V(G)$

~~If S is a vertex cover, then $V \setminus S$ is an independent set~~ Then:

S is a vertex cover $\iff$ $V \setminus S$ is an independent set

Def^n (matching): $G = (V, E)$ and $E' \subseteq E$

We say that E' is a matching if

$$e \cap e' = \phi, \quad \forall e \neq e' \in E'$$

PTO →

Lemma: Let G be a graph and $E' \subseteq E$ be a "maximal matching" in G. Let α be the size of smallest vertex cover of G. Then:

$$\alpha \leq 2|E'|$$

pf: Let $S = V(E') :=$ vertices participating in the matching E'

Claim: S is a vertex cover of G

Complete the proof (Exercise!)

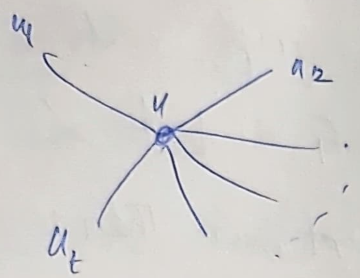
Defn (Dominating set): Let G be a graph. $S \subseteq V(G)$ is called a "dominating set" if $\forall u \in G$, atleast one of the following two statements is true:

① $u \in S$

② $\exists \{u, v\} \in E(G)$ with $v \in S$

NOTE: ② $\iff N_G[u] \cap S \neq \phi$

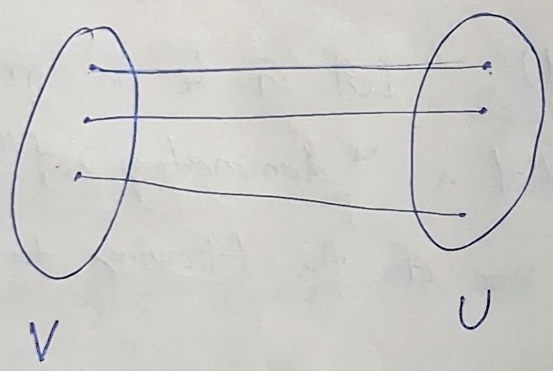
see next page → "$N_G[u]$"



$N_G(u) := \{ v : \{u, v\} \in E(G) \}$ → "open neighbourhood of u"

$N_G[u] := N_G(u) \cup \{u\}$ → "closed neighbourhood of u"

Defn (Bipartite graph): $G = (V \cup U, E)$



Paths, Cycles, Tours, Walks

$G = (V, E)$

Path : $P = (v_0\, e_1\, v_1\, e_2 \ldots \ldots e_t\, v_t)$, where

$v_1, v_2, \ldots, v_t$ are distinct vertices of G

and $e_i = \{v_{i-1}, v_i\}$, $\forall i \in \{1, 2, \ldots, t\}$

we say that P is a path from v_0 to v_t

$\boxed{\text{Remark : "length" of } P \text{ is } t}$

~~Cycle : Cycle is a path $C = (v_0\, e_1\, v_1\, e_2 \ldots e_t\, v_t)$, where $v_t = v_0$ and $v_0, v_1, \ldots, v_{t-1}$ are distinct~~

Cycle : $C = (v_0\, e_1\, v_1\, e_2 \ldots \ldots v_{t-1}\, e_t\, v_t)$, where

$e_i = \{v_{i-1}, v_i\}$, $\forall i \in \{1, 2, \ldots, t\}$,

$v_t = v_0$ and $v_0, v_1, \ldots, v_{t-1}$ are distinct

walk : $W = (v_0\, e_1\, v_1 \ldots \ldots e_t\, v_t)$, where

$e_i = \{v_{i-1}, v_i\}$, $\forall i \in \{1, 2, \ldots, t\}$

Tour : Tour is a walk with all the edges distinct

"Distance" in a graph

$\forall u, v \in G$, we define

$$d_G(u,v) = \begin{cases} 0, & \text{if } u = v \\ \text{length of the shortest path connecting } u \text{ and } v, & \text{o.w} \end{cases}$$

NOTE : $d_G(\ ,\ )$ is a metric space

i.e, ① $d_G(u,v) = 0 \iff u = v$

② $d_G(u,v) = d_G(v,u)$, $\forall u, v \in V(G)$

③ $d_G(u,v) \le d_G(u,w) + d_G(w,v)$, $\forall u, v, w \in V(G)$

$G = (V, E)$

Suppose $u \underset{G}{\sim} v$ if $d_G(u,v) < \infty$

We can check that $\underset{G}{\sim}$ is an equivalence relation on V

equivalently, $u \underset{G}{\sim} v$ if $\exists$ a walk b/w u and v of finite length

Lemma: $\underset{G}{\sim}$ forms an equivalence relation on $V(G)$

Lets look at $V(G)/\underset{G}{\sim}$ (the equivalence classes)

$= [u_0], [u_1], \ldots$

"connected components"

~~Defn: If $V(G)/\sim_G$~~

Defn: If $V(G)/\underset{G}{\sim}$ is a singleton set, then we say that G is "connected"

Adjacency matrix representation:

$$G = (\{1,2,\ldots,n\}, E)$$

$$A_G = (a_{ij})_{1 \le i,j \le n}$$

$$a_{ij} := \begin{cases} 1, & \text{if } \{i,j\} \in E \\ 0, & \text{o.w} \end{cases}$$

$\boxed{\text{NOTE: sometimes, we'll call } A_G \text{ as simply } A}$

(i) <u>NOTE</u> : $A^2 = A \cdot A$

$$= \{ a_{ij}^{(2)} \}_{1 \le i, j \le n}$$

$$a_{ij}^{(2)} = \sum_{l=1}^{n} a_{il} \, a_{lj}$$

= no. of walks of "length" 2

b/w i and j

- - - - - - - - - - - - -

$$A^K = A^{K-1} \cdot A$$

$$= \{ a_{ij}^{(K)} \}_{1 \le i, j \le n}$$

Lecture 8

Def^n (degree sequence):

Graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ s.t $\deg(v_1) \le \deg(v_2) \le \ldots \le \deg(v_n)$

$(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ is called a "degree sequence"

FACT1: $\sum_{i=1}^{n} \deg(v_i) = 2|E|$

COROLLARY: In any graph, we only have even no. of vertices with odd degree.

FACT2: In any graph, we'll always have atleast two distinct vertices u and v with $\deg(u) = \deg(v)$

Pf (Hint): use PHP

- -

motivation: $G = (V, E)$, $D_G = (\deg(v_1), \ldots, \underset{\overset{\shortparallel}{d}}{\deg(v_n)})$

$$d_i' := \begin{cases} \deg(v_i), & \text{if } i < n-d \\ \deg(v_i) - 1, & \text{if } n-d \le i \le n-1 \end{cases}$$

$D' = (d_1', \ldots, d_{n-1}')$

D_G is a valid degree sequence $\Leftrightarrow$ D' is a valid degree sequence

Thm: $D = (d_1, \dots, d_n)$ and $D' = (d_1', \dots, d_{n-1}')$

where:

$$d_i' := \begin{cases} d_i, & \text{if } i < n - d_n \\ d_i - 1, & \text{if } n - d_n \leq i \leq n-1 \end{cases}$$

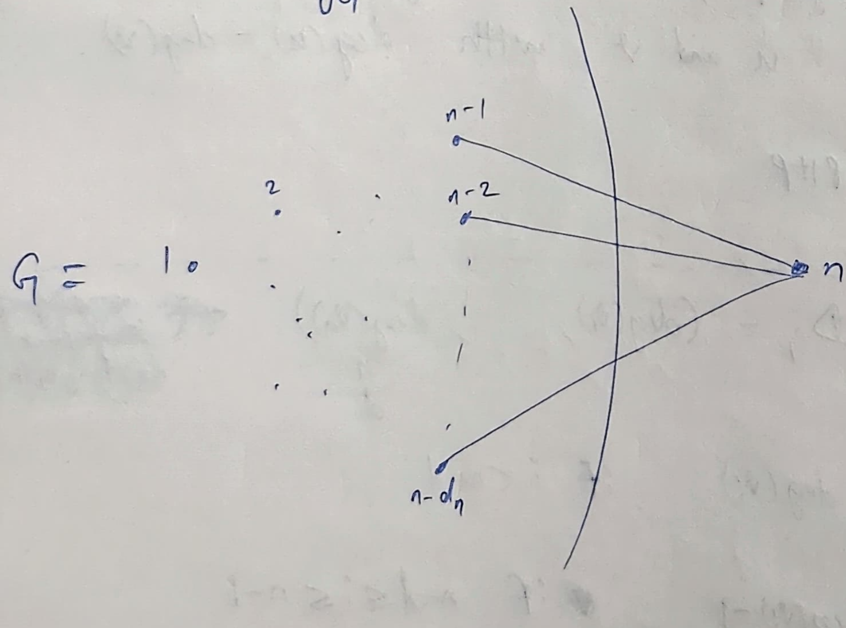
D is a "valid" degree sequence

$\iff D'$ is a valid degree sequence

pf: ($\Leftarrow$):

Suppose D' is a valid degree sequence on $n-1$ vertices

$\deg_{G'}(i) = d_i'$



Degree sequence of G is D ✓

($\Rightarrow$): [using "max-min trick"]

G: $\overset{\bullet}{1}$ $\overset{\bullet}{2}$ $\cdots$ $\overset{\bullet}{n-d_n-1}$ $\cdots$ $\overset{\bullet}{n}$

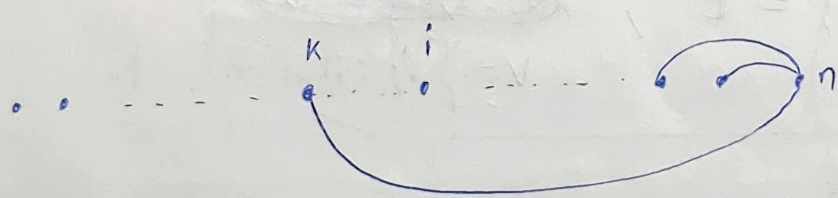
$I(G) := \max \{ i : i$ is NOT a neighbour of n in $G \}$

$\hat{G} := \underset{\substack{G \text{ with} \\ \text{deg. seq. } \Delta}}{\arg\min} \; I(G)$

Claim: $\hat{G}$ is the required graph

(i.e, $I(\hat{G}) = n - d_n - 1$)

pf: Suppose to the contrary let $I(\hat{G}) \geq n - d_n$



Complete the proof $\boxed{\text{Exercise}}$

Eulerian Graphs (EG)

Eulerian Tour

Thm: (Characterizing EG's):

$G = (V, E)$ is an Eulerian Graph (EG)

$\iff$ G is connected and degree of each vertex is even

$v_0 \, e_1 \, v_1 \, \ldots \, e_m \, v_m$, where $v_m = v_0$,

$e_1, e_2, \ldots, e_m$ are all distinct

$\{e_1, \ldots, e_m\} = E$, $V = \{v_i : i \in \{1, \ldots, m\}\}$

pf: ($\Rightarrow$):

Clearly, G has to be connected

Since vertex $\neq v_0$, we enter and also leave. Thus, their degree has to be even. $\therefore$ $\deg(v_0)$ will also be even ✓

$(\Leftarrow)$:

$v_0' \, e_1' \cdots\cdots e_\ell' \, v_\ell'$

Take a tour of max. length
(max. no. of edges in the tour)

claim 1

even

First thing to show is that $v_0' = v_\ell'$

$v_0' \, e_1' \cdots\cdots e_\ell' \, v_\ell'$ and $v_0' = v_\ell'$

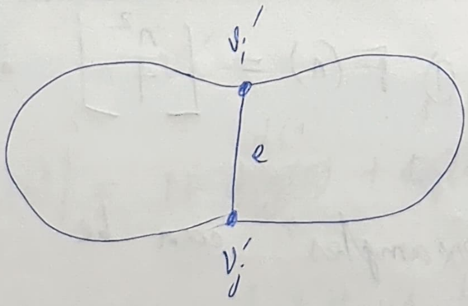
claim 2: $V = \{ v_i' : i \in \{0, 1, \ldots, \ell\} \}$

connectedness

If $E(T) = E$, then we are done

So, Suppose $E \setminus E(T) \neq \phi$

Let $e \in E \setminus E(T)$



claim: $E(T) = E$

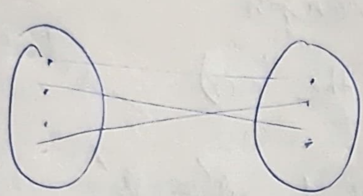
Complete the pf. exercise

$\square$

$G = (V, E)$

Suppose G does not have a Δ

Then, how many edges can G have ?



$n/2$ $\qquad$ $n/2$ $\qquad$ $\frac{n^2}{4}$

"Mantel's theorem"

"Kovari-Sos-Turan theorem"

NOTATIONS : $F_{\Delta}(n)$ = max. no. of edges in an n vertex graph with no Δ's

$G = (V = \{1, 2, \ldots, n\}, E)$, $|E| = F_{\Delta}(n)$

Thm (Mantel's theorem): ① $F(n) = \left\lfloor \frac{n^2}{4} \right\rfloor$

② Also, all "extremal examples" can be completely characterized

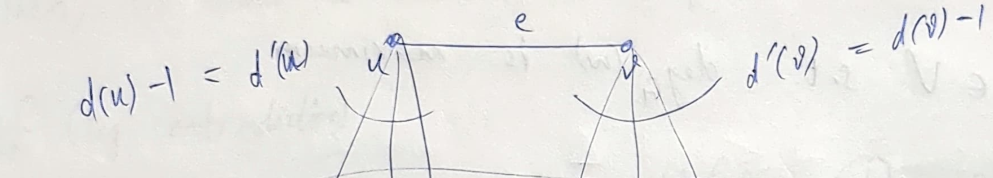
Pf. ①: (using induction on n):

Base case ($n = 1, 2, 3, 4$): Trivial ✓

Suppose the theorem is true for $n = n$

To show: the theorem holds for $n = n+2$

Let G be a Δ-free graph on $(n+2)$ vertices.



Let $G' = V \setminus \{u, v\}$ be the induced subgraph on V'

Now, $|E(G)| = 1 + \underbrace{\quad}_{d'(u)} + \underbrace{\quad}_{d'(v)} + |E(G')|$

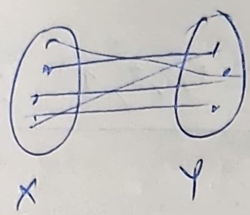
Apply induction hypothesis to complete the proof

$\boxed{\text{Exercise}}$

□

pf: 2 K_n

$K_{X,Y}$



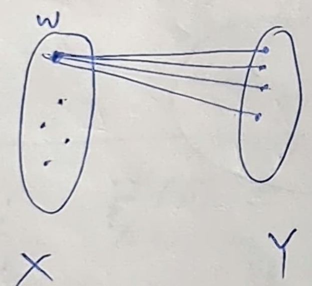
claim:

for any Δ-free graph $G=(V,E)$,

$\exists$ a two partition of $V = X \cup Y$

s.t $\deg_G(u) \leq \deg_{K_{X,Y}}(u)$, $\forall u \in V$

pf: $w \in V$ s.t $\deg_G(w)$ is maximum



$X = \{w\} \cup \{V \setminus N(w)\}$

neighbours of w

Complete the proof $\boxed{\text{Exercise}}$

$\square$

Lecture 9

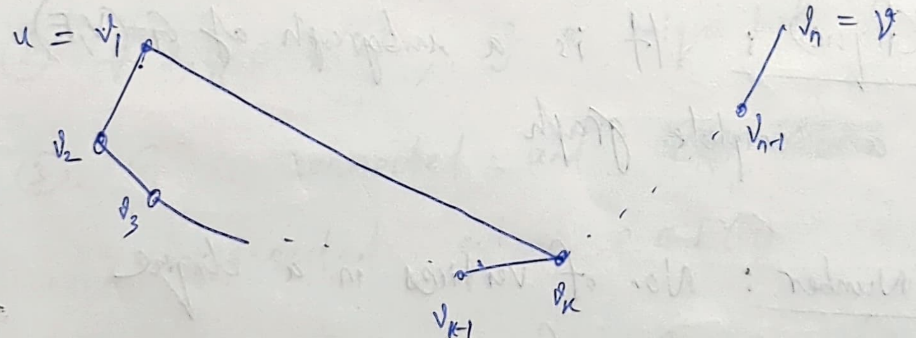
Hamiltonian Cycle (path) : in a graph G is a cycle (path) containing all vertices of G

Theorem : Let $G = (V, E)$, $|V| \geq 3$ and $\deg(v) \geq \frac{|V|}{2}$, $\forall v \in V$ $\Big\}$ — (*)

Then, G has a hamiltonian cycle

pf (by contradiction)

Suppose $\exists$ a graph G satisfying (*) s.t G is not a hamiltonian cycle



Complete the proof $\boxed{\text{Exercise}}$

K-coloring :

$G = (V, E)$, $f : V \to \{1, 2, \ldots, K\}$ s.t

$f(x) \neq f(y)$, $\forall \{x, y\} \in E$

Chromatic Number ($\chi(G)$) :

min. no. $\chi(G)$ s.t a coloring

$f : V \to \{1, 2, \ldots, \chi(G)\}$ exists

Exercise : $\chi(G) \leq 1 + \Delta$, where

$$\Delta = \max_{v \in V(G)} \deg(v)$$

Def^n (Clique) : It is a subgraph of $G = (V, E)$ which is a complete graph

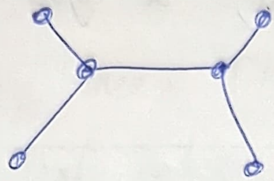
Clique Number : No. of vertices in a clique with maximum no. of vertices

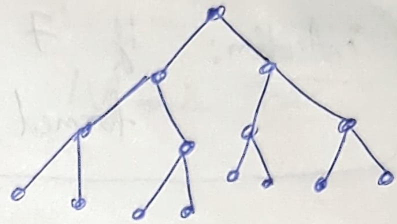
(denoted by $\omega(G)$)

Exercise : Prove/Disprove : $\omega(G) \leq \chi(G)$

Def^n (Tree): A connected acyclic graph

Examples:





("Binary Tree")

Def^n (Forest): A graph whose every connected component is a tree

Theorem: TFAE:

1. G is a tree
2. $\forall x, y \in V(G)$, $\exists$ a unique path from x to y
3. G is connected and $G-e$ is disconnected where $e \in E(G)$
4. G is acyclic and $G+e$ contains a cycle where $e \notin E(G)$
5. G is connected and has $|V|-1$ edges

pf: ① $\Rightarrow$ ② :

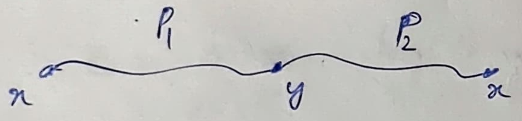
intuition: if $\exists$ 2 paths, a cycle is formed ($\Rightarrow\Leftarrow$)

Let $x, y \in V(G)$

Suppose $\exists$ 2 paths from x to y :

P_1 : $x = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \cdots \xrightarrow{e} y$

P_2 : $x = u_0 \xrightarrow{e_1'} u_1 \xrightarrow{e_2'} u_2 \rightarrow \cdots \rightarrow y$



② $\Rightarrow$ ③ :

③ $\Rightarrow$ ④ :

④ $\Rightarrow$ ① :

Exercise

Def^n (leaf nodes) : vertices with degree 1 in a tree

Lemma : Every tree $T = (V, E)$, $|V| \geq 2$ has atleast 2 leaf nodes

pf : Take the path of longest length :

$v_0 \; v_1 \; v_2 \; - - - \; v_t$

claim : $\deg(v_0) = 1$

pf : Suppose to the contrary, let $\deg(v_0) \geq 2$

Complete the proof $\boxed{\text{Exercise}}$

Exercise : Let v be a leaf node. Then :

T is a tree $\iff$ $T - v$ is a tree

Theorem (① $\iff$ ⑤ in the previous theorem) :

T is a tree $\iff$ T is connected and has $|V| - 1$ edges

pf : $(\Rightarrow)$

We'll prove this by induction on no. of vertices

~~Case~~ By induction hypothesis,

any tree T with $(n-1)$ vertices

has $(n-2)$ edges

Consider a tree T with n vertices

Remove a leaf node v from T

Look at $T - v$

Complete the proof $\boxed{\text{Exercise}}$

$(\Leftarrow)$:

Here also, we'll prove this using induction

on no. of vertices

By IH, any connected graph

with $(n-1)$ vertices having $(n-2)$ edges

is a tree.

To prove: G has n vertices and $(n-1)$ edges

$\implies G$ is a tree

By "Handshaking lemma",

Total degree in $G = 2n-2$

Complete the proof $\boxed{\text{Exercise}}$

HINT:

claim: $\exists u \in G(V)$

s.t $\deg(u) = 1$

and . . . . .

Exercise: $T = (V, E)$ is acyclic and has K edges

Then:

① $K = |V| - 1 \Rightarrow T$ is a tree

② $K < |V| - 1 \Rightarrow T$ has $(|V| - K)$ connected components

---

Def^n (Tree Isomorphism):

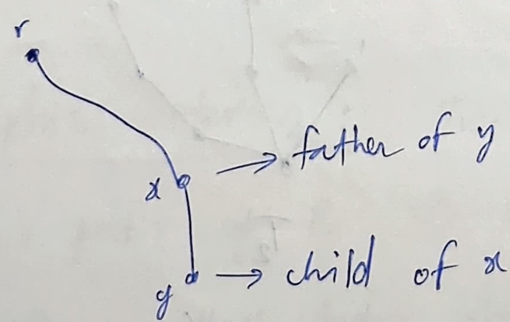
$T_1 \cong T_2$ if $\exists$ a bijective function $f : V(T_1) \to V(T_2)$

(i.e, if $\{x, y\} \in E(T_1) \iff \{f(x), f(y)\} \in E(T_2)$)

Def^n (rooted tree): A tree with a vertex which is root

[denoted by (T, r)]

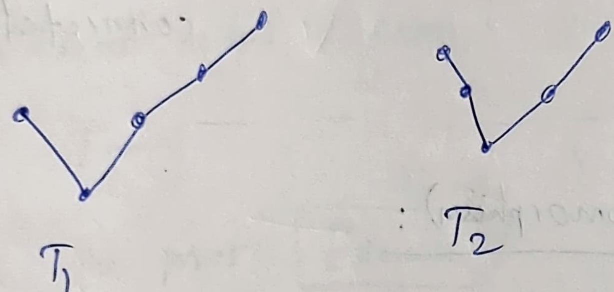
$\{x, y\} \in E$



Rooted Isomorphism $[(T_1, r_1) \cong' (T_2, r_2)]$:

$T_1 \cong T_2$ and $f(r_1) = r_2$

Example:

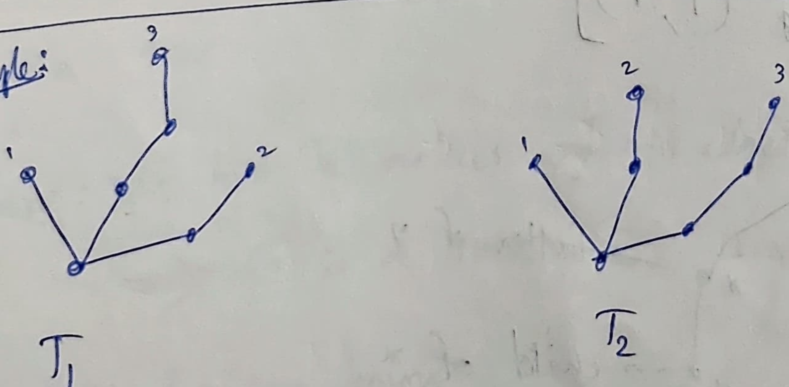


$T_1 \cong T_2$ but $T_1 \not\cong' T_2$

Planted Isomorphism $(T_1 \cong'' T_2)$:

Rooted Isomorphism + a drawing of T

Example:

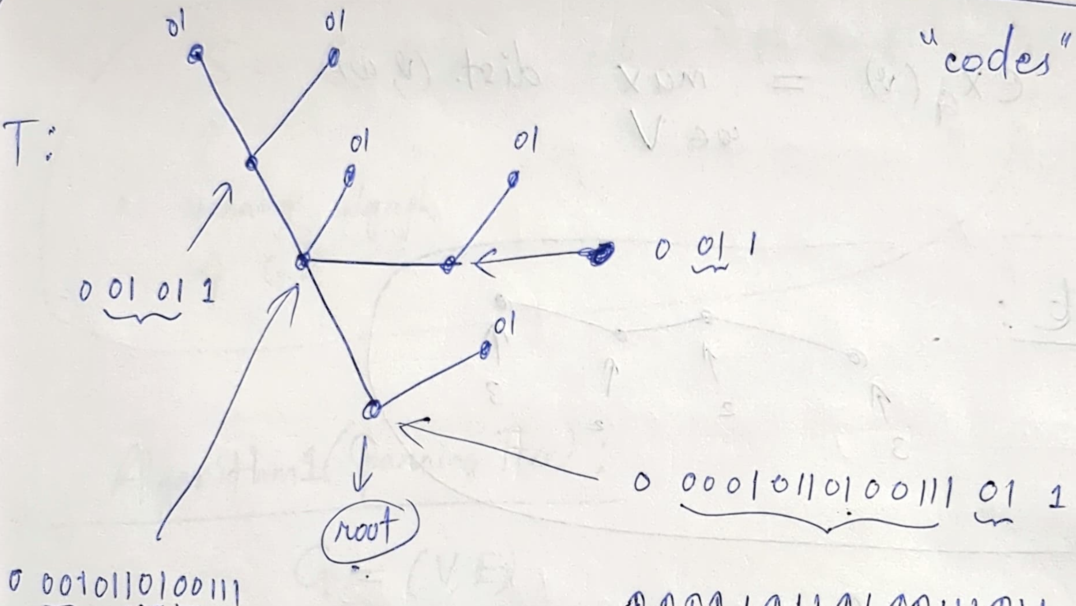


"linear ordering"

$T_1 \cong' T_2$, but $T_1 \not\cong'' T_2$

Remark: $\cong'' \Rightarrow \cong' \Rightarrow \cong$

(i.e, PI $\Rightarrow$ RI $\Rightarrow$ I)



↑↑↑↑↓↑↓↓↑↓↑↑↓↓↓↑↓↓

Can reconstruct T using this

Lexicographic Ordering: Two codes A and B s.t

① A is a prefix of B $\Rightarrow A \leq B$

② j is the smallest index where bits differ

$A_j < B_j \Rightarrow A \leq B$

$A_j > B_j \Rightarrow B \leq A$

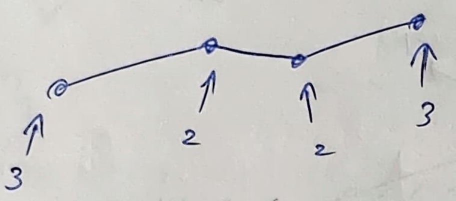
Def^n (Eccentricity):

for a vertex v of a graph G,

eccentricity of v in G is given by:

$$ex_G(v) = \max_{w \in V} \text{dist.}(v, w)$$

Example:



Def^n ("Center" of a graph):

A vertex with minimum eccentricity

NOTATION: $C(G) = \{\text{all centers of } G\}$

Lemma: T is a tree $\Rightarrow |C(T)| \leq 2$

Furthermore, if $C(T) = \{u, v\}$, then $\{u, v\} \in E(T)$

Def^n (Spanning Subgraph) :

A subgraph of a graph $G=(V,E)$ is called "spanning" if no. of vertices of the subgraph is equal to $|V|$

$S = (V, E')$ where $E' \subseteq E$

a spanning subgraph of G

Algorithm 1 (Spanning Tree) :

$G = (V, E)$

order the edges $e_1, e_2, \ldots, e_m$

$E_0 = \phi$

Calculate E_i from E_{i-1} :

$$E_i = \begin{cases} E_{i-1} \cup \{e_i\}, & \text{if } (V, E_{i-1} \cup \{e_i\}) \text{ has no cycle} \\ E_{i-1}, & \text{o.w} \end{cases}$$

Stop condition : If some E_i has $n-1$ edges or $i=m$

Correctness of Algorithm 1 :

If Algorithm 1 produces a graph T :

① If T has $(n-1)$ edges, then T is a spanning tree.

② If T has $< (n-1)$ edges, then G has $(n-k)$ connected components

Exercise : ② T has $(n-k)$ connected components

Consequence : Every connected graph has a spanning tree.

Lecture 10

$G = (V, E)$

Def^n 1 (k-connectedness) : Let $n \geq k+1$ and G be a graph on n vertices.

① G is said to be "vertex k-connected" if $\forall S \subseteq V$ with $|S| = k-1$, $G[V \setminus S]$ is connected

② G is said to be "edge k-connected" if $\forall F \subsetneq E$ with $|F| = k-1$, the subgraph $G' = (V, E \setminus F)$ is connected.

Remark : We'll be characterizing vertex 2-connected graphs.

Q^n: Let G be a connected graph. What happens (in terms of no. of connected components) when we remove a vertex from G and when we remove an edge from G?

NOTE: From now, we'll write vertex k-connected as just k-connected.

Theorem (Characterization of vertex 2-connected graphs):

A graph $G = (V, E)$ is 2-connected

$\iff \forall$ distinct vertices $u, v \in G$,

$\exists$ two vertex disjoint paths b/w u and v

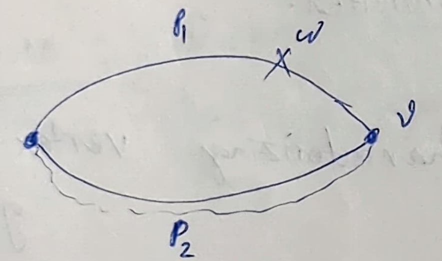
pf:

($\Leftarrow$):

Let $w \in V$. Consider $G' = G[V \setminus \{w\}]$

Take any two vertices $u, v \in G$

We know that $\exists$ two vertex disjoint paths P_1 and P_2 in G



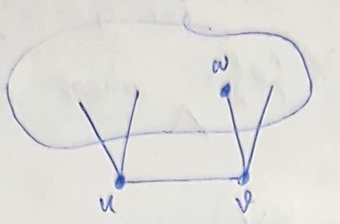
($\Rightarrow$):

First observe the following claim about 2-connected graphs.

Claim: ① Let G be 2-connected and "e" be an edge in G. Then, the subgraph $G \setminus \{e\}$ is connected.

② G cannot have a degree one vertex

$e = \{u, v\}$



We've to show that $\forall u \neq v$, $\exists$ a cycle C containing both u and v

Base Case ($d(u,v) = 1$, i.e, $\{u,v\} \in E$): From Claim, we know that $\exists$ a path P connecting u and v that doesn't use the edge $e = \{u, v\}$. Then, $C = P \cup \{e\}$ ✓

Induction hypothesis: If $d(w_1, w_2) = k-1$, then $\exists$ a cycle C that contains w_1 and w_2 ✓

Now, suppose $d(u, v) = k$

Complete the proof $\boxed{\text{Exercise}}$

$\square$

PTO →

GRAPH OPERATIONS

① **Edge deletions:** $G \setminus \{e\}$ (or $G - \{e\}$), where $e \in E$

② **Edge addition:** $G + e$, where $e \in \binom{V}{2} \setminus E$

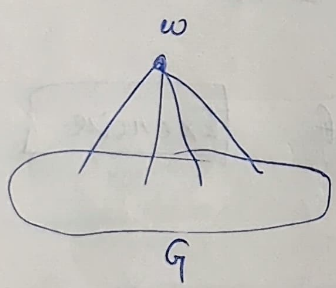
③ **Vertex ~~addition~~ deletion:** $G[V \setminus \{w\}]$, where $w \in V$

④ **Cone / Vertex addition:** for $w \notin V$,

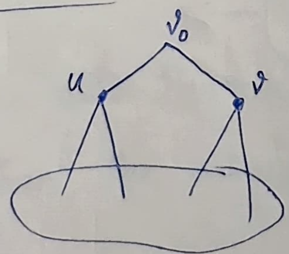
$G' = (V', E')$, where:

$$V' = V \cup \{w\}$$

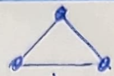
$$E' = E \cup \{\{w, u\} : u \in V\}$$



⑤ **Edge subdivision:**



Theorem:



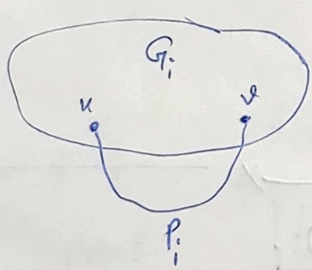
edge addition and edge subdivision

$$G \cong G'$$

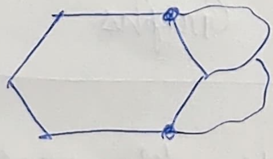
2-connected graph

In other words: Any 2-connected graph is isomorphic to

pf: Consider a sequence $G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_k = G$



$G_i \cup P_i = G_{i+1}$



G_0

Claim: ② If $G_i \neq G$, then $\exists$ a path P_i connecting two vertices of G and it is internally vertex disjoint from G_i

① Each G_i is 2-connected

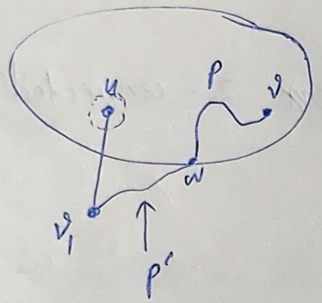
If $V_{G_i} = V_G$, we are done (check!)

So, assume $V_{G_i} \neq V_G$

Then, $\exists \{u, v_1\} \in E_G$ s.t $u \in G_i$

and $v_1 \in V_G \setminus V_{G_i}$

G_i :



$P_{i+1} = P' + \{u, v_1\}$

Complete the proof $\boxed{\text{Exercise}}$

$\square$

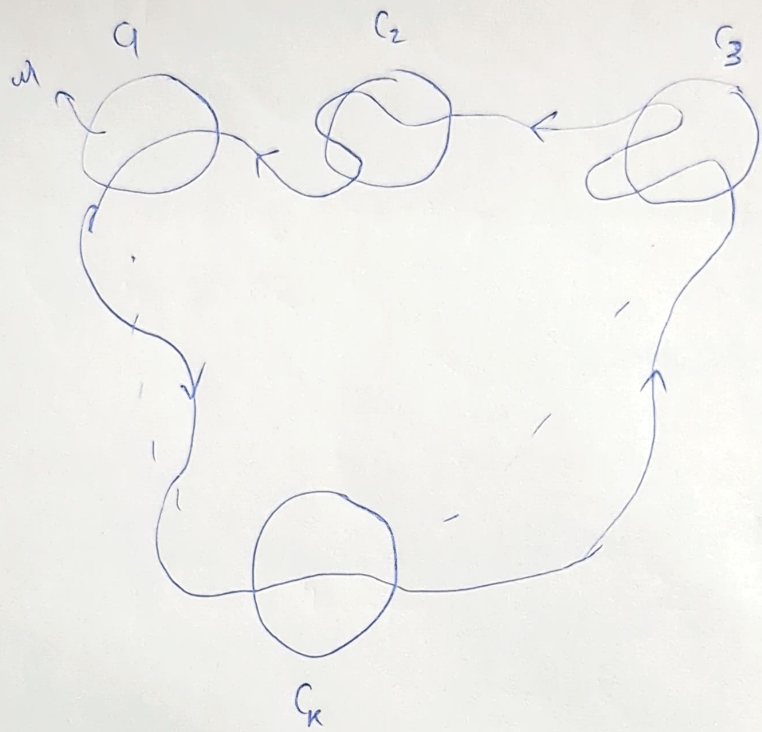
Recall (Lecture 9):

Hamiltonian Graphs

Theorem: Let a bipartite graph $G = (A \cup B, E)$ be Hamiltonian. Then, $|A| = |B|$

Lemma: Let $G' = (V', E')$ be a Hamiltonian graph and $S \subseteq V'$. Then, $G'[V' \setminus S]$ has at most $|S|$ connected components.

Pf:



Use "charging" idea to complete the proof $\boxed{\text{Exercise}}$

□

Lecture 11 [Lecture 9 continued]

Recall (Lecture 9):

Algo 1 (Spanning Tree)

$G = (V, E)$, $|V| = n$, $|E| = m$

Successively constructing $E_0, E_1, \ldots \subseteq E$:

Order the edges $e_1, e_2, \ldots, e_m$

$E_0 = \phi$

Given E_{i-1}:

$$E_i = \begin{cases} E_{i-1} \cup \{e_i\}, & \text{if } (V, E_{i-1} \cup \{e_i\}) \text{ is acyclic} \\ E_{i-1}, & \text{o.w} \end{cases}$$

Stop: $|E_i| = n-1$ for some i or $i = m$

Algo 2 (Spanning Tree)

$G = (V, E)$, $|V| = n$, $|E| = m$

Successively construct $V_0, V_1, \ldots \subseteq V$; $E_0, E_1, \ldots \subseteq E$:

$V_0 = \{v\}$, $E_0 = \{\phi\}$,

given V_{i-1}, E_{i-1}

find $x_i \in V_{i-1}$, $y_i \in V \setminus V_{i-1}$ s.t $\{x_i, y_i\} \in E$

$V_i = V_{i-1} \cup \{y_i\}$

$E_i = E_{i-1} \cup \{x_i, y_i\}$

Stop: No such edge $\{x_i, y_i\}$ can be found

Correctness of Algo 1:

Algo 1 gives a subgraph T of G

Claims $\left\{ \begin{array}{l} \text{i) If } |E(T)| = (n-1) \Rightarrow T \text{ is a spanning tree} \\ \text{ii) If } |E(T)| = k < (n-1) \Rightarrow G \text{ is disconnected and has } (n-k) \text{ connected components} \end{array} \right.$

Theorems (HW's in Lecture 9):

① T is acyclic and has $(n-1)$ edges $\Rightarrow$ T is a tree

② T is acyclic and has $< (n-1)$ edges $\Rightarrow$ T is disconnected and has $(n-K)$ connected components

Pf (Claim ii>)

T has $(n-k)$ connected components

Since T is a spanning sub graph of G, need to prove this first too

the no. of connected components in $G \leq n-k$

Complete the rest of the proof $\boxed{\text{Exercise}}$

Correctness of Algo 2 :

Algo 2 gives a subgraph T

Claims

i> If $|E(T)| = (n-1) \Rightarrow$ T is a spanning tree

ii> If $|E(T)| = k < (n-1)$ edges $\Rightarrow$ G is disconnected and T is the spanning tree of the connected component of G with vertex v

Complete the proof $\boxed{\text{Exercise}}$

Q^n: Which algorithm (Algo 1/Algo 2) gives us more information about the graph ?

Sol^n: Both algo's give the same amount of information (HINT: Think about arrays in case of algo 2)

Also, note that the running times of algo 1 and algo 2 are the same

---

COROLLARY

Every connected graph has a spanning tree

Pf: Exercise!

Prim's and Kruskal algorithms:

Defn (weighted graph): $G = (V, E)$ with a function $w : E \longrightarrow \mathbb{R}^+ \cup \{0\}$

Qn: G is a connected weighted graph. Find a connected spanning subgraph T with minimum

$$w(T) := \sum_{e \in E(T)} w(e)$$

Defn (Minimum spanning tree [MST]): A spanning tree with minimum weight in a weighted graph G

Lets modify "Algo 1" for MST :

discussed previously for spanning tree

just change this line :

order the edges $e_1, e_2, \ldots, e_m$ s.t

$$w(e_1) \leq w(e_2) \leq \cdots \leq w(e_m)$$

This modified version of Algo1 is known as

"Kruskal's algorithm"

Now, lets modify "Algo 2" for MST :

just change this line :

find $x_i \in V_{i-1}$, $y_i \in V \setminus V_{i-1}$ s.t $\{x_i, y_i\} \in E'$,
~~take the edge~~ where E' is the edge with minimum weight from the set $\{\{x_i, y_i\} \in E : x_i \in V_{i-1} \text{ and } y_i \notin V_{i-1}\}$

This modified version of Algo 2 is called

"Prim's algorithm"

<u>Lemma</u>: $G = (V, E)$, G' and G'' are spanning subgraphs of G s.t. $G' = (V, E')$ is disconnected, $G'' = (V, E'')$ is acyclic, and $|E''| > |E'|$

Then, $\exists\, e \in E''$ which connects two connected components of G'

<u>pf</u>: G' has s connected components,

$$|E'| \geq |V| - s$$

$$|E'' \cap E'| \leq |V| - s$$

Complete the proof $\boxed{\text{Exercise}}$

<u>Correctness of Kruskal's algorithm:</u>

Kruskal's gives a spanning tree T

Let T' be any other spanning tree

<u>To show</u>: $w(T) \leq w(T')$

$T : e_1, e_2, \dots, e_{n-1}$

$T' : e_1', e_2', \dots, e_{n-1}'$

ordered by weight, $w(e_i) \leq w(e_i')$, $\forall i$

Suppose to the contrary, let $w(e_i) > w(e_i')$ for some i (take i to be the smallest such index)

$E' = \{e_1, e_2, \dots, e_{i-1}\}$

$E'' = \{e_1', e_2', \dots, e_{i-1}', e_i'\}$

$|E''| > |E'|$

Thus, by Lemma, $\exists e \in E''$ which does NOT form a cycle with edges in E'

$w(e) \leq w(e_i') < w(e_i)$

So, the algorithm chooses edge e_i over e ($\Rightarrow \Leftarrow$)

Thus, Kruskal's algo is correct ✓

Correctness of Prim's algorithm :

Prim's algo gives us a spanning tree T,

$E(T) = \{e_1, e_2, \dots, e_{n-1}\}$

Let T be an MST

Let T' be an MST s.t $e_1, e_2, \ldots, e_k \in E(T')$

and $e_{k+1} \notin E(T')$, where k is the largest

from Prim's algo : $w(e_{k+1}) \leq w(e)$

Now, consider $T'' = (T' + e_{k+1}) - e$

(Check that T'' is a spanning tree)

$$w(T'') = w(T') + w(e_{k+1}) - w(e)$$
$$\leq w(T')$$

$$\Rightarrow w(T'') = w(T') \xrightarrow{\because} \boxed{T' \text{ is an MST}}$$

Using this, we can arrive at a contradiction $\boxed{\text{check!}}$

Thus, Prim's algo is correct ✓

Lecture 12

midsem format (Total marks = 60), (Time = 2 hrs.)

Group A (30 marks)	Group B (30 marks)
(iii) 4 questions	(iii) 4 questions
(ii) Exact theorems covered in class	(ii) new problems
(i) No step marking ✗	(i) step marking ✓

Hall's theorem

Konig-E thm.

Dilworth's thm.

→ all Equivalent theorems

Important

Matching in Bipartite Graphs :

__Def^n :__ $G = (V, E)$

$$M_t = \{e_1, e_2, \dots, e_t\} \in E$$

- M_t is a "matching" if elements of M_t are pairwise disjoint

-

~~$G = (A \cup B, E)$ where A, B are both independent sets~~

Def^n (perfect matching) :

A matching M_t is a "perfect matching" if

$$e_1 \cup e_2 \cup \dots \cup e_t = V$$

Def^n (Edge cover of a set) :

$\{e_1, \dots, e_s\} = S \subseteq E$ is an edge cover of $T \subseteq V$ if

$$T \subseteq \bigcup_{i=1}^{s} e_i$$

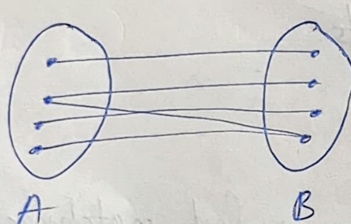
Def^n (d-matching) : $S \subseteq E$ is a "d-matching" if max. degree of a vertex in graph $G' = (V, S)$ is at most d

Q^n: $\underbrace{P_1, P_2, \ldots, P_{2n}}_{\text{points}} \in \mathbb{R}^2$

Is there a way of pairing two points and connecting a line b/w them s.t no lines intersect?

~~Sol^n: HINT~~

Q^n: $G = (A \cup B, E)$, where A, B are both independent sets in G



Hall

Q^n: $\begin{array}{cccc} a_1 & a_2 & & a_T \\ \uparrow & \uparrow & & \uparrow \\ A_1, & A_2, & \ldots, & A_T \end{array}$, where

$A_i \subseteq [n]$

s.t $a_i \neq a_j$, $\forall i \neq j$

"SDR" (System of Distinct Representative)

Thm (Hall's theorem) :

$G = (A \cup B, E)$ has a matching that covers A $\iff$ $|N(S)| \geq |S|$, $\forall S \subseteq A$

where $N(S) = \bigcup_{u_i \in S} N(u_i)$

Pf :

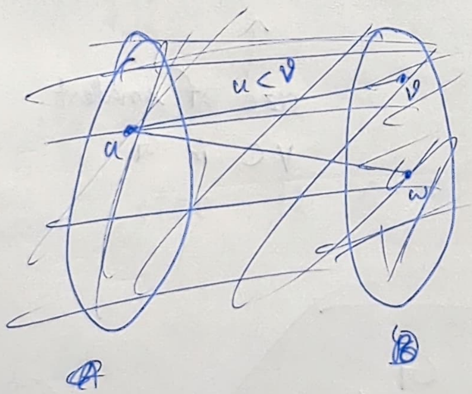
$(\Rightarrow)$:

Easy (Check!)

$(\Leftarrow)$:

$S \subseteq V$ is an independent set

$\iff$ $\overline{S}$ is a VC



Observations : ① Length of a chain can be atmost 2

② B is an antichain

③ B is a largest antichain

~~Suppose ∃ an antichain D of size |B|+1~~

$\exists$ a chain decomposition (by Dilworth's thm.)

and ① with $|A|$ many chains of length 2

and $|B| - |A|$ many chains of length 1

Complete the proof $\boxed{\text{Exercise}}$

$\square$

Theorem (Konig's theorem):

$G = (A \cup B, E)$, where A, B are indep. sets in G

Then,

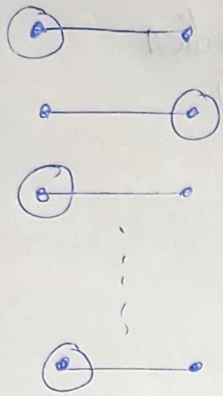
$$\underbrace{\nu(G)}_{\text{max. size matching in } G} = \underbrace{\tau(G)}_{\text{size of smallest VC in } G}$$

pf: for all graphs G,

$$\underset{\underset{M}{\shortparallel}}{\nu(G)} \leq \underset{\underset{k}{\shortparallel}}{\tau(G)}$$ — ⓘ

Let $M = \nu(G)$, $k = \tau(G)$

and n = no. of vertices in G



$$\underbrace{(n-2M)}_{\text{singleton}} + \underbrace{M}_{\text{length 2 chains}} \leftarrow \text{length of chain decomposition}$$

$$= n - M$$

i.e, length of chain decomposition $= n - M$

Since the smallest VC size is k, the largest antichain size is $(n-k)$

$$\Rightarrow (n-k) \geq (n-M)$$

$$\Rightarrow k \leq M \quad \text{— (II)}$$

(I), (II) $\Rightarrow k = M$

i.e, $\tau(G) = \nu(G)$ ✓

□

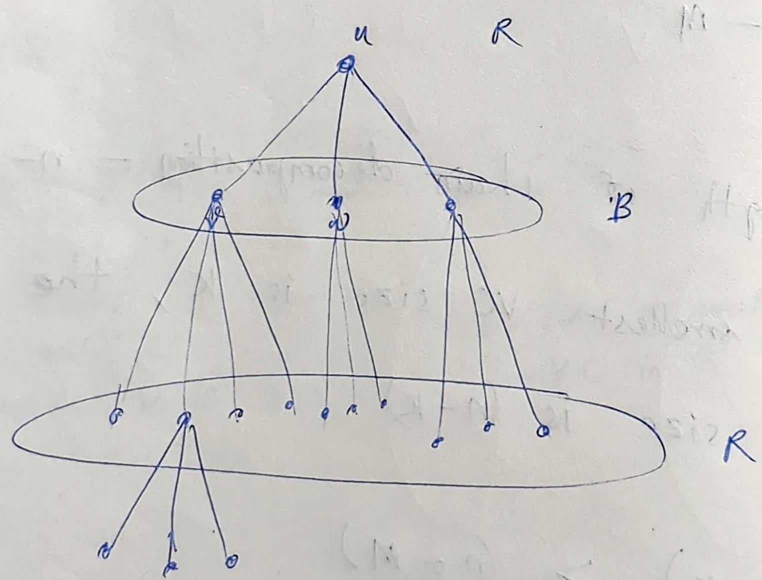
Theorem (Odd cycle characterization of Bipartite graphs)

---

pf: Assume G is connected

Suppose G doesn't have an odd cycle

and G is bipartite



Lecture 13

Midsem discussion

Group B :

BQ 1) G has n vertices, m edges and T Δ's

To show : $T \geq \frac{m}{3n}(4m - n^2)$

Pf:

$\deg(i) = d_i$, $\deg(j) = d_j$

T_{ij} = no. of Δ's containing edge ij

Then, $T_{ij} \geq (d_i - 1) + (d_j - 1) - (n-2)$ → Check!

$= d_i + d_j - n$

Now, $\sum_{(i,j)\in E} T_{ij} = 3T$

$\Rightarrow 3T \geq \sum_{(i,j)\in E} (d_i + d_j - n)$

NOTE:

$$\sum_{(i,j)\in E} (d_i + d_j) - mn = \sum_{i\in V} d_i^2 - mn$$

$$\geq \frac{\left(\sum d_i\right)^2}{n} - mn \longrightarrow \text{using CS inequality}$$

$$= \frac{4m^2}{n} - mn = \frac{m}{n}(4m - n^2)$$

$\Rightarrow \boxed{T \geq \frac{m}{3n}(4m - n^2)}$ ✓

BQ2) $A = (A_{ij})_{1\le i,j\le n}$ s.t. $A_{ij} \in \{0,1\}$, $\forall\, i,j$

Sum of entries of any row/column of A is k

<u>To show:</u> A can be written as a sum of k permutation matrices

<u>sol^n:</u> sum of entries of any row/column of A is k

i.e, $A\underset{\sim}{1} = k\,\underset{\sim}{1}$, where $\underset{\sim}{1} = \begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}_{n\times 1}$

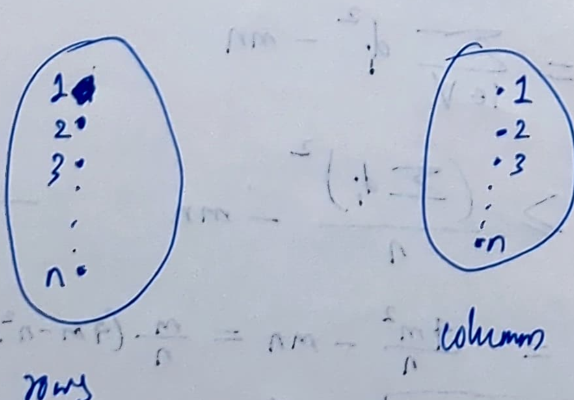
and $\underset{\sim}{1}^T A = k\,\underset{\sim}{1}^T$

We'll use induction on k now

<u>Base case:</u>

Suppose true for $(k-1)$

$$A_{n\times n} = \begin{array}{c|c} & \text{col } 1 \;\; \text{col } 2 \; \cdots \; \text{col } n \\ \hline \begin{array}{c}\text{row } 1\\ \text{row } 2\\ \vdots\\ \text{row } n\end{array} & \left[\begin{array}{ccc} & & \\ & & \\ & & \\ & & \end{array}\right]_{n\times n}\end{array}$$



→ bipartite graph

By Hall's theorem, $\exists$ a perfect matching

Example

$$A = \begin{pmatrix} 0 & 1 & \boxed{1} \\ \boxed{1} & 0 & 1 \\ 1 & \boxed{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

permutation matrix embedded in A

use induction hypothesis here

Exercise!

BQ 3) $A_1, A_2, \ldots, A_n \longrightarrow$ distinct subsets of $\mathbb{N}$

To show: $\exists$ a subset $X \subseteq \mathbb{N}$ with: $|X| \leq n-1$

s.t $A_i \cap X \neq A_j \cap X$, $\forall i \neq j$

Sol^n: Use (strong) induction on n

Base case $(n=2)$ ✓

Suppose true for $2, 3, \ldots, n-1$

PTO $\longrightarrow$

$A_1, A_2, \ldots, A_n$

$\exists x \in \bigcup_{i \in [n]} A_i \setminus \bigcap_{i=1}^{n} A_i$

divide the collection into two sets

$\{A_1, A_2, \ldots, A_j\}$, $\{A_{j+1}, \ldots, A_n\}$

contains x doesn't contain x

size k_1 size k_2

$k_1, k_2 \leq n-1$

Complete the proof Exercise

BQ4) To prove: $\forall$ posets $(X, \leq)$, $\exists$ an embedding into the ordered set $(2^X, \subseteq)$

Pf:

Lecture 14

Generating Functions

$a_0, a_1, a_2, \ldots, a_r$

$g(x) = a_0 + a_1 x + \cdots + a_r x^r$

where a_r = no. of ways in which r objects are selected from n objects

$g(x) = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{r} x^r$

Q) Find the generating function for a_r, the no. of ways to select r balls from 3 green, 3 blue, 3 red balls

Sol^n: $e_1 \to$ green, $e_2 \to$ blue, $e_3 \to$ red

$e_1 + e_2 + e_3 = r$

$x^{e_1} \, x^{e_2} \, x^{e_3}$ $\quad (x^0 + x^1 + x^2 + x^3)^3$

$e_1 + e_2 + e_3 = r$

Q) Use a generating function to model the problem of counting all selection of 6 objects chosen from 3 types of objects with repitition upto 4 objects of each type

$$(1+x+x^2+\dots+x^4)^3$$

find the coefficient of x^6 here

=

Polynomial Expansions :

① $\dfrac{1-x^{n+1}}{1-x} = 1+x+x^2+\dots+x^n$

② $\dfrac{1}{1-x} = (1-x)^{-1} = 1+x+x^2+\dots$

③ $(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{n}x^n$

④ $(1-x^n)^n = 1-\binom{n}{1}x^n+\binom{n}{2}x^{2n}-\dots$

$\dots+(-1)^n$

⑤ $\dfrac{1}{(1-x)^n} = (1-x)^{-n} = 1+\binom{1+n-1}{1}x$

$+\binom{2+n-1}{2}x^2+\dots$

$+\binom{r+n-1}{r}x^r+\dots$

Note: If $h(x) = f(x)\,g(x)$, where $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$, then

$$h(x) = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \cdots$$

Then, coefficient of $x^r = a_r b_0 + a_{r-1} b_1 + a_{r-2} b_2 + \cdots + a_0 b_r$

Q) Use generating functions ~~defined~~ to find the no. of ways to collect ₹ 15 from 20 distinct people if each of the first 19 people can give a rupee and the 20th person gives either 0, 1 or 5

Sol^n^:

$$(1+x)^{19}\,(1+x+x^5) \leftarrow \text{HINT}$$

Q) Verify the binomial identity:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

Sol^n^:

$$(1+x)^{2n} = (1+x)^n\,(1+x)^n$$

$$a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}$$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

Q) find the generating function for a_r, the no. of ways to express r as a sum of distinct positive integers

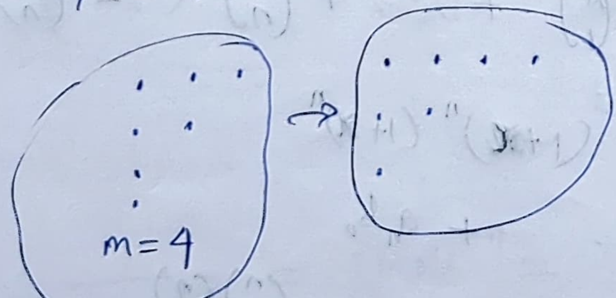
sol^n: $(1+x)(1+x^2)(1+x^3)\ldots(1+x^{r-1})(1+x^r)\ldots$

coefficient of x^r in

Q) Show that the no. of partitions of an integer r as a sum of m positive integers is equal to the no. of partitions of r as a sum of +ve integers, the largest of which is m

sol^n: "Ferrer's diagram"

$7 = 1+1+2+3$



Exponential Generating function

Q) a_r, $g(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots + a_r \frac{x^r}{r!} + \cdots$

Find the exponential GF of a_r, where a_r = no. of r arrangements without repitition of n objects.

Sol^n: $(1+x)^n \rightarrow \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$

no. of arrangements $\frac{n!}{(n-r)!}$

find the coefficient of $\frac{x^r}{r!}$ in $(1+x)^n$

$A(x) = \sum a_n x^n$, $B(x) = \sum b_n x^n$, $C(x) = \sum c_n x^n$

① If $b_n = d \cdot a_n$, then $B(x) = d\,A(x)$
(Here, d is a constant)

② If $c_n = \sum_{i=0}^{n} a_i\, b_{n-i}$, then $C(x) = A(x)\,B(x)$

③ If $b_n = a_{n-k}$ except $b_i = 0$ for $i < k$,
then $B(x) = x^k A(x)$

Q> Find a generating function $f(x)$ with

$a_r = 2r^2$

Sol^n: $(1-x)^{-1} = 1+x+x^2+ \cdots$

$$\frac{x \times 1}{(1-x)^2} = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x\,(1+2x+3x^2+\cdots)$$

$$= x+2x^2+3x^3+\cdots$$

$$x \times \frac{d}{dx}\frac{x}{(1-x)^2} = x\left[x \cdot \frac{1}{(1-x)^3} + \frac{1}{(1-x)^2}\right]$$

$$= x\left[1+2\cdot 2x+3\cdot 3x+\cdots\right]$$

~~Multiply by 2~~

Exercise!

Theorem: If $h(x)$ is a generating function where a_r is the coefficient of x^r, then

$$h^*(x) = \frac{h(x)}{1-x}$$ is a generating function of the sum of the a_r's

$$h^*(x) = a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + \cdots + \left(\sum_{j=0}^{r} a_j\right)\cdot x^r + \cdots$$

$$h^*(x) = f(x)\,h(x),$$

$$f(x) = \frac{1}{1-x}$$

Lecture 15

Recurrence relations

sequence $\{a_n\}_{n \geq 1}$ $\quad a_n : \mathbb{N} \to \mathbb{R}$

$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_1)$$

1) <u>computing factorial of n:</u>

$$a_n = n\, a_{n-1}, \quad a_0 = 1$$

2) <u>Dividing the plane by lines:</u>

Given n lines in a plane s.t any 2 are intersecting and any 3 are not intersecting at a common point

To find: How many regions are there in the plane?

$n = 0$, $a_0 = 1$

$n = 1$, $a_1 = 2$

$n = 2$, $a_2 = 4$

$n = 3$, $a_3 = 7$

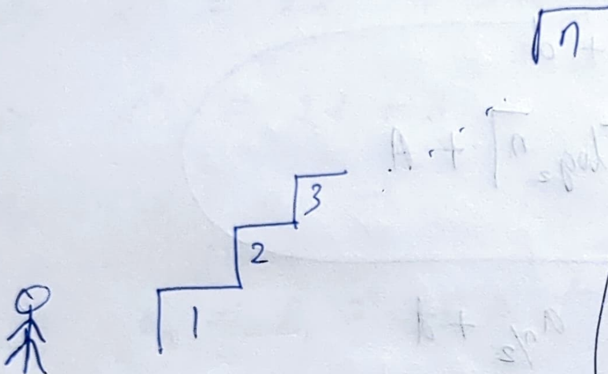
Given n, what is a_n?

$a_1, a_2, \dots, a_{n-1}$ are given

$$a_n = a_{n-1} + n \;;\; n \geq 1$$

$$a_0 = 1$$

3) Climbing Stairs problem :



$$a_1 = 1$$

$$a_2 = 2$$

$$a_3 = 3$$

$$\vdots$$

$$a_n = a_{n-1} + a_{n-2}$$

" Fibonacci "

---

① Divide and Conquer relation :

$$a_n = c\, a_{n/2} + f(n)$$

c	$f(n)$	a_n
$c = 1$	d	$d\lceil \log_2 n \rceil + A$
$c = 2$	d	$An - d$
$c > 2$	dn	$A\, n^{\log_2 c} + \left(\frac{2d}{2-c}\right) n$

$c = 2$	dn	$dn\,(\lceil \log_2 n \rceil + A)$

d is a constant, A is a constant you need to calculate from initial condition

We'll see one proof:

$$a_n = a_{n/2} + d$$

Then, $a_n = d \lceil \log_2 n \rceil + A$

Pf: $a_n = a_{n/2} + d$

$$= \left(d \lceil \log_2 n/2 \rceil + A\right) + d$$

$$= \left(d \lceil \log_2 n - 1 \rceil + A\right) + d$$

$$= d \lceil \log_2 n \rceil \cancel{- d} + A \cancel{+ d}$$

② Linear Recurrence relations:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r}$$

General technique: Take $a_n = \alpha^n$, $\forall n \geq 1$

We get: $\alpha^n = c_1 \alpha^{n-1} + c_2 \alpha^{n-2} + \cdots + c_r \alpha^{n-r}$

$$\Rightarrow \alpha^r - c_1 \alpha^{r-1} - c_2 \alpha^{r-2} - \cdots - c_r = 0$$

"characteristic eqn"

Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be the roots of the above eq^n. General sol^n will be

$$a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \dots + A_r \alpha_r^n$$

Example 1: $a_n = 2a_{n-1} + 3a_{n-2}$; $a_0 = a_1 = 1$

$$\Rightarrow \alpha^n = 2\alpha^{n-1} + 3\alpha^{n-2}$$

$$\Rightarrow \alpha^2 = 2\alpha + 3$$

$$\Rightarrow \alpha = 3, -1$$

Thus, $a_n = A_1 3^n + A_2 (-1)^n$

from $a_0 = a_1 = 1$, we get that $A_1 = A_2 = \frac{1}{2}$

Example 2: $a_n = a_{n-1} + a_{n-2}$; $a_0 = a_1 = 1$

$$\Rightarrow a_n = A_1 \left(\frac{\sqrt{5}+1}{2}\right)^n + A_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where $A_1 = \left(\frac{1+\sqrt{5}}{2}\right)\frac{1}{\sqrt{5}}$, $A_2 = \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$

"Fibonacci"

③ Inhomogeneous Recurrence relation :

$$a_n = c\, a_{n-1} + f(n)$$

Solve $a_n = c\, a_{n-1}$, $a_n = Ac^n$

get any sol^n of $a_n = c\, a_{n-1} + f(n) \rightarrow$ "particular sol^n"

"general sol^n": $a_n = Ac^n + p(n)$

$f(n)$	particular sol^n
d, $(c \neq 1)$	B
dn	$B_1 n + B_0$
dn^2	$B_2 n^2 + B_1 n + B_0$
ed^n	Bd^n

$\rightarrow$ if $c = d$, $p(n) = Bnd^n$

Example : $a_n = 2a_{n-1} + 1$ / $a_1 = 1$

sol^n of $a_n = 2a_{n-1} \longrightarrow a_n = A2^n$

particular sol^n, take $f(n) = B$

general sol^n $a_n = A2^n + B$

$B = 2B + 1 \Rightarrow B = -1$

"Tower of Hanoi"

Example : $a_n = 3a_{n-1} - 4n + 3\times 2^n$, $a_1 = 8$

Solve $a_n = 3a_{n-1} \Rightarrow a_n = A\, 3^n$,

Solve $a_n = 3a_{n-1} - 4n$ (for particular sol^n)

$$p(n) = B_1 n + B_0$$

$$\Rightarrow B_1 n + B_0 = 3\left[B_1(n-1) + B_0\right] - 4n$$

$$= (3B_1 - 4)n + 3(B_0 - B_1)$$

Comparing coefficients, we have:

$$3B_1 - 4 = B_1 \quad , \quad 3(B_0 - B_1) = B_0$$

$$\Rightarrow \boxed{B_1 = 2 \quad , \quad B_0 = 3}$$

particular sol^n for $\left.\begin{array}{l} a_n = 3a_{n-1} + 3\times 2^n \\ p(n) = B\,2^n \end{array}\right\} \Rightarrow \boxed{B = -6}$

Thus, general sol^n : $a_n = A\,3^n + (B_1 n + B_0) - B\,2^n$

$$= A\,3^n + (2n+3) - 6\times 2^n$$

Find A using $a_1 = 8$, $A = 5$

x ——————— x

~~Example : $a_n = c\, a_{n-1}$~~

Recall : coefficient of x^r in $\frac{1}{(1-x)^n}$

is : $\binom{r+n-1}{r}$

"negative binomial"

Given : $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_0)$

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$$

→ come up with a closed form of $g(x)$

→ find coefficient of x^n in $g(x)$

Example (Recall: Dividing the plane) :

$$a_n = a_{n-1} + n \ ; \quad n \geq 1$$

$$a_0 = 1$$

$$g(x) = a_0 + a_1 x + a_2 x + \ldots\ldots$$

$$g(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} (a_{n-1} + n)\, x^n$$

$$= \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} n x^n$$

$$= x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + (x + 2x^2 + 3x^3 + \cdots)$$

$$= x \sum_{m=0}^{\infty} a_m x^m + x(1 + 2x + 3x^2 + \cdots)$$

$$= x\, g(x) + \frac{x}{(1-x)^2}$$

$$\Rightarrow g(x) - 1 = x\, g(x) + \frac{x}{(1-x)^2}$$

$$\Rightarrow g(x) = \frac{1}{1-x} + \frac{x}{(1-x)^3}$$

coefficient of x^n in $\left(\frac{1}{1-x} + \frac{x}{(1-x)^3}\right)$

$= 1 +$ coefficient of x^{n-1} in $\frac{1}{(1-x)^3}$

$$= 1 + \binom{n+1}{2}$$

Example (Fibonacci) :

$$a_n = a_{n-1} + a_{n-2} \; ; \quad n \geq 1$$

$$a_0 = a_1 = 1$$

$$g(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} \left[a_{n-1} x^n + a_{n-2} x^n \right]$$

$$= x \sum_{m=1}^{\infty} a_m x^m + x^2 \sum_{m=0}^{\infty} a_m x^m$$

$$= x\left(g(x) - a_0\right) + x^2 g(x)$$

$$\Rightarrow g(x) = \frac{1}{1-x-x^2}$$

$$g(x) = \frac{1}{(1-\alpha_1 x)(1-\alpha_2 x)} \; ; \quad \alpha_1 = \frac{1+\sqrt{5}}{2}, \quad \alpha_2 = \frac{1-\sqrt{5}}{2}$$

$$= \frac{A}{(1-\alpha_1 x)} + \frac{B}{(1-\alpha_2 x)}$$

" partial fraction "

$$\Rightarrow 1 = A(1-\alpha_2 x) + B(1-\alpha_1 x)$$

Take $x = \frac{1}{\alpha_2}$ $\Rightarrow B = \dfrac{1}{1-\frac{\alpha_1}{\alpha_2}}$

$$= \frac{1}{1-\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)} = \frac{-\alpha_2}{\sqrt{5}}$$

$x = \frac{1}{\alpha_1}$ $\Rightarrow A = \dfrac{\alpha_1}{\sqrt{5}}$

Lecture 16

Mathematical programming :

- Linear programs (LP)
- Integer linear programs (ILP)
- Strong Duality of LP (statement only)
- Results from "Combinatorial convex geometry"

Integer Linear Programs (ILP) :

NOTATIONS :

$x \in \mathbb{R}^n$

$\| $

$(x_1, x_2, \dots, x_n)$ where $x_i \in \mathbb{R}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1}$$

$\bar{x} \in \mathbb{R}^n \longrightarrow$ particular sol^n

$$G = (V, E)$$

$$V = [n]$$

want to compute the size of the smallest vertex cover

assign variables to the vertices of G:

$$x_1, x_2, \ldots, x_i, \ldots, x_n \in \{0,1\}$$

vertex 1   vertex i   vertex n

ILP for VC:

$$\text{min. } \sum_{i=1}^{n} x_i$$

subject to $x_i \in \{0,1\}, \quad \forall i \in [n],$

$$x_i + x_j \geq 1, \quad \forall \{i,j\} \in E$$

M1

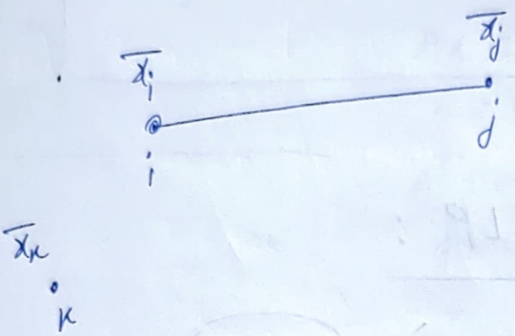
Optimal sol^n vs Optimal value:

$$Opt_{M1}$$

$$\sum_{i=1}^{n} (\bar{x}_{M1})_i = Opt_{M1}$$

Linear programs

Fractional Vertex Cover (VC) :



$\bar{x}_i \in [0, 1]$

ILP for FVC :

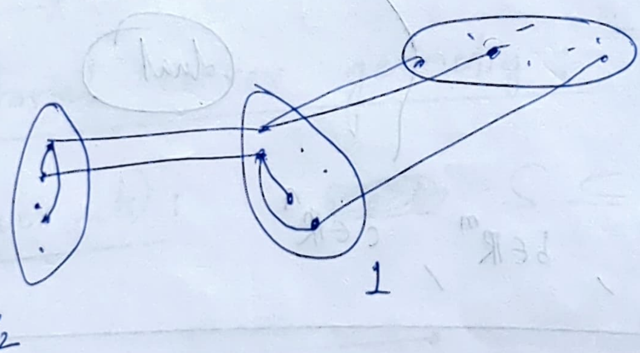
$$\left.\begin{aligned} &\min \sum_{i=1}^{n} x_i \\ &\text{s.t. } x_i \in [0,1], \ \forall i \in [n] \\ &\quad x_i + x_j \geq 1, \ \forall \{i,j\} \in E \end{aligned}\right\} \text{— M2}$$

Theorem 1 :

$\exists \ \bar{y}$, an optimal sol^n to M2 with the following property:

$\forall i \in [n]$, $\bar{y}_i \in \{0, \tfrac{1}{2}, 1\}$

Pf :



Def^n (ILP/LP) :

$x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$

$$\left.\begin{aligned} &\min. \ \langle c, x \rangle \\ &\text{subject to } \ x \in \mathbb{R}^n \\ &\qquad\qquad\quad Ax \leq b \end{aligned}\right\} \text{ LP}$$

In case of ILP, $x \in \mathbb{Z}^n$

NOTE:

$\bar{x} \leq \bar{y}$

$\bar{x}_i \leq \bar{y}_i, \ \forall i$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad \langle c, x \rangle = \sum_{i=1}^{n} c_i x_i$$

LP relaxation of ILP :

$$\left.\begin{array}{ll} \min & \langle c, x \rangle \\ \text{subject to} & Ax \leq b \end{array}\right\} \text{— RILP}$$

$$\left.\begin{array}{ll} \max & \langle c, x \rangle \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}\right\} \text{— PP} \qquad \left.\begin{array}{ll} \min. & \langle b, y \rangle \\ \text{subject to} & A^T y \geq c \\ & y \geq 0^{[n]} \end{array}\right\} \text{— DP}$$

primal $\qquad\qquad$ dual

$$A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c \in \mathbb{R}^n$$

Theorem 2 (Strong Duality) :

w.r.t PP and DP we can only have one of the following 4 possibilities :

1. Both PP and DP are not "feasible"
2. PP infeasible and DP unbounded
3. PP unbounded and DP infeasible
4. Both feasible and $Opt_{PP} = Opt_{DP}$

$$\min \quad 0$$

subject to $\langle c, x\rangle = \langle b, y\rangle$

$$Ax \leq b$$
$$x \geq 0$$
$$A^T y \geq c$$
$$y \geq 0$$

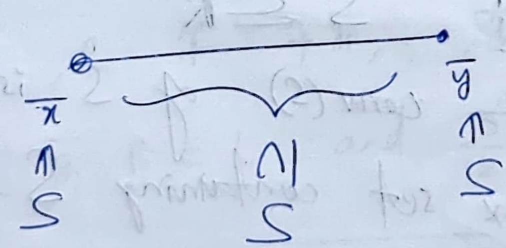
$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \bar{z} = (\bar{x}, \bar{y})$$

Theorem (weak duality):

$$\text{Opt}(PP) \geq \text{Opt}(DP)$$

Combinatorial convex geometry:

(convex sets): $S \subseteq \mathbb{R}^n$



NOTE: ~~Sphere~~ Hollow sphere is NOT convex, but solid sphere is

(Halfspaces) :

$$A x \leq b$$

$$\begin{bmatrix} — & A_1 & — \\ & \vdots & \\ — & A_m & — \end{bmatrix} \begin{pmatrix} \\ x \\ \\ \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\langle A_i , x \rangle \leq b_i \quad , \forall i$$

Defn (convex set) :

$S \subseteq \mathbb{R}^n$ is a "convex set" if $\forall \bar{x}, \bar{y} \in S$ and $\bar{\lambda} \in [0,1]$, we have :

$$\bar{z}_{\bar{\lambda}} = \bar{\lambda} \bar{x} + (1-\bar{\lambda}) \bar{y} \in S$$

Defn (convex hull) : $S \subseteq \mathbb{R}^n$

Then, "convex hull" conv(S) of S is the smallest convex set containing S

~~Theorem: Let~~ $\cancel{S = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}^n}$

Def^n (convex combination):

Let $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$

$b \in \mathbb{R}^n$ is said to be a convex combination of $a_1, a_2, \ldots, a_m$ if $\exists\ \bar{\lambda}_1, \ldots, \bar{\lambda}_m \in [0,1]$

s.t $\sum_{i=1}^{m} \bar{\lambda}_i a_i = b$ and $\sum_{i=1}^{m} \bar{\lambda}_i = 1$

Theorem: Let $S = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}^n$

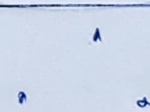
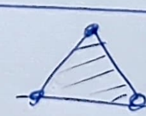
Then, $\operatorname{Conv}(S) = \left\{ b : b \text{ is a convex combination of } a_1, a_2, \ldots, a_m \right\}$

Def^n (affine combination):

Let $P_1, P_2, \ldots, P_m \in \mathbb{R}^n$

$b \in \mathbb{R}^n$ is said to be an affine combination of $P_1, P_2, \ldots, P_m$ if $\exists\ \bar{\lambda}_1, \ldots, \bar{\lambda}_m \in \mathbb{R}$

s.t $\sum_{i=1}^{m} \bar{\lambda}_i P_i = b$ and $\sum_{i=1}^{m} \bar{\lambda}_i = 1$

Example: convex combination of  is 

Def^n (affine hull): Let $P_1, P_2, \ldots, P_m \in \mathbb{R}^n$ . . . . check!

NOTATION : $\operatorname{Aff\,hull}(\{P_1, P_2, \ldots, P_m\})$

Def^n (affine dependence) :

Let $P_1, P_2, \ldots, P_m \in \mathbb{R}^n$

We say $P_1, P_2, \ldots, P_m$ are "*affinely dependent*" if $\exists\ \bar{\lambda}_i$ with all $\bar{\lambda}_i$ not equal to 0 and satisfy the following :

$$\sum_{i=1}^{m} \bar{\lambda}_i P_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \bar{\lambda}_i = 0$$

Theorem : TFAE :

1. $P_0, P_1, \ldots, P_{m-1}$ are affinely dependent
2. vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_{m-1}$ are linearly dependent, where $\underset{\sim}{v}_i = P_i - P_0$, $\forall i$
3. Any point in Aff hull $(\{P_0, P_1, \ldots, P_{m-1}\})$ have more than one representation

Pf : Exercise !

Lecture 17

Finite versions
- Radon's theorem
- Helly's theorem
- Caratheodory's theorem

all equivalent theorems

|||

" Strong Duality of LP " ~~Representation theorem for polytopes~~

Helly's theorem:

Let $C_1, \dots, C_m$ be convex sets in $\mathbb{R}^n$

If $\forall$ $(n+1)$ convex sets from the above family intersects, then the whole family has a non-empty intersection

(i.e, $\bigcap_{j=1}^{n+1} C_{i_j} \neq \phi$, $\forall\, i_j \in [m] \Rightarrow \bigcap_{j=1}^{m} C_j \neq \phi$)

Radon's theorem:

Let $P_1, P_2, \dots, P_{n+2}$ be distinct points in $\mathbb{R}^n$.

Then, $\exists$ a partition $S_1 \cup S_2$ of the above points

s.t $\quad Conv(S_1) \bigcap Conv(S_2) \neq \phi$

~~pf~~: Since we have $(n+2)$ points in $\mathbb{R}^n$, we know by the last theorem of Lecture 16, that they are affinely dependent

PTO →

So, $\exists\ \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{n+2}$ (not all zero)

s.t $\displaystyle\sum_{i=1}^{n+2} \bar{\lambda}_i P_i = 0$ and $\displaystyle\sum_{i=1}^{n+2} \bar{\lambda}_i = 0$

$$S_1 = \{ P_i : \bar{\lambda}_i > 0 \}$$

$$S_2 = \{ P_j : \bar{\lambda}_j \leq 0 \}$$

Now, $\displaystyle\sum_{i=1}^{n+2} \bar{\lambda}_i = 0$

$$\Rightarrow 0 \neq \sum_{P_i \in S_1} \bar{\lambda}_i = -\sum_{P_j \in S_2} \bar{\lambda}_j = \Lambda \text{ (say)}$$

Now, $\displaystyle\sum_{i=1}^{n+2} \bar{\lambda}_i P_i = 0$

$$\Rightarrow \underbrace{\frac{1}{\Lambda} \sum_{P_i \in S_1} \bar{\lambda}_i P_i}_{\in\ \mathrm{Conv}(S_1)} = \underbrace{\frac{1}{\Lambda} \sum_{P_j \in S_2} (-\bar{\lambda}_j) P_j}_{\in\ \mathrm{Conv}(S_2)}$$

$\square$

Remark :

$P_1, P_2, \ldots, P_m \in \mathbb{R}^n$

$$\hat{V}_i := \begin{bmatrix} | \\ P_i \\ | \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$$

"lifts to $\mathbb{R}^{n+1}$"

$\hat{V}_1, \ldots, \hat{V}_m$ lin. indep. in $\mathbb{R}^{n+1}$

$\iff P_1, P_2, \ldots, P_m$ affinely indep. in $\mathbb{R}^n$

Check!

pf (Helly's theorem) :

$C_1, C_2, \ldots, C_m \in \mathbb{R}^n$

$$\bigcap_{j=1}^{n+1} C_{i_j} \neq \phi \ , \ \forall \ i_j \in [m]$$

To prove: $\bigcap_{j=1}^{m} C_j \neq \phi$

We'll prove this using induction on m

Base case : $m = n+1$ ✓ trivial

Suppose it is true for m

<u>ETS</u> : true for $m+1$

$$C_1, C_2, \ldots, C_{m+1}$$

$$S_i := \{C_j : j \neq i\}, \quad \forall i \in [m+1]$$

from induction hypothesis, $\exists a_i \in \mathbb{R}^n$ s.t
$a_i \in C_j$, $\forall C_j \in S_i$

Let $A := \{a_1, a_2, \ldots, a_{m+1}\}$

Since $|A| \geq n+2$, from Radon's theorem,
$\exists$ a partition $A_1 \cup A_2$ of A s.t
$$Conv(A_1) \cap Conv(A_2) \neq \phi$$

i.e, $\exists \bar{x} \in Conv(A_1) \cap Conv(A_2)$

~~Take some~~ $\cancel{C_i}$ ~~s.t~~ $\cancel{a_i \in C_j}$ ~~where~~ $\cancel{j \neq i}$

~~WLOG, assume that~~ $\cancel{a_i \in A_1}$

~~Then, observe that~~ $\cancel{a_j \in C_j, \forall a_j \in A_2}$

$$A_1 = \{a_{i_1}, \ldots, a_{i_t}\}$$

$$A_2 = \{a_{j_1}, \ldots, a_{j_\ell}\}$$

$$S_{i_j}$$

$$C_{i_j} \longrightarrow a_{i_j}$$

Here, $[m+1] = \{i_1, \ldots, i_t, j_1, \ldots, j_\ell\}$

Now, $i_j \notin \{j_1, \ldots, j_\ell\}$

$\Rightarrow C_{i_j} \in S_{j_r}, \ \forall r \in [\ell]$

$\Rightarrow a_{j_r} \in C_{i_j}, \ \forall r \in [\ell]$

$\Rightarrow \mathrm{Conv}(A_2) \subseteq C_{i_j}$

$\Rightarrow \bar{x} \in C_{i_j}$ ✓

$\square$

NOTE (Radon's theorem):

$$S = \{P_1, P_2, \ldots, P_m\}$$

$$m \geq \text{aff-dim}(S) + 1$$

"affine dimension"

Caratheodory's theorem :

Let $S = \{P_1, P_2, \dots, P_m\}$ be a set of m points in $\mathbb{R}^n$.

Then, all the points in Conv(S) can be written as a convex combination of atmost $(n+1)$ points from S

Pf (by induction on m)

Base case ($m = n+1$) : trivial

(using the Fact that Conv(S) = set of all convex combinations of S)

Suppose true for m

ETS : true for $m+1$

Let $\bar{x} \in$ Conv(S) (arbitrary)

We know that $\bar{x} = \sum_{i=1}^{m} \bar{\lambda}_i P_i$,

where $\bar{\lambda}_i \geq 0$ and $\sum_{i=1}^{m} \bar{\lambda}_i = 1$

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$$

Consider the following LP :

PTO →

$$\begin{bmatrix} | & & | \\ P_1 & \cdots & P_m \\ | & & | \\ 1 & & 1 \end{bmatrix} \lambda = \begin{pmatrix} 1 \\ x \\ | \\ 1 \end{pmatrix}$$

$$\lambda \geq 0$$

LP

We know that LP has a feasible sol^n. $\bar{\lambda}$

The "support" of $\bar{\lambda}$ is given by:

$$Supp(\bar{\lambda}) = \{ i : \bar{\lambda}_i > 0 \}$$

If $|Supp(\bar{\lambda})| = n+1$, done ✓

So, let $|Supp(\bar{\lambda})| > (n+1)$

i.e, $|\{ P_i : i \in Supp(\bar{\lambda}) \}| > (n+1)$

$\exists \bar{\mu}_i$ s.t $\bar{\mu} = (\mu_1, \mu_2, \dots) \neq 0$ and

$$\sum_{i \in Supp(\bar{\lambda})} \bar{\mu}_i P_i = 0 \text{ and } \sum_{i \in Supp(\bar{\lambda})} \bar{\mu}_i = 0$$

Now, let $\beta_\varepsilon := \bar{\lambda} - \varepsilon \bar{\mu}$, $\varepsilon \geq 0$,

Complete the proof!

$\square$

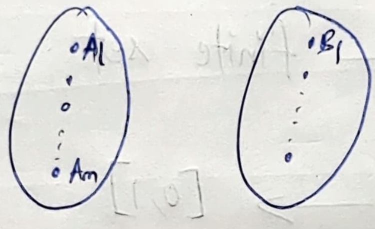
Lecture 18

- ☒ Probabilistic Method
- ☒ Planar graphs
- ☒ Algebraic Techniques

First Moment Methods :

Problem 1 : Sperner's Lemma

Problem 2 :



where A_i, B_i are subsets of $\mathbb{N}$ with

a) $|A_i| = k$, b) $|B_i| = l$, c) $B_i \cap A_i = \phi$,

d) $A_i \cap B_j \neq \phi$, $i \neq j$

Problem 3 : Existence of "large" independent set in "sparse" graphs

Problem 4 : Existence of "small" dominating set in "dense" graph

Problem 5 : Existence of a tournament with many Hamiltonian paths

Problem 6 : Vector balancing problem

$v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ s.t $\|v_i\| = 1$

$v = v_1 \mp v_2 \mp \cdots \mp v_n$ s.t $\|v\|$ will have "small" norm.

To show : $\exists\, \varepsilon_i \in \{\pm 1\}$ s.t $\left\| \sum_{i=1}^{n} \varepsilon_i v_i \right\| \le \sqrt{n}$

Finite Probability Space :

$\Omega \leftarrow$ finite set

$$P : 2^{\Omega} \longrightarrow [0,1]$$

$$P(\phi) = 0$$

$$P(\Omega) = 1$$

$$P(A \cup B) = P(A) + P(B)$$

Example (Hypercube / Hamming cube) :

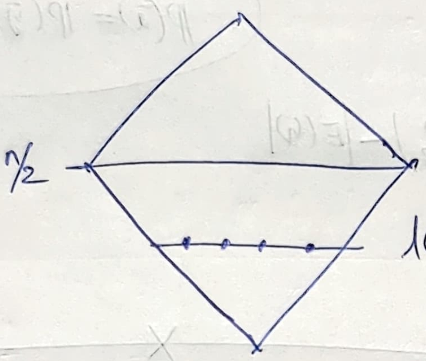
$$\Omega = \{0,1\}^n$$

$\bar{x} \in \{0,1\}^n$, $P(\bar{x}) = \frac{1}{2^n}$

$P_{1/2}(A) = \frac{|A|}{2^n}$, $\forall A \subseteq \{0,1\}^n$

$$\mathbb{P}_q(\bar{x}) = (1-q)^{n-|\bar{x}|} \; q^{|\bar{x}|} \quad , \quad |\bar{x}| = \# \text{ one's in } \bar{x}$$

$$\mathbb{P}_q(A) = \sum_{\bar{x} \in A} \mathbb{P}_q(\bar{x})$$



level(K) = $\left\{\begin{array}{l}\text{all points from } \{0,1\}^n \\ \text{with } K\text{-many ones}\end{array}\right\}$

Random Permutation :

$\Omega_n := \{$ set of all permutations on n numbers $\}$

$= S_n$

$\pi : [n] \longrightarrow [n]$ is a permutation

$\mathbb{P}(\pi) = \frac{1}{A!}$, $A \subseteq \Omega_n$, $\mathbb{P}(A) = \frac{|A|}{n!}$

PTO →

Random Graphs :

$G(n, p)$

$\Omega \longrightarrow$ set of all graphs with same weight

$\bar{x} \in \Omega,$
$P(\bar{x}) = P(\bar{y}), \ \forall \bar{x}, \bar{y} \in \Omega$

$$P(G) = \frac{1}{2^{\binom{n}{2}}}$$

$G(n, p)$

$$P(G) = p^{|E(G)|} (1-p)^{\binom{n}{2} - |E(G)|}$$

Expectation :

$f : \Omega \longrightarrow \mathbb{R}$

$$E[f] = \sum_{w \in \Omega} f(w) \, P(w)$$

$$P[f \leq a] = P[S]$$

$$S = \{ w \in \Omega : f(w) \leq a \}$$

Lemma 1: If $E[f] = c$, then $\exists w \in \Omega$ s.t $f(w) \geq c$

Similarly, $\exists w'$ s.t $f(w') \leq c$

Lemma 2 (Markov's inequality):

Let $f : \Omega \to R$ be a non-negative function.

Then, $P[f \geq a] \leq \frac{E[f]}{a}$, $\forall a > 0$

pf: $$E[f] = \sum_{w \in \Omega} f(w) P(w)$$

$$= \sum_{w : f(w) < a} f(w) P(w) + \sum_{w : f(w) \geq a} f(w) P(w)$$

$$\geq a \sum_{w ; f(w) \geq a} P(w) = a P[f \geq a]$$

$\square$

Independence:

$A_1, A_2, \ldots, A_n$ be n events. We say that they are "mutually independent" if

$$P\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} P[A_i], \quad \forall I \subseteq [n]$$

NOTE: " k-wise independent" with $|I| \leq k$

Theorem (Existence of a Tournament with many Hamiltonian paths):

Pf: K_n T

$$P[T] = \frac{1}{2^{\binom{n}{2}}}$$

$\Omega \to$ collection of all tournament on the vertex set $[n]$

$f : \Omega \to \mathbb{R}$ s.t

$f(T) :=$ no. of Hamiltonian paths in T

ETS: $E[f]$ is large

$$E[f] = \sum_{T \in \Omega} f(T) \frac{1}{2^{\binom{n}{2}}}$$

$$= \frac{1}{2^{\binom{n}{2}}} \sum_{T \in \Omega} f(T)$$

$$= \frac{1}{2^{\binom{n}{2}}} \sum_{\substack{\text{`H'} \\ \text{Hamiltonian} \\ \text{path}}} \sum_{\substack{T : T \text{ containing} \\ H}} 1$$

$$\leq \frac{1}{2^{\binom{n}{2}}} \sum_{\substack{\text{`H'} \\ \text{Hamiltonian} \\ \text{path}}} 2^{\binom{n}{2} - (n-1)}$$

$$= n! \; 2^{-(n-1)}$$

$$= 2^{\theta(n \log n) - (n-1)}$$

$$= 2^{\theta(n \log n)}$$ ✓

Indicator random variable:

$$\mathbb{1}_E = \begin{cases} 1, & \text{if } \omega \in E \\ 0, & \text{o.w} \end{cases}$$

Lemma: $E[\mathbb{1}_A] = P[A]$

Lemma: $f, g : \Omega \longrightarrow \mathbb{R}$

$$\mathbb{E}[\lambda f + \mu g] = \lambda \mathbb{E}[f] + \mu \mathbb{E}[g]$$

ALT proof (using Indicator variables):

$\mathbb{1}_H$:= indicator random variable of H being present in your tournament

$$\mathbb{1}_H(T) = \begin{cases} 1, & \text{if } H \text{ is subgraph of } T \\ 0, & \text{o.w} \end{cases}$$

$$f = \sum_{\substack{\text{"H" Hamiltonian} \\ \text{path}}} \mathbb{1}_H$$

$$\mathbb{E}[f] = \sum_{\substack{\text{"H" Hamiltonian} \\ \text{path}}} P(\mathbb{1}_H = 1)$$

$$= \sum_H \frac{2^{\binom{n}{2} - (n-1)}}{2^{\binom{n}{2}}}$$

Lecture 19

~~Property~~ **Def^n (Independent random variables):**

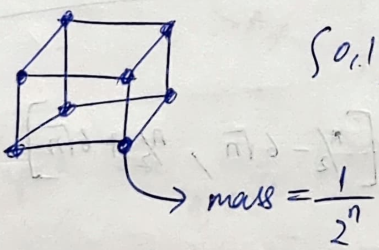
$X_1, X_2, \ldots, X_n$ are (mutually) independent if $\forall I \subseteq [n]$ and $\forall \omega_i$ we have:

$$\mathbb{P}\left[\bigcap_{i \in I} (X_i = \omega_i)\right] = \prod_{i \in I} \mathbb{P}[X_i = \omega_i]$$

Lemma: $X \perp\!\!\!\perp Y \Rightarrow E(XY) = E(X)\,E(Y)$

Lemma: $E[\mathbb{1}_A] = \mathbb{P}(A)$

Application 1:



$\{0,1\}^n$ Hamming Cube

Let $X :=$ no. of 1's

Then, $E(X) = \sum_{i=0}^{n} i\, \mathbb{P}(X=i)$

Aside:
Verify that $E(X) = \sum_{\omega \in \Omega} X(\omega)\, \mathbb{P}(\omega) = \sum_{\alpha \in S} \alpha\, \mathbb{P}(X=\alpha)$

$$= \sum_{i=0}^{n} i \frac{\binom{n}{i}}{2^n}$$

$$= \frac{1}{2^n} \sum_{i=0}^{n} i \binom{n}{i} = \frac{1}{2^n} \cdot n\, 2^{n-1} = n/2$$

ALT

Let $\mathbb{1}_i :=$ event that the ith component of the point in $\{0,1\}^n$ is 1.

Then, $X = \sum_{i=1}^{n} \mathbb{1}_i$

$$\Rightarrow E(X) = \sum_{i=1}^{n} E(\mathbb{1}_i) = n \cdot \frac{1}{2} = n/2$$

Remark:

$$\Omega(n) = [n/2 - 6\sqrt{n},\ n/2 + 6\sqrt{n}]$$

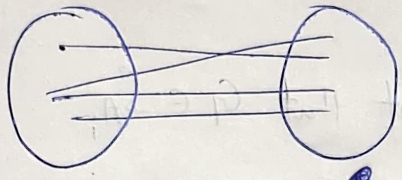
Application 2 (Random Graphs):

$\Omega = \{$family of graphs on the vertex set $[n]\}$

$$G \in \Omega, \quad P(G) = \frac{1}{2^{\binom{n}{2}}}$$

NOTE: $|\Omega| \simeq 2^{n^2/2}$

Lets look at bipartite graphs



$A \cup \overline{A} = [n]$

$\overline{A} = [n] \setminus A$

$S_A := \{$ no. of bipartite graphs on $[n]$ that has $A, \overline{A}$ as a bipartition $\}$

$$|S_A| = 2^{|A|(n-|A|)} \leq 2^{n^2/4}$$

NOTATIONS :

$A_1, A_2, \ldots\ldots, A_{2^n}$

$S_{A_1}, S_{A_2}, \ldots\ldots, S_{A_{2^n}}$

$$\text{no. of bipartite graphs} \leq \sum_{A_i \in 2^{[n]}} |S_{A_i}|$$

$$\leq 2^{n + n^2/4} \quad \left(<< |\Omega| = 2^{n^2/2} \right)$$

Lets prove this

PTO $\longrightarrow$

Lemma : $\mathbb{P}[G \text{ is bipartite}] \leq \dfrac{2^{n+n^2/4}}{2^{n^2/2}} = 2^{n-n^2/4}$

pf : Let $E_i :=$ event that $G \in S_{A_i}$

Then, $\bigcup_{i=1}^{2^n} E_i$ = event that G is bipartite

$\quad\quad\quad\; \|$

$\quad\quad\;\; E$ (say)

Now, $\mathbb{P}(E) = \mathbb{P}\left[\bigcup_{i=1}^{2^n} E_i\right]$

$$\leq \sum_{i=1}^{2^n} \mathbb{P}[E_i]$$

$\boxed{\begin{array}{l} \underline{\text{NOTE}}: \text{ for r.v's } X, Y, \\ \text{①} \;\; X \leq Y : \;\; := X(\omega) \leq Y(\omega), \; \forall \omega \in \Omega \\ \text{②} \;\; X \leq Y \Rightarrow E(X) \leq E(Y) \end{array}}$

Let $\mathbb{1}_{Bip} :=$ indicator r.v that G is bipartite

Now, for $A_1, \dots, A_{2^n}$, define :

$\mathbb{1}_{A_i} :=$ indicator r.v that G is bipartite with bipartition $A_i, \overline{A_i}$

Then, $\mathbb{1}_{B_{ip}} \leq \sum_{i=1}^{2^n} \mathbb{1}_{A_i}$

$\left(\text{i.e., } \mathbb{P}\left[\bigcup_i A_i\right] \leq \sum_i \mathbb{P}(A_i)\right)$

Complete the proof

$\square$

Q1) $S_n := \{\pi : [n] \longrightarrow [n] \text{, where } \pi \text{ is 1-1}\}$ "permutation"

$\mathbb{P}[\pi] = \frac{1}{n!}, \ \forall \pi \in S_n$

Fix k indices (say, $i_1, i_2, \ldots, i_k \in [n]$)

$\mathbb{P}\left[\pi(i_k) > \pi(i_j), \ \forall j \leq k-1\right] = ?$

sol^n : $\frac{1}{k}$ Check!

Q2) Size of the largest independent set in a graph G $= ?$

sol^n : Let $\Delta(G) :=$ largest degree in G

PTO $\longrightarrow$

Observation : $\alpha(G) \geq \dfrac{n}{\Delta(G)+1}$

size of the largest indep. set

But this is not a very good bound

Turan, Ramsey type problem

Theorem : $\alpha(G) \geq \sum_{i \in [n]} \dfrac{1}{d(i)+1}$
(Cara-Tuza)

pf : Let π be a random permutation

$\pi(1), \pi(2), \ldots, \pi(i), \ldots, \pi(n)$

$\boxed{S \to \text{indep. set (considering a greedy way to construct it)}}$

Let $\mathbb{1}_i$ = indicator r.v that i was chosen from S

Then, $|S| = \sum_{i=1}^{n} \mathbb{1}_i$

$\Rightarrow \mathbb{E}[|S|] = \sum_{i=1}^{n} \mathbb{P}[i \in S]$

$= \sum_{i=1}^{n} \dfrac{1}{d(i)+1}$ (using Q1)

□

Q3> [Shearer]

Let G be a Δ-free d-regular graph

Then, $\alpha(G) \gtrsim \dfrac{n \log d}{d}$

Pf : Check paper by "Shearer"

Recall : $E(XY) = E(X)\, E(Y)$

Theorem : Let $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n$ be unit vectors in $\mathbb{R}^N$

Then, $\exists\, \varepsilon_i \in \{\pm 1\}$ s.t

$$\left\| \sum_{i=1}^{n} \varepsilon_i \underset{\sim}{v}_i \right\|_2 \le \sqrt{n}$$

Pf : $\forall i$, let $X_i := \begin{cases} -1, & \text{w.p } 1/2 \\ +1, & \text{w.p } 1/2 \end{cases}$

Let $V := \sum_{i=1}^{n} X_i \underset{\sim}{v}_i$

Then, $\|V\|_2^2 = \sum_{i=1}^{n} \|\underset{\sim}{v}_i\|^2 + \sum_{i \ne j} \langle \underset{\sim}{v}_i, \underset{\sim}{v}_j \rangle X_i X_j$

$= n + \sum_{i \ne j} \langle \underset{\sim}{v}_i, \underset{\sim}{v}_j \rangle X_i X_j$

$$\Rightarrow \mathbb{E}\left[\|V\|_2^2\right] = n + \sum_{i \neq j} \langle v_i, v_j \rangle \, \mathbb{E}(X_i X_j)$$

$$= n \quad \left(\because \mathbb{E}(X_i X_j) = \mathbb{E}(X_i)\mathbb{E}(X_j) = 0 \right)$$

$\Rightarrow \exists \, \varepsilon_i \in \{\pm 1\}$ s.t

$$\left\| \sum \varepsilon_i v_i \right\|_2 \leq \sqrt{n} \quad (\text{or } \geq \sqrt{n})$$

$\square$

Defn (dominating set): Let $G = ([n], E)$ and let $S \subseteq [n]$.

Then, S is called a "dominating set" if

$$S \cap (i \cup N[i]) \neq \phi, \quad \forall i \in [n]$$

~~"Alteration"~~

NOTATION: $\omega(G) =$

Theorem: Let G be a d-regular graph. Then:

$$\alpha(G) \lesssim \frac{n \log d}{d}$$

pf: $p \in [0,1]$

$1 \quad 2 \quad \cdots \quad n$

$S \leftarrow$ include each vertex "i" with probability "p"

"Alteration"

$C \leftarrow$ "correction set" (vertices included in S)

Let $S_{new} = S \cup C$

$$\Rightarrow |S_{new}| = |S| + |C|$$

$$\Rightarrow E(|S_{new}|) = E(|S|) + E(|C|)$$

$$= np + n(1-p)^{d+1}$$

$\mathbb{1}_i \leftarrow i$ is not dominated by S

$$|C| = \sum_{i=1}^{n} \mathbb{1}_i$$

$$\Rightarrow E(|C|) = n(1-p)^{d+1}$$

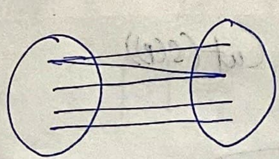
want to find n s.t $E(|S_{new}|)$ is minimum.

Complete the proof

Lecture 20

$G = ([n], E)$

$S \subseteq [n]$



$\bar{S} = [n] \setminus S$

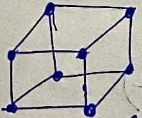
$Cut(S) :=$ no. of edges with one end point in S and other in $\bar{S}$

$$\text{Min-Cut} := \min_{S \subseteq [n]} Cut(S)$$

$$\text{Max-Cut} := \max_{S \subseteq [n]} Cut(S)$$

" Goeman – Williams "

" SDP and Rounding Techniques "



$\{0,1\}^n$ "Hyperaube"

$p = (0, 1, \ldots) \longrightarrow \boxed{\text{weight} = \frac{1}{2^n}}$

$p = (p_1, p_2, \ldots, p_n)$

$S(p) = \{i : p_i = 1\} \subseteq [n]$

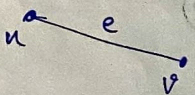
Let $X : \{0,1\}^n \longrightarrow \mathbb{Z}$ s.t

$$X(p) = \mathrm{Cut}_G(S(p))$$

Then, $E(X)$ = average over all cuts of G

$$= \frac{1}{2^n} \sum_{p \in \{0,1\}^n} \mathrm{Cut}(S(p))$$

Goal: $E(X) \geq \frac{|E|}{2}$



$\mathbb{1}_e :=$ indicator r.v. that e goes across $S, \bar{S}$

Then, $X = \sum_{e \in E} \mathbb{1}_e$

$$\Rightarrow E(X) = \sum_{e \in E} \mathbb{P}(e \text{ in cut})$$

$$= \sum_{e \in E} \frac{1}{2} \qquad \left(\because P(e \text{ not in cut}) = 1/2 \right)$$

Check!

$$= |E|/2 \quad \checkmark$$

A Classical problem in Extremal Set Theory

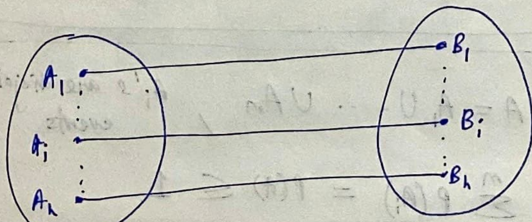
Theorem (Erdos–Ko–Rado): Let S be a collection of subsets of $[n]$ with each set in S of size k. If $\forall A, B \in S$ we have $A \cap B \neq \phi$, then

$$|S| \leq \binom{n-1}{k-1}$$

Here, $n \geq 2k$

(k, l) – systems

Bipartite graph with partite sets $\{A_1, \ldots, A_h\}$ and $\{B_1, \ldots, B_h\}$, where $A_i, B_i \subseteq \mathbb{N}$ s.t:



① $|A_i| = k$ and $|B_j| = l$, $\forall i, j$

② $A_i \cap B_i = \phi$, $\forall i$

③ $A_i \cap B_j \neq \phi$, $\forall i \neq j$

motivation for "matroid theory"

Then, how large can h be?

Theorem (Bollobas) : for (k,l)-systems,

$$h \leq \binom{k+l}{l}$$

Theorem (LYN inequality for bipartite systems)

Modify condition ① to :

① $|A_i| = a_i$, $\forall i$ and $|B_j| = b_j$, $\forall j$

Then, $$\sum_{i=1}^{h} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1$$

Remark : This is a stronger result than "Bollobas"

Pf : Idea: $A = A_1 \cup \dots \cup A_m$, A_i's are disjoint events

Then, $\sum_{i=1}^{m} P(A_i) = P(A) \leq 1$

Relabel the elements of A_i's and B_j's s.t
(WLOG)

$[n] = \left(\bigcup_{i=1}^{h} A_i\right) \cup \left(\bigcup_{j=1}^{h} B_j\right)$, Let π be a random permutation of $[n]$

$\pi(1), \pi(2), \ldots, \pi(n)$

Let $E_i :=$ event that elements of A_i come before elements of B_i in π

Then, $P(E_i) = \frac{1}{\binom{a_i+b_i}{a_i}}$ → check!

$\boxed{\text{NOTE: } E_i \text{ and } E_j \text{ are disjoint events, } \forall i \neq j \quad (\text{i.e., } E_i \cap E_j = \phi, \ \forall i \neq j)}$

Now, $\sum_{i=1}^{h} P(E_i) \leq 1$

$$\Rightarrow \sum_{i=1}^{h} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1$$ ✓

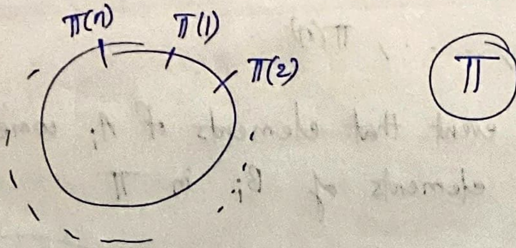
□

Pf (Erdos - Ko - Rado) due to "katona"

Given $S \subseteq \binom{[n]}{k}$, (a subset of collection of k-sized subsets of $[n]$)

$\forall A, B \in S$, $A \cap B \neq \phi$

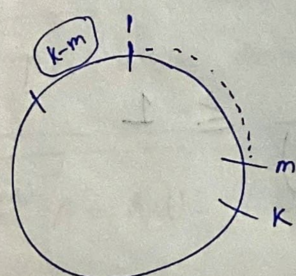
Let Π be a random cyclic permutation of $[n]$



Let $A_{\Pi,l} := \{\Pi(l+1), \dots, \Pi(l+k)\}$, $0 \le l \le n-1$

$$\left|\{A_{\Pi,l} : 0 \le l \le n-1\} \cap S\right| \le k$$

WLOG, assume $A_{\Pi,0} \in S$



$A_{\Pi,m}$

Let Π be a random cyclic permutation of $[n]$
and i be a random number between $[n]$

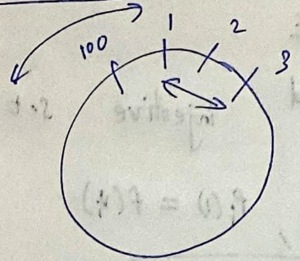
Then, $P(A_{\Pi,i} \in S) \le \frac{k}{n} \rightarrow$ Check!

$$\Rightarrow \frac{|S|}{\binom{n}{k}} \le \frac{k}{n}$$

$$\Rightarrow |S| \le \frac{k}{n} \cdot \binom{n}{k} = \binom{n-1}{k-1} \checkmark$$

$\square$

example :



$\{2,3,4\}$, $\{3,4,5\}$, $\{99,100,1\}$, $\{100,1,2\}$

disjoint

So, lets pair them up :

$\{\{2,3,4\}, \{99,100,1\}\}$, $\{\{3,4,5\}, \{100,1,2\}\}$

Graph Embedding / Drawing

$$G = ([n], E)$$

"Embedding in $\mathbb{R}^d$" :

$$f : [n] \longrightarrow \mathbb{R}^d$$

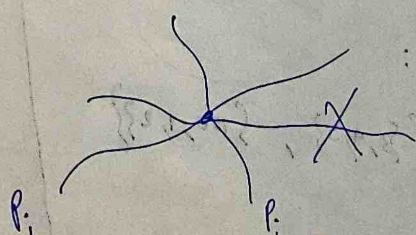
$e_i = \{u_i, v_i\} \in E$

$p_i : [0,1] \longrightarrow \mathbb{R}^d$ injective s.t

$p_i(0) = f(u_i)$, $p_i(1) = f(v_i)$

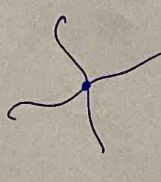
Also, check? "proper Embedding"

Embedding in $\mathbb{R}^2$: "planarity"



: a point where 3 paths pass through internally

don't want



"Crossing"

Lecture 21

Graph Drawing / Planarity

Def^n (Graph Drawing): Given a graph $G = (V, E)$, a "drawing" of G is a family of maps s.t.:

1. $f : V \longrightarrow \mathbb{R}^d$ → injective
2. for each edge $e \in E$, $g_e : [0,1] \longrightarrow \mathbb{R}^d$ → continuous injective maps

   $g_e \rightarrow$ "curves"

3. $f(V) \cap g_e((0,1)) = \phi$, $\forall e \in E$

Def^n (Planar graph): A graph G is a "planar graph" if it can be drawn in $\mathbb{R}^2$ s.t curves associated with edges in G don't intersect in the interior

Theorem (Wagner): A graph G is planar IFF it does not contain K_5 or $K_{3,3}$ as its "minor"

Theorem (Kurotowski's thm.): A graph G is planar IFF it does not contain a "subdivision" of K_5 or $K_{3,3}$

Remark: Wagner's thm. and Kurotowski's thm. are equivalent

Theorem (Fary): All planar graphs have a non-intersecting straight line drawing in $\mathbb{R}^2$

Theorem (Pach): Suppose $G = ([n], E)$ is planar

Then, $f : [n] \rightarrow [2n-1] \times [2n-1]$ and still have a straight line drawing of G

Q?) Given G, find a drawing in $\mathbb{R}^2$ that has the smallest no. of crossings

Theorem: Every graph has a straight line drawing that is non-intersecting in $\mathbb{R}^3$

Pf: Let $G = (V, E)$

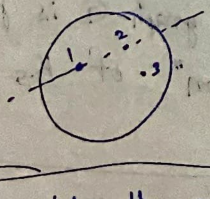
and $f : V \rightarrow B(0,1)$

Ball with centre 0 and radius 1

$V = [n] = \{1, 2, \ldots, n\}$

$f(1), f(2), f(3), f(4), \ldots, f(m), f(m+1), \ldots$

induction



Complete the proof!

ALT pf:

$\exists\ f : [n] \longrightarrow \overbrace{[N] \times [N] \times [N]}^{G_N} \subseteq \mathbb{R}^3$ s.t

the points $\{f(1), \dots, f(n)\}$ satisfy the property of the previous pf

> Claim: $\forall$ planes H in $\mathbb{R}^3$, we have:
>
> $$|\{G_N \cap H\}| \leq c N^2,$$
>
> where $c > 0$ is some constant.

$f(1), f(2), f(3), f(4), \dots, f(m), f(m+1), \dots$

$$c \binom{m}{3} N^2 < N^3$$

$$N = c n^3$$

Complete the proof!

$$G = ([n], E)$$

Theorem: Let $G = ([n], E)$ be a planar graph. $\textcircled{*}$

Then, $|E| \leq 3n - 6$

COROLLARY : PTO $\rightarrow$

Def^n (crossing number): Given a graph G, the "crossing no." of G, denoted $Cr(G)$, is the minimum no. of crossing with which you can draw G in $\mathbb{R}^2$.

Q) Let $G = ([n], E)$ s.t $|E| \geq 8n$
(i.e, G is far from planar)
Then, what is the lower bound of $Cr(G)$?

Theorem (Leytm, Sharir, ...):

If $|E| \geq 8n$, then $|Cr(G)| \gtrsim \frac{m^3}{n^2}$,

where $m = |E|$ and $n = |V|$

COROLLARY (to ⊛): $Cr(G) \geq m - 3n + 6$

Remark: COROLLARY $\Rightarrow$ K_5 is NOT planar
Check!

Jordan's curve theorem: Let C be a "closed curve" in $\mathbb{R}^2$
Then, $\mathbb{R}^2 \setminus C$ has exactly two "connected components": with one bounded and one unbounded, and the "boundary" of both the components is C

we mean maximally connected component

Def^n (closed curve) : $f : [0,1] \to \mathbb{R}^2$

s.t $f(0) = f(1)$

$\therefore$

(i.e, $f : S^1 \to \mathbb{R}^2$)

circle'

Def^n (connected component) :

Def^n (boundary): $S \subseteq \mathbb{R}^2$

$\overline{S} \to$ smallest closed set containing S

$S^\circ \to$ largest open set containing S

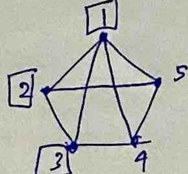
$\partial S := \overline{S} \setminus S^\circ$

$S^2 := \{ \bar{x} \in \mathbb{R}^3 : \|\bar{x}\| = 1 \} \subseteq \mathbb{R}^3$

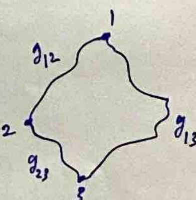
$\uparrow$

sphere

K_5 $\nearrow$



$\xrightarrow{\mathbb{R}^2}$



Lecture 22 (Buddha)

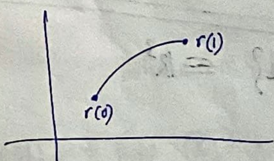
Planar Graphs

Recap:

Def^n (Arc) : Let $\gamma : [0,1] \to \mathbb{R}^2$ be an injective, cts.

Then "arc" is the set $\{ r(x) : x \in [0,1] \}$

Def^n (Jordan Curve) : A closed, simple curve (i.e, an arc with $r(0) = r(1)$)



Def^n (Jordan Curve Theorem [JCT]) :

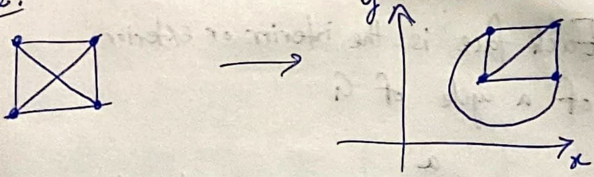
Def^n ("Drawing" of a graph $G=(V,E)$) :

an injective function $b: V \to \mathbb{R}^2$ and arcs α_e, $\forall e \in a(E)$ s.t

$$e = \{x, y\} \Rightarrow \alpha_e(0) = b(x) \text{ and } \alpha_e(1) = b(y)$$

Def^n (planar graph) : A graph is "planar" if $\exists$ a drawing that has arcs which don't intersect except at end-points

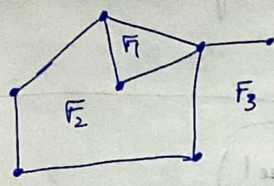
Example:



planar

Def^n (Face of a planar graph) : A set in a drawing of a planar graph $A \subseteq \mathbb{R}^2$ is called "face" of the graph if $\forall \bar{x}, \bar{y} \in A$, $\exists$ an arc $\alpha \subseteq A$ with end point $\bar{x}, \bar{y}$ and α doesn't intersect with the arcs of the planar graph

Example:



$F_3 \rightarrow$ unbounded face (outer face)

$F_1, F_2 \rightarrow$ bounded face (inner face)

NOTE: Let $e_1, e_2, \ldots, e_n$ be the edges of a cycle of a planar graph

$\alpha_{e_1}, \alpha_{e_2}, \ldots, \alpha_{e_n}$ will form a Jordan Curve

Lemma: Each face of a planar graph of G lies inside or outside some cycle of G if $\exists$ a cycle in a planar graph

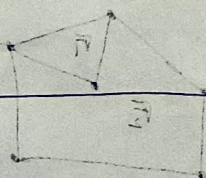
Pf: Exercise!

Property : Each face is the interior or exterior of a cycle of G

Theorem : Let G be a 2-vertex connected planar graph, then every face in the graph is a region of some cycle of G

pf : $G=(V,E)$, $|V|\geq 3$

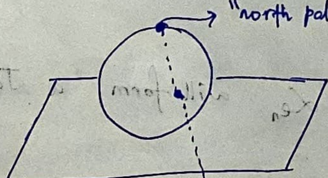
A graph is 2-vertex connected $\iff$ It can be created from a Δ from a sequence of edge subdivision and edge addition.



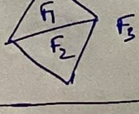
Exercise!

Def^n (Stenographic projection) :

Q^n > Can I draw a planar graph onto a sphere?



Q^n >



Can we "exchange" F_1 and F_3 ?

Theorem (Euler's formula): Let $G=(V,E)$ be a connected planar graph. Then:

$$|V|-|E|+F=1$$, where F = no. of faces in the planar graph (excluding the unbounded face)

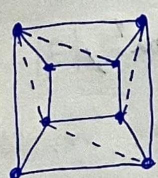
pf:

method 1

Use induction on $|E|$ (Exercise!)

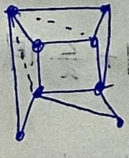
method 2 ("combinatorial argument"):

motivation:



"triangulation"

The formula $(|V|-|E|+|F|=1)$ remains same after triangulation



$\hookrightarrow$ still doesn't change formula

Complete the proof!

"Regular Polyhedron" : ① all faces have same no. of edges

② same no. of faces/edges should meet at each vertex

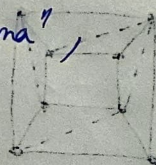
"Platonic Solids" : set of all regular polyhedrons

Proposition: There're only 5 platonic solids in 3-D

Pf: Let a = no. of edges in each face

b = no. of edges at each vertex

Then, by "Handshaking Lemma",



$$aF = 2|E|$$

and $$b|V| = 2|E|$$

Now, Euler's theorem $\Rightarrow |V| - |E| + F = 2$

$$\Rightarrow \frac{2}{b}|E| - |E| + \frac{2}{a}|E| = 2$$

$$\Rightarrow \frac{1}{|E|} = \frac{1}{a} + \frac{1}{b} - \frac{1}{2}$$

Now, $\frac{1}{|E|} = \frac{1}{a} + \frac{1}{b} - \frac{1}{2}$

$\Rightarrow (a \geq 3,\ b \geq 3)$ and $(a \leq 5,\ b \leq 5)$

i.e, $3 \leq a, b \leq 5$

$\boxed{\text{NOTE: } |(a,b)| = 9}$

only 5 sol^n's : $a = b = 3 \rightarrow$ tetrahedron

$a = 4, b = 3 \rightarrow$ cuboid

$a = 3, b = 4 \rightarrow$ octahedron

$a = 3, b = 5 \rightarrow$

$a = 5, b = 3 \rightarrow$

$\square$

Theorem :

Let $G = (V, E)$, $|V| \geq 3$ be a planar graph. Then:

$$|E| \leq 3|V| - 6$$

The equality holds for maximal planar graph

pf : Keep adding edges in G, until it becomes maximal planar

This maximal graph has each of its bounded faces a Δ

no. of bounded faces

Now, "Hand Shaking Lemma" $\Rightarrow 3(F-1) = 2|E|$

Now, apply Euler's formula (for planar graphs)

Exercise!

$\square$

Lecture 23 (various applications)

Let $G = (V, E)$ s.t $|V| = n$, $|E| = m$

Thm (Euler's thm.): If G is planar, then

$$m \leq 3n - 6$$

$\square$

Def^n: Let H, G be graphs.
We say G contains a "subdivision" of H if
$\exists$ a sequence of subdivision of

~~$G_0 \to G_1 \to \cdots \to G_t$~~

$$\underset{\substack{\| \\ H}}{H_0} \to H_1 \to \cdots \to H_t$$

s.t $\exists$ an injective homomorphism of H_t into G

Theorem (Kuratowski):
G is planar $\iff$ It does not contain subdivision of K_5 or $K_{3,3}$

Theorem (Wagner):
G is planar $\iff$ It doesn't contain K_5 or $K_{3,3}$ as its minor

$\square$

Def^n (minor) : H is a minor of G if H can be obtained by deleting edges, vertices and collapsing degree 2 vertices

Corollary (to Euler's thm.) : $Cr(G) \geq m - 3n + 6$

$$Cr(G) = \Omega(n^2)$$

To show : $m = \Omega(n^2) \Rightarrow Cr(G) \gtrsim n^4$

Crossing Lemma : If $m \geq 4n$, then

$$Cr(G) \gtrsim m^3/n^2$$

pf's : "Leyton - 80's", "Sharir - Pach - Zigh : 90's"

$$G(V, E)$$

Let V_p = the random vertex set selected by tossing independent coins (w.p p) on each vertex

and E_p = edges of the induced graph $G[V_p]$

Consider $G_p = (V_p, E_p)$

We're now considering the induced drawing of G_p

$$E(|V_p|) = np$$

$m \geq 4n$

$Cr(G_p)$ = no. of crossing in the induced drawing

$$|E_p| = \sum_{e \in E} \mathbb{1}_e$$

$$\Rightarrow E(|E_p|) = \sum_{e \in E} P(e \in E_p) = mp^2$$

Now, $E(Cr(G_p)) = p^4\, Cr(G)$ → Check!

Now, using (Corollary to Euler),

$$p^4\, Cr(G) \geq mp^2 - 3np + 6$$

$$Cr(G_p) \geq |E_p| - 3|V_p| + 6$$

$$\Rightarrow \quad Cr(G) \geq \frac{m}{p^2} - \frac{3n}{p^3} + \frac{6}{p^4}, \quad \forall p \in (0,1]$$

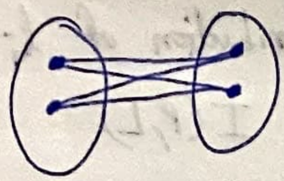
<u>Theorem</u>: $m \geq 4n \Rightarrow Cr(G) \gtrsim m^3/n^2$

Let $L = \{\text{lines in } \mathbb{R}^2\}$, $|L| = m$, $|P| = n$

$P = \{\text{points in } \mathbb{R}^2\}$

and $I(P, L) := \{(p, l) : p \in P,\ l \in L,\ p \in l\}$

NOTE:



"Kovari – Sos – Turan"

does n't contain $K_{2,2}$

$$|E| \lesssim n^{3/2}$$

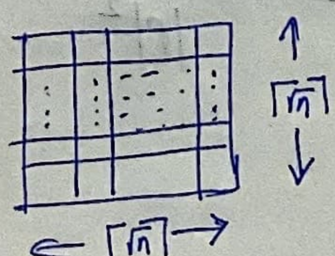
<u>Corollary (Kovari – Sos – Turan):</u> Let $|P| = |L| = n$

Then, $|I(P,L)| \lesssim n^{3/2}$

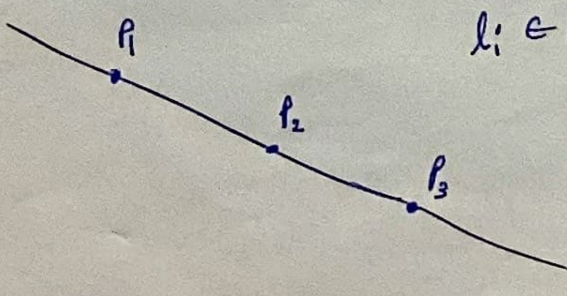
<u>Theorem (Szemeredi – Trotter):</u> If P is a set of n points in $\mathbb{R}^2$ and L is a set of m lines in $\mathbb{R}^2$, then

$$|I(P,L)| \lesssim (mn)^{2/3} + m + n$$

NOTE:



<u>pf:</u>



$l_i \in L$

$P = \{\text{vertices}\}$

Let $e(l_i) =$

Note that, $e(l_i) + 1$ = contribution of l_i to $I(P, L)$.

Now, $|E| = \sum_i e(l_i)$

$$= \sum_{l \in L} [e(l) + 1] - \sum_{l \in L} 1$$

$$= I(P, L) - |L|$$

Assume $|E| \geq 4|P|$

($\because$ $|E| < 4|P| \Rightarrow I(P,L) - |L| < 4|P|$

$\Rightarrow I(P,L) \lesssim |L| + |P| = n + m$ ✓)

Now, by "Crossing Lemma",

$$|L|^2 \geq Cr(G) \gtrsim \frac{\left(|I(P,L)| - |L|\right)^3}{|P|^2}$$

$\square$