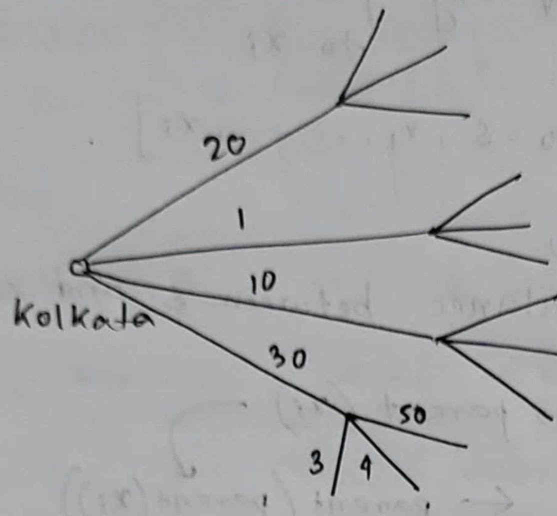


Weighted graphs but +ve weights single source shortest path.

⇒ Dijkstra's Algorithm (G, s, l)

Input: $G = (V, E)$

positive edge lengths $\rightarrow l(u,v) \quad \forall (u,v) \in E$

source vertex; $s \in V$, $|V| = n+1$

Output: For all vertices u reachable from s,

dist[u] = distance of u from S in G.
(shortest distance).

Data Structure : Priority Queue implemented using heaps.

Algo for all $v \in V$:
  dist[V] = ∞
  parent[V] = nil
  dist[s] = 0
  R = { } # known region
  while R ≠ V :
    pick the vertex $v \notin R$, with the smallest dist().
    Add v to R
    for all edges $(v, z) \in E$:

if $dist(z) > dist(v) + l[v,z]$;

$dist[z] = dist[v] + l[v,z]$

$parent[z] = v$.

Termination :→

The while loop runs for atmost $n+1$ iterations.

why? Each time in the while loop, one vertex $v \in V \setminus R$ is added to R and is stopped when $R = V$.

Lemma [Monotonicity Lemma]

Let the order in which vertices are added to R be $v_0, v_1, \ldots, v_n$ where $v_0 = s$, then

$$dist[v_0] \leq dist[v_1] \leq \ldots \leq dist[v_n].$$

where $dist[v]$ is the distance value at the end of the algorithm.

Proof by contradiction :→ Let v_{i+1} be the first vertex for which this is violated,

$$dist[v_{i+1}] < dist[v_i]$$

By our def^n, v_i was added to R first.

when v_i was added; $dist[v_i] \leq dist[v_{i+1}]$ — 1

Case-1 Say $(v_i, v_{i+1}) \notin E$

That is there doesn't exist an edge between v_i, v_{i+1}

So, from v_i, at next iteration $dist[v_{i+1}]$ is not updated.

$\therefore dist[v_i] \leq dist[v_{i+1}]$ (will hold).

[From 1]

Case-2 Let $(v_i, v_{i+1}) \in E$,

$$dist[v_{i+1}] = dist[v_i] + l[v_{i+1}, v_i]$$

$$\Rightarrow dist[v_{i+1}] \geq dist[v_i] \quad [\because \geq 0]$$

- $dist[u]$ is the distance computed when u is added to R.

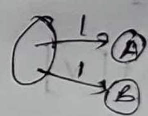
" Using next Lemma this Lemma can be proved ".

Lemma :→ Once a vertex enters R, its distance label and parent is unchanged.

Fix some vertex $x \in R$

Let u be the first vertex added to R after 'x' which can potentially change $dist[x]$.

This means that $(u, x) \in E$.



If $dist_i[x]$ is changed then,

$$dist_i[x] > dist_j[u] + \underbrace{l[u, x]}_{\geq 0}$$

This is contradiction.

- x was added before u to R
- When x was added;

$$dist[x] \leq dist[u]$$

Corollary :→ $x \neq s$

" Parent at the end of algorithm "

[Not parent at each step].

$$dist[x] = dist[parent[x]] + l[parent[x], x]$$

Lemma-3 (Shortest Path Lemma) :→

Let $P = (x_0 = s, x_1, \ldots, x_k)$ be any path,

Let p_i be the subpath $(x_0, \ldots, x_i)$

$$dist[x_i] \leq l[P_i]$$

Proof by contradiction :-

Lets take a path p and x_i be the first vertex in P, for which this is violated ;

$$dist[x_i] > l[P_i] \quad \text{—(1)} \quad ; \quad dist[x_{i-1}] \leq l[P_{i-1}]$$

$$dist[x_{i-1}] \leq l[P_{i-1}] = l[P_i] - l[x_{i-1}, x_i] \leq l[P_i] < dist[x_i]$$

$\Downarrow$ [from (1)]

$$\Rightarrow dist[x_{i-1}] < dist[x_i]$$

$\Rightarrow$ x_{i-1} went into R first

$$dist[x_i] = \min \left\{ dist[x_i], \underbrace{dist[x_{i-1}] + l[x_{i-1}, x_i]}_{} \right.$$

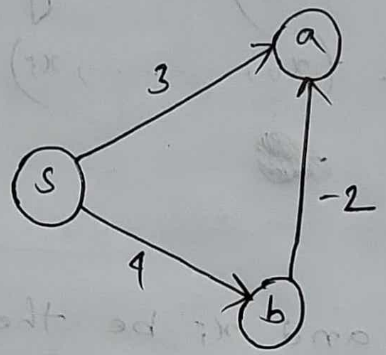
$$\leq l[P_{i-1}] + l[x_{i-1}, x_i]$$

$$= l[P_i]$$

$\therefore$ $dist[x_i] \leq l[P_i]$

$\therefore$ Contradiction.

Bellman Ford ⇒



update(z)

$$dist(z) = \min\{dist(z),\ dist[u] + l[u,z]\}$$

(I) dist(z) is shortest ~~path~~ to s, if u is the last node in the shortest path to z and dist[u] is correctly set.

(II) More updates don't harm you.

Bellman-ford (G, l, s)

for all $v \in V$

  dist(v) = ∞

  parent(v) = nil

dist(s) = 0

repeat |V| - 1 times:

  for all $(u,v) \in E$:

    $dist[v] = \min\{dist[v],\ dist[u] + l[u,v]\}$

for -ve weight cycles

for all $(u,v) \in E$:

  if $dist[v] > dist[u] + l[u,v]$

    return "Negative cycle"

Proof of correctness

(i) Bellman-Ford detects negative cycles reachable from S.

(ii) If there are no negative cycles, dist at the end of the algo is the true shortest distance.

$dist_i[u] \rightarrow$ distance estimated for u from s at the end of i^{th} iteration

Proof for (i) $\exists$ neg cycle, $\exists v$ s.t.

$$dist_n[v] > dist_n[u] + l[u,v]$$

for some $(u,v) \in E$

Neg cycle C

$$v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k \rightarrow v_0 \Rightarrow \sum_{i=0}^{k-1} l[v_i, v_{i+1}] < 0$$

By contradiction Assume that $\forall v$

$$dist_n[v] \le dist_n[u] + l[u,v]$$

$$\Rightarrow \underbrace{\sum_{i=1}^{k} dist_n[v] \le \sum_{i=1}^{k} dist_n[v_{i-1}]}_{\text{Both are same}} + \underbrace{\sum_{i=1}^{k} l[v_{i-1}, v_i]}_{\ge 0}$$

all v_{i-1} are connected to v_i

$$0 \le \sum_{i=1}^{k} l[v_{i-1}, v_i] \quad \text{(Contradiction).}$$

Proof for (ii)

$dist_k[v]$ = Shortest dist. from s to v using $\le k$ edges

atmost n-1 edges

shortest path

v_0 v_n

No, neg cycles

Base-case $k = 0$

$dist_0[s] = 0$ $dist_0[v] = \infty$

$\because$ All other vertices are not connected at $k=0$. Trivially True.

Induction hypothesis

$dist[k-1] = l[Q]$

at the k^{th} iteration.

Also study
DAG,
Topological Sort

$$dist_k[V] = \min \left\{ \begin{array}{c} dist_{k-1}[V] \\ \geq l[P] \end{array}, \underbrace{dist_{k-1}[V] + l[u,v]}_{l(P)} \right\}$$

$dist_k[V] \leq l[P]$