General form of 2×2 Unitary Matrix

8-k

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix} = I$$

(1) $\det(A) = e^{i\theta}$

(2) Other 4 constraints

⟨1⟩ Assignment -1

(a) Find the generalized form of 2×2 unitary matrix. $U(2)$

(b) Extend this for 3×3 unitary matrix. $U(3)$.

(c) Write down a code to check whether a matrix is unitary or not. $(n \times n)$.
(Don't use inbuilt libraries).

$\boxed{\text{Th. 5.22}}$ $x^3 + a_0x^2 + a_1x + a_2$

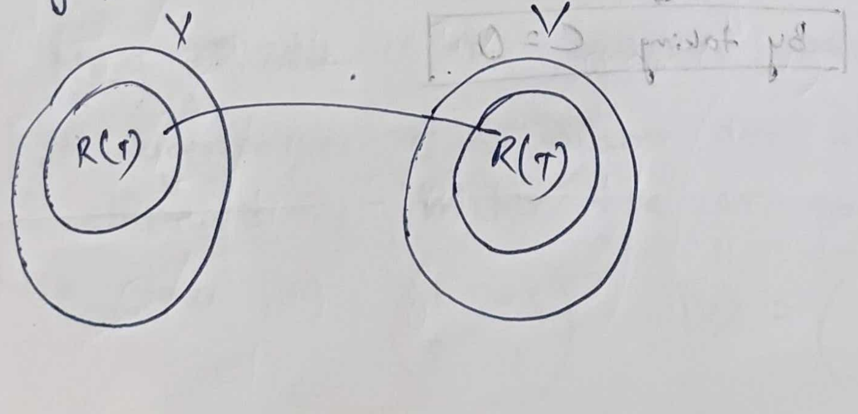
$$T^2\left(T^3 + a_0T^2 + a_1T + a_2\right) = 0$$

(Q) $AB = BA$? ⇒ Difficult to tell

But if $A = f(B)$ then yes it's satisfied.
or $B = f(A)$.

A B also commute.
| |
$f(T)$ $g(T)$

Range(T) is invariant ; what does that mean?



$M(T) \cdot e_1$

$$\overset{T}{\begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{5\times 5} \overset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{5\times 1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_2.$$

$T^5 e_1 = -3e_1 + 6e_2$

$T^5 e_1 = -3e_1 + 6Te_1$

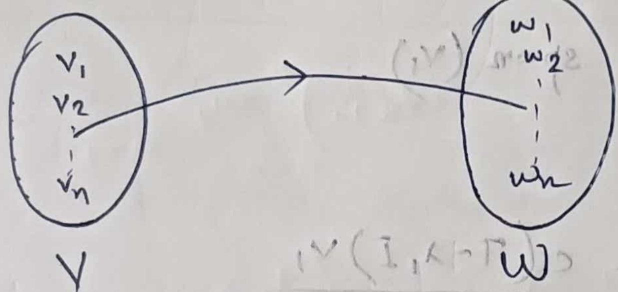
$\Rightarrow 3e_1 - 6Te_1 + T^5 e_1 = 0$

$\Rightarrow (3 - 6T + T^5)\, e_1 = 0$

$\Rightarrow 3 - 6z + z^5$

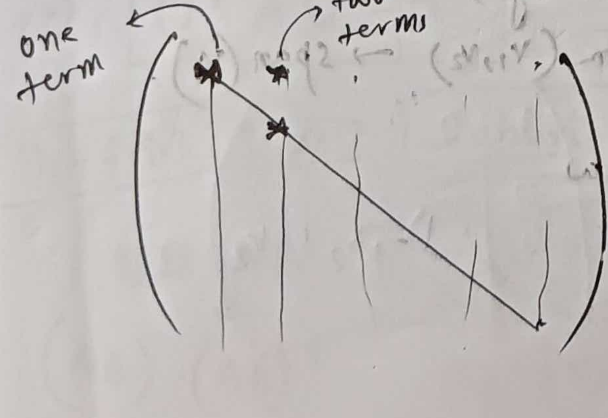
∴ Upper Triangular Matrices :- 03/10/24

- If its not mentioned we take standard basis.



$$T(v_i) = \sum_{j=1}^{m} a_{ij} w_j$$

$\boxed{\text{Th. 5.39}}$ $\quad T(v_1) = \sum_{j=1}^{n} a_{ij} v_j$



$T(v_1) = c v_1$

$T(v_2) = a v_1 + b v_2$

Th. 5.40 For a 2×2 matrix.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$ Find the eigenvalues? $\Rightarrow$ These are $\boxed{a, c}$

$$\begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$\cancel{\lambda_2 v_2 = a v_1 + \lambda_2 v_2}$

$(T - \lambda_1 I) v_1 = 0$

$(T - \lambda_1 I) \cdots (T - \lambda_m I) v_1 = 0$

[Multiplying both sides by ....]
& then using commutativity

$(T - \lambda_1 I) v_1 = 0$

$\underbrace{(T - \lambda_2 I)(T - \lambda_1 I)}_{f(T)\ g(T)} v_1 = 0$

commutativity is justified

So, $(T - \lambda_1 I)(T - \lambda_2 I) v_1 = 0$

$\because (T - \lambda_2 I) v_2 \in \text{span}(v_1)$.

$\therefore (T - \lambda_2 I) v_2 = c v_1$

$(T - \lambda_1 I)(T - \lambda_2 I) v_2 = c (T - \lambda_1 I) v_1$

$= c \times 0 = 0$

Th. (5.41) $T(v_1) = \lambda_1 v_1$.

$(T - \lambda_1 I) v_1 = 0$: span $(v_1) \to$ span (0)

Similarly;

$(T - \lambda_2 I) v_2 = a_1 v_1$ span $(v_1, v_2) \to$ span (v_1).

$S(u) = S(w)$, $u \neq w$

$S(u - w) = 0$

$(T - \lambda_k I)$ # 0

(Th-5.44) → Later

$\boxed{\text{Corollary - 6.4.2}}$ Given $A^{+} = A$

Given; $Ax_i = \lambda_i x_i$

$Au_i = \lambda_i u_i$; $u_i = \frac{x_i}{||x_i||}$

[Multiplying by $\frac{1}{||x_i||}$.]

Given all λ_i's are distinct

[All u_i's are orthonormal]

$\therefore U = (u_1\ u_2\ u_3 \dots u_n)$.

$\therefore U$ is unitary matrix.

For a square matrix; $UU^{+} = I$ only this is sufficient. We don't require; $U^{+}U = I$

$AU = (Au_1\ Au_2 \dots Au_n)$ (Q). why it is non singular

$= (\lambda_1 u_1\ \lambda_2 u_2 \dots \lambda_n u_n)$

$$= (u_1\ u_2 \dots u_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$$

$AU = UD$

$\Rightarrow \boxed{U^{-1}AU = D}$ $\boxed{\because U^{-1} \equiv U^{+}}$

$\Rightarrow \boxed{U^{+}AU = D}$

Product of two Hermitian matrix is not necessarily hermitian

AB ; $A^{+} = A$

$(AB)^{+}$

$= B^{+}A^{+}$

$= BA$.

$\boxed{\text{Th-6.4.3}}$ [VVI] [Schur's theorem]

AB Unitary $\boxed{AA^{+} = I = A^{+}A}$

$(AB)\cdot(AB)^{+} = A\underline{B\,B^{+}}A^{+} = AIA^{+} = I$

→ Product of unitary matrix is unitary.

(Q) A has all real eigen values. $\boxed{\text{For any arbitrary matrix}}$

B " " "

AB " " " " ? (T/F)?

2) $\boxed{\text{Think}}$

- Even (Normal ~~matrix~~ Operator) is Diagonal. $\boxed{AA^+ = A^+A}$

$\boxed{\text{Th-6.4.3}}$

$$W = (w_1 \quad w_2 \cdots w_k \quad w_{k+1})$$

$$W^H A W = (W^H A w_1 \quad \cdots W^H A w_2 \cdots W^H A w_{k+1})$$

$$W^H w_1 = \underbrace{(w_1 \quad w_2 \cdots\cdots w_{k+1})^H}_{n \times n} \underbrace{w_1}_{n \times 1}$$

$$= \begin{pmatrix} w_1^H w_1 \\ w_2^H w_1 \\ \vdots \\ w_{k+1}^H w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1$$

$\boxed{W^+ = W^H}$

$$V^+ W^+ A W V$$

$$= V^+ \left(\begin{array}{c|c} \lambda_1 & * \quad * \quad * \cdots * \\ \hline 0 & \\ 0 & M \\ \vdots & \\ 0 & \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ 0 & V_1 \\ \vdots & \\ 0 & \end{array}\right)$$

$$= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ 0 & & V_1^+ & \\ \vdots & & & \\ 0 & & & \end{array}\right) \left(\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & MV_1 & \\ 0 & & & \end{array}\right)$$

$$= \left(\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & V_1^+ M V_1 & \\ 0 & & & \end{array}\right) = \left(\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & T_1 & \\ 0 & & & \end{array}\right) = T$$

Th. 6.4.9 **Spectral Theorem**

If A is Hermitian, then there exists a unitary matrix U that diagonalizes A.

$\Rightarrow$ w

The Real Schur Decomposition

- we will do for some ~~...~~ special, real, ..., stochastic matrix.