Thm 5 (Caratheodory Thm).

Let $P_1, P_2, \dots P_n$ be pts from $\mathbb{R}^d$. Let $P \in Conv(\{P_1, \dots P_n\})$ then $\exists$ $(d+1)$ pts from P_1 to P_n whose convex hull also contains P.

Thm 5':- Let $P_1, P_2, \dots P_n$ be pts from $\mathbb{R}^d$. Let $P \in Conv(\{P_1, P_2, \dots P_n\})$. Then $\exists$ a affine independent subset of $\{P_1, P_2, \dots P_n\}$ whose convex hull also contains P.

Proof (Thm 5'):-

Let S be the smallest in terms of size, subset of $\{P_1, P_2, \dots P_n\}$ s.t $P \in Conv(S)$. We will show that S is affinely independent. ~~This follows from the fact that $\forall$ finite subsets of~~ if Q is ~~finite subset of~~ $\mathbb{R}^d$ ~~and Q' is the largest affinely ind subset of~~

Suppose S is not affinely independent. Then $\exists \lambda_i$ not all zero s.t

$$\sum_{P_i \in S} \lambda_i P_i = 0 \text{ and } \sum_{P_i \in S} \lambda_i = 0.$$

$$\exists\ \mu_i \geq 0 \text{ s.t } \sum_{P_i \in S} \mu_i P_i = P \text{ and } \sum_{P_i \in S} \mu_i = 1.$$

Assume $\epsilon > 0$ and very very small s.t $\mu_i(\epsilon) = \mu_i - \epsilon \lambda_i \geq 0$. $\forall i$

$$\sum_{\forall P_i \in S} \mu_i(\epsilon) P_i = P. \text{ and } \sum_{\forall P_i \in S} \mu_i(\epsilon) = 1.$$

Now as one of $M_i(\epsilon)$ becomes '0' I can throw out that $M_i P_i$.
so I can write P as a convex hull of subset S. This contradicts our assumption so it can't be true.

Linear Program (LP) & Integer LP.

LP (General form)

<u>Inputs</u>:- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$.

LP(1)

$$\begin{cases} \max \langle c, x \rangle \text{ s.t } Ax \leq b. \\ x \leq y \Rightarrow x_i \leq y_i \ \forall i. \end{cases}$$

Things we are interested about (LP1):

(1) <u>Optimal value</u>:- Opt LP1.

(2) <u>Optimal point</u> :- P ~~opt~~ (L1).

(3) <u>Feasibility of (LP1)</u>:-

$$Ax \leq b \iff \underbrace{\langle a_i, x \rangle \leq b_i}_{\text{halfspaces}} \ \forall i \in [m].$$

Bounded space in this case is polyhedron.

Linear program in "standard form" | "Equational form"

$$\max \ \langle c, x \rangle \text{ s.t } Ax = b, x \geq 0$$

Lemma 8: Every LP can be converted into an equivalent equational form LP.

Proof: LP2:

$$\max \langle c, x\rangle$$
$$\text{s.t } Ax \le b.$$

Equivalent (LP3).

$$\max \langle c, x\rangle.$$
$$\text{s.t } Ax + y = b.$$
$$y \geq 0.$$

$x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.

$x_i = \mu_i - \mu_i'$, $\mu \geq 0$, $\mu' \geq 0$.

n [ ]

n [ ]

m [ ]

Equivalent (LP4)

$\cancel{A(\mu-\mu')}$

$$\max \langle c, \mu-\mu'\rangle$$
$$\text{s.t } A(\mu-\mu') + y = b.$$
$$\mu \geq 0, \mu' \geq 0, y \geq 0.$$

$$Z = (\mu, \mu', y) \in \mathbb{R}^{2n+m}$$ ⇒ columnwise.

$$\max \langle \tilde{C}, Z\rangle \text{ subject to } \hat{A} Z = \tilde{b},\ Z \geq 0.$$

$$\hat{A} = \left[\, A \mid -A \mid I_m \,\right].$$

$[A \ -A$

$= A\mu - A\mu$

$= A(\mu -$

$$\tilde{C} = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} \in \mathbb{R}^{2n+m}.$$

$$\tilde{b} = b.$$

Lemma 8:- Every LP can be converted into an equivalent equational form LP.

Proof:- LP2:

$$\max \langle c, x \rangle$$
$$\text{s.t} \quad Ax \le b.$$

Equivalent (LP3).

$$\max \quad \langle c, x \rangle$$
$$\text{s.t} \quad Ax + \cancel{B}\, y = b.$$
$$y \ge 0.$$

$x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.

$x_i = \mu_i - \mu_i'$, $\mu \ge 0$, $\mu' \ge 0$.

$n\,[\;]\,\cancel{[\;]}$
$n\,[\;]$
$m\,[\;]$

Equivalent (LP4)

$\boxed{\cancel{A(\mu - \mu')}}$

$$\max \langle c, \mu - \mu' \rangle$$
$$\text{s.t} \quad A(\mu - \mu') + y = b.$$
$$\mu \ge 0, \ \mu' \ge 0, \ y \ge 0.$$

$z = (\mu, \mu', y) \in \mathbb{R}^{2n+m}$ $\Rightarrow$ columnwise.

$\max \langle \hat{c}, z \rangle$ subject to $\hat{A} z = \hat{b}$, $z \ge 0$.

$$\hat{A} = \left[\begin{array}{c|c|c} A & -A & I_m \end{array} \right].$$

$$\hat{c} = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} \in \mathbb{R}^{2n+m}.$$

$$\hat{b} = b.$$

$$\begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} \mu \\ \mu' \\ y \end{bmatrix}$$
$$= A\mu - A\mu' + y.$$
$$= A(\mu - \mu') + y.$$

Duality of LP

Primal	Dual
$\max \langle c, x \rangle$	$\min \langle b, y \rangle$
s.t $Ax = b$	s.t $A^T y \leq c$
$x \geq 0$.	$y \in \mathbb{R}^m$.
$x \in \mathbb{R}^n$	(DP5)
(LP5)	

Lemma 9 (Weak duality):- Let x_0 be a feasible solⁿ to (LP5) and y_0 be a feasible solⁿ to (DP5). Then,

$$\langle c, x_0 \rangle \leq \langle b, y_0 \rangle$$

$$\Rightarrow OPT_{LP5} \leq OPT_{DP5}.$$

Prove it.

Strong Duality.- LP satisfies strong duality.

Ex:- Derive the dual of the following LP.

1) $\max \langle c, x \rangle$. s.t $Ax \leq b$, $x \geq 0$.