

Linear Programs & Duality

Primal (LP₁)

$$\begin{aligned} \min & \langle c, x \rangle \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual (DP₁)

$$\begin{aligned} \max & \langle b, y \rangle \\ \text{s.t.} & A^T y \leq c \end{aligned}$$

Weak Duality: Let, x_0 & y_0 be feasible sol'n to (LP₁) & (DP₁). Then $\langle c, x_0 \rangle \geq \langle b, y_0 \rangle$
This implies $OPT_{LP_1} \geq OPT_{DP_1}$

Mathematical
Programs

Lagrangian Dual
of MP.

Weak duality always
hold.

Thm: (Strong Duality Theorem)

Given LP₁ & DP₁, exactly one of the 4 possibilities always holds:

- (a) Both (LP₁) & (DP₁) aren't feasible.
- (b) $OPT_{LP_1} = -\infty$ and DP₁ isn't feasible.
- (c) $OPT_{DP_1} = \infty$ and LP₁ isn't feasible.
- (d) $-\infty < OPT_{LP_1} = OPT_{DP_1} < \infty$.

(2)

Thm: Feasibility checking & Optimally solving an LP is "polynomially equivalent".

Pf: Optimal sol'n \Rightarrow Feasibility checking.
Let's try to prove the converse.

Let, Alg be a feasibility checking algo.

- Step 1**: First feed (LP_1) & (DP_1) sep. to alg. Take care of cases a, b, c of thm (2)
- Step 2**: Only do if a, b, c doesn't apply.

That is both (LP_1) & (DP_1) are feasible. Now find a feasible solⁿ to the following LP using Alg.

Not trying to optimize but just satisfying the constraints.

$$\begin{array}{l} \min 0 \\ \text{s.t., } Ax = b \\ A^T y \leq c \\ \langle c, x \rangle = \langle b, y \rangle \\ x \geq 0. \end{array}$$

This problem has a feasible solⁿ. (Why?)

This is a combined system involving primal constraints ($Ax = b, x \geq 0$) and dual constraints ($A^T y \leq c$) with additional equality constraint ($\langle c, x \rangle = \langle b, y \rangle$)

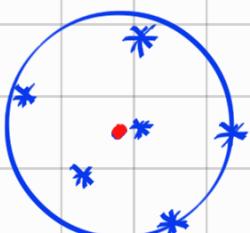
Examples of LPs

Eg 1: (Helly Thm. for half-spaces)

H_1, H_2, \dots, H_n be half-spaces in \mathbb{R}^d satisfy $(d+1)$ property.
Find a pt. P that lies in all of them. Assume that
 $H_i = \{x : \langle a_i, x \rangle - b_i \geq 0\}$.

$$\begin{array}{l} \min 0 \\ \text{s.t., } \langle a_i, x \rangle - b_i \geq 0, \forall i \in [n]. \end{array}$$

Eg. 2: (Facility locating Problem)



Find centre of the smallest ball containing all the *'s.

$$\begin{array}{l} \min r \\ \text{s.t., } d(c, p_i) \leq r, \\ r \geq 0, c \in \mathbb{R}^2. \end{array}$$

Circumscribing circle for pts.

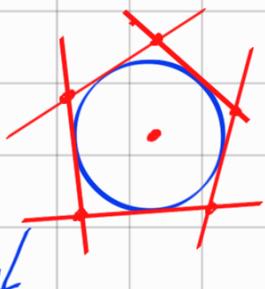
$p_1, p_2, \dots, p_n \in \mathbb{R}^d$

$$\min \Omega$$

$$\text{s.t., } \|p_i - c\|^2 = \|p_i - c\|^2; \forall i \in [n].$$

$\leftarrow \min \Omega \Rightarrow \text{No optimiz'n}$
 just constraints
 satisfaction.

Eg. 3:



Find centre & radius of the largest circle that can be inscribed inside the polytope.

$$\langle a_i, x \rangle - b_i \leq 0; i \in [n].$$

$$z = (c, r) \in \mathbb{R}^{n+1}$$

$$\max r$$

$$\text{s.t., } \langle c, a_i \rangle - b_i \leq 0;$$

$$r \geq 0;$$

$$\left\{ \frac{|\langle c, a_i \rangle - b_i|}{\|a_i\|} \geq r \right.$$

$$\left. \rightarrow r \leq \frac{b_i - \langle c, a_i \rangle}{\|a_i\|} \right.$$

LP is ; $\max r$

$$\text{s.t., } 0 \leq r \leq \frac{b_i - \langle c, a_i \rangle}{\|a_i\|} \quad \forall i \in [n]$$

$$\langle a_i, c \rangle - b_i \leq 0 \quad \forall i \in [n].$$

Eg. 4 (Linear Classification)

$x_1, \dots, x_n \in \mathbb{R}^N$

\downarrow \downarrow
 $x_1, \dots, x_n \in \{\pm 1\}$.

We want to separate all the points labeled +1 from the points labeled -1 via a hyperplane.

Goal is to find $\langle a, x \rangle - b = 0$ that sep. +1 pts. & -1 pts.
 that sep +1 & -1 pts.

$$\begin{aligned} \min \quad & 0 \\ \text{s.t., } & l_i (\langle a_i, \bar{x}_i \rangle - b_i) \geq 1 \quad \forall i \in [n]. \end{aligned}$$

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Integer LP (ILP)

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ \text{s.t., } & Ax \leq b \\ & x_i \in \mathbb{Z}; i \in [n]. \end{aligned}$$

MIS Problem $G = ([n], E)$

Find the largest independent set in G .

$$\max \sum_{i=1}^n x_i \quad \text{s.t., } \begin{aligned} & x_i + x_j \leq 1 \quad \forall \{i, j\} \in E \quad 0: \text{not included.} \\ & 0 \leq x_i \leq 1 \quad \forall i \in [n]. \end{aligned}$$

(i $\mapsto x_i \in \{0, 1\}$. x_i : indicator variable $\rightarrow 1$: vertex included in IS)

maximize vertex included.

"Set of vertices that cover all edges"

between vertex i and vertex j which share an edge either 1 or none will be present in the IS -

Vertex Cover Problem

Find the smallest set S of vertices of $G = ([n], E)$ s.t., $\forall C \in E$, we have $C \cap S \neq \emptyset$.

Find smallest independent set in G .

$$i \mapsto x_i \in \{0, 1\}.$$

$$\begin{aligned} \min \sum_{i=1}^n x_i \quad \text{s.t., } & x_i + x_j \geq 1; \forall \{i, j\} \in E \\ & 0 \leq x_i \leq 1; \forall i \in [n] \\ & x_i \in \mathbb{Z}, \forall i \in [n]. \end{aligned}$$

Fractional ind. set & Fractional vertex cover problem.

$V = [n]$; $V_k = \{v : x_i = k\} \quad k \in \{0, \frac{1}{2}, 1\}$.

