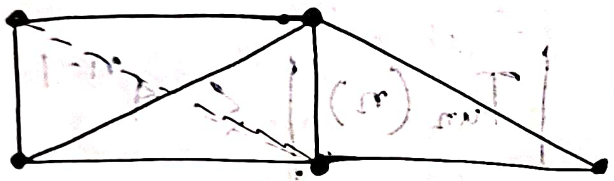
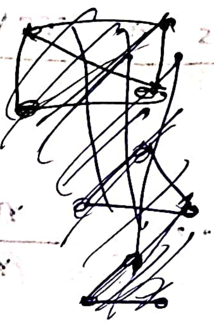
Hall's Theorem → It completely defines a matching in a bipartite graph.

Perfect matching → A matching that covers all the vertex of the graph. As there are odd no. of vertices so there will be always a vertex that would be covered.

(Covers vertex with minimum no. of edges).





For a given Graph, $G = (V, E)$ we define

$q(G)$ = no. of odd components in G.

$\boxed{\text{Lemma-1}}$

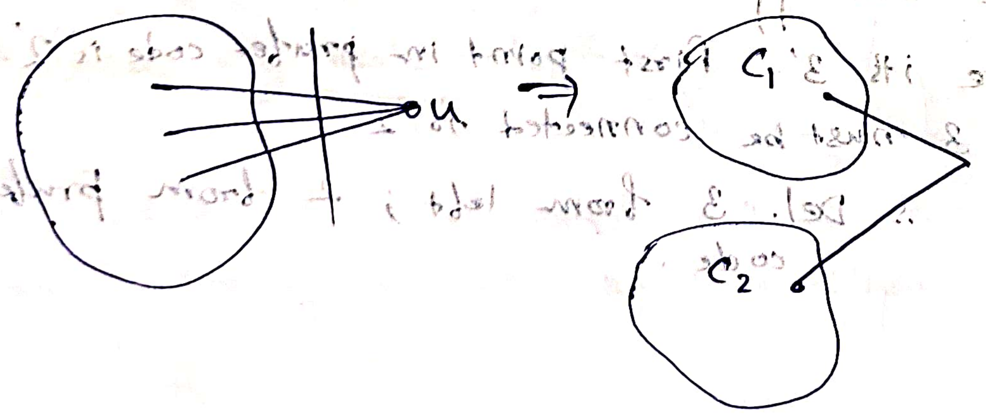
If G has perfect Matching then $q(G \setminus S) \leq |S|$

$\forall\, S \subseteq V(G)$

$\boxed{\text{Proof}}$:- Lets, assume G has a perfect matching. Let $G - u$,

[Odd component = Odd no. of vertices in components],

has more than two odd components.

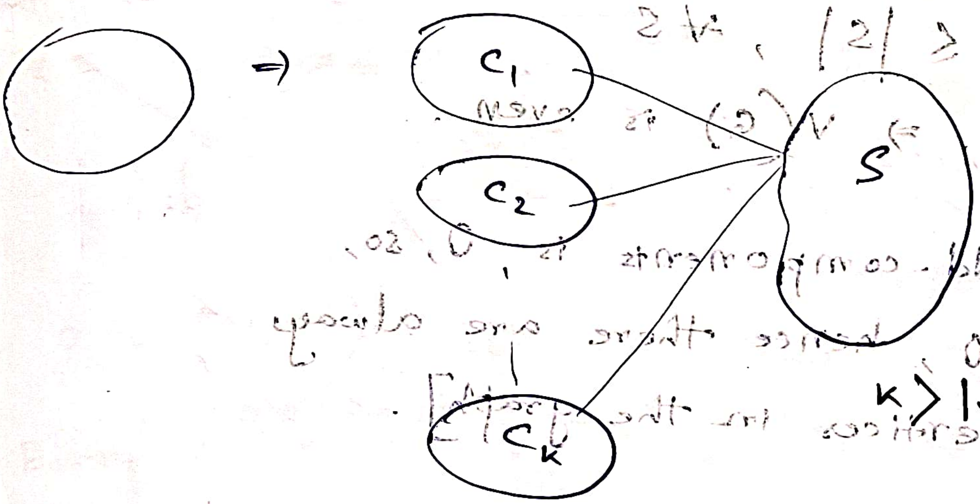


Now; if i remove u, there will two odd components c_1, c_2.

Now, if i join 'u', then either in C_1 or in C_2 there will be a vertex that won't be covered.

- General case.

G has perfect matching, let $G \setminus S$ has more than $|S|$ odd components.



G will have more than k connected components.

There will be a left one which will not be covered. (By pigeonhole principle)

Th: (Tutte's Theorem) ⇒ G has a perfect matching iff $q(G \setminus S) \leq |S|$, $\forall S \subseteq V(G)$ —

- (⇐) if $q(G \setminus S) \leq |S|$, $\forall S \subseteq V(G)$ then G has a perfect matching.

(By contradiction) G does not have a perfect Matching and G is an edge.

- <u>maximal</u> in perfect Matching.

~~[Suppose I have perfect Matching~~

i.e. $G' = G \cup \{e\}$ has a perfect matching.

$$q(G'\setminus S) \le q(G\setminus S), \forall S$$

where $G' = G \cup \{e\}$, $\boxed{V(G) = V(G')}$

$\therefore G'$ has a perfect matching.

$$q(G'\setminus S) \le |S|, \forall S$$

take $S = \phi$, $\Rightarrow V(G)$ is even.

[As no. of odd components is 0, so, $q(G'\setminus S) = 0$, hence there are always even no. of vertices in the graph].

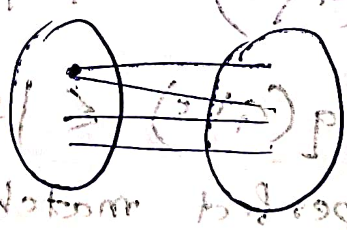
[As in G' we are adding an edge so the no. of vertex are not changing, hence $V(G)$ is even].

Consider a subset $U \subseteq V(G)$ ~~which is~~ ~~connected to every other vertex in $G\setminus U$.~~ where each vertex has degree $|V|-1$.

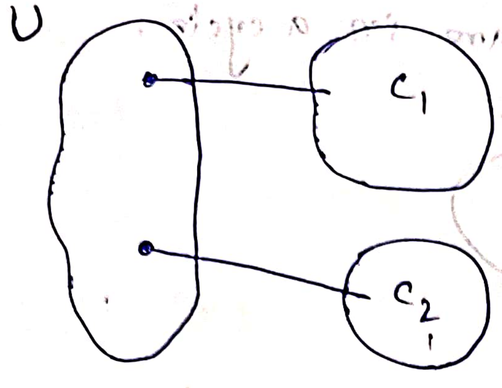
Now consider ~~$G\setminus U$~~ $G\setminus U$

<u>Case(i)</u> - $G\setminus U$ is a disjoint union of cliques

Consider all those cliques which have odd components.



As if a clique has even no. of vertices then it's already perfectly matched.

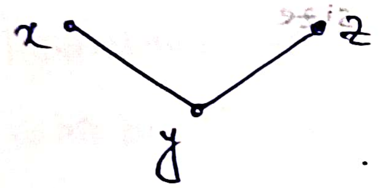


Odd components.

$[\because q(G \setminus U) \leq |U|]$

Then i can connect them with U so, there will be a perfect matching.

Case-2 – $G \setminus U$ is not a disjoint union of cliques.



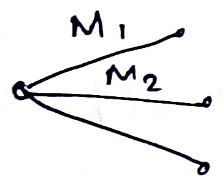
$G \cup \{e\}$ is a perfect matching.

$M_1 = G \cup \{x, z\}$

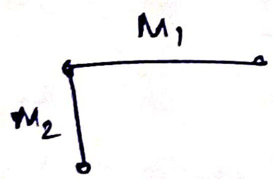
$M_2 = G \cup \{y, w\}$

$F = (V, E')$, $\boxed{E' = M_1 \Delta M_2}$

Every vertex in F has degree either 0 or 2.

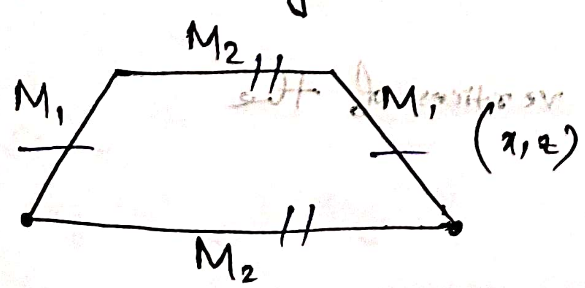


It can't be 3 or more.



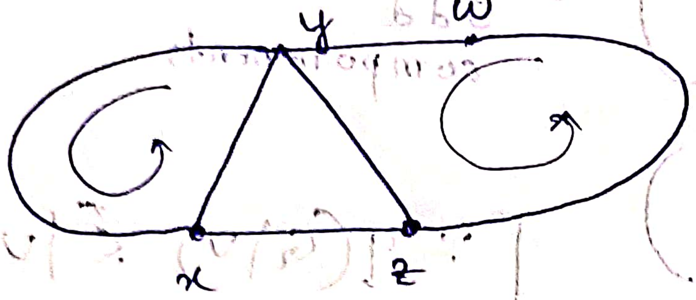
"It can't be 1 also"

"So, F is a union of disjoint cycles or isolated vertices". cycle containing (x, z) or (y, w).



I will simply consider either M_2 or M_1.

Both (x, z) & (y, w) are in a cycle.



one of them has to be even cycle; take that and take alternating path from other one.

Th: (Berge-tutte) the largest matching in a n-vertex graph has size

$$\frac{1}{2}\left[n + \min_{S \subseteq V}\left(|S| - q(G \setminus S)\right)\right].$$

[Hint: G has a perfect matching of size k iff $G^{(k)}$ obtained by adding $n-2k$ vertices has a perfect matching.

$q(G\setminus S) \leq |S|$

$\Rightarrow |S| - q(G\setminus S) \geq 0$.

$\Rightarrow \min_{S \subseteq V}\left(|S| - q(G\setminus S)\right) = 0$.

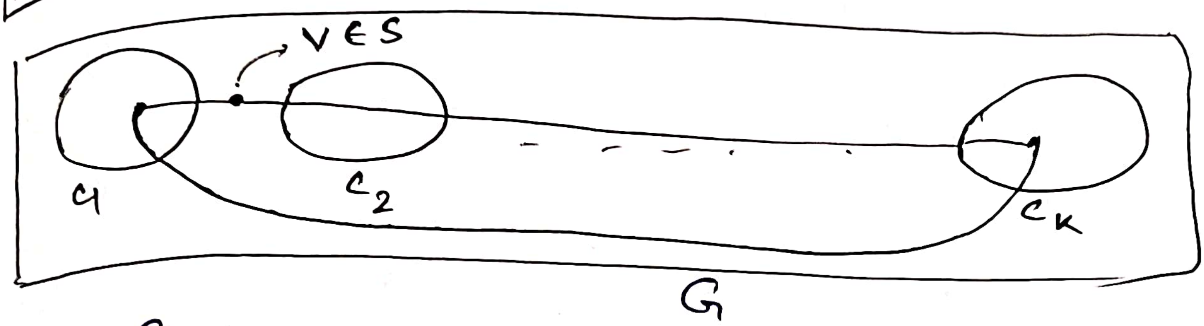
It will be $n/2$

Hamiltonian Cycle:

A cycle containing all the vertices of the graph.

Th :- G is hamiltonian then, for any set $S \subseteq V$ the graph $G \setminus S$ has atmost $|S|$ connected components.

Proof :- Let $G \setminus S$ has 'k' connected components.



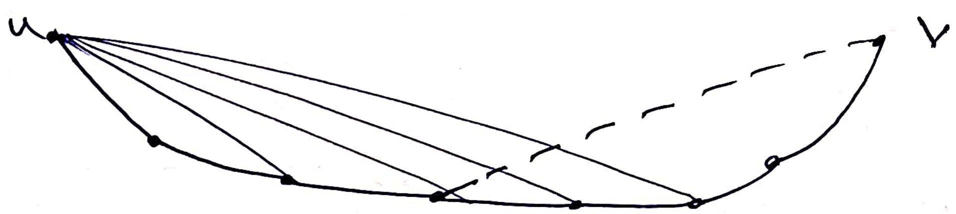
As G is a hamiltonian cycle, so, $\exists$ $v_1, v_2, \dots, v_k$ vertices where each $v_i \in S$
$i \in [1,k]$

Also observe $k \leq |S|$ upper bounded by $|S|$.

Th :- If G with $n \geq 3$, if $\deg(v) \geq n/2$ then G is hamiltonian.

Pf :- [Proof by Contradiction].
[Maximal edge connected].

Let G doesn't have a hamiltonian cycles but $G' = G \cup \{e\}$ does where $e = (u,v)$.



Since $\deg(u)$ & $\deg(v) \geq n/2$.
So, u & v must have atleast a common vertex; to which both u & v are connected.