Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
Date: August 21, 2023

Problem Set 1

Refer to the main textbooks for this course [VLW01, MN09] for the definitions.

Question 1. Write the negation of the following:

1) Every complete bipartite graph is not planar.
2) If $G = (v, e)$ is a directed graph s.t. for any two vertices $u, v \in V$ at most one of (u, v) and $(v, u) \in E$ then $\exists w \in V$ s.t. the number of vertices that are at distance 2 from w is at least the number of vertices that are at distance 1 from w.
3) There exists $x \in \mathbb{R}$ such that for every $\epsilon > 0$ there exist $N_\epsilon \in \mathbb{N}$ such that for every $n \geq N_\epsilon$ we have $|x_n - x| < \epsilon$. (Note: This is precisely the definition of convergence of a real sequence)
4) In each tree in the garden, we can find a branch on which all of the leaves are green. (Hint: Write using quantifiers)

Question 2. Write the contrapositive statement:

1) If we have good crops then either it rained well or the fertilization worked properly.
2) If a tournament has a cycle then it must have a 3-cycle.

Question 3. Prove or disprove:

1) A finite sum of distinct irrational numbers can never be a rational number.
2) If p, q are two primes, $\sqrt{pq}$ is irrational.

Question 4. Check the logical equivalence of the following: $(s \to r) \wedge (q \to r) = (s \wedge q) \to r$

Question 5. Check whether the following proposition is a tautology or a contradiction or none: $(p \wedge q) \vee (\neg p \vee (p \wedge \neg q))$

Question 6. Prove that:

1) $3n^5 + 5n^3 + 7n$ is divisible by 15, for every natural number n.
2) If $2^n - 1$ is a prime, n is a prime.

Question 7. Prove that, at a party with at least 2 people, there are at least two people who know the same number of people (not necessarily the same people), provided every person is invited by someone who knows him.

Question 8. If $[k] = \{1, \ldots, k\}$ then calculate the following:

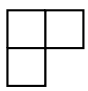
1) The number of monotonically increasing functions from $[m]$ to $[n]$, where $n \geq m$.
2) The number of monotonically decreasing functions from $[m]$ to $[n]$, where $n \geq m$.

Question 9. Let p and q be two distinct primes, find the remainder when $p^q + q^p$ is divided by pq. (Hint: Little Theorem)

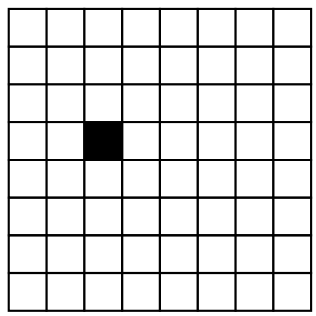
Question 10. The rules of double chess are exactly those of regular chess (all the pieces and their moves are identical), except that in double chess either player (Black or White) plays consequently two moves (instead of just one move permitted in regular chess) in each of their alternate turns. Show that there exists a strategy for white which guarantees him at least a tie.

Question 11. Take any real number $d \in \mathbb{R}$. Suppose all points in the plane $\mathbb{R}^2$ are colored either red or blue. Prove that there are two points at distance d that are of the same color.

Question 12. For any $n \in \mathbb{N}$, prove that, the $2^n \times 2^n$ checkerboard with any square removed can be tiled by L-shaped triominos.



A L-shaped triomino



A 8×8 checkerboard with one square removed

Question 13. Consider a $n \times n$ binary matrix $B = (b_{ij})_{1 \le i,j \le n}$ (i.e., $b_{ij} \in \{0, 1\}$). The $\mathsf{flip}(i, j)$ operation inverts all the $2n - 1$ cells of row i and column j, where inverting a cell means replacing a 0 by 1 or vice versa. Can we get the zero matrix (where every element is 0) from B by performing flip operations?

Question 14. Does there exist irrational numbers x and y, such that x^y is rational?

Question 15. Let $a_1, a_2, \cdots, a_n$ be positive reals such that $a_1 a_2 \cdots a_n = 1$. Then prove that

$$a_1 + a_2 + \cdots + a_n \ge n$$

Question 16. n cars are placed at distinct positions on a closed track. The total amount of fuel in all their tanks combined is exactly equal to the amount of fuel one car would consume to make a round trip off the track (come back to its starting point). Prove that there is a car that can make the round trip by taking fuel from all the other cars that it encounters on this trip.

Question 17. Are there integers $x, y, z, t \in \mathbb{Z}$, such that $8x^4 + 4y^4 + 2z^4 = t^4$?

Question 18. Prove that every graph can be written as an edge-disjoint union of cycles and trees.

Question 19. Can there be a connected graph with more than two vertices, such that the degrees of its vertices are mutually distinct?

Question 20. Prove that either a graph or its complement is connected.

Question 21. Every graph with n vertices and more than $n^2/4$ edges must contain a triangle.

Question 22. State and prove the fundamental theorem of arithmetic. (Hint: Strong induction)

Question 23. Suppose every road in West Bengal is one way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly or via at most one other city. (Hint: Think about graphs or induct on the number of cities)

Question 24. Let x be any real number such that $x + \frac{1}{x} \in \mathbb{Z}$. Prove that

$$x^n + \frac{1}{x^n} \in \mathbb{Z}$$

for any $n \in \mathbb{N}$. (Hint: need some strength in induction)

Question 25. There are n students in each of the three schools. Any student has altogether $n+1$ friends from the other two schools, note that friendship is a symmetric relation. Prove that one can select one student from each school so that three selected students know each other. (Hint: Consider the student with the maximum number of friends)

References

[MN09] Jirí Matousek and Jaroslav Nesetril. *Invitation to Discrete Mathematics (2. ed.)*. Oxford University Press, 2009.

[VLW01] Jacobus Hendricus Van Lint and Richard Michael Wilson. *A Course in Combinatorics*. Cambridge University Press, 2001.

Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
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Problem Set 2

For the following definitions

- *Composition* $R \circ T$ between two relations R and T,
- *Inverse* R^{-1} of a relation R, and
- *Matrix represention* A_R of a relation R

refer to Sections 1.5 and 1.6 from [MN09]. Also, see the following definitions related to partially ordered sets from Chapter 3 from [MN09]

- *Minimal/maximal* element
- *Minimum/maximum* element
- *Supremum/infimum* element
- *Embedding* of one partially ordered set into another partially order set

Question 1. **(a)** Prove that a smallest element, if it exists, is determined uniquely.
(b) Prove that for a linearly ordered set, a minimal element is also the smallest element.
(c) Prove or disprove: If a partially ordered set $(X, \preceq)$ has a single minimal element, then it is a smallest element as well.

Question 2. Prove that a relation R on a set X satisfies $R \circ R^{-1} = \{(x, x) : x \in X\}$ if and only if R is reflexive and antisymmetric.

Question 3. A set C of propositions is called *logically closed* if it satisfies both the following conditions:

1) C contains all *tautologies* (propositions with truth tables containing only T in the output column)
2) (*Modus ponens*) For any propositions p and q, if both p and $p \implies q$ are in C, then so is q.

If C is a logically closed set, then prove the following:

1) If the propositions $p_1, p_2, \cdots, p_n$ are in C, and $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \implies q$ is a tautology, then q is also in C.
2) If both the propositions p and $\neg p$ are in C, then C contains every proposition.

Question 4. Show that for every ordered set $(X, \preceq)$ there exists an embedding into the ordered set $\left(2^X, \subseteq\right)$.

Question 5. Prove the associativity of composing relations: if R, S, T are relations such that $(R \circ S) \circ T$ is well defined, then $R \circ (S \circ T) = (R \circ S) \circ T$.

Question 6. Let k be some fixed positive integer and $P(n)$ be a mathematical statement that satisfies the following properties:

1) All of $P(0), P(1), \cdots, P(k-1)$ are true;
2) $P(n) \implies P(n+k)$, for any $n \geq 0$.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

Question 7. Consider the Fibonacci numbers $\{F_n\}_{n\geq 0}$, defined as $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n, \text{ for all } n \geq 1.$$

Question 8. Suppose a set contains $n \geq a^2 + 1$ elements. Prove that, one of the following statements has to be true:

1) At least $a + 1$ elements are identical.
2) At least $a + 1$ elements are pairwise distinct.

Question 9. **(a)** Prove that for any relation R, the relation $T := R \cup (R \circ R) \cup (R \circ R \circ R) \cup \ldots$ (the union of all multiple compositions of R) is transitive.

(b) Prove that any transitive relation containing R as a subset also contains T.

(c) Prove that if $|X| = n$, then

$$T = R \cup (R \circ R) \cup \ldots \cup \underbrace{(R \circ R \circ \cdots \circ R)}_{(n-1)\ times}.$$

Question 10. Suppose there are $n \geq 4$ spies, each of whom knows a secret not known to the others. They communicate in pairs (simultaneous communication between more than two people is not allowed, as such gatherings among spies might get noticed and reveal their identities) and in each communication, two spies share each other's secrets. Show that $2n-4$ communications are *sufficient* before each of them knows everything.

Question 11. Suppose $\tau(n)$ denotes the number of positive divisors of an integer $n \in \mathbb{N}$. Prove that $n^{\tau(n)}$ is a perfect square for all $n \in \mathbb{N}$.

Question 12. Describe all relations on a set X that are equivalences and also partial orderings at the same time.

Question 13. For every finite ordered set $P = (X, \preceq)$ define

$$\alpha(P) := \max\{|A| \,:\, A \text{ is an independent set in } X\}, \text{ and}$$
$$\omega(P) := \max\{|A| \,:\, A \text{ is a chain in } X\}.$$

Show that

$$\alpha(P) \cdot \omega(P) \geq |X|.$$

Question 14. Let k, ℓ be natural numbers. Then every sequence of real numbers of length $k\ell + 1$ contains a nondecreasing subsequence of length $k + 1$ or a decreasing subsequence of length $\ell + 1$.

Question 15. Let the sets $A_1, \ldots, A_m$ be m distinct subsets of $[n]$. Show that if for $i \neq j$ we have $A_i \cap A_j \neq \emptyset$ then $m \leq 2^{n-1}$.

References

[MN09] Jirí Matousek and Jaroslav Nesetril. *Invitation to Discrete Mathematics, Second Edition*. Oxford University Press, 2009.

Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
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Problem Set 3

Question 1. Show that every tree of n vertices, with $n \geq 2$, has at least two degree one vertices.

Question 2. Prove that a tree on n vertices has exactly $(n-1)$ edges.

Question 3. Prove that if a graph G is connected and has exactly $(n-1)$ edges then G is a tree.

Question 4. Show that every graph on at least two vertices contains two vertices of equal degree.

Question 5. A tree where every vertex has degree 3 or 1 (except the root that has degree 2 and degree zero is the tree is of size 1) is called a *binary tree.* Prove that a binary tree on n vertices has at least $\lfloor n/2 \rfloor + 1$ number of leaves (Hint: induction pivoting at the root).

Question 6. Let $G = (V, E)$ be an undirected graph. Prove that if all its edge weights are distinct, then it has a unique minimum spanning tree.

Question 7. Let $G = (V, E)$ be a connected graph with t many vertices of add degree where $t > 0$. Show that the edge set of G can be written as a disjoint union (in terms of edges) of t many trails.

Question 8. Prove that a regular bipartite graph of degree at least 2 does not contain a bridge.

Question 9. Prove that either G or $\overline{G}$ is connected.

Question 10. Show that every connected graph $G = (V, E)$ on at least two vertices contains two distinct vertices u and v such that both $G[V \setminus \{u\}]$ and $G[V \setminus \{v\}]$ are connected.

Question 11. *A walk is a sequence of vertices and edges of a graph. An Eulerian walk, in an undirected graph is a walk that uses each edge exactly once.*

Prove that, an undirected connected graph has an Eulerian walk if and only if exactly zero or two vertices have odd degree.

Question 12. Prove that there is a tournament T with n players and at least $n!2^{-(n-1)}$ Hamiltonian paths.

Question 13. If G is a graph with n vertices such that any two non-adjacent vertices of G have degree sum at least n, then prove that G contains a Hamiltonian cycle.

Question 14. Prove that, $\mathcal{Q}_n, n \geq 2$, (hypercube graph of dimension n) contains a Hamiltonian cycle.

Question 15. Prove that a graph is bipartite if and only if the graph has no odd cycle.

Question 16. If G is a graph such that all vertices have degree more than 2 then G has a cycle.

Question 17. If G is a graph such that all the vertices have even degree then prove that G can be written as a union of edge-disjoint cycles.

Question 18. Let T, T' be two spanning trees of a connected graph G. For an edge $e \in E(T) \setminus E(T')$, prove that there exists an edge $e' \in E(T') \setminus E(T)$ such that both $T' + e - e'$ and $T + e' - e$ are spanning trees of G.

Question 19. Let $T_1, \dots, T_k$ be subtrees of a tree. Assume that each pair (T_i, T_j) shares a vertex. Then prove that all the $T_i, i \in [k]$ share a common vertex.

Question 20. Show that a graph G is 2-connected if and only if for any three vertices u, v, w in G there exists a path from u to v that also contains w.

Question 21. Let G be a simple graph with n vertices and $(n-2)$ edges. Prove that

(a) Either G has an isolated vertex or G has two components that are nontrivial trees.
(b) G is a subgraph of $\bar{G}$, using (a).

Question 22. Construct a graph $G = (V, E)$ on n vertices satisfying the following two conditions:

- $|E| = \binom{n-1}{2} + 1$, and
- G is not a Hamiltonian graph.

Question 23. Prove that, every n vertex tree other than $K_{1,n-1}$ is contained in its complement.

Question 24. Let S be an n element set and $A_1, \dots, A_n$ be n distinct subsets of S. Prove that, $\exists x \in S$ such that $A_1 \cup \{x\}, \dots, A_n \cup \{x\}$ are distinct.

Question 25. Let G be a connected graph and $e \in E(G)$. Prove that

1) e is a cut-edge if and only if e belongs to every spanning tree.
2) e is a loop if and only if e belongs to no spanning tree.

Question 26. In a party there are $2n$ participants, where n is a natural number. Some participants shake hands with other participants. It is known that there do not exist three participants who have shaken hands with each other. Prove that the total number of handshakes is not more than n^2.

Question 27. A vertex cover in a graph G is a set of vertices that hits every edge. Prove that the complement of a vertex cover is an independent set and vice versa.

Question 28. In a T20 cricket tournament, n teams participate. Each team plays against all the other teams exactly once and each match has a winner (no ties). Prove that, one can order the teams in such a way, say $t_1, \dots, t_n$ such that t_i defeats $t_{i+1}, \forall i \in [n]$.

Question 29. A directed graph is called strongly connected if for all vertices of u, v there is a directed path from u to v and also a directed path from v to u. Prove that every strongly connected tournament with $n \geq 3$ vertices has a directed Hamiltonian cycle.

Question 30. We are given two square sheets of paper with an area 2003. Suppose we divide each of these papers into 2003 polygons, each of area 1. (The divisions for the two piece of papers may be distinct.) Then we place the two sheets of paper directly on top of each other. Show that we can place 2003 pins on the pieces of paper so that all 4006 polygons have been pierced.

Question 31. Consider an 8×8 chessboard with the property that on each column and each row there are exactly n pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column.

Question 32. Define $d = (d_1, d_2, \ldots, d_{2k})$ by $d_{2i-1} = d_{2i} = i,\ \forall i \in [k]$. Prove that d is graphic.

Question 33. Prove that, any graph G contains a *bipartite* subgraph G' such that $V(G) = V(G')$ and $2|E(G')| \geq |E(G)|$.

Question 34. Prove that, every graph G has a matching of size at least $\frac{|V(G)|}{1+\Delta(G)}$, where $\Delta(G)$ is the maximum degree of a vertex in G.

Question 35. Prove or disprove: Every tree has at most one perfect matching.

Question 36. Prove that for every perfect matching M in $\mathcal{Q}_k$ (hypercube graph of dimension k) and every coordinate $i \in [k]$, there are an even number of edges in M whose endpoints differ in coordinate i. Using this, count the number of edges in $\mathcal{Q}_3$.

Question 37. For $k \geq 2$, prove that $\mathcal{Q}_k$ has at least $2^{(2^{k-2})}$ perfect matchings.

Question 38. The *weight* of a vertex in $\mathcal{Q}_k$ is the number of 1s in its label. Prove that for every perfect matching in $\mathcal{Q}_k$, the number of edges matching words of weight i to words of weight $i+1$ is $\binom{k-1}{i}$, for $0 \leq i \leq k-1$.

Question 39. In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

Question 40. Prove that if all the vertices of a bipartite graph have the same degree, then it has a perfect matching.

Question 41. Let $S_1, S_2, \ldots, S_m$ be sets. A transversal is an ordered set $(s_1, s_2, ..., s_m)$ such that each of the s_i's are different and $s_i \in S_i$. Prove that there exists a transversal if and only if the union of any k sets has at least k elements.

Question 42. Prove the equivalence of Konig's theorem and Hall's theorem.

Question 43. Suppose that a regular deck of 52 playing cards has been dealt into 13 piles of 4 cards each. Show that there is a way to select one card from each pile, such that you have one card from every rank (ace, $2, \ldots,$ king).

Question 44. To n people are to be assigned n different houses. Each person ranks the houses in some order (with no ties). After the assignment is made, it is observed that every other assignment assigns at least one person to a house that person ranked lower than in the given assignment. Prove that at least one person received his/her top choice in the given assignment.

Question 45. An $m \times n$ Latin rectangle is an $m \times n$ matrix $M = (m_{ij})$ whose entries are integers and satisfy the following two conditions:

- $1 \leq m_{ij} \leq n$.
- No two entries in any row or in any column are equal. Note that this implies that $m \leq n$.

If $m = n$, then the Latin rectangle is a Latin square. Let M be an $m \times n$ Latin rectangle with $m < n$. Then M can be extended to a Latin square by the addition of $n - m$ new rows.

Question 46. Let $S = \{1, 2, \ldots, kn\}$, and suppose $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are both partitions of S into n sets of size k. Then there exists a set T of size n such that every intersection $T \cap A_i$ and $T \cap B_i$ has cardinality exactly 1.

Question 47. (Gale-Shapley stable marriage protocol) Suppose there are equal numbers of men and women, and each person ranks the opposite group according to preference. A series of rounds ensue. In each round:

1) Every currently-not-engaged man simultaneously proposes to the top choice among those remaining on his list.
2) Then each woman compares her incoming proposals to the person she is currently engaged to. She selects the best of these (possibly dumping her current engagement), and is now engaged to the new person.
3) All rejected men permanently remove the corresponding women from their list.

Show that when this process terminates, everybody has gotten married. Next, show that the resulting marriage is stable.

Question 48. Got a graph G let $\nu(G)$ and $\tau(G)$ denote the size of the maximum matching and the size of the smallest vertex cover in G. Show that $\nu(G) \leq \tau(G) \leq 2\nu(G)$.

Question 49. Using the above result design a 2-factor approximation algorithm for computing vertex cover in a graph.

Question 50. Show that König Theorem and Dilworth's Thoorem are equivalent.

Question 51. Show that in a connected graph any two paths of maximum length share at least one vertex.

Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
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Problem Set 4

Question 1. Use a generating function to model the problem of counting all selections of six objects chosen from three types of objects with repetition of up to four objects of each type. Also model the problem with unlimited repetition.

Question 2. Find a generating function for a_r, the number of ways to distribute r identical objects into five distinct boxes with an even number of objects not exceeding 10 in the first two boxes and between three and five in the other boxes.

Question 3. Use a generating function for modeling the number of 5-combinations of the letters M, A, T, H in which M and A can appear any number of times but T and H appear at most once. Which coefficient in this generating function do we want?

Question 4. Build a generating function for a_r, the number of integer solutions to the following equations:

1) $e_1 + e_2 + e_3 + e_4 + e_5 = r,\ 0 \leq e_i \leq 5.$
2) $e_1 + e_2 + e_3 + e_4 = r,\ 2 \leq e_i \leq 7,\ e_1$ even, e_2 odd.

Question 5.

1) Use a generating function for modeling the number of different election outcomes in an election for class president if 25 students are voting among four candidates. Which coefficient do we want?
2) Suppose each student who is a candidate votes for herself or himself. Now what is the generating function and the required coefficient?
3) Suppose no candidate receives a majority of the vote. Repeat part (1).

Question 6. Find a generating function $g(x, y, z)$ whose coefficient of $x^r y^s z^t$ is the number of ways eight people can each pick two different fruits from a bowl of apples, oranges, and bananas for a total of r apples, s oranges, and t bananas.

Question 7. How many ways are there to get a sum of 25 when 10 distinct dice are rolled?

Question 8. Use generating functions to show that:

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}.$$

Question 9. Use the equation

$$\frac{(1-x^2)^n}{(1-x)^n} = (1+x)^2$$

to show that

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k} \binom{n+m-2k-1}{n-1} = \binom{n}{m},$$

where $m \leq n$ and m is even.

Question 10. If $g(x)$ is the generating function for a_k, then show that

$$\frac{g^{(k)}(0)}{k!} = a_k.$$

Question 11. A probability generating function $P_X(t)$ for a discrete random variable X has a polynomial expansion in which p_r, the coefficient of t^r, is equal to the probability that $X = r$.

1) If X is the number of heads that occur when a fair coin is flipped n times, show that $P_X(t) = \frac{(1+t)^n}{2^n}$.
2) If X is the number of heads that occur when a biased coin is flipped n times with probability p of heads (and $q = 1 - p$), show that $P_X(t) = (q + pt)^n$.
3) If X is the number of times a fair coin is flipped until the fifth head occurs, find $P_X(t)$.
4) Repeat (3) until the m^{th} head occurs and the probability of a head is p.

Question 12. Use Ferrers diagram to show that the number of partitions of an integer r as a sum of m positive integers is equal to the number of partitions of r as a sum of positive integers, the largest of which is m.

Question 13.

1) Show that the number of partitions of 10 into distinct parts (integers) is equal to the number of partitions of 10 into odd parts by listing all partitions of these two types.
2) Show algebraically that the generating function for partitions of r into distinct parts equals the generating function for partitions of r into odd parts, and hence the numbers of these two types of partitions are equal.

Question 14. Show with generating functions that every positive integer has a unique decimal representation.

Question 15. Let $R(r, k)$ denote the number of partitions of the integer r into k parts.

1) Show that $R(r, k) = R(r - 1, k - 1) + R(r - k, k)$.
2) Show that $\sum_{k=1}^{r} R(n - r, k) = R(n, r)$.

Question 16. Use a Ferrers diagram to show that the number of partitions of an integer into parts of even size is equal to the number of partitions into parts such that each part occurs an even number of times.

Question 17. Show that the number of partitions of n is equal to the number of partitions of $2n$ into n parts.

Question 18. Show that the number of partitions of $r + k$ into k parts is equal to:

1) The number of partitions of $r + \binom{k+1}{2}$ into k distinct parts.
2) The number of partitions of r into parts of size $\leq k$.

Question 19.

1) A partition of an integer r is self-conjugate if the Ferrers diagram of the partition is equal to its own transpose. Find a one-to-one correspondence between the self-conjugate partitions of r and the partitions of r into distinct odd parts.
2) The largest square of dots in the upper left-hand corner of a Ferrers diagram is called the Durfee square of the Ferrers diagram. Find a generating function for the number of self-conjugate partitions of r whose Durfee square is size k (that is, a $k \times k$ array of dots). (Hint: Use a one-to-one correspondence between these and the partitions of $r - k^2$ into even parts of size at most $2k$.)

Question 20. Find the exponential generating function for the number of ways to distribute r distinct objects into five different boxes when $b_1 < b_2 \leq 4$, where b_1, b_2 are the numbers of objects in boxes 1 and 2, respectively.

Question 21. Find an exponential generating function for the number of distributions of r distinct objects into n different boxes with exactly m nonempty boxes.

Question 22. Find a generating function with $a_r = r(r+2)$ (do not add together generating functions for r^2 and for $2r$).

Question 23. If $h(x)$ is the generating function for a_r, what is the coefficient of x_r in $h(x)(1-x)$ (give your answer in terms of the a_r's)?

Question 24. If $h(x)$ is the generating function for a_r, find the generating function for $s_r = \sum_{k=r+1}^{\infty} a_k$, assuming all s_r's are finite and $a_r \to 0$ as $r \to \infty$.

Question 25. The Lucas Sequence $1, 3, 4, 7, 11, 18, 29, \ldots$ is defined by $a_1 = 1, a_2 = 3, a_n = a_{n-1} + a_{n-2}$. Give a closed-form expression for a_n. Also, prove that $a_n = O(1.75^n)$.

Question 26. Show that the binomial sum

$$s_n = \binom{n+1}{0} + \binom{n}{1} + \binom{n-2}{2} + \ldots$$

satisfies the Fibonacci relation.

Question 27. Verify the following identities for Fibonacci numbers (F_i is the ith Fibonacci number). Here $F_0 = F_1 = 1$.

a) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$
b) $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$
c) $\sum_{k=0}^{n} F_{2k} = F_{2n+1}$
d) $F_n F_{n+2} = F_{n+1}^2 + (-1)^n$
e) $F_1 - F_2 + F_3 - \cdots - F_{2n} = F_{2n-1}$

Question 28. Show that if F_n is the nth Fibonacci number in the Fibonacci sequence starting $F_0 = F_1 = 1$, then

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2}$$

Question 29. A ternary sequence is a sequence composed of 0s, 1s and 2s. Find a recurrence relation for a_n, the number of n-digit ternary sequences without any occurrence of the subsequence "012".

Question 30. Each day Angela eats lunch at a deli, ordering one of the following: chicken salad, a tuna sandwich, or a turkey wrap. Find a recurrence relation for the number of ways for her to order lunch for the n days if she never orders chicken salad three days in a row.

Question 31. Find a recurrence relation for the number of ways to divide an n-gon into triangles with noncrossing diagonals.

Question 32. Find a system of recurrence relations for computing the number of n-digit binary sequences with an even number of 0s and an even number of 1s.

Question 33. Find a system of recurrence relations for computing the number of n-digit binary sequences with exactly one pair of consecutive 0s.

Question 34. Find a recurrence relation for a_n, the number of ways to place parentheses to multiply the n numbers $k_1 \times k_2 \times k_3 \times k_4 \times \cdots \times k_n$ on a calculator. there is only one way to calculate $k_1 \times k_2$, so $a_2 = 1$. there are two ways to calculate $k_1 \times k_2 \times k_3$, which are $(k_1 \times k_2) \times k_3$ and $k_1 \times (k_2 \times k_3)$, so $a_3 = 2$. assume $a_1 = 1$.

Question 35. Solve the following recurrence relations:

a) $a_n = 3a_{n-1} + 4a_{n-2}$, $a_0 = a_1 = 1$
b) $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$, $a_0 = a_1 = 1$, $a_2 = 2$

Question 36. If the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has a general solution $a_n = A_1 3^n + A_2 6^n$, find c_1 and c_2.

Question 37. Solve the following recurrence relations:

a) $a_n = a_{n-1} + n(n-1)$, $a_0 = 3$
b) $a_n = 2a_{n-1} + (-1)^n$, $a_0 = 2$

Question 38. Find and solve a recurrence relation for the number of different square subboards of any size that can be drawn on an $n \times n$ chessboard.

Question 39. Solve the following recurrence relation $a_n = -na_{n-1} + n!$, $a_0 = 1$.

Question 40. Solve the recurrence relations using generating functions.

a) $a_n = a_{n-1} + 2$, $a_0 = 1$
b) $a_n = 3a_{n-1} - 2a_{n-2} + 2$, $a_0 = a_1 = 1$

Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
Date: October 18, 2023

Problem Set 5

Refer to the main textbooks for this course [VLW01, MN09, Rei12, Mat02] for the definitions.

Question 1. Consider the graph $G = ([n], E)$. Formulate the following optimization problems from graph theory as Integer Linear Programs (ILP):

a. Compute an *independent set* in G of the largest size.
b. Compute the *smallest vertex* cover in G.
c. Compute the smallest *dominating set* in G.
d. Compute the largest size *matching* in G.
e. Assuming G is connected, compute a *spanning tree* of G.

Question 2. (Hitting Set Problem) Let $S_1, \ldots, S_m$ be subsets of $[n]$. Write an IPL to compute a subset $S \subseteq [n]$ of the smallest size that satisfies the following: for all $i \in [m]$,

$$S_i \cap S \neq \emptyset.$$

Question 3. (Set Cover Problem) Let $\mathcal{N} = \{S1, \ldots, S_m\}$ be subsets of $[n]$. Write an IPL to compute a subset $\mathcal{M} \subseteq \mathcal{N}$ of the smallest size that satisfies

$$\bigcup_{S_i \in \mathcal{M}} S_j = [n].$$

Question 4. (Largest Ball in a Polyhedron) Let $P = \bigcap_{i=1}^{m} H_i$ where H_i's are halfspaces in $\mathbb{R}^n$. Write a linear program (LP) for finding the radius of the largest ball contained in P.

Question 5. (Existence of Center Point) For a set S of n points in $\mathbb{R}^d$, we say $c \in \mathbb{R}^d$ is a *center point* of S if all halfspaces containing c also contains at least $\frac{n}{d+1}$ points of S. Using Helly Theorem show the existence of a center point for all finite collections of points in $\mathbb{R}^d$.

Question 6. (Minkowski Sum) Let A and B be convex sets in $\mathbb{R}^n$, and define

$$C := \{a + b \mid a \in A, \text{and}\, b \in B\}.$$

Show that C is also a convex set.

Question 7. Let $K \subseteq \mathbb{R}^n$ be a convex set and $S = \{C_1, \ldots, C_m\}$ be a collection of $m \geq n+1$ convex sets in $\mathbb{R}^n$. If for every subcollection $C_{i_1}, \ldots, C_{i_{n+1}}$ of size $n+1$ of S there exists a translate of K that intersects all of them then there exists a translate of K that intersects all the sets in S.

Question 8. Let S be a finite collection of segments in $\mathbb{R}^2$, such that any three segments of S can be intersected by a single line. Show that there exists a line segment that intersects all the line segments in S.

Question 9. Suppose there are n lines in $\mathbb{R}^2$ such that every three of them can be intersected with a unit circle. Prove that all of them can be intersected with a unit circle.

Question 10. Let $P_1, \ldots, P_n \subset \mathbb{R}^2$ be rectangles with sides parallel to the coordinate axes, such that every two rectangles from the above collection intersect. Show that all rectangles have a nonempty intersection, that is,

$$\bigcap_{i=1}^{n} P_n \neq \emptyset.$$

Question 11. Let $P \in \mathbb{R}^2$ be a finite set of points, not all on the same line. Then there exists a line containing exactly two points in P. (Remark: This important result is called Sylvester–Gallai Theorem.)

Question 12. What happens to the conclusion of the above theorem if P is not a finite set?

Question 13. Let L be a finite set of lines in the plane, not all going through the same point. Prove that there exists a point $p \in \mathbb{R}^2$ contained in exactly two lines.

Question 14. Let $P_1, \ldots, P_N \subset R^n$ be any convex sets, and $z \in \operatorname{conv}(P_1 \cup \cdots \cup P_n)$. Then there exists a subset $I \subset [N]$, $|I| = n+1$, such that $z \in \operatorname{conv}(\bigcup_{i \in I} P_i)$.

References

[Mat02] Jirí Matousek. *Lectures on Discrete Geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer, 2002.

[MN09] Jirí Matousek and Jaroslav Nesetril. *Invitation to Discrete Mathematics, Second Edition*. Oxford University Press, 2009.

[Rei12] Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[VLW01] Jacobus Hendricus Van Lint and Richard Michael Wilson. *A Course in Combinatorics*. Cambridge University Press, 2001.

Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
Date: November 26, 2023

Problem Set 6

Refer to the main textbooks for this course [VLW01, MN09, Rei12, Mat02] for the definitions.

Question 1. Let $\mathcal{M}$ be a collection of subsets of a finite set X such that for all $S \in \mathcal{M}$ we have $|S| = k$. We say that $\mathcal{M}$ is 2-colorable if each point of X can be colored either *red* or *blue* in such a way that no set of $\mathcal{M}$ has all points red or all points white. Show that if $|\mathcal{M}| < 2^{k-1}$ then $\mathcal{M}$ is 2-colorable.

Question 2. Let $A_1, \ldots, A_n$ be n events in a finite probability space. Show that

$$\mathbb{P}[A_1 \cup \cdots \cup A_n] = \sum_{1 \le i \le n} \mathbb{P}[A_i] - \sum_{1 \le i < j \le n} \mathbb{P}[A_i \cap A_j] + \cdots + (-1)^{n+1} \mathbb{P}[A_1 \cap \cdots \cap A_n].$$

Question 3. Let $A_1, \ldots, A_n$ be n events in a finite probability space, and k be an odd natural number with $1 \le k \le n$. Show that

$$\mathbb{P}[A_1 \cup \cdots \cup A_n] \le \sum_{1 \le i \le n} \mathbb{P}[A_i] - \sum_{1 \le i < j \le n} \mathbb{P}[A_i \cap A_j] + \cdots + (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbb{P}[A_{i_1} \cap \cdots \cap A_{i_k}].$$

Question 4. Let $A_1, \ldots, A_n$ be n events in a finite probability space, and k be an even natural number with $1 \le k \le n$. Show that

$$\mathbb{P}[A_1 \cup \cdots \cup A_n] \ge \sum_{1 \le i \le n} \mathbb{P}[A_i] - \sum_{1 \le i < j \le n} \mathbb{P}[A_i \cap A_j] + \cdots + (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbb{P}[A_{i_1} \cap \cdots \cap A_{i_k}].$$

Question 5. Show that if A, B are independent events in some finite probability space then show that their complements, $\bar{A}$ and $\bar{B}$, are also independent.

Question 6. Let $G \in \mathcal{G}(n, 1/2)$ then show that

$$\lim_{n \to \infty} \mathbb{P}[G \text{ is connected}] = 1.$$

Question 7. Let $(\Omega, \mathbb{P})$ be a finite probability space. Suppose that n independent events $A_1, A_2, \ldots, A_n \subseteq \Omega$ exist such that $0 < \mathbb{P}[A_i] < 1$ for each $i \in [n]$. Show that $|\Omega| \ge 2^n$.

Question 8. Let $\pi : [n] \to [n]$ be a uniformly random permutation on $[n]$. Show that for any k distinct number $i_1, i_2, \ldots, i_k \in [n]$ we have

$$\mathbb{P}[\pi(i_1) < \min \{\pi(i_j) \ : \ 2 \le j \le k\}] = \frac{1}{k}.$$

Question 9. Let $G = ([n], E)$ be a connected graph with $|E| = m$, and we put a weight on the edges of G independently and uniform at random from the interval $[2m^2]$. Then the probability that the weighted graph G has a unique Minimum Spanning Tree is > 0.

Question 10. Let $\mathcal{F}$ be a collection of subsets of $[n]$, and for any two distinct sets A and B in $\mathcal{F}$ we have $A \nsubseteq B$. Using random permutation show that

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1.$$

References

[Mat02] Jirí Matousek. *Lectures on Discrete Geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer, 2002.

[MN09] Jirí Matousek and Jaroslav Nesetril. *Invitation to Discrete Mathematics, Second Edition*. Oxford University Press, 2009.

[Rei12] Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[VLW01] Jacobus Hendricus Van Lint and Richard Michael Wilson. *A Course in Combinatorics*. Cambridge University Press, 2001.

Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
Date: November 29, 2023

Problem Set 7

Refer to the main textbooks for this course [VLW01, MN09, Rei12, Mat02] for the definitions.

Question 1. Let $G = ([n], E)$ be a planar graph. Show that any crossing-free drawing of G has at most $2n - 4$ faces.

Question 2. Let $G = ([n], E)$ be a triangle-free planar graph. Show that any crossing-free drawing of G has at most $n - 2$ faces.

Question 3. Find a planar graph all of whose vertices have degree 5.

Question 4. Show that every planar graph can be colored with 6 colors.

Question 5. Let G be a planar Eulerian graph, and consider a crossing-free planar drawing of G. Show that there exists a closed Eulerian tour that never crosses itself in the considered drawing (it may touch itself at vertices but it never "crosses over to the other side").

Question 6. Show that the graph $K_{3,3}$ is not planar using the Jordan Curve Theorem.

Question 7. Let G be a planar graph. Suppose there exists a planar drawing of G such that every face of G in the drawing is a region of a cycle of G then show that G is 2-connected.

Question 8. Let $G = ([n], E)$ be a graph with $|E| \geq 4n$. Show that every planar drawing of G has at least $\Omega\left(m^3/n^2\right)$ crossings.

Question 9. Show that every planar graph can be colored with 5 colors.

Question 10. For all $n \geq 5$, any planar drawing of K_n will have at least $\frac{1}{5}\binom{n}{4}$ many crossings.

Question 11. Let $G = ([n], E)$ be a triangle-free planar graph. Show that $|E| \leq 2n - 4$.

Question 12. Let L be a set of m lines in $\mathbb{R}^2$ and P be a set of n lines in $\mathbb{R}^2$. Show that

$$|\mathcal{I}(L, P)| = O\left((mn)^{2/3} + m + n\right),$$

where $\mathcal{I}(L, P) := \{(p, \ell) \mid p \in P, \ \ell \in L, \text{ and } p \in \ell\}$.

References

[Mat02] Jirí Matousek. *Lectures on Discrete Geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer, 2002.

[MN09] Jirí Matousek and Jaroslav Nesetril. *Invitation to Discrete Mathematics, Second Edition*. Oxford University Press, 2009.

[Rei12] Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[VLW01] Jacobus Hendricus Van Lint and Richard Michael Wilson. *A Course in Combinatorics*. Cambridge University Press, 2001.

Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
Date: October 9, 2023

Assignment 1

This assignment has to be submitted by October 20, 2023. Details of the submission procedure will be given in the next lecture.

Question 1. Let $A_1, \ldots, A_n$ be n distinct subsets of $[n]$. Prove that $\exists x \in [n]$ such that $A_1 \cup \{x\}, \ldots, A_n \cup \{x\}$ are distinct. [10]

Question 2. Prove the equivalence of Konig's theorem and Hall's theorem. [10]

Question 3. Let k, ℓ be natural numbers. Then every sequence of real numbers of length $k\ell + 1$ contains a nondecreasing subsequence of length $k + 1$ or a decreasing subsequence of length $\ell + 1$. [10]

Question 4. Let $T_1, \ldots, T_k$ be subtrees of a tree. Assume that each pair (T_i, T_j) shares a vertex. Then prove that all the $T_i, i \in [k]$ share a common vertex. [10]

Instructor: Arijit Ghosh
Course: Discrete Mathematics (M.Tech CS, 2023-24)
Date: November 26, 2023

Assignment 2

This assignment has to be submitted by December 10, 2023.

Question 1. Let $A_1, \ldots, A_n$ be n events in a discrete probability space. Show that

$$\mathbb{P}\left[A_1 \cup \cdots \cup A_n\right] = \sum_{1 \leq i \leq n} \mathbb{P}\left[A_i\right] - \sum_{1 \leq i < j \leq n} \mathbb{P}\left[A_i \cap A_j\right] + \cdots + (-1)^{n+1} \mathbb{P}\left[A_1 \cap \cdots \cap A_n\right].$$

[10]

Question 2. Let L and P denote the set of m lines and n points in $\mathbb{R}^2$. Using *crossing lemma* show that

$$\left|\mathcal{I}\left(L, P\right)\right| = O\left((mn)^{2/3} + m + n\right),$$

where $\mathcal{I}\left(L, P\right) := \left\{(p, \ell) \in P \times L \ : \ p \in P,\ \ell \in L, \text{ and } p \in \ell\right\}$. [10]

INDIAN STATISTICAL INSTITUTE

Mid Semester Examination

M.Tech CS, 2023-2024 (Semester – I)

Discrete Mathematics

Date: 22 September 2023 | Maximum Marks: 60 | Duration: 2 hours

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General comment. Answer as much as you can, but the maximum you can score from both Group-A and Group-B is 30.

Notations and definitions. $\mathbb{N}$ denotes the set of *natural numbers*. Given any set X, 2^X denotes the collection of all subsets of X.

Formally, given two partially ordered sets (posets) $(S, \leq)$ and $(T, \preceq)$ a function $f : S \to T$ is an *embedding* if for all x and y in S

$$x \leq y \text{ if and only if } f(x) \preceq f(y)$$

A $n \times n$ matrix $M = (M_{ij})_{1 \leq i,j \leq n}$ is a *permutation matrix* if $M_{ij} \in \{0, 1\}$, and every entry row and column of M contains exactly one 1.

Group-A

(AQ1) Show the equivalence between equivalence relations on a set Ω and different partitions of Ω. [10]

(AQ2) (a) Show that every connected graph contains a spanning tree.

(b) Let G be a connected graph on n vertices. If the number of edges of G is $n - 1$ then G is a tree.

[5+5 = 10]

(AQ3) Let $(X, \leq)$ be a poset where X is a finite set. If the length of any chain in X is less than $m + 1$ then X can be written as a union of m antichains. [10]

(AQ4) Let $G = (A \cup B, E)$ be a bipartite graph. Show that the size of the minimum vertex cover of G is equal to the size of the largest matching in G. (Can use Dilworth's Theorem for posets.) [10]

Group-B

(BQ1) Let G be a graph with n vertices, m edges, and T triangles. Show that

$$T \geq \frac{m}{3n}(4m - n^2).$$

[10]

(BQ2) Let $A = (A_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix with $A_{ij} \in \{0, 1\}$, for all $1 \leq i, j \leq n$. If the sum of entries of any row or column of A is k then A can be written as a sum of k many permutation matrices.

[10]

(BQ3) Let $A_1, \ldots, A_n$ be distinct subsets of $\mathbb{N}$. Show that there exists a subset X of $\mathbb{N}$ of size at most $n - 1$ such that for all $i \neq j$, we have $A_i \cap X \neq A_j \cap X$. [10]

(BQ4) Show that for every ordered set $(X, \preceq)$ there exists an embedding into the ordered set $(2^X, \subseteq)$.

poset

[10]

INDIAN STATISTICAL INSTITUTE

End Semester Examination

M.Tech CS, 2023-2024 (Semester – I)

Discrete Mathematics

Date: 4 December 2023 Maximum Marks: 100 Duration: 3 hours

General comment. Answer as much as you can, but the maximum you can score in Group-A and Group-B is 60 and 40 respectively.

Notations and definitions. $\mathbb{N}$ and $\mathbb{R}$ denote the set of *natural numbers* and *real numbers* respectively. Given any set X, 2^X denotes the collection of all subsets of X.

For all $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$.

For a graph $G = (V, E)$, we say

- $S \subseteq V$ is an *independent set* if for all $e \in E$ we have $e \not\subseteq S$, and
- $U \subseteq V$ is a *vertex cover* if for all $e \in E$ we have $e \cap U \neq \emptyset$.

Group-A

(AQ1) Let L be a set of n lines in $\mathbb{R}^2$ and P be a set of m points in $\mathbb{R}^2$. Define

$$\mathcal{I}(P, L) := \{(p, \ell) \ : \ p \in P, \ \ell \in L, \text{and}\, p \in \ell\}.$$

Show that

(a) $\mid \mathcal{I}(P, L) \mid = O\left((mn)^{2/3} + m + n\right)$

(b) For all $n \in \mathbb{N}$ there exists a set of n points P' and n lines L' with

$$|\mathcal{I}(P, L)| = \Omega\left(n^{4/3}\right).$$

[10+5 = 15]

(AQ2) Let $C_1, \ldots, C_n$ be n convex sets in $\mathbb{R}^d$ with $n \geq d + 1$. Show that if any $d + 1$ convex sets from the above collection have a nonempty intersection then the whole collection have a nonempty intersection. [15]

(AQ3) Let $(\Omega, \leq)$ be a partially ordered finite set. Show that the minimum number m of disjoint chains which together contain all elements of Ω is equal to the maximum number M of elements in an antichain of $(\Omega, \leq)$. [15]

(AQ4) Write Integer Linear Programs for the following two problems:

(a) Compute the size of the largest independent set in $G = ([n], E)$.

(b) Compute the size of the smallest vertex cover in the graph $G = ([n], E)$.

[7+8 =15]

(AQ5) Let $\mathcal{G}(n)$ be the number of graphs with the vertex set $[n]$, and assume that $\mathcal{B}(n)$ denotes the number of bipartite graphs with the vertex set $[n]$. Show that

$$\lim_{n \to \infty} \frac{\mathcal{B}(n)}{\mathcal{G}(n)} = 0.$$

[15]

Group-B

(BQ1) Let G be a triangle-free planar graph on n vertices. Show that G can be colored with 4 colors. [10]

(BQ2) Let $G := ([n], E)$ be a weighted connected graph, and the weight function $\omega : E \to \mathbb{R}^{\geq 0}$ satisfies the following: for any pair of distinct edges e_1 and e_2 in E we have $\omega(e_1) \neq \omega(e_2)$. Show that minimum spanning tree (MST) of G is unique.

[10]

(BQ3) Let $A_1, \ldots, A_n$ be n events in a discrete probability space. Using indicator random variables show that

$$\mathbb{P}\left[\bigcup_{1\leq i\leq n} A_i\right] = \sum_{1\leq i\leq n} \mathbb{P}[A_i] - \sum_{1\leq i<j\leq n} \mathbb{P}[A_i \cap A_j] + \sum_{1\leq i<j<k\leq n} \mathbb{P}[A_i \cap A_j \cap A_k] \ldots$$

$$\cdots + (-1)^{n+1}\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n]$$

[10]

(BQ4) Use the equation

$$\frac{(1-x^2)^n}{(1-x)^n} = (1+x)^2$$

to show that

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{n}{k} \binom{n+m-2k-1}{n-1} = \binom{n}{m},$$

where $m \leq n$ and m is even.

[10]

(BQ5) For a set S of n points in $\mathbb{R}^d$, we say $c \in \mathbb{R}^d$ is a *center point* of S if all halfspaces containing c also contains at least $\frac{n}{d+1}$ points of S. Show the existence of a center point for all finite collections of points in $\mathbb{R}^d$.

[10]

INDIAN STATISTICAL INSTITUTE

Comprehensive Examination

M.Tech CS, 2023-2024 (Semester – I)

Discrete Mathematics

Date: 29 July 2024 | Maximum Marks: 100 | Duration: 3 hours

General comment. Answer as much as you can, but the maximum you can score in Group-A and Group-B is 60 and 40 respectively.

Notations and definitions. $\mathbb{N}$ and $\mathbb{R}$ denote the set of *natural numbers* and *real numbers* respectively. Given any set X, 2^X denotes the collection of all subsets of X.

For all $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$.

For a graph $G = (V, E)$, we say

- $S \subseteq V$ is an *independent set* if for all $e \in E$ we have $e \not\subseteq S$, and
- $U \subseteq V$ is a *vertex cover* if for all $e \in E$ we have $e \cap U \neq \emptyset$.

Group-A

(AQ1) For a set S of n points in $\mathbb{R}^d$, we say $c \in \mathbb{R}^d$ is a *center point* of S if all halfspaces containing c also contains at least $\frac{n}{d+1}$ points of S. Show the existence of a center point for all finite collections of points in $\mathbb{R}^d$. [15]

(AQ2) Let $(\Omega, \leq)$ be a partially ordered finite set. Show that the minimum number m of disjoint chains which together contain all elements of Ω is equal to the maximum number M of elements in an antichain of $(\Omega, \leq)$. [15]

(AQ3) Let $A_1, \ldots, A_n$ be n events in a discrete probability space. Using indicator random variables show that

$$\mathbb{P}\left[\bigcup_{1\leq i\leq n} A_i\right] = \sum_{1\leq i\leq n} \mathbb{P}[A_i] - \sum_{1\leq i<j\leq n} \mathbb{P}[A_i \cap A_j] + \sum_{1\leq i<j<k\leq n} \mathbb{P}[A_i \cap A_j \cap A_k] \ldots$$

$$\cdots + (-1)^{n+1}\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n]$$

[15]

(AQ4) Every graph on n vertices with more than $n^2/4$ edges must contain a triangle. [15]

(AQ5) A graph G is called 2-connected if it has at least 3 vertices, and by deleting any single vertex we obtain a connected graph. Show that a graph G is 2-connected if and only if there exists, for any two vertices of G, a cycle in G containing these two vertices. [15]

Group-B

(BQ1) Let $A_1, \ldots, A_n$ be n distinct subsets of $[n]$. Prove that $\exists x \in [n]$ such that $A_1 \cup \{x\}, \ldots, A_n \cup \{x\}$ are distinct. [10]

(BQ2) Let $G \in \mathcal{G}(n, 1/2)$ then show that

$$\lim_{n\to\infty} \mathbb{P}[G \text{ is connected}] = 1.$$

[10]

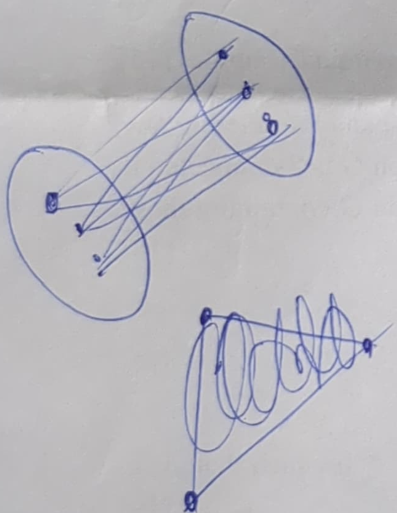
(BQ3) Let $\mathcal{F}$ be a collection of subsets of $[n]$, and for any two distinct sets A and B in $\mathcal{F}$ we have $A \not\subseteq B$. Show that

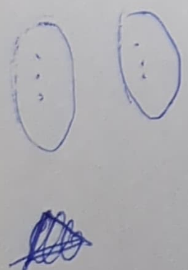
$$\sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1.$$

[10]

(BQ4) Show that the number of partitions of n is equal to the number of partitions of $2n$ into n parts. [10]

(BQ5) Show that the graph $K_{3,3}$ is not planar using the Jordan Curve Theorem. [10]





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