A1. Let w and g be two image arrays $(N \times N)$

and $W = \sum_{n=1}^{N} N_n w V_n$ be vector version of image w

and $G = \sum_{n=1}^{N} N_n g V_n$ be vector version of image g.

Here, V_n is a column vector with n^{th} row 1 and rest all 0

N_n is a $N^2 \times N$ matrix with N matrices of order $N \times N$ stacked on each other with nth matrix as I and rest all 0.

Now,

$$\sum_{n=1}^{N} N_n (\alpha w + \beta g) V_n$$

$$= \sum_{n=1}^{N} N_n (\alpha w V_n + \beta g V_n)$$

$$= \sum_{n=1}^{N} (N_n \alpha w V_n + N_n \beta g V_n)$$

$$= \sum_{n=1}^{N} N_n \alpha w V_n + \sum_{n=1}^{N} N_n \beta g V_n$$

Here, α, β are constants. So,

$$\sum_{n=1}^{N} N_n (\alpha w + \beta g) V_n = \alpha \sum_{n=1}^{N} N_n w V_n + \beta \sum_{n=1}^{N} N_n g V_n$$

$$= \alpha W + \beta G$$

A2. For any entry in an $N \times N$ image, we can do a bitwise and with a mask to obtain whether a particular value is set or not.

So, for each bit plane we will have a mask such that we get to know that particular bit is set or not. This is to be done for all the pixels.

Mask for bit plane 1 (LSB) = 1 (0000 0001)

Mask for bit plane 2 = 2 (00000010)

Mask for bit plane 3 = 4 (00000100)

Mask for bit plane 4 = 8 (00001000)

Mask for bit plane 5 = 16 (00010000)

Mask for bit plane 6 = 32 (00100000)

Mask for bit plane 7 = 64 (01000000)

Mask for bit plane 8 (MSB) = 128 (10000000)

We will iterate over all pixels, do a bitwise and with the mask of the respective plane to get the value ~~of bit of~~ as 1 or 0 for that pixel and the plane.

We can store the results in N X N array for each bit plane.

A3. Let no. of pixels with r_k intensity level be n_k

Total no. of pixels = MN

$r_k \in [0, L-1]$ (L intensity levels)

$$P_r(r_k) = \frac{n_k}{MN}$$ (Probability of occurrence of intensity level r_k)

Now, s_k = intensity level in the output image corresponding to r_k intensity level of input image

Then, $$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} P_r(r_j), \quad k = 0, 1, \ldots L-1$$

Upon applying second round of histogram equalization, let the corresponding intensity level be u_k. Then,

$$u_k = T(s_k) = (L-1)\sum_{j=0}^{k} P_s(s_j)$$

where $$P_s(s_k) = \frac{n_k}{MN}$$

Now, since every pixel with value r_k is mapped to s_k, we can say that $n_k = n'_k$

Therefore,

$$u_k = (L-1)\sum_{j=0}^{k} p_s(s_j) = (L-1)\sum_{j=0}^{k} \frac{n'_k}{MN}$$

$$= (L-1)\sum_{j=0}^{k} \frac{n_k}{MN} = (L-1)\sum_{j=0}^{k} p_r(r_j)$$

$$= s_k$$

Therefore, we can say that a second pass of the histogram equalization would give us same result as the first pass.

A4. $p_r(r) = Ae^{-r}$; $r \in [0,b]$

$p_s(s) = Bse^{-s^2}$; $s \in [0,b]$

Now,

$$z = T(r) = b\int_0^r p_r(w)\,dw$$

$$= b\int_0^r Ae^{-w}\,dw = Ab\left[\frac{e^{-w}}{-1}\right]_0^r$$

$$= Ab(1-e^{-r})$$

Now,

$$G(s) = b\int_0^s p_s(v)\,dv$$

$$= \cancel{b\int_0^s Bse^{-s^2}},\ b\int_0^s Bve^{-v^2}\,dv$$

$$= \frac{Bb}{2}\int_0^s 2ve^{-v^2}\,dv$$

$$= \frac{Bb}{2}\left[\frac{e^{-v^2}}{-1}\right]_0^s$$

$$= \frac{Bb}{2}(1-e^{-s^2})$$

Now, $z = G(s)$

$\Rightarrow s = G^{-1}(z)$

$= G^{-1}(T(r))$

$\therefore AK(1-e^{-r}) = \frac{BK}{2}(1-e^{-s^2})$

$\Rightarrow \frac{2A}{B}(1-e^{-r}) = 1-e^{-s^2}$

$\Rightarrow e^{-s^2} = 1 - \frac{2A}{B}(1-e^{-r})$

taking log both sides

$-s^2 = \ln\left(1-\frac{2A}{B}(1-e^{-r})\right)$

$\Rightarrow s = \left(-\ln\left(1-\frac{2A}{B}(1-e^{-r})\right)\right)^{1/2}$

$= \left(\ln B - \ln\left(B - 2A(1-e^{-r})\right)\right)^{1/2}$

A5. $p_r(r) = 2r/(L-1)^2$; for $0 \le r \le L-1$

$= 0$; o/w

(a) Histogram equalization

$s = T(r) = (L-1)\int_0^r p_r(v)\,dv$

$= \cancel{\frac{2}{(L-1)}\int_0^r v\,dv}$

$= (L-1)\int_0^r \frac{2v}{(L-1)^2}\,dv$

$= \frac{2}{(L-1)}\int_0^r v\,dv$

$= \frac{2}{(L-1)}\left[\frac{v^2}{2}\right]_0^r = \frac{r^2}{(L-1)}$

(b) $P_z(z) = \frac{3z^2}{(L-1)^3}$; $0 \le z \le L-1$

$= 0$; o/w

~~$G(z) = (L-1) \int_0^z P_{z}(z$~~

$$G(z) = (L-1) \int_0^z P_z(v)\,dv$$

$$= (L-1) \int_0^z \frac{3v^2}{(L-1)^3}\,dv$$

$$= \frac{3}{(L-1)^2} \int_0^z v^2\,dv = \frac{z^3}{(L-1)^2}$$

Now, $G(z) = s \Rightarrow z = G^{-1}(s)$

We have calculated s in part (a)

$\therefore s = \frac{z^3}{(L-1)^2} \Rightarrow z = (s(L-1)^2)^{1/3}$ is the required f^n

(c) $z = (s(L-1)^2)^{1/3}$

$$= \left(\frac{r^2 (L-1)^2}{(L-1)}\right)^{1/3} = \cancel{r^2}$$

$$= ((L-1) r^2)^{1/3}$$ is the required of transformat^n function

A6. Commutative Property of Convolution

$$f * g = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau \quad -①$$

$$g * f = \int_{-\infty}^{+\infty} g(\tau)\, f(t-\tau)\, d\tau \quad -②$$

Let $t - \tau = \theta$

$\Rightarrow -d\tau = d\theta \quad -③$

$$f * g = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_{+\infty}^{-\infty} f(t-\theta)\, g(\theta)\, (-d\theta) \quad (\text{from } 3)$$

$$= \int_{-\infty}^{+\infty} g(\theta)\, f(t-\theta)\, d\theta$$

$$= g * f$$

Associative property of Convolution.

$$f * (g * h) = \int_{-\infty}^{+\infty} f(\tau)\, (g * h)(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} f(\tau) \int_{-\infty}^{+\infty} g(\theta)\, h(t-\tau-\theta)\, d\theta\, d\tau$$

$$(f * g) * h = \int_{-\infty}^{+\infty} (f * g)(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(\theta)\, g(\tau-\theta)\, d\theta \right) h(t-\tau)\, d\tau$$

let $\tau - \theta = \phi$ $\qquad d\tau = d\phi$ — 4

$\cancel{d\tau = d\phi \text{ — } 4}$ $\quad -d\theta = d\phi$ — 5

$$\therefore (f * g) * h = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(\theta)\, g(\tau-\theta)\, d\theta \right) h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} \left(\int_{+\infty}^{-\infty} f(\tau-\phi)\, g(\phi)\, (-d\phi) \right) h(t-\tau)\, d\tau \quad (\text{from } 4)$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(\tau-\phi)\, g(\phi)\, d\phi \right) h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(\theta)\, g(\phi)\, d\phi \right) h(t-\theta-\phi)\, d\phi \quad (\text{from } 5)$$

$$= \int_{-\infty}^{+\infty} f(\theta) \left(\int_{-\infty}^{+\infty} g(\phi)\, h(t-\theta-\phi)\, d\phi \right) d\theta$$

$$= f * (g * h)$$

A7. Let $w_1 \in \mathbb{R}_{m\times 1}$

$w_2 \in \mathbb{R}_{1\times n}$

2D convolution between w_1 and w_2 can be defined by

$$(w_1 * w_2)(x,y) = \sum_s \sum_t w_1(s,t) \cdot w_2(x-s, y-t)$$

Now, deducing the limits of the summation from the dimensions of w_1, w_2.

Choosing zero-based indexing, $t=0$ and

$x - s = 0 \Rightarrow s = x$

$$\therefore (w_1 * w_2)(x,y) = \sum_{s=x} \sum_{t=0} w_1(s,t) \cdot w_2(x-s, y-t)$$

$$= w_1(x,0) \cdot w_2(0,y)$$

$$= (w_1 \cdot w_2)_{m\times n}(x,y)$$

A8. Let $f(x) = \frac{1}{\sqrt{2\pi}\sigma_f} e^{-\frac{(x-\mu_f)^2}{2\sigma_f^2}}$

$g(x) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\frac{(x-\mu_g)^2}{2\sigma_g^2}}$ be two Gaussian function

Now, $f \cdot g = \frac{1}{2\pi\sigma_f\sigma_g} e^{-\frac{1}{2}\left[\frac{(x-\mu_f)^2}{\sigma_f^2} + \frac{(x-\mu_g)^2}{\sigma_g^2}\right]}$ — ①

And, $\frac{(x-\mu_f)^2}{\sigma_f^2} + \frac{(x-\mu_g)^2}{\sigma_g^2} = \frac{x^2 - 2x\mu_f + \mu_f^2}{\sigma_f^2} + \frac{x^2 - 2x\mu_g + \mu_g^2}{\sigma_g^2}$

$$= \frac{\sigma_g^2 x^2 - 2x\mu_f\sigma_g^2 + \mu_f^2\sigma_g^2 + \sigma_f^2 x^2 - 2x\mu_g\sigma_f^2 + \mu_g^2\sigma_f^2}{\sigma_f^2\sigma_g^2}$$

$$= \frac{(\sigma_g^2+\sigma_f^2)x^2 - 2x(\mu_f\sigma_g^2+\mu_g\sigma_f^2) + \mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{\sigma_f^2\sigma_g^2}$$

Dividing by $(\sigma_g^2+\sigma_f^2)$ in ~~a~~ numerator and denominator, we get:

$$\frac{(x-\mu_f)^2}{\sigma_f^2} + \frac{(x-\mu_g)^2}{\sigma_g^2} = \frac{x^2 - 2x\left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_g^2+\sigma_f^2}\right) + \frac{\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2}{\sigma_g^2+\sigma_f^2}}{\frac{\sigma_f^2\sigma_g^2}{\sigma_g^2+\sigma_f^2}}$$

Upon solving, we get

$$\sigma_{fg} = \sqrt{\frac{\sigma_f^2 \sigma_g^2}{\sigma_f^2 + \sigma_g^2}} \quad \text{and} \quad \mu_{fg} = \frac{\mu_f \sigma_g^2 + \mu_g \sigma_f^2}{\sigma_f^2 + \sigma_g^2}$$

Finally, we get

$$f \cdot g = \frac{1}{\sqrt{2\pi}\, \sigma_{fg}} e^{-\frac{1}{2}\left(\frac{x - \mu_{fg}}{\sigma_{fg}}\right)^2}$$

For convolution,

$$\cancel{f \otimes g} \; f * g\,(t) = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\, \sigma_f} e^{-\frac{(\tau-\mu_f)^2}{2\sigma_f^2}} \cdot \frac{1}{\sqrt{2\pi}\, \sigma_g} e^{-\frac{(t-\tau-\mu_g)^2}{2\sigma_g^2}}\, d\tau$$

$$= \frac{1}{2\pi \sigma_f \sigma_g} \int_{-\infty}^{+\infty} \exp\left\{ -\frac{(\tau-\mu_f)^2}{2\sigma_f^2} - \frac{(t-\tau-\mu_g)^2}{2\sigma_g^2} \right\} d\tau$$

$$= \frac{1}{2\pi \sigma_f \sigma_g} \int_{-\infty}^{+\infty} \exp\left\{ -\frac{(\tau^2 + \mu_f^2 - 2\tau\mu_f)}{2\sigma_f^2} - \frac{(t^2 + \tau^2 + \mu_g^2 - 2t\tau + 2\tau\mu_g - 2t\mu_g)}{2\sigma_g^2} \right\}$$

$$= \frac{1}{2\pi \sigma_f \sigma_g} \int_{-\infty}^{+\infty} \exp\left\{ \frac{-2\sigma_g^2\tau^2 - 2\sigma_g^2\mu_f^2 + 4\tau\mu_f\sigma_g^2 - 2t^2\sigma_f^2 - 2\tau^2\sigma_f^2 - 2\sigma_f^2\mu_g^2 + 4t\tau\sigma_f^2 - 4\tau\mu_g\sigma_f^2 + 4t\mu_g\sigma_f^2}{4\sigma_f^2\sigma_g^2} \right\}$$

A9.

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\cdot\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}$$

$$= \frac{\partial f}{\partial x'}\cos\theta + \frac{\partial f}{\partial y'}\sin\theta$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\cos\theta + \frac{\partial f}{\partial y'}\sin\theta\right)$$

$$= \cos\theta\,\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\right) + \sin\theta\,\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right)$$

$$= \cos\theta\,\frac{\partial^2 f}{\partial x'^2}\frac{\partial x'}{\partial x} + \sin\theta\,\frac{\partial^2 f}{\partial y'^2}\frac{\partial y'}{\partial x}$$

$$= \cos^2\theta\,\frac{\partial^2 f}{\partial x'^2} + \sin^2\theta\,\frac{\partial^2 f}{\partial y'^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}\cdot\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\cdot\frac{\partial y'}{\partial y}$$

$$= \frac{\partial f}{\partial x'}(-\sin\theta) + \frac{\partial f}{\partial y'}\cos\theta$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[-\sin\theta\,\frac{\partial f}{\partial x'} + \cos\theta\,\frac{\partial f}{\partial y'}\right]$$

$$= -\sin\theta\,\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x'}\right) + \cos\theta\,\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)$$

$$= -\sin\theta\,\frac{\partial^2 f}{\partial x'^2}\frac{\partial x'}{\partial y} + \cos\theta\,\frac{\partial^2 f}{\partial y'^2}\frac{\partial y'}{\partial y}$$

$$= \sin^2\theta\,\frac{\partial^2 f}{\partial x'^2} + \cos^2\theta\,\frac{\partial^2 f}{\partial y'^2}$$

$$\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= \cos^2\theta\,\frac{\partial^2 f}{\partial x'^2} + \sin^2\theta\,\frac{\partial^2 f}{\partial y'^2} + \sin^2\theta\,\frac{\partial^2 f}{\partial x'^2} + \cos^2\theta\,\frac{\partial^2 f}{\partial y'^2}$$

$$= \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

A10. Let K be the Kernal such that $\sum_{m,n} K(m,n) = 1$

Let f be the image (input) and

g be the image we get after convolving f with K

$$g(x,y) = \sum_{m,n} K(m,n) f(x-m, y-n)$$

Applying summation on both sides.

$$\sum_{x,y} g(x,y) = \sum_{x,y} \sum_{m,n} K(m,n) f(x-m, y-n)$$

Because of linearity of summation, we can interchange the order of summation. So;

$$\sum_{x,y} g(x,y) = \sum_{m,n} \sum_{x,y} K(m,n) f(x-m, y-n)$$

$$= \sum_{m,n} K(m,n) \sum_{x,y} f(x-m, y-n)$$

$$\because \sum_{m,n} K(m,n) = 1$$

$$\therefore \sum_{x,y} g(x,y) = \sum_{x,y} f(x-m, y-n)$$

A11. let K be the kernel such that $\sum_{m,n} K(m,n) = 0$

let f be the image (input) and

g be the image we get after convolving f by K

$$g(x,y) = \sum_{m,n} K(m,n) f(x-m, y-n)$$

Applying summation on both sides

$$\sum_{x,y} g(x,y) = \sum_{x,y} \sum_{m,n} K(m,n) f(x-m, y-n)$$

Because the linearity of summation, We can interchange the order of summation. So,

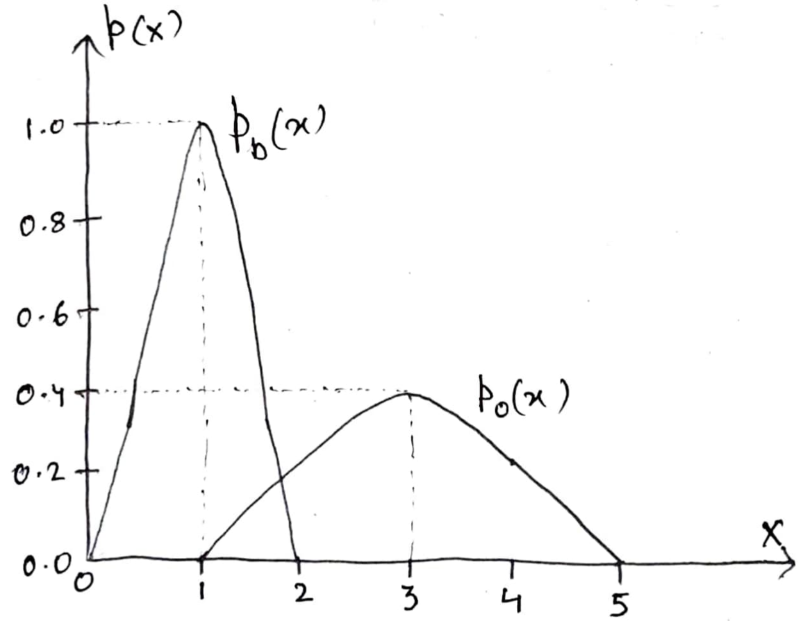
$$\sum_{x,y} g(x,y) = \sum_{m,n} \sum_{x,y} K(m,n) f(x-m, y-n)$$

$$= \sum_{m,n} K(m,n) \sum_{x,y} f(x-m, y-n)$$

$\because \sum_{m,n} K(m,n) = 0$

$\therefore \sum_{x,y} g(x,y) = 0$

A12.



Now, $\theta = \frac{1}{3}$

$$p_o(x) = \frac{\pi}{4} \cos \frac{(x-1)\pi}{2}$$

$$p_b(x) = \frac{\pi}{8} \cos \frac{(x-3)\pi}{4}$$

So, $\theta p_o(x) = (1-\theta) p_b(x)$

$$\Rightarrow \frac{1}{3} \cdot \frac{\pi}{4} \cos \frac{(x-1)\pi}{2} = \frac{2}{3} \cdot \frac{\pi}{8} \cos \frac{(x-3)\pi}{4}$$

$$\Rightarrow \cos \frac{(x-1)\pi}{2} = \cos \frac{(x-3)\pi}{4}$$

$$\Rightarrow \frac{(x-1)\pi}{2} = \pm \frac{(x-3)\pi}{4}$$

Now, $\frac{(x-1)\cancel{\pi}}{\cancel{2}} = \frac{(x-3)\cancel{\pi}}{\cancel{4}2}$

$\Rightarrow 2x-2 = x-3$

$\Rightarrow x = -1$ (Rejected as $x>0$)

$\frac{(x-1)\cancel{\pi}}{\cancel{2}} = -\frac{(x-3)\cancel{\pi}}{\cancel{4}2}$

$\Rightarrow 2x-2 = -x+3$

$\Rightarrow 3x = 5$

$\Rightarrow x = \frac{5}{3}$

$\therefore \frac{5}{3}$ is the threshold for minimum error.

Now, fraction of misclassified object pixels by optimal thresholding $= \int_{5/3}^{2} \frac{\pi}{4} \cos \frac{(x-1)\pi}{2} dx$

Let $t = x-1 \Rightarrow dt = dx$

$x = 5/3 \Rightarrow t = \frac{2}{3}$

$x = 2 \Rightarrow t = 1$

$$\therefore \int_{5/3}^{2} \frac{\pi}{4} \cos \frac{(x-1)\pi}{2} dx = \frac{\pi}{4} \int_{2/3}^{1} \cos \frac{t\pi}{2} dt$$

$$= \frac{\pi}{4}_{2} \left[\frac{\sin \frac{t\pi}{2}}{\frac{\pi}{2}} \right]_{2/3}^{1}$$

$$= \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{2-1.7}{4} = \frac{0.3}{4} = 7.5\%$$

A13. given : line $y = x \Rightarrow f(y,x) = f(x,y)$

$$m_{ij} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^i y^j f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^i y^j f(y,x)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^j x^i f(y,x)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^j x^i f(y,x)\,dy\,dx$$

$$= m_{ji} \quad \forall\, i,j$$

A14. $p_o = \frac{1}{2\sigma_o} \exp\left(-\frac{|x-\mu_o|}{\sigma_o}\right)$

$p_b = \frac{1}{2\sigma_b} \exp\left(-\frac{|x-\mu_b|}{\sigma_b}\right)$

$\mu_o = 60,\ \mu_b = 40,\ \sigma_o = 10,\ \sigma_b = 5,\ \theta = \frac{2}{3}$

We know, $\theta\, p_o(t) = (1-\theta)\, p_b(t)$

$$\frac{\theta}{2\sigma_o} \exp\left(-\frac{|t-\mu_o|}{\sigma_o}\right) = \frac{(1-\theta)}{2\sigma_b} \exp\left(-\frac{|t-\mu_b|}{\sigma_b}\right)$$

$$\Rightarrow \frac{\cancel{2}}{2\times 3\times \cancel{10}} \exp\left(-\frac{|t-60|}{10}\right) = \frac{1}{2\times 3\times 5} \exp\left(-\frac{|t-40|}{5}\right)$$

$$\Rightarrow +\frac{|t-60|}{\cancel{10}\,2} = +\frac{|t-40|}{\cancel{5}} \quad -(1)$$

Case-1 $(t<40)$

eq (1) changes to

$$\cancel{-(-(t-60))} + \frac{(t-60)}{2} = +(t-40)$$

$\Rightarrow t-60 = 2t-80$

$\Rightarrow t = 20$

Case-2 $(40<t<60)$

eq (1) changes to

$$-\frac{(t-60)}{2} = (t-40)$$

$\Rightarrow -t+60 = 2t-80$

$\Rightarrow 3t = 140 \Rightarrow t = \frac{140}{3} = 46.67 \approx 47$

Case-3 $(t>60)$

eq (1) changes to

$$\frac{(t-60)}{2} = (t-40)$$

$\Rightarrow t-60 = 2t-80 \Rightarrow t = 20$ (rejected)

$\therefore t_1 = 20$ and $t_2 = 47$

A15. We know that translation can be denoted by

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

and rotation be denoted by

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Translation followed by rotation:

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{Transl}^n} \begin{bmatrix} x+a \\ y+b \end{bmatrix} \xrightarrow{\text{Rot}^n} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

$$= \begin{bmatrix} x\cos\theta + a\cos\theta - y\sin\theta - b\sin\theta \\ x\sin\theta + a\sin\theta + y\cos\theta + b\cos\theta \end{bmatrix}$$

Rotation followed by Translation

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{Rot}^n} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$$\Big\downarrow \text{translat}^n$$

$$\begin{bmatrix} x\cos\theta - y\sin\theta + a \\ x\sin\theta + y\cos\theta + b \end{bmatrix}$$

Since, $\begin{bmatrix} x\cos\theta + a\cos\theta - y\sin\theta - b\sin\theta \\ x\sin\theta + a\sin\theta + y\cos\theta + b\cos\theta \end{bmatrix} \neq \begin{bmatrix} x\cos\theta - y\sin\theta + a \\ x\sin\theta + y\cos\theta + b \end{bmatrix}$

Thus, translation and rotation don't commute

A16. Translation followed by Scaling

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{Translat}^n} \begin{bmatrix} x+a \\ y+b \end{bmatrix} \xrightarrow{\text{Scaling}} \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

$$= \begin{bmatrix} cx+ca \\ cy+cb \end{bmatrix}$$

Scaling followed by translation

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{Scaling}} \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} \xrightarrow{\text{Transl}^n} \begin{bmatrix} cx+a \\ cy+b \end{bmatrix}$$

Since, $\begin{bmatrix} cx+ca \\ cy+cb \end{bmatrix} \neq \begin{bmatrix} cx+a \\ cy+b \end{bmatrix}$

$\therefore$ Scaling and translation don't comute

Now,

Rotation followed by scaling

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{Rotat}^n} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$\downarrow$ Scaling

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} cx\cos\theta - cy\sin\theta \\ cx\sin\theta + cy\cos\theta \end{bmatrix}$$

Scalling followed by rotation

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{Scaling}} \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} cx \\ cy \end{bmatrix}$$

$$= \begin{bmatrix} cx\cos\theta - cy\sin\theta \\ cx\sin\theta + cy\cos\theta \end{bmatrix}$$

Since, $\begin{bmatrix} cx\cos\theta - cy\sin\theta \\ cx\sin\theta + cy\cos\theta \end{bmatrix} = \begin{bmatrix} cx\cos\theta - cy\sin\theta \\ cx\sin\theta + cy\cos\theta \end{bmatrix}$

Thus, Rotation and scaling commute

A18. Let A be a symmetric matrix of size $n \times n$.

Let $\lambda_1, \dots, \lambda_n$ be the eigen values and $v_1, \dots, v_n$ be the corresponding eigen vectors

Now, let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

Then, we need to show that principal axis is along v_1

Let x be a linear combination of $v_1, \dots, v_n$

$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

Mulplying A both sides, we get

$Ax = A(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$

$= c_1 A v_1 + c_2 A_2 v_2 + \dots + c_n A_n v_n$

$= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_n \lambda_n v_n$

The direction v_1 maximizes Ax due to the largest eigenvalue λ_1. Therefore, the principal axis is aligned with v_1