LEC-11

30/08/2024

Probability inequalities

I Markov Inequality: Suppose X is a nonnegative random variable. Then $P(X \geq a) \leq \frac{E(X)}{a}$, for any $a > 0$.

proof: (Discrete Case) $E(X) = \sum_{x} x P(X=x)$

$$= \sum_{x \geq a} x P(X=x) + \sum_{x < a} x P(X=x)$$

$$\geq \sum_{x \geq a} x P(X=x)$$

$$\geq a \underbrace{\sum_{x \geq a} P(X=x)}_{\Downarrow \; P(X \geq a)}$$

https://www.youtube.com/watch?v=e-nAr3MkAIl

Alt: Define $Y = \begin{cases} 1 & \text{if } X \geq 0 \\ 0 & \text{o/w} \end{cases}$ $\quad Z = 1-Y$, $\quad Y+Z=1$

$$\mu = E(X) = E(X(Y+Z))$$

$$= E(XY) + E(XZ)$$

$$\geq E(XY) \quad [\because X, Z \geq 0] \longleftarrow (i)$$

$XY \geq aY$

i.e. $E(XY) \geq a E(Y)$

i.e. $E(XY) \geq a P(X \geq a)$

i.e. $P(X \geq a) \leq \frac{E(XY)}{a} \leq \frac{E(X)}{a}$ using (i)

II Chebyshev Inequality: Let X be a random variable with finite variance. Then $\forall \epsilon > 0$ $P[|X - E(X)| \geq \epsilon] \leq \frac{Var(X)}{\epsilon^2}$

proof: $P(|X - E(X)| \geq \epsilon)$

$$= P((X - E(X))^2 \geq \epsilon^2)$$

$$\leq \frac{E(X - E(X))^2}{\epsilon^2}$$

$$= \frac{Var(X)}{\epsilon^2}$$

Alt: $Y = \begin{cases} 1, & (X-E(X))^2 \geq \epsilon \\ 0, & \text{o/w} \end{cases}$

Eg: X be a random variable that represents the systolic blood pressure of the population of 18-74 year in India. X has mean 129 mm Hg and s.d 19.8 mm Hg. Obtain a bound on the probability that systolic blood pressure of the population will assume values between 89.4 mm Hg and 168.6 mm Hg.

$$P(|X - 129.0| \leq 39.6)$$

$$= 1 - (P(|X - 129.0| \geq 39.6))$$

$$\geq 1 - \frac{Var(X)}{(39.6)^2}$$

$$= 1 - \frac{(19.8)^2}{(39.6)^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$\frac{89.4 + 168.6}{2} = \frac{258.0}{2} = 129.0$

$129.0 - 89.4 = 39.6$

Limit Theorems.

I. Convergence in Probability:

II Convergence in Distribution:

• **Definition**: $\{X_n\}_{n\geq}$ be a sequence of random variables and X be another random variable. Then X_n converges in probability X $\left(X_n \xrightarrow{P} X\right)$ if

$$P\left(|X_n - X| \geq \epsilon\right) \to 0 \quad \text{as } n \to \infty \; : \; \forall \epsilon > 0.$$

or, $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0 \quad \forall \epsilon > 0.$

~~We need to check~~,

$$S_n = \sum_{i=1}^{n} X_i$$

• **Properties**:

I. $\left.\begin{matrix} X_n \xrightarrow{P} X \\ Y_n \to Y \end{matrix}\right\} \quad X_n \pm Y_n \xrightarrow{P} X \pm Y$

II $\left.\begin{matrix} X_n \xrightarrow{P} X \\ Y_n \xrightarrow{P} Y \end{matrix}\right\} \quad \begin{matrix} X_n Y_n \xrightarrow{P} XY \\ \frac{X_n}{Y_n} \xrightarrow{P} \frac{X}{Y} \end{matrix}$ (assuming Y_n or Y never takes 0).

Proof: $X_n + Y_n \xrightarrow{P} X+Y$

$$P\left(|(X_n+Y_n) - (X+Y)| \geq \epsilon\right)$$

$$= P\left[|((X_n - X) + (Y_n - Y))| \geq \epsilon\right]$$

$$\leq P\left(\left(|X_n - X| + |Y_n - Y|\right) \geq \epsilon\right)$$

$$\leq P\left(|X_n - X| \geq \epsilon/2 \text{ or } |Y_n - Y| \geq \epsilon/2\right)$$

$$\leq P\left(|X_n - X| \geq \epsilon/2\right) + P\left(|(Y_n - Y)| \geq \epsilon/2\right)$$

$$\to 0 \text{ as } n \to \infty \quad \left(\text{since } X_n \xrightarrow{P} X,\; Y_n \xrightarrow{P} Y\right)$$

$$\left[P(|X_n - X| \geq \epsilon) \leq \frac{E(X_n - X)^2}{\epsilon^2}\right] \text{ using Markov Ineq}$$

$\square$

Similarly prove others.

Eg: Suppose $\{X_n\}$ is exponential (n)

Then show that $X_n \xrightarrow{P} 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x, \lambda > 0 \\ 0 & ; o/w \end{cases}$$

$$P(|X_n - 0| \geq \epsilon) = P(|X_n| \geq \epsilon)$$

$$\leq \frac{E(X_n)^2}{\epsilon^2} = \frac{1}{n^2 \epsilon^2} \to 0 \text{ as } n \to \infty$$

Alt. $P(|X_n - 0| \geq \epsilon)$ use CDF Approach

• **Weak Law of Large Numbers: (WLLN)**

Let $\{X_n\}$ be a i.i.d sequence of random variables with mean μ and finite variance. Then

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu \quad \text{as } n \to \infty$$

or $\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \epsilon) = 0$

• **Convergence in distribution.**

$\{X_n\}_{n\geq}$ be a sequence of random variables with CDF F_n. X be another random variable with CDF F.

We say that X_n converges to X in distribution if for all continuity point x of F : $\left(X_n \xrightarrow{D} X\right)$

$$F_n(x) \to F(x) \quad \text{as } n \to \infty$$

• **Eg**: $X_1, X_2, \ldots \; ; \; X_i \sim Bin\left(i, \frac{\lambda}{i}\right)$

Then $X_n \xrightarrow{D} \text{Poisson}(\lambda)$

Eg: $X_1, X_2, \ldots$ be a sequence of random variables.

$$F_{X_n}(x) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{nx} & x > 0 \\ 0 & o/w \end{cases}$$

Show that $X_n \xrightarrow{D}$ Exponential (1)

proof: $\lim_{n \to \infty} F_{X_n}(x)$

$$= \lim_{n \to \infty} 1 - \left(1 - \frac{1}{n}\right)^{nx}$$

$$= 1 - \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{nx}$$

$$e^{\left(1 - \frac{1}{n} - 1\right) n x}$$

$$e^{-\frac{1}{n} \cdot n x} \Rightarrow \underline{e^{-x}}.$$

$$= 1 - e^{-x} \Leftarrow \text{CDF of Exponential } (1)$$

Remark: I Convergence in probability implies converges in distribution. (Converse may not be true always)

II. Suppose $X_n \xrightarrow{D} c$ for some constant c.

Then $X_n \xrightarrow{P} c$. (Check this proof)