Endsem

AQ 1) $\mathcal{L} = \{l_1, \dots, l_n\} \longrightarrow n$ lines in $\mathbb{R}^2$

$\mathcal{P} = \{P_1, \dots, P_m\} \longrightarrow m$ points in $\mathbb{R}^2$

$I(\mathcal{P}, \mathcal{L}) := \{(p, l) : p \in \mathcal{P},\ l \in \mathcal{L},\ p \in l\}$

a) To show: $|I(\mathcal{P}, \mathcal{L})| = O((mn)^{2/3} + m + n)$

pf: WLOG assume that every line contains atleast one point and every point lies in some line (i.e, no isolated vertices or lines)

$\because$ $\boxed{\text{For } m' < m \text{ and } n' < n, \text{ we've: } O((m'n')^{2/3} + m' + n') = O((mn)^{2/3} + m + n)}$

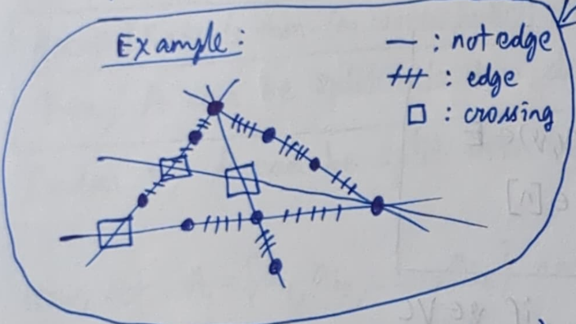
Let m_j = no. of points on l_j

Then, $|I(\mathcal{P}, \mathcal{L})| = \sum_{j=1}^{n} m_j$

Now, define a graph $G(V, E)$:

$V = \mathcal{P}$

$(u, v) \in E \iff u, v$ correspond to consecutive points along a line of $\mathcal{L}$



A line l_j contributes $(m_j - 1)$ to $|E|$ $\Rightarrow |E| = \sum_{j=1}^{n} (m_j - 1) = |I(\mathcal{P}, \mathcal{L})| - n$

Also, $|V| = m$

$\boxed{\text{Recall (Crossing Lemma): } G(V, E),\ |E| \geq 4|V| \Rightarrow Cr(G) = \Omega\left(\frac{|E|^3}{|V|^2}\right)}$

case I $(|E| < 4|V|)$: $|E| = |I(\mathcal{P}, \mathcal{L})| - n < 4|V| = 4m \Rightarrow |I(\mathcal{P}, \mathcal{L})| < 4m + n < 4m + 4n$
$= O(m + n)$
$= O((mn)^{2/3} + m + n)$ ✓

case II $(|E| \geq 4|V|)$:

Crossing Lemma $\Rightarrow Cr(G) = \Omega\left(\frac{(|I(\mathcal{P}, \mathcal{L})| - n)^3}{m^2}\right)$

Also, $Cr(G) \leq \binom{n}{2} = O(n^2)$

$\because$ need 2 lines for a crossing

So, we've: $\frac{(|I(\mathcal{P}, \mathcal{L})| - n)^3}{m^2} \leq k \cdot n^2$, for some constant k

$\Rightarrow |I(\mathcal{P}, \mathcal{L})| - n \leq k'(mn)^{2/3} + n < k'[(mn)^{2/3} + n] = O((mn)^{2/3} + n) = O((mn)^{2/3} + m + n)$ ✓

□

b) <u>To show</u>: $\forall n \in \mathbb{N}$, $\exists \mathcal{P}', \mathcal{L}'$ with $|\mathcal{P}'| = |\mathcal{L}'| = n$ s.t $|I(\mathcal{P}', \mathcal{L}')| = \Omega(n^{4/3})$

Pf: Let $n = 4k^3$, for some $k \in \mathbb{N}$ and let

$$\mathcal{P}' = \{(x,y) : x \in [k], y \in [4k^2]\}$$

$$\mathcal{L}' = \{y = ax+b : a \in [2k], b \in [2k^2]\}$$

NOTE
$4k^3 = k \times 4k^2 \rightarrow \mathcal{P}'$
$= 2k \times 2k^2 \rightarrow \mathcal{L}'$

Here, $|\mathcal{P}'| = |\mathcal{L}'| = 4k^3 = n$

Now, note that $\forall x \in [k]$, $ax+b \leq ak+b \leq 2k \cdot k + 2k^2 = 4k^2$

(i.e, every line of $\mathcal{L}'$ contains a point $(x,y) \in \mathcal{P}'$, $\forall x \in [k]$)

Thus, $|I(\mathcal{P}', \mathcal{L}')| \geq 4k^3 \cdot k = n \cdot \left(\frac{n}{4}\right)^{1/3} = c \cdot n^{4/3}$

$$\Rightarrow |I(\mathcal{P}', \mathcal{L}')| = \Omega(n^{4/3})$$

AQ4) a) For $v \in [n]$, let $x_v := \begin{cases} 1, & \text{if } v \in IS \\ 0, & \text{o.w} \end{cases}$

$$\boxed{\begin{aligned} &\max. \sum_{i=1}^{n} x_i \\ &\text{subject to } x_u + x_v \leq 1, \quad \forall (u,v) \in E \\ &\qquad x_v \in \{0,1\}, \quad \forall v \in [n] \end{aligned}}$$

b) For $v \in [n]$, let $x_v := \begin{cases} 1, & \text{if } v \in VC \\ 0, & \text{o.w} \end{cases}$

$$\boxed{\begin{aligned} &\min. \sum_{i=1}^{n} x_i \\ &\text{subject to } x_u + x_v \geq 1, \quad \forall (u,v) \in E \\ &\qquad x_v \in \{0,1\}, \quad \forall v \in [n] \end{aligned}}$$

AQ2> [Helly thm. (on convex sets)]

Let $C_1, \ldots, C_n$ be n convex sets in $\mathbb{R}^d$ with $n \geq d+1$. If any $(d+1)$ of the above sets have a non-empty intersection, then all n of the sets have a non-empty intersection

pf (induction on n): Base case ($n = d+1$): trivial ✓

IH: Suppose true for n ($n \geq d+2$)

To show: true for $n+1$ ($n+1 \geq d+2$)

$C_1, \ldots, C_{n+1} \rightarrow$ convex sets in $\mathbb{R}^d$ ($n \geq d+1$)

s.t any $(d+1)$ of the C_i's have a non-empty intersection

IH $\Rightarrow \bigcap_{\substack{i=1 \\ i \neq j}}^{n+1} C_i \neq \phi, \ \forall j \in [n+1]$, so let $a_j \in \bigcap_{\substack{i=1 \\ i \neq j}}^{n+1} C_i, \ \forall j \in [n+1]$ —— (*)

($a_j \in \mathbb{R}^d$)

Now, $A = \{a_1, a_2, \ldots, a_{n+1}\}$ is a set of atleast $(d+2)$ points in $\mathbb{R}^d$

Recall [Radon's thm. (on convex hulls)]: Let A be a set of atleast $(d+2)$ points in $\mathbb{R}^d$. Then, A can be split into two disjoint sets $A_1 \cup A_2$ s.t $\text{Conv}(A_1) \cap \text{Conv}(A_2) \neq \phi$

Radon $\Rightarrow$ A can be split into two disjoint sets $A_1 \cup A_2$ s.t $\text{Conv}(A_1) \cap \text{Conv}(A_2) \neq \phi$

(say, $x \in \text{Conv}(A_1) \cap \text{Conv}(A_2)$)

Now, let $A_1 = \{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$ and $A_2 = \{a_{j_1}, a_{j_2}, \ldots, a_{j_l}\}$,

where $\{i_1, \ldots, i_t, j_1, \ldots, j_l\} = [n+1]$

Fix $\alpha \in [t]$. Then, $i_\alpha \notin \{j_1, \ldots, j_l\}$

$\Rightarrow a_{j_r} \in C_{i_\alpha}, \ \forall r \in [l]$ ⟶ using (*)

$\Rightarrow \text{Conv}(A_2) \subseteq C_{i_\alpha}$

$\Rightarrow x \in C_{i_\alpha}, \ \forall \alpha \in [t]$

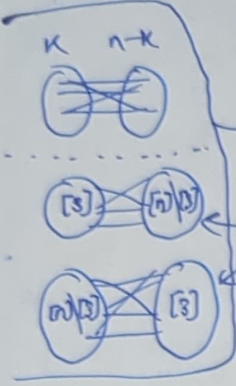
Similarly, $x \in C_{j_\beta}, \ \forall \beta \in [l]$

$\therefore \ x \in \bigcap_{i=1}^{n+1} C_i$

□

AQ5) $G(n)$ = no. of graphs with vertex set $[n]$ $= 2^{\binom{n}{2}}$

$B(n)$ = no. of bipartite graphs with vertex set $[n]$



$$\leq \sum_{K=1}^{n-1} \binom{n}{K} 2^{K(n-K)}$$

over counting

Thus, $$\frac{B(n)}{G(n)} \leq \frac{2^{\frac{n^2}{4}} \sum_{K=1}^{n-1} \binom{n}{K}}{2^{n(n-1)/2}}$$

$$< \frac{2^{\frac{n^2}{4}} \cdot 2^{n}}{2^{\frac{n^2}{2}} \cdot 2^{-n/2}} = \frac{1}{2^{\frac{n^2}{4} - \frac{3n}{2}}}$$

$$\binom{n}{1} + \dots + \binom{n}{n-1} < \binom{n}{0} + \dots + \binom{n}{n} = 2^n$$

i.e, $$0 \leq \frac{B(n)}{G(n)} < \frac{1}{2^{\frac{n^2}{4} - \frac{3n}{2}}} \longrightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow \lim_{n \to \infty} \frac{B(n)}{G(n)} = 0$$

Squeeze thm.

□

BQ1) $G(V, E) \longrightarrow \Delta$-free planar graph, $|V| = n$

To show : G is 4-colorable

pf (induction on n): Base Case ($n \le 4$) ~~($n=1,2,3$)~~ : trivial ✓

IH : Suppose true for $(n-1)$

Now, consider a Δ-free planar graph $G(V, E)$ with $|V| = n$.

Claim : $\deg(v) \le 3$, for some $v \in V$

Pf : Suppose not, i.e, let $\deg(v) \ge 4$, $\forall v \in V$

Then, $\sum_{v \in V} \deg(v) = 2|E| \xrightarrow{\because}$ Handshake

$\Rightarrow 2|E| \ge 4|V| \Rightarrow |E| \ge 2|V|$ — I

Now, Euler $\Rightarrow |V| - |E| + |F| = 2$

$\Rightarrow 2 = |V| - |E| + |F| \le |V| - |E| + \frac{|E|}{2} = |V| - \frac{|E|}{2}$ $\because$

$\Rightarrow |E| \le 2|V| - 4$ — II

I, II $\Rightarrow$ contradiction

G is Δ-free

$|E| \ge \frac{4|F|}{2}$

$\Rightarrow |F| \le \frac{|E|}{2}$

So, let $v \in V$ s.t $\deg(v) \le 3$

Then, IH $\Rightarrow G' = G \setminus \{v\}$ is 4-colorable

Now, add v to G'

Clearly, we can color v as $\deg(v) \le 3$ and we've 4 colors

□

BQ3> [PIE] $\quad A_1, A_2, \dots, A_n \longrightarrow n$ events

To show: $\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{1\le i\le n} \mathbb{P}[A_i] - \sum_{1\le i<j\le n} \mathbb{P}[A_i\cap A_j] + \sum_{1\le i<j<k\le n} \mathbb{P}[A_i\cap A_j\cap A_k] - \dots + (-1)^{n+1}\mathbb{P}\left[\bigcap_{i=1}^{n} A_i\right]$

pf: Let $A = \bigcup_{i=1}^{n} A_i$ and for any event B, let $\mathbb{1}_B := \begin{cases} 1, & \text{if } B \text{ occurs} \\ 0, & \text{o.w} \end{cases}$

Claim: $1 - \mathbb{1}_A = \prod_{i=1}^{n} (1 - \mathbb{1}_{A_i})$

pf: $\text{LHS} = 1 \iff \mathbb{1}_A = 0 \iff A = \bigcup_{i=1}^{n} A_i$ doesn't occur

$\iff A_i$'s don't occur, $\forall i \in [n]$

$\iff \mathbb{1}_{A_i} = 0, \ \forall i \in [n] \iff \text{RHS} = 1$

Claim $\Rightarrow 1 - \mathbb{1}_A = \prod_{i=1}^{n} (1 - \mathbb{1}_{A_i}) = 1 - \sum_{1\le i\le n} \mathbb{1}_{A_i} + \sum_{1\le i<j\le n} \mathbb{1}_{A_i}\mathbb{1}_{A_j} - \sum_{1\le i<j<k\le n} \mathbb{1}_{A_i}\mathbb{1}_{A_j}\mathbb{1}_{A_k} + \dots - (-1)^{n+1} \mathbb{1}_{A_1}\mathbb{1}_{A_2}\cdots\mathbb{1}_{A_n}$

$\because$

Idea:

$(1-x_1)(1-x_2)(1-x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_2x_3 + x_3x_1 - x_1x_2x_3$

Just put $x_i = \mathbb{1}_{A_i}$

$\Rightarrow \mathbb{1}_A = \sum_{1\le i\le n} \mathbb{1}_{A_i} - \sum_{1\le i<j\le n} \mathbb{1}_{A_i}\mathbb{1}_{A_j} + \sum_{1\le i<j<k\le n} \mathbb{1}_{A_i}\mathbb{1}_{A_j}\mathbb{1}_{A_k} - \dots + (-1)^{n+1}\mathbb{1}_{A_1}\mathbb{1}_{A_2}\cdots\mathbb{1}_{A_n}$

$\Rightarrow \mathbb{E}(\mathbb{1}_A) = \mathbb{E}(\quad\downarrow\quad)$

$\Rightarrow \mathbb{P}(A) = \sum_{1\le i\le n} \mathbb{P}(A_i) - \sum_{1\le i<j\le n} \mathbb{P}(A_i\cap A_j) + \sum_{1\le i<j<k\le n} \mathbb{P}(A_i\cap A_j\cap A_k) - \dots + (-1)^{n+1}\mathbb{P}\left[\bigcap_{i=1}^{n} A_i\right]$

$\because$

$\mathbb{1}_{B_1}\mathbb{1}_{B_2}\cdots\mathbb{1}_{B_t} = \mathbb{1}_{B_1\cap B_2\cap\dots\cap B_t}$

$\square$

1) $A_1, \dots, A_n \longrightarrow$ distinct subsets of $[n]$

To show: $\exists x \in [n]$ s.t $A_1 \cup \{x\}, \dots, A_n \cup \{x\}$ are distinct

Pf: First we'll show that $\exists x \in [n]$ s.t $A_1 \setminus \{x\}, \dots, A_n \setminus \{x\}$ are distinct

Suppose not, i.e, suppose $\forall x \in [n]$ $\exists i, j \in [n]$ $(i \neq j)$ s.t $A_i \setminus \{x\} = A_j \setminus \{x\}$

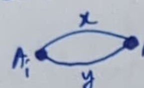
Define a graph $G(V, E)$:

① $V = \{A_1, \dots, A_n\}$

② For each $x \in [n]$, randomly select exactly one pair (A_i, A_j) satisfying $A_i \setminus \{x\} = A_j \setminus \{x\}$ and join A_i, A_j by an edge

NOTE: $|V| = |E| = n$

Claim 1: G doesn't have multi-edges

Pf: we'll show that  is not possible

Suppose $A_i \setminus \{x\} = A_j \setminus \{x\}$ and $A_i \setminus \{y\} = A_j \setminus \{y\}$

Then, $A_i \setminus \{x\} = A_j \setminus \{x\} \implies x \in A_i, x \notin A_j$ (WLOG) $\because A_i \neq A_j$

$\implies x \in A_i \setminus \{y\}, x \notin A_j \setminus \{y\}$

$(\Rightarrow\Leftarrow)$

Claim 2: G has a cycle

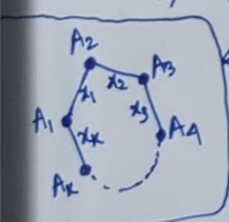
Pf: Suppose not, i.e, let G be acyclic

If G has k connected components, then each component is a tree and thus G has $(n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1) = (n - k)$ edges

$(\Rightarrow\Leftarrow)$ as $|E| = n$

WLOG, say $A_1 A_2 \dots A_k A_1$ is a cycle of G

Suppose $A_1 \setminus \{x_1\} = A_2 \setminus \{x_1\}$, $A_2 \setminus \{x_2\} = A_3 \setminus \{x_2\}, \dots, A_k \setminus \{x_k\} = A_1 \setminus \{x_k\}$



NOTE: x_i's are distinct

Now, $A_1 \setminus \{x_1\} = A_2 \setminus \{x_1\} \implies x_1 \notin A_1, x_1 \in A_2$ (WLOG)

Then, $A_2 \setminus \{x_2\} = A_3 \setminus \{x_2\} \implies x_1 \in A_3$ $(\because x_1 \in A_2)$

$A_3 \setminus \{x_3\} = A_4 \setminus \{x_3\} \implies x_1 \in A_4$ $(\because x_1 \in A_3)$

$\vdots$

$A_k \setminus \{x_k\} = A_1 \setminus \{x_k\} \implies x_1 \in A_1$ $(\because x_1 \in A_k)$

$(\Rightarrow\Leftarrow)$ as $x_1 \notin A_1$

$\therefore A_1 \setminus \{x\}, A_2 \setminus \{x\}, \dots, A_n \setminus \{x\}$ are distinct

PTO →

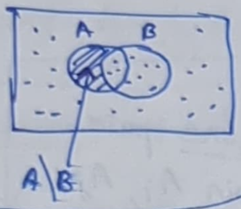
Now, $A_1, \dots, A_n$ distinct $\Rightarrow A_1^c, \dots, A_n^c$ distinct

$\Rightarrow \exists x \in [n]$ s.t $A_1^c \setminus \{x\}, \dots, A_n^c \setminus \{x\}$ distinct

$\Rightarrow (A_1^c \setminus \{x\})^c, \dots, (A_n^c \setminus \{x\})^c$ distinct

$\Rightarrow A_1 \cup \{x\}, \dots, A_n \cup \{x\}$ distinct

$\because$ NOTE: $(A \setminus B)^c = A^c \cup B$



$\square$

10) LYM $\mathcal{F} \to$ collection of subsets of $[n]$; $A \not\subseteq B$, $\forall A, B \in \mathcal{F}$ $(A \neq B)$

To show: $\displaystyle\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$

pf 1 (without using random permutation):

Alternate Formulation: Consider the poset $P = (2^{[n]}, \subseteq)$ and let $\mathcal{F}$ be an antichain in P

To show: $\displaystyle\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$

Lets look at the maximal chains of P. They're of the form:

$$\phi \subseteq \{x_1\} \subseteq \{x_1, x_2\} \subseteq \{x_1, x_2, x_3\} \subseteq \dots \subseteq \{x_1, \dots, x_n\} = \{1, 2, \dots, n\}, \text{ where } x_i \in [n], \forall i$$

$\Rightarrow$ no. of maximal chains in $P = n!$

Now, let $F \in \mathcal{F}$ (arbitrary element [set] of antichain $\mathcal{F}$)

How many maximal chains contain F? (Ans: $|F|!\,(n-|F|)!$)

$$\underbrace{\phi \subseteq \dots \subseteq}_{|F|!} F \underbrace{\subseteq \dots \subseteq \{1, 2, 3, \dots, n\}}_{(n-|F|)!}$$

Now, observe that only one element of $\mathcal{F}$ can be present in a (maximal) chain

Thus, $\displaystyle\sum_{F \in \mathcal{F}} |F|!\,(n-|F|)! \leq n!$, i.e, $\displaystyle\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$ $\square$

---

pf 2 (using random permutation): Let σ be a random permutation of $\{1, 2, \dots, n\}$

Consider the family of sets $P_\sigma := \{\{\sigma(1)\}, \{\sigma(1), \sigma(2)\}, \{\sigma(1), \sigma(2), \sigma(3)\}, \dots, \{\sigma(1), \sigma(2), \dots, \sigma(n)\}\}$, where $\{\sigma(1), \sigma(2), \dots, \sigma(n)\} = \{1, 2, \dots, n\}$

Define a r.v $X :=$ no. of common sets of $\mathcal{F}$ and P_σ

First, lets look at X as the no. of sets of P_σ that belong to $\mathcal{F}$

$X \leq 1$ ($\because$ two sets of P_σ can't be in $\mathcal{F}$ as one contains the other)

Now, lets look at X as the no. of sets of $\mathcal{F}$ that belong to P_σ

Note that $X = \displaystyle\sum_{F \in \mathcal{F}} \mathbb{1}_F$, where $\mathbb{1}_F := \begin{cases} 1, & \text{if } F \in P_\sigma \\ 0, & \text{o.w} \end{cases}$

$$\Rightarrow \mathbb{E}(X) = \mathbb{E}\Big(\sum_{F \in \mathcal{F}} \mathbb{1}_F\Big) = \sum_{F \in \mathcal{F}} \mathbb{P}(F \in P_\sigma) \overset{\because}{=} \sum_{F \in \mathcal{F}} \frac{|F|!\,(n-|F|)!}{n!} = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}$$

$$\underbrace{\{\sigma(1)\} \subseteq \dots \subseteq}_{|F|!} F \underbrace{\subseteq \dots \subseteq \{\sigma(1), \dots, \sigma(n)\}}_{(n-|F|)!}$$

Now, $X \leq 1 \Rightarrow \mathbb{E}(X) \leq 1 \Rightarrow \displaystyle\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$

$\square$

1> $\mathcal{M} \to$ collection of K-element subsets of X

$\mathcal{M}$ is 2-colorable IFF $\nexists$ monochromatic set in $\mathcal{M}$

To show: $\mathcal{M}$ is 2-colorable

pf: Randomly assign R/B color to each point of X (i.e, color R/B w.p $1/2$ each)

$$\begin{aligned}
\mathbb{P}(\mathcal{M} \text{ is 2-colorable}) &= \mathbb{P}(\nexists \text{ monochromatic set in } \mathcal{M}) \\
&= 1 - \mathbb{P}(\exists \text{ monochromatic set in } \mathcal{M}) \\
&= 1 - \mathbb{P}\Big(\bigcup_{S \in \mathcal{M}} \{S \text{ is monochromatic}\}\Big) \\
&\geq 1 - \sum_{S \in \mathcal{M}} \underbrace{\mathbb{P}(S \text{ is monochromatic})}_{= 2/2^K} \\
&= 1 - |\mathcal{M}| \cdot \frac{2}{2^K} \\
&= 1 - \frac{|\mathcal{M}|}{2^{K-1}} > 0
\end{aligned}$$

$\because$ Union Bound

$\mathbb{P}(A_1 \cup \cdots \cup A_n) \leq \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n)$

Thus, $\exists$ a coloring of X where $\mathcal{M}$ is 2-colorable

□

5> To show: $A \perp\!\!\!\perp B \Rightarrow \bar{A} \perp\!\!\!\perp \bar{B}$

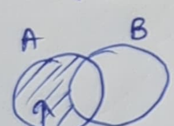
pf: Claim: $A \perp\!\!\!\perp B \Rightarrow A \perp\!\!\!\perp \bar{B}$

pf: $\mathbb{P}(A \cap \bar{B}) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$ $\because$ $A = \underbrace{(A \cap B) \cup (A \cap \bar{B})}_{\text{mutually disjoint}}$

$= \mathbb{P}(A) - \mathbb{P}(A)\,\mathbb{P}(B)$ $\because$ $A \perp\!\!\!\perp B$

$= \mathbb{P}(A)(1 - \mathbb{P}(B))$

$= \mathbb{P}(A)\,\mathbb{P}(\bar{B})$



$A \cap \bar{B}$

$A \perp\!\!\!\perp B \Rightarrow A \perp\!\!\!\perp \bar{B}$, i.e, $\bar{B} \perp\!\!\!\perp A \Rightarrow \bar{B} \perp\!\!\!\perp \bar{A}$, i.e, $\bar{A} \perp\!\!\!\perp \bar{B}$

□

14) $k, l \in \mathbb{N}$, $\{a_i\}_{i=1}^{kl+1} \to$ sequence of reals

To show: $\{a_i\}_{i=1}^{kl+1}$ contains a non-decreasing sequence of length $(k+1)$ or a decreasing sequence of length $(l+1)$

Pf: Let $X = \{(a_j, j) : 1 \le j \le kl+1\}$ be a set and $\le_p$ be a relation on X defined as: $(a_i, i) \le_p (a_j, j)$ IFF $a_i \le a_j$ and $i \le j$

Claim: $\le_p$ is a partial ordering on X (i.e, $(X, \le_p)$ is a poset)

pf: 1) reflexive: $(a_i, i) \le_p (a_i, i)$, $\forall i \in [kl+1]$ ✓

2) antisymmetric: $(a_i, i) \le_p (a_j, j)$, $(a_j, j) \le_p (a_i, i) \Rightarrow (a_i, i) = (a_j, j)$ ✓

3) transitive: $(a_i, i) \le_p (a_j, j)$, $(a_j, j) \le_p (a_k, k) \Rightarrow (a_i, i) \le_p (a_k, k)$ ✓

Clearly, the chains of $(X, \le_p)$ are just non-decreasing subsequences of $\{a_i\}_{i=1}^{kl+1}$

Also note that the antichains of $(X, \le_p)$ are just decreasing subsequences of $\{a_i\}_{i=1}^{kl+1}$

$\because$ Let $(a_i, i), (a_j, j) \in$ some antichain

Then, $a_i \le a_j$ and $i > j$ OR $a_i > a_j$ and $i \le j$

$a_j \ge a_i$ $\longrightarrow$ $\qquad$ $a_i > a_j$ $\longrightarrow$

Result: A finite poset of size atleast $(kl+1)$ contains either a chain of length $(k+1)$ or an antichain of length $(l+1)$

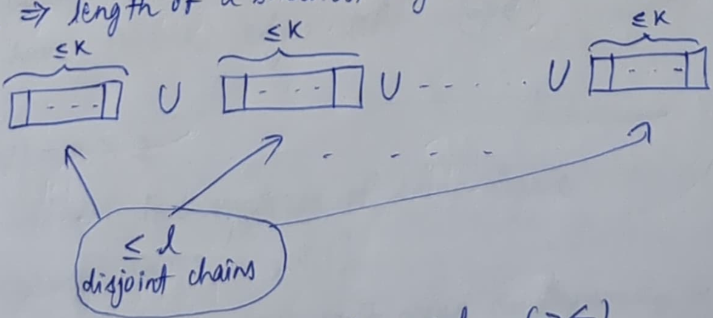
pf: If $\exists$ a chain of length $(k+1)$, then done ✓

Suppose not, i.e, suppose all chains have length $\le k$

ETS: $\exists$ an antichain of length $(l+1)$

Suppose not, i.e, suppose all antichains have length $\le l$

Dilworth $\Rightarrow$ length of a smallest disjoint chain decomposition $\le l$



Then, total elements in poset $\le kl$ $(\Rightarrow\Leftarrow)$

Using Result on the poset $(X, \le_p)$, we're done

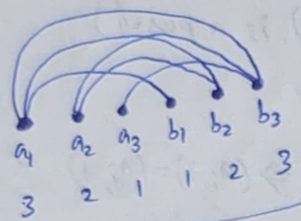
□

3.2) $d = (d_1, d_2, \dots, d_{2k})$, where $d_1 = d_2 = 1$, $d_3 = d_4 = 2$, $\dots$, $d_{2k-1} = d_{2k} = k$

i.e, $d = (1, 1, 2, 2, 3, 3, \dots, k, k)$

Define $G(V, E)$ to be : $\boxed{\begin{array}{l} V = \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k\} \\ (a_i, b_j) \in E \Leftrightarrow i \le j \end{array}}$

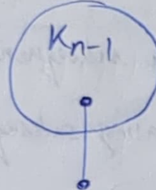
Example: $(1,1,2,2,3,3)$



Clearly, $\forall i \in [k]$, $\deg(a_i) = k - i + 1$
and $\deg(b_i) = i$

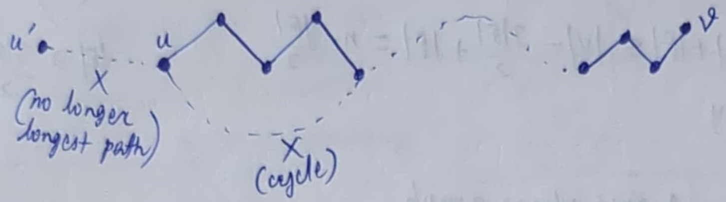
□

2.2)

$G =$ 

1) <u>To show</u>: Every tree of n vertices has atleast two degree 1 vertices $(n \geq 2)$

<u>Pf</u>: Consider a longest path (say from u to v):



<u>Claim</u>: $\deg(u) = \deg(v) = 1$

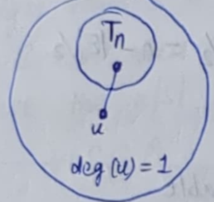
□

---

2) <u>To prove</u>: A tree on n vertices has $(n-1)$ edges

<u>pf (induction on n)</u>: we'll use Q1)

T_{n+1} :



$(n-1)+1 = n$ edges

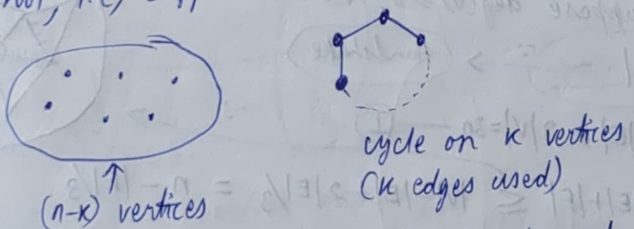
□

---

3) <u>To prove</u>: G connected, $|V| = n$, $|E| = n-1 \implies G$ is a tree

<u>pf</u>: <u>ETS</u> : G is acyclic

Suppose not, i.e, suppose G has a cycle of length k $\quad 3 \leq k \leq n-1$



we need atleast $(n-k-1)+1 = (n-k)$ edges to connect G,

but we only have $(n-1-k)$ edges left $(\Rightarrow\Leftarrow)$

□

---

4) $G = (V, E)$, $|V| = n \geq 2$

<u>To show</u>: G has atleast two vertices of same degree

<u>pf</u>: possible degrees : $\{0, 1, 2, \ldots, n-1\}$

Note that the degrees 0 and $(n-1)$ can't occur simultaneously in G

Thus, actual possible degree sets are :

① $\{0, 1, 2, \ldots, n-2\}$ $\longrightarrow$

② $\{1, 2, 3, \ldots, n-1\}$ $\longrightarrow$

In both cases, by PHP we're done

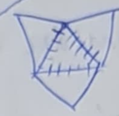
□

1) $G = ([n], E) \to$ planar graph, <u>To show:</u> $|F| \leq 2n-4$

<u>sol^n:</u> Euler $\Rightarrow |V| - |E| + |F| = 2$

$\Rightarrow 2 = |V| - |E| + |F| \leq |V| - \frac{3|F|}{2} + |F| = n - \frac{|F|}{2}$

$\Rightarrow |F| \leq 2n-4$



$\because |E| \geq \frac{3|F|}{2}$

2), 11) $G = ([n], E) \to \Delta$-free planar graph

<u>To show:</u> (2) $|F| \leq n-2$, (11) $|E| \leq 2n-4$

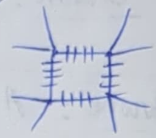
<u>sol^n:</u> Euler $\Rightarrow |V| - |E| + |F| = 2$

$\Rightarrow 2 = |V| - |E| + |F| \leq |V| - 2|F| + |F| = n - |F|$

$\Rightarrow |F| \leq n-2$

Again, Euler $\Rightarrow 2 = |V| - |E| + |F| \leq |V| - |E| + |E|/2 = n - |E|/2$

$\Rightarrow |E| \leq 2n-4$



$\because |E| \geq \frac{4|F|}{2} = 2|F|$

$\Rightarrow |F| \leq |E|/2$

4) <u>To show:</u> Every planar graph is 6-colorable

pf (induction on $n = |V|$): <u>Base case ($n \leq 6$): trivial</u>

<u>IH</u>: Suppose true for $(n-1)$

Now, consider a graph $G(V, E)$ with $|V| = n$ (WLOG assume G is connected)

<u>claim:</u> $\deg(v) \leq 5$, for some $v \in V$

pf: Suppose not, i.e, suppose $\deg(v) \geq 6$, $\forall v \in V$

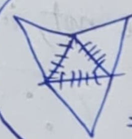
Then, $\sum_{v \in V} \deg(v) = 2|E| \xrightarrow{\because}$ Handshake

$\Rightarrow 2|E| \geq 6|V| \Rightarrow |E| \geq 3|V| = 3n$ — (I)

Now, Euler $\Rightarrow 2 = |V| - |E| + |F| \leq |V| - |E| + 2|E|/3 = n - |E|/3$

$\Rightarrow |E| \leq 3n - 6$ — (II)

(I), (II) $\Rightarrow$ contradiction



$\because |E| \geq \frac{3|F|}{2}$

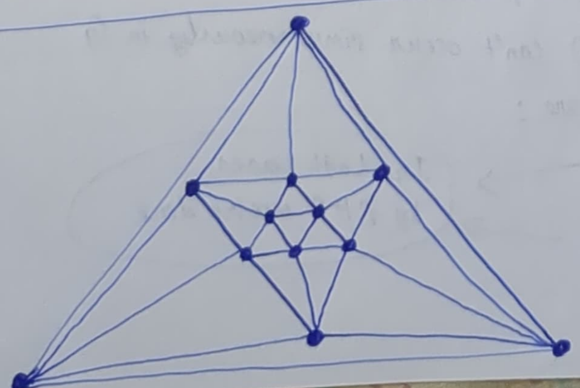
$\Rightarrow |F| \leq \frac{2|E|}{3}$

So, let $v \in V$ s.t $\deg(v) \leq 5$

IH $\Rightarrow G' = G \setminus \{v\}$ is 6-colorable

Now, add v to G'. Clearly, we can color v as $\deg(v) \leq 5$ and we've 6 colors □

3)



page 198 [Matousek]

Hall's theorem: Let $G = (A \cup B, E)$ be a bipartite graph. Then:

G contains an A-perfect matching $\Leftrightarrow |S| \leq |N(S)|, \forall S \subseteq A$
(i.e, a matching that covers A)

40) $G(A \cup B, E) \longrightarrow K$-regular

To prove: G has a perfect matching

pf: Claim: $|A| = |B|$

pf: There're $K \cdot |A|$ edges "outgoing" from A and $K \cdot |B|$ edges "recieved" by B
$\Rightarrow K|A| = K|B| \Rightarrow |A| = |B|$

Now, let $S \subseteq A$

No. of "outgoing" edges from $S = K \cdot |S|$

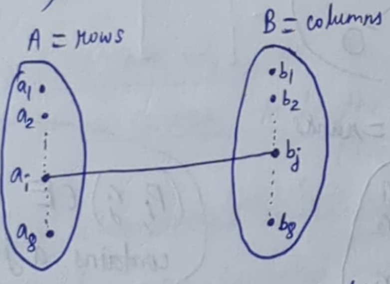
Since each vertex in B can "recieve" atmost K edges, atleast $|S|$ vertices are required to recieve the $K|S|$ outgoing edges from S

Thus, $|N(S)| \geq |S|, \forall S \subseteq A$

So, Hall $\Rightarrow G$ has an A-perfect matching $\Rightarrow G$ has a perfect matching ($\because |A| = |B|$) □

31) For $1 \leq i \leq 8$, a_i = row i and b_i = column i

G(A∪B, E):



$(a_i, b_j) \in E$ IFF $(i,j)^{th}$ square has a piece

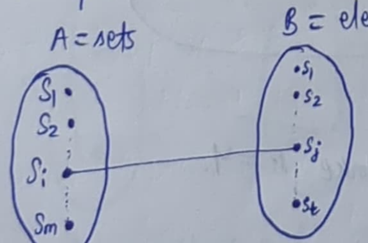
NOTE: G is n-regular

To show: G has a perfect matching

pf: By Q40), we know that a regular bipartite graph has a perfect matching □

41) Let $t = |S_1 \cup S_2 \cup \cdots \cup S_m|$

G(A∪B, E):



$(S_i, s_j) \in E$ IFF $s_j \in S_i$

By Hall, we're:

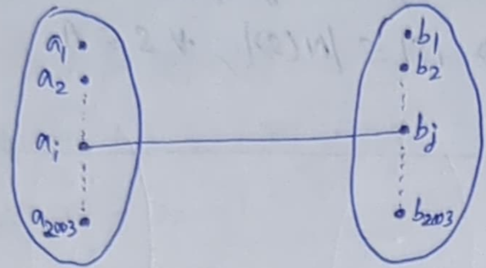
G has an A-perfect matching $\Leftrightarrow |S| \leq |N(S)|, \forall S \subseteq A$

i.e, $\exists$ a traversal $\Leftrightarrow$ union of any k sets has $\geq k$ elements, $\forall k \in [m]$ □

30)

$A = \text{sheet } 1$ $\qquad$ $B = \text{sheet } 2$

$G(A \cup B, E)$:



$(a_i, b_j) \in E$ IFF polygons a_i and b_j overlap

ETS : G has a perfect matching

pf : Let $S \subseteq A$

suppose $|S| = k$, i.e, suppose S contains k polygons (each of area 1)

Then, $\text{area}(S) = k$

Now, $\text{area}(N(S)) \geq \text{area}(S) = k$

$\Rightarrow |N(S)| \geq k$ ($\because$ each polygon has area 1)

Thus, we've that $|S| \leq |N(S)|$, $\forall S \subseteq A$

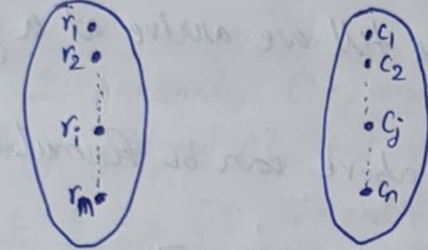
Hall $\Rightarrow$ G has an A-perfect matching

$\Rightarrow$ G has a perfect matching ($\because |A| = |B|$) $\square$

39)

$G(A \cup B, E)$:

A = rows $\quad$ B = columns



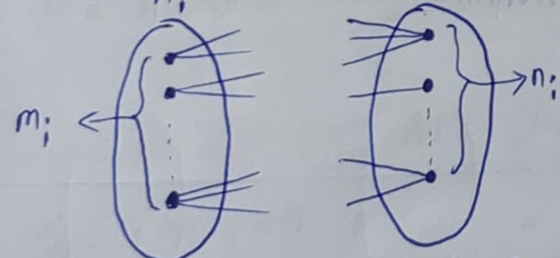
$(r_i, c_j) \in E$ IFF $(i,j)^{th}$ element of array > 0

NOTE: $\deg(v) \geq 1$, $\forall v \in A \cup B$
(i.e, no isolated vertex in G)

To show: $m = n$

pf: Let weight of an edge be the corresponding array element
Define weight of a vertex to be the sum of its (row/col) elements
(i.e, sum of its edge weights)

Suppose G has k connected components $k \geq 1$
Since G doesn't have an isolated vertex, the connected components are bipartite graphs themselves
Consider the ith connected component of G:



NOTE: Here, every vertex has the same weight (say w_i)

Then, weight "going out" of A_i = weight "coming into" B_i

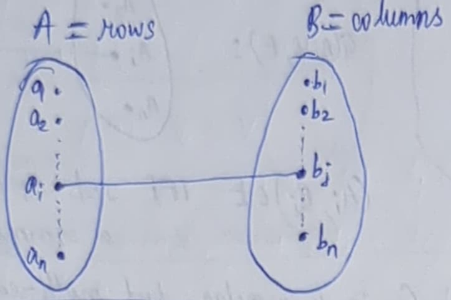
$$\Rightarrow m_i \cdot w_i = n_i \cdot w_i$$

$$\Rightarrow m_i = n_i \quad \Rightarrow \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} n_i \quad \Rightarrow \boxed{m = n}$$

□

BQ2) For $1 \le i \le n$, let a_i = row i and b_i = column i

$G(A \cup B, E)$:



$(a_i, b_j) \in E$ IFF $a_{ij} = 1$

NOTE: G is k-regular

Result (Q40) in PS3: A k-regular bipartite graph has a perfect matching

Thus, by Result, G has a perfect matching (say M)
Now, remove the edges of M from G to obtain a $(k-1)$-regular graph (say G')
Keep on repeating the above process till we arrive at a graph with no edges
In matrix terms, what we've done above can be formulated as:

$$A_{n\times n} - \underbrace{[\quad]_{n\times n} - [\quad]_{n\times n} - \cdots\cdots - [\quad]_{n\times n}}_{k \text{ many permutation matrices}} = O_{n\times n}$$

zero matrix

$$\Rightarrow A = \underbrace{[\quad] + [\quad] + \cdots\cdots + [\quad]}_{k \text{ many permutation matrices}}$$

$\square$

12) Pick a random tournament T. In particular, let T be a directed complete graph with vertices $\{1,2,\dots,n\}$ and edge directions are random (i.e, prob. $\frac{1}{2}$ in either direction)

Define a r.v $X :=$ no. of Ham Paths in T

Note that $X = \sum_{\sigma \in S_n} \mathbb{1}_\sigma$, where $\mathbb{1}_\sigma := \begin{cases} 1, & \text{if } \sigma \text{ is a Ham Path in } T \\ 0, & \text{o.w} \end{cases}$

> NOTATION: $i \to j \Rightarrow i$ defeats j
>
> $\sigma = (\sigma(1), \dots, \sigma(n))$ is a Ham Path in T if $\sigma(1) \to \sigma(2) \to \dots \to \sigma(n)$

$$\text{Now, } \mathbb{E}(X) = \sum_{\sigma \in S_n} \mathbb{E}(\mathbb{1}_\sigma) = \sum_{\sigma \in S_n} \underbrace{\mathbb{P}(\sigma \text{ is a Ham Path in } T)}_{= \frac{1}{2^{n-1}}} = \frac{n!}{2^{n-1}}$$

$\Rightarrow X \geq \frac{n!}{2^{n-1}}$ for some tournament T

□

---

33) Assign 0 or 1 randomly (i.e, w.p $\frac{1}{2}$) to each vertex of $G(V,E)$

Let $V_0 = \{\text{vertices assigned } 0\}$ and $V_1 = \{\text{vertices assigned } 1\}$

For $e \in E$, let $\mathbb{1}_e := \begin{cases} 1, & \text{if end vertices of } e \text{ have different values} \\ 0, & \text{o.w} \end{cases}$

Now, let $X :=$ no. of edges b/w V_0 and V_1

Then, $X = \sum_{e \in E} \mathbb{1}_e$

$$\Rightarrow \mathbb{E}(X) = \sum_{e \in E} \mathbb{E}(\mathbb{1}_e) = \sum_{e \in E} \underbrace{\mathbb{P}(\text{end vertices of } e \text{ have different values})}_{= \frac{1}{2}} = \frac{|E|}{2}$$

$\Rightarrow X \geq \frac{|E|}{2}$ for some assignment of 0's and 1's

(i.e, $\exists$ a bipartite subgraph $G'(V_0 \cup V_1, E')$ of G s.t $|E'| \geq \frac{|E|}{2}$

□

ALT (induction on $|V(G)|$)

Base Case ($|V(G)| = 1$) : • ✓

IH: Suppose true for $|V(G)| - 1$

To show: True for $|V(G)|$

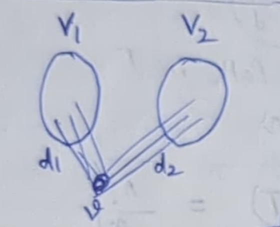
(PTO) →

Pick any $v \in V(G)$ and consider the graph $H = G \setminus \{v\}$

IH $\Rightarrow$ H contains a bipartite subgraph H' s.t

$$V(H) = V(H') \quad \text{and} \quad |E(H')| \geq |E(H)|/2 = \frac{|E(G)| - \deg_G(v)}{2}$$

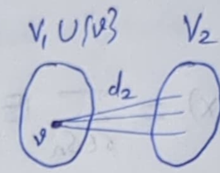
Suppose $V(H') = V(H) = V_1 \cup V_2$

H': 

For $i = 1, 2$, let d_i = no. of edges from v to V_i

Then, $d_1 + d_2 = \deg_G(v)$

WLOG, say, $d_2 \geq \deg_G(v)/2$

Add v to V_1 to obtain G': 

Now, $|E(G')| = |E(H')| + d_2$

$$\geq \frac{|E(G)| - \deg_G(v)}{2} + \frac{\deg_G(v)}{2} = |E(G)|/2$$

and $V(G') = V(G)$ $\square$

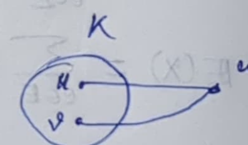
9> <u>To show</u>: Either G or $\overline{G}$ is connected

<u>pf</u>: If G is connected, we're done. Suppose not, i.e, suppose G is not connected.

Let $u, v \in \overline{G}$

<u>case I ($(u,v) \notin E$)</u>: Then, $(u,v) \in \overline{E}$ ✓

<u>case II ($(u,v) \in E$)</u>: Suppose u, v lie in some connected component K of G. Let $w \in G \setminus K$



Then, $(u, w) \in \overline{E}$ and $(w, v) \in \overline{E}$

$\Rightarrow$ $u - w - v$ is a path from u to v in $\overline{G}$ and hence u and v are connected ✓

$\square$

21) $G(V,E)$, $|V| = n$, $|E| = m > n^2/4$ To show: G contains a Δ

pf: Suppose not, i.e, suppose G doesn't contain a Δ

Let $(x,y) \in E$ (arbitrary)

Since G doesn't contain a Δ, every vertex of G is connected to atmost one of x and y. Thus, $\deg(x) + \deg(y) \leq (n-2)+1+1 = n$

$$\Rightarrow \sum_{(x,y)\in E} [\deg(x) + \deg(y)] \leq \sum_{(x,y)\in E} n = mn \quad \text{— (I)}$$

Also, $$\sum_{(x,y)\in E} [\deg(x) + \deg(y)] = \sum_{v\in V} (\deg(v))^2 \underset{\text{CS inequality}}{\geq} \frac{1}{n}\left(\sum_{v\in V} \deg(v)\right)^2 \underset{\text{Handshake}}{=} \frac{4m^2}{n} \quad \text{— (II)}$$

(I), (II) $\Rightarrow \frac{4m^2}{n} \leq mn \Rightarrow m(n^2-4m) \geq 0 \Rightarrow n^2 - 4m \geq 0 \Rightarrow m \leq \frac{n^2}{4}$ ($\Rightarrow \Leftarrow$)

□

BQ1) $G(V,E)$, $|V| = n$, $|E| = m$, T = no. of Δ's in G

To show: $T \geq \frac{m}{3n}(4m - n^2)$

pf: Let $(x,y) \in E$ (arbitrary)

Let T_{xy} := no. of Δ's containing edge (x,y)

Then, $T_{xy} \geq [\deg(x)-1] + [\deg(y)-1] - (n-2)$ $\because$ $|A\cap B| = |A| + |B| - |A\cup B|$

$= \deg(x) + \deg(y) - n$

$$\Rightarrow \underbrace{\sum_{(x,y)\in E} T_{xy}}_{= 3T} \geq \sum_{(x,y)\in E} [\deg(x) + \deg(y) - n]$$

$$\Rightarrow 3T \geq \sum_{(x,y)\in E} [\deg(x) + \deg(y)] - mn$$

$$= \sum_{v\in V} (\deg(v))^2 - mn$$

$$\geq \frac{1}{n}\left(\sum_{v\in V} \deg(v)\right)^2 - mn \quad \because \text{CS inequality}$$

$$= \frac{1}{n} \times (2m)^2 - mn$$

$$= \frac{4m^2}{n} - mn = \frac{m}{n}(4m - n^2)$$

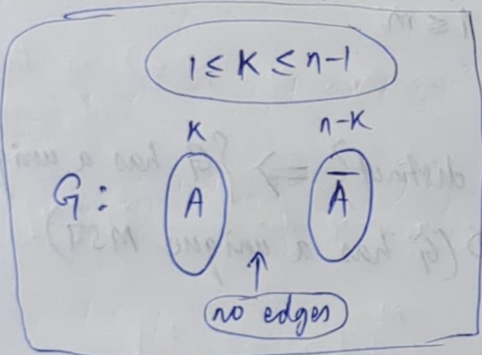
$$\Rightarrow \boxed{T \geq \frac{m}{3n}(4m - n^2)}$$

□

$n \to \infty$

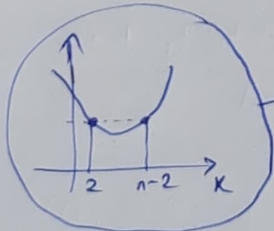
6) $G \in \mathcal{G}(n, \tfrac{1}{2})$ <u>To show:</u> $\lim_{n\to\infty} \mathbb{P}[G \text{ is connected}] = 1$

<u>pf:</u> $\{G \text{ is disconnected}\} \Rightarrow \left\{ \begin{array}{l} \exists \text{ a partition } (A \cup \bar{A}) \text{ of vertices of } G \\ \text{with no edges b/w } A \text{ and } \bar{A} \end{array} \right\}$



Thus, $$\mathbb{P}(G \text{ is disconnected}) \le \frac{\sum_{K=1}^{n-1} \binom{n}{K} 2^{\binom{K}{2}} \cdot 2^{\binom{n-K}{2}}}{2^{\binom{n}{2}}} = \frac{\sum_{K=1}^{n-1} \binom{n}{K} 2^{\binom{K}{2}+\binom{n-K}{2}}}{2^{\binom{n}{2}}}$$

$\because$ union bound



$\because$ $$\le \frac{2^{\binom{2}{2}+\binom{n-2}{2}} \cdot 2^{n}}{2^{\binom{n}{2}}}$$

$$= \frac{2^{n^2 - 3n/2 + 4}}{2^{n^2 - n/2}} = \frac{1}{2^{n-4}}$$

$\to 0$ as $n \to \infty$

BQ2) $G \to$ weighted connected graph with all edge weights distinct

To show: G has a unique MST

pf: Suppose not, i.e, let T and T' be 2 distinct MST's of G

Let e be the min. wt. edge in $T \setminus T'$ and e' be the min. wt. edge in $T' \setminus T$

WLOG, suppose $wt(e) < wt(e')$

Now, $T' \cup \{e\}$ contains exactly one cycle (say C)

Let $e'' \in C \setminus T$

Also, since $e \in T$, thus $e'' \neq e$ and hence $e'' \in T' \setminus T$

$\boxed{\text{NOTE: } wt(e) < wt(e') < wt(e'')}$

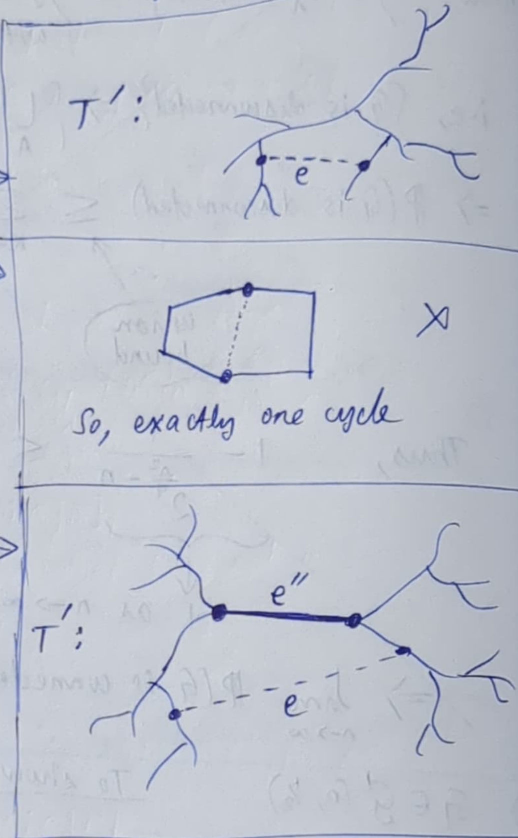
Now, consider the spanning tree $T'' = T' + e - e''$

(It may be possible that $T'' = T$)

Then, $wt(T'') = wt(T') + \underbrace{wt(e) - wt(e'')}_{<0}$

$\Rightarrow wt(T'') < wt(T')$

($\Rightarrow\Leftarrow$) as T' is an MST



$\square$

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9) $G = ([n], E) \to$ weighted connected graph, $|E| = m$

$\omega_i \overset{iid}{\sim} U(\{1, 2, \dots, 2m^2\})$, $1 \le i \le m$

To show: $\mathbb{P}(G \text{ has a unique MST}) > 0$

pf: We know that $\{\text{edge wt's of } G \text{ distinct}\} \Rightarrow \{G \text{ has a unique MST}\}$

$\Rightarrow \underbrace{\mathbb{P}(\text{edge wt's of } G \text{ distinct})}_{>0} \le \mathbb{P}(G \text{ has a unique MST})$

($\because \omega_i = i$, $\forall 1 \le i \le m$ is a possible weighting of G)

$\square$

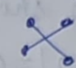
8) $G = ([n], E)$, $|E| = m > 4n$ To show: $Cr(G) = \Omega\left(\frac{m^3}{n^2}\right)$

pf: Construct H, a random induced subgraph of G

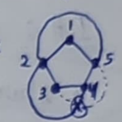
pick any vertex of G and include it in H w.p $p > 0$

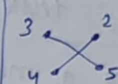
NOTE:
$|V(H)| \sim Bin(n, p)$
$|E(H)| \sim Bin(m, p^2)$

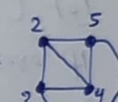
Then, $\mathbb{E}(|V(H)|) = np$, $\mathbb{E}(|E(H)|) = mp^2$, $\mathbb{E}(Cr(H)) \leq Cr(G) \cdot p^4$



Example:

$G = K_5 =$ 



$H = K_4 =$ 

Recall (Result): For any graph G, $Cr(G) \geq |E| - 3|V| + 6$

$\Rightarrow Cr(H) \geq |E(H)| - 3|V(H)| + 6 \geq |E(H)| - 3|V(H)|$

$$\Rightarrow \underbrace{\mathbb{E}(Cr(H))}_{\leq\, Cr(G)p^4} \geq \underbrace{\mathbb{E}(|E(H)|)}_{=\, mp^2} - 3\,\underbrace{\mathbb{E}(|V(H)|)}_{=\, np}$$

$\Rightarrow Cr(G)\, p^4 \geq mp^2 - 3np \Rightarrow Cr(G) \geq \dfrac{mp - 3n}{p^3}$

NOTE: $m > 4n$
$\Rightarrow 0 \leq \frac{4n}{m} < 1$

Taking $p = \frac{4n}{m}$, we've: $Cr(G) \geq \dfrac{4n - 3n}{64n^3} \times m^3 = \dfrac{m^3}{64n^2}$

$\therefore Cr(G) = \Omega\left(\frac{m^3}{n^2}\right)$ □

AQ3) [Dilworth] Let $(\Omega, \leq)$ be a finite poset and let

m = length of a smallest disjoint chain decomposition of Ω

M = length of a longest antichain in Ω

Then, $m = M$

pf: Note that $m \geq M$ ($\because$ a chain has atmost 1 element of an antichain)

So, enough to show that $m \leq M$. We'll prove this using strong induction on $|\Omega|$

Base case ($|\Omega| = 0$) : trivial

IH: Suppose true for $0, 1, 2, \dots, |\Omega|-1$

To show: true for $|\Omega|$

Let C be a maximal chain in Ω

If every antichain in $\Omega \setminus C$ has length $\leq M-1$, then by IH on $\Omega \setminus C$, we're done

So, assume that $\{a_1, \dots, a_M\}$ is an antichain in $\Omega \setminus C$

Now, define $S^- := \{x \in \Omega : x \leq a_i \text{ for some } i \in [M]\}$

Since C is a maximal chain, the maximum element of C is not in S^-

and thus by IH on S^-, we've:

$$S^- = S_1^- \cup S_2^- \cup \dots \cup S_M^-,$$

where S_i^-'s are disjoint chains and $a_i \in S_i^-$ (WLOG)

claim: a_i is the maximum element of S_i^-, $\forall i \in [M]$

pf: Suppose not, i.e, suppose $\exists x \in S_i^-$ s.t $a_i < x$

Now, $x \in S_i^- \subseteq S^- \Rightarrow \exists j \in [M]$ s.t $x \leq a_j$

Thus, we've that $a_i < x \leq a_j \Rightarrow a_i < a_j$ ($\Rightarrow\Leftarrow$) as they're not comparable

Analogously define $S^+ := \{x \in \Omega : a_i \leq x \text{ for some } i \in [M]\}$

and do the same as above here.

Then, we've that $\Omega = S^- \cup S^+ = \bigcup_{i=1}^{M} [S_i^- \cup S_i^+]$

and hence $m \leq M$

$\square$

AQ3) [Dilworth Dual (Mirsky)] Let $(X, \leq)$ be a finite poset

If the length of any chain in $X \leq m$, then X can be written as a union of m antichains

pf (induction on m):

Base Case ($m=1$): trivial

IH: Suppose true for $m-1$ ($m \geq 2$)

To show: true for m

Let M be the set of maximal elements of X. Then, M is an antichain

If $\{x_1 < x_2 < \cdots < x_m\}$ were a chain in $X \setminus M$, it would also be a maximal chain in X and hence $x_m \in M$ ($\Rightarrow\Leftarrow$)

So, assume that every chain in $X \setminus M$ has length $\leq (m-1)$

Then, IH on $X \setminus M \Rightarrow X \setminus M$ can be written as a union of $(m-1)$ antichains

$\Rightarrow X$ can be written as a union of m antichains □

AQ4) [Konig] $G = (A \cup B, E) \rightarrow$ bipartite graph, $|A \cup B| = n$

Let m = size of max. matching of G and k = size of min. VC of G

To show: $m = k$

pf: Consider the poset $(A \cup B, \leq)$, where for $u, v \in A \cup B$:

$$u \leq v \iff u \in A,\ v \in B,\ (u,v) \in E \quad \text{(OR)} \quad u = v$$

Each chain in this poset has length either 1 or 2. A smallest disjoint chain decomposition must include as many chains of length 2 as possible.

Thus, length of a smallest chain decomposition of $A \cup B = \underbrace{m}_{\text{length 2 chains}} + \underbrace{(n-2m)}_{\text{length 1 chains}} = n - m$ — (I)

Here, antichain is basically an independent set in G. Since VC and IS are complements, we've:

length of a longest antichain of $A \cup B$ = size of a max. IS of G

$= n - $ (size of a min. VC of G)

$= n - k$ — (II)

Using Dilworth, (I), (II) $\Rightarrow n - m = n - k \Rightarrow \boxed{m = k}$

□

Hall (using Dilworth) $G = (A \cup B, E) \longrightarrow$ bipartite graph.

To show: G has an A-perfect matching $\Leftrightarrow |S| \leq |N(S)|, \forall S \subseteq A$

pf:

($\Rightarrow$):

trivial

($\Leftarrow$):

Let $|A| = a$, $|B| = b \geq a$

Consider the poset $(A \cup B, \leq)$, where for $u, v \in A \cup B$:

$$u \leq v \Leftrightarrow u \in A, v \in B, (u,v) \in E \text{ (OR) } u = v$$

Each chain in this poset has length either 1 or 2. A smallest disjoint chain decomposition must include as many chains of length 2 as possible (i.e, any smallest chain decomposition must contain a max. matching). Suppose, size of a max. matching $= m$. Then, we've:

$$\text{length of a smallest chain decomposition} = \underbrace{m}_{\text{length 2 chains}} + \underbrace{(a+b-2m)}_{\text{length 1 chains}} = a+b-m$$

An antichain is basically an independent set in G

Let $\{a_1, a_2, \dots, a_k, b_1, \dots, b_l\}$ be a longest antichain of length $M = k+l$

Then, $N(\{a_1, \dots, a_k\}) \subseteq B \setminus \{b_1, \dots, b_l\}$

$$\Rightarrow k \leq b - l \Rightarrow k + l \leq b, \text{ i.e, } M \leq b$$

Now, Dilworth $\Rightarrow a + b - m = M \leq b$

$\Rightarrow m \geq a \Rightarrow m = a$ and hence, G has an A-perfect matching $\square$

BQ4) <u>To show:</u> For every poset $(X, \leq)$, $\exists$ an embedding into the poset $(2^X, \subseteq)$

pf: <u>Def^n (embedding):</u> Let $(X, \leq)$ and $(X', \leq')$ be posets.
A map $f: X \to X'$ is called an embedding of $(X, \leq)$ into $(X', \leq')$ if:

① f is 1-1
② $f(x) \leq' f(y) \iff x \leq y$

Let $f: X \to 2^X$ s.t $f(x) = \{y \in X : y \leq x\}$, $\forall x \in X$

<u>Claim:</u> f is an embedding

pf: ① <u>f is 1-1:</u> Suppose $f(x) = f(y)$

$x \leq x \Rightarrow x \in f(x) = f(y) \Rightarrow x \leq y$ — (I)

$y \leq y \Rightarrow y \in f(y) = f(x) \Rightarrow y \leq x$ — (II)

(I), (II) $\Rightarrow x = y$ (by antisymmetry of $\leq$)

② <u>$f(x) \subseteq f(y) \iff x \leq y$:</u>

<u>($\Rightarrow$):</u>

$x \in f(x) \subseteq f(y) \Rightarrow x \in f(y) \Rightarrow x \leq y$ ✓

<u>($\Leftarrow$):</u>

Let $z \in f(x)$

$\Rightarrow z \leq x$

Now, $z \leq x$ and $x \leq y \Rightarrow z \leq y$ (by transitivity of $\leq$)

$\Rightarrow z \in f(y)$

$\therefore f(x) \subseteq f(y)$ ✓

□