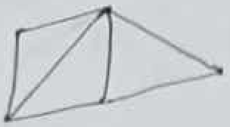
Hall's Theorem →



For a giun Graph, $G = (V, E)$ and $S \subseteq V(G)$, we define -

$q(G)$ = no. of odd components in G.

Lemma

If G is a perfect matching, then
$$q(G \setminus S) \leq |S| \quad \forall \; S \subseteq V(G)$$

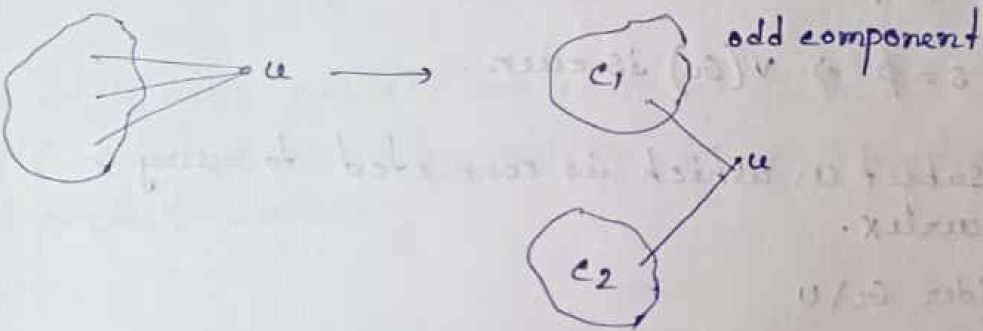
Proof G has a perfect matching

Let, $G - u$ has more than two odd components.

Remoning one vertex u, make two odd component.

Then total verties odd + odd + u

it always odd. hence not a perfect matching.



Now, G/S has more than S odd component.

No. of components = $S+1$

according to pigeon hole principal, There must be two component adjecent to one particular vertex, say V.

Then, we can consider only one. Hence, it won't be perfect matching.

So, $q(G \setminus S) \leq |S| \quad \forall \; S \subseteq V(G)$.

Theorem (tutte's Theorem) → G has a perfect matching iff $q(G \setminus S) \leq |S| \ \forall \ S \subseteq V(G)$.

Proof

(i) First part is done.

(ii) $q(G \setminus S) \leq |S| \ \forall \ S \subseteq V(G) \longrightarrow G$ is a perfect matching.

(By contradiction) G doesn't have perfect matching. and G is edge maximal in perfect matching i.e. $G' = G \cup \{e\}$ has a perfect matching.

$q(G' \setminus S) \leq q(G \setminus S) \quad \forall S$

$\rightarrow G' = G \cup \{e\} \quad V(G) = V(G')$

G' has a perfect matching

$q(G' \setminus S) \leq |S| \quad \forall S$

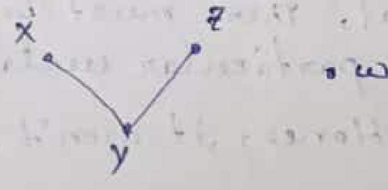
take $S = \phi \Rightarrow V(G)$ is even.

consider a subset U, which is connected to every other vertex.

Now, consider $G \setminus U$

case (i) – $G \setminus U$ is a disjoint union of cliques.

case (ii) – $G \setminus U$ is not a disjoint union of cliques.



$G \cup \{e\}$ is a perfect matching

$M_1 = G \cup (x, z)$

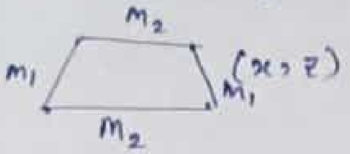
$M_2 = G \cup (y, w)$

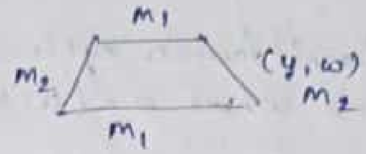
$F = (V, E') \quad E' = M_1 \Delta M_2$

F has degree either 0 or 2.

so, F is a union of disjoint ~~cliques~~ cycles or isolated vertices.

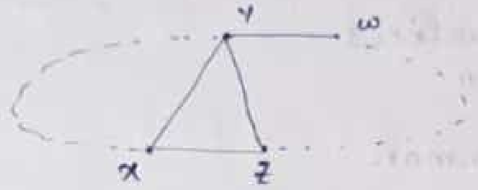
cycles containing (x, z) or (y, ω)





remove M_2,   remove M_1.

both (x, z) and (y, ω) are in a cycle.



Theorem (~~Ba~~ tutte's Theorem) $\longrightarrow$ the largest matching in a n-vertex graph has size

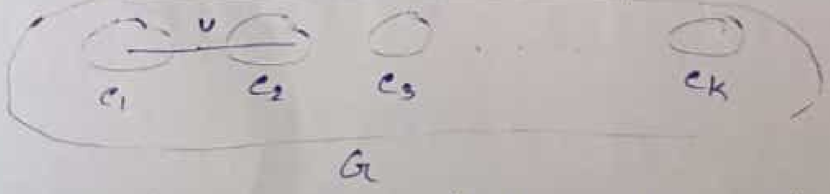
$$\frac{1}{2}\left[n + \min_{S \subseteq V} (|S| - q(G \setminus S))\right]$$

Hint $\rightarrow$ G has a perfect matching of size K iff $G^{(k)}$ obtained by adding $(n-2K)$ vertices have perfect matching.

Hamiltonian cycle $\rightarrow$ A ~~edge~~ cycle containing all the vertices of the graph.

Theorem G is a hamiltonian then for any set $S \subseteq V$ the graph $G \setminus S$ has atmost $|S|$ connected components.

Pf let, $G \setminus S$ has K connected components.



Going from one component to another, should pass through a vertex.

Thm If ~~Graph~~ G has $n \geq 3$ if degree $(v) \geq n/2$ then G is hamiltonian.

Pf G doesn't have a hamiltonian cycle but $G' = G \cup \{e\}$ does where $e = (u, v)$.

u connected to atleast $n/2$ vertices
v " " " $n/2$ "

There must be a vertex in common.
So, hamitonial cycle.

Thm if $\deg(u) + dg(v) \geq n$ then graph is hamiltonian