Problem set 1

2) $A \Rightarrow B \xleftrightarrow{\text{contrapositive}}$ not $B \Rightarrow$ not A

i.e, $p \to q = (\neg q) \to (\neg p)$

① Recall (De Morgan):

$$\neg(p \vee q) = (\neg p) \wedge (\neg q)$$

where $\neg$: NOT, $\vee$: OR, $\wedge$: AND

If it didn't rain well and fertilization didn't work properly, then we don't have good ~~bad~~ crops.

② If a tournament doesn't have a 3-cycle, then it doesn't have a cycle

3) ① FALSE counter: $(2+\sqrt{3})+(2-\sqrt{3}) = 4$

② case I ($p = q$):

Here, $\sqrt{pq} = p \notin \mathbb{Q}^c$

FALSE

case II ($p \neq q$):

Suppose $\sqrt{pq}$ is rational

Then, let $\sqrt{pq} = \frac{m}{n}$ for some $m, n \in \mathbb{N}$ s.t $(m,n) = 1$

Now, $pq = \frac{m^2}{n^2} \in \mathbb{N}$

$\Rightarrow n^2pq = m^2$ —— $\circledast$

$\Rightarrow p \mid m^2 \Rightarrow p \mid m$ —— I

$\Rightarrow m = kp$ for some $k \in \mathbb{N}$

$\circledast \Rightarrow n^2pq = k^2p^2$

$\Rightarrow n^2q = k^2p$

$\Rightarrow p \mid n^2 \Rightarrow p \mid n$ —— II

I, II $\Rightarrow p \mid (m,n) = 1 \Rightarrow p = 1$

$(\Rightarrow\Leftarrow)$

Thus, $\sqrt{pq}$ is irrational, whenever $p \neq q$

TRUE

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6) ① To prove: $15 \mid 3n^5 + 5n^3 + 7n$, $\forall n \in \mathbb{N}$

i.e, $3 \mid 5n^3 + 7n$ and $5 \mid 3n^5 + 7n$, $\forall n \in \mathbb{N}$

pf: Claim 1: $3 \mid n(5n^2+7)$, $\forall n \in \mathbb{N}$

pf: $n \equiv 0 \pmod 3$ ✓

$n \equiv 1 \pmod 3 \Rightarrow n(5n^2+7) \equiv 12 \equiv 0 \pmod 3$ ✓

$n \equiv -1 \pmod 3 \Rightarrow n(5n^2+7) \equiv -12 \equiv 0 \pmod 3$ ✓

Claim 2: $5 \mid n(3n^4+7)$, $\forall n \in \mathbb{N}$

pf: $n \equiv 0 \pmod 5$ ✓

$n \equiv \pm 1 \pmod 5 \Rightarrow 3n^4+7 \equiv 10 \equiv 0 \pmod 5$ ✓

$n \equiv \pm 2 \pmod 5 \Rightarrow 3n^4+7 \equiv 55 \equiv 0 \pmod 5$ ✓

$\square$

② To prove: $2^n - 1$ is prime $\Rightarrow n$ is prime

(i.e, n is not prime $\Rightarrow 2^n-1$ is not prime)

pf: Suppose $n = a \cdot b$, where $a, b \in \{2, 3, \dots, n-1\}$

Then, $2^n - 1 = (2^a)^b - 1$

$= (2^a - 1)(1 + 2 + 2^2 + \dots + 2^{a(b-1)})$

$\Rightarrow \underbrace{2^a - 1}_{\substack{\text{non-trivial factor} \\ (\because a \notin \{1, n\})}} \mid 2^n - 1 \Rightarrow 2^n - 1$ is not prime

$\square$

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7) n people: $x_1, x_2, \dots, x_n$ $\quad n \geq 2$

each person x_i knows $\underbrace{y_i \in \{1, 2, 3, \dots, n-1\}}_{(n-1)\text{ possibilities}}$ no. of people

PHP $\Rightarrow$ 2 people must know the same no. of people

8) $f : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$ $n \geq m$

Claim: no. of monotonically increasing such f's

= no. of " decreasing " "

$= \binom{n+m-1}{m}$

pf: Idea: If f is monotone, it is completely determined by how many values of $j \in \{1, 2, \ldots, m\}$ are assigned to equal $k \in \{1, 2, \ldots, n\}$

To see why, consider the example:

$f : \{1, 2, 3, 4, 5\} \longrightarrow \{1, 2, 3, 4, 5, 6\}$

s.t two values are assigned to 2, one value is assigned to 4 and two values are assigned to 5

If f is MI, it is given by:

$f(1) = f(2) = 2$, $f(3) = 4$, $f(4) = f(5) = 5$

and if f is MD, it is given by:

$f(1) = f(2) = 5$, $f(3) = 4$, $f(4) = f(5) = 2$

Let x_k $(1 \leq k \leq n)$ be the no. of values of $j \in \{1, 2, \ldots, m\}$ s.t $f(j) = k$

Then, $x_1 + x_2 + x_3 + \cdots + x_n = m$, where $x_i \geq 0, \forall i$

0 [] 00 [] . . . . . . 0 [] 0

$n-1$

no. of sol^n's $= \binom{n-1+m}{m}$

$\therefore$ no. of MI such f's = no. of MD such f's

$= \binom{n-1+m}{m}$

$\square$

9) Recall (Fermat's Little theorem [FLT]):

If p is a prime no. and $a \in \mathbb{N}$, then:

$a^p \equiv a \pmod{p}$

FLT $\Rightarrow p^q \equiv p \pmod{q} \Rightarrow q \mid p^q + q^p - p - q$ — (I)

FLT $\Rightarrow q^p \equiv q \pmod{p} \Rightarrow p \mid p^q + q^p - p - q$ — (II)

Since $(p, q) = 1$, (I), (II) $\Rightarrow pq \mid p^q + q^p - p - q$

$\Rightarrow \boxed{p^q + q^p \equiv p + q \pmod{pq}}$

10) Suppose to the contrary, assume that White doesn't have a non-losing strategy

(i.e, $\exists$ a winning strategy for Black) [2nd player]

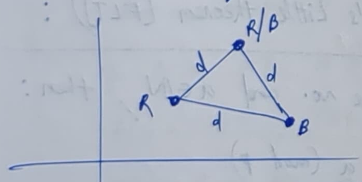
Note that White can "pass" the 1st round (i.e, by playing one of its knights and then moving it back to its original position)

Now, White is in the same situation as Black (i.e, it is the 2nd player now) and thus should have a winning strategy

($\Rightarrow\Leftarrow$)

$\square$

11) Consider an equilateral Δ of side d in $\mathbb{R}^2$:

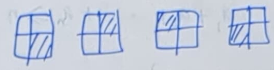


3 points, 2 colours $\Rightarrow$ apply PHP ✓

A simple argument without using PHP is also fine

12) We'll use induction on n

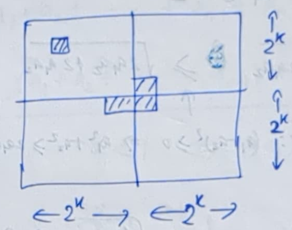
Base case ($n=1$):



✓

Suppose the statement is true for $n = k$

Consider a $2^{k+1} \times 2^{k+1}$ checkerboard:



Apply induction hypothesis on the 4 $2^k \times 2^k$ checkerboards

14) Yes: $e^{\log_e 2} = 2$,

$\sqrt{2}^{2\log_2 3} = \sqrt{2}^{\log_{\sqrt{2}^2} 3^2} = \sqrt{2}^{\log_{\sqrt{2}} 3} = 3$

Remark: Let $x = \sqrt{2}^{\sqrt{2}}$

If $x \in \mathbb{Q}$, we are done ✓

If $x \notin \mathbb{Q}$, take $y = \sqrt{2}$

Then, $x^y = \sqrt{2}^2 = 2 \in \mathbb{Q}$ ✓

15) $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ s.t $a_1 a_2 \cdots a_n = 1$

To prove : $a_1 + a_2 + \cdots + a_n \geq n$, $\forall n \geq 2$ —— I

Pf (By induction on n)

Base case ($n=2$):

$a_1, a_2 \in \mathbb{R}^+$ s.t $a_1 a_2 = 1$

$$a_1 + a_2 = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2} \geq \sqrt{2a_1 a_2 + 2a_1 a_2}$$

($\because (a_1 - a_2)^2 \geq 0 \Rightarrow a_1^2 + a_2^2 \geq 2a_1 a_2$)

$$= \sqrt{4a_1 a_2} = 2 \quad \checkmark$$

Suppose I is true for $n = n$

Now, let $a_1, a_2, \dots, a_n, a_{n+1} \in \mathbb{R}^+$ s.t $a_1 a_2 \cdots a_n a_{n+1} = 1$

ETS : $a_1 + a_2 + \cdots + a_n + a_{n+1} \geq n+1$

case I ($a_i = 1$, $\forall i \in \{1, 2, \dots, n+1\}$):

$a_1 + a_2 + \cdots + a_{n+1} = n+1 \geq n+1$ ✓

case II ($a_j < 1$ for some $j \in \{1, 2, \dots, n+1\}$):

Then, $\exists k \in \{1, 2, \dots, n+1\} \setminus j$ s.t $a_k > 1$

WLOG, let $j = n$, $k = n+1$ (i.e, $a_n < 1$, $a_{n+1} > 1$)

Now, consider the following n numbers :

$a_1, a_2, \dots, a_{n-1}, a_n a_{n+1}$

Since $a_1 a_2 \cdots a_{n-1} a_n a_{n+1} = 1$, by induction hypothesis :

$a_1 + a_2 + \cdots + a_{n-1} + a_n a_{n+1} \geq n$ —— (*)

Now, $a_1 + a_2 + \cdots + a_{n-1} + a_n + a_{n+1}$

$= (a_1 + a_2 + \cdots + a_{n-1}) + (a_n + a_{n+1})$

NOTE : $a_1 < 1 \Leftrightarrow 1 - a_1 > 0$

$a_{n+1} > 1$

$\Rightarrow (1 - a_n) a_{n+1} > 1 - a_n$

$\Rightarrow a_{n+1} - a_n a_{n+1} > 1 - a_n$

$\Rightarrow \boxed{a_n + a_{n+1} > 1 + a_n a_{n+1}}$

$> a_1 + a_2 + \cdots + a_{n-1} + 1 + a_n a_{n+1}$

$= (a_1 + a_2 + \cdots + a_{n-1} + a_n a_{n+1}) + 1$

$\geq n + 1$ (using —(*))

as required

$\therefore$ I is true $\forall n \geq 2$ and equality holds IFF $a_i = 1$, $\forall i$

17) $8x^4 + 4y^4 + 2z^4 = t^4$; $x, y, z, t \in \mathbb{Z}$ — ⊛

LHS is even $\Rightarrow t^4$ is even $\Rightarrow t$ is even

$\therefore$ Let $t = 2t_1$, $t_1 \in \mathbb{Z}$

⊛ $\Rightarrow 8x^4 + 4y^4 + 2z^4 = 16t_1^4$

$\Rightarrow 4x^4 + 2y^4 + z^4 = 8t_1^4$ — I

$\Rightarrow z^4 = 8t_1^4 - 4x^4 - 2y^4$ is even

$\Rightarrow z$ is even

Let $z = 2z_1$, $z_1 \in \mathbb{Z}$

I $\Rightarrow 4x^4 + 2y^4 + 16z_1^4 = 8t_1^4$

$\Rightarrow 2x^4 + y^4 + 8z_1^4 = 4t_1^4$ — II

$\Rightarrow y^4 = 4t_1^4 - 2x^4 - 8z_1^4$ is even

$\Rightarrow y$ is even

Let $y = 2y_1$, $y_1 \in \mathbb{Z}$

II $\Rightarrow 2x^4 + 16y_1^4 + 8z_1^4 = 4t_1^4$

$\Rightarrow x^4 + 8y_1^4 + 4z_1^4 = 2t_1^4$ — III

$\Rightarrow x^4 = 2t_1^4 - 8y_1^4 - 4z_1^4$ is even

$\Rightarrow x$ is even

Let $x = 2x_1$, $x_1 \in \mathbb{Z}$

III $\Rightarrow 16x_1^4 + 8y_1^4 + 4z_1^4 = 2t_1^4$

$\Rightarrow 8x_1^4 + 4y_1^4 + 2z_1^4 = t_1^4$; $x_1, y_1, z_1, t_1 \in \mathbb{Z}$

where $x_1 = \frac{x}{2}$, $y_1 = \frac{y}{2}$, $z_1 = \frac{z}{2}$, $t_1 = \frac{t}{2}$

i.e, if (x, y, z, t) is a solⁿ of ⊛,
then $\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{t}{2}\right)$ is also a solⁿ of ⊛ $\Big\}$ — IV

~~case I ($x=y=z=t=0$):~~

Clearly, $(0,0,0,0)$ is a solⁿ of ⊛

~~case II~~: Suppose (x, y, z, t) is a non-zero solⁿ set of ⊛

Then, by IV, $\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}, \frac{t}{2^k}\right)$ must also be a solⁿ of ⊛, $\forall k \in \mathbb{N}$

But this is impossible since no integer is divisible by arbitrarily large powers of 2

$\therefore (0,0,0,0)$ is the only solⁿ of ⊛

22) Fundamental Theorem of Arithmetic :

Every integer $n \geq 2$ can be represented uniquely as a product of prime numbers (upto the order of the factors)

pf: Two parts: ① product of primes
② uniqueness

part 1 (product of primes) :

We'll use (strong) induction on n

Base case $(n=2)$: $2 = 2 \rightarrow$ prime ✓

Suppose true for $n = 2, 3, \ldots, k$

Consider $n = k+1$

If $(k+1)$ is prime, we are done ✓

If not, then $(k+1)$ is a product of smaller numbers $\in \{2, 3, \ldots, k\}$ and by induction hypothesis, both are product of primes

Thus, $(k+1)$ is a product of primes ✓

part 2 (uniqueness): We'll again strongly induct on n

Base case $(n=2)$: $2 = 2$ ← unique factorization

Suppose true for $n = 2, 3, \ldots, k$

Consider $n = k+1$

If $(k+1)$ is prime, then done ✓

Suppose not:

By part 1, let $(k+1) = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$, where p_i's and q_i's are primes $\in \{2, 3, \ldots, k\}$

All we have to do is check that p_1 is one of the q_j's and then we are done by the induction hypothesis (cancel $p_1 = q_j$ and what is left is uniquely the product of primes, so all the other p_i's and q_i's must match up)

If $p_1 = q_1$, we are done ✓

Suppose not (i.e, let $p_1 < q_1$ [WLOG])

Now, let $q_1 - p_1 = r_1 r_2 \cdots r_t$ be the unique prime factorization of $q_1 - p_1$ (by induction hypothesis since $0 < q_1 - p_1 < k+1$),

where r_i's are primes

Now, let $m := r_1 r_2 \cdots r_t \, q_2 q_3 \cdots q_s$

$$\Rightarrow m = (q_1 - p_1)\, q_2 q_3 \cdots q_s = (k+1) - p_1 q_2 q_3 \cdots q_s$$

Since $2 \le m \le k$, by induction hypothesis, m is uniquely the product of primes

Now, $p_1 \mid (k+1)$, $p_1 \mid p_1 q_2 q_3 \cdots q_s$

$$\Rightarrow p_1 \mid (k+1) - p_1 q_2 q_3 \cdots q_s = m, \text{ i.e, } p_1 \mid m$$

Since $p_1 \mid m$, the unique prime factorization of $m = r_1 r_2 \cdots r_t \, q_2 q_3 \cdots q_s$ must contain p_1

Thus, either $p_1 = r_i$ for some $i \in \{1, 2, \dots, t\}$

OR $p_1 = q_j$ for some $j \in \{2, 3, \dots, s\}$

If $p_1 = q_j$ for some $j \in \{2, 3, \dots, s\}$, we are done ✓

Suppose not, i.e, let $p_1 = r_i$ for some $i \in \{1, 2, \dots, t\}$

Since $q_1 - p_1 = r_1 r_2 \cdots r_t \Rightarrow p_1 \mid q_1$

$(\Rightarrow\Leftarrow)$ since $p_1 < q_1$

□

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24) $x \in \mathbb{R}$ s.t $\boxed{x + \frac{1}{x} \in \mathbb{Z}}$

To prove: $x^n + \frac{1}{x^n} \in \mathbb{Z}$, $\forall n \in \mathbb{N}$ — (*)

pf: We'll use strong induction on n

Base case ($n = 1, 2$): $n = 1$ ✓

$n = 2$: $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 \in \mathbb{Z}$ ✓

Suppose (*) is true for $n = 1, 2, \dots, k$

Now, $\left(x + \frac{1}{x}\right)^{k+1}$

$$= x^{k+1} + \frac{1}{x^{k+1}} + a_k\left(x^k + \frac{1}{x^k}\right) + a_{k-1}\left(x^{k-1} + \frac{1}{x^{k-1}}\right) + \cdots + a_1\left(x + \frac{1}{x}\right) + a_0, \quad a_i \in \mathbb{N} \cup \{0\}$$

$$\Rightarrow x^{k+1} + \frac{1}{x^{k+1}} \in \mathbb{Z}$$ (using induction hypothesis)

□

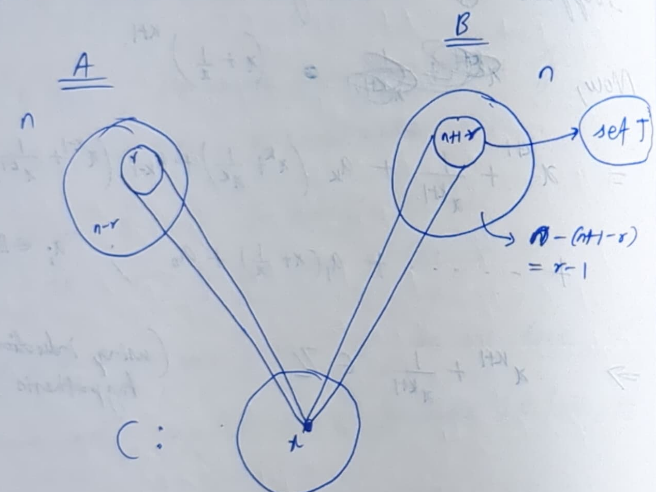
25) schools: $\boxed{A}$ $\boxed{B}$ $\boxed{C}$

n students n students n students

~~Consider a student~~

Out of 3n students, consider a student ~~(say x)~~ with minimum no. of friends from another school

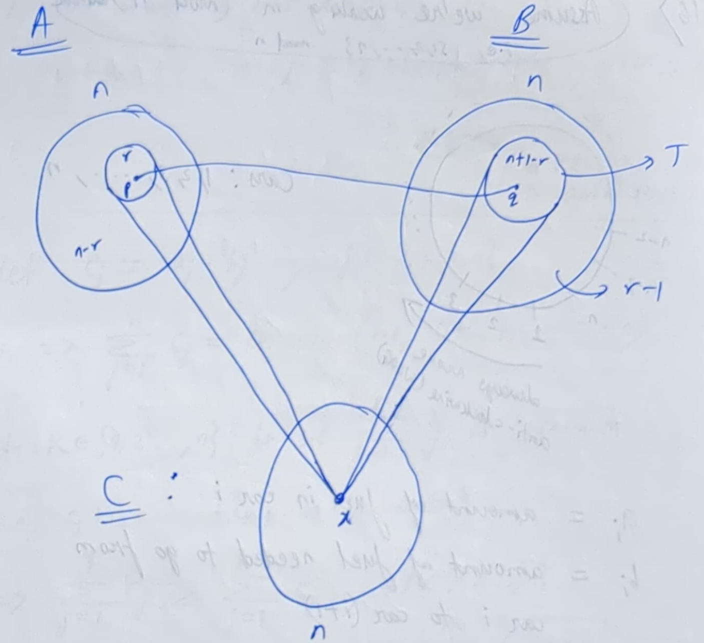
Say $x \in C$ has min. no. of friends (say r) from school A (WLOG)



Suppose $p \in A$ is a friend of $x \in C$

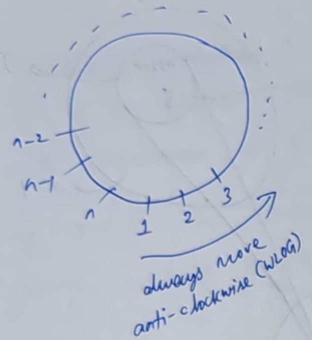
Since x has min. no. of friends r from another school, p must have $\geq r$ friends from school B

Since $|B \setminus T| = r-1$, p must have a friend from T, say $q \in B$



Thus, $p \in A$, $q \in B$ and $x \in C$ know each other □

16) Assume we're working in (mod n) universe
i.e, $\{1,2,\dots,n\}$ mod n



cars : $1, 2, 3, \dots, n$

a_i = amount of fuel in car i

b_i = amount of fuel needed to go from car i to car $(i+1)$

Given that : $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ ——— $(*)$

want to find $k \in \{1,2,\dots,n\}$ s.t :

$$\boxed{a_k \geq b_k}$$

$$(a_k - b_k) + a_{k+1} \geq b_{k+1} \iff \boxed{a_k + a_{k+1} \geq b_k + b_{k+1}}$$

$$(a_k - b_k) + (a_{k+1} - b_{k+1}) + a_{k+2} \geq b_{k+2} \iff \boxed{a_k + a_{k+1} + a_{k+2} \geq b_k + b_{k+1} + b_{k+2}}$$

$\vdots$

$$\boxed{a_k + a_{k+1} + a_{k+2} + \dots + a_n + a_1 + a_2 + \dots + a_{k-1} \geq b_k + b_{k+1} + b_{k+2} + \dots + b_n + b_1 + b_2 + \dots + b_{k-1}}$$

i.e, we want to find $k \in \{1,2,\dots,n\}$ s.t

$$a_k + a_{k+1} + \dots + a_l \geq b_k + b_{k+1} + \dots + b_l, \quad \forall l \in \{1,2,\dots,n\}$$

($\forall l \geq k$ in (mod n) sense)

Let $c_j := a_j - b_j$, $j \in \{1,2,\dots,n\}$

$(*) \Rightarrow \sum_{j=1}^{n} c_j = 0$ (i.e, $c_1 + c_2 + \dots + c_n = 0$)

Let $k \in \{1,2,\dots,n\}$ be s.t $\sum_{j=1}^{k-1} c_j$ is minimum

(i.e, $c_1 + c_2 + \dots + c_{k-1}$ is minimum)

$\Rightarrow \sum_{j=1}^{l} c_j \geq \sum_{j=1}^{k-1} c_j$, $\forall l \in \{1,2,\dots,n\}$ ($\forall l \geq k$ in (mod n) sense)

$\Rightarrow \sum_{j=k}^{l} c_j \geq 0$, $\forall l \in \{1,2,\dots,n\}$

Example: Let $k = 7$. Then for $l = 2$:

$$\sum_{j=1}^{2} c_j \geq \sum_{j=1}^{6} c_j$$

$$\Rightarrow c_1 + c_2 + \dots + c_n + c_1 + c_2 \geq c_1 + c_2 + \dots + c_6$$

$$\Rightarrow c_7 + c_8 + \dots + c_{n-1} + c_n + c_1 + c_2 \geq 0$$

$\Rightarrow \sum_{j=k}^{l} (a_j - b_j) \geq 0$, $\forall l \in \{1,2,\dots,n\}$

$\Rightarrow a_k + a_{k+1} + \dots + a_l \geq b_k + b_{k+1} + \dots + b_l$, $\forall l \in \{1,2,\dots,n\}$

as required ✓

20) Let $G=(V,E)$ be a graph

To prove: Either G or $\overline{G}$ is connected

complement of G

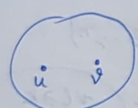
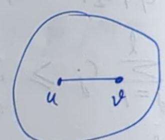
Pf: Suppose G is disconnected

ETS: $\overline{G}$ is connected

Let $\overline{G}=(V,\overline{E})$

Let $u, v \in V$ (arbitrary)

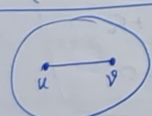
case I ($\{u,v\} \notin E$):

G:  $\Rightarrow$ $\overline{G}$: 

$\{u,v\} \in \overline{E}$

$\Rightarrow$ u, v are connected in $\overline{G}$ ✓

case II ($\{u,v\} \in E$):

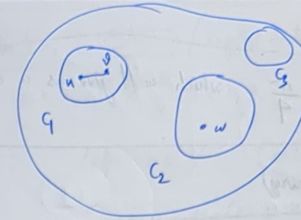
G: 

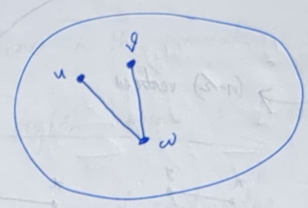
Since G is disconnected, it has atleast two connected components (say C_1, C_2)

Suppose $u, v \in C_1$

Let $w \in C_2$ (arbitrary)

Note that $\{u,w\} \notin E$, $\{v,w\} \notin E$

G: 

$\overline{G}$: 

i.e, $\{u,w\} \in \overline{E}$, $\{v,w\} \in \overline{E}$

i.e, u and v are connected in $\overline{G}$ ✓

$\therefore$ Any two vertices in $\overline{G}$ have a path (infact a path of length one or two) b/w them in $\overline{G}$

Thus, $\overline{G}$ is connected □

21) $G=(V,E)$, $|V|=n$, $|E| > \frac{n^2}{4}$

To prove: G contains a △

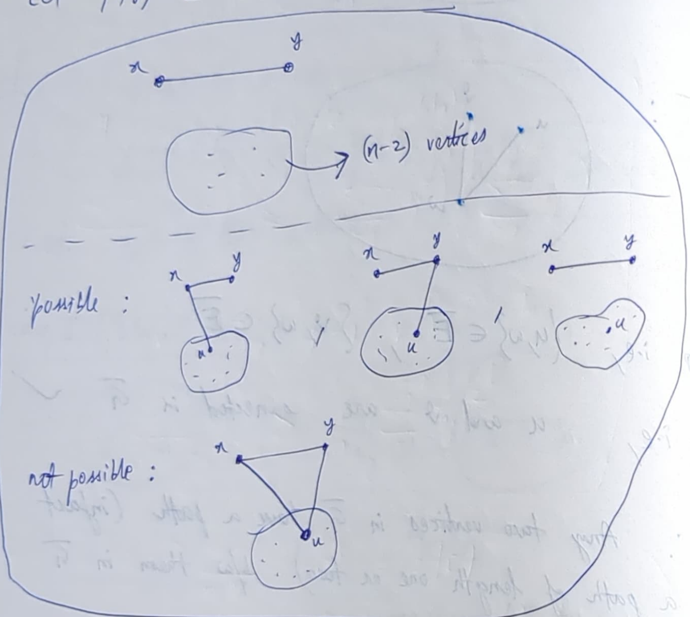
Pf: Suppose not (i.e, let G doesn't contain a △)

PTO →

Let $|E| = m$

We'll show that $m \leq \frac{n^2}{4}$ which will give us a contradiction

Let $\{x, y\} \in E$ (arbitrary)



Since G doesn't have a $\triangle$, every vertex of G is connected to atmost one of x and y

Thus, $\deg(x) + \deg(y) \leq (n-2) + 1 + 1 = n$

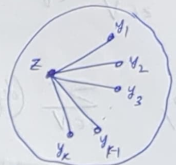
$$\Rightarrow \sum_{\{x,y\} \in E} \left(\deg(x) + \deg(y)\right) \leq \sum_{\{x,y\} \in E} n = mn$$

i.e, $\displaystyle\sum_{\{x,y\} \in E} \left(\deg(x) + \deg(y)\right) \leq mn$ —— $\circledast$

Claim 1 : $\displaystyle\sum_{\{x,y\} \in E} \left(\deg(x) + \deg(y)\right) = \sum_{z \in V} \left(\deg(z)\right)^2$

pf : Let $z \in V$ (arbitrary)

Suppose $\deg(z) = k$



Hence, in the summation $\displaystyle\sum_{\{x,y\} \in E} (\deg(x) + \deg(y))$,

$\deg(z)$ appears k times (corresponding to each $\{z, y_i\} \in E$)

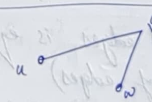
i.e, $z \in V$ contributes $\underbrace{\deg(z) + \deg(z) + \cdots + \deg(z)}_{k \text{ times}}$

$$= k \cdot \deg(z) = \deg(z) \cdot \deg(z) = (\deg(z))^2$$

Since $z \in V$ was arbitrary, we have :

$$\sum_{\{x,y\} \in E} \left(\deg(x) + \deg(y)\right) = \sum_{z \in V} \left(\deg(z)\right)^2$$

Example :



Here, $\displaystyle\sum_{\{x,y\} \in E} \left(\deg(x) + \deg(y)\right) = [\deg(u) + \deg(v)] + [\deg(v) + \deg(w)]$

$$= (1+2) + (2+1)$$

$$= 1^2 + 2^2 + 1^2$$

$$= \sum_{z \in V} (\deg(z))^2$$

Using claim 1, we get:

$$\sum_{z\in V}(\deg(z))^2 = \sum_{\{x,y\}\in E}\left(\deg(x)+\deg(y)\right) \le mn \quad \text{(by } \circledast\text{)}$$

i.e, $\sum_{z\in V}(\deg(z))^2 \le mn$ —— I

Recall (Cauchy-Schwartz inequality):

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \le \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

COROLLARY (Taking $b_j = 1, \forall j$):

$$\sum_{j=1}^{n} a_j^2 \ge \frac{1}{n}\left(\sum_{j=1}^{n} a_j\right)^2$$

Claim 2: $\sum_{x\in V}\deg(x) = 2|E|$

(i.e, sum of all edges is equal to twice the no. of edges)

pf: Every edge of the graph contributes 2 to the degree sum

Now, $\sum_{z\in V}(\deg(z))^2 \ge \frac{1}{n}\left(\sum_{z\in V}\deg(z)\right)^2$ → using CS inequality

$= \frac{1}{n}\times(2m)^2$ → by Claim 2

$= \frac{4m^2}{n}$

i.e, $\sum_{z\in V}(\deg(z))^2 \ge \frac{4m^2}{n}$ —— II

I, II $\Rightarrow mn \ge \frac{4m^2}{n}$

$\Rightarrow m(n^2-4m) \ge 0$

$\Rightarrow n^2-4m \ge 0$

$\Rightarrow m \le \frac{n^2}{4}$

$(\Rightarrow\Leftarrow)$

$\therefore$ G contains a Δ □

BQ 1> [midsem]

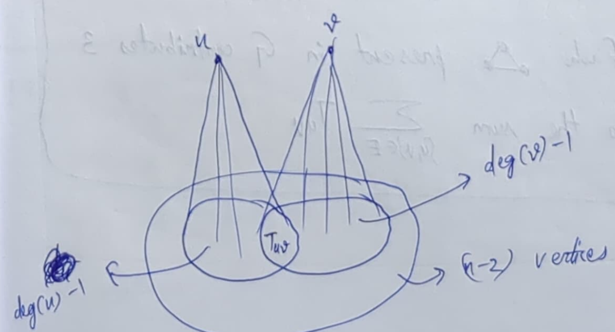
$G = (V, E)$, $|V| = n$, $|E| = m$

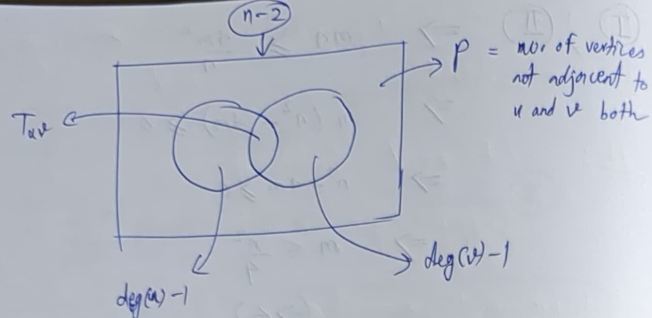
G has T Δ's

To show: $T \ge \frac{m}{3n}(4m-n^2)$

pf: Let $\{u, v\} \in E$ (arbitrary)

Let T_{uv} := no. of Δ's in G containing the edge $\{u,v\}$





$$(n-2) = P + \left[(\deg(u)-1) - T_{uv}\right] + \left[(\deg(v)-1) - T_{uv}\right] + T_{uv}$$

$$\Rightarrow (n-2) = P + (\deg(u)+\deg(v)-2) - T_{uv}$$

$$\Rightarrow T_{uv} = \left[\deg(u)+\deg(v)-n\right] + P$$

$$\geq \deg(u)+\deg(v)-n \quad (\because P \geq 0)$$

i.e, $T_{uv} \geq \deg(u)+\deg(v)-n$ —— (*)

Claim: $\displaystyle\sum_{\{u,v\}\in E} T_{uv} = 3T$

Pf: Each $\triangle$ present in G contributes 3 to the sum $\displaystyle\sum_{\{u,v\}\in E} T_{uv}$

Thus, from claim and (*), we have:

$$3T = \sum_{\{u,v\}\in E} T_{uv} \geq \sum_{\{u,v\}\in E} (\deg(u)+\deg(v)-n)$$

$$= \left(\sum_{\{u,v\}\in E} \deg(u)+\deg(v)\right) - mn$$

$$= \left[\sum_{z\in V} (\deg(z))^2\right] - mn$$

see previous answer

$$\geq \frac{\left(\sum_{z\in V} \deg(z)\right)^2}{n} - mn$$

(by CS inequality)

$$= \frac{(2m)^2}{n} - mn$$

$$= \frac{4m^2}{n} - mn = \frac{m}{n}(4m-n^2)$$

$$\Rightarrow T \geq \frac{m}{3n}(4m-n^2)$$

□

19) $G = (V,E)$: connected graph, $|V| = n > 2$

degree of any vertex of G $\in \{1, 2, \ldots, n-1\}$

(0 not possible as G is connected)

n vertices, (n-1) possible degrees $\Rightarrow$ two vertices have same degree (by PHP)

Ans: No