

by Ramachandran & Tsokos.

Focus of this course: Understanding basic concepts and developing intuition so that you can apply these to solve some problems in your area of research.

Assignments : 30%

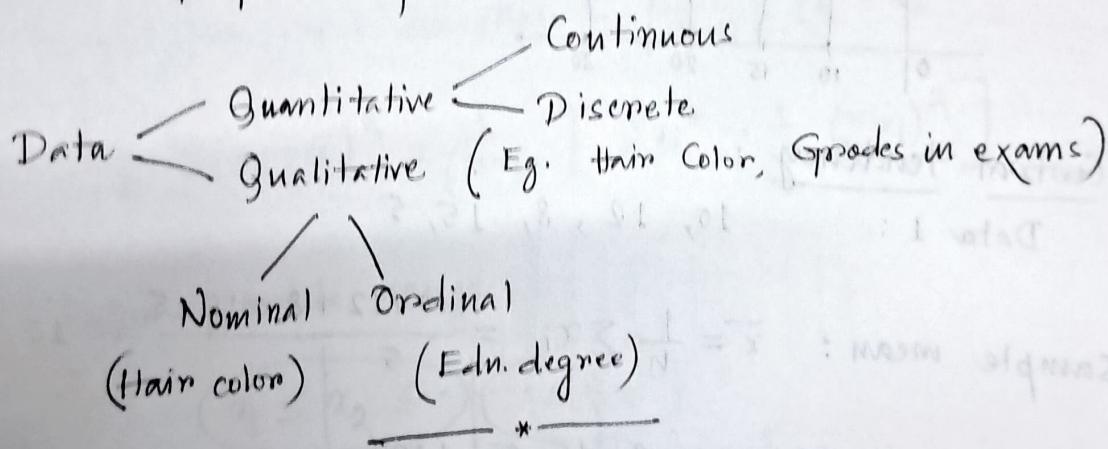
Mid-term : 20%

End-semester: 50%

→ (Class tests, take-home assignments)

## • Descriptive statistics.

### 1. Graphical representation of data.



Population

Sample

A subset of population

(very often it is randomly selected)

Quantitative data :

10, 11, 8, 7, 12, 16, 15, 25, 18, 7, 6, ...

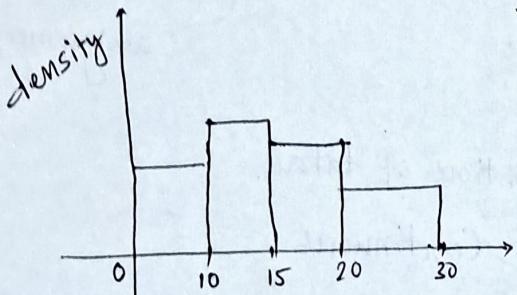
### • Histogram:

Class intervals :  $[0, 10)$ ,  $[10, 15)$ ,  $[15, 20)$ ,  $[20, 30)$

Class intervals	Frequency	Relative Frequency	density
[0, 10)	$f_1$	$f_1/n$	$(f_1/n)/w_1$
[10, 15)	$f_2$	$f_2/n$	$(f_2/n)/w_2$
[15, 20)	$f_3$	$f_3/n$	$(f_3/n)/w_3$
[20, 30)	$f_4$	$f_4/n$	$(f_4/n)/w_4$
	$\frac{n}{\text{sample size}}$	$\frac{1}{\text{sample size}}$	

$w_i = i^{\text{th}}$  class width density

Height of  $i^{\text{th}}$  class =  $\frac{(f_i/n)}{w_i}$



### ① Central tendency :

Data 1 : 10, 12, 8, 15, 5

② Sample mean :  $\bar{x} = \frac{1}{n} \sum x_i = \frac{10+12+8+15+5}{5} = 10$

Data 2 : 10, 12, 8, 15, 5, 100

Sample mean =  $\frac{150}{6} = 25$

— Order the data from smallest to largest.

Data 1      5, 8, 10, 12, 15  
                   ↑ sample median

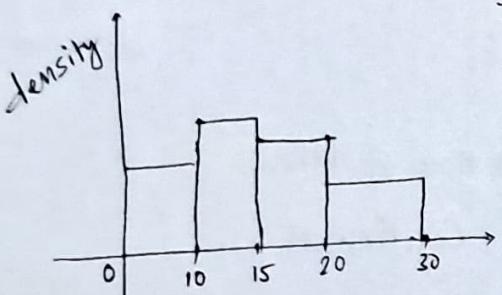
Data 2      5, 8, 10, 12, 15, 100

↑ sample median =  $\frac{10+12}{2} = 11$

Class intervals	Frequency	Relative Frequency	density
[0, 10)	$f_1$	$f_1/n$	$(f_1/n)/w_i$
[10, 15)	$f_2$	$f_2/n$	$(f_2/n)/w_2$
[15, 20)	$f_3$	$f_3/n$	$(f_3/n)/w_3$
[20, 30)	$f_4$	$f_4/n$	$(f_4/n)/w_4$
	$\overbrace{\hspace{10em}}^n$	$\overbrace{\hspace{10em}}^1$	

$w_i = i\text{-th class width density}$  (sample size)

Height of  $i$ -th class =  $(f_i/n)/w_i$



### • Central tendency:

Data 1 : 10, 12, 8, 15, 5

• Sample mean :  $\bar{x} = \frac{1}{n} \sum x_i = \frac{10+12+8+15+5}{5} = 10$

Data 2 : 10, 12, 8, 15, 5, 100

Sample mean =  $\frac{150}{6} = 25$

— Order the data from smallest to largest.

Data 1 5, 8, 10, 12, 15  
↑ sample median

Data 2 5, 8, 10, 12, 15, 100

↑ sample median =  $\frac{10+12}{2} = 11$

— Sample median is more robust than sample mean.

Calculating median: Order  $x_{(1)} \leq x_{(2)} \leq x_{(3)} \leq \dots \leq x_{(n)}$

$$\begin{cases} n \text{ is odd} & \text{sample median } x_{\frac{n+1}{2}} \\ n \text{ is even} & \text{sample median} = \frac{1}{2} (x_{\frac{n}{2}} + x_{\frac{n}{2}+1}) \end{cases}$$

• Trimmed mean — Trim a% of numbers from each end.

• Sample variance —  $\{x_1, x_2, \dots, x_n\} \leftarrow \text{sample}$

Use  $\bar{x}$ : sample mean as center

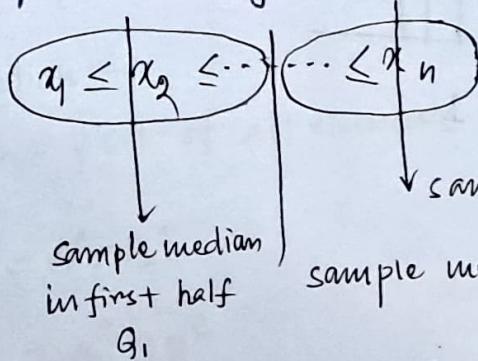
$$(x_1 - \bar{x})^2, (x_2 - \bar{x})^2, \dots, (x_n - \bar{x})^2$$

$$\text{sample variance} = \frac{1}{(n-1)} \sum (x_i - \bar{x})^2$$

$$\text{sample standard deviation} = \sqrt{\frac{1}{(n-1)} \sum (x_i - \bar{x})^2}$$

$$\frac{1}{(n-1)} \sum (x_i - \bar{x})^2 = \left( \frac{1}{n-1} \right) \left[ \sum x_i^2 - \frac{1}{n} (\sum x_i)^2 \right]$$

• Inter-quartile range:



$$\text{Inter-quartile range} = Q_3 - Q_1$$

Eg:  $8, 9, 11, 15, 18, 20, 21$

$$Q_2 = 15$$

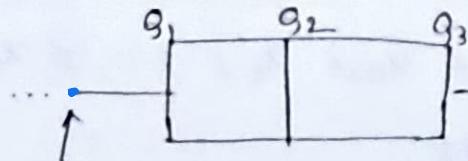
$$Q_1 = 10$$

$$Q_3 = 19$$

$$IQR = 19 - 10 = 9$$

1<sup>st</sup> quartile  $\rightarrow$  median of 1<sup>st</sup> half  
2<sup>nd</sup> quartile  $\rightarrow$  median of 2<sup>nd</sup> half  
3<sup>rd</sup> quartile  $\rightarrow$  median of 3<sup>rd</sup> half

## Boxplot:



min point that  
is not an outlier

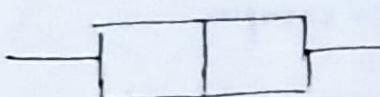
max point that is not  
an outlier

## Outliers:

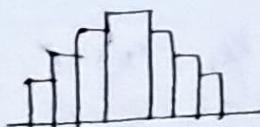
$$> Q_3 + 1.5 \times IQR$$

or

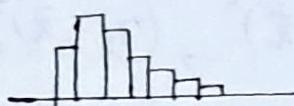
$$< Q_1 - 1.5 \times IQR$$



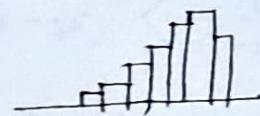
symmetric



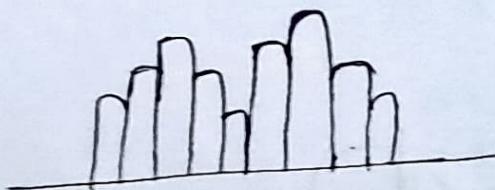
right skewed



left skewed



-Bimodal distribution



Reading assignment — Chapter 1.

Sample space: Set of all possible outcomes in an experiment.

LEC-02

26/07/24

Eg. • Tossing a coin.  $S = \{H, T\}$

• Rolling a dice  $S = \{1, 2, 3, 4, 5, 6\}$

• Max temp tomorrow  $S = (10, 50)$

Event: A subset of the sample space.

$A = \text{the roll of a dice results in an odd number.}$

$A = \{1, 3, 5\}$ , a subset of  $\{1, 2, 3, 4, 5, 6\}$

Event operations:

$S = \text{Sample Space}$

• Complement Suppose  $A$  is an event.  $A \subseteq S$ .

$A^c / \bar{A}$  (Complement of the event)  $= S \setminus A$

= {all elements of the sample space which are not in  $A$ }

• Union: Let  $A$  and  $B$  are two events.

$A \cup B = \{\text{set of all elements either in } A \text{ or in } B \text{ or in both}\}$

• Intersection -

$A \cap B = \{\text{set of all elements in both } A \text{ and } B\}$

$$\bullet (A \cup B)^c = A^c \cap B^c$$

$$\bullet (A \cap B)^c = A^c \cup B^c$$

$$\bullet A = (A \cap B) \cup (A \cap B^c)$$

• If  $U \cap V = \emptyset$ , then  $U$  and  $V$  are said to be disjoint.

• Probability:  $P(A)$  is a function. Domain = {set of events}

Range =  $[0, 1]$

$P(A) \geq 0$  for any event  $A$   
 $\Rightarrow P(S) = 1$        $S = \text{sample space}$   
 $\Rightarrow$  If  $A_1, A_2, \dots$  are mutually disjoint events,

then  $P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$

•  $P(\emptyset) = 0$

Take  $A_1 = \emptyset$ , any event such that  $P$ .

$A_2 = A_3 = \dots = \emptyset$

$P(\emptyset) = P(A) + P(B) + P(C) + \dots$

This is possible when  $P(\emptyset) = 0$

•  $P(A^c) = 1 - P(A)$

$A \cup A^c = S \Rightarrow P(A \cup A^c) = P(S) = 1$

$A_1 = A, \quad A_2 = A^c, \quad A_3 = A_4 = \dots = \emptyset$

$P(\emptyset) = P(A) + P(A^c) + P(\emptyset) + P(\emptyset) + \dots$

$\Rightarrow 1 = P(A) + P(A^c)$

$\Rightarrow P(A^c) = 1 - P(A)$

• If  $A$  and  $B$  are two disjoint events, then

$P(A \cup B) = P(A) + P(B)$ .

•  $A_1, A_2, \dots, A_n$  disjoint events.

$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$

If  $A, B$  are any two events,  $P(A \cup B) = P(A) + P(B) - P(AB)$

$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$$

$$\therefore P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$$

Also  $P(A) = P(A \cap B^c) + P(A \cap B)$

$$P(B) = P(A^c \cap B) + P(A \cap B)$$

Substituting,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

•  $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C)$   
 $- P(B \cap C) + P(A \cap B \cap C)$

•  $P\left(\bigcup_{i=1}^n A_i\right) = \sum_1^n P(A_i) - \sum_{i < j} P(A_i \cap A_j) + \sum_{i < j < k} P(A_i \cap A_j \cap A_k)$   
 $- \dots + (-1)^{n+1} P(A_1 \cap A_2 \cap \dots \cap A_n)$

### Counting Techniques:

1. Sample with replacement, order is important.

— How many ways can we select  $k$  objects out of  $n$ ?  $\rightarrow n^k$ .

2. Sample without replacement, order is important.

$$n(n-1) \dots (n-k+1) = \frac{n!}{(n-k)!}$$

= # of permutations

3. Sample without replacement, order is not important.

$$\frac{n!}{(n-k)! k!} = \# \text{ no of combinations}$$

$$= \binom{n}{k};$$

( $n$  choose  $k$ )

## Axioms

i)  $P(A) \geq 0$  for any event  $A$

ii)  $P(S) = 1$   $S = \text{sample space}$

iii) If  $A_1, A_2, \dots$  are mutually disjoint events,

then  $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ .

•  $P(\emptyset) = 0$

Take  $A_1 = A$ , any event other than  $\emptyset$ .

$$A_2 = A_3 = \dots = \emptyset$$

$$P(A) = P(A) + P(\emptyset) + P(\emptyset) + \dots$$

This is possible when  $P(\emptyset) = 0$

•  $P(A^c) = 1 - P(A)$

$$A \cup A^c = S \Rightarrow P(A \cup A^c) = P(S) = 1$$

$$A_1 = A, \quad A_2 = A^c, \quad A_3 = A_4 = \dots = \emptyset$$

$$P(S) = P(A) + P(A^c) + P(\emptyset) + P(\emptyset) + \dots$$

$$\Rightarrow 1 = P(A) + P(A^c)$$

$$\Rightarrow P(A^c) = 1 - P(A)$$

• If  $A$  and  $B$  are two disjoint events, then

$$P(A \cup B) = P(A) + P(B).$$

•  $A_1, A_2, \dots, A_n$  disjoint events.

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$$

• If  $A, B$  are any two events,  $P(A \cup B) = P(A) + P(B) - P(AB)$

$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$$

$$\therefore P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$$

Also  $P(A) = P(A \cap B^c) + P(A \cap B)$

$$P(B) = P(A^c \cap B) + P(A \cap B)$$

Substituting,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

•  $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C)$   
 $\quad \quad \quad \quad \quad - P(B \cap C) + P(A \cap B \cap C)$

•  $P\left(\bigcup_{i=1}^n A_i\right) = \sum_1^n P(A_i) - \sum_{i < j} P(A_i \cap A_j) + \sum_{i < j < k} P(A_i \cap A_j \cap A_k)$   
 $\quad \quad \quad \quad \quad - \dots + (-1)^{n+1} P(A_1 \cap A_2 \cap \dots \cap A_n)$

### Counting Techniques:

1. Sample with replacement, order is important.

— How many ways can we select  $k$  objects out of  $n$ ?  $\rightarrow n^k$ .

2. Sample without replacement, order is important.

$$n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

$= \# \text{ of permutations}$

3. Sample without replacement, order is not important.

$$\frac{n!}{(n-k)! k!} = \# \text{ no of combinations}$$
$$= \binom{n}{k} ;$$

$(n \text{ choose } k)$

4. Sample with replacement, order is not important.

— = # of ways we can arrange  
(n-i) sticks and k stars. Order is not important.  
 $= \binom{n+k-1}{k}$

—  $x_1 + x_2 + \dots + x_n = k$ ,  $x_i > 0$ ,  $x_i$  integer

# no of solutions

• Suppose there are 6 black balls and 4 white balls.

You randomly select 2 balls without replacement. What is the probability that one of them is black and other is white?

— 
$$\frac{\binom{6}{1} \binom{4}{1}}{\binom{10}{2}}$$

• Conditional Probability.

$P(A|B)$  (conditional probability of A given B)

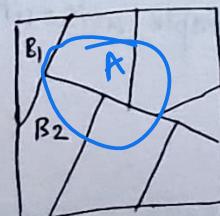
Defn  $P(A|B) = \frac{P(A \cap B)}{P(B)}$ ,  $P(B) > 0$

$\Rightarrow P(A \cap B) = P(A|B) P(B)$  } Multiplication rules  
=  $P(B|A) P(A)$  } (provided  $P(A) > 0$ )

$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_K)$

$\Rightarrow P(A) = \sum_{i=1}^k P(A \cap B_i)$

$= \sum_{i=1}^k P(A|B_i) P(B_i)$



$S = B_1 \cup B_2 \cup \dots \cup B_K$   
 $B_i \cap B_j = \emptyset \quad i \neq j$

$$P(A) = \sum_{i=1}^k P(A|B_i) P(B_i) \quad (\text{Law of total probability})$$

• Toss a coin, where  $P(\text{Head}) = 0.6$

Toss a coin, if head select a ball from bag 1, if tail from bag 2.

$$\begin{array}{c} 6B \\ 4H \end{array} \quad \begin{array}{c} 3B \\ 5H \end{array}$$

$$B_1 \quad B_2$$

$$\begin{aligned} P(\text{selected ball is white}) &= P(A) = P(A|B_1)P(B_1) + \\ &\quad P(A|B_2)P(B_2) \\ &= \left(\frac{1}{10}\right)(0.6) + \left(\frac{5}{8}\right)(0.4) \end{aligned}$$

$$\text{Also, } P(B_j | A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{P(A)},$$

(Bayes Theorem)

2.2.3 A pair of six-sided balanced dice are rolled.

[LEC-03]

31/07/24

$$\mathcal{S} = \{(1,1), (1,2), \dots, (6,6)\} \quad |\mathcal{S}| = 36$$

$$\begin{aligned} (a) \quad A &= \{x_1 + x_2 = 8\} \\ &= \{(2,6), (3,5), (4,4), (5,3), (6,2)\} \end{aligned}$$

$$|A| = 5$$

$$\therefore P(A) = P(x_1 + x_2 = 8) = \frac{|A|}{|\mathcal{S}|} = \frac{5}{36}$$

$$\begin{aligned} (b) \quad B &= \{x_1 + x_2 = 6 \text{ or } 9\} \\ &= \{(1,5), (5,1), (2,4), (4,2), (3,3), (3,6), (6,3), (4,5), \\ &\quad (5,4)\} \end{aligned}$$

$$\therefore P(B) = \frac{9}{36} = \frac{1}{4}$$

$$(2.2.13) (A) B = A \cup (B \cap A^c)$$

$$B \cup \emptyset \cup \dots = A \cup (B \cap A^c) \cup \emptyset \cup \dots$$

$$P(B) + P(\emptyset) + \dots = P(A) + P(B \cap A^c) + P(\emptyset) + \dots$$

$$\Rightarrow P(B) + 0 + 0 + \dots = P(A) + P(A^c \cap B) + 0 + \dots$$

Since  $P(\emptyset) \neq 0$

$$\Rightarrow P(B) \geq P(A).$$

(need to be proved)

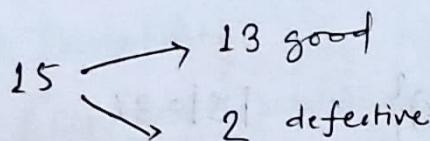
$$(2.2.16) \quad P(A) = 0.17, \quad P(B) = 0.46, \quad A \cap B = \emptyset$$

$$(a) \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$(b) \quad P(A^c) = 1 - P(A)$$

$$(c) \quad P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1$$

2.3.12



$$(a) \quad P(\text{none of the selected apples is defective})$$

$$= \frac{\binom{13}{4}}{\binom{15}{4}}$$

$$(b) \quad P(\text{at least one of the selected apples is defective})$$

$$= 1 - \frac{\binom{13}{4}}{\binom{15}{4}}$$

$$\underline{2.3.19} \quad (a) \quad |S| = 6^5$$

Def<sup>n</sup>: A and B are said to be independent  
iff  $P(A \cap B) = P(A) \cdot P(B)$  ,  $\Leftrightarrow P(A|B) = P(A)$

2.4.5

$$(1) P(A^c|B) = \frac{P(A^c \cap B)}{P(B)}$$

$$B = (A \cap B) \cup (A^c \cap B)$$

$$= \frac{P(B) - P(A \cap B)}{P(B)}$$

$$P(B) \left( 1 - \frac{P(A \cap B)}{P(B)} \right)$$

$$= \frac{P(B)(1 - P(A))}{P(B)}$$

$$= \frac{P(B)(1 - P(A))}{P(B)} \quad (\because A \text{ and } B \text{ are independent})$$

$$= 1 - P(A)$$

$$= P(A^c)$$

2.4.13 D : Disease + : detected disease

$$P(+|D) = 0.98$$

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+)}$$

$$P(+|D^c) = 0.005$$

$$= \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)}$$

$$= \frac{0.98 \times 0.02}{(0.98)(0.02) + (0.005)(0.98)}$$

$$P(D) = 0.02$$

2.4.20

$P(1 \text{ has chosen} \mid \text{at least one white ball has been removed})$

$\boxed{\begin{matrix} 2W \\ 2B \end{matrix}}$

$$= \frac{P(\text{at least one white ball was removed} \mid 1 \text{ has chosen}) \cdot P(1 \text{ has chosen})}{P(\text{at least one white ball was removed})}$$

$$= \frac{\frac{1}{4} \cdot \frac{2}{4}}{\frac{1}{4} \cdot \frac{2}{4} + \frac{1}{4} \cdot \frac{5}{6} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{5}{6} + \frac{1}{4} + \frac{1}{4}} = \frac{3}{20}$$

$$(2.2.13) \quad (A) \quad P(B) = A \cup (B \cap A^c)$$

$$B \cup \emptyset \cup \dots = A \cup (B \cap A^c) \cup \emptyset \cup \dots = P(A) + P(B \cap A^c) + P(\emptyset) + \dots$$

$$P(B) + P(\emptyset) + \dots = P(A) + P(B \cap A^c) + P(\emptyset) + \dots$$

$$\Rightarrow P(B) + 0 + 0 \dots = P(A) + P(A^c \cap B) + 0 + \dots$$

Since  $P(\emptyset) = 0$   
(need to be proved)

$$\Rightarrow P(B) \geq P(A).$$

$$(2.2.16) \quad P(A) = 0.17, \quad P(B) = 0.46, \quad A \cap B = \emptyset$$

$$(a) \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$(b) \quad P(A^c) = 1 - P(A)$$

$$(c) \quad P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1$$

2.3.12 (a)

15	$\nearrow$	13 good
	$\searrow$	2 defective

$$(a) \quad P(\text{none of the selected apples is defective})$$

$$= \frac{\binom{13}{4}}{\binom{15}{4}}$$

$$(b) \quad P(\text{at least one of the selected apples is defective})$$

$$= 1 - \frac{\binom{13}{4}}{\binom{15}{4}}$$

2.3.19 (a)  $|S| = 6^5$

(b) • Def<sup>n</sup>: A and B are said to be independent  
iff  $P(A \cap B) = P(A) \cdot P(B)$  ,  $\Leftrightarrow P(A|B) = P(A)$

2.4.5

$$(1) P(A^c|B) = \frac{P(A^c \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)}$$

$$= \frac{P(B)(1 - P(A))}{P(B)}$$

$$= 1 - P(A)$$

$$= P(A^c)$$

$B = (A \cap B) \cup (A^c \cap B)$

2.4.13 D: Disease + : detected disease

$$P(+|D) = 0.98$$

$$P(+|D^c) = 0.005$$

$$P(D) = 0.02$$

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+|D) \cdot P(D) + P(+|D^c) \cdot P(D^c)}$$

$$= \frac{0.98 \times 0.02}{(0.98 \times 0.02) + (0.005 \times 0.98)}$$

$$= \frac{0.98 \times 0.02}{(1 - 0.02)}$$

2.4.20

$P(1 \text{ was chosen} | \text{at least one white ball was removed})$

$\begin{cases} 2H \\ 2B \end{cases}$

$$= \frac{P(\text{at least one white ball was removed} | 1 \text{ was chosen}) \cdot P(1 \text{ was chosen})}{P(\text{at least one white ball was removed})}$$

$$= \frac{\frac{1}{4} \cdot \frac{2}{4}}{\frac{1}{4} \cdot \frac{2}{4} + \frac{1}{4} \cdot \frac{5}{6} + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{5}{6} + 1 + 1} = \frac{3}{20}$$

## Random Variables

Discrete random variable : Probability mass function

(pmf) of  $X = P(X=x)$ ,  $x \in S$ .

$$\underline{\text{Ex}} \quad \text{Rolling a die} \quad P(X=i) = \frac{1}{6} \quad i=1, 2, 3, 4, 5, 6 \\ (\text{pmf. of } X)$$

$$\boxed{E(X) = \sum_{x \in S} x \cdot P(X=x)} \quad (M_X) \\ (\text{or mean of } X)$$

$$\underline{\text{Ex}} \quad E(X) = \sum_{i=1}^6 i \cdot \frac{1}{6} = 3.5$$

$$\boxed{V(X) = V(X) = \sum_{x \in S} (x - M_X)^2 P(X=x)}$$

Variance of the random variable  $X$

$$= \sum x^2 P(X=x) - (\sum x P(X=x))^2$$

$$\sum (x - M_X)^2 P(X=x)$$

$$= \sum (x^2 - 2M_X x + M_X^2) P(X=x)$$

$$= \sum x^2 P(X=x) - 2M_X (\sum x P(X=x)) + M_X^2 (\sum P(X=x))$$

$$= \sum x^2 P(X=x) - 2M_X^2 + M_X^2$$

$$= \sum x^2 P(X=x) - M_X^2$$

$$= \sum x^2 P(X=x) - (\sum x P(X=x))^2$$

$$\bullet E(h(x)) = \sum_{x \in S} h(x) P(X=x)$$

$$\bullet E(X^2) = \sum x^2 P(X=x)$$

$$\bullet E(ax) = a E(X)$$

$$\bullet E(X+b) = E(X) + b$$

$$\bullet E(ax+b) = a E(X) + b$$

$$\left| \begin{array}{l} V(ax) = a^2 V(X) \\ V(X+b) = V(X) \\ V(ax+b) = a^2 V(X) \end{array} \right.$$

LEC-01

02/08/24

$S/\Omega \leftarrow$  sample space

$\omega \in \Omega$ .

RV:  $X : \Omega \rightarrow \mathbb{R}$  (mapping the sample space (outcomes) to Real Number).

$$X(\omega) : \Omega \rightarrow \mathbb{R}$$

Eg  $\Omega = \{ HH, HT, TH, TT \}$   $X$  : No of H's

$$X(\omega_1) = 2, X(\omega_2) = 1, X(\omega_3) = 1, X(\omega_4) = 0$$

Eg Toss a coin until we get a Head

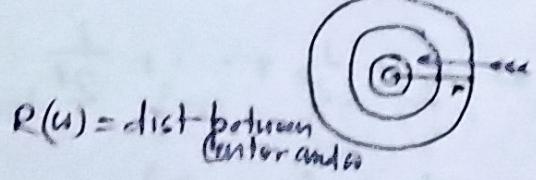
$$\Omega = \{ H, TH, TTH, \dots \}$$

$$X(\omega_1) = 0, X(\omega_2) = 1, X(\omega_3) = 2, \dots$$

Eg

$$0 \leq R(\omega) \leq R$$

$$R(\omega) \in [0, r]$$



$R \leftarrow$  a continuous random variable.

$$P(X=1) = P(X^{-1}(\{1\}))$$

- Discrete RV: A random variable taking only countable (finite or infinite) number of isolated points  $x_1, x_2, \dots$  with  $P[X=x_i] > 0$  is called a discrete random variable.

$P(X=x_i) > 0$        $x_i$ : mass point/jump point

- PMF (Probability Mass function)

$$f/P : \mathbb{R} \rightarrow [0, 1]$$

$$\text{Ex: Suppose } P(x) = \begin{cases} 2^{-x} & \text{if } x \in \{1, 2, \dots\} \\ 0 & \text{o/w} \end{cases}$$

Compute  $P(X \in A)$  where i)  $A = \{1, 2, 3\}$

$$\text{(ii)} \quad A = [5, 8]$$

$$\begin{aligned} \text{Soh} \text{ i) } P(X \in \{1, 2, 3\}) & \quad \text{(iii)} \quad A = \{1, 3, 5\} \\ & = P(X \in \{1\}) + P(X \in \{2\}) + P(X \in \{3\}) \\ & = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \end{aligned}$$

$$\text{(ii) } P(X \in [5, 8])$$

$$= P(X=1) + P(X=2) + \dots + P(X=8)$$

$$= \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^8}$$

$$\text{(iii) } P(X \in \{1, 3, 5\})$$

$$= P(X=1) + P(X=3) + P(X=5) = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5}$$

$$(iv) P(X \in \{1, 3, 5, \dots\})$$

$$= P(X \in \Omega \setminus \{2, 4\}) = 1 - \frac{1}{2^2} - \frac{1}{2^4}$$

$$= P(X=1) + P(X=3) + P(X=5) + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}} = \frac{2}{3}$$

All def<sup>n</sup> of pmf  $P: \mathbb{R} \rightarrow [0, 1]$  is a pmf iff it satisfies

i. Countable set  $S \subseteq \mathbb{R}$  s.t.  $\forall x \notin S \quad P(X=x)=0$

ii.  $\forall x \in S, \quad P(x) > 0$ .

iii.  $\sum_{x \in S} P(x) = 1$

Eg  $P(x) = \begin{cases} c \cdot \frac{\theta^x}{x}, & x=1, 2, \dots \\ 0 & \text{o/w} \end{cases}$

$$\sum_{x=1}^{\infty} c \frac{\theta^x}{x} = 1.$$

$$c \cdot \sum_{n=1}^{\infty} \frac{\theta^n}{n} = 1$$

For what values of  $\theta$  and  $c$ ,  $P(x)$  is a pmf?

$$S = \{1, 2, \dots\}$$

$$\sum_{x \in S} P(x) = 1 \Rightarrow \sum_{n=1}^{\infty} c \cdot \frac{\theta^n}{n} = 1$$

$$\begin{cases} c \frac{\theta}{1} > 0 \Rightarrow c\theta > 0 \\ c \frac{\theta^2}{2} > 0 \Rightarrow c\theta^2 > 0 \\ \dots \\ \Rightarrow c > 0, \theta > 0 \end{cases}$$

$$\Rightarrow c \ln(1-\theta) = 1. \quad (\text{if } 1-\theta \neq 0)$$

$$\Rightarrow c = \frac{1}{-\ln(1-\theta)}$$

Eg Find the value of  $c$  for which  $f(x) = \begin{cases} c \left(\frac{2}{3}\right)^x; & x=1, 2, \dots \\ 0 & \text{o/w} \end{cases}$  is a pmf.

soln.  $f(1) > 0 \Rightarrow c > 0$

$$\text{and } \sum_{n=1}^{\infty} f(n) = 1 \Rightarrow \sum_{n=1}^{\infty} c \cdot \left(\frac{2}{3}\right)^n = 1$$

$$\Rightarrow c \cdot \frac{1}{2/3} \cdot \frac{1}{1 - 2/3} = 1$$

$$\Rightarrow c = 1/2$$

• Continuous random Variable : A r.v  $X$  is said to be continuous random variable if it takes any value from its range of variation.

Remark : For a continuous r.v  $P(X=x) = 0 \ \forall x$

• PDF (probability density function)

$$f: \mathbb{R} \rightarrow [0, \infty) \text{ if } \forall a, b \in \mathbb{R}, P(X \in [a, b]) = \int_a^b f(x) dx$$

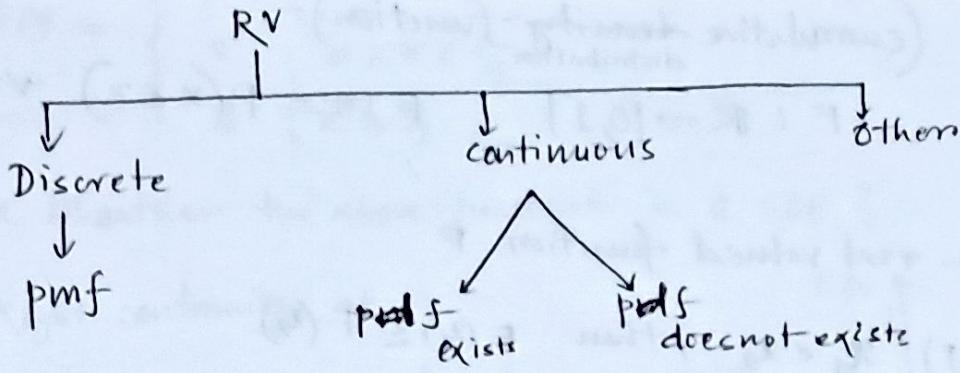
$$\text{Ex- } f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

Compute  $P(X \in A)$ , where  $A$  is  
 (i)  $[2, 5]$   
 (ii)  $\{3\}$   
 (iii)  $[2, 3)$

$$\text{Soh} : \text{(i) } P(X \in [2, 5]) = \int_2^5 e^{-x} dx \\ = -e^{-5} + e^{-2}$$

$$\text{(ii) } P(X \in \{3\}) = P(X=3) = 0$$

$$\text{(iii) } P(X \in [2, 3)) = \int_2^3 e^{-x} dx = -e^{-3} + e^{-2}$$



All  $f: \mathbb{R} \rightarrow [0, \infty)$  is a pmf of some random variable  $X$

iff (I)  $\forall x \quad f(x) \geq 0$

$$(II) \quad \int_{-\infty}^{+\infty} f(x) = 1$$

Eg  $f(x) = \begin{cases} x/2 & , 0 \leq x \leq 1 \\ 1/2 & , 1 \leq x \leq 2 \\ \frac{3-x}{2} & , 2 \leq x \leq 3 \\ 0 & \text{o/w} \end{cases}$

Check whether  $f(x)$  is a pmf.

Eg:  $f(x) = \begin{cases} k & \text{if } x \in [0, 1/e] \\ 0 & \text{o/w.} \end{cases}$  is a pdf. Find  $k$ .

(Cumulative distribution function)

$$F: \mathbb{R} \rightarrow [0, 1], \quad F(x) = P(X \leq x)$$

A real valued function  $F$

(I)  $x_1 < x_2$ , then  $F(x_1) \leq F(x_2)$

(i.e.,  $F$  is monotonically non-decreasing)

(II)  $F(-\infty) = \lim_{x \rightarrow -\infty} F(x) = 0$

(III)  $F(\infty) = \lim_{x \rightarrow \infty} F(x) = 1$

(IV)  $F$  is right continuous.

$$F(x+0) = \lim_{k \rightarrow 0^+} F(x+k) = F(x)$$

Remark I.  $F$  is not necessarily continuous.

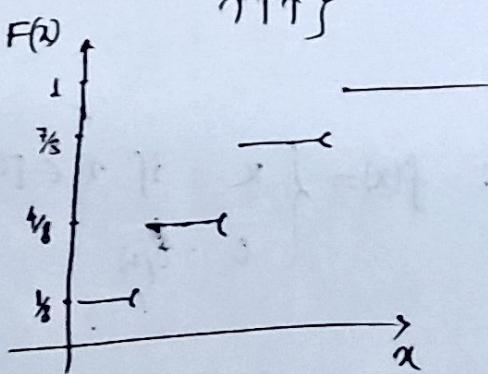
II. A necessary and sufficient condition

for a r.v.  $X$  or its CDF  $F$  to be continuous  
at  $x=x$  if  $P(X=x)=0$

Eg  $X := \# \text{ heads}$

$$\Omega = \{ HHH, HHT, HTH, THH, THT, TTH, TTT \}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{8} & 0 < x \leq 1 \\ \frac{4}{8} & 1 < x \leq 2 \\ \frac{7}{8} & 2 < x \leq 3 \\ 1 & x \geq 3 \end{cases}$$



$$F(x) = \begin{cases} 0 & ; x < 0 \\ x & ; 0 \leq x < \frac{1}{2} \\ 1 & ; x \geq \frac{1}{2} \end{cases}$$

Check whether the above function is a CDF?

Sohy Right continuity at 0 :

$$\lim_{h \rightarrow 0^+} F(0+h) = \lim_{h \rightarrow 0^+} h = 0 = F(0)$$

Right continuity at  $\frac{1}{2}$ :

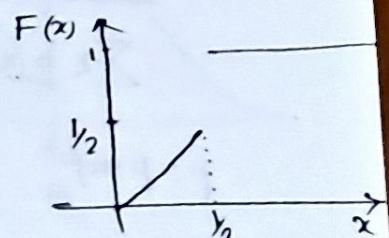
$$\lim_{h \rightarrow 0^+} F\left(\frac{1}{2}+h\right) = \lim_{h \rightarrow 0^+} 1 = 1 = F\left(\frac{1}{2}\right)$$

~~Right continuity at  $+$~~   
 $\therefore F$  is right continuous.

$$\text{Also } \lim_{x \rightarrow -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} F(x) = 1$$

$F$  is also monotonically nondecreasing.

Hence  $f$  is a CDF.



# Expectation and Variance of Random Variables

LEC-05

06/08/24

Discrete

$$\sum x p(x)$$

$x \in S$

p: pmf

Continuous

$$P(x < X \leq x+h) \quad , \quad h > 0$$

$$= P(X \leq x+h) - P(X \leq x)$$

$$= F_X(x+h) - F_X(x)$$

$$-\frac{1}{h} P(x < X \leq x+h) = \frac{1}{h} (F_X(x+h) - F_X(x))$$

$$\lim_{h \downarrow 0} (P(x < X \leq x+h)) = \lim_{h \downarrow 0} \frac{1}{h} (F_X(x+h) - F_X(x))$$

$$= \frac{d}{dx} F_X(x) \quad (\text{pdf})$$

$$\begin{aligned} \frac{d}{dx} F(x) &= f(x) \\ F(x) &= \int_{-\infty}^x f(t) dt \\ F_X(x) &= P(X \leq x) \end{aligned}$$

[NOTE] pdf gives the information about the randomness of X.

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx$$

$$V(X) = E(X^2) - (E(X))^2 = E((X - \mu_X)^2) = \int_{-\infty}^{+\infty} (x - \mu_X)^2 f(x) dx$$

• Quantiles:

*(Assume  $X$  is continuous)*

$q$  — quantile of  $X$  is  $m$  such that

$$P(X \leq m) = q$$

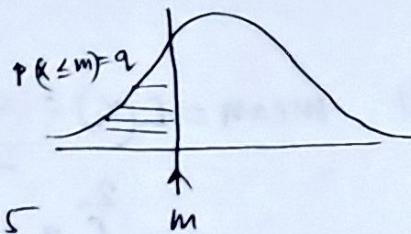
$\downarrow$   
( $100q$  percentile)

0.5-quantile of  $X$

$$m \text{ such that } P(X \leq m) = 0.5$$

$$P(X > m) = 0.5$$

$$\stackrel{\text{II}}{=} P(X \geq m) \quad (\because X \text{ is continuous})$$



• Eg: pdf of  $X$  is  $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Calculate the median of  $X$ .

— Let  $m$  be the median of  $X$ .

$$\text{Then } \int_{-\infty}^m f(x) dx = 0.5 \Rightarrow \int_0^m e^{-x} dx = 0.5$$

$$\Rightarrow -e^{-x} \Big|_0^m = 0.5 \Rightarrow 1 - e^{-m} = 0.5$$

$$\Rightarrow e^{-m} = \frac{1}{2} \Rightarrow -m = \ln \frac{1}{2} \Rightarrow m = -\ln \frac{1}{2} = m_2.$$

$(0.5-q \equiv 50 \text{ percentile})$

• Suppose the pdf of  $X$  is  $f(x) = \begin{cases} cx^3, & 1 \leq x \leq 2 \\ 0, & \text{o/w} \end{cases}$

i) find  $c$ .

ii) Find the mean and median of  $X$ .

# Expectation and Variance of Random Variables

LEC-05

06/08/24

Discrete

$$\sum x p(x)$$

$x \in S$

p: pmf

Continuous

$$P(x < X \leq x+h), h > 0$$

$$= P(X \leq x+h) - P(X \leq x)$$

$$= F_X(x+h) - F_X(x)$$

$$\begin{aligned} \frac{d}{dx} F(x) &= f(x) \\ F(x) &= \int_{-\infty}^x f(t) dt \\ F_X(x) &= P(X \leq x) \end{aligned}$$

$$-\frac{1}{h} P(x < X \leq x+h) = \frac{1}{h} (F_X(x+h) - F_X(x))$$

$$\lim_{h \downarrow 0} (P(x < X \leq x+h)) = \lim_{h \downarrow 0} \frac{1}{h} (F_X(x+h) - F_X(x))$$

$$= \frac{d}{dx} F_X(x) \quad (\text{pdf})$$

[NOTE] pdf gives the information about the randomness of X.

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx$$

$$V(X) = E(X^2) - (E(X))^2 = E((X - \mu_X)^2) = \int_{-\infty}^{+\infty} (x - \mu_X)^2 f(x) dx$$

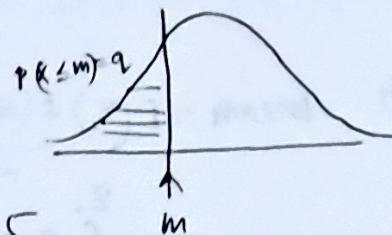
• Quantiles:

(Assume  $X$  is continuous)

$q$  — quantile of  $X$  is  $m$  such that

$$P(X \leq m) = q$$

↓  
(100q percentile)



0.5-quantile of  $X$

$m$  such that  $P(X \leq m) = 0.5$

$$P(X > m) = 0.5$$

$$\text{II } P(X > m) \quad (\because X \text{ is continuous})$$

• Eg: pdf of  $X$  is  $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Calculate the median of  $X$ .

— Let  $m$  be the median of  $X$ .

$$\text{Then } \int_{-\infty}^m f(x) dx = 0.5 \Rightarrow \int_0^m e^{-x} dx = 0.5$$

$$\Rightarrow -e^{-x} \Big|_0^m = 0.5 \Rightarrow 1 - e^{-m} = 0.5$$

$$\Rightarrow e^{-m} = \frac{1}{2} \Rightarrow -m = \ln \frac{1}{2} \Rightarrow m = -\ln \frac{1}{2} = m_2.$$

(0.5-q ≡ 50 percentile)

• Suppose the pdf of  $X$  is  $f(x) = \begin{cases} cx^2, & 1 \leq x \leq 2 \\ 0, & \text{o/w} \end{cases}$

i) find  $c$ .

ii) Find the mean and median of  $X$ .

soln. i)  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \int_1^2 cx^2 dx = 1 \Rightarrow \frac{c}{3} (8-1) = 1 \Rightarrow c = \frac{3}{7}$$

(ii) mean =  $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_1^2 \frac{3}{7} x^3 dx = \frac{3}{7 \cdot 4} (16-1) = \frac{15}{28}$$

Let  $m$  be the median.

$$P(X \leq m) = \frac{1}{2}$$

$$\Rightarrow \int_1^m \frac{3}{7} x^3 dx = \frac{1}{2} \Rightarrow \frac{1}{7} (m^3 - 1) = \frac{1}{2}$$

$$\Rightarrow m^3 = \frac{7}{2} + 1 = \frac{9}{2} \Rightarrow m = \left(\frac{9}{2}\right)^{\frac{1}{3}}$$

$P(a \leq X \leq b) = F_X(b) - F_X(a^-)$

$$P(X > b) = 1 - P(X \leq b) = 1 - F_X(b)$$

$$P(X \geq b) = 1 - P(X < b) = 1 - F_X(b^-)$$

CDF  
 (left limit of  $f(x)$  at  $b$ )

### Joint distribution of $(X, Y)$

Suppose, both  $X$  and  $Y$  are discrete R.V.

Joint pmf of  $(X, Y)$ :

$$p(x, y) = P(X=x \text{ and } Y=y), \quad x \in S_X, y \in S_Y$$

$$\sum_x \sum_y P(x,y) = 1, \quad P(x,y) \geq 0$$

$$P(A) = \sum_{x \in A} \sum_{y \in S_Y} P(x,y)$$

$$P(X \in A | Y \in B) = \frac{P((X \in A) \cap (Y \in B))}{P(Y \in B)}$$

$$= \frac{\sum_{x \in A} \sum_{y \in B} P(x,y)}{\sum_{x \in S_X} \sum_{y \in B} P(x,y)}$$

$$E(h(x,y)) = \sum_{x \in S_X} \sum_{y \in S_Y} h(x,y) P(x,y)$$

$$E(X) = \sum_{x \in S_X} \sum_{y \in S_Y} x P(x,y)$$

Eg: pmf of  $(X,Y)$

$y \backslash x$	0	1	2
0	0.1	0	0.4
1	0.2	0.1	0.2

Calculate (i)  $E(X^2)$

$$(ii) P(X=2 | Y=1)$$

$$(iii) P(Y=0)$$

$$E(X^2) = \sum x^2 \cdot P(x,y) \Rightarrow 0 \times 0.3 + 1 \times (0+0.1) + 4(0.4+0.2) = 0.1 + 2.4$$

$$(ii) P(X=2 | Y=1) = \frac{P(X=2 \text{ and } Y=1)}{P(Y=1)} = \frac{0.2}{0.5} = \boxed{0.4}$$

$$(iii) P(Y=0) = 0.5$$

$$\text{Soln. i) } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_1^2 cx^2 dx = 1 \Rightarrow \frac{c}{3} (8-1) = 1 \Rightarrow c = \frac{3}{7}$$

$$\text{(ii) mean} = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_1^2 \frac{3}{7} x^3 dx = \frac{3}{7 \cdot 4} (16-1) = \frac{45}{28}$$

Let  $m$  be the median.

$$P(X \leq m) = \frac{1}{2}$$

$$\Rightarrow \int_1^m \frac{3}{7} x^3 dx = \frac{1}{2} \Rightarrow \frac{1}{7} (m^3 - 1) = \frac{1}{2}$$

$$\Rightarrow m^3 = \frac{7}{2} + 1 = \frac{9}{2} \Rightarrow m = \left(\frac{9}{2}\right)^{\frac{1}{3}}$$

$$P(a \leq X \leq b) = F_X(b) - F_X(a-)$$

$$P(X > b) = 1 - P(X \leq b) = 1 - F_X(b)$$

$$P(X \geq b) = 1 - P(X < b) = 1 - F_X(b-)$$

(left limit of  $f(x)$   
at  $b$ )

### Joint distribution of $(X, Y)$

→ Suppose, both  $X$  and  $Y$  are discrete R.V.

Joint p.m.f of  $(X, Y)$ :

$$P(x, y) = P(X=x \text{ and } Y=y), \quad \begin{array}{l} x \in S_X \\ y \in S_Y \end{array}$$

$$\sum_x \sum_y P(x,y) = 1, \quad P(x,y) \geq 0.$$

$$P(A) = \sum_{x \in A} \sum_{y \in S_Y} P(x,y)$$

$$P(X \in A | Y \in B) = \frac{P((X \in A) \cap (Y \in B))}{P(Y \in B)}$$

$$= \frac{\sum_{x \in A} \sum_{y \in B} P(x,y)}{\sum_{x \in S_X} \sum_{y \in B} P(x,y)}$$

$$E(h(X,Y)) = \sum_{x \in S_X} \sum_{y \in S_Y} h(x,y) P(x,y)$$

$$E(X) = \sum_{x \in S_X} \sum_{y \in S_Y} x P(x,y)$$

Eg: pmf of  $(X,Y)$

<del>y</del>	<del>x</del>	0	1	2
0	0.1	0	0.4	
1	0.2	0.1	0.2	

Calculate (i)  $E(X^2)$

$$(ii) P(X=2 | Y=1)$$

$$(iii) P(Y=0)$$

d)

$$(ii) P(X=2 | Y=1) = \frac{P(X=2 \text{ and } Y=1)}{P(Y=1)} = \frac{0.2}{0.5} = 0.4$$

$$(iii) P(Y=0) = 0.5$$

→ Suppose both  $X$  and  $Y$  are continuous

Joint pdf of  $(X, Y)$  :  $f(x, y)$

$$P(X \in A \text{ and } Y \in B) = \int_{x \in A} \int_{y \in B} f(x, y) dy dx.$$

$$P(X \in A | Y \in B) = \frac{P(X \in A \text{ and } Y \in B)}{P(Y \in B)}$$

$$= \frac{\int \int f(x, y) dy dx}{\int \int f(x, y) dy dx}$$
$$\underset{x \in S_X, y \in S_Y}{\underset{x \in S_X, y \in S_Y}{}}$$

$$\bullet E(h(x, y)) = \int \int x f(x, y) dy dx \quad \int \int h(x, y) f(x, y) dy dx$$
$$\underset{x \in S_X, y \in S_Y}{\underset{x \in S_X, y \in S_Y}{}}$$

$$\bullet E(X) = \int \int x f(x, y) dy dx$$
$$\underset{x \in S_X, y \in S_Y}{\underset{x \in S_X, y \in S_Y}{}}$$

Joint pdf of  $(X, Y)$  :

$$f(x, y) = \begin{cases} c(x + y^2), & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0, & \text{o/w} \end{cases}$$

Find  $c$ . Calculate  $P(X \leq 0.5 | Y \leq 0.6)$

$$- \int_{x=0}^1 \int_{y=0}^1 c(x + y^2) = 1$$

$$\Rightarrow \int_0^1 c\left(x + \frac{1}{3}\right) = 1 \Rightarrow c \left(\frac{x^2}{2} + \frac{x}{3}\right) \Big|_{x=0}^1 = 1$$

$$\Rightarrow c \cdot \left(\frac{1}{2} + \frac{1}{3}\right) = 1 \Rightarrow c = \frac{6}{5}$$

$$\begin{aligned}
 P(X \leq 0.5 | Y \leq 0.6) &= \frac{P(X \leq 0.5 \text{ and } Y \leq 0.6)}{P(Y \leq 0.6)} \\
 &= \frac{\int_{x=0}^{1/2} \int_{y=0}^{3/5} 6/5 (x+y^2) dy dx}{\int_{x=0}^1 \int_{y=0}^{3/5} 6/5 (x+y^2) dy dx} \\
 &= \frac{\frac{1}{2} \int_{x=0}^{1/2} \frac{3x}{5}}{\int_{x=0}^1 \int_{y=0}^{3/5} 6/5 (x+y^2) dy dx}
 \end{aligned}$$

- Calculate  $P(X \leq 0.5 | Y = 0.6)$
- Calculate pdf of  $X$ , given  $Y = 0.6$
- Conditional distribution of  $X$ , given  $y$

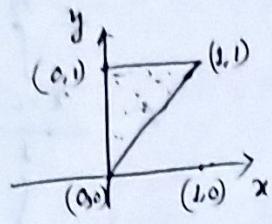
$$\frac{\frac{6}{5} (x + 0.6^2)}{\int_0^1 6/5 (t + 0.6^2) dt}$$

→ To make it a pdf we divide by this

$$\begin{aligned}
 f_{X|Y} &= (x | y^*) \\
 &= \frac{f(x, y^*)}{\int f(x, y^*) dx} \\
 &\quad x \in S_X
 \end{aligned}$$

$$f(x,y) = \begin{cases} cx^2y^2 & 0 \leq x \leq y \leq 1 \\ 0 & \text{o/w} \end{cases}$$

Find  $c$ , and  $P(X \leq 0.5)$ .



$$-\int_{x=0}^1 \int_{y=x}^1 cx^2y^2 dy dx$$

$$\Rightarrow \frac{c}{3} \int_0^1 x(1-x^3) dx = 1 \Rightarrow \frac{c}{3} \int_0^1 (x-x^4) dx = 1$$

$$\Rightarrow \frac{c}{3} \left( \frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1 = 1$$

$$\Rightarrow \frac{c}{3} \left( \frac{1}{2} - \frac{1}{5} \right) = 1 \Rightarrow c = 10$$


---

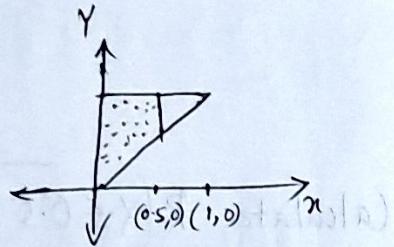
$$P(X \leq 0.5)$$

$$= \int_{x=0}^{1/2} \int_{y=x}^1 10x^2y^2 dy dx$$

$$= \frac{10}{3} \int_0^{1/2} (x-x^4) dx$$

$$= \frac{10}{3} \left( \frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^{1/2} = \frac{10}{3} \left( \frac{1}{8} - \frac{1}{32 \times 5} \right)$$

$$= \frac{10}{24} \left( 1 - \frac{1}{20} \right) = \frac{10 \times 19}{24 \times 20} = \frac{19}{48}$$



## Generating Function.

(Moment Generating Function / MGF)

I.  $n^{\text{th}}$  order raw moment is  $E(X^n)$ II.  $n^{\text{th}}$  order central moment is  $E.(X - E(X))^n$ 

MGF: Suppose  $X$  is a random variable. Then MGF of  $X$  is defined as

$$M_X(t) = E(e^{tX}) \quad t \in \mathbb{R}$$

- We say that MGF of  $X$  exist if  $M_X(t)$  is finite for all  $t \in (-a, a)$ , ( $a$  is some positive constant) for some  $a$ .

• Lemma: If the expectation does not exists in an open nbhd of 0, then we say that MGF doesn't exist.

• Lemma: It generates all the moments of  $X$

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= E\left[1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots\right] \\ &= 1 + tE(X) + \frac{t^2}{2!} E(X^2) + \dots \end{aligned}$$

↑ expansion of  $e^{tX}$ .

$$\therefore E(X^n) = \frac{d^n}{dt^n}(M_X(t))$$

• MGF uniquely determines the distribution, if exists.

• Eg:  $X \sim f(x) = \frac{1}{2} e^{-|x|}$ ,  $x \in \mathbb{R}$ . Find the MGF of  $X$ .

Soln  $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{2} e^{-|x|} dx$

$$\begin{aligned}
 &= \frac{1}{2} \int_{-\infty}^{\infty} e^{tx} c^x dx + \frac{1}{2} \int_0^{\infty} e^{tx} c^{-x} dx \\
 &= \frac{1}{2t+1} e^{(t+1)x} \Big|_{-\infty}^0 + \frac{1}{2} \frac{e^{t(x-1)}}{t} \Big|_0^{\infty} \\
 &= \frac{1}{2t+1} + \frac{1}{2t} = \frac{1}{2} \left[ \frac{e^{(t+1)x}}{(t+1)} \Big|_{-\infty}^0 + \frac{e^{(t-1)x}}{(t-1)} \Big|_0^{\infty} \right] \\
 &= \frac{1}{2} \cdot \frac{c^0}{1} + \\
 &= \frac{1}{2} \left[ \int_{-\infty}^0 e^{(t+1)x} dx + \int_0^{\infty} e^{(t-1)x} dx \right] \\
 &= \frac{1}{2} \left[ \int_0^{\infty} e^{-(t+1)x} dx + \int_0^{\infty} e^{(t-1)x} dx \right]
 \end{aligned}$$

The value of the integration will be finite if

$$-(t+1) < 0 \quad \text{and} \quad (t-1) < 0$$

$$\Rightarrow -1 < t < 1$$

Homework: Compute  $E(X)$  and  $\text{Var}(X)$  from  $M_X(t)$ .

o **Theorem:** Let  $X$  and  $Y$  be two independent random variable with MGF  $M_X(t)$  and  $M_Y(t)$ , respectively.

Define  $Z = X+Y$ . Then  $M_Z(t) = M_X(t) \cdot M_Y(t)$ .

$$\rightarrow Z = X_1 + X_2 + \dots + X_n$$

$$M_Z(t) = \prod_{i=1}^n M_{X_i}(t)$$

Some Standard distributions:

I. Bernoulli's R.V : If a random variable has two possible outcomes (say 0 and 1), then this random variable is known as Bernoulli's RV.

$$x \sim \text{Bernoulli's RV}$$

$$\text{If } P(x) = \begin{cases} (1-p) & \text{if } x=0 \\ p & \text{if } x=1 \\ 0 & \text{o/w} \end{cases}$$

$$= p^x (1-p)^{1-x}, x \in \{0, 1\}$$

$$E(X) = p$$

$$V(X) = (1-p)$$

$$M_X(t) = (1-p) + pe^t$$

$$M_X(t) = E(e^{tx})$$

$$= e^{t \cdot 0} \cdot (1-p) + e^{t \cdot 1} \cdot p$$

$$= (1-p) + pe^t.$$

Suppose we perform  $n$  independent Bernoulli trials (with outcome 0 and 1).

$\binom{n}{r}$  ← exactly  $r$  many success

Binomial Distribution:

$$Y_1, Y_2, \dots, Y_n \sim \text{Bernoulli}(p)$$

$$X = \sum_{i=1}^n Y_i, \quad \# \text{ of success in } n \text{ trials.}$$

$X \sim \text{Binomial Random variable} \quad [\text{Binomial}(n, p)]$

— proposition:  $E(X) = np$

$$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0, 1, 2, \dots, n \\ 0 & \text{o/w.} \end{cases}$$

$$X = \sum_{i=1}^n Y_i \Rightarrow E(X) = \sum_{i=1}^n E(Y_i) = np$$

$$\begin{aligned}
 &= \frac{1}{2} \int_{-\infty}^0 e^{tx} e^x dx + \frac{1}{2} \int_0^{\infty} e^{tx} e^{-x} dx \\
 &= \frac{1}{2(t+1)} e^{(t+1)x} \Big|_{-\infty}^0 + \frac{1}{2} \frac{e^{t(x-1)}}{t} \Big|_0^{\infty} \\
 &= \frac{1}{2t} + \frac{1}{2t} = \frac{1}{2} \left[ \frac{e^{(t+1)\infty}}{(t+1)} \Big|_{-\infty}^0 + \frac{e^{(t-1)0}}{(t-1)} \Big|_0^{\infty} \right] \\
 &= \frac{1}{2} \cdot \frac{e^{\infty}}{t+1} + \\
 &= \frac{1}{2} \left[ \int_{-\infty}^0 e^{(t+1)x} dx + \int_0^{\infty} e^{(t-1)x} dx \right] \\
 &= \frac{1}{2} \left[ \int_0^{\infty} e^{-(t+1)x} dx + \int_0^{\infty} e^{(t-1)x} dx \right]
 \end{aligned}$$

The value of the integration will be finite if

$$-(t+1) < 0 \quad \text{and} \quad (t-1) < 0$$

$$\Rightarrow -1 < t < 1$$

Homework: Compute  $E(X)$  and  $\text{Var}(X)$  (from  $M_X(t)$ ).

o Theorem: Let  $X$  and  $Y$  be two independent random variable with MGF  $M_X(t)$  and  $M_Y(t)$ , respectively.

Define  $Z = X+Y$ . Then  $M_Z(t) = M_X(t) \cdot M_Y(t)$ .

$$\rightarrow Z = X_1 + X_2 + \dots + X_n$$

$$M_Z(t) = \prod_{i=1}^n M_{X_i}(t)$$

• Some Standard distributions:

I. Bernoulli's R.V. : If a random variable has two possible outcomes (say 0 and 1), then this random variable is known as Bernoulli's RV.

$$X \sim \text{Bernoulli's RV}$$

$$\text{If } P(X) = \begin{cases} (1-p) & \text{if } x=0 \\ p & \text{if } x=1 \\ 0 & \text{o/w} \end{cases}$$

$$= p^x (1-p)^{1-x}, x \in \{0, 1\}$$

$$E(X) = p$$

$$M_X(t) = E(e^{tX})$$

$$V(X) = (1-p)$$

$$= e^{t \cdot 0} \cdot (1-p) + e^{t \cdot 1} \cdot p$$

$$M_X(t) = (1-p) + pe^t.$$

$$= (1-p) + pe^t.$$

— Suppose we perform  $n$  independent Bernoulli trials (with outcome 0 and 1).

$\binom{n}{r}$  ← exactly  $r$  many success

Binomial Distribution:

$$Y_1, Y_2, \dots, Y_n \in \text{Bernoulli}(p)$$

$$X = \sum_{i=1}^n Y_i, \quad \# \text{ of success in } n \text{ trials.}$$

$X \sim \text{Binomial Random variable} \quad [\text{Binomial}(n, p)]$

— proposition:  $E(X) = np$

$$X = \sum Y_i \Rightarrow E(X) = \sum_{i=1}^n E(Y_i) = np$$

$$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0, 1, 2, \dots, n \\ 0 & \text{o/w.} \end{cases}$$

→ PMF of Bin. dist:  $(p+q)^n$

$$V(X) = \sum_{i=1}^n V(Y_i) = np(1-p)$$

$$M_X(t) = E(e^{tx})$$

$$= \sum_{i=0}^n e^{ti} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= (pe^t + (1-p))^n$$

$$M_X(t) = \prod_{i=1}^n M_{Y_i}(t)$$

$$= (1-p + pe^t)^n$$

$$\sum_{i=0}^n n_{ci} (e^t \cdot p)^i (1-p)^{n-i}$$

$p \rightarrow e^t p, q \rightarrow 1 - p$

Example: A and B play a game in which A's choice of winning is  $\frac{2}{3}$ . In a series of 8 games what is the probability that A will win at least 6 games.

Just like PMF of binomial dist:

— let  $X \sim \#$  of times A wins.

Then the required probability =  $P(X \geq 6)$

$$= P(X=6) + P(X=7) + P(X=8)$$

$$(e^t p + q)^n$$

where  $q = (1-p)$ .

Example: Two dice are tossed together a number of times in succession. Find the minimum number of throws to ensure that the probability of obtaining 6 on both dice at least one is greater than 0.5.

ans = 25

— let  $X = \#$  times  $(6, 6)$

$$P(X < 1) = P(X=0)$$

$$= \binom{n}{0} \left(\frac{1}{36}\right)^0 \left(\frac{35}{36}\right)^n < 0.5$$

• Poisson Distribution:  $X \sim \text{Poisson}(\lambda)$

$$P(X=x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \quad x=0, 1, 2, \dots$$

$$E(X) = \lambda$$

$$V(X) = \lambda$$

$$M_X(t) = e^{\lambda(e^t - 1)}$$

• Theorem:  $X \sim \text{Binomial}(n, p)$ . Suppose  $n \rightarrow \infty$ ,  $p \rightarrow 0$ , and  $np = \lambda$ .

$$\text{Then } \lim_{n \rightarrow \infty} P(X=x) \rightarrow e^{-\lambda} \frac{\lambda^x}{x!}$$

$$\lim_{n \rightarrow \infty} P(X=x) = \lim_{n \rightarrow \infty} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \lim_{n \rightarrow \infty} \frac{n(n-1)\cdots(n-x+1)}{x!} p^x (1-p)^{n-x}$$

$$= \lim_{n \rightarrow \infty} \frac{n(n-1)\cdots(n-x+1)}{x!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n \rightarrow \infty} \frac{n(n-1)\cdots(n-x+1)}{x!} \cdot \left(\frac{\lambda/n}{1-\lambda/n}\right)^x \cdot \underbrace{\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n}_{e^{(\lambda/n)(1-\lambda/n)}} \cdot \underbrace{\lim_{n \rightarrow \infty} (1-\lambda/n)^{-1}}_{e^{\lambda}}$$

$$= \frac{\lambda^x \cdot e^{-\lambda}}{x!} \lim_{n \rightarrow \infty} \frac{n(n-1)\cdots(n-x+1)}{(n-\lambda)^x} \rightarrow 1 \text{ as } n \rightarrow \infty$$

$$= e^{-\lambda} \cdot \frac{\lambda^x}{x!}$$

Eg An individual suffers an adverse reaction from a particular drug is known to be 0.001. Determine the probability that out of 2000 individuals

i) Exactly three will suffer

$$X \sim \text{Bin}(2000, 0.001)$$

ii) More than two will suffer.

$$P(X=3) = e^{-\lambda} \cdot \frac{\lambda^3}{3!} \quad \lambda = 2000 \times 0.001.$$

## Continuous distribution

i) Normal / Gaussian Distribution

A r.v  $X \sim$  Gaussian dist

$$f(x) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} ; \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma^2 > 0$$

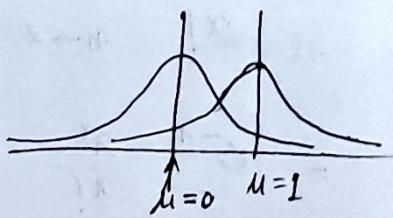
$$\text{If } \mu=0, \sigma^2=1$$

↳ standard normal.

$$E(X) = \mu$$

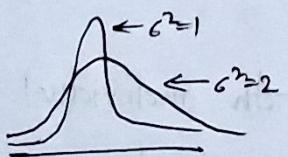
$$V(X) = \sigma^2$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$



Properties:  $Z \sim N(0, 1)$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, z \in \mathbb{R}$$



CDF of  $N(0,1) := \Phi$

$$\text{I. } \lim_{x \rightarrow \infty} \Phi(x) = 1$$

$$\Phi(z) = P(Z \leq z)$$

$$\text{II. } \lim_{x \rightarrow -\infty} \Phi(x) = 0$$

Proof of IV.

$$1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \quad \text{III. } \lim_{x \rightarrow 0} \Phi(0) = \frac{1}{2}$$

$$\text{IV. } \Phi(-x) = 1 - \Phi(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt, \text{ Let } y = -x \rightarrow y^2 = x^2, \\ dy = -dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-x}^{\infty} e^{-y^2/2} (-dy) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^x e^{-y^2/2} dy = \Phi(-x)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\checkmark \mu=0, \sigma^2=1$$

| LEC-07 |

$N(\mu, \sigma^2) \rightarrow \text{Normal dist.}$

13/08/24

$$\text{pdf} := \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < \infty$$

$\curvearrowleft N(0,1)$

$$\text{cdf} \quad \Phi(x) = \int_{-\infty}^x \phi(t) dt$$

$$\text{pdf of } N(\mu, \sigma^2), \quad f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

Property:

$$1. \text{ If } x \sim N(\mu, \sigma^2), \text{ then } \frac{x-\mu}{\sigma} \sim N(0,1)$$

$$\text{soln} \quad P\left(\frac{x-\mu}{\sigma} \leq t\right) = P(X \leq (\mu + \sigma t))$$

$$= \int_{-\infty}^{\mu+\sigma t} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(\omega-\mu)^2} d\omega$$

$$= \int_{-\infty}^t \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2}} \cdot \sigma dy$$

$$\text{Take } y = \frac{\omega-\mu}{\sigma}$$

$$= \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = \Phi(t).$$

This is true for any  $t \in (-\infty, \infty)$

$\therefore$  The cdf of  $\left(\frac{X-\mu}{\sigma}\right)$  is same as  $\Phi$ .

$$\therefore \frac{X-\mu}{\sigma} \sim N(0,1).$$

### • MGF of $N(0,1)$

$$M_Z(t) = E(e^{tZ}) \quad \text{where } Z \sim N(0,1)$$

$$= \dots = e^{\frac{t^2}{2}}$$

$$\text{proof: } E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tw} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(w-t)^2}{2}} \cdot e^{\frac{t^2}{2}} dw$$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= e^{\frac{t^2}{2}}$$

$y = w - t$   
 $dy = dw$

from  $-\infty \rightarrow \infty$  the area under the curve = 1.

### • Find the mgf of $N(\mu, \sigma^2)$ .

$$M_X(t) = E(e^{tx}) \quad \text{where } X \sim N(\mu, \sigma^2)$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Property 2. If  $X \sim N(\mu, \sigma^2)$ , then  $(aX + b) \sim N(a\mu + b, a^2\sigma^2)$

proof: Use mgf. [If  $M_X(t) = M_Y(t)$  for all  $t$  in a small open neighbourhood containing 0, then  $F_X = F_Y$ ]

$$\begin{aligned} M_{aX+b}(t) &= E(e^{taX+tb}) \\ &= e^{tb} \cdot e^{\mu ta + \frac{1}{2}\sigma^2 t^2 a^2} \\ &= e^{(a\mu+b)t + \frac{1}{2}\sigma^2 t^2 a^2} = \text{MGF of } N(a\mu+b, a^2\sigma^2) \end{aligned}$$

Property 3: If  $X \sim N(\mu_1, \sigma_1^2)$ ,  $Y \sim N(\mu_2, \sigma_2^2)$ ,  $X$  and  $Y$  are independent, then  $(a_1 X + a_2 Y) \sim N(a_1 \mu_1 + a_2 \mu_2, a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2)$

proof: using mgf.

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = \Phi(t).$$

This is true for any  $t \in (-\infty, \infty)$

$\therefore$  The cdf of  $\left(\frac{X-\mu}{\sigma}\right)$  is same as  $\Phi$ .

$$\therefore \frac{X-\mu}{\sigma} \sim N(0,1).$$

• MGF of  $N(0,1)$

$$M_Z(t) = E(e^{tZ}) \quad \text{where } Z \sim N(0,1)$$

$$= \dots = e^{\frac{t^2}{2}}$$

$$\text{proof: } E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tw} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(w-t)^2}{2}} \cdot e^{\frac{t^2}{2}} dw$$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \quad \begin{aligned} y &= w-t \\ dy &= dw \end{aligned}$$

$$= e^{\frac{t^2}{2}}$$

• Find the mgf of  $N(\mu, \sigma^2)$ .

$$M_X(t) = E(e^{tx}) \quad \text{where } X \sim N(\mu, \sigma^2)$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

ptr1 word file

Property 2. If  $X \sim N(\mu, \sigma^2)$ , then  $(aX + b) \sim N(a\mu + b, a^2\sigma^2)$

proof: Use mgf. [If  $M_X(t) = M_Y(t)$  for all  $t$  in a small open neighbourhood containing 0, then  $F_X = F_Y$ ]

$$\begin{aligned} M_{aX+b}(t) &= E(e^{taX+tb}) \\ &= e^{tb} \cdot e^{\mu ta + \frac{1}{2} \sigma^2 t^2 a^2} \\ &= e^{(a\mu + b)t + \frac{1}{2} \sigma^2 t^2 a^2} \end{aligned}$$

= MGF of  $N(a\mu + b, a^2\sigma^2)$

Property 3: If  $X \sim N(\mu_1, \sigma_1^2)$ ,  $Y \sim N(\mu_2, \sigma_2^2)$ ,  $X$  and  $Y$  are independent, then  $(a_1 X + a_2 Y) \sim N(a_1 \mu_1 + a_2 \mu_2, a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2)$

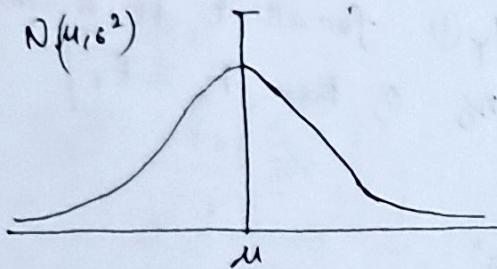
proof: using mgf.

$$X \sim N(\mu, \sigma^2)$$

$$\mathbb{E}(X) = \mu$$

$$\mathbb{E}(X^2) = \mu^2 + \sigma^2$$

$$\text{Var}(X) = (\mu^2 + \sigma^2) - \mu^2 \\ = \sigma^2$$



$\mu$  is mean, median, mode

1. Suppose  $X \sim N(5, \sigma^2)$ . Calculate  $P(X \leq 7)$ .

$$\cdot P(X \leq 7) = P\left(\frac{X-5}{\sigma} \leq \frac{7-5}{\sigma}\right)$$

cdf of  $N(0,1)$   
 $\Phi$

$$= P\left(Z \leq \frac{1}{\sigma}\right)$$

$$Z = \frac{X-5}{\sigma} \sim N(0,1)$$

$$= \Phi(0.25)$$

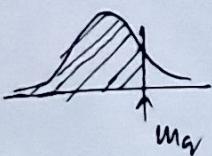
$$\cdot P(X \geq 7) = 1 - \Phi(0.25)$$

$$\cdot P(3 \leq X \leq 6) = \Phi\left(\frac{6-5}{\sigma}\right) - \Phi\left(\frac{3-5}{\sigma}\right)$$

very useful.  
(by property of  
CDF).

Find  $q$ -quantile of  $N(0,1)$ .

$$\Phi(m_q) = q, \quad m_q = \Phi^{-1}(q).$$



Find  $q$ -quantile of  $N(\mu, \sigma^2)$ .

$$P(X \leq m_q) = q \Rightarrow P\left(Z \leq \frac{m_q - \mu}{\sigma}\right) = q$$

$$\Rightarrow \Phi\left(\frac{m_q - \mu}{\sigma}\right) = q$$

$$\Rightarrow m_q = \sigma \Phi^{-1}(q) + \mu$$

### • Linear Combination Random Variables.

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$E(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = \sum_{i=1}^n a_i E(x_i)$$

$$V(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = \sum_{i=1}^n a_i^2 V(x_i) \quad \text{if } x_i \text{'s are independent}$$

• Example: Suppose, the pmf of  $X$  is as follows.

$$P(X=-1) = \frac{1}{3}, \quad P(X=0) = \frac{1}{3}, \quad P(X=1) = \frac{1}{3}$$

Suppose  $x_1, x_2, x_3$  are independent and each has the same distribution of  $X$ . Find the distribution of  $(x_1 + x_2 + x_3)$

Soln: Take  $T = x_1 + x_2 + x_3$

Let's try to find the pdf of  $T$ .

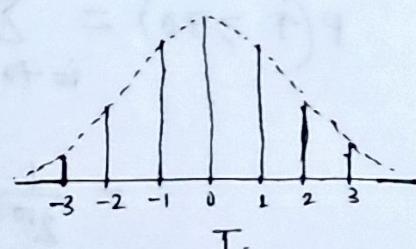
$$P(T=-3) = P(x_1 = -1, x_2 = -1, x_3 = -1) = \left(\frac{1}{3}\right)^3$$

$$P(T=-2) = 3 \cdot \left(\frac{1}{3}\right)^3 \quad (\text{since there are 3 different ways})$$

$$P(T=-1) = 6 \cdot \left(\frac{1}{3}\right)^3 \quad P(T=2) = 3 \cdot \left(\frac{1}{3}\right)^3$$

$$P(T=0) = 7 \cdot \left(\frac{1}{3}\right)^3 \quad P(T=3) = \left(\frac{1}{3}\right)^3$$

$$P(T=1) = 6 \cdot \left(\frac{1}{3}\right)^3$$

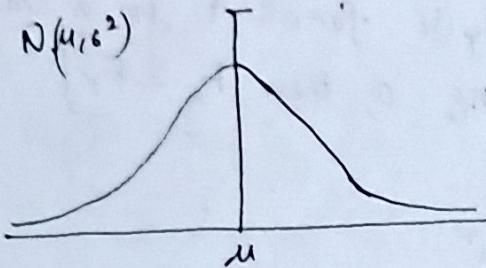


$$x \sim N(\mu, \sigma^2)$$

$$E(x) = \mu$$

$$E(x^2) = \mu^2 + \sigma^2$$

$$\text{Var}(x) = (\mu^2 + \sigma^2) - \mu^2 \\ = \sigma^2$$



$\mu$  is mean, median, mode

1. Suppose  $x \sim N(5, \sigma^2)$ . Calculate  $P(x \leq 7)$ .

$$\cdot P(x \leq 7) = P\left(\frac{x-5}{\sigma} \leq \frac{7-5}{\sigma}\right)$$

$$= P\left(Z \leq \frac{1}{\sigma}\right)$$

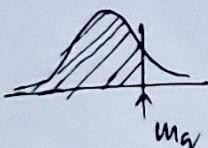
$$= \Phi(0.25)$$

| cdf of  $N(0,1)$  |  
|  $\Phi$  |

$$\cdot P(x > 7) = 1 - \Phi(0.25)$$

$$\cdot P(3 \leq x \leq 6) = \Phi\left(\frac{6-5}{\sigma}\right) - \Phi\left(\frac{3-5}{\sigma}\right)$$

Find  $q$ -quantile of  $N(0,1)$ .



$$\Phi(m_q) = q, \quad m_q = \Phi^{-1}(q).$$

Need  
Review

Find  $q$ -quantile of  $N(\mu, \sigma^2)$ .

$$P(x \leq m_q) = q \Rightarrow P\left(Z \leq \frac{m_q - \mu}{\sigma}\right) = q$$

$$\Rightarrow \Phi\left(\frac{m_q - \mu}{\sigma}\right) = q$$

$$\Rightarrow m_q = \sigma \Phi^{-1}(q) + \mu$$

### • Linear Combination Random Variables.

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$E(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = \sum_{i=1}^n a_i E(x_i)$$

$$V(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = \sum_{i=1}^n a_i^2 V(x_i) \quad \text{if } x_i \text{'s are independent}$$

• Example: Suppose, the pmf of  $X$  is as follows.

$$P(X=-1) = \frac{1}{3}, \quad P(X=0) = \frac{1}{3}, \quad P(X=1) = \frac{1}{3}$$

Suppose  $x_1, x_2, x_3$  are independent and each has the same distribution of  $X$ . Find the distribution of  $(x_1 + x_2 + x_3)$

Soln: Take  $T = x_1 + x_2 + x_3$

Let's try to find the pdf of  $T$ .

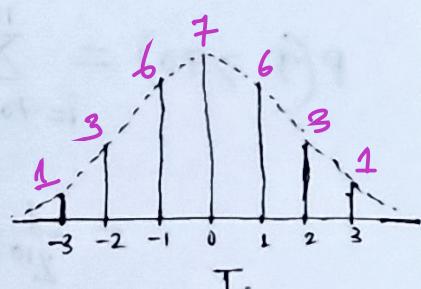
$$P(T=-3) = P(x_1 = -1, x_2 = -1, x_3 = -1) = \left(\frac{1}{3}\right)^3$$

$$P(T=-2) = 3 \cdot \left(\frac{1}{3}\right)^3 \quad (\text{since there are 3 different ways})$$

$$P(T=-1) = 6 \cdot \left(\frac{1}{3}\right)^3 \quad P(T=2) = 3 \cdot \left(\frac{1}{3}\right)^3$$

$$P(T=0) = 7 \cdot \left(\frac{1}{3}\right)^3 \quad P(T=3) = \left(\frac{1}{3}\right)^3$$

$$P(T=1) = 6 \cdot \left(\frac{1}{3}\right)^3$$



NOTE If  $x_1, x_2, \dots, x_n$  is a random sample from a distribution, with  $\xrightarrow{\text{(}} \text{ } (x_i \text{ 's are iid)}$

mean  $\mu$  and variance  $\sigma^2$  (finite);

if  $n$  is large. (finite)

then the approximate distribution of  $T = \sum_{i=1}^n x_i$   $\left| \begin{array}{l} E(x_i) = \mu \\ V(x_i) = \sigma^2 \end{array} \right.$   
is  $N(\mu n, n\sigma^2)$ .

So the approx distribution of  $\bar{x} = \frac{1}{n} \sum x_i$  is  $N\left(\mu, \frac{\sigma^2}{n}\right)$

This is called Central Limit Theorem (CLT).

Ordinary ( $n > 30$ ) is considered large enough.

But if the original dist is Normal, then  $n$  is much lesser than 30.

$$\text{Ex} \quad x_i = \begin{cases} 0 & \text{prob } (1-p) \\ 1 & \text{prob } p \end{cases}$$

$x_i$ 's are independent.  $T = \sum_{i=1}^n x_i$

$T = \sum_{i=1}^n x_i \sim \text{Binomial}(n, p)$ . (exact distribution)

Prob: Suppose you toss a fair coin 100 times independently.  
What is the probability that you get at least 70 head counts?

Soln: Suppose  $p = P(\text{Head}) = 0.5$ .

$$P(T \geq 70) = \sum_{i=70}^{100} \binom{100}{i} (0.5)^i (0.5)^{100-i}$$

$$= \frac{1}{2^{100}} \sum_{i=70}^{100} \binom{100}{i}$$

Let's apply CLT to approximate the same probability.

By CLT  $T \sim N(np, np(1-p))$

i.e.,  $N\left(100 \times \frac{1}{2}, 100 \times \frac{1}{2} \times \frac{1}{2}\right)$ , i.e.  $N(50, 25) = N(50, 5^2)$

$$\begin{aligned}
 & P(T > 70) \\
 &= 1 - P(T \leq 69.5) \\
 &= 1 - \Phi\left(\frac{69.5 - 50}{5}\right) \\
 &= 1 - \Phi(3.9...) \\
 &\approx 0.
 \end{aligned}
 \quad \text{Handwritten notes: } \boxed{\begin{aligned} P(T \leq a) &\approx P(T \leq a + 0.5) \\ P(T > a) &\approx P(T > a - 0.5) \end{aligned}}$$

$$\begin{aligned}
 & 1 - P\left(\frac{T - \mu}{\sigma} \leq \frac{69.5 - 50}{5}\right) \\
 &= 1 - P\left(Z \leq \frac{69.5 - 50}{5}\right) \\
 &\stackrel{d}{=} 1 - \Phi\left(z \leq \frac{69.5 - 50}{5}\right)
 \end{aligned}$$

### Example 3.5.6

A soft-drink vending machine is set so that the amount of drink dispensed is a random variable with a mean of 8 ounces and a standard deviation of 0.4 ounces. What is the approximate probability that the average of 36 randomly chosen fills exceed 8.1 ounces?

#### Solution

From the CLT,  $((\bar{X} - 8)/(0.4/\sqrt{36})) \sim N(0, 1)$ . Hence, from the normal table,

$$\begin{aligned}
 P\{\bar{X} > 8.1\} &= P\left\{Z > \frac{8.1 - 8.0}{\frac{0.4}{\sqrt{36}}}\right\} \\
 &= p\{Z > 1.5\} = 0.0668.
 \end{aligned}$$