Practice Questions

Random Variables

Question 1

-4

$$P(X = x) = \begin{cases} 1/9 & \text{if x is an integer in the range [-4, 4],} \\ 0 & \text{Otherwise} \end{cases}$$

Let $Y = |X|$. Find out $P(Y = 2)$

Question 1

$$P(X = x) = \begin{cases} 1/9 & \text{if x is an integer in the range [-4, 4],} \\ 0 & \text{Otherwise} \end{cases}$$

Let $Y = |X|$. Find out $P(Y = 2)$

$Y = |X|$

X	P(X)	Y
-4	1/9	4
-3	1/9	3
-2	1/9	$\boxed{2}$
-1	1/9	1
0	1/9	0
1	1/9	1
2	1/9	$\boxed{2}$
3	1/9	3
4	1/9	4

Y	$P(Y=k)$
0	$1/9$
1	$2/9$
2	$2/9$
3	$2/9$
4	$\frac{1}{9}+\frac{1}{9} = \frac{2}{9}$

$x=0$

$x=-3$

$x=3$

-4

4

$P(Y=2) = 2/9$

$$p_Y(y) = \begin{cases} 2/9, & \text{if } y = 1, 2, 3, 4, \\ 1/9, & \text{if } y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

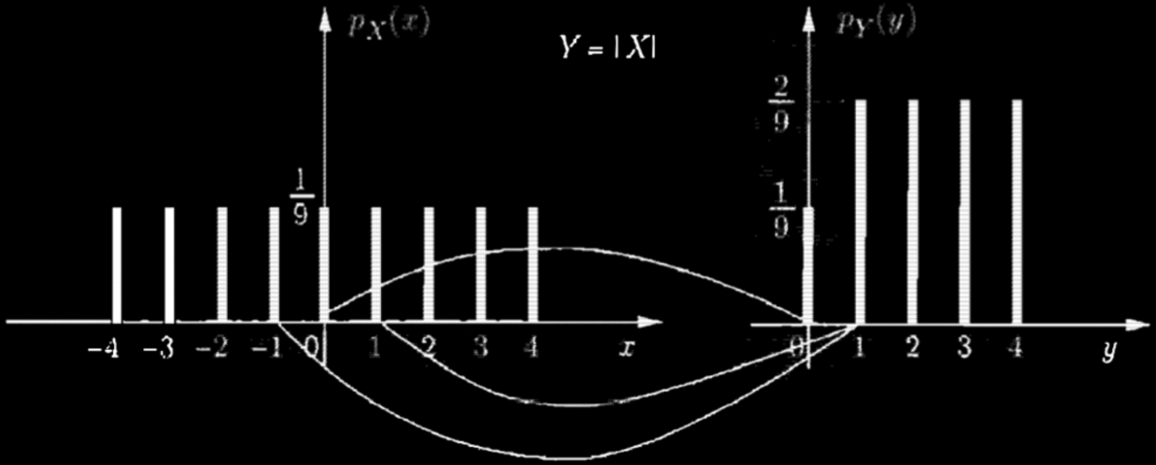


Figure 2.6: The PMFs of X and $Y = |X|$ in Example 2.1.

Question 2

indep.

Let X and Z are two random variables.

$Y = X \oplus Z$. Find $P(Y = y)$.

	$X = 0$	$X = 1$
$P(X = x)$	p	$1 - p$

	$Z = 0$	$Z = 1$
$P(Z = z)$	ϵ	$1 - \epsilon$

X $\quad P(X=0) = p \qquad P(X=1) = 1-p$

Z $\quad P(Z=0) = \epsilon \qquad P(Z=1) = 1-\epsilon$

$Y = X \oplus Z$

$P(Y=0) = ?$

$P(Y=1) = ?$

Method 1 (Recommended)

$X \quad P(X=0) = p \qquad P(X=1) = 1-p$

$Z \quad P(Z=0) = \epsilon \qquad P(Z=1) = 1-\epsilon$

$Y = X \oplus Z$

$P(Y=0) = ?$

$P(Y=1) = ?$

X	Z	Y	
0	0	0	$p\epsilon$
0	1	1	
1	0	1	
1	1	0	$(1-p)(1-\epsilon)$

$P(X=1, Z=1) = P(X=1) \cdot P(Z=1)$

$= (1-p) \cdot (1-\epsilon)$

Method 2

$$P(Y=0) = P\big((X=0,\ Z=0) \text{ or } (X=1,\ Z=1)\big)$$

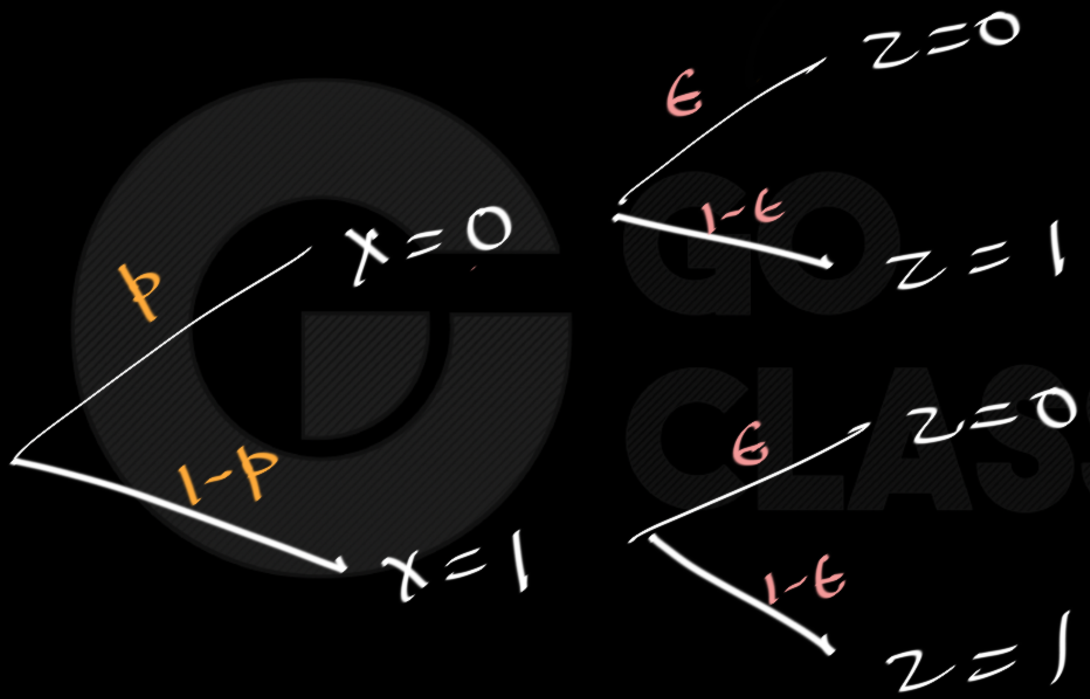
$$= P(X=0,\ Z=0) + P(X=1, Z=1)$$

becoz X and Z are indep.

$$= P(X=0)\cdot P(Z=0) + P(X=1)\cdot P(Z=1)$$

$$= p\cdot\epsilon + (1-p)(1-\epsilon)$$

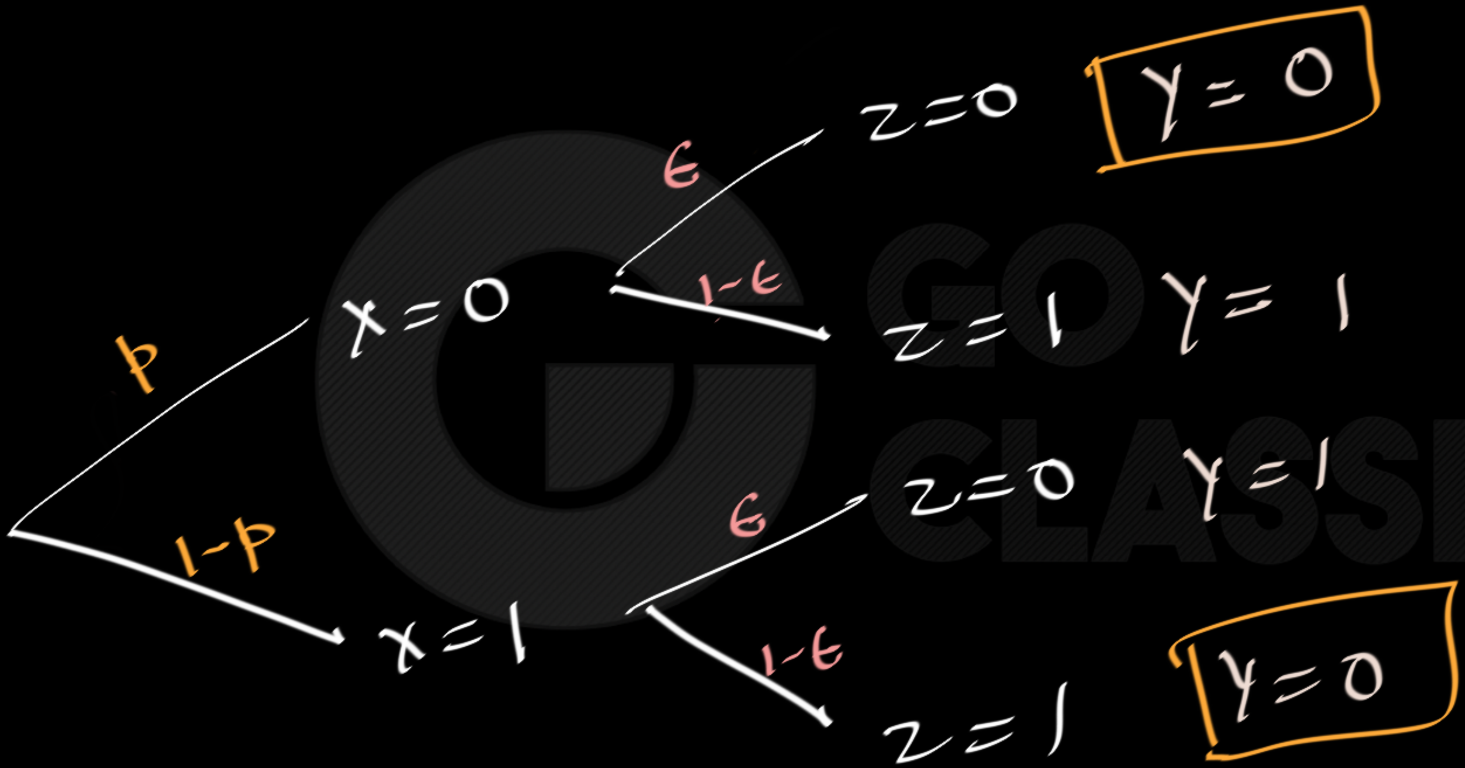
Method 3 : Tree method



$P(z=0 \mid x=0)$

$= P(z=0)$

$P(z=0) = \epsilon$



$p\epsilon$ +

$(1-p)(1-\epsilon)$

Variation

$$P(Y=0 \mid X=0) = ?$$

$$P(X \oplus Z = 0 \mid X=0)$$

Method 1

x	z	y
0	0	0
0	1	1
1	0	1
1	1	0

$P(y=0 \mid x=0) = ?$

$P(x \oplus z = 0 \mid x=0)$

(0 XOR Z = Z)

$\Rightarrow P(z=0 \mid x=0)$

$= p$

after applying x=0 in y=0 we cannot discard the x=0 in 'given' part.

X	Z	Y	
0	0	0	$p\epsilon$
0	1	1	$p(1-\epsilon)$
1	0	1	
1	1	0	

$$P(Y=0 \mid X=0) = ?$$

$$\frac{p\epsilon}{p\epsilon + p(1-\epsilon)}$$

(distributing the probability mass after reducing the domain)

$x=0, z=0$

$\downarrow$

$p\epsilon$

$\downarrow$

$\frac{p\epsilon}{p} = \epsilon$

$x=0, z=1$

$\downarrow$

$p(1-\epsilon)$

$\downarrow$

$\frac{p(1-\epsilon)}{p} = 1-\epsilon$

$x=1, z=0$

$\downarrow$

$(1-p)\epsilon$

$\downarrow$

0

$x=1, z=1$

$\downarrow$

$(1-p)(1-\epsilon)$

$\downarrow$

0

x	z	y	
0	0	0	ϵ
0	1	1	$1-\epsilon$
1	0	1	
1	1	0	

new probabilities after reduced domain.

$$P(y=0 \mid x=0) = ?$$

$$\downarrow$$

$$\epsilon$$

Method 3 tree method

$P(Y=0 \mid X=0)$

$= \epsilon$

Method 4 : using plain formula.

$$P(Y=0 \mid X=0) = \frac{P(Y=0, X=0)}{P(X=0)}$$

$P(X=0) \leftarrow p$

we are given that x,z are indep.
Also, we know x,y are dependent (since y = x XOR z)

$$P(Y=0, X=0) \neq P(Y=0) \cdot P(X=0)$$

$$P(Y=0, X=0) \neq P(Y=0) \cdot P(X=0)$$

$$P(Y=0, X=0)$$

$$\downarrow$$

$$X \oplus Z = 0$$

$$P(X \oplus Z = 0, X = 0) \Rightarrow P(Z=0, X=0)$$

$$P(Z=0, X=0) = P(Z=0) \cdot P(X=0)$$

$$\epsilon \cdot P$$

Method 4 : using plain formula.

$$P(Y=0 \mid X=0) = \frac{P(Y=0, X=0)}{P(X=0)}$$

$P(X=0) \leftarrow p$

$$= \frac{\epsilon \cdot p}{p} = \epsilon$$

Question 3

Bi-variate RV

Consider two random variables X and Y with joint probability distribution given in the table below.

	$Y=0$	$Y=1$	$Y=2$
$X=0$	1/6	1/4	1/8
$X=1$	1/8	1/6	1/6

$P(X = 0, Y = 1) \quad =$

$P(X = 0, Y \leq 1) \quad =$

$P(X = 0) \quad =$

$P(X = 1) \quad =$

$P(Y = 0) \quad =$

$P(Y = 1) \quad =$

$P(Y = 2) \quad =$

$P(Y = 1 \mid X = 0) \quad =$

Are X and Y independent ?

Question 3

Consider two random variables X and Y with joint probability distribution given in the table below.

	$Y=0$	$Y=1$	$Y=2$
$X=0$	1/6	1/4	1/8
$X=1$	1/8	1/6	1/6

pmf of X

pmf of Y

$P(X=0, Y=1) = 1/4$

$P(X=0, Y\le 1) = 1/6 + 1/4 = 5/12$

$P(X=0) = 1/6 + 1/4 + 1/8$

$P(X=1) = 1/8 + 1/6 + 1/6$

$P(Y=0) = 1/6 + 1/8$

$P(Y=1) =$

$P(Y=2) =$

$P(Y=1 \mid X=0) =$

Are X and Y independent ?

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2)$$

using the intution (Reduced domain)

$$P(Y=1 \mid X=0) = \frac{1/4}{13/24} = 6/13$$

	Y=0	Y=1	Y=2
X=0	1/6	1/4	1/8
X=1	1/8	1/6	1/6

we know, $\left(\overset{x=0}{1/6 + 1/4 + 1/8}\right) + \left(\overset{x=1}{1/8 + 1/6 + 1/6}\right) = 1$

$\therefore\ 1/6 + 1/4 + 1/8 = \left(1 - \frac{11}{24}\right)$

$\therefore\ P(x=0) = 13/24$

$1/6 + 1/4 + 1/8 \Rightarrow$ Summation.

$$\frac{P(x=0)}{13/24} = 1$$

reducing domain context

(plain formula:)

$$P(Y=1 \mid X=0) = \frac{P(Y=1, X=0)}{P(X=0)} =$$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	1/6	1/4	1/8
$X=1$	1/8	1/6	1/6

$$\frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13}$$

Are X and Y independent ?

	Y=0	Y=1	Y=2
X=0	1/6	1/4	1/8
X=1	1/8	1/6	1/6

$\Rightarrow$ X & Y are dependent variable

$\frac{1}{6} =$

$P(X=0, Y=0) \neq P(X=0) \cdot P(Y=0)$

$\frac{13}{24} \cdot \frac{7}{24}$

$P(X=0, Y=1) = P(X=0) \cdot P(Y=1)$

$P(X=0, Y=2) = P(X=0) \cdot P(Y=2)$

	Y=0	Y=1	Y=2
X=0	1/6	1/4	1/8
X=1	1/8	1/6	1/6

Joint prob distribution

[very powerful distribution]

$P(X=0)$ $P(Y=0 \mid X=0)$

in machine learning: generative models

trying to learn joint prob. distribution

$P(\text{mail}_1, \text{Spam})$

$P(\text{mail}_1, \text{Not Spam})$

Congratulations you won lottery

$\underline{P(\text{Spam} \mid \text{mail})} > \underline{P(\text{Not spam} \mid \text{mail})}$

we want to generate Not spam mails

(Data)

random words

$$\underbrace{P(\underline{\text{Spam}} \mid \text{mail})}_{\to\ \underline{0.8}} > \underbrace{P(\text{Not } \underline{\text{Spam}} \mid \text{mail})}_{\downarrow\ 0.2}$$

Question 4

I toss a coin twice and define X to be the number of heads I observe. Then, I toss the coin two more times and define Y to be the number of heads that I observe this time.

Find P(X<2) and (Y>1))

last two tosses

	X	Y
H H	2	2
H T	1	1
T H	1	1
T T	0	0

$P(X<2, Y>1)$

	X	Y
HHHH	2	2
HHHT	2	1

$$\frac{12}{16} = \frac{3}{4}$$

TT ← 4

HT ← 4

TH ← 4

HH

$\frac{4}{16}$

$$P(X<2, Y>1) = P(X<2) \cdot P(Y>1)$$

$$\frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$$

Question 5

$P(x=1) + P(x=2) + P(x=3) + P(x=4)$ (for x)

$P(y=1) + P(y=2) + P(y=3) + P(y=4)$ (for Y).

Given joint distribution

a. Find distribution of X and Y.

b. Find $P(Y = 1|X = 1)$

		X = 1	X = 2	X = 3	X = 4
Y	1	1/16	0	1/8	1/16
	2	1/32	1/32	1/4	0
	3	0	1/8	1/16	1/16
	4	1/16	1/32	1/16	1/32

Question 5

$P(x=1)$

Given joint distribution

a. Find distribution of X and Y.

b. Find $P(Y = 1|X = 1)$

take out $x = 2,3,4$ ∵ we're given $x=1$.

		X			
		1	2	3	4
Y	1	1/16	0	1/8	1/16
	2	1/32	1/32	1/4	0
	3	0	1/8	1/16	1/16
	4	1/16	1/32	1/16	1/32

$P(Y=1|X=1) = \frac{1/16}{5/32} = 2/5$

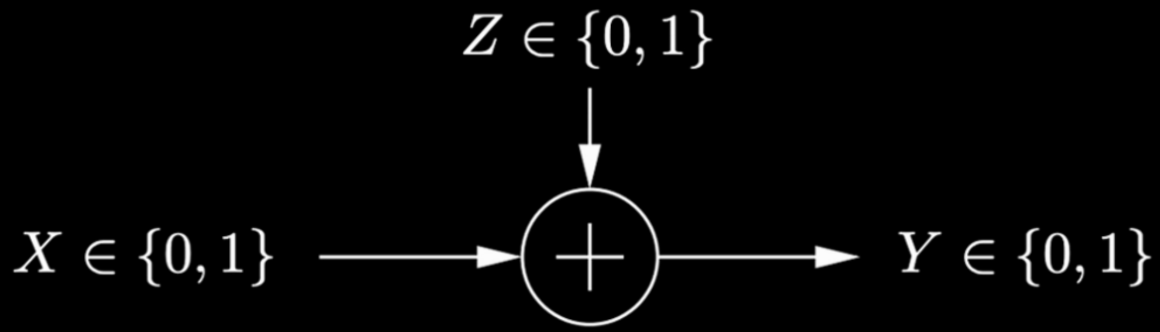
$5/32$ ← normalizing factor $(1/16 + 1/32 + 0 + 1/16)$

or $\frac{P(X=1 \cap Y=1)}{P(X=1)} = \frac{1/16}{5/32} = 2/5$

Question 6

$\boxed{\text{MIT Book}}$

Consider the following binary communication channel



$P(X = 0) = p$
$P(Z = 0) = \epsilon$

$Y = (X + Z) \mod 2 = X \oplus Z$, and X and Z are independent. Find $P(Y = 0)$

X: 0, 1 $\quad$ Z: 0 $\rightarrow p\epsilon$, 1 $\rightarrow (1-p)(1-\epsilon)$

$P(y=0) = p\epsilon + (1-p)(1-\epsilon)$

$P(X = 0) = p$
$P(Z = 0) = \epsilon$

$Y = X \oplus Z$

$P(Y=0) \Rightarrow P(X \oplus Z = 0) = P(X=0, Z=0) + P(X=1, Z=1)$

indep

$p \cdot \epsilon + (1-p)(1-\epsilon)$

X	Z	Y
0	0	0
0	1	1
1	0	1
1	1	0

$p\epsilon$

$P(Y=0)$

$= p\epsilon$

$+ (1-p)(1-\epsilon)$

$P(Y = 0 \mid X = 0)$

$P(Y = 1 \mid X = 0)$

$P(Y = 0 \mid X = 1)$

$P(Y = 1 \mid X = 1)$

$P(X=0) = p$

$P(Z=0) = \epsilon$

$Y = X \oplus Z$

$P(Y=0 \mid X=0) = P(X \oplus Z = 0 \mid X=0) = P(0 \oplus Z = 0 \mid X=0)$

This is the silly mistake that is often committed. $\rightarrow$ $P(0 \oplus Z = 0) = P(Z=0)$

$\hookrightarrow P(0 \oplus Z = 0 \mid x = 0) = \epsilon = P(Z=0 \mid x=0) = \epsilon.$

Question 7: GATE 2017

Suppose X_i for $i = 1, 2, 3$ are independent and identically distributed random variables whose probability mass functions are $Pr[X_i = 0] = Pr[X_i = 1] = \frac{1}{2}$ for $i = 1, 2, 3$. Define another random variable $Y = X_1 X_2 \oplus X_3$, where $\oplus$ denotes XOR. Then $Pr[Y = 0 \mid X_3 = 0] =$_____.

$P(X_2=0) = P(X_2=1) = P(X_1 = 0) = P(X_1=1) = 1/2$

Method 1

$$Y = x_1 x_2 \oplus x_3$$

$$P(x_i = 1) = P(x_i = 0)$$
$$= 1/2$$

$$P(Y = 0 \mid x_3 = 0)$$

$$= P(x_1 x_2 \oplus x_3 = 0 \mid x_3 = 0)$$

$$= P(x_1 x_2 = 0 \mid x_3 = 0) = P(x_1 x_2 = 0)$$

$$=$$

$$P(x_1x_2=0 \mid x_3=0) = \frac{P(x_1x_2=0,\ x_3=0)}{P(x_3=0)} = \frac{P(x_1x_2=0)\cdot \cancel{P(x_3=0)}}{\cancel{P(x_3=0)}} = P(x_1x_2=0)$$

$$P(x_1x_2=0) = P(x_1=0,\ x_2=0) + P(x_1=0,\ x_2=1) + P(x_1=1,\ x_2=0)$$

$$P(x_1=0,\ x_2=0) \rightarrow \tfrac{1}{2}\cdot\tfrac{1}{2}$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2}$$

$$= \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4}$$

$$= \tfrac{3}{4}$$

Method 2 ← Reduced domain

(remove all cases where $x_3 = 1$) → Don't forget to NORMALIZE

$1/8 + 1/8 + 1/8 + 1/8 = 4/8 = 1/2$

↘ Normalization by $1/2$

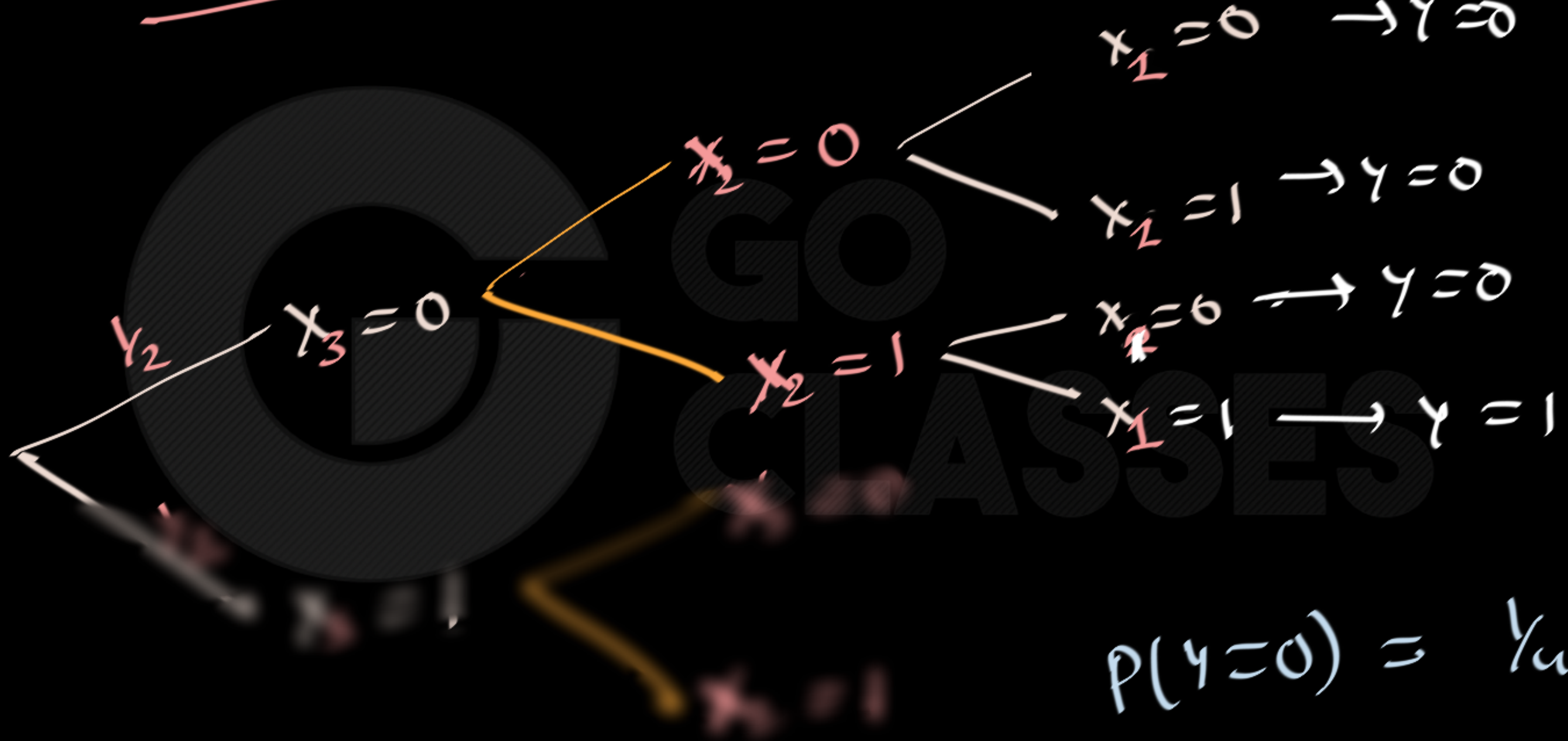
x_1	x_2	x_3	y		
0	0	0	0	$\to 1/8$	$\to 1/4$
0	1	0	0	$\to 1/8$	$\to 1/4$
1	0	0	0	$\to 1/8$	$\to 1/4$
1	1	0	1	$\to 1/8$	$\to 1/4$

$y = x_1 x_2 \oplus x_3$

$P(y=0 \mid x_3 = 0)$

$= 1/4 + 1/4 + 1/4 = 3/4$

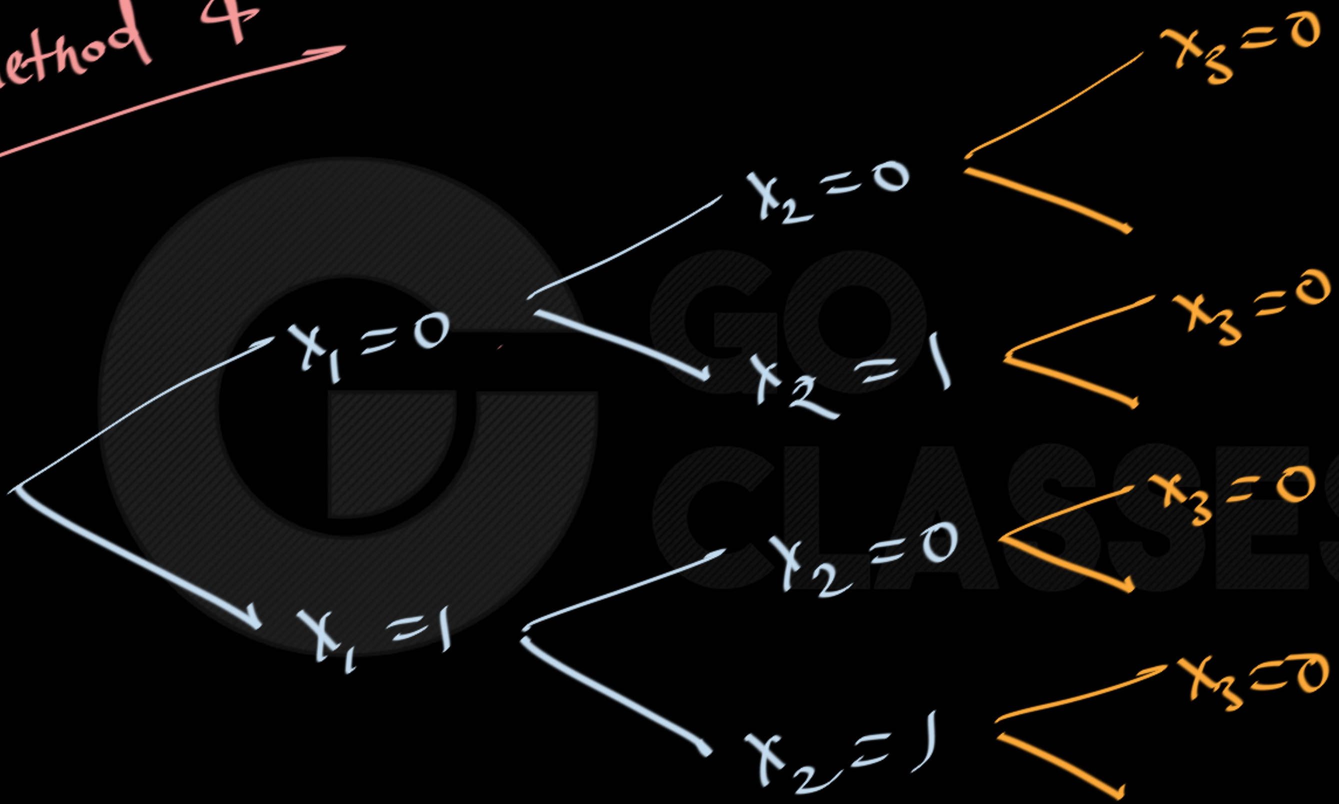
Method 3



$$P(Y=0) = 1/4 + 1/4 + 1/4$$

$$= 3/4$$

Method 4



$P(Y=0 \mid X_3=0)$

$$\frac{P(Y=0, X_3=0)}{P(X_3=0)}$$

$$\frac{P(y=0,\ X_3=0)}{P(X_3=0)} = \frac{P(X_1X_2 \oplus X_3=0,\ X_3=0)}{P(X_3=0)}$$

$$= \frac{P(X_1X_2=0,\ X_3=0)}{P(X_3=0)}$$

$$= \frac{P(X_1X_2=0)\cdot \cancel{P(X_3=0)}}{\cancel{P(X_3=0)}} = P(X_1X_2=0)$$

$$= 3/4$$

Method 3

x_1	x_2	x_3	y	
0	0	0	0	$\to 1/8$
0	0	1	1	$\to 1/8$
0	1	0	0	$\to 1/8$
0	1	1	1	
1	0	0	0	
1	0	1	1	
1	1	0	1	
1	1	1	0	

$$P(Y=0 \mid X_3=0)$$

$$\downarrow$$

$$\frac{P(Y=0, X_3=0)}{P(X_3=0)}$$

$$\frac{3/8}{1/2} = 3/4$$

Expectation

- Expectation v/s Average
- Recursive expectation

- Descrete random variable
- Cont. " "