Matching :- ("vertex cover problem") ; $V = [n]$

$= (V, E)$

Def^n 1 :- (Matching) :- Let G be a graph. We say $M \subseteq E$ to be a matching in G if $\forall\, e_1 \neq e_2 \in M$, we have $e_1 \cap e_2 = \phi$. [e_1 and e_2 are pair of vertex].

eg :-

Def^n 2 :-

Computational Question :-

Input : $G = (V, E)$.

Goal : Compute largest size matching in G.

$w : E \longrightarrow \mathbb{R}$

Goal 1' :- Compute a matching in G with largest weight.

ILP for computing largest Matching :-

Input : $V = [n]$, $A \leftarrow$ Incidence matrix of G.

$e \leftarrow x_e \in \{0,1\}$.

$$\max \sum_{e \in E} x_e$$

$$\text{s.t} \quad \sum_{e \ni i} x_e \leq 1, \ \forall i \in [n] = V.$$

$$x_e \in \{0,1\}\ \forall\, e \in E.$$

~~Suppose~~ let $m = |E|$

$$C = \mathbb{1}^m = (1, \dots, 1) \in \mathbb{R}^m.$$

$$b = \mathbb{1}^n \in \mathbb{R}^n.$$

$$\left.\begin{array}{l} \max \langle 1^n, x \rangle \\ \text{sub to } Ax \le \mathbb{1}^n \\ 0^m \le x \le \mathbb{1}^m. \end{array}\right\}$$

Fractional matching L.P

~~,~~ $x \in \mathbb{Z}^n$.

ILP for vertex cover ~~problem~~ :-

$$i \leftarrow y_i \in \{0, 1\}$$

$$\min \sum_{i=1}^{n} y_i$$

$$\text{s.t } y_i + y_j \ge 1 \quad \forall e = \{i, j\} \in E$$

$$0 \le y_i \le 1 \quad \forall i \in [n]$$

$$y_i \in \mathbb{Z}.$$

matrix form :-

$$\left.\begin{array}{l} \min \langle 1^n, y \rangle \\ \text{s.t } A^T y \ge \mathbb{1}^n \\ 0^n \le y \le \mathbb{1}^n. \end{array}\right\}$$

Fractional VC L.P

$y \in \mathbb{Z}^n$

$$\left.\begin{array}{l} \min \langle 1^n, y \rangle \\ \text{s.t } A^T y \ge \mathbb{1}^n \\ 0^n \le y \le \mathbb{1}^n \end{array}\right\}$$

Fractional V.C L.P

fractional matching and vertex cover is dual of each other.

$$\max \langle 1^m, x \rangle$$
$$\text{s.t. } Ax \leq 1^n$$
$$0^m \leq x.$$

primal

$\updownarrow$

$$\min \langle 1^n, y \rangle$$
$$\text{s.t. } A^T y \geq 1^m.$$
$$0^n \leq y.$$

dual

Question 3: What relation do we have between $\tau(G)$ and $\nu(G)$?

$\nu(G) \to$ size of largest matching.

Size of the smallest ~~vertex~~ vertex cover $\to \tau(G)$

Th^m 4: $\nu(G) \leq \tau(G) \leq 2\nu(G)$

$\tau^f(G)$ matching for fractional case

Th^m 5: $\tau^f(G) \leq \tau(G) \leq 2\tau^f(G)$

Proof: let $x^{f*} \in \mathbb{R}^n$ be a optimal V.C sol^n.

i —— j : $x_i^{f*} + x_j^{f*} \geq 1$.

from x^f we construct

$$\bar{x}_i = \begin{cases} 1 & \text{if } x_i^{f^*} \geq 1/2 \\ 0 & \text{o/w.} \end{cases} \quad \in \{0,1\}^n.$$

$$A^T \bar{x} \geq 1.$$

$$\bar{x}_i + \bar{x}_j \geq 1.$$

Therefore, we can say $\bar{x}$ is a ~~feasiad~~ feasible sol^n to the ILP for vertex cover.

$$\sum_{i=1}^{n} \bar{x}_i \leq \sum_{i=1}^{n} \left(2 x_i^{f^*}\right) = 2\,\tau^f(G)$$

□

Th^m 6: (Hall's Theorem): Let G be a Bipartite graph s.t $\forall\, S \subseteq A$, we have $|N(S)| \geq |S|$

$G = (A \sqcup B, E)$

Then $\exists$ a matching $M \subseteq E$ s.t all the vertices of A are covered by M.

$$\boxed{N(S) = \bigcup_{u \in S} N(u).}$$

Equivalent Th^m in terms of vertex cover matching

Th^m 7: (Konig's Th^m): Let $G = (A \sqcup B, E)$. Then

$$\tau(G) = \nu(G).$$

Another equivalent form

Th^m 8 :- Dilworth's Th^m for posets.

For General graph :- $\tau^f(G) \le \tau(G) \ge \nu(G) \le \nu^f(G)$

$=$

For Bipartite graph :- everything will same

Proof :- (Hall's Th^m) :- We will use induction on $|A|$.

Base case :- $|A| = 1$. True.

Ind^n hyp :- Result holds for $|A| = n$.

Ind^n step :- $|A| = n+1$.

Subcase 1 :- Suppose for all $S \subsetneq A$ we have $|N(S)| \ge |S| + 1$.

Then take an edge $(a,b) \in E$. and Remove

$G \setminus \{a,b\} = G'$.

$G' = (A' \sqcup B', E')$ where

$A' = A \setminus \{a\}$ $B' = B \setminus \{b\}$

~~$E' = E \setminus \{a,b\}$.~~

E' = induced edges on $A' \sqcup B'$.

G' satisfies Hall's cond^n.

$\forall\ S' \subseteq A'$, $|N_{G'}(S')| \ge |N(S')| - 1$

$\ge |S'| + 1 - 1$

$= |S'|$

Since size of $A' = n$ ~~Then~~ Therefore we can use Ind^n Hypothesis to show that $\exists$ a matching M' in G' that covers A'.

Now observe that $M = M' \cup \{(a,b)\}$ is a matching in G and it covers A.

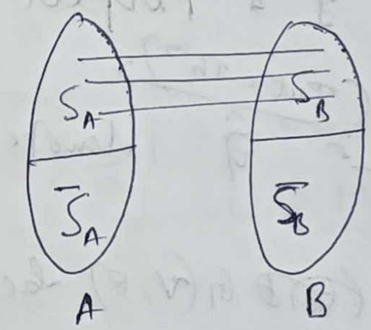
subcase II:
Now, we have to consider the case where $\exists S_A \subsetneq A$.
s.t $|N(S_A)| = |S_A|$.

Let $N(S_A) = S_B \subseteq B$.

$$G'' = (S_A \sqcup S_B, E'')$$

where E'' induced edges on the set $S_A \sqcup S_B$.

As $|S_A| \leq n$. Therefore $\exists$ a matching M'' in G'' that covers S_A.



$|S_A| = |S_B|$

$\tilde{G} = (\bar{S}_A \sqcup \bar{S}_B, \tilde{E})$ satisfies Hall's cond^n.

$\forall\ \tilde{S} \subseteq \bar{S}_A$, we have

$$|N_{\tilde{G}}(\tilde{S})| \geq |\tilde{S}|$$

$$|N_G(S_A \cup \tilde{S})| = |S_B| + |N_{\tilde{G}}(\tilde{S})|$$

$$= |S_A| + |N_{\tilde{G}}(\tilde{S})|$$

$$\geq |S_A| + |\tilde{S}|$$

$$\therefore |N_{\tilde{G}}(\tilde{S})| \geq |\tilde{S}|$$ $\square$

Problem 1 :- Let $G = (A \sqcup B, E)$ ~~set~~ and $d \in \mathbb{N} \cup \{0\}$ Suppose $\forall\ S \subseteq A$, we have $|N(S)| \geq |S| - d$. Then $\exists$ a matching $M \subseteq E$ that covers at least $|A| - d$ many vertices of A.

Prob 2 :- Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{N}$. Find necessary and sufficient cond^n for an existence of "unique rep^n" for each A_i.

unique rep^n ~~mean~~ means -

$$\begin{array}{ccc} s_1, & \ldots, & s_n \in \mathbb{N}. \\ \uparrow & & \uparrow \\ A_1 & & A_n \end{array}$$

Prob 3 :- Let $G = (A \sqcup B, E)$ be a bipartite graph that is d-regular where $d \geq 1$. Show that $\exists$ a Perfect matching in G.

Prob 4 (Tutte Th^m) :- denote : $q(G)$ = # of connected components with odd no. of vertices.

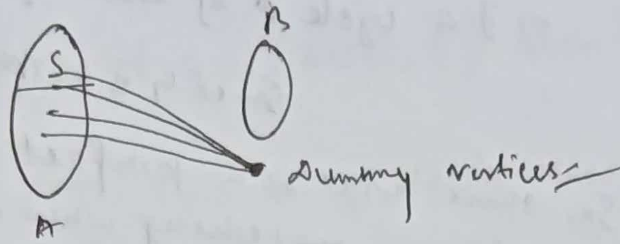
Let $G(V, E)$ be a graph with a perfect matching. Show that $\forall\ S \subseteq V$, we have

$$\cancel{q(G|_S) \leq |S|}.$$

$$q(G_{(V \setminus S)}) \leq |S|.$$

sol^n :- ①   Given G is Bipartite graph. $G = (A \sqcup B, E)$

and $\forall S \subseteq A$, $|N(S)| \geq |S| - d$, $d \in \mathbb{N} \cup \{0\}$



$$\left| N_G(S) \right| \geq |S| - (d-1)$$

Then remove the dummy vertices. Since dummy vertices contribute at most one edges in matching, So, After removing the dummy vertices. $|N_G(S)| \geq |S| - (d-1) - 1$

$\therefore |N_G(S)| \geq |S| - d$.

② Hall's Th^m :- $G = (A \sqcup B, E)$ s.t $\forall S \subseteq A$, we have $|N_G(S)| \geq |S|$. Then $\exists$ a matching $M \subseteq E$ s.t M covers all the vertices in A.

(Necessarily)

~~$N_G(A) \geq |A_i|$~~

$\forall I \in [n]$.

$$\left| \bigcup_{i \in I} A_i \right| \geq |I|$$

let $A = (A_1, \ldots, A_n)$

and $B = \{x \mid x \in A_i\}$.

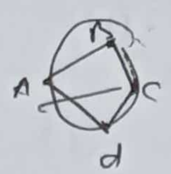
Then use Hall's Th^m.

Q. ~~D is doubly~~

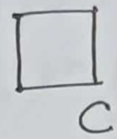
③ G is 2-regular graph

$\Rightarrow \exists$ a cycle of even length,

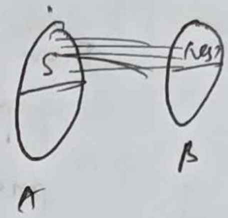
Since G is bipartite.



So there will be a perfect matching when d is even.



But when d = odd then. By Hall's theorem it's guaranteed that there exists a matching



S have $|S| \times d$ edges

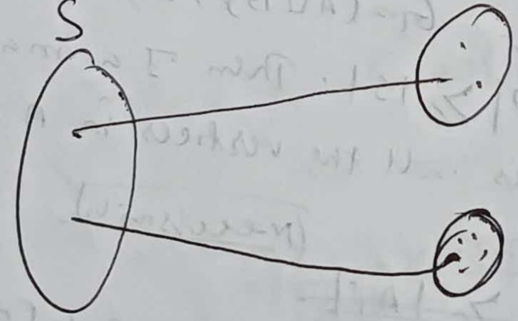
and N(S) have $|N(S)| \times d$ →

$\Rightarrow \quad |S| \times d \leq |N(S)| \times d;$

$\Rightarrow |N(S)| \geq |S|;$

Then take $d-1$ which is even. use the previous argument.

④



Thm :- (**Tutte Thm** :- let $G = (V, E)$. Then G has a perfect matching iff $\forall\, S \subseteq V$, $q(G \setminus S) \leq |S|$.

Thm :- (**König Thm**) :- let $G = (A \sqcup B, E)$ Then $\tau(G) = \nu(G)$

Proof :- We will prove this using Hall's Theorem.

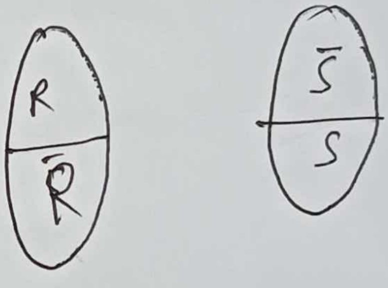
Observe that $\tau(G) \geq \nu(G)$ is always true.

To complete this proof it sufficient to show that $\exists$ a matching $M \subseteq E$ s.t $|M| = \tau(G)$.

Let $U \subseteq V$ be a vertex cover of size $\tau(G)$ in G.

And $R = A \cap U$ and $S = B \cap U$

$\exists$ a matching in $G_{R \sqcup \bar{S}}$ of size $|R|$. Similarly, $\exists$ a matching of size $|S|$ in the graph $G_{\bar{R} \sqcup S}$



$\forall\, T \subseteq S$

$|N_{G'}(T)| \geq |T|$