

Lecture 1

examples

Discrete objects : \mathbb{N} , \mathbb{Q} , collection of Nations, etc.

Defⁿ(set) : collection of "distinct" objects from a fixed universe

$$x \in S \subseteq \Omega$$

an element of set S a set universe

Defⁿ (multiset) : Elements come with multiplicity

Example :

$$\Omega = \mathbb{N}$$

$$S = \{1, 2, 3, 10, 4\} \subseteq \Omega$$

a set

$$\{ \text{e.g. } T = \{1, 1, 1, 2, 3, 4, 5, 4\} \}$$

a multiset

Defⁿ (frequency) : No. of times a particular element occurs in a multiset

Set Operations :

$$A \cup B := \{x \in \Omega : x \in A \text{ or } x \in B\}$$

$$A \cap B := \{x \in \Omega : x \in A \text{ and } x \in B\}$$

$$\overline{A} := \{x \in \mathbb{R} : x \notin A\}$$

Properties:

$$\textcircled{1} \quad A \cup (B \cup C) = (A \cup B) \cup C$$

$$\textcircled{2} \quad A \cup B = B \cup A \rightarrow \text{"commutativity"}$$

$$A \cap B = B \cap A$$

$$\textcircled{3} \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\textcircled{4} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{\overline{A} \cap \overline{B}}$$

Defⁿ (product of sets):

$$A \times B := \{(x, y) : x \in A, y \in B\}$$

Defⁿ (power set): for a given set S ,

power set of S , denoted S , is defined as

$${}^S := \{ \text{subsets of } S \}$$

(i.e., power set of S is the collection of all subsets of $S\} =: 2^S$)

NOTATION: $\{1, 2, 3, \dots, n\} =: [n]$

Let $S = \{1, 2, 3, \dots, n\} = [n]$

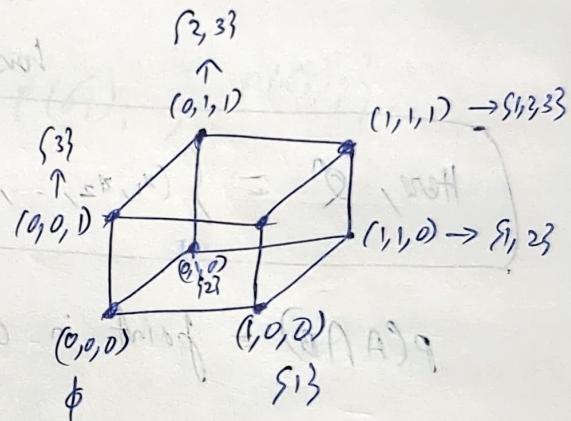
and let $\sigma \in 2^{[n]}$

$$p(\sigma) = (0, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$$

'point'
(not probability)

"boolean-cube"

example: $n = 3$



Now, let $\sigma_1, \sigma_2 \in 2^{[n]}$

Suppose $\sigma = \sigma_1 \Delta \sigma_2 = S/A$

$$p(\sigma_1), p(\sigma_2) \xrightarrow{(A/A) \cup (S/A)} S \Delta A$$

$$\begin{matrix} x \\ y \end{matrix} \in \mathbb{R}^n$$

$$x \oplus y = (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in \mathbb{R}^n$$

Suppose $\sigma = \sigma_1 \cup \sigma_2$

$$p(\sigma_1 \cup \sigma_2) = x \oplus y = \text{max} \{x_1, y_1\}, \max \{x_2, y_2\}, \dots, \max \{x_n, y_n\}$$

"Domination": = A point "dominates" another if it is coordinatewise bigger

(check!)

$$[n] = \{1, 2, \dots, n\} = 2 \text{ for}$$

i.e., (x_1, x_2, \dots, x_n) dominates (y_1, \dots, y_n)

if $x_i \geq y_i$, $\forall i = 1, \dots, n$

$$\Rightarrow (x_1, x_2, \dots, x_n) = (y_1, \dots, y_n)$$

Defⁿ: $p(A \cup B)$ = point in \mathbb{Q}^n that dominates both $p(A)$ and $p(B)$ and has the smallest no. of 1's

$$\text{Here, } \mathbb{Q}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \{0, 1\}\}$$

$p(A \cap B)$ = point in \mathbb{Q}^n that

NOTE:

$$A \setminus B = \{x \in A : x \notin B\}$$

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

NOTATION: $|A|$ = size of A

"A" = (a_1, a_2, \dots, a_n) (i.e., no. of distinct elements of set A)

$$|\sigma_1 \wedge \sigma_2| = ?$$

~~So, $\sigma_i = \sum_{j=1}^n a_{ij} x_j$~~

$\sigma_1, \sigma_2 \in \mathbb{R}$

$$\langle \sigma_1, \sigma_2 \rangle = \sum_{i=1}^n a_i \cdot b_i$$

So,

$$|\sigma_1 \wedge \sigma_2| = \langle p(\sigma_1), p(\sigma_2) \rangle$$

Now,

$$|\sigma_1 \vee \sigma_2| = ?$$

$$|\sigma_1 \vee \sigma_2| = \max \{ p(\sigma_1), p(\sigma_2) \}$$

After sets no particular $\sigma \in A \times \mathbb{R}$ $\supseteq 2$

$\sigma \sim \sigma' \wedge \text{rest } 2 \ni (\sigma, \tau)$ \forall

σ ist kettbar zu τ

$(\Rightarrow, \wedge) : \text{Ampel}$

(\Rightarrow, \wedge)

$x \sim y \wedge A \ni x \wedge \text{rest} \ni y \wedge A \ni y$: mittlerer mitte

: mittlerer unterhalb ①

$x \sim y \Leftarrow y \sim x \wedge A \ni y$

: mittlerer oberhalb ②

$x \sim y \Leftarrow x \sim y \wedge y \sim x$

Lecture 2

Recall: $A \times B := \{(x, y) : x \in A, y \in B\}$

Special case (when $B = A$)

$$\langle (x_1), (x_2) \rangle$$

"ordered pairs"

$\{x, y\} \rightarrow$ unordered pair

In general, $A_1 \times A_2 \times \dots \times A_n = \{(x_1, x_2, \dots, x_n) : x_i \in A_i, i \in \mathbb{N}\}$

Special case (when $B = A$)

Defⁿ (relation):

$S \subseteq A \times A$ is a relation on the set A

If $(x, y) \in S$, then $\overset{\text{we say}}{\sim} x \sim y$

x is related to y

Example: (\mathbb{N}, \leq)

$(\mathbb{R}, <)$

Defⁿ of reflexive relation: $\forall x \in A, x \sim x$

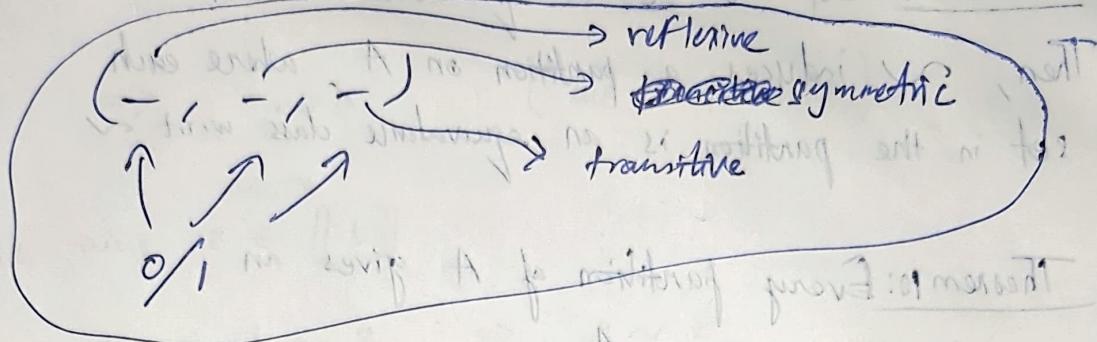
② Symmetric relation:

$x, y \in A$ s.t. $x \sim y \Rightarrow y \sim x$

③ Transitive relation:

$x \sim y, y \sim z \Rightarrow x \sim z$

Qⁿ: Construct & different examples of Relations



Defⁿ (Equivalence relation): \sim is an equivalence relation if it is reflexive, symmetric and transitive.

Defⁿ (class):

Let \sim be a relation. Then:

$$[a] := \{b \in A : a \sim b\}$$

class of a

Note: If \sim is an equivalence relation, then $[a]$ is called the "equivalence class of a"

Defⁿ (partition): $A \leftarrow$ ground set / base set

Let A_1, \dots, A_k, \dots be collection of non-empty

subsets of A s.t. $A_i \cap A_j = \emptyset$ for $i \neq j$

$$\text{and } A = \bigcup_{i=1}^k A_i$$

Qⁿ: $A = [n] = \{1, 2, 3, \dots, n\}$ $[\cdot], [\cdot], [\cdot]$ (\cdot, \cdot) \vdash

How many k sized partitions of A are there?

Hint: Start with $k=2$ $\{\}, \{\}\cup\{\}$ \vdash

containing empty set \vdash

$\{\} \neq \{\} \wedge \{\} \neq \{\}$ \vdash

$\{\} \wedge \{\} \neq \{\}$ \vdash

Theorem 9: Let \sim be an equivalence relation.

Then, \sim induces a partition on A , where each set in the partition is an equivalence class w.r.t \sim .

Theorem 10: Every partition of A gives an equivalence relation on A .

Pf (10): Let $\{A_\beta\}_{\beta \in I}$ be a partition of A .

Let's define the following relation:

$(x, y) \in A \times A$, $x \sim y$ if $\exists \beta \in I$ s.t $x \in A_\beta$ and $y \in A_\beta$

Reflexive: $x \sim x$

Symmetric: $x \sim y \Rightarrow y \sim x$

Transitive: $(x \sim y), y \sim z \Rightarrow x \sim z$

Here, we need partitioning for proof

Check:

Pf (Thm. 9): $[a], [b]$

As long as \sim is reflexive, $\{[a]\}_{a \in A}$

will ~~satisfy~~ satisfy $\bigcup_{a \in A} [a] = A$

We just need pairwise disjointness

Suppose $[a] \cap [b] \neq \emptyset$

Thus, $\exists y \in [a] \cap [b]$

Now, $a \sim y$ and $b \sim y$ (equivalently, y is "symmetric")

$\Rightarrow a \sim b$ (due to transitivity)

Take any $z \in [b]$

$a \sim b$, $b \sim z \Rightarrow a \sim z$

$\Rightarrow z \in [a]$

X

X

X

Propositional Logic

~~p, q~~ two statements

~~p, q~~ from a set to a set

Defⁿ (Function): A function $f: A \rightarrow B$ is a relation $f \subseteq A \times B$ s.t each element of A is related to exactly one element of B

Qⁿ: How many relations are there $f \subseteq A \times B$?

Qⁿ: How many functions are there from A to B ?

$$\underline{\text{Solⁿ$$

$A(R) B$

$$\underline{R} \subseteq (A \times B)$$

$$\underline{\text{Solⁿ$$

Qⁿ: If $B = A$, how many equivalence relations do we have on A ?

A	$f(x) \in B$
x_1	$f(x_1)$
\vdots	\vdots
x_n	$f(x_n)$

"Truth-table"
of f

Defⁿ (Boolean function):

$$f: \{0,1\}^n \rightarrow \{0,1\}$$

\circlearrowleft can also
be $\{-1, +1\}$, X ,
and other
things as well
(with 2 elements)

\circlearrowleft n -variable Boolean function

Defⁿ (Types of functions): Let $f: D \rightarrow R$ be a function

① Injective (1-1):

$$\forall y \in R, |f^{-1}(y)| \leq 1$$

② Surjective (onto):

$$\forall y \in R, f^{-1}(y) \neq \emptyset$$

③ Bijection (1-1, onto):

Injective and Surjective

Propositional Logic

		p	v	q	$p \vee q$	$p \wedge q$	$\neg q / \bar{q}$
		p	v	q	T	F	T
		T	F	F	T	F	F
		F	T	F	T	F	T
		F	F	T	F	T	F
		T	T	F	T	F	T

V : OR , A : AND ,

\neg : NOT

$$(p \Rightarrow q) = f(p, q)$$

$$(p \Leftrightarrow q)$$

$$p \Leftrightarrow q$$

Properties :

$$\begin{aligned} ① \quad p \vee (q \vee r) &= (p \vee q) \vee r \\ p \wedge (q \wedge r) &= (p \wedge q) \wedge r \end{aligned} \quad \left. \begin{array}{l} \text{associativity} \\ \text{distributivity} \end{array} \right\}$$

$$\begin{aligned} ② \quad p \wedge (q \vee r) &= (p \wedge q) \vee (p \wedge r) \\ p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r) \end{aligned} \quad \left. \begin{array}{l} \text{distributivity} \\ \text{de Morgan} \end{array} \right\}$$

$$\begin{aligned} ③ \quad \neg(p \vee q) &= (\neg p) \wedge (\neg q) \\ \neg(p \wedge q) &= (\neg p) \vee (\neg q) \end{aligned} \quad \left. \begin{array}{l} \text{de Morgan} \\ \text{distributivity} \end{array} \right\}$$

Set theoretic formulation:

ω, A, B, C

$$1_A : \omega \rightarrow \{0, 1\}$$

$$1_A := \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{o.w.} \end{cases}$$

$$1_A \cdot 1_B = 1_{A \cap B}$$

$$(p \Rightarrow q)$$

If p is true,
then q is true

$$\neg p \vee q \quad (\neg p) \vee q = (p \leftarrow q)$$

$$p \leftrightarrow q$$

definition $\left\{ \begin{array}{l} \neg \neg (p \vee q) = (p \vee \neg q) \vee q \\ \neg \neg (p \wedge q) = (p \wedge \neg q) \wedge q \end{array} \right. \quad ①$

definition $\left\{ \begin{array}{l} (p \wedge q) \vee (p \wedge r) = (p \vee q) \wedge r \\ (p \vee q) \wedge (p \vee r) = (p \wedge q) \vee r \end{array} \right. \quad ②$

axioms $\left\{ \begin{array}{l} (p \top) \wedge (q \top) = (p \vee q) \top \\ (p \top) \vee (q \top) = (p \wedge q) \top \end{array} \right. \quad ③$

Lecture 3

$\exists x \ P(x)$

$\forall x \ P(x)$

$$\neg (\exists x \ P(x)) = \forall x \ \neg P(x)$$

base 2/00 (i)
2/1 (1+0) nett, 2/2m 2I (ii)

$$\neg (\forall x \ P(x)) = \exists x \ \neg P(x)$$

base 2/00 (i)
 $N=2$, nett

Induction

To prove: $P(n) \quad \forall n \in \mathbb{N}$

Base case ($n=1$)

Induction hypothesis: Assume that the formula is true for some $n \geq 1$

Induction step: we show that the formula is true

for $n+1$

Mathematical Induction follows from the "well ordering" axiom

for natural numbers $(\rightarrow \Leftarrow)$

"Every non-empty subset of \mathbb{N} has a least element"

Let's write all off the above formally:

(PTO) \rightarrow

[Assume $0 \in \mathbb{N}$]

\Rightarrow S exists]

Theorem 1 (Mathematical Induction):

Let $S \subseteq \mathbb{N}$ s.t

i) $0 \in S$ and

ii) If $m \in S$, then $(m+1) \in S$

Then, $S = \mathbb{N}$

Pf: Suppose to the contrary, let $S \subsetneq \mathbb{N}$

~~Then, $\exists A \subseteq \mathbb{N} \setminus S$~~

Consider $\bar{S} := \mathbb{N} \setminus S$

(clearly, $\bar{S} \neq \emptyset$)

Since $\bar{S} \neq \emptyset$, by WOP, it has a least element
least element, say m

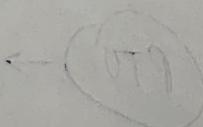
Note that $(m-1) \in \mathbb{N}$ and $(m-1) \in \bar{S}$

But then, by ii), $(m-1)+1 \in \bar{S}$

i.e. $m \in \bar{S}$

($\Rightarrow \Leftarrow$)

$\therefore S = \mathbb{N}$



Partial Ordering (non-~~empty~~) "N/A"

(A, \sim)

$S \subseteq A \times A$

$a \sim b$ if $(a, b) \in S$

\sim is called a "partial ordering" if \sim is reflexive, transitive over A

① \sim is reflexive (i.e., $a \sim a$, $\forall a \in A$)

② $a \sim b, b \sim c \Rightarrow a \sim c$

③ $a \sim b, b \sim a \Rightarrow a = b$

NOTE: partial ordering is usually denoted by \leq (instead of \sim)

i.e., ① $a \leq a$

② $a \leq b, b \leq c \Rightarrow a \leq c$

③ $a \leq b, b \leq a \Rightarrow a = b$

Examples of partial orderings

④ (\mathbb{N}, \leq)

less
equal to

(\mathbb{R}, \leq)

But not a total ordering

Defⁿ (Total ordering):

(A, \leq) is a "total ordering" if it is a partial ordering and $\forall a, b \in A$, either $a \leq b$ or $b \leq a$

Defⁿ (predecessor)

Let (A, \leq) $A \times A \supseteq$ (w, A)

We say $a \triangleleft b$ if:

① ~~a < b~~ $a < b$

② $\exists c \in S_{ab} \subset A$ s.t. $a \leq c \leq b$

Lemma: Let (A, \leq) be a partial ordering with $|A| < \infty$. If $a < b$, then $\exists c \in A$ with $t \in \mathbb{N}$ s.t.

$$a \triangleleft c \triangleleft \dots \triangleleft c \triangleleft b$$

Pf: Suppose $\exists c$ s.t. $a \leq c \triangleleft b$

If $c = a$, we are done

If not, recursively apply the above logic

Apply induction on t (the length of the chain) (w.r.t.)

(Exercise!).

Writing $a \triangleleft b$ as "preorder between" and $a \geq b$ as "postorder".

Defⁿ (chain) :

Let (A, \leq) be a partial ordering.

A chain in (A, \leq) is of the following form:

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n$$

where $a_i \in A$, $\forall i \in [n]$

Defⁿ (anti-chain) :

~~Defⁿ~~ Let (A, \leq) be a partial ordering

An antichain is a subset $S \subseteq A$ where no two distinct elements from S are related.

$$\{b_1, \dots, b_k\}$$

$$b_i \neq b_j$$

$$b_i \neq b_{i+1}$$

Check: "Dilworth's Theorem"

Example: $1, 2, \dots, n^2 + 1 = [n^2 + 1]$

Consider any sequence $a_1, a_2, \dots, a_{n^2+1}$ set

① $a_i \in [n^2 + 1]$

② $a_i \neq a_j$, $\forall i \neq j$

Does \exists a monotone subsequence of length $(n+1)$?

(i.e., $b_1 < b_2 < \dots < b_{n+1}$ or $b_1 > b_2 > \dots > b_{n+1}$)

~~if set~~:

Yes

: (infacts) "Pd"

$a_1, a_2, a_3, \dots, a_i, \dots, a_{n+1} \geq n$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$(1, a_1) \quad (2, a_2) \quad (i, a_i) \quad (n+1, a_{n+1})$

$$A = \{ (i, a_i) : i \in [n+1] \}$$

$$(i, a_i) \leq (j, a_j)$$

if $i \leq j$ and $a_i \leq a_j$

Take a min. disjoint chain decomposition
of A

If \exists a chain in the decomposition of length $\geq n+1$, we are done

By Dilworth's theorem, \exists an antichain

$$(j_1, a_{j_1}), \dots, (j_m, a_{j_m})$$

where $j_1 < j_2 < \dots < j_m$

But we know that $(j_1, a_{j_1}) \neq (j_2, a_{j_2}) \neq \dots$

They can have an increasing subsequence
as required

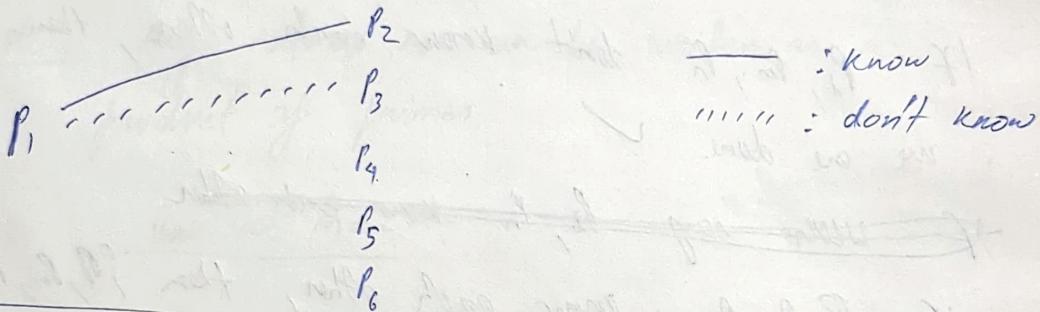
"Check" Hall's theorem

Lecture 4

Proof by case analysis

Example: Out of 6 people, ~~there~~ either there are 3 people who know each other or 3 people who don't know each other

set : $P_1, P_2, P_3, P_4, P_5, P_6$

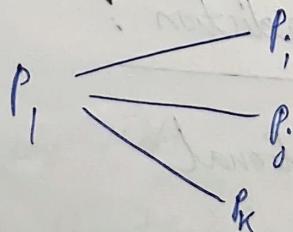


Consider two cases:

case I : (~~at least~~ at least three — edges b/w P_1 and rest)

case II : (at most two — edges b/w P_1 and rest)

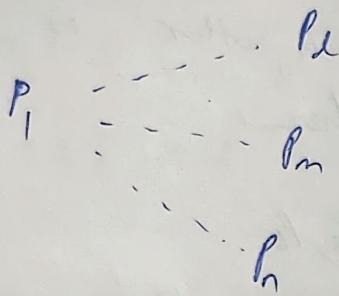
Case I :



If P_i, P_j, P_k don't know each other, then we are done ✓

Suppose w_{204} , P_i and P_k know each other. Then, $\{P_i, P_j, P_k\}$ know each other and we are done. ✓

Case II :



If P_1, P_m, P_h don't know each other, then we are done ✓

~~If w_{204} says P_i, P_n know each other~~

If P_i, P_m, P_n know each other, then $\{P_i, P_m, P_n\}$ know each other and we are done ✓

If, say, P_i and P_m don't know each other, then, $\{P_i, P_k, P_m\}$ don't know each other and we are done ✓

Proof by contradiction :

Lemma : $\sqrt{2}$ is irrational

Lemma : Let $p \in \mathbb{R}^{>0}$. If p is irrational then \sqrt{p} is also irrational

Equivalent statement ("contrapositive"): $a \Rightarrow b \Leftrightarrow \neg b \Rightarrow \neg a$

\sqrt{p} rational $\Rightarrow p$ rational

Disprove by counter-example:

Strong Induction:

Example: Any natural no. can be written as a product of primes.

: Base case ($n=2$): ✓

Suppose $2, 3, \dots, n$ can be written as a product of primes.

complete the proof

Asymptotics:

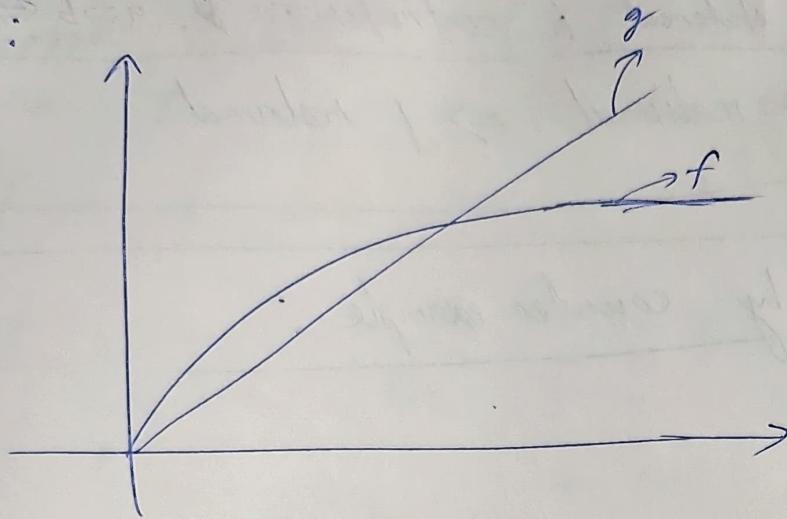
Let $f, g: \mathbb{N} \rightarrow \mathbb{N}$ ~~$f(i) \geq g(i)$~~

Big-O: We say that

$f(n) = O(g(n))$ if $\exists c > 0$ and $n_0 \in \mathbb{N}$ s.t

$f(n) \leq c g(n), \forall n \geq n_0$

Example:



Alt defⁿ (big-O): $f(n) = O(g(n))$ if $\exists c > 0$

$$\text{s.t } \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} \leq c$$

Defⁿ (big-Omega): We say that $f(n) = \Omega(g(n))$

if $\exists c > 0$ and $n_0 \in \mathbb{N}$ s.t

$$f(n) \geq c g(n), \forall n \geq n_0$$

Defⁿ (Theta): We say $f(n) = \Theta(g(n))$ if

$$f = O(g) \text{ and } f = \Omega(g)$$

Example: $f(n) = n!$, $f'(n) = \log(n!)$

$$g(n) = n^n, g'(n) = \log(n^n)$$

Relations b/w f, g, f', g' ? Exercise

~~(small-o)~~ : We say $f = o(g)$ if $\forall \varepsilon > 0$

$\exists n_0 \in \mathbb{N}$ s.t. $g(n) \geq \varepsilon f(n), \forall n \geq n_0$,

~~Def (small-o)~~ (small-o):

~~We say~~ Note: $f = o(g) \Rightarrow f = O(g)$

but converse may not be true

? : In the previous example, is $f = o(g)$?

X -

X -

Back to Lecture 3 :

A partial ordering (P, \leq) is called a "poset" if:

① $a \leq a, \forall a \in P$

② $a \leq b, b \leq c \Rightarrow a \leq c$

③ $a \leq b, b \leq a \Rightarrow a = b$

chain: a_1, \dots, a_t

$$a_1 \leq a_2 \leq \dots \leq a_t$$

anti-chain: a_1, \dots, a_t

$$a_i \not\leq a_j \quad \text{if } i \neq j$$

chain decomposition:

$$c_1, c_2, \dots, c_l$$

(here,
("length of
decomposition" = l))

Dilworth's theorem:

Let (P, \leq) be a poset ~~and~~, $n = \text{length of the}$
 finite

smallest disjoint chain decomposition of (P, \leq)

and $M = \text{length of the longest antichain in } P$

Then, $m = M$

pf: $m < M$ is not possible (check!)

$\therefore m \geq M$

We'll use induction on $|P|$ to prove this

Let C be a maximal chain in P

Now, consider $P \setminus C$

Note that $|P \setminus C| < |P|$

Suppose length of maximal antichain in $P \setminus C$ is almost $M-1$

By induction as

case I (length of maximal antichain in $P \setminus C$ is almost $M-1$):

Then, from induction hypothesis, \exists a disjoint chain decomposition of $P \setminus C$ of length almost $M-1$

To this decomposition, add C to get the bound for P

case II (\exists an antichain $\{a_1, a_2, \dots, a_m\}$ in $P \setminus C$):

$$S^- = \{x \in P : \exists i \in [M] \text{ with } x \leq a_i\}$$

$$S^+ = \{x \in P : \exists i \in [M] \text{ with } x \geq a_i\}$$

Note that $a_i \in S^-$ and $a_i \in S^+$, $\forall i$

and thus, $S^- \cap S^+ \neq \emptyset$

Consider a chain decomposition of S^- :

$$S^- : s_1^-, s_2^-, \dots, s_m^-$$

$$\downarrow \\ a_i$$

Suppose $x \in s_i^-$ and $x > a_i$

Since $x \in s_i^-$, $\exists j$ s.t. $x \leq a_j$

($\Rightarrow \Leftarrow$)

Claim: a_1, a_2, \dots, a_m
are maximal elements
of $s_1^-, s_2^-, \dots, s_m^-$

Similarly, a_1, a_2, \dots, a_m are the minimal elements of s^+, t^+, \dots, m^+

Combining both, we are done ✓

□

$$\left(\binom{[n]}{2}, \leq \right) \subseteq$$

A_1, \dots, A_t are subsets of $[n]$

so $A_i \neq A_j$, if $i \neq j$

Equivalent: What is the length of the maximal antichain
in the poset?

Lecture 5

Pigeonhole principle (PHP) : $\left\{ \begin{array}{l} \text{generalized PHP} \\ \text{Buckets} \hookrightarrow n \\ \text{balls} \leftarrow \text{kntr}, 1 \leq r \leq k \end{array} \right.$

Example :
(Recall Lecture 3)

$$\left\{ a_1, a_2, \dots, a_{n^2+1} \right\} \rightsquigarrow \left\{ 1, 2, \dots, n^2+1 \right\}$$

↑
all distinct

some permutation of

Let $\pi(a_i) :=$ length of the longest increasing subsequence starting at a_i

Consider $\{\pi(a_1), \pi(a_2), \dots, \pi(a_{n^2+1})\}$

NOTE:
 $1 \leq \pi(a_j) \leq n$
 $\forall 1 \leq j \leq n^2+1$

By PHP, we know that $\exists i \in [n]$ s.t. i appears atleast $(n+1)$ times in the following sequence:

$$\pi(a_1), \pi(a_2), \dots, \pi(a_{n^2+1})$$

$\downarrow \quad \downarrow$

$$\pi(a_1) = i \quad \pi(a_2) = i$$

$a_1 > a_2$

Example: $\{A_1, A_2, \dots, A_m\}$, where

① $A_i \subseteq [n]$, $\forall i$ and

② $A_i \subseteq A_j$, $\forall i \neq j$

How large can m be wrt n ?

(Equivalent Q'?: How large can an antichain
be in the poset $([n], \subseteq)$?)

Theorem 2: Let (Ω, \leq) be a poset,
where Ω is a finite set. If (Ω, \leq)
does not contain a chain of length $m+1$
then Ω can be partitioned into $\lceil m \rceil$ antichains

pf: Let M be all the maximal elements

from Ω

Now, consider $\Omega \setminus M$

$\tilde{\Omega}''$

Note that $\tilde{\Omega}$ does not contain any chain
of length m

Now, apply induction on m (Exercise!)

Theorem (Sperner's theorem):

$$\rightarrow m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

pf: Direct consequence of "LYM(B)-inequality"

Theorem (LYM(B) - inequality):

$\{A_1, \dots, A_m\}$ be a collection satisfying the conditions of Sperner's theorem.

Let $\forall k \in [n]$, λ_k be the no. of sets of size k in F .

$$\text{Then, } \sum_{k \in [n]} \frac{\lambda_k}{\binom{n}{k}} \leq 1$$

Remark: Using the fact that $\sum \lambda_k = m$

and $\binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$, we can prove Sperner's thm.

from LYM(B) - inequality.

if LYM inequality: What can we say about maximal chains in $(2^{[n]}, \subseteq)$?

$$\emptyset \subseteq S_1 \subseteq S_2 \subseteq \dots \subseteq [n]$$

How many maximal chains are there in $(2^{[n]}, \subseteq)$?

$$n!$$

③ $A \subseteq [n]$ with $|A|=k$

How many maximal chains can A be part of?

$$k! (n-k)!$$

$$\emptyset \subseteq \dots \subseteq A \subseteq \dots \subseteq [1]$$

$$\{1, 2, \dots, n\} \quad (\text{wlog})$$

$$F = \{A_1, \dots, A_m\}$$

↑
antichain

$$\Pi(A_i) = \{C : C \text{ is a maximal chain and } A_i \in C\}$$

Observe that $\Pi(A_i) \cap \Pi(A_j) = \emptyset$, $\forall i \neq j$

$$\text{Thus, } \sum_{i \in [m]} |\Pi(A_i)| \leq n! \quad \text{--- } \times$$

$$\text{Also, } |\Pi(A_i)| = k! (n-k)! \quad \text{where } k = |A_i|$$

$$\sum_{i \in [m]} |\Pi(A_i)| = \sum_{k \in [n]} \sum_{\substack{|A_i|=k, \\ A_i \in F}} k! (n-k)!$$

$$= \sum_{k \in [n]} d_k k! (n-k)!$$

Now, we can use — (1) to get the required inequality

□

(1) a_1, a_2, \dots, a_n

$$a_{i_1}, a_{i_2}, \dots, a_{i_n} \rightarrow n!$$

Suppose k_i ~~objects~~ objects of type i

't' types of objects

$$\text{Let } \sum_{i=1}^t k_i = n$$

$$\frac{n!}{k_1! \cdot \dots \cdot k_t!}$$

$a_1, \dots, a_n \rightarrow n$ distinct objects

$$\text{ways of selecting } k \text{ objects} = \binom{n}{k} = \frac{n!}{k! (n-k)!}$$

How many ways of selecting k objects?

Exercise

② No. of non-negative sol's to

$$\sum_{i=1}^k x_i = n ?$$

o [] o [] o [] ... - - - - -

No. of ways ~~ways~~ of arranging $(k-1)$ []'s

and n balls $= \binom{n+k-1}{n}$

③ Property: $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$

If: $S = \{1, 2, \dots, n\}$

No. of ways of choosing k elements ~~sets~~ sets
out of S :

i) method 1 (direct): $\binom{n}{k}$

ii) method 2: $\binom{n-1}{k-1} + \binom{n-1}{k}$

↑
No. of
~~sets~~
sets containing 1

↑
No. of
~~sets~~
sets not
containing 1

$$\textcircled{4} \quad (1+x)^n = \sum_{i \geq 0} \binom{n}{i} x^i$$

COROLLARIES: $\textcircled{i} \quad \sum_{i \geq 0} \binom{n}{i} = 2^n$

↑
no. of ~~all~~ possible
subsets of S_1, \dots, S_n

Recall (before)
 $\phi: 2^{\binom{n}{2}} \rightarrow \{0, 1\}^n$
 bijection
 use this to
 prove \textcircled{i} , \textcircled{ii}

$$\textcircled{ii} \quad \sum_{i \text{ even}} \binom{n}{i} = 2^{n-1} = \sum_{i \text{ odd}} \binom{n}{i}$$

$$\textcircled{5} \quad (x+y)^n = \sum_{i \geq 0} \binom{n}{i} x^i y^{n-i}$$

$$\textcircled{6} \quad (x_1 + x_2 + \dots + x_t)^n = \sum_{\substack{k_1 \geq 0 \\ \sum_{i=1}^t k_i = n}} \binom{n}{k_1 k_2 \dots k_t} x_1^{k_1} x_2^{k_2} \dots x_t^{k_t}$$

, where $\binom{n}{k_1 k_2 \dots k_t} = \frac{n!}{k_1! k_2! \dots k_t!}$

$$\binom{n}{k_1 k_2 \dots k_t} = \sum_{i=1}^t \binom{n-1}{k_1 k_1-1 \dots k_i}$$

$$\textcircled{7} \quad \sum_{i \geq 0} \binom{n}{i}^2 = \binom{2n}{n}$$

If:

$$\text{HINT: } \sum_{i \geq 0} \binom{n}{i}^2 = \sum_{i \geq 0} \binom{n}{i} \binom{n}{n-i}$$

no. of ways of choosing n elements out of $2n$ elements

\textcircled{8}



n elements

no. of ways $\rightarrow (n-1)!$

Theorem (Erdos-Ko-Rado theorem):

$\{A_1, A_2, \dots, A_m\}$, where $A_j \in [n]$, $\forall j$

s.t. $A_i \cap A_j \neq \emptyset$, $\forall i, j$ and

$$|A_i| \leq k \leq \frac{n}{2}$$

$$\text{Then, } m \leq \binom{n-1}{k-1}$$

Lecture 6

Asymptotic Notations

$F_n := \text{no. of primes} \leq n$

$\{a_n\}_{n \in \mathbb{N}}$

$\{b_n\}_{n \in \mathbb{N}}$

$a_n \sim b_n \quad \text{if} \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$

NOTE: $F_n \sim \lceil \frac{n}{\log n} \rceil$

Some estimates:

FACT 1: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

$H_n = \Theta(\log n)$

Let $S_k = \frac{1}{2^{k-1}+1} + \dots + \frac{1}{2^k}$

$\frac{\log_2 n}{2} \leq H(n) \leq \log_2 n + 1$

FACT 2 : $1+x \leq e^x$

FACT 3 (Stirling) :

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n$$

FACT 4 : $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$

$$\Rightarrow (n-1)! \leq e\left(\frac{n}{e}\right)^n \leq n!$$

$$\Rightarrow n! = en\left(\frac{n}{e}\right)^n (1+o(1))$$

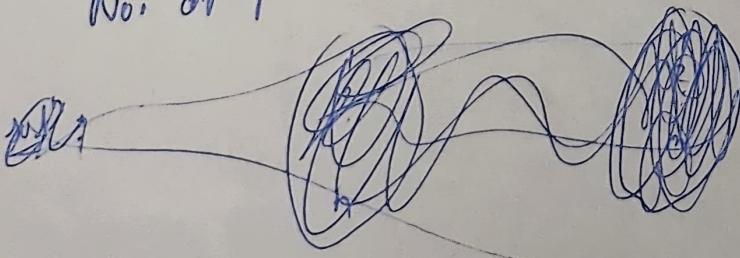
FACT 5 : $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$

Pf: Hint: $\sum_{0 \leq i \leq k} \binom{n}{i} x^i \leq (1+x)^n$

Q) Let $m \leq n$, $f: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, m\}$

where f is surjective

No. of possible such $f = ?$



Q2) How many ways can we partition $\{1, 2, \dots, n\}$ into m subsets?

\downarrow
unordered

Q3) $\Pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be
an injective map
(i.e., Π is a "permutation")

Let $D(n) = \{\Pi : \Pi(i) \neq i, \forall i \in \{1, 2, \dots, n\}\}$

$$|D(n)| = ?$$

Q4) Let $m \leq n$. How many non-decreasing functions are there from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$?

(Here, "non-decreasing" $\rightarrow f(i) \geq f(j), \forall i \geq j$)

Principle of Inclusion and Exclusion (PIE) :

Let A_1, A_2, \dots, A_n be subsets of Ω

want : $|A_1 \cup A_2 \cup \dots \cup A_n| = ?$

$$\text{we know } |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Theorem : Let A_1, A_2, \dots, A_n be subsets of a finite universe Ω . Then,

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^n |A_i| - \sum_{i < j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} \cancel{\boxed{|A_1 \cap A_2 \cap \dots \cap A_n|}}$$

Pf (using induction on n):

Base case ($n=2$) :

~~check!~~

Exercise

$$\left| \underbrace{A_1 \cup A_2 \cup \dots \cup A_n}_{B} \cup A_{n+1} \right|$$

ALT pf:

Let $a \in A_1 \cup A_2 \cup \dots \cup A_n$

WLOG assume that $a \in A_1 \cap A_2 \cap \dots \cap A_j$

and $a \notin A_k, \forall k \geq j+1$

$$1 - \left[j - \binom{j}{2} + \binom{j}{3} - \binom{j}{4} + \dots \right] = 0$$

check!

ALT PT:

Let $f_i : \Omega \rightarrow \{0, 1\}$ s.t

$$f_i(a) = \begin{cases} 1, & \text{if } a \in A \\ 0, & \text{o.w.} \end{cases}$$

where $1 \leq i \leq n$

Suppose $F : \Omega \rightarrow \{0, 1\}$ s.t

$$F(x) := \prod_{i=1}^n (1 - f_i(x))$$

Then, $F(a) = \begin{cases} 0, & \text{if } a \in \bigcup_{i=1}^n A_i \\ 1, & \text{o.w.} \end{cases}$

~~0~~ $\sum_{a \in \bigcup_{i=1}^n A_i} F(a) = 0$

Look at $\prod_{i \in S \subseteq [n]} f_i(a)$

Exercise!

Solutions to Q1, 2, 3 :

Q1) $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, m\}$ / $n \leq m$

f is surjective (onto)

We'll use PIE to solve this

Let $A_i = \{f : f^{-1}(i) = \phi\}$

$f \in A_1 \cup A_2 \cup \dots \cup A_m \rightarrow$ not surjective

No. of such f = total no. of f 's
from $\{1, \dots, n\}$ to $\{1, \dots, m\}$

- $|A_1 \cup A_2 \cup \dots \cup A_m|$ \rightarrow (no. of non
surjective f 's)

$$= m^n - |A_1 \cup A_2 \cup \dots \cup A_m|$$

NOTE: $|A_1 \cap A_2| = (m-2)^n$

$$|A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_k}| = (m-k)^n$$

Complete the solution

Exercise

Q27 $\{1, 2, \dots, n\}$ $\xrightarrow{\text{partitioning}} M$ subsets

method 1 (recursive):

$$\{1, 2, \dots, n\} \rightarrow \{1\}, \{2, 3, \dots, n\}$$



$$\binom{n}{m} = m \binom{n-1}{m} + \binom{n-1}{m-1}$$

Q3)

" Rearrangement "

$\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a 1-1 map

Let $A_i := \{\pi : \pi(i) = i\}$

$$\pi \in A_1 \cup A_2 \cup \dots \cup A_n$$

~~REMEMBER~~ $|D(A)| = n! - |A_1 \cup A_2 \cup \dots \cup A_n|$

NOTE:

$$|A_1 \cap A_2| = (n-2)!$$

$$|A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}|$$

NOTE: $|A_1 \cap A_2| = (n-2)!$

$$|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = (n-k)!$$

Complete the sol'n

Exercise

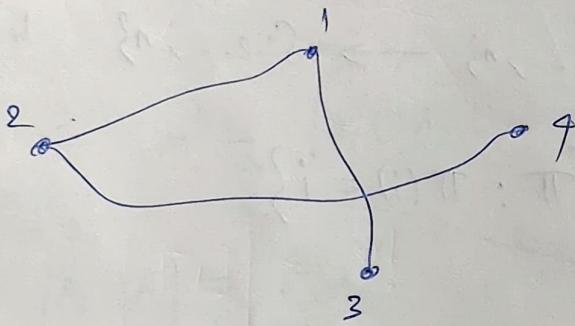
Remark : $D(n) \sim \frac{n!}{e}$

Graph Theory

$$G = (V, E)$$

$$E \subseteq \binom{V}{2}$$

Example : $V = \{1, 2, 3, 4\}$



Def (Graph Homomorphism) : Suppose we have

two graphs $G = (V, E)$ and $G' = (V', E')$

A map $f: V \rightarrow V'$ is called a homomorphism if $\{u, v\} \in E \Rightarrow \{f(u), f(v)\} \in E'$

Defⁿ (injective homomorphism) : 1-1 homomorphism

NOTE: injective homomorphism is also called "embedding")

Defⁿ (surjective homomorphism) : onto homomorphism

Defⁿ (Isomorphism) : both injective and surjective homomorphism

Defⁿ (Isomorphism) : Let $G = (V, E)$ and $G' = (V', E')$

be two graphs

$f: V \rightarrow V'$ is called an isomorphism if

f is ~~not~~ injective, surjective and

$$\{u, v\} \in E \Leftrightarrow \{f(u), f(v)\} \in E'$$

Remark: f can be an injective and surjective homomorphism but NOT an isomorphism

(the 3rd condition is also required)

Defⁿ (Automorphism) :

Lecture 7

Let $F(n) := \{ \text{all graphs whose vertex set is } \{1, 2, \dots, n\} \}$

NOTE: $|F(n)| = 2^{\binom{n}{2}}$

Suppose $G \sim G'$ if G and G' are isomorphic

Theorem: \sim as defined above is an equivalence relation on $F(n)$

Pf: Exercise

$$[G] \in F(n)/\sim$$



"equivalence class containing G "

Note that

$$|[G]| \leq n!$$

Observation: $|F(n)/\sim| \geq \frac{2^{\binom{n}{2}}}{n!}$

NOTATIONS: For a graph G ,

$$V(G) = \{ \text{vertices of } G \}$$

$$E(G) = \{ \text{edges of } G \}$$

Defⁿ (Subgraph): Let G, G' be two graphs.

We say that G' is a subgraph of G if:

$$V(G') \subseteq V(G) \text{ and } E(G') \subseteq E(G)$$

Defⁿ (induced subgraph): for two graphs G, G' ,

G' is called an "induced subgraph" of G if:

$$V(G') \subseteq V(G) \text{ and } E(G') = E(G) \cap \binom{V(G')}{2}$$

~~Example~~

NOTATION:

$$S \subseteq V(G)$$

$G[S] :=$ induced subgraph with vertex set S

Defⁿ (Independent set): $G = (V, E)$

$S \subseteq V$ is called an "independent set" if

$G[S]$ has no edges

Qⁿ: Given G , what is the size of the largest independent set in G ?

(this is an "NP-complete" problem)

Defⁿ (vertex cover):

$G = (V, E)$ and $S \subseteq V$

We say S is a "vertex cover"

if $e \cap S \neq \emptyset$, $\forall e \in E$

(i.e., atleast one end point of any edge of G
is in S)

Lemma: Let G be a graph and $S \subseteq V(G)$

~~If S is a vertex cover, then $V \setminus S$~~
~~is an independent set~~ Then:

S is a vertex cover $\Leftrightarrow V \setminus S$ is an
independent set

Defⁿ (matching): $G = (V, E)$ and $E' \subseteq E$

We say that E' is a matching if

$e \cap e' = \emptyset$, $\forall e \neq e' \in E'$

(PTO) →

Lemma: Let G be a graph and $E' \subseteq E$ be a "maximal matching" in G . Let α be the size of smallest vertex cover of G . Then:

$$\alpha \leq 2 |E'|$$

Pf: Let $S = V(E')$:= vertices participating in the matching E'

Claim: S is a vertex cover of G

Complete the proof (Exercise!)

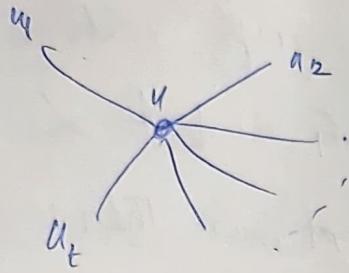
Defⁿ (Dominating set): Let G be a graph
 $S \subseteq V(G)$ is called a "dominating set" if
 $\forall u \in G$, atleast one of the following two statements is true:

$$\textcircled{1} \quad u \in S$$

$$\textcircled{2} \quad \exists \{u, v\} \in E(G) \text{ with } v \in S$$

NOTE: $\textcircled{2} \Leftrightarrow N_G[u] \cap S \neq \emptyset$

see next page $\rightarrow "N_G[u]"$



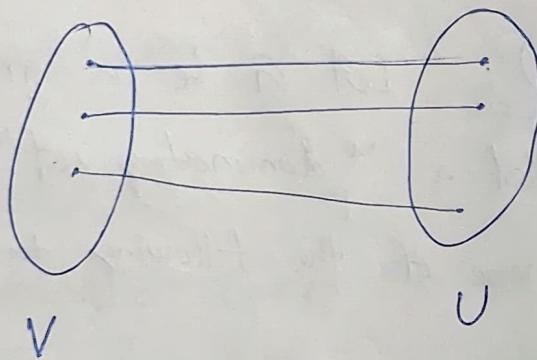
$$N_G(u) := \{v : \{u, v\} \in E(G)\}$$

"open neighbourhood of u "

$$N_G[u] := N_G(u) \cup \{u\}$$

"closed neighbourhood of u "

Def' (Bipartite graph): $G = (V \cup U, E)$



Paths, Cycles, Tours, Walks

$$G = (V, E)$$

Path : $P = (v_0 e_1 v_1 e_2 \dots e_t v_t)$, where

v_1, v_2, \dots, v_t are distinct vertices of G

and $e_i = \{v_{i-1}, v_i\}$, $\forall i \in \{1, 2, \dots, t\}$

We say that P is a path from v_0 to v_t

Remark : "length" of P is t

Cycle : Cycle is a path $C = (v_0 e_1 v_1 e_2 \dots e_t v_t)$,

where $v_t = v_0$ and v_0, v_1, \dots, v_{t-1} are distinct

Cycle : $C = (v_0 e_1 v_1 e_2 \dots v_{t-1} e_t v_t)$, where

$e_i = \{v_{i-1}, v_i\}$, $\forall i \in \{1, 2, \dots, t\}$

$v_t = v_0$ and v_0, v_1, \dots, v_{t-1} are distinct

walk : $W = (v_0 e_1 v_1 \dots e_t v_t)$, where

$e_i = \{v_{i-1}, v_i\}$, $\forall i \in \{1, 2, \dots, t\}$

Tour : Tour is a walk with all the edges distinct

(2) "Distance" in a graph

$\forall u, v \in G$, we define

$$d_G(u, v) = \begin{cases} 0, & \text{if } u = v \\ \text{length of the} \\ \text{shortest path} \\ \text{connecting } u \text{ and } v \end{cases}, \quad \text{or } w$$

NOTE: $d_G(,)$ is a metric space

$$\text{i.e., } \textcircled{1} \quad d_G(u, v) = 0 \iff u = v$$

$$\textcircled{2} \quad d_G(u, v) = d_G(v, u), \quad \forall u, v \in V(G)$$

$$\textcircled{3} \quad d_G(u, v) \leq d_G(u, w) + d_G(w, v),$$

$$\forall u, v, w \in V(G)$$

$$G = (V, E)$$

Suppose $u \sim v$ if $d_G(u, v) < \infty$

we can check that \sim is an equivalence relation on V

equivalently, $u \sim v$ if there is a walk b/w u and v of finite length

Lemma: \sim_G forms an equivalence relation on $V(G)$

(~~REPROVE~~)

Let's look at $V(G)/\sim_G$ (the equivalence classes)

$$= [u_0], [u_1], \dots$$

↑ ↑
"connected
components"

Defn: If $V(G)/\sim_G$

Defn: If $V(G)/\sim_G$ is a singleton set,
then we say that G is "connected"

Adjacency matrix representation:

$$G = (\{1, 2, \dots, n\}, E)$$

$$A_G = (a_{ij})_{1 \leq i, j \leq n}$$

$$a_{ij} := \begin{cases} 1, & \text{if } \{i, j\} \in E \\ 0, & \text{otherwise} \end{cases}$$

Note: Sometimes, we'll call A_G as simply A

$$\text{(P)NOTE : } A^2 = A \cdot A$$

$$= \{ a_{ij}^{(2)} \}_{1 \leq i,j \leq n}$$

$$a_{ij}^{(2)} = \sum_{l=1}^n a_{il} a_{lj}$$

= no. of walks of "length" 2
b/w i and j

$$A^K = A^{K-1} \cdot A$$

$$= \{ a_{ij}^{(K)} \}_{1 \leq i,j \leq n}$$

Lecture 8

Defⁿ (degree sequence):

Graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$
s.t. $\deg(v_1) \leq \deg(v_2) \leq \dots \leq \deg(v_n)$

$(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$ ~~is~~ is called a

"degree sequence"

FACT 1: $\sum_{i=1}^n \deg(v_i) = 2|E|$

COROLLARY: In any graph, we ~~can~~ only have
even no. of ~~vertices~~ vertices with odd degree

FACT 2: In any graph, we'll always have at least
two distinct vertices u and v with $\deg(u) = \deg(v)$

If Hint: use PHTP

Motivation: $G = (V, E)$, $D_G = (\deg(v_1), \dots, \deg(v_n))$ ~~length~~ length

$$d'_i := \begin{cases} \deg(v_i) & , \text{if } i < n-d \\ \deg(v_i)-1 & , \text{if } n-d \leq i \leq n-1 \end{cases}$$

$$D' = (d'_1, \dots, d'_{n-1})$$

D_G is a valid degree sequence $\Leftrightarrow D'$ is a valid
degree sequence

Thm: $D = (d_1, \dots, d_n)$ and $D' = (d'_1, \dots, d'_{n-1})$

where :

$$d'_i := \begin{cases} d_i & \text{if } i < n-d_n \\ d_{i-1} & \text{if } n-d_n \leq i \leq n-1 \end{cases}$$

D is a "valid" degree sequence

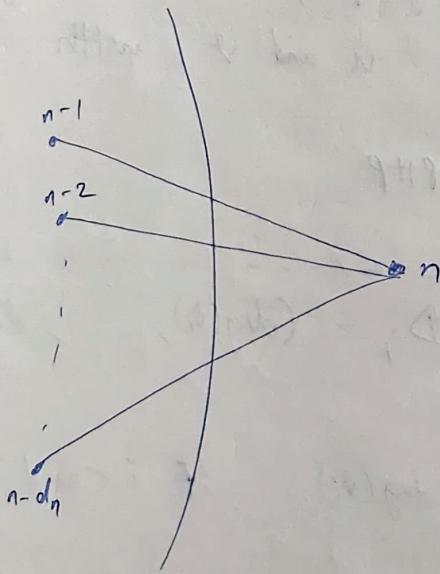
$\Leftrightarrow D'$ is a valid degree sequence

Pf: \leftarrow :

Suppose D' is a valid degree sequence
on $n-1$ vertices

$$\deg_{G'}(i) = d'_i$$

$$G = \begin{matrix} & 1 & 2 & \dots & n-1 & n \\ & \vdots & \vdots & \ddots & \vdots & \end{matrix}$$



Degree sequence of G is D ✓

(\Rightarrow): [using "max-min trick"]

$G: \quad 1 \quad 2 \quad \dots \quad n-d_h-1 \quad \dots \quad n$

$I(G) := \max \{ i : i \text{ is NOT a neighbour of } n \text{ in } G \}$

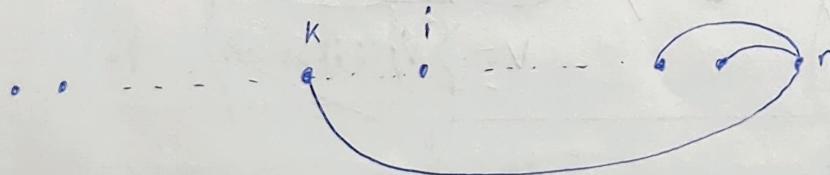
$\hat{G} := \arg \min_{G \text{ with deg. seq. } D} I(G)$

$\text{deg. seq. } D$

claim: \hat{G} is the required graph

(i.e., $I(\hat{G}) = n - d_h - 1$)

pf: Suppose to the contrary let $I(\hat{G}) \geq n - d_h$



Complete
the proof

Exercise

Eulerian Graphs (EG)

Eulerian Tour

Thm: (Characterizing EG's):

$G = (V, E)$ is an Eulerian Graph (EG)

$\Leftrightarrow G$ is connected and degree of each vertex is even

$v_0 e_1 v_1 \dots e_m v_m$, where $v_m = v_0$
 e_1, e_2, \dots, e_m are all distinct

$\{e_1, \dots, e_m\} = E$, 
 $V = \{v_i : i \in \{1, \dots, m\}\}$

Pf: \Rightarrow :

Clearly, G has to be connected

Since vertex $\neq v_0$, we enter and also leave. Thus, their degree has to be even.  $\therefore \deg(v_0)$ will also be even ✓



(\Leftarrow):

$v'_0 e'_1 \dots e'_l v'_l$



Take a tour of max. length

(max. no. of edges in the tour)

Claim 2

First thing to show is that $v'_0 = v'_l$

even

$v'_0 e'_1 \dots e'_l v'_l$ and $v'_0 = v'_l$

claim 2: $V = \{v'_i : i \in \{0, 1, \dots, l\}\}$

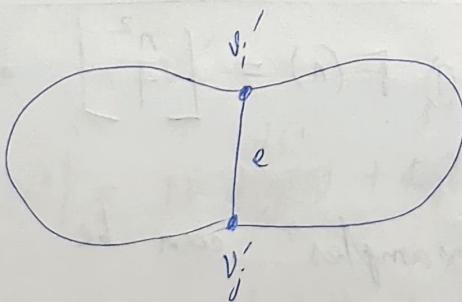
connected

If $E(T) = E$, then we are done

So, suppose $E \setminus E(T) \neq \emptyset$

Let $e \in E \setminus E(T)$

claim: $E(T) = E$

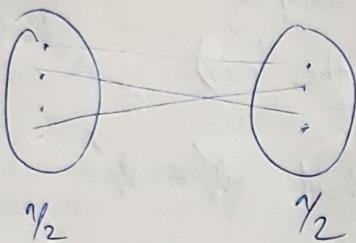


Complete the # Exercise



$$G = (V, E)$$

Suppose G does not have a Δ
Then, how many edges can G have?



$$\frac{n^2}{4}$$

THEOREM.

"Mantel's theorem"

"Kovari-Sos-Turán theorem"

NOTATIONS: $F_\Delta(n) = \max$ no. of edges
in an n vertex
graph with no Δ 's

$$G = (V = \{1, 2, \dots, n\}, E), |E| = F_\Delta(n)$$

Thm (Mantel's theorem): ① $F_\Delta(n) = \left\lfloor \frac{n^2}{4} \right\rfloor$

② Also, all "extremal examples" can be
completely characterized

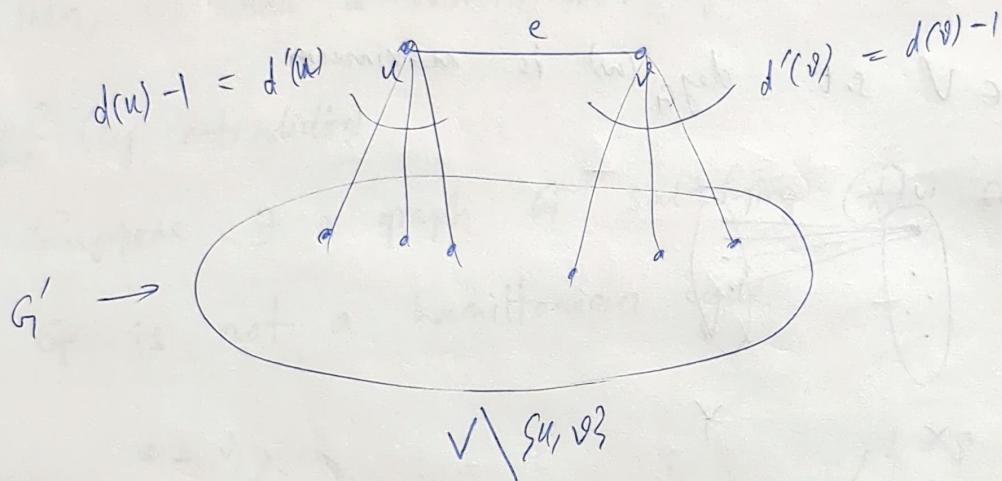
① ~~(using induction on n)~~ :

Base case ($n=1, 2, 3, 4$) : Trivial ✓

Suppose the theorem is true for $n = n$

To show: the theorem holds for $n = n+2$

Let G be a ~~triangle~~-free graph on $(n+2)$ vertices.



Let $G' = V \setminus \{u, v\}$ be the induced subgraph on V'

$$\text{Now, } |E(G)| = 1 + \cancel{d(u)} + \cancel{d(v)} + |E(G')|$$

Apply induction hypothesis to complete the proof

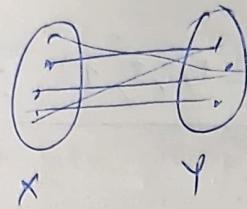
Exercise

□

#f: ②

K_n

$K_{X,Y}$



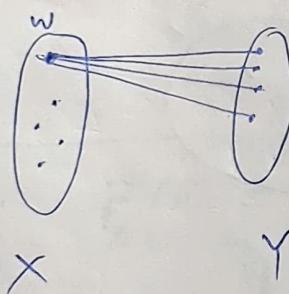
claim:

for any Δ -free graph $G = (V, E)$,

if a two partition of $V = X \cup Y$

s.t. $\deg_G(w) \leq \deg_{K_{X,Y}}(w)$, $\forall w \in V$

f: $w \in V$ s.t. $\deg_G(w)$ is maximum



$$X = \{w\} \cup \left\{ v \mid \bigcap_{\text{neighbours of } w} \right\}$$

neighbours of w

Complete the proof Exercise



Lecture 9

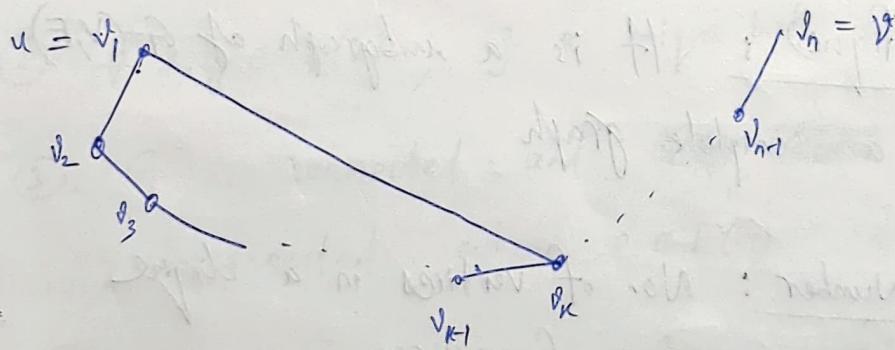
Hamiltonian Cycle (path) : in a graph G is a cycle (path) containing all vertices of G

Theorem : Let $G = (V, E)$, $|V| \geq 3$ } - \otimes
and $\deg(v) \geq \frac{|V|}{2}$, $\forall v \in V$.

Then, G has a hamiltonian cycle

Pf (by contradiction)

Suppose \exists a graph G satisfying \otimes s.t
 G is not a hamiltonian cycle



Complete the proof Exercise

K-coloring :

$G = (V, E)$, $f: V \rightarrow \{1, 2, \dots, k\}$ s.t.

$$f(x) \neq f(y), \quad \{x, y\} \in E$$

Chromatic Number ($\chi(G)$) :

min. no. $\chi(G)$ s.t. a coloring

$$f: V \rightarrow \{1, 2, \dots, \chi(G)\} \text{ exists}$$

Exercise : $\chi(G) \leq 1 + \Delta$, where

$$\Delta = \max_{v \in V(G)} \deg(v)$$

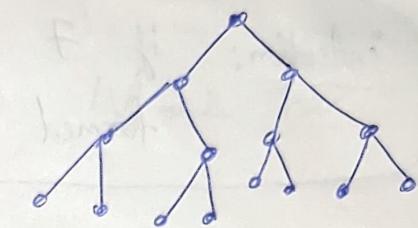
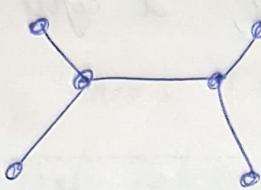
Defn (Clique) : It is a subgraph of $G = (V, E)$
which is a complete graph

Clique Number : No. of vertices in a clique
with maximum no. of vertices
(denoted by $\omega(G)$)

Exercise : Prove / Disprove : $\omega(G) \leq \chi(G)$

Defⁿ (Tree) : A connected acyclic graph

Examples :



("Binary Tree")

Defⁿ (Forest) : A graph whose every connected component is a tree

Theorem : TFAE :

① G is a tree

② $\forall x, y \in V(G)$, \exists a unique path from x to y

③ G is connected and $\cancel{G-e}$ is disconnected where $e \in E(G)$

④ G is acyclic and $G+e$ contains a cycle where $e \notin E(G)$

⑤ G is connected and has $|V|-1$ edges

Pf: $\textcircled{1} \Rightarrow \textcircled{2}$:

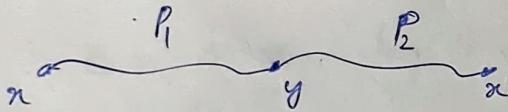
intuition: if \exists 2 paths, a cycle is formed (\Leftrightarrow)

~~Lemma~~ Let $x, y \in V(G)$

Suppose \exists 2 paths from x to y :

P_1 : ~~$x = v_0$~~ $x = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \dots \xrightarrow{e} y$

P_2 : $x = u_0 \xrightarrow{e'_1} u_1 \xrightarrow{e'_2} u_2 \dots \rightarrow y$



$\textcircled{2} \Rightarrow \textcircled{3}$:

$\textcircled{3} \Rightarrow \textcircled{4}$:

$\textcircled{4} \Rightarrow \textcircled{1}$:

Exercise

Defⁿ (leaf nodes): vertices with degree 1
in a tree

Lemma: Every tree $T = (V, E)$, $|V| \geq 2$ has
at least 2 leaf nodes

Pf: Take the path of longest length:

$$v_0 \ v_1 \ v_2 \ \dots \ v_t$$

claim: $\deg(v_0) = 1$

Pf: Suppose to the contrary, let $\deg(v_0) \geq 2$

Complete the proof

Exercise

Exercise: Let v be a leaf node. Then:

T is a tree $\Leftrightarrow T - v$ is a tree

Theorem ($\textcircled{1} \Leftrightarrow \textcircled{5}$ in the previous theorem):

T is a tree $\Leftrightarrow T$ has $|V| - 1$ edges

Pf:

(\Rightarrow)

We'll prove this by induction on
no. of vertices

~~Suppose~~ By induction hypothesis,
any tree T with $(n-1)$ vertices
has $(n-2)$ edges

Consider a tree T with n vertices

Remove a leaf node v from T

Look at $T - v$

Complete the proof Exercise

(\Leftarrow):

Here also, we'll prove this using induction
on no. of vertices

By IH, any connected graph
with $(n-1)$ vertices having $(n-2)$ edges
is a tree.

To prove: G has n vertices and $(n-1)$ edges
 $\Rightarrow G$ is a tree

By "Handshaking lemma"

$$\text{Total degree in } G = 2n - 2$$

Complete the proof

Exercise

AIM:
claim: $\exists u \in G(V)$
s.t $\deg(u) = 1$
and ...

Exercise: $T = (V, E)$ is acyclic and has k edges

Then :

① $k = |V| - 1 \Rightarrow T$ is a tree

② $k < |V| - 1 \Rightarrow T$ has $(|V| - k)$

connected components

Defⁿ (Tree Isomorphism):

$T_1 \cong T_2$ if \exists a bijective function $f: V(T_1) \rightarrow V(T_2)$

(i.e., if $\{x, y\} \in E(T_1) \Leftrightarrow \{f(x), f(y)\} \in E(T_2)$)

Defⁿ (rooted tree): A tree with a vertex

which is root

[denoted by (T, r)]

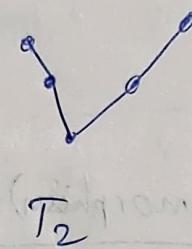
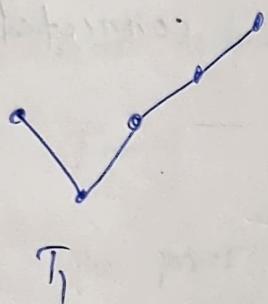
$\{x, y\} \in E$

x → father of y
 y → child of x

Rooted Isomorphism $[T_1, r_1] \cong' [T_2, r_2]$:

$$T_1 \cong T_2 \text{ and } f(r_1) = r_2$$

Example:

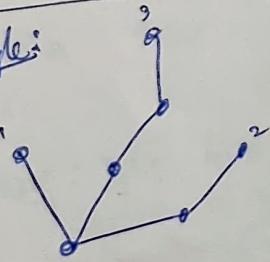


$$T_1 \cong T_2 \text{ but } T_1 \not\cong'' T_2$$

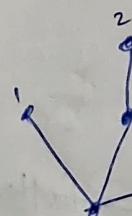
Planted Isomorphism ($T_1 \cong'' T_2$) :

Rooted Isomorphism + a drawing of T

Example:



T_1



T_2

$$T_1 \cong' T_2, \text{ but } T_1 \not\cong'' T_2$$

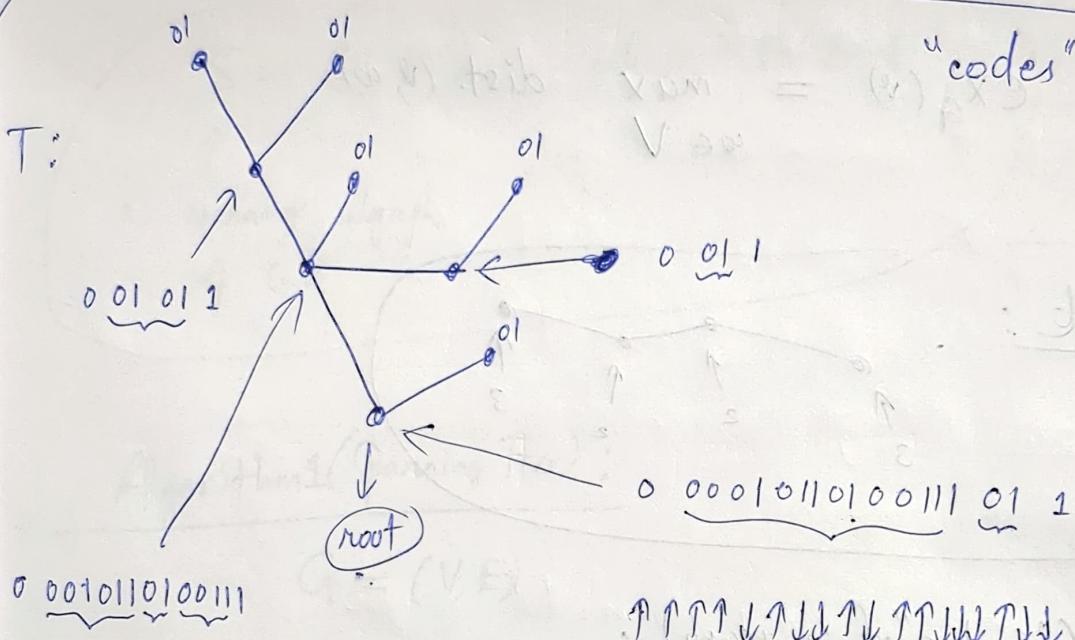
"linear ordering"

Remark:

$$\cong'' \rightarrow \cong' \rightarrow \cong$$

(i.e., $P\mathbb{I} \Rightarrow R\mathbb{I} \Rightarrow \mathbb{I}$)

T:



Can reconstruct T
using this

Lexicographic Ordering: Two codes A and B set

① A is a prefix of B $\Rightarrow A \leq B$

② j is the smallest index where bits differ

$$A_j < B_j \Rightarrow A \leq B$$

$$A_j > B_j \Rightarrow B \leq A$$

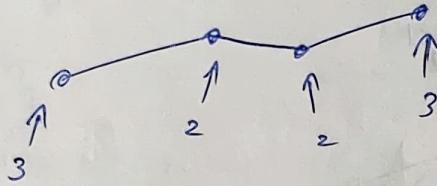
Defⁿ (Eccentricity) :

for a vertex v of a graph G ,

eccentricity of v in G is given by:

$$ex_G(v) = \max_{w \in V} \text{dist.}(v, w)$$

Example :



Defⁿ ("center" of a graph) :

A vertex with minimum eccentricity

NOTATION : $C(G) = \{\text{all centers of } G\}$

Lemma : T is a tree $\Rightarrow |C(T)| \leq 2$

Furthermore, if $C(T) = \{u, v\}$, then

$$\{u, v\} \in E(T)$$

Defⁿ (Spanning Subgraph) : $G = (V, E)$

A subgraph of a graph G is called "spanning" if no. of vertices of the subgraph is equal to $|V|$.

$$S = (V, E') \quad \text{where } E' \subseteq E$$

↑
a spanning subgraph
of G

Algorithm 1 (Spanning Tree) :

$$G = (V, E)$$

order the edges e_1, e_2, \dots, e_m

$$E_0 = \emptyset$$

Calculate E_i from E_{i-1} :

$$E_i = \begin{cases} E_{i-1} \cup \{e_i\}, & \text{if } (V, E_{i-1} \cup \{e_i\}) \text{ has} \\ & \text{no. cycle} \\ E_{i-1}, & \text{o.w} \end{cases}$$

Stop condition : If some E_i has $n-1$ edges
or $i = m$

Correctness of Algorithm1 :

If Algorithm1 produces a graph T :

① If T has $(n-1)$ edges, then T is a spanning tree.

② If T has $< (n-1)$ edges, then G has $(n-k)$ connected components

Exercise: ② T has $(n-k)$ connected components

Consequence: Every connected graph has a spanning tree

Lecture 10

$G = (V, E)$

Defⁿ 1 (k -connectedness) : Let $n \geq k+1$ and $\text{Graph } G$ be a graph on n vertices.

① G is said to be "vertex k -connected" if

$\forall S \subseteq V$ with $|S| = k-1$, $G[V \setminus S]$ is connected

② G is said to be "edge k -connected" if

$\forall F \subsetneq E$ with $|F| = k-1$, the subgraph $G' = (V, E \setminus F)$ is connected.

Remark : We'll be characterizing vertex 2-connected graphs.

Q: Let G be a connected graph. What happens (in terms of no. of connected components) when we remove a vertex from G and when we remove an edge from G ?

NOTE: From now, we'll ~~not~~ write vertex k -connected as just k -connected.

Theorem (Characterization of vertex 2-connected graphs):

A graph $G = (V, E)$ is 2-connected

\iff \forall distinct vertices $u, v \in G$,

\exists two vertex disjoint paths P_1, P_2 from u and v

pf:

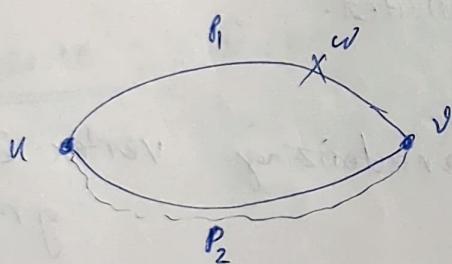
(\Leftarrow):

Let $w \in V$. Consider $G' = G[V \setminus \{w\}]$

Take any two vertices $u, v \in G'$

We know that \exists two vertex disjoint paths

P_1 and P_2 in G'



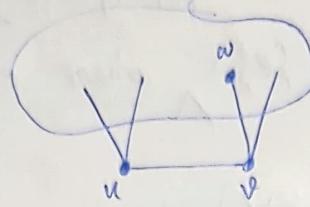
(\Rightarrow):

First observe the following claim about 2-connected graphs.

Claim: ① Let G be 2-connected and "e" be an edge in G . Then, the subgraph $G \setminus \{e\}$ is connected.

② G cannot have a degree one vertex

$$e = \{u, v\}$$



We've to show that $\neq u \neq v$, \exists a cycle C containing both u and v

Base Case \bullet ($d(u, v) = 1$, i.e., $\{u, v\} \in E$): From Claim, we know that \exists a path P connecting u and v that doesn't use the edge $e = \{u, v\}$. Then, $C = P \cup \{e\}$

Induction hypothesis: If $d(w_1, w_2) = k-1$, then \exists a cycle C ~~containing~~ that contains w_1 and w_2

Now, suppose $d(u, v) = k$

Complete the proof

Exercise

□

$\times - - - - - X$

PTO →

GRAPH OPERATIONS

(or $G - \{e\}$),

① Edge deletions: $G \setminus \{e\}$, where $e \in E$

② Edge addition: ~~$G + e$~~ ,
where $e \in \binom{V}{2} \setminus E$

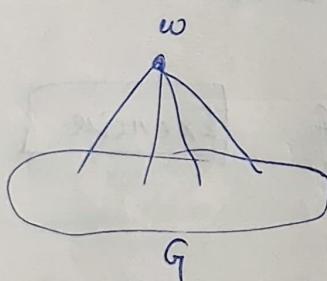
③ Vertex deletion: $G[V \setminus \{w\}]$, where $w \in V$

④ Cone / vertex addition: for $w \notin V$,

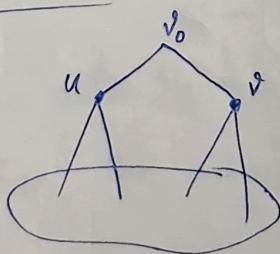
$$G' = (V', E') \text{, where:}$$

$$V' = V \cup \{w\}$$

$$E' = E \cup \{(w, u) : u \in V\}$$



⑤ Edge subdivision:



Theorem:

$$G \underset{\substack{\text{2-connected} \\ \text{graph}}}{\cong} G'$$



edge addition and edge subdivision



G

\cong

G'



2-connected
graph

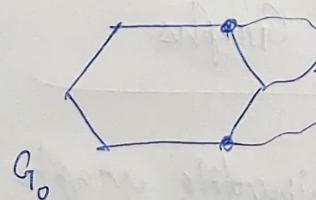
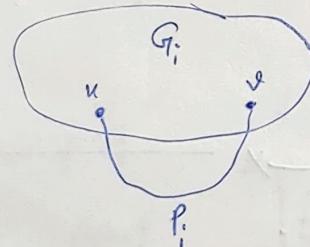


In other words: Any 2-connected graph is
isomorphic to

Pf: Consider a sequence $G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$

such that every G_i is 2-connected

$$G_i \cup P_i = G_{i+1}$$



Claim: ② If $G_i \neq G$, then \exists a path P_i connecting two vertices of G_i and it is internally vertex disjoint from G_i .

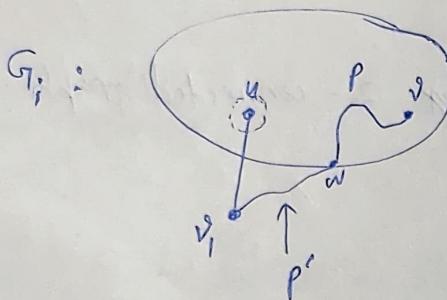
① Each G_i is 2-connected

If $V_{G_i} = V_G$, we are done (check!)

So, assume $V_{G_i} \neq V_G$

Then, $\exists \{u, v\} \in E_G$ s.t. $u \in G_i$

and $v \in V_G \setminus V_{G_i}$



$$P_{i+1} = P' + \{u, v\}$$

Complete the proof Exercise

□

Recall (Lecture 9) :

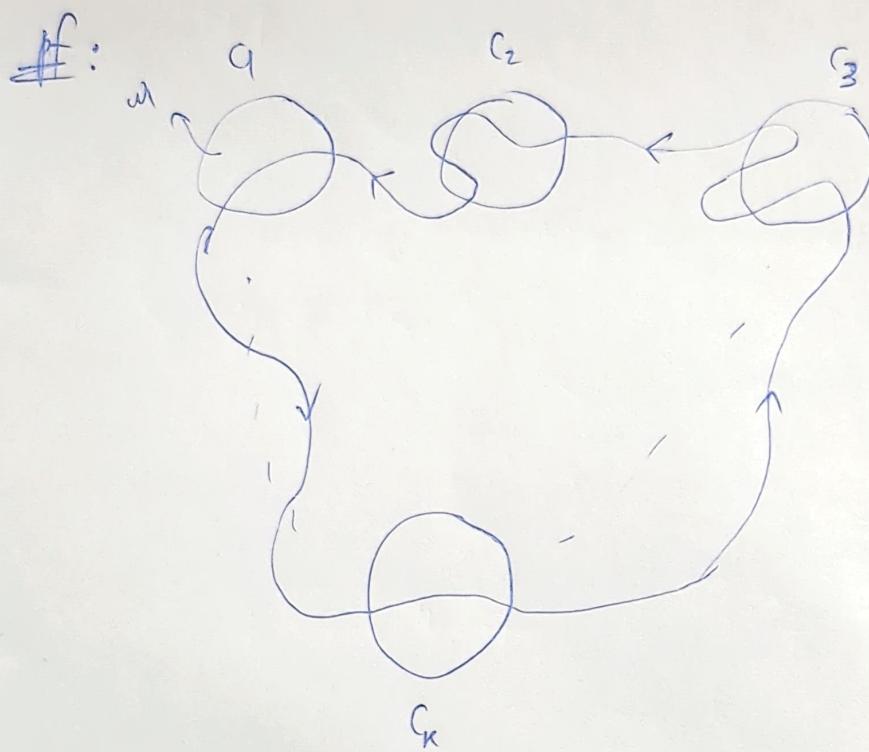
Hamiltonian Graphs

Theorem: Let $\textcircled{1}$ a bipartite graph

$G = (A \cup B, E)$ be Hamiltonian.

Then, $|A| = |B|$

Lemma: Let $G' = (V', E')$ be a Hamiltonian graph and $S \subseteq V'$. Then, $G'[V' \setminus S]$ has at most $|S|$ connected components.



Use "charging" idea to complete
the proof

Exercise

□

Lecture 11 [Lecture 9 continued]

Recall (Lecture 9) :

Algo 1 (Spanning Tree)

$$G = (V, E), |V| = n, |E| = m$$

Successively constructing $E_0, E_1, \dots \subseteq E$:

Order the edges e_1, e_2, \dots, e_m

$$E_0 = \emptyset$$

Given E_{i-1} :

$$E_i = \begin{cases} E_{i-1} \cup \{e_i\}, & \text{if } (V, E_{i-1} \cup \{e_i\}) \\ & \text{is acyclic} \\ E_{i-1}, & \text{o.w.} \end{cases}$$

Stop : $|E_i| = n-1$ for some i or $i=m$

Algo 2 (Spanning Tree)

$G = (V, E)$, $|V| = n$, $|E| = m$

Successively construct $V_0, V_1, \dots \subseteq V$; $E_0, E_1, \dots \subseteq E$:

$V_0 = \{\emptyset\}$, $E_0 = \{\emptyset\}$,

given V_{i-1}, E_{i-1}

find $x_i \in V_{i-1}$, $y_i \in V \setminus V_{i-1}$ s.t. $\{x_i, y_i\} \in E$

$V_i = V_{i-1} \cup \{y_i\}$

$E_i = E_{i-1} \cup \{x_i, y_i\}$

Stop: No such edge $\{x_i, y_i\}$ can be found

Correctness of Algo 1:

Algo 1 gives a subgraph T of G

- Claims
- i) If $|E(T)| = (n-1) \Rightarrow T$ is a spanning tree
 - ii) If $|E(T)| < (n-1) \Rightarrow G$ is disconnected and has $(n-K)$ connected components

Theorems (HW's in Lecture 9):

① T is acyclic and has $(n-1)$ edges $\Rightarrow T$ is a tree

② T is acyclic and has $< (n-1)$ edges $\Rightarrow T$ is disconnected and has $(n-K)$ connected components

PF (Claim ii))

T has $(n-k)$ connected components

Since T is a spanning ^{sub}graph of G

the no. of connected components in G $\leq n-k$

need to prove this first too

Complete the rest of the proof

Exercise

Correctness of Algo 2:

Algo 2 gives a subgraph T

Claims { i) If $|E(T)| = (n-1) \Rightarrow T$ is a spanning tree
ii) If $|E(T)| = k < (n-1)$ edges $\Rightarrow G$ is disconnected and T is the spanning tree of the connected component of G with vertex v

Complete the prof

Exercise

Q: Which algorithm (Algo 1/Algo 2) gives us more information about the graph?

A: Both algo's give the same amount of information (Hint: Think about arrays in case of algo 2)

Also, note that the running times of algo 1 and algo 2 are the same

X

X

COROLLARY

Every connected graph has a spanning tree

If:

Exercise!

X

X

Prim's and Kruskal algorithms :

Defⁿ (weighted graph) : $G = (V, E)$, with

a function $w: E \rightarrow \mathbb{R}^+ \cup \{0\}$

Q?
Exercise: G is a ^{connected} weighted graph. Find a connected spanning subgraph T with minimum weight

$$w(T) := \sum_{e \in E(T)} w(e)$$

Defⁿ (minimum spanning tree [MST]):

A spanning tree with minimum weight in a weighted graph G

Let's modify "Algo 1" for MST:

just change this line: →

order the edges e_1, e_2, \dots, e_m s.t

$$w(e_1) \leq w(e_2) \leq \dots \leq w(e_m)$$

discussed previously for spanning tree

This modified version of Algo 1 is known as

"Kruskal's algorithm"

Now, let's modify "Algo 2" for MST:

just change this line:

find $x_i \in V_{i-1}, y_i \in V \setminus V_{i-1}$ s.t $\{x_i, y_i\} \in E'$,

~~the edge~~ where E' is the edge with

minimum weight from the set $\{ \{x_i, y_i\} \in E : x_i \in V_{i-1} \text{ and } y_i \notin V_{i-1} \}$

This modified version of Algo 2 is called

"Prim's algorithm"

Lemma: $G = (V, E)$, G' and G'' are spanning subgraphs of G s.t. $G' = (V, E')$ is disconnected, $G'' = (V, E'')$ is acyclic and $|E''| > |E'|$

Then, $\exists e \in E''$ which connects two connected components of G'

Pf: G' has s connected components,

$$|E'| \geq |V| - s$$

$$|E'' \cap E'| \leq |V| - s$$

Complete the proof Exercise

Correctness of Kruskal's algorithm:

Kruskal's gives a spanning tree T

Let T' be any other spanning tree

To show: $w(T) \leq w(T')$

$T: e_1, e_2, \dots, e_{n-1}$ } ordered by weight,
 $T': e'_1, e'_2, \dots, e'_{n-1}$ } $w(e_i) \leq w(e'_i), \forall i$

Suppose to the contrary,
let $w(e_i) > w(e'_i)$ for
some i (take i to be the
smallest such index)

$$E' = \{e_1, e_2, \dots, e_{i-1}\}$$

$$E'' = \{e'_1, e'_2, \dots, e'_{i-1}, e'_i\}$$

$$|E''| > |E'|$$

Thus, by Lemma, $\exists e \in E''$ which
~~does~~ does NOT form a cycle
with edges in E'

$$w(e) \leq w(e'_i) < w(e_i)$$

~~So, the algorithm chooses edge e'_i over e~~ \Rightarrow

Thus, Kruskal's algo is correct ✓

Correctness of Prim's algorithm :

Prim's algo gives us a spanning tree T ,

$$E(T) = \{e_1, e_2, \dots, e_{n-1}\}$$

Let T be an MCT

Let T' be an MST s.t $e_1, e_2, \dots, e_k \in E(T')$
and $e_{k+1} \notin E(T')$, where k is the largest

from Prim's algo : $w(e_{k+1}) \leq w(e)$

Now, consider $T'' = (T' + e_{k+1}) - e$

(Check that T'' is a spanning tree)

$$\begin{aligned} w(T'') &= w(T') + w(e_{k+1}) - w(e) \\ &\leq w(T') \end{aligned}$$

$$\Rightarrow w(T'') = w(T')$$

$\therefore T'$ is an MST

Using this, we can arrive at a
contradiction check!

Thus, Prim's algo is correct ✓

Lecture 12

midsem format (Total marks = 60) , (Time = 2 hrs.)

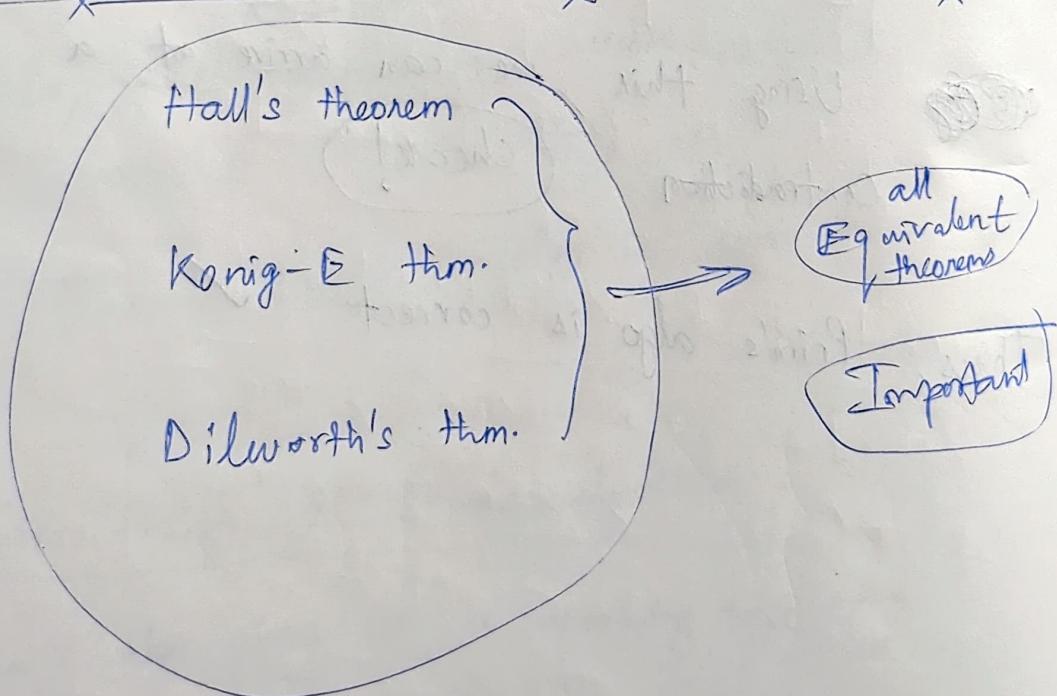
Group A (30 marks)

- ① 4 questions
- ② Exact theorems covered in class
- ③ NO step marking X

Group B (30 marks)

- ④ 4 questions
- ⑤ new problems

- ⑥ step marking ✓



Matching in Bipartite Graphs :

Defn: $G = (V, E)$

$$M_t = \{e_1, e_2, \dots, e_t\} \subseteq E$$

$\circledcirc M_t$ is a "matching" if elements of M_t are pairwise disjoint

~~$G = (A \cup B, E)$, where A, B are both independent sets~~

Defn (perfect matching) :

A matching M_t is a perfect matching if

$$e_1 \cup e_2 \cup \dots \cup e_t = V$$

Defn (Edge cover of a set) :

$\{e_1, \dots, e_s\} \subseteq E$ is an edge cover of $T \subseteq V$ if

$$T \subseteq \bigcup_{i=1}^s e_i$$

Defn (d -matching) : $S \subseteq E$ is a " d -matching" if max. degree of a vertex in graph $G' = (V, S)$ is at most d

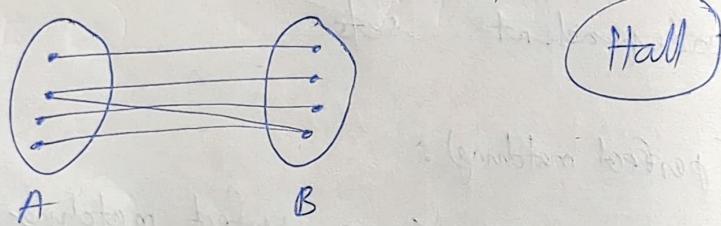
\mathcal{Q}^n : $P_1, P_2, \dots, P_n \in \mathbb{R}^2$

$\underbrace{\qquad\qquad\qquad}_{\text{points}}$

Is there a way of pairing two points and connecting a line ^{between them}, s.t. no lines intersect?

~~DEFINITION~~

\mathcal{Q}^n : $G = (A \cup B, E)$, where A, B are both independent sets in G



Hall

\mathcal{Q}^n :

a_1, a_2, \dots, a_r
 $\uparrow \quad \uparrow \quad \uparrow$
 A_1, A_2, \dots, A_r , where
 $A_i \subseteq [n]$

"SDR"
 (System of
 Distinct
 Representative)

s.t. $a_i \neq a_j, i \neq j$

Thm (Hall's theorem) :

$G = (A \cup B, E)$ has a matching
that covers $A \Leftrightarrow |N(S)| \geq |S|, \forall S \subseteq A$

$$\text{where } N(S) = \bigcup_{u_i \in S} N(u_i)$$

Pf :

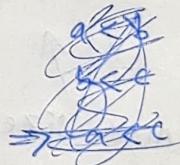
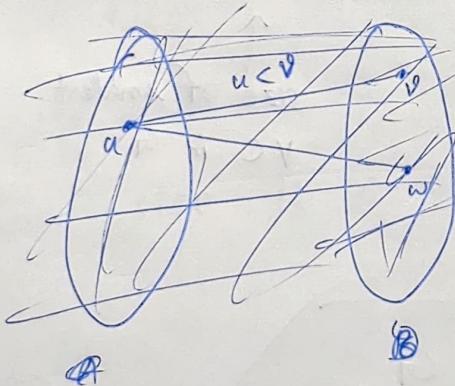
(\Rightarrow) :

Easy (check!)

(\Leftarrow) :

$S \subseteq V$ is an independent set

$\Leftrightarrow \overline{S}$ is a VC



Observations: ① Length of a chain can be atmost 2

② B is an antichain

③ B is a largest ~~subset~~ antichain

~~of antichain~~ Suppose ~~is an antichain~~ $|B| > |C \cap G|$

\exists a chain decomposition (by Dilworth's thm.)
 and ① with $|A|$ many chains of length 2
 and $|B| - |A|$ many chains of length 1

Complete the proof

Exercise

□

Theorem (Konig's theorem) :

$G = (A \cup B, E)$, where A, B are indep. sets in G

Then, $\forall G \text{ such that } V \geq 2$

$$v(G) = \gamma(G)$$

\uparrow

max. size
matching in G

\uparrow

size of smallest
VC in G

pf: ② for all graphs G ,

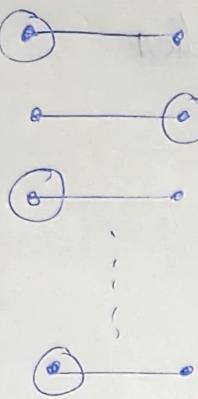
$v(G) \leq \gamma(G)$

\parallel \parallel

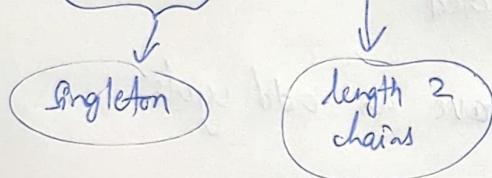
I

Let $M = v(G)$, $K = \gamma(G)$

and $n = \text{no. of vertices in } G$



$$(n - 2M) + M \leftarrow \text{length of chain decomposition}$$



$$= n - M$$

i.e., length of chain decomposition = $n - M$

Since the smallest VC size is k , the largest anti-chain size is $(n - k)$

$$\Rightarrow (n - k) \geq (n - M)$$

$$\Rightarrow k \leq M \quad \text{--- (I)}$$

$$(I), (II) \Rightarrow k = M$$

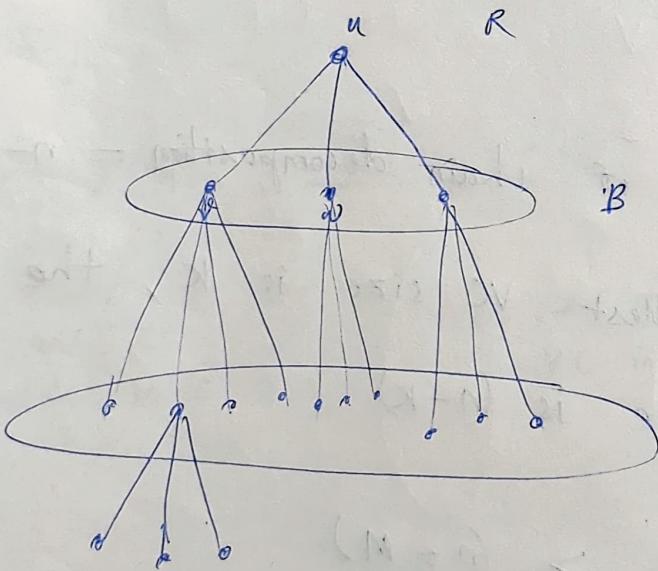
$$\text{i.e., } \cancel{\chi(G)} = \chi(G) \quad \checkmark$$

□

Theorem (Odd cycle characterization of Bipartite graphs)

Pf: Assume G is connected

Suppose G doesn't have an odd cycle
and G is bipartite



Lecture 13

Midsem discussion

Group B :

B&1) G has n vertices, m edges and T Δ 's

To show: $T \geq \frac{m}{3n} (4m - n^2)$



$$\deg(i) = d_i, \deg(j) = d_j$$

T_{ij} = no. of Δ 's containing edge ij

$$\text{Then, } T_{ij} \geq (d_i - 1) + (d_j - 1) - (n - 2)$$

$$= d_i + d_j - n$$

$$\text{Now, } \sum_{(ij) \in E} T_{ij} = 3T$$

$$\Rightarrow 3T \geq \sum_{(ij) \in E} (d_i + d_j - n)$$

$$\left[\sum_{(ij) \in E} (d_i + d_j) \right] - mn = \sum_{i \in V} d_i^2 - mn$$

$$\geq \frac{\left(\sum d_i \right)^2}{n} - mn$$

$$= \frac{4m^2}{n} - mn = \frac{m}{n} (4m - n^2)$$

using CS inequality

$$\Rightarrow T \geq \frac{m}{3n} (4m - n^2) \quad \checkmark$$

$$BQ2) A = (A_{ij})_{1 \leq i,j \leq n} \text{ s.t. } A_{ij} \in \{0, 1\}, \forall i, j$$

Sum of entries of any row/column of A is k

To show: A can be written as a sum of k permutation matrices

Soln: Sum of entries of any row/column of A is k

i.e., $A \underset{\text{sum of entries}}{\sim} \underset{\text{rows}}{1^T} = k \underset{\text{columns}}{1} \underset{\text{where }}{\sim} \underset{\text{rows}}{1^T}$ where $\underset{\text{columns}}{1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$

and $\underset{\text{rows}}{1^T} A = k \underset{\text{columns}}{1}$

We'll use induction on K now, $: b = (i)$

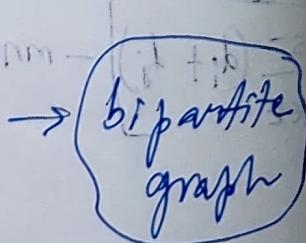
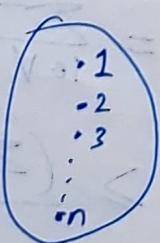
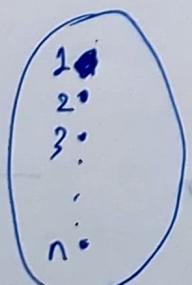
Base case:

$$\text{if } a = (1) \text{ then } a = \underset{\text{rows}}{1^T}$$

$$(1) = (1-1) + (1-0) + (1-0) \leq \underset{\text{rows}}{1^T}$$

Suppose true for $(k-1)$

$$A_{mn} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \underset{n \times n}{=} \underset{m \times n}{=}$$



$$(a_{11} - 1) + \dots + (a_{1n} - 1) = m - \underset{n \times 1}{\text{columns}}$$

$$(a_{21} - 1) + \dots + (a_{2n} - 1) \leq T \leq$$

By Hall's theorem, if a matching

perfect matching

Example

$$A = \begin{pmatrix} 0 & 1 & \boxed{1} \\ \boxed{1} & 0 & 1 \\ 1 & \boxed{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

permutation matrix embedded
in A

use
induction
hypothesis
here

Exercise!

B&Q3) $A_1, A_2, \dots, A_n \rightarrow$ distinct subsets of \mathbb{N}

To show: \exists a subset $X \subseteq \mathbb{N}$ with $|X| \leq n-1$ s.t. $A_i \cap X \neq A_j \cap X$, $\forall i \neq j$

Solⁿ: Use ^(strong) induction on n

Base case ($n=2$) ✓

Suppose true for $1, 2, \dots, n-1$

(PTO) \rightarrow

A_1, A_2, \dots, A_n

$$\text{Fix } x \in \bigcup_{i \in [n]} A_i \setminus \bigcap_{i=1}^n A_i$$

divide the collection into two sets

$$\{A_1, A_2, \dots, A_j\}, \{A_{j+1}, \dots, A_n\}$$

↑
size k_1
contains x
not mapped
to \mathbb{N}

↑
size k_2
doesn't contain x
to \mathbb{N}

$$k_1, k_2 \leq n-1$$

Complete the proof

Exercise

B&Q4) To prove: If \mathcal{P} posets (X, \leq) , \mathcal{F} an embedding into the ordered set $(2^X, \subseteq)$

if:

no mistakes

$(S = \mathbb{N})$ \rightarrow \mathbb{N}

← \mathbb{N}

Lecture 14

Generating Functions

$a_0, a_1, a_2, \dots, a_r$

$$g(x) = a_0 + a_1 x + \dots + a_r x^r / (1 - x)^{r+1}$$

where a_r = no. of ways in which r objects are selected from n objects

$$g(x) = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{r} x^r$$

Q) find the generating function for a_r , the no. of ways to select r balls from 3 green, 3 blue, 3 red balls

$$\text{SOLN: } e_1 \rightarrow \text{green}, e_2 \rightarrow \text{blue}, e_3 \rightarrow \text{red}$$

$$e_1 + e_2 + e_3 = r$$

$$e_1 + e_2 + e_3 = r$$

$$x^{e_1} \cdot x^{e_2} \cdot x^{e_3} + \dots = (1+x)^{e_1} \cdot (1+x)^{e_2} \cdot (1+x)^{e_3} = \frac{1}{(1-x)^{e_1+e_2+e_3}}$$

$$+ x^{e_1+e_2+e_3} + \dots$$

$$+ x^{e_1+e_2+e_3} + \dots$$

Q) Use a generating function to model the problem of counting all selection of 6 objects chosen from 3 types of objects with repetition upto 4 objects of each type

$$(1+x+x^2+\dots+x^4)^3 = \dots + x^6 + \dots = (b)$$

find the coefficient of ~~x^6~~ here

Polynomial Expansions :

$$\textcircled{1} \quad \frac{1-x^{n+1}}{1-x} = 1+x+x^2+\dots+x^n$$

$$\textcircled{2} \quad \frac{1}{1-x} = (-x)^{-1} = 1+x+x^2+\dots$$

$$\textcircled{3} \quad (1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{r} x^n$$

$$\textcircled{4} \quad (-x^n)^r = 1 - \binom{n}{1} x^n + \binom{n}{2} x^{2n} - \dots + (-1)^n$$

$$\textcircled{5} \quad \frac{1}{(1-x)^n} = (-x)^{-n} = 1 + \binom{1+n-1}{1} x + \binom{2+n-1}{2} x^2 + \dots + \binom{r+n-1}{r} x^r + \dots$$

Note: If $h(x) = f(x)g(x)$, where $f(x) = a_0 + a_1x + a_2x^2 + \dots$
 $g(x) = b_0 + b_1x + b_2x^2 + \dots$, then
 $h(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \dots$
Then, coefficient of $x^r = a_r b_0 + a_{r-1}b_1 + a_{r-2}b_2 + \dots + a_0b_r$

Q) Use generating functions ~~defined~~ to find
the no. of ways to collect ₹15 from 20
distinct people if each of the first 19 people
can give a rupee and the 20th person
gives either 0, 1 or 5

$$(1+x)^{19} (1+x+x^5) \leftarrow \text{HINT}$$

Soln:

Q) Verify the binomial identity:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$$

Soln: $(1+x)^{2n} = (1+x)^n (1+x)^n$

$$\begin{aligned} & a_0b_n + a_1b_{n-1} + \dots + a_nb_0 \\ &= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n}\binom{n}{0} \\ &= \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 \end{aligned}$$

Q) find the generating function for a_r ,
 the no. of ways to express r as a
 sum of distinct positive integers

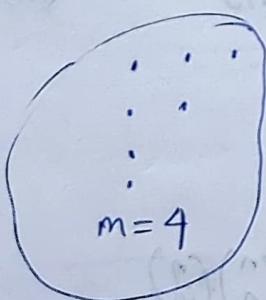
$$\text{sol}^n: \quad (1+x)(1+x^2)(1+x^3)\dots(1+x^{r-1})(1+x^r)\dots$$

coefficient of x^r in

Q) Show that the no. of partitions of an integer r as a sum of m positive integers
 is equal to the no. of partitions of r
 as a sum of r integers, the largest
 of which is m

solⁿ: "Ferrer's diagram"

$$7 = 1+1+2+3$$



Exponential Generating Function

Q) a_r , $g(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots + a_r \frac{x^r}{r!} + \dots$

find the exponential GF of a_r ,
 where $a_r = \text{no. of } r \text{ arrangements without}$
 repetition of n objects.

Sol: $(1+x)^n \rightarrow \binom{n}{r} = \frac{n!}{(n-r)! r!}$

$\left[\frac{1}{(n-r)!} + \frac{1}{(n-r+1)!} + \dots + \frac{1}{(n-1)!} + \frac{1}{n!} \right] x^n = \frac{n!}{(n-r)!} \frac{1}{r!} x^r$
 no. of arrangements

find the coefficient of $\frac{x^r}{r!}$ in $(1+x)^n$

$$A(x) = \sum a_n x^n, \quad B(x) = \sum b_n x^n, \quad C(x) = \sum c_n x^n$$

① If, $b_n = d \cdot a_n$, then $B(x) = d A(x)$
 (Here, d is a constant)

② If $c_n = \sum_{i=0}^n a_i b_{n-i}$, then $C(x) = A(x) B(x)$

③ If $b_n = a_{n-k}$ except $b_i = 0$ for $i < k$,
 then $B(x) = x^k A(x)$

Q) Find a generating function $f(x)$ with

$$a_0 = 2x^2$$

Soln: $(1-x)^{-1} = 1+x+x^2+\dots$

$$\frac{ax}{(1-x)^2} = \frac{d}{dx} \left(\frac{1}{1-x} \right) = x(1+2x+3x^2+\dots)$$
$$= x+2x^2+3x^3+\dots$$

$$ax \cdot \frac{d}{dx} \frac{x}{(1-x)^2} = a \cdot \left[x \cdot \frac{1}{(1-x)^3} + \frac{1}{(1-x)^2} \right]$$
$$= a [1+2 \cdot 2x+3 \cdot 3x+\dots]$$

~~Multiplication~~

Exercise!

Theorem: If $h(x)$ is a generating function where a_r is the coefficient of x^r , then

$$h^*(x) = \frac{h(x)}{1-x}$$
 is a generating function

of the sum of the a_r 's

$$h^*(x) = a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + \dots + \left(\sum_{j=0}^r a_j x^j \right) x^r + \dots$$

$$h^*(x) = f(x) h(x),$$

$$f(x) = \frac{1}{1-x} \quad \textcircled{B}$$

Lecture 15

Recurrence relations

sequence $\{a_n\}_{n \geq 1}$ $a_n : \mathbb{N} \rightarrow \mathbb{R}$

$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_1)$$

1) computing factorial of n:

$$a_n = n a_{n-1}, \quad a_0 = 1$$

2) Dividing the plane by lines:

Given n lines in a plane s.t. any 2 are intersecting and any 3 are not intersecting at a common point

To find: How many regions are there in the plane?

$$n=0, \quad a_0 = 1$$

$$n=1, \quad a_1 = 2$$

$$n=2, \quad a_2 = 4$$

$$n=3, \quad a_3 = 7$$

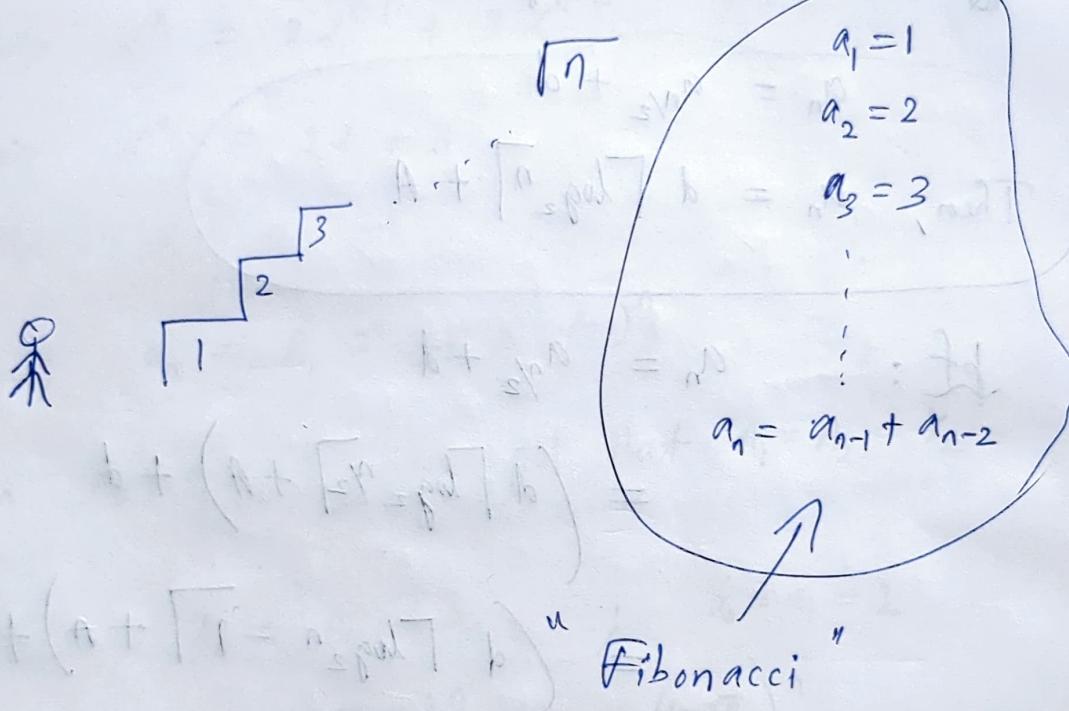
Given n , what is a_n ?

a_1, a_2, \dots, a_{n-1} are given

$$a_n = a_{n-1} + n \quad ; \quad n \geq 1$$

$$a_0 = 1$$

3) Climbing Stairs problem :



① Divide and Conquer relation :

$$a_n = c a_{n/2} + f(n)$$

c	f(n)	a_n
$c=1$	d	$d \lceil \log_2 n \rceil + A$
$c=2$	d	An $A n - d$
$c > 2$	$\Theta(dn)$	$A \Theta(n \log_2 c) + \left(\frac{2d}{2-c}\right)n$

$$c = 2$$

$$\frac{dn}{d\ln n} \left(\ln n + \left\lceil \log_2 n \right\rceil + A \right)$$

d is a constant, A is a constant you need to calculate from initial condition

We'll see one proof:

$$a_n = a_{n/2} + d$$

$$\text{Then, } a_n = d \left\lceil \log_2 n \right\rceil + A$$

$$\text{pf: } a_n = a_{n/2} + d$$

$$= \left(d \left\lceil \log_2 n/2 \right\rceil + A \right) + d$$

$$= \left(d \left\lceil \log_2 n - 1 \right\rceil + A \right) + d$$

$$= d \left\lceil \log_2 n \right\rceil - d + A + d$$

② Linear Recurrence relations:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r}$$

General technique: Take $a_n = \lambda^n$, $\forall n \geq 1$

$$\text{We get: } \lambda^n = c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_r \lambda^{n-r}$$
$$\Rightarrow \lambda^r - c_1 \lambda^{r-1} - c_2 \lambda^{r-2} - \dots - c_r = 0$$

→ "Characteristic eqⁿ"

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the roots of the above eqⁿ. General solⁿ will be

$$a_n = A_1 \lambda_1^n + A_2 \lambda_2^n + \dots + A_r \lambda_r^n$$

Example 1: $a_n = 2a_{n-1} + 3a_{n-2}$; $a_0 = a_1 = 1$

$$\Rightarrow \lambda^n = 2\lambda^{n-1} + 3\lambda^{n-2}$$

$$\Rightarrow \lambda^2 = 2\lambda + 3$$

$$\Rightarrow \lambda = 3, -1$$

Thus, $a_n = A_1 3^n + A_2 (-1)^n$

from $a_0 = a_1 = 1$ we get that $A_1 = A_2 = \frac{1}{2}$

Example 2: $a_n = a_{n-1} + a_{n-2}$; $a_0 = a_1 = 1$

$$\Rightarrow a_n = A_1 \left(\frac{\sqrt{5}+1}{2} \right)^n + A_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

where $A_1 = \left(\frac{1+\sqrt{5}}{2} \right) \frac{1}{\sqrt{5}}$, $A_2 = \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)$

"Fibonacci"

③ In homogeneous Recurrence relation :

$$a_n = c a_{n-1} + f(n)$$

Solve $a_n = c a_{n-1}$, $a_n = A c^n$

get any solⁿ of $a_n = c a_{n-1} + f(n) \rightarrow$ "particular solⁿ"

"general solⁿ": $a_n = A c^n + p(n)$

$f(n)$	particular sol ⁿ
$d, (c \neq 1)$	B
$d n$	$B_1 n + B_0$
$d n^2$	$B_2 n^2 + B_1 n + B_0$
$e^{d n}$	$B d^n$

if $c=d$, $p(n) = B n d^n$

Example : $a_n = 2 a_{n-1} + 1$, $a_1 = 1$

solⁿ of $a_n = 2 a_{n-1} \rightarrow a_n = A 2^n$

particular solⁿ, take $f(n) = B$

general solⁿ $a_n = A 2^n + B$

$$B = 2 B + 1 \Rightarrow B = -1$$

"Tower of Hanoi"

$$\text{Example : } a_n = 3a_{n-1} - 4n + 3x^{2^n} / a_1 = 8$$

$$\text{Solve } a_n = 3a_{n-1} \Rightarrow a_n = A 3^n,$$

$$\text{Solve } a_n = 3a_{n-1} - 4n \quad (\text{for particular soln})$$

$$p(n) = B_1 n + B_0$$

$$\Rightarrow B_1 n + B_0 = 3 [B_1(n-1) + B_0] - 4n$$

$$\Rightarrow (3B_1 - 4)n + 3(B_0 - B_1)$$

Comparing coefficients, we have:

$$3B_1 - 4 = B_1 \quad , \quad 3(B_0 - B_1) = B_0 = (1) p$$

$$\Rightarrow \boxed{B_1 = 2, B_0 = 3}$$

$$\text{particular soln for } a_n = 3a_{n-1} + 3x^{2^n} \quad \left. \right\} \Rightarrow \boxed{B = -6}$$

$$\text{Thus, general soln : } a_n = A 3^n + (B_1 n + B_0) - B 2^n \\ = A 3^n + (2n+3) - 6x^{2^n}, 0$$

$$\text{find } A \text{ using } a_1 = 8, A = 5$$

Example: $a_n = c a_{n-1}$

Recall: coefficient of x^r in $\frac{1}{(1-x)^n}$

$$\text{is } \binom{n+r-1}{r}$$

"negative binomial"

Given

$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_0)$$

$$g(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

→ come up with a closed form of $g(x)$

→ find coefficient of x^n in $g(x)$

Example (Recall: Dividing the plane):

$$a_n = a_{n-1} + n ; n \geq 1$$

$$a_0 = 1$$

$$g(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

$$g(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} (a_{n-1} + n) x^n$$

$$= \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} -nx^n$$

$$= x \sum_{n=1}^{\infty} [a_{n-1} x^{n-1} + \dots] + (x + 2x^2 + 3x^3 + \dots)$$

$$= x \sum_{m=0}^{\infty} a_m x^m + x (1 + 2x + 3x^2 + \dots)$$

$$= x g(x) + \frac{x}{(1-x)^2}$$

$$\Rightarrow g(x) - 1 = xg(x) + \frac{x}{(1-x)^2} \quad \text{--- (1)}$$

$$\Rightarrow g(x) = \frac{1}{1-x} + \frac{x}{(1-x)^3}$$

coefficient of x^n in $\left(\frac{1}{1-x} + \frac{x}{(1-x)^3}\right)$

$$= 1 + \text{coefficient of } x^{n-1} \text{ in } \left(\frac{1}{1-x} + \frac{1}{(1-x)^3}\right)$$

$$= 1 + \binom{n+1}{2} \quad \text{"without notation"}$$

Example (Fibonacci):

$$a_n = a_{n-1} + a_{n-2} ; \quad n \geq 1$$

$$a_0 = a_1 = 1$$

$$\begin{aligned}
 g(x) - a_0 - a_1 x &= \sum_{n=2}^{\infty} a_n x^n \\
 &= \sum_{n=2}^{\infty} \left[a_{n-1} x^n + a_{n-2} x^n \right] \\
 &= x \sum_{m=1}^{\infty} a_m x^m + x^2 \sum_{m=0}^{\infty} a_m x^m \\
 &= x(g(x) - a_0) + x^2 g(x)
 \end{aligned}$$

$$\Rightarrow g(x) = \frac{1}{1-x-x^2} + (x)p = 1 - (x)p$$

$$g(x) = \frac{1}{(1-\alpha_1 x)(1-\alpha_2 x)} ; \quad \alpha_1 = \frac{1+\sqrt{5}}{2}, \quad \text{and} \quad \alpha_2 = \frac{1-\sqrt{5}}{2}$$

$$= \frac{A}{(1-\alpha_1 x)} + \frac{B}{(1-\alpha_2 x)}$$

"partial fraction"

$$\Rightarrow 1 = A(1-\alpha_2 x) + B(1-\alpha_1 x)$$

$$\text{Take } x = \frac{1}{\alpha_2} \Rightarrow B = \frac{1}{1 - \frac{\alpha_1}{\alpha_2}}$$

$$= \frac{1}{1 - \left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)} = \frac{-\alpha_2}{\sqrt{5}}$$

~~$x = \frac{1}{\alpha_1}$~~ $\Rightarrow A = \frac{\alpha_1}{\sqrt{5}}$

Lecture 16

(E, V) = P

Mathematical programming :

- Linear programs (LP)
- Integer linear programs (ILP)
- Strong Duality of LP (statement only)
- Results from "Combinatorial convex geometry"

Integer Linear Programs (ILP) :

~~NOTATIONS:~~

$$x \in \mathbb{R}^n$$

$$(x_1, x_2, \dots, x_n) \quad \text{where } x_i \in \mathbb{R}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad n \times 1$$

$$\bar{x} \in \mathbb{R}^n \rightarrow \text{particular soln}$$

$$G = (V, E)$$

$\begin{matrix} n \\ [n] \end{matrix}$

want to compute the size of the smallest vertex cover
assign variables to the vertices of G :

$$x_1, x_2, \dots, x_i, \dots, x_n \in \{0, 1\}$$

vertex 1

vertex i

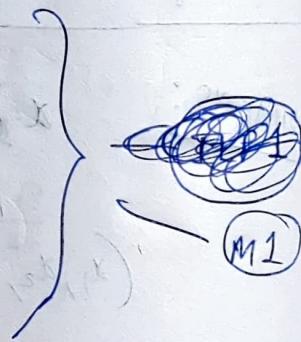
vertex n

ILP for VC:

$$\min. \sum_{i=1}^n x_i$$

$$\text{subject to } x_i \in \{0, 1\}, \forall i \in [n]$$

$$x_i + x_j \geq 1, \forall \{i, j\} \in E$$



Optimal solⁿ vs Optimal value :

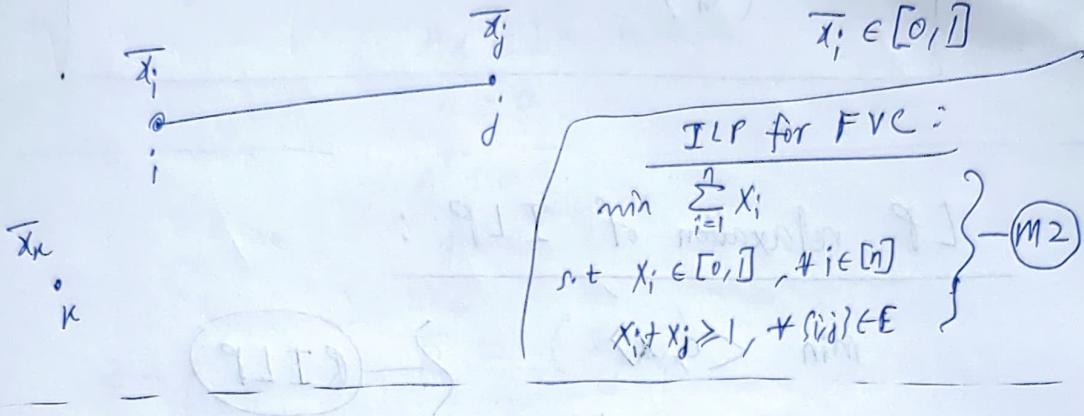
~~Optimal solution~~

Opt_{M1}

$$\sum_{i=1}^n (\bar{x}_{M1})_i = \cancel{\text{Optimal}} \quad Opt_{M1}$$

Linear programs

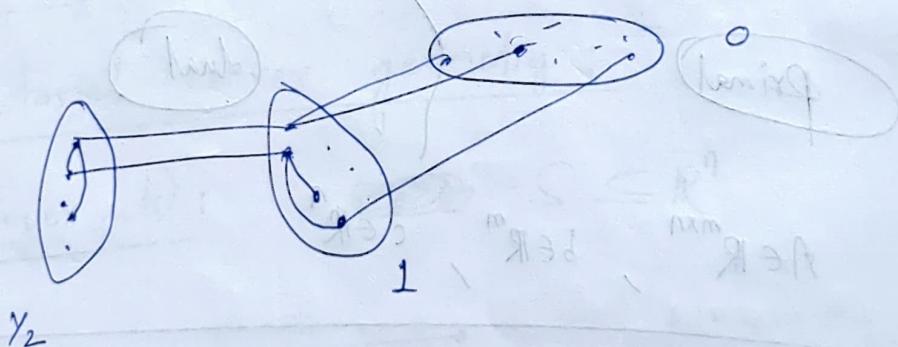
Fractional Vertex Cover (VC) :



Theorem 1 :

If \bar{y} is an optimal solⁿ to M2 with the following property:

$$\forall i \in [n], \bar{y}_i \in \{0, \frac{1}{2}, 1\}$$



Defⁿ (ILP/LP) :

$$x \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n$$

$$\begin{aligned} & \text{min. } \langle c, x \rangle \\ & \text{subject to } x \in \mathbb{R}^n, Ax \leq b \end{aligned}$$

NOTE:

$$\bar{x} \leq \bar{y}$$

$$\bar{x}_i \leq \bar{y}_i, \forall i$$

In case of ILP, $x \in \mathbb{Z}^n$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad \langle c, x \rangle = \sum_{i=1}^n c_i x_i$$

LP relaxation of ILP :

$$\begin{aligned} \min & \quad \langle c, x \rangle \\ \text{subject to} & \quad Ax \leq b \end{aligned} \quad \left\} \text{RILP} \right.$$

$$\begin{aligned} \max & \quad \langle c, x \rangle \\ \text{subject to} & \quad Ax \leq b \\ & \quad x \geq 0 \end{aligned} \quad \left\} \text{PP} \right.$$

$$\begin{aligned} \min & \quad \langle b, y \rangle \\ \text{subject to} & \quad A^T y \geq c \\ & \quad y \geq 0 \end{aligned} \quad \left\} \text{DP} \right.$$

$$A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c \in \mathbb{R}^n$$

Theorem 2 (Strong Duality) :

Given PP and DP we can only have one of the following 4 possibilities:

- ① Both PP and DP are not "feasible"
- ② PP infeasible and DP unbounded
- ③ PP unbounded and DP infeasible
- ④ Both feasible and $\text{Opt}_{\text{PP}} = \text{Opt}_{\text{DP}}$

min. 0

subject to $\langle c, x \rangle = \langle b, y \rangle$

$$Ax \leq b$$

$$x \geq 0$$

$$A^T y \geq c$$

$$y \geq 0$$

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \bar{z} = (\bar{x}, \bar{y})$$

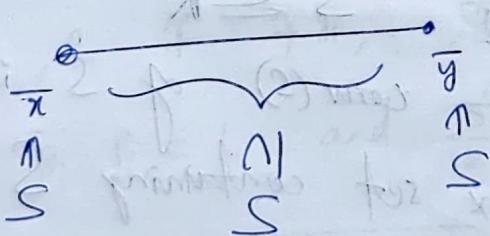
Theorem (weak duality):

$$\text{Opt}_{\text{PP}} \geq \text{Opt}_{\text{DPP}}$$

Combinatorial convex geometry:

(convex sets):

$$S \subseteq \mathbb{R}^n$$



NOTE:

Hollow sphere is NOT convex,
but solid sphere is

(Halfspaces) :

$$A\alpha \leq b$$

$$\begin{bmatrix} - & A_1 & - \\ \vdots & \vdots & \vdots \\ - & A_m & - \end{bmatrix} \begin{pmatrix} \alpha \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\langle a_i, \alpha \rangle \leq b_i, \forall i$$

Defⁿ (convex set) :

$S \subseteq \mathbb{R}^n$ is a "convex set" if
 $\forall \bar{x}, \bar{y} \in S$ and $\bar{\lambda} \in [0, 1]$, we have:

$$\bar{z}_{\bar{\lambda}} = \bar{\lambda} \bar{x} + (1-\bar{\lambda}) \bar{y} \in S$$

Defⁿ (convex hull) : $S \subseteq \mathbb{R}^n$

Then, "convex hull" $\text{conv}(S)$ of S is
the smallest convex set containing S

~~Theorem: Let $S = \{a_1, a_2, \dots, a_n\} \subseteq \mathbb{R}^n$~~

Defⁿ (convex combination):

Let $a_1, a_2, \dots, a_m \in \mathbb{R}^n$

$b \in \mathbb{R}^n$ is said to be a convex combination

of a_1, a_2, \dots, a_m if $\exists \bar{\lambda}_1, \dots, \bar{\lambda}_m \in [0, 1]$

s.t. $\sum_{i=1}^m \bar{\lambda}_i a_i = b$ and $\sum_{i=1}^m \bar{\lambda}_i = 1$

Theorem: Let $S = \{a_1, a_2, \dots, a_m\} \subseteq \mathbb{R}^n$

Then, $\text{Conv}(S) = \left\{ b : b \text{ is a convex combination of } a_1, a_2, \dots, a_m \right\}$

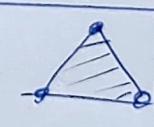
Defⁿ (affine combination):

Let $p_1, p_2, \dots, p_m \in \mathbb{R}^n$

$b \in \mathbb{R}^n$ is said to be an affine combination

of p_1, p_2, \dots, p_m if $\exists \bar{\lambda}_1, \dots, \bar{\lambda}_m \in \mathbb{R}$

s.t. $\sum_{i=1}^m \bar{\lambda}_i p_i = b$ and $\sum_{i=1}^m \bar{\lambda}_i = 1$

Example: convex combination of  is

Defⁿ (affine hull): Let $p_1, p_2, \dots, p_m \in \mathbb{R}^n$ check!

NOTATION: $\text{Aff Hull}(\{p_1, p_2, \dots, p_m\})$

Defⁿ (affine dependence) :

Let $p_1, p_2, \dots, p_m \in \mathbb{R}^n$

We say p_1, p_2, \dots, p_m are "affinely dependent" if
 $\exists \bar{\lambda}_i$ with all $\bar{\lambda}_i$ not equal to 0 and
satisfy the following:

$$\sum_{i=1}^m \bar{\lambda}_i p_i = 0 \quad \text{and} \quad \sum_{i=1}^m \bar{\lambda}_i = 0$$

Theorem : TFAE:

- ① p_0, p_1, \dots, p_{m-1} are affinely dependent
- ② vectors v_1, v_2, \dots, v_{m-1} are linearly dependent,
where $v_i = p_i - p_0$, i
- ③ Any point in $\text{Aff hull}(\{p_0, p_1, \dots, p_{m-1}\})$ have
more than one representation

Exercise!

Lecture 17

Finite
versions

- Radon's theorem
- Helly's theorem
- Caratheodory's theorem

III

all equivalent
theorems

"Strong Duality of LP"

~~Representation theorem
for polytopes~~

Helly's theorem:

Let C_1, \dots, C_m be convex sets in \mathbb{R}^n .
If ~~(n+1)~~ convex sets from the above family intersects,
then the whole family has a non-empty intersection.

i.e., $\bigcap_{j=1}^{n+1} C_j \neq \emptyset$, $\forall i_j \in [m] \Rightarrow \bigcap_{j=1}^m C_j \neq \emptyset$

Radon's theorem:

Let P_1, P_2, \dots, P_{n+2} be distinct points in \mathbb{R}^n .

Then, if a partition $S_1 \cup S_2$ of the above points

s.t. $\text{Conv}(S_1) \cap \text{Conv}(S_2) \neq \emptyset$

pf: Since we have $(n+2)$ points in \mathbb{R}^n ,
by the last theorem of Lecture 16, we know
that they are affinely dependent

(PTO) →

$\zeta_0, \exists \bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_{n+2}$ (not all zero)

s.t. $\sum_{i=1}^{n+2} \bar{\lambda}_i p_i = 0$ and $\sum_{i=1}^{n+2} \bar{\lambda}_i = 0$

$$\zeta_1 = \{p_i : \bar{\lambda}_i > 0\}$$

$$\zeta_2 = \{p_j : \bar{\lambda}_j \leq 0\}$$

Now, $\sum_{i=1}^{n+2} \bar{\lambda}_i = 0$

$$\Rightarrow 0 \neq \sum_{p_i \in \zeta_1} \bar{\lambda}_i = - \sum_{p_j \in \zeta_2} \bar{\lambda}_j = \perp \text{ (say)}$$

Now, $\sum_{i=1}^{n+2} \bar{\lambda}_i p_i = 0$

$$\Rightarrow \frac{1}{\perp} \sum_{p_i \in \zeta_1} \bar{\lambda}_i p_i = \frac{1}{\perp} \sum_{p_j \in \zeta_2} (-\bar{\lambda}_j) p_j$$

\cap
 $\text{Conv}(\zeta_1)$

\cap
 $\text{Conv}(\zeta_2)$

Remark:

$$P_1, P_2, \dots, P_m \in \mathbb{R}^n$$

$$\hat{v}_i := \begin{bmatrix} 1 \\ p_i \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$$

"lifts to \mathbb{R}^{n+1} "

$\hat{v}_1, \dots, \hat{v}_m$ lin. indep. in \mathbb{R}^{n+1}

$\Leftrightarrow P_1, P_2, \dots, P_m$ affinely indep. in \mathbb{R}^n

check!

pf (Helly's theorem):

$$C_1, C_2, \dots, C_m \subset \mathbb{R}^n$$

$$\bigcap_{j=1}^{n+1} C_{ij} \neq \emptyset, \quad \forall i \in [m]$$

To prove: $\bigcap_{j=1}^m C_j \neq \emptyset$

We'll prove this using induction on m

Base case: $m = n+1$

✓ trivial

Suppose it is true for m

ETS : true for $m+1$

c_1, c_2, \dots, c_{m+1}

$S_i := \{c_j : j \neq i\}, \forall i \in [m+1]$

from induction hypothesis, $\exists a_i \in \mathbb{R}^d$ s.t

$a_i \in c_j, \forall j \in S_i$

Let $A := \{a_1, a_2, \dots, a_{m+1}\}$

Since $|A| \geq m+2$, from Radon's theorem,

\exists a partition $A_1 \cup A_2$ of A s.t

$\text{Conv}(A_1) \cap \text{Conv}(A_2) \neq \emptyset$

i.e., $\exists \bar{x} \in \text{Conv}(A_1) \cap \text{Conv}(A_2)$

~~Take some c_j s.t $a_i \in c_j$ where $j \neq i$~~

~~wlog, assume that $a_i \in A_1$~~

~~Then, observe that $a_j \in c_j, \forall j \in A_2$~~

$$A_1 = \{a_{i1}, \dots, a_{i4}\}$$

$$A_2 = \{a_{j1}, \dots, a_{jl}\}$$

$$C_{ij} \rightarrow a_{ij}$$

$$S_{ij}$$

Here, $[m+1] = \{i_1, \dots, i_t, j_1, \dots, j_r\}$

Now, $i_j \notin \{j_1, \dots, j_r\}$

$$\Rightarrow C_{ij} \in S_{j_r}, \forall r \in [l]$$

$$\Rightarrow a_{ij} \in C_{ij}, \forall r \in [l]$$

$$\Rightarrow \text{Conv}(A_2) \subseteq C_{ij}$$

$$\Rightarrow \bar{x} \in C_{ij} \quad \checkmark$$

Note (Radon's theorem):

$$S = \{P_1, P_2, \dots, P_m\}$$

$$m \geq \text{aff-dim}(S) + 1$$

"affine dimension"

Caratheodory's theorem :

Let $S = \{P_1, P_2, \dots, P_m\}$ be a set of m points in \mathbb{R}^n .

Then, all the points in $\text{Conv}(S)$ can be written as a convex combination of atmost $(n+1)$ points from S

pf (by induction on m)

base case ($m = n+1$) : trivial

(using the fact that $\text{Conv}(S) = \text{set of all convex combinations of } S$)

Suppose true for m

ETs : true for $m+1$

Let $\bar{x} \in \text{Conv}(S)$ (arbitrary)

We know that $\bar{x} = \sum_{i=1}^m \bar{\lambda}_i P_i$

where $\bar{\lambda}_i \geq 0$ and $\sum_{i=1}^m \bar{\lambda}_i = 1$

$$\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_m)$$

Consider the following LP :

PTO \rightarrow

$$\begin{bmatrix} 1 & & & \\ p_1 & \cdots & p_m & \\ 1 & & & \\ 1 & & & \end{bmatrix} \lambda = \begin{pmatrix} 1 \\ x \\ \vdots \\ 1 \end{pmatrix}$$

~~LP~~

$$\lambda \geq 0$$

We know that \textcircled{LP} has a feasible solⁿ. $\bar{\lambda}$

The "support" of $\bar{\lambda}$ is given by:

$$\text{Supp}(\bar{\lambda}) = \{i : \bar{\lambda}_i > 0\}$$

If $|\text{Supp}(\bar{\lambda})| = n+1$, done ✓

So, let $|\text{Supp}(\bar{\lambda})| > (n+1)$

$$\text{i.e., } |\{p_i : i \in \text{Supp}(\bar{\lambda})\}| > (n+1)$$

$\exists \bar{\mu}_i$ s.t. $\bar{\mu} = (\mu_1, \mu_2, \dots) \neq 0$ and

$$\sum_{i \in \text{Supp}(\bar{\lambda})} \bar{\mu}_i p_i = 0 \quad \text{and} \quad \sum_{i \in \text{Supp}(\bar{\lambda})} \bar{\mu}_i = 0$$

Now, let $\beta_\varepsilon := \bar{\alpha} - \varepsilon \bar{\mu}$, $\varepsilon \geq 0$

Complete the proof!

□

$$0 \leq R$$

For the almost surely

⑪ last word

and with R to "trigger" at

$$\{0 < \bar{t} : \eta\} = \{(\bar{R})_{\eta \wedge 2}\}$$

$$(H_0) = \{(\bar{R})_{\eta \wedge 2}\}$$

$$(H_0) \leq \{(\bar{R})_{\eta \wedge 2}\} + b$$

$$(H_0) \leq \{(\bar{R})_{\eta \wedge 2} + b\}$$

the $\eta \wedge 2$ is a stopping time

$\eta \wedge 2 = 3$ from $\eta \wedge 2 = 1$

$(H_0) \leq \{(\bar{R})_3 + b\}$

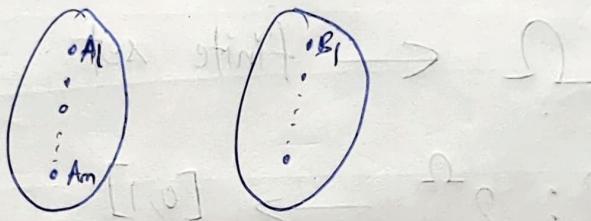
Lecture 18

- ◻ Probabilistic Method
- ◻ Planar graphs
- ◻ Algebraic Techniques

First Moment Methods:

Problem 1: Sperner's Lemma

Problem 2:



- where A_i, B_j are subsets of \mathbb{N} with
- a) $|A_i| = k$,
 - b) $|B_j| = l$,
 - c) $B_j \cap A_i = \emptyset$
 - d) $A_i \cap B_j \neq \emptyset$, if $j < i$

Problem 3: Existence of "large" independent set
in "sparse" graphs

Problem 4: Existence of "small" dominating set in
"dense" graph

Problem 5: Existence of a tournament with many
Hamiltonian paths

Problem 6: Vector balancing problem

$$v_1, v_2, \dots, v_n \in \mathbb{R}^n \text{ s.t. } \|v_i\| = 1$$

$v = v_1 + v_2 + \dots + v_n$ s.t. $\|v\|$ will have "small" norm.

$$\text{To show: } \exists \epsilon_i \in \{\pm 1\} \text{ s.t. } \left\| \sum_{i=1}^n \epsilon_i v_i \right\| \leq \sqrt{n}$$

Finite Probability Space

$\Omega \leftarrow$ finite set.

$$P: 2^\Omega \rightarrow [0, 1]$$

$$P(\emptyset) = 0$$

$$P(\Omega) = 1$$

$$P(A \cup B) = P(A) + P(B)$$

Example (Hypercube / Hamming cube):

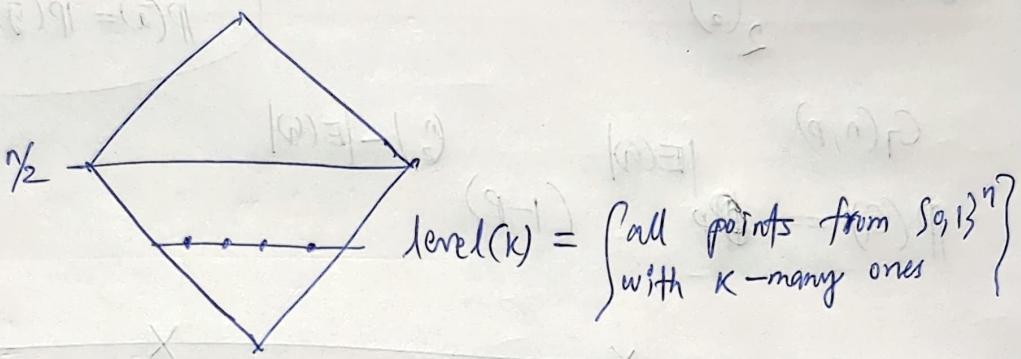
$$\Omega = \{0, 1\}^n$$

$$\bar{x} \in \{0, 1\}^n, \quad P(\bar{x}) = \frac{1}{2^n}$$

$$P_{x_2}(A) = \frac{|A|}{2^n}, \quad A \subseteq \{0, 1\}^n$$

$$P_q(\bar{x}) = (1-q)^{n-|\bar{x}|} q^{|\bar{x}|} \quad |\bar{x}| = \#\text{ ones in } \bar{x}$$

$$P_q(A) = \sum_{\bar{x} \in A} P_q(\bar{x})$$



Random Permutation:

$$\Omega_n := \left\{ \text{set of all permutations on } n \text{ numbers} \right\} \\ = S_n$$

$\pi : [n] \rightarrow [n]$ is a permutation

$$P(\pi) = \frac{1}{n!}, \quad A \subseteq \Omega_n, \quad P(A) = \frac{|A|}{n!}$$

(pto) \rightarrow

Random Graphs :

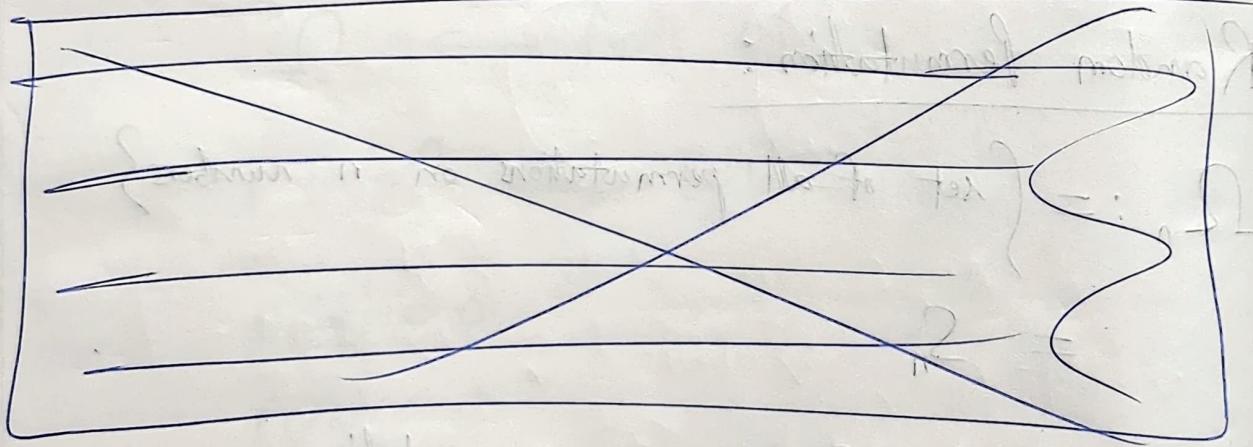
$G(n, p)$

$\Omega \rightarrow$ set of all graphs with same weight

$$P(G) = \frac{1}{2^{\binom{n}{2}}}$$

$$\bar{a} \in \mathcal{R}, \\ P(\bar{a}) = P(\bar{b}), \forall \bar{a}, \bar{b} \in \mathcal{R}$$

$$G(n, p) \quad |E(G)| \quad \binom{n}{2} - |E(G)| \\ P(G) = p^{|E(G)|} (1-p)^{\binom{n}{2} - |E(G)|}$$



Expectation :

$$\therefore f: \Omega \rightarrow \mathbb{R}$$

$$E[f] = \sum_{w \in \Omega} f(w) P(w)$$

$$P[f \leq a] = P[S]$$

$$S = \{w \in \Omega : f(w) \leq a\}$$

Lemma 1: If $E[f] = c$, then $\exists w \in \Omega$

s.t. $f(w) \geq c$

Similarly, $\exists w'$ s.t. $f(w') \leq c$

Lemma 2 (Markov's inequality):

Let $f: \Omega \rightarrow \mathbb{R}$ be a non-negative function.

Then, $P[f \geq a] \leq \frac{E[f]}{a}$, $\forall a > 0$

$$\text{pf: } E[f] = \sum_{w \in \Omega} f(w) P(w)$$

$$= \sum_{w: f(w) < a} f(w) P(w) + \sum_{w: f(w) \geq a} f(w) P(w)$$

$$\geq a \sum_{w: f(w) \geq a} P(w) = a P[f \geq a]$$

□

Independence:

A_1, A_2, \dots, A_n be n events. We say that they are "mutually independent" if

$$P\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} P[A_i], \quad \forall I \subseteq [n]$$

NOTE: "k-wise independent" with $|I| \leq k$

Theorem (Existence of a Tournament with many Hamiltonian paths)

Pf: K_n T

$$P[T] = \frac{1}{2^{\binom{n}{2}}} = [n \leq t]$$

$\Omega \rightarrow$ collection of all Tournament on the vertex set $[n]$

$$f: \Omega \rightarrow \mathbb{R} \text{ s.t.}$$

$f(T) :=$ no. of Hamiltonian paths in T

ETS: $E[f]$ is large

$$E[f] = \sum_{T \in \Omega} f(T) \cdot \frac{1}{2^{\binom{n}{2}}}$$

$$= \frac{1}{2^{\binom{n}{2}}} \sum_{T \in \Omega} f(T)$$

$$= \frac{1}{2^{\binom{n}{2}}} \sum_{\substack{H \\ \text{Hamiltonian path}}} \sum_{\substack{T: T \text{ contains} \\ H}} 1$$

$$\leq \frac{1}{2^{\binom{n}{2}}} \sum_{\substack{H \\ \text{Hamiltonian path}}} 2^{\binom{n-1}{2} - (n-1)}$$

$$= n! 2^{\binom{n}{2} - (n-1)} \quad \checkmark$$

Indicator random variable:

$$\mathbb{1}_E = \begin{cases} 1, & \text{if } \omega \in E \\ 0, & \text{o.w.} \end{cases}$$

Lemma: $E[\mathbb{1}_A] = P[A]$

Lemma: $f, g : \Omega \rightarrow \mathbb{R}$

$$\mathbb{E}[af + \mu g] = a\mathbb{E}[f] + \mu\mathbb{E}[g]$$

ALT proof (using Indicator variables):

1_H := indicator random variable
of H being present in
your tournament

$$1_H(T) = \begin{cases} 1, & \text{if } H \text{ is subgraph of } T \\ 0, & \text{o.w.} \end{cases}$$

$$f = \sum_{\text{"H" Hamiltonian path}} 1_H$$

$$\mathbb{E}[f] = \sum_{\text{"H" Hamiltonian path}} P(1_H = 1)$$

$$= \sum_H \frac{\binom{n}{2} - (n-1)}{2}$$

Lecture 19

Property 10:

Defⁿ (Independent random variables):

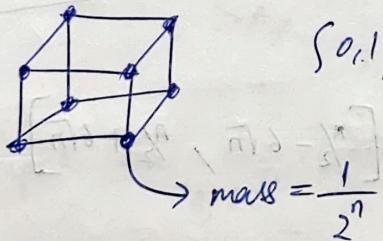
x_1, x_2, \dots, x_n are (mutually) independent if
+ $I \subseteq [n]$ and $\forall w_i$ we have:

$$P\left[\bigcap_{i \in I} (X_i = w_i)\right] = \prod_{i \in I} P[X_i = w_i]$$

Lemma: $X \perp\!\!\!\perp Y \Rightarrow E(XY) = E(X)E(Y)$

Lemma: $E[1_A] = P(A)$

Application 1:



$$\{0,1\}^n$$

$$\rightarrow \text{mass} = \frac{1}{2^n}$$

Let $X := \text{no. of } 1's$

Then, $E(X) = \sum_{i=0}^n i \cdot P(X=i)$

Aside:

Verify that $E(X) = \sum_{w \in \Omega} X(w) P(w) = \sum_{x \in S} x \cdot P(X=x)$

$$= \sum_{i=0}^n i \cdot \frac{\binom{n}{i}}{2^n}$$

$$= \frac{1}{2^n} \sum_{i=0}^n i \binom{n}{i} = \frac{1}{2^n} \cdot n 2^{n-1} = \frac{n}{2} \quad \checkmark$$

Let $\mathbb{1}_i :=$ event that the i th component of the point in $\{0,1\}^n$ is 1

Then, $X = \sum_{i=1}^n \mathbb{1}_i$

$$\Rightarrow E(X) = \sum_{i=1}^n E(\mathbb{1}_i) = n \cdot \frac{1}{2} = \frac{n}{2} \quad \checkmark$$

Remark :

$$\Omega(n) = \left[\frac{n}{2} - 6\sqrt{n}, \frac{n}{2} + 6\sqrt{n} \right]$$

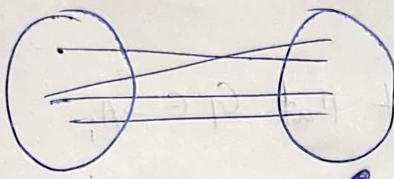
Application 2 (Random Graphs):

$\Omega = \{ \text{family of graphs on the vertex set } [n] \}$

$$G \in \Omega, \quad P(G) = \frac{1}{2^{\binom{n}{2}}}$$

NOTE : $|\Omega| \approx 2^{\frac{n^2}{2}}$

Let's look at bipartite graphs



$$A \cup \bar{A} = [n]$$

$$\text{stating } A \text{ as } \{A\} \text{ and } \bar{A} = [n] \setminus A = \{\bar{A}\}$$

$S_A := \left\{ \text{no. of bipartite graphs on } [n], \text{ that has } A, \bar{A} \text{ as a bipartition} \right\}$

$$|S_A| = 2^{|A|(n-|A|)} \leq \frac{n^{n/2}}{2} \quad (\text{why?})$$

NOTATIONS:

$$A_1, A_2, \dots, A_{2^n}$$

$$S_{A_1}, S_{A_2}, \dots, S_{A_{2^n}}$$

$$\begin{aligned} \text{no. of bipartite graphs} &\leq \sum_{A_i \in \mathcal{P}} |S_{A_i}| \quad (Y \geq X) \quad \text{①} \\ &\leq 2^{n + n^{n/2}} \quad (\text{why?}) \end{aligned}$$

stating of P from V or whatever = $\frac{n^{n/2}}{2}$

Lemma

Let's prove this

PTO

$$\underline{\text{Lemma}} : \mathbb{P}[G \text{ is bipartite}] \leq \frac{2^{\frac{n}{2} + \frac{n^2}{4}}}{2^{\frac{n^2}{2}}} = 2^{\frac{n - n^2}{4}}$$

Pf: Let $E_i :=$ event that $G \in S_{A_i}$

Then, $\bigcup_{i=1}^{2^n} E_i =$ event that G is bipartite

$\mathbb{P}(E)$ (say) $=$ $\sum_{i=1}^{2^n} \mathbb{P}[E_i]$

$$\begin{aligned} \text{Now, } \mathbb{P}(E) &= \mathbb{P}\left[\bigcup_{i=1}^{2^n} E_i\right] \\ &\leq \sum_{i=1}^{2^n} \mathbb{P}[E_i] \end{aligned}$$

NOTE: for r.v's X, Y ,

- ① $X \leq Y$ $\Leftrightarrow X(w) \leq Y(w), \forall w \in \Omega$
- ② $X \leq Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y)$

Let $1_{\text{Bip}} :=$ indicator r.v that G is bipartite

Now, for A_1, \dots, A_{2^n} , define:

$1_{A_i} :=$ indicator r.v that G is bipartite with bipartition A_i, \bar{A}_i

$$\text{Then, } \frac{1}{\Pr_{\text{Bip}}} \leq \sum_{i=1}^{2^n} \Pr_{A_i}$$

$$\left(\text{i.e., } \Pr[\cup A_i] \leq \sum_i \Pr(A_i) \right)$$

Complete the proof

□

$$Q1) S_n := \{\pi: [n] \rightarrow [n], \text{ where } \pi \text{ is 1-1}\}$$

$$\Pr[\pi] = \frac{1}{n!}, \quad * \forall \pi \in S_n \subset (\mathbb{R})$$

"permutation"

Fix k indices (say, $i_1, i_2, \dots, i_k \in [n]$)

$$\Pr[\pi(i_k) > \pi(i_j), \quad * j \leq k-1] = ?$$

sol:

$$\frac{1}{k}$$

check!

Q2) Size of the largest independent set in a graph G

$$[2, 3] \times [3] = ?$$

sol: Let $\Delta(G) :=$ largest degree in G

PTO →

$$\text{Observation: } \lambda(G) \geq \frac{n}{\Delta(G) + 1}$$

size of the
largest indep. set

But this is not a very good bound

Turán, Ramsey type problem

$$\text{Theorem: } \lambda(G) \geq \sum_{i \in [n]} \frac{1}{d(i)+1} = \frac{1}{\bar{d}} \quad (\text{Caro-Tuza})$$

pf: Let π be a random permutation

$\pi(1), \pi(2), \dots, \pi(i), \dots, \pi(n)$
 $S \rightarrow$ indep. set (considering a greedy way)
 construct it

Let $\mathbb{1}_i =$ indicator r.v that i was chosen from S

$$\text{Then, } |S| = \sum_{i=1}^n \mathbb{1}_i$$

$$\Rightarrow E[|S|] = \sum_{i=1}^n P[i \in S]$$

$$= \sum_{i=1}^n \frac{1}{d(i)+1} \quad (\text{using } \alpha_1)$$

Q3) [Shearer] \rightarrow d-regular
 Let G be a Δ -free ~~regular~~ graph

$$\text{Then, } \chi(G) \geq \frac{n \log d}{d}$$

Pf : Check paper by "shearer"

$$\text{Recall : } E(XY) = E(X)E(Y)$$

Theorem : Let v_1, \dots, v_n be unit vectors in \mathbb{R}^N

Then, $\exists \varepsilon_i \in \{\pm 1\}$ s.t $n \geq 2$ such that

$$\left\| \sum_{i=1}^n \varepsilon_i v_i \right\|_2 \leq \sqrt{n}$$

Pf : $\forall i$, let $x_i := \begin{cases} -1, & \omega \cdot p \neq \frac{1}{2} \\ +1, & \omega \cdot p = \frac{1}{2} \end{cases}$

$$\text{Let } V := \sum_{i=1}^n x_i v_i$$

$$\begin{aligned} \text{Then, } \|V\|_2^2 &= \sum_{i=1}^n \|v_i\|^2 + \sum_{i \neq j} \langle v_i, v_j \rangle x_i x_j \\ &= n + \sum_{i \neq j} \langle v_i, v_j \rangle x_i x_j \end{aligned}$$

$$\Rightarrow \mathbb{E}[\|V\|_2^2] = n + \sum_{i \neq j} \langle v_i, v_j \rangle \mathbb{E}(X_i X_j)$$

$$= n \quad (\because \mathbb{E}(X_i X_j) = \mathbb{E}(X_i) \mathbb{E}(X_j) = 0)$$

$$\Rightarrow \exists \varepsilon; \quad \varepsilon \in \{+1\} \text{ s.t. } \| \varepsilon_i v_i \|_2 \leq \sqrt{n} \quad (\text{or } \geq \sqrt{n})$$

$$(Y) \exists (X) \exists = (Y X) \exists$$

Defⁿ (dominating set): Let $G = ([n], E)$

and let $S \subseteq [n]$
Then, S is called a "dominating set" if

$$S \cap (\{i \cup N[i]\}) \neq \emptyset \quad \forall i \in [n]$$

~~Iteration~~

NOTATION: $w(G) =$

Theorem: Let G be a d -regular graph. Then:

$$\omega(G) \leq \frac{n \log d}{d}$$

with std proof

Pf: $p \in [0, 1]$

$i = 1, 2, \dots, n$

$S \leftarrow$ include each vertex " i " with probability " p "

"Alteration"

$C \leftarrow$ "correction set" (vertices included in S)

Let $S_{\text{new}} = S \cup C$

$$\Rightarrow |S_{\text{new}}| = |S| + |C|$$

$$\begin{aligned}\Rightarrow E(|S_{\text{new}}|) &= E(|S|) + E(|C|) \\ &= np + n(1-p)^{d+1}\end{aligned}$$

$\mathbb{1}_i \leftarrow i$ is not dominated by S

$$|C| = \sum_{i=1}^n \mathbb{1}_i$$

$$\Rightarrow E(|C|) = n(1-p)^{d+1}$$

want to find n s.t. $E(f_{\text{new}})$ is minimum.

Complete the proof

$$[1, 0] \ni q$$

At the "i" vertex there should be $\rightarrow 2$
"q" probabilities

"minimum"

(2 at bottom vertex) too many $\rightarrow 0$

$$0 \cup 2 = \text{near} 2 + 3$$

$$|D| + |E| = \{\text{near} 2\} \leftarrow$$

$$(|D|)3 + (|E|)3 = (\{\text{near} 2\})3 \leq$$

$$\text{Hence } (|D|)3 + (|E|)3 =$$

2 at bottom vertex $\rightarrow 1$ $\rightarrow 1$

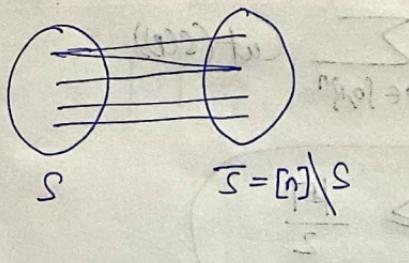
$$\sum_{i=1}^n \frac{1}{i} = |D|$$

$$\text{Hence } (n-1)!! = (|D|)3 \leq$$

Lecture 20

$$G = ([n], E)$$

$$S \subseteq [n]$$



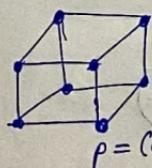
$\text{Cut}(S) :=$ no. of edges
with one end point
in S and other
in \overline{S}

$$\text{Min-cut} := \min_{S \subseteq [n]} \text{Cut}(S)$$

$$\text{Max-cut} := \max_{S \subseteq [n]} \text{Cut}(S)$$

"Goeman - Williams"

"SDP and Rounding Techniques"



$$(0, 1)^n$$

"Hypercube"

$$\text{weight} = \frac{1}{2^n}$$

$$p = (p_1, p_2, \dots, p_n)$$

$$S(p) = \{i : p_i = 1\} \subseteq [n]$$

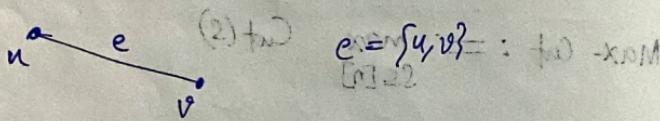
Let $X: \{0, 1\}^n \rightarrow \mathbb{Z}$ s.t

$$x(p) = \text{Cut}_G(s(p))_{[n]} = p$$

Then, $E(X) = \text{average over all cuts of } G$

$$\text{weights } f_0 \cdot s_n = \sum_{\substack{\text{string trees are also} \\ \text{partial trees}}} \frac{1}{2^n} \sum_{p \in S_0, i \in n} \text{Cut}(SCP)$$

$$\underline{\text{Goal:}} \quad E(X) \geq \frac{|E|}{2}$$



η_e := indicator variable that e goes across S, \bar{S}

$$\text{Then, } X = \sum_{e \in E} 1_e$$

$$\Rightarrow E(X) = \sum_{e \in E} P_e(e \text{ in cut})$$

$$= \sum_{e \in E} \frac{1}{2} \left(P(e \text{ not in cut}) - P(e \text{ in cut}) \right) = \frac{|E|}{2}$$

A Classical problem in Extremal Set Theory

Theorem (Erdos-Ko-Rado) : Let S be a

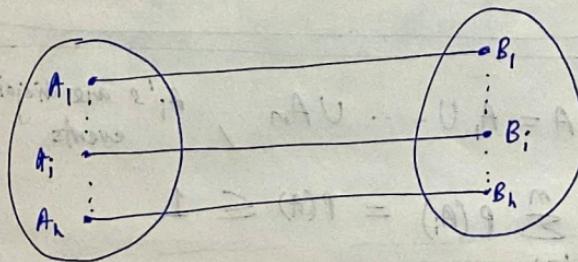
collection of subsets of $[n]$ with each set in S of size k . If $\forall A, B \in S$ we have $A \cap B \neq \emptyset$,

then $|S| \leq \binom{n-1}{k-1}$

Here, $n \geq 2k$

(k, l) - systems

Bipartite graph with partite sets $\{A_1, \dots, A_k\}$ and $\{B_1, \dots, B_l\}$, where $A_i, B_j \subseteq \mathbb{N} \neq \emptyset$ s.t:



① $|A_i| = k$ and $|B_j| = l$, $\forall i, j$

② $A_i \cap B_j = \emptyset$, $\forall i \neq j$

③ $A_i \cap B_j \neq \emptyset$, $\forall i \neq j$

Then, how large can h be?

motivation
for
"matroid theory"

Theorem (Bollobas) : for (k, l) -systems,

$$h \leq \log \binom{k+l}{l}$$

Theorem (LYN inequality for bipartite systems)

Modify condition ① to :

$$\textcircled{1} \quad |A_i| = a_i \text{ and } |B_j| = b_j$$

Then, $\sum_{i=1}^h \frac{1}{(a_i + b_i)} \leq 1$

Remark: This is a stronger result than "Bollobas"

Pf : Idea: $A = A_1 \cup \dots \cup A_m$, A_i 's are disjoint events

$$\text{then, } \sum_{i=1}^m P(A_i) = P(A) \leq 1$$

Relabel the elements of A_i 's and B_j 's s.t

$$(\text{wlog}) \quad [n] = \left(\bigcup_{i=1}^h A_i \right) \cup \left(\bigcup_{j=1}^h B_j \right)$$

Let π be a random permutation of $[n]$

$$\pi(1), \pi(2), \dots, \pi(n)$$

Let $E_i :=$ event that elements of A_i come before elements of B_i in π

Then, $P(E_i) = \frac{1}{\binom{a_i+b_i}{a_i}}$ check!

NOTE: E_i and E_j are disjoint events, $\forall i \neq j$
(i.e., $E_i \cap E_j = \emptyset$, $\forall i \neq j$)

Now, $\sum_{i=1}^h P(E_i) \leq 1$

$$\Rightarrow \sum_{i=1}^h \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1$$

if (Erdos - Ko - Rado)

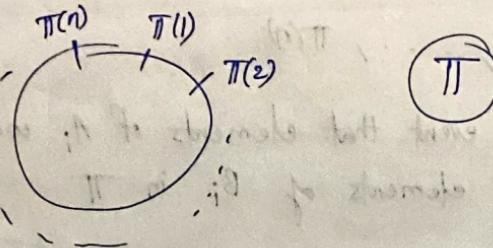
due to "Katona"

Given $S \subseteq \binom{[n]}{k}$ (a subset of

collection of k -sized subsets of $[n]$)

$\forall A, B \in S, A \cap B \neq \emptyset$

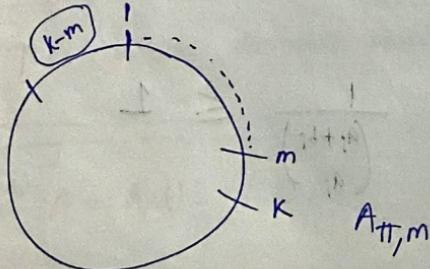
Let π be a random cyclic permutation of $[n]$



Let $A_{\pi, l} := \{\pi(l+1), \dots, \pi(l+k)\}$, $0 \leq l \leq n-1$

$$\left| \{A_{\pi, l} : 0 \leq l \leq n-1\} \cap S \right| \leq k$$

WLOG, assume $A_{\pi, 0} \in S$



Let π be a random cyclic permutation of $[n]$
and i be a random number between $[n]$

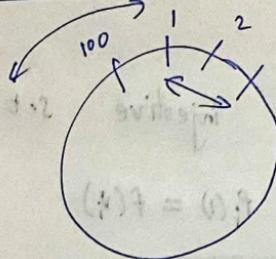
Then, $P(A_{\pi, i} \in S) \leq \frac{k}{n} \rightarrow \text{Check!}$

$$\Rightarrow \frac{|S|}{\binom{n}{k}} \leq \frac{k}{n}$$

$$\Rightarrow |S| \leq \frac{k}{n} \cdot \binom{n}{k} = \binom{n-1}{k-1} \quad \checkmark$$

□

Example :



$\{2, 3, 4\}, \{3, 9, 5\}, \{99, 100, 1\}, \{100, 1, 2\}$

disjoint

So, let's pair them up :

$\{2, 3, 4\}, \{99, 100, 1\} , \{3, 9, 5\}, \{100, 1, 2\}$

From book

"pizza"



Graph Embedding / Drawing

$$G = ([n], E)$$

"Embedding in \mathbb{R}^d ":

$$f: [n] \longrightarrow \mathbb{R}^d$$

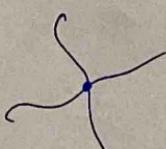
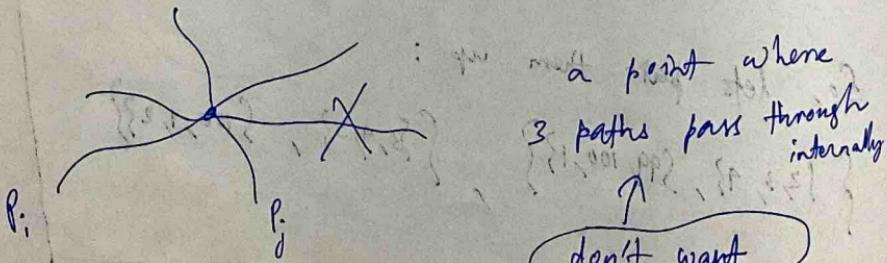
$$e_i = \{u_i, v_i\} \in E$$

$$p_i: [0, 1] \longrightarrow \mathbb{R}^d \quad \text{injective s.t.}$$

$$p_i(0) = f(u_i), \quad p_i(1) = f(v_i)$$

(Also, check "proper embedding")

Embedding in \mathbb{R}^2 : "planarity"



"Crossing"

Lecture 21

Graph Drawing / Planarity

Defⁿ (Graph Drawing): Given a graph $G = (V, E)$, a "drawing" of G is a family of maps [s.t. f]

① $f: V \rightarrow \mathbb{R}^d$
 \rightarrow injective

② for each edge $e \in E$, $g_e: [0, 1] \rightarrow \mathbb{R}^d$

$g_e \rightarrow$ "curves"

\rightarrow continuous injective maps

③ $f(v) \cap g_e([0, 1]) = \emptyset$ s.t. $e \in E$

Defⁿ (Planar graph): A graph G is a "planar graph".

if it can be drawn in \mathbb{R}^2 s.t. curves associated with edges in G don't intersect in the interior

Theorem (Wagner): A graph G is planar IFF it does not contain K_5 or $K_{3,3}$ as its "minor".

Theorem (Kuratowski's thm.): A graph G is planar IFF it does not contain a "subdivision" of K_5 or $K_{3,3}$.

Remark: Wagner's thm. and Kuratowski's thm. are equivalent

Theorem (Fary): All planar graphs have a non-intersecting straight line drawing in \mathbb{R}^2

Theorem (Pach): Suppose $G = (V, E)$ is planar. Then, $f: [n] \rightarrow [2n-1] \times [2n-1]$ and still have a straight line drawing of G

Q^n> Given G , find a drawing in \mathbb{R}^2 that has the smallest no. of crossings

Theorem: Every graph has a straight line drawing that is non-intersecting in \mathbb{R}^3

~~if~~ Let $G = (V, E)$

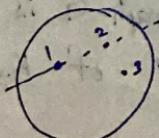
and $f: V \rightarrow B(0, 1)$

Ball with centre 0
and radius 1

$$V = [n] = \{1, 2, \dots, n\}$$

$$f(1), f(2), f(3), f(4), \dots, f(m), f(m+1), \dots$$

induction



Complete the proof!

ALT Pf: $\{g_N\}$

$\exists f : [n] \rightarrow [N] \times [N] \times [N] \subseteq \mathbb{R}^3$ s.t

the points $\{f(1), \dots, f(n)\}$ satisfy the property
of ~~#~~ the previous #

Claim: ~~$\forall N \in \mathbb{N}$~~ + planes H in \mathbb{R}^3 , we have:

$$|\{g_N \cap H\}| \leq c N^2$$

where $c > 0$ is some constant.

$f(1), f(2), f(3), f(4), \dots, f(m), f(m+1), \dots$

$$c \binom{m}{3} N^2 < N^3$$

$$N = c n^3$$

Complete the proof!

$$G = ([n], E)$$

Theorem: Let $G = ([n], E)$ be a planar graph.

$$\text{Then, } |E| \leq 3n - 6$$

COROLLARY:

PTO \Rightarrow

Defⁿ (crossing number): Given a graph G , the "crossing no." of G , denoted $Cr(G)$, is the minimum no. of crossing with which you can draw G in \mathbb{R}^2 .

Q1) Let $G = ([n], E)$ s.t $|E| \geq 8n$

(i.e., G is far from planar)

Then, what is the lower bound of $Cr(G)$?

Theorem (Leytm, Sharir, ...):

If $|E| \geq 8n$, then $|Cr(G)| \geq \frac{m^3}{n^2}$,

where $m = |E|$ and $n = |V|$

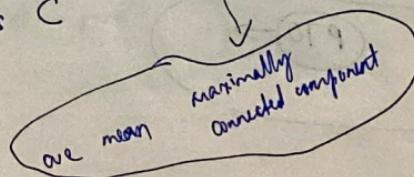
COROLLARY (to ④): $Cr(G) \geq m - 3n + 6$

Remark: COROLLARY $\Rightarrow K_5$ is NOT planar

Check!

Jordan's theorem: Let C be a "closed curve" in \mathbb{R}^2

Then, $\mathbb{R}^2 \setminus C$ has exactly two "connected components": with one bounded and one unbounded and the "boundary" of both the components is C



Defⁿ (closed curve) : $f : [0, 1] \rightarrow \mathbb{R}^2$
s.t. $f(0) = f(1)$

(i.e., $f : S^1 \xrightarrow{\text{'circle'}} \mathbb{R}^2$)

Defⁿ (connected component) :

Defⁿ (boundary) : $S \subseteq \mathbb{R}^2$

$\bar{S} \rightarrow$ smallest closed set containing S

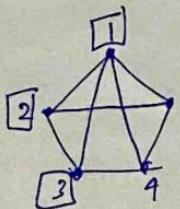
$S^\circ \rightarrow$ largest open set containing S

~~∅~~ $\partial S := \bar{S} \setminus S^\circ$

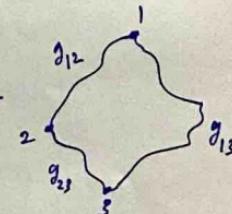
$$S^2 := \{\bar{x} \in \mathbb{R}^3 : \|\bar{x}\| = 1\} \equiv \mathbb{R}^3$$

\uparrow
sphere

\curvearrowright
 K_5



$\xrightarrow{\mathbb{R}^2}$



Lecture 22 (Buddha)

Planar Graphs

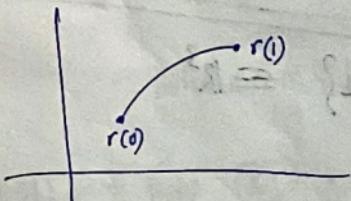
Recap:

Defⁿ (Arc): Let $r: [0, 1] \rightarrow \mathbb{R}^2$ be

an injective, ~~cont.~~

Then "arc" is the set $\{r(x) : x \in [0, 1]\}$

Defⁿ (Jordan Curve): A closed, simple ~~curve~~
(i.e., an arc with $r(0) = r(1)$)



Defⁿ (Jordan Curve Theorem [JCT]):

Defⁿ ("Drawing" of a graph $G = (V, E)$):

an injective function $b: V \rightarrow \mathbb{R}^2$ and

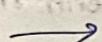
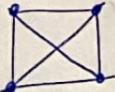
arcs ω_e , $x, y \in G(E)$ s.t

$$e = \{x, y\} \Rightarrow \omega_e(0) = b(x) \text{ and } \omega_e(1) = b(y)$$

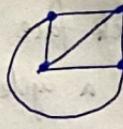
Defⁿ (planar graph): A graph is "planar" if

if a drawing that has arcs which don't intersect
except at end points

Example:



y

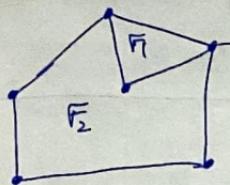


planar

Defn (Face of a planar graph): A set in a drawing

of a planar graph $A \subseteq \mathbb{R}^2$ is called "face" of the graph if $\forall \bar{x}, \bar{y} \in A$, if an arc $\alpha \subseteq A$ with end point \bar{x}, \bar{y} and α doesn't intersect with the arcs of the planar graph

Example:



$F_3 \rightarrow$ unbounded face
(outer face)

$F_1, F_2 \rightarrow$ bounded face
(inner face)

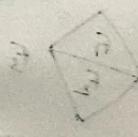
NOTE: Let e_1, e_2, \dots, e_n be the edges of a cycle of a planar graph

e_1, e_2, \dots, e_n will form a Jordan Curve

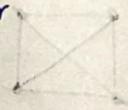
Lemma: Each face of a planar graph of G lies inside or outside some cycle of G if \exists a cycle in a planar graph

Pf:

Exercise!



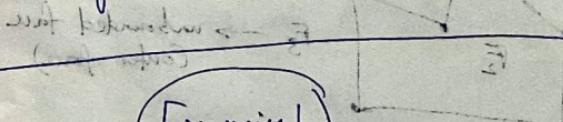
Property: Each face is the interior or exterior of a cycle of G



Theorem: Let G be a 2-vertex connected planar graph, then every face in the graph is a region of some cycle of G

pf: A graph is 2-vertex connected

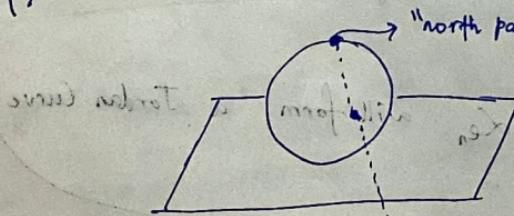
\Leftrightarrow It can be created from a Δ from a sequence of edge subdivision and edge addition.



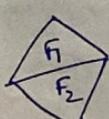
Exercise!

Defⁿ (Stenographic projection):

Qⁿ) Can I draw a planar graph onto a sphere?



Qⁿ)



F₃

Can we "exchange" F₁ and F₃?

Theorem (Euler's formula): Let $G = (V, E)$ be a connected planar graph. Then:

$$|V| - |E| + F = 1 \quad \text{where } F = \text{no. of faces in the planar graph (excluding the unbounded face)}$$

pf:

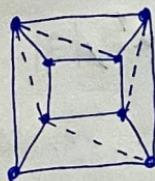
method 1

Use induction on $|E|$

(Exercisel)

method 2 (combinatorial argument)

motivation:

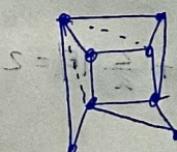


"triangulation"

|E| \leftarrow The formula $(|V| - |E| + |F| = 1)$

|E| \leftarrow remains same after triangulation

$$S = E + |V| - |V| \leftarrow \text{method 2 works}$$



still doesn't change formula

Complete the proof!

$$\frac{1}{2} - \frac{1}{d} + \frac{1}{n} = \frac{1}{|V|}$$

$$(2 \geq d, 2 \geq n) \xrightarrow{\text{but}} (2 \leq d, 2 \leq n)$$

$$(2 \leq d, n \geq 2) \xrightarrow{\square}$$

$$P = \{e, n\} \vdash \neg \exists m$$

$$2 \geq d, n \geq 2$$

"Regular Polyhedron" : ① all faces have same no. of edges
 ② same no. of faces/edges should meet at each vertex

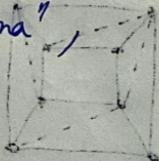
"Platonic Solids" : set of all regular polyhedrons

Proposition : There're only 5 platonic solids in 3-D

Pf : Let $a = \text{no. of edges in each face}$
 $b = \text{no. of edges at each vertex}$

Then, by "Handshaking Lemma",

$$(1 = |V| + |E| - |F|) \text{ where } F = 2|E|$$



$$\text{and } \frac{1}{b}|V| = 2|E|$$

Now, Euler's theorem $\Rightarrow |V| - |E| + F = 2$

$$\text{above graph shows } \Rightarrow \frac{1}{b}|V| - |E| + \frac{2}{a}|E| = 2$$

$$\Rightarrow \frac{1}{|E|} = \frac{1}{a} + \frac{1}{b} - \frac{1}{2}$$

Now, $\frac{1}{|E|} = \frac{1}{a} + \frac{1}{b} - \frac{1}{2}$ (using wt. of edges)

$$\Rightarrow (a \geq 3, b \geq 3) \text{ and } (a \leq 5, b \leq 5)$$

i.e., $3 \leq a, b \leq 5$

Note: $(a, b) = 9$

- only 5 sol's :
- $a = b = 3 \rightarrow$ tetrahedron
 - $a = 4, b = 3 \rightarrow$ cuboid
 - $a = 3, b = 4 \rightarrow$ octahedron
 - $a = 3, b = 5 \rightarrow$
 - $a = 5, b = 3 \rightarrow$
-

□

Theorem:

Let $G = (V, E)$, $|V| \geq 3$ ~~be a~~ be a planar graph. Then:

$$|E| \leq 3|V| - 6$$

The equality holds for maximal planar graph

Pf: Keep adding edges in G , until it becomes maximal planar

This maximal graph has each of its ~~no.~~ bounded faces a Δ

Now, "Hand Shaking Lemma" $\Rightarrow 3(F) = 2|E|$

Now, apply Euler's formula (for planar graphs)

Exercise!

□

Lecture 23 (various applications)

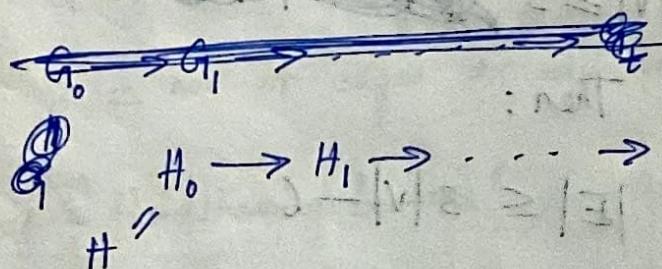
Let $G = (V, E)$ s.t $|V| = n$, $|E| = m$

Theorem (Euler's thm.): If G is planar, then

$$m \leq 3n - 6$$

Defⁿ: Let H, G be graphs.

We say G contains a "subdivision" of H if
there is a sequence of subdivision of



s.t there is an injective homomorphism of H_t into G

Theorem (Kuratowski):

G is planar \Leftrightarrow It does not contain subdivision
of K_5 or $K_{3,3}$

Theorem (Wagner):

G is planar \Leftrightarrow It doesn't contain K_5
or $K_{3,3}$ as its minor

Defⁿ (minor) : H is a minor of G if H can be obtained by deleting edges, vertices and collapsing degree 2 vertices

Corollary (to Euler's formula) : $C_r(G) \geq m - 3n + 6$

$$C_r(G) = \Omega(n^2)$$

To show : $m = \Omega(n^2) \Rightarrow C_r(G) \geq n^4$

Crossing Lemma : If $m \geq 4n$, then

$$C_r(G) \geq \frac{m^3}{n^2}$$

pf's : "Leyton '80's", "Shanbhag - Pach - Zigh : 90's"

$$G(V, E)$$

Let V_p = the random vertex set selected by tossing independent coins (w.p p) on each vertex

and E_p = edges of the induced graph $G[V_p]$

$$\text{Consider } G_p = (V_p, E_p)$$

We're now considering the induced drawing of G_p

$$E(|V_p|) = np$$

$m \geq 4n$

$\text{Gr}(G_p) = \text{no. of crossing}$
in the induced
drawing

$$|E_p| = \sum_{e \in E} \mathbb{1}_e$$

$$\Rightarrow E(|E_p|) = \sum_{e \in E} P(e \in E_p) = mp^2$$

Now, $E(\text{Gr}(G_p)) = p^4 G(G)$ \rightarrow Check!

Now, using Corollary to Euler,

$$p^4 \text{Gr}(G) \geq mp^2 - 3np + 6$$

\uparrow

$$\text{Gr}(G_p) \geq |E| - 3|V_p| + 6$$

$$\Rightarrow \text{Gr}(G) \geq \frac{m}{p^2} - \frac{3n}{p^3} + \frac{6}{p^4}, \quad p \in (0, 1]$$

Theorem: $m \geq 4n \Rightarrow \text{Gr}(G) \geq m^3/n^2$

Let $L = \{ \text{lines in } \mathbb{R}^2 \}$

$$|L| = m, |P| = n$$

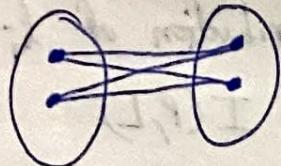
$$P = \{ \text{points in } \mathbb{R}^2 \}$$

$$I(P, L) := \{ (p, l) : p \in P, l \in L, p \in l \}$$

and

$I^P = \{ (l, p) : p \in I(l) \}$

NOTE:



"Kovari - Sos - Turan"

does n^4 contain $K_{2,2}$

$$|E| \lesssim n^{3/2}$$

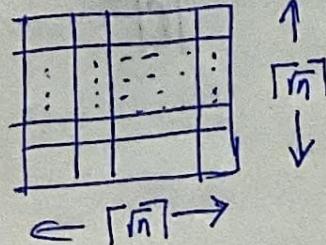
Corollary (Kovari - Sos - Turan): Let $|P|=|L|=n$

$$\text{Then, } |I(P, L)| \lesssim n^{3/2}$$

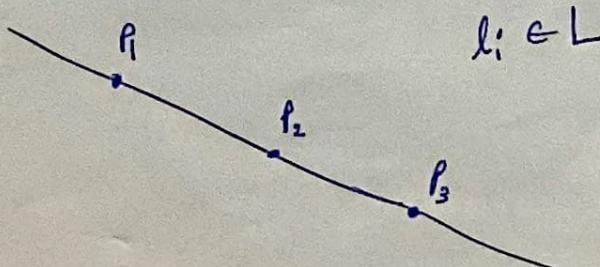
Theorem (Szemerédi-Trotter): If P is a set of n points in \mathbb{R}^2 and L is a set of m lines in \mathbb{R}^2 , then

$$|I(P, L)| \lesssim (mn)^{2/3} + m + n$$

NOTE:



PF:



$$P = \{\text{vertices}\}$$

Let $e(l_i) =$

Note that, $e(l_i) + 1 =$ contribution of l_i
to $I(P, L)$

Now, $|E| = \sum_i e(l_i)$ Let's prove this

$$= \sum_{l \in L} [e(l) + 1] - \sum_{l \in L} 1$$

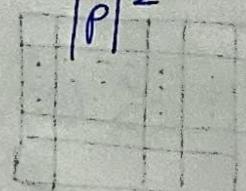
$$n = |L| = |U| = I(P, L) = |L| \quad (\text{since } L \text{ is a simple path})$$

Assume $|E| \geq 4|P|$

$$\begin{aligned} \because |E| < 4|P| &\Rightarrow I(P, L) - |U| < 4|P| \\ &\Rightarrow I(P, L) \leq |U| + |P| = n + m. \quad \checkmark \end{aligned}$$

Now, by "Crossing Lemma", (n) > (1/3)|E|^3

$$|L|^2 \geq G(G) \geq \frac{(|I(P, L)| - |U|)^3}{|P|^2}$$



□

$\{w_i, v_j\} = ?$

$= (10_0 + 1_0)$