Linear Programs & Duality

Primal (LP1)

$\min \langle c, x\rangle$

s.t.; $Ax = b$

$x \geq 0$

Dual (DP1)

$\max \langle b, y\rangle$

s.t., $A^T y \leq c$

Weak Duality: Let, x_0 & y_0 be feasible sol^n to (LP1) & (DP1). Then $\langle c, x_0\rangle \geq \langle b, y_0\rangle$

This implies $OPT_{LP1} \geq OPT_{DP1}$.

$$\underbrace{\text{Mathematical Programs} \qquad \text{Lagrangian Dual of MP.}}_{\text{Weak duality always hold.}}$$

Thm: (Strong Duality Theorem)

Given LP1 & DP1, exactly one of the 4 possibilities always holds:

(a.) Both (LP1) & (DP1) aren't feasible.

(b.) $OPT_{LP1} = -\infty$ and DP1 isn't feasible.

(c.) $OPT_{DP1} = \infty$ and LP1 isn't feasible.

(d.) $-\infty < OPT_{LP1} = OPT_{DP1} < \infty$. ——(2)

Thm: Feasibility checking & Optimally solving an LP is "polynomially equivalent".

Pf: Optimal sol^n $\Rightarrow$ Feasibility checking.

Let's try to prove the converse.

Let, Alg be a feasibility checking algo.

$\boxed{\text{Step 1}}$: First feed (LP1) & (DP1) sep. to alg. Take care of cases a, b, c of thm (2)

$\boxed{\text{Step 2}}$: Only do if a, b, c doesn't apply.

That is both (LP1) & (DP1) are feasible. Now find a feasible sol^n to the following LP using Alg.

$$\begin{aligned} \min\ & 0 \\ \text{s.t., } & Ax = b \\ & A^T y \leq c \\ & \langle c, x\rangle = \langle b, y\rangle \\ & x \geq 0. \end{aligned}$$

This problem has a feasible sol^n. (Why?)

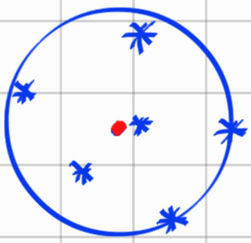
<u>Examples of LPs</u>

<u>Eg 1:</u> (Helly Thm. for half-spaces)

$H_1, H_2, \ldots, H_n$ be half-spaces in $\mathbb{R}^d$ satisfy $(d+1)$ property. Find a pt. p that lies in all of them. Assume that $H_i = \{x : \langle a_i, x\rangle - b_i \geq 0\}$.

$$\begin{aligned} \min\ & 0 \\ \text{s.t., } & \langle a_i, x\rangle - b_i \geq 0, \ \forall\, i \in [n]. \end{aligned}$$

<u>Eg. 2:</u> (Facility locating Problem)



Find centre of the smallest ball containing all the $*$'s.

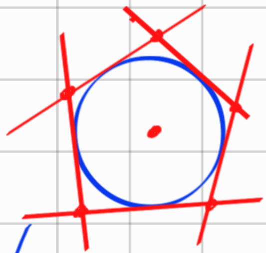
$$\begin{aligned} \min\ & r \\ \text{s.t., } & d(c, p_i) \leq r; \\ & r \geq 0;\ c \in \mathbb{R}^2. \end{aligned}$$

Circumscribing circle for pts.

$p_1, p_2, \dots, p_n \in \mathbb{R}^d$

$$\min \; 0$$
$$\text{s.t., } \|p_i - c\|^2 = \|p_1 - c\|^2 \; ; \; \forall \; i \in [n].$$

Eg. 3:



Find centre & radius of the largest circle that can be inscribed inside the polytope.

$\langle a_i, x\rangle - b_i \le 0 \; ; \; i \in [n].$

$z = (c, r) \in \mathbb{R}^{n+1}$

$$\max \; r$$
$$\text{s.t., } \langle c, a_i\rangle - b_i \le 0 \; ;$$
$$r \geq 0 \; ;$$
$$\frac{|\langle c, a_i\rangle - b_i|}{\|a_i\|} \geq r$$
$$r \le \frac{b_i - \langle c, a_i\rangle}{\|a_i\|}$$

LP is ; $\max \; r$

$$\text{s.t., } 0 \le r \le \frac{b_i - \langle c, a_i\rangle}{\|a_i\|} \quad \forall \; i \in [n]$$
$$\langle a_i, c\rangle - b_i \le 0 \quad \forall \; i \in [n].$$

Eg 4: (Linear Classification)

$x_1, \dots, x_n \in \mathbb{R}^N$

$\downarrow \quad\quad \downarrow$

$l_1 \quad\quad l_n \in \{\pm 1\}.$

We want to separate all the points labeled +1 from the points labeled −1 via a hyperplane.

Goal is to find $\langle a, x\rangle - b = 0$ that sep. +1 pts. & −1 pts.
that sep +1 & −1 pts.

$$\min \; 0 \; ;$$
$$\text{s.t.,} \; l_i\left(\langle a, \bar{x}_i \rangle - b_i\right) \geq 1 \quad \forall i \in [n].$$

Integer LP (ILP)

$$\min \; \langle c, x \rangle$$
$$\text{s.t.,} \; Ax \leq b$$
$$x_i \in \mathbb{Z} \; ; i \in [n].$$

MIS Problem $G = ([n], E)$

Find the largest independent set in G.

$$i \mapsto x_i \in \{0, 1\}.$$

$$\max \sum_{i=1}^{n} x_i \quad \text{s.t.,} \quad x_i + x_j \leq 1 \quad \forall \{i, j\} \in E$$
$$0 \leq x_i \leq 1 \quad \forall \; i \in [n].$$
$$x_i \in \mathbb{Z}$$

Vertex Cover Problem

Find the smallest set S of vertices of $G = ([n], E)$ s.t., $\forall \; c \in E$, we have $c \cap S = \phi$.

Find smallest independent set in G.

$$i \longmapsto x_i \in \{0, 1\}.$$

$$\min \sum_{i=1}^{n} x_i \quad \text{s.t.,} \quad x_i + x_j \geq 1 \; ; \forall (i, j) \in E$$
$$0 \leq x_i \leq 1 \; ; \forall \; i \in [n]$$
$$x_i \in \mathbb{Z}, \forall \; i \in [n].$$

Fractional ind. set & Fractional vertex cover problem.

$V = [n]$; $V_k = \{ i : x_i = k \}$ $k \in \{0, \frac{1}{2}, 1\}$.