Assignment 1: Matrix Calculus and Algebra Assignment

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January 19, 2025

Instructions

Answer all questions. Show all work to receive full credit. Use clear and concise reasoning. You may use computational tools to verify your results but provide analytical solutions where required.

Questions

1. Given a matrix $A \in \mathbb{R}^{3\times 3}$ with entries $a_{ij} = i + j$, compute the determinant of A and determine if A is invertible.

2. Let $B \in \mathbb{R}^{2\times 2}$ be a matrix such that $B = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors of B.

3. For the matrix $C \in \mathbb{R}^{4\times 4}$ where $c_{ij} = \delta_{ij}$ (the Kronecker delta), verify if C is an orthogonal matrix.

4. Determine the null space and column space of the matrix $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

   Visualize these spaces as a subspace in $\mathbb{R}^3$ and sketch them schematically.

5. Solve the overdetermined system $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ using the least squares method.

6. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Visualize the effect of T on the unit circle in $\mathbb{R}^2$.

7. Show that the rank of a matrix plus the nullity of the matrix equals the number of its columns using the matrix $E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

8. Compute the gradient of $f(x, y) = x^2 + y^2 + 2xy$ and evaluate it at the point $(1, -1)$.

9. For the function $g(x, y, z) = e^{x+y} + \sin(z)$, compute the Jacobian matrix at the point $(0, 0, \pi/2)$.

10. Given $h(x) = \|Ax - b\|^2$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, compute the gradient of $h(x)$.

11. Let $f(x, y) = x^2y + y^3$. Compute the Hessian matrix of f and evaluate it at $(1, 1)$.

12. If $A \in \mathbb{R}^{n\times n}$ is symmetric, show that $x^T Ax$ is a scalar-valued quadratic form. Compute the gradient with respect to $x \in \mathbb{R}^n$.

13. For the transformation $T(x) = Ax$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, describe the geometric effect of T on vectors in $\mathbb{R}^2$.

14. Verify that the eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

15. Consider the matrix $G = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Determine if G can be diagonalized. If yes, find the diagonal matrix.

16. For $H \in \mathbb{R}^{n\times n}$, prove that $H^T H$ is positive semi-definite.

17. Given the quadratic function $q(x) = x^T Ax + b^T x + c$, where $A \in \mathbb{R}^{n\times n}$ is symmetric, find the critical points and determine their nature (minima, maxima, or saddle points).

18. Let $J(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$ represent the Jacobian matrix of $\mathbf{f}(x,y)$. Compute $J(x,y)$ for $\mathbf{f}(x,y) = \begin{bmatrix} x^2 y \\ \sin(x+y) \end{bmatrix}$.

19. Let $f(x,y) = x^2 + 2y^2 + 4xy - 6x - 8y + 15$.

    (a) Compute $\nabla f(x,y)$ and find all critical points by solving $\nabla f(x,y) = 0$.

    (b) Compute the Hessian matrix H of $f(x,y)$. Using the Hessian, determine the nature (minimum, maximum, or saddle point) of each critical point.

    (c) Verify that the critical point(s) satisfy the second-order necessary conditions for optimality.

20. Consider the quadratic function $f(x) = \frac{1}{2}x^T Qx - b^T x$, where $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix and $b \in \mathbb{R}^n$.

    (a) Derive the update rule for gradient descent $x^{(k+1)} = x^{(k)} - \alpha \nabla f(x^{(k)})$.

    (b) Show that the gradient descent algorithm converges to the optimal solution if $0 < \alpha < \frac{2}{\lambda_{\max}(Q)}$, where $\lambda_{\max}(Q)$ is the largest eigenvalue of Q.

    (c) Prove that the optimal step size for gradient descent in this case is $\alpha = \frac{2}{\lambda_{\max}(Q)+\lambda_{\min}(Q)}$.