Thm 5 (Caratheodory Thm).

Let $P_1, P_2, \ldots P_n$ be pts from $\mathbb{R}^d$. Let $P \in \text{Conv}(\{P_1, \ldots P_n\})$ then $\exists\ (d+1)$ pts from P_1 to P_n whose convex hull also contains P.

Thm 5':- Let $P_1, P_2, \ldots P_n$ be pts from $\mathbb{R}^d$. Let $P \in \text{Conv}(\{P_1, P_2, \ldots P_n\})$. Then $\exists$ a affine independent subset of $\{P_1, P_2, \ldots P_n\}$ whose convex hull also contains P.

Proof (Thm 5'):-

Let S be the smallest in terms of size, subset of $\{P_1, P_2, \ldots P_n\}$ s.t. $P \in \text{Conv}(S)$. We will show that S is affinely independent. ~~This follows from the fact that $\forall$ finite subsets of~~ if Q is finite ~~subset of $\mathbb{R}^d$ and Q' is the largest affinely ind subset of~~

Suppose S is not affinely independent. Then $\exists\ \lambda_i$ not all zero s.t

$$\sum_{P_i \in S} \lambda_i P_i = 0 \text{ and } \sum_{P_i \in S} \lambda_i = 0.$$

$$\exists\ \mu_i \geq 0 \text{ s.t } \sum_{P_i \in S} \mu_i P_i = P \text{ and } \sum_{P_i \in S} \mu_i = 1.$$

Assume $\epsilon > 0$ and very very small s.t $\underbrace{\mu_i - \epsilon\lambda_i}_{\mu_i(\epsilon)} \geq 0$. $\forall i$

$$\sum_{\forall P_i \in S} \mu_i(\epsilon) P_i = P. \text{ and } \sum_{\forall P_i \in S} \mu_i(\epsilon) = 1.$$

Now as one of $\mu_i(\epsilon)$ becomes '0' I can throw out that $\mu_i P_i$.

so I can write P as a convex hull of subset S. Which contradicts our assumption so it can't be true.

Linear Program (LP) & Integer LP.

LP (general form)

Inputs :- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$.

LP(1)

$$\begin{cases} \max. \langle c, x \rangle \; (c^T x) \text{ s.t } Ax \leq b. \\ x \leq y \Rightarrow x_i \leq y_i \; \forall i. \end{cases}$$

Things we are interested about (LP1):

(1) Optimal value :- Opt LP1.

(2) Optimal point :- P (L1).

(3) Feasibility of (LP1) :-

$$Ax \leq b \iff \underbrace{\langle a_i, x \rangle \leq b_i}_{\text{halfspaces}} \quad \forall i \in [m].$$

Bounded space in this case is polyhedron.

Linear program in "standard form" | "Equational form"

$$\max \langle c, x \rangle \text{ s.t } Ax = b, \; x \geq 0$$

Lemma 8: Every LP can be converted into an equivalent equational form LP.

Proof: LP2:

$$\max \langle c, x\rangle \quad \text{s.t } Ax \le b.$$

(See next page for explanations).

Equivalent (LP3).

$$\max \langle c, x\rangle$$
$$\text{s.t } Ax + \cancel{B}\, y = b.$$
$$y \geq 0.$$

$x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

$x_i = \mu_i - \mu_i'$, $\mu \geq 0$, $\mu' \geq 0$.

$n\, [\,]$
$n\, [\,]$
$m\, [\,]$

Equivalent (LP4)

$\boxed{\cancel{A(\mu-\mu')}}$

$$\max \langle c, \mu-\mu'\rangle$$
$$\text{s.t } A(\mu-\mu') + y = b.$$
$$\mu \geq 0, \mu' \geq 0, y \geq 0.$$

$$z = (\mu, \mu', y) \in \mathbb{R}^{2n+m} \Rightarrow \text{columnwise.}$$

$$\max \langle \tilde{c}, z\rangle \text{ subject to } \hat{A} z = \tilde{b},\ z \geq 0.$$

$$\hat{A} = \left[\begin{array}{c|c|c} A & -A & I_m \end{array}\right].$$

$[A \;\; -A \;\; ?$

$= A\mu - A\mu$

$= A(\mu - \mu$

$$\tilde{c} = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} \in \mathbb{R}^{2n+m}.$$

$$\tilde{b} = b.$$

Lemma 8:- Every LP can be converted into an equivalent equational form LP.

Proof:- LP2:

$$\max \langle c, x \rangle$$
$$\text{s.t} \quad Ax \le b.$$

Equivalent (LP3).

$$\max \quad \langle c, x \rangle.$$
$$\text{s.t} \quad Ax + y = b.$$
$$y \ge 0.$$

introduce a slack variable to make $\le$ to $=$

$x \in \mathbb{R}^n, \; y \in \mathbb{R}^m.$

$x_i = \mu_i - \mu_i', \quad \mu \ge 0, \; \mu' \ge 0.$

n [ ] ~~[ ]~~
n [ ]
m [ ]

Now $\because$ x_i can be +ve/−ve
$\therefore$ take $x_i = \mu_i - \mu_i'$
where μ_i, μ_i' are +ve.

Equivalent (LP4)

~~$A(\mu - \mu')$~~

$$\max \langle c, \mu - \mu' \rangle \rightarrow \max(\langle c, \mu \rangle - \langle c, \mu' \rangle) \; ①$$
$$\text{s.t} \quad A(\mu - \mu') + y = b.$$
$$\mu \ge 0, \; \mu' \ge 0, \; y \ge 0.$$

$z = (\mu, \mu', y) \in \mathbb{R}^{2n+m} \Rightarrow$ columnwise.

$\max \langle \hat{c}, z \rangle$ subject to $\hat{A} z = \hat{b}$, $z \ge 0$.

$$\begin{array}{c} \text{cols} \\ \hat{A} = \left[\begin{array}{c|c|c} A & -A & I_m \end{array} \right]. \end{array}$$

$$[A \;\; -A \;\; I] \begin{bmatrix} \mu \\ \mu' \\ y \end{bmatrix} =$$
$$= A\mu - A\mu' + y = b$$
$$= A(\mu - \mu') + y = b$$

$$\hat{c} = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} \in \mathbb{R}^{2n+m}.$$

$$\hat{b} = b.$$

we can write ① in compact form if $c = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}$

$\because \max \langle \hat{c}, z \rangle = \max(\langle c, \mu \rangle - \langle c, \mu' \rangle + \langle 0, y \rangle$

Duality of LP

Primal	Dual
$\max \langle c, x\rangle$	$\min \langle b, y\rangle$
s.t $Ax = b$ $\quad b^t$	s.t $A^T y \le c$
$x \ge 0$.	$y \in \mathbb{R}^m$.
$x \in \mathbb{R}^n$	(DP5)
(LP5)	

new variable y.

(Weak duality).

Lemma 9 :- Let x_0 be a feasible sol^n to (LP5) and y_0 be a feasible sol^n to (DP5). Then,

$$\langle c, x_0\rangle \le \langle b, y_0\rangle.$$

$$\Rightarrow OPT_{LP5} \le OPT_{DP5}.$$

Prove it. (proved in laptop notes.)

Strong Duality.- LP satisfies strong duality.

Ex :- Derive the dual of the following LP.

1) $\max \langle c, x\rangle$. s.t $Ax \le b$, $x \ge 0$.