Multivariate Calculus

(for Machine Learning)

Single-Variable Functions

Recall that the derivative of a single-variable function $f(x)$ at $x = a$ is defined as follows:

$$f'(a) = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This definition is designed specifically to tell us the instantaneous rate of change of f at $x = a$.

With this motivation in mind, how might we define the derivative of, say, a two-variable function $f(x, y)$? Could we design it to give us the instantaneous rate of change of $f(x, y)$ at a point, (a, b)?

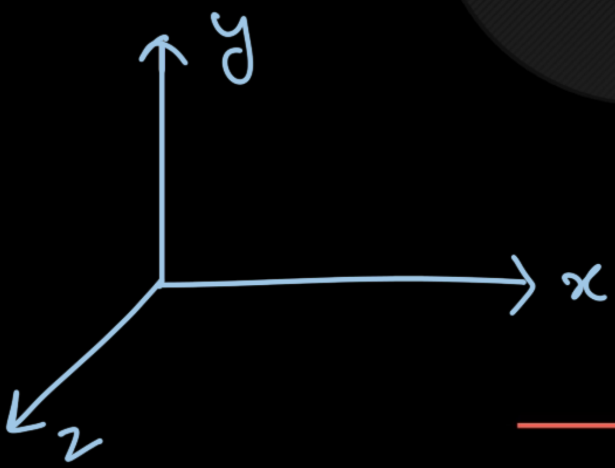
Functions of several variables

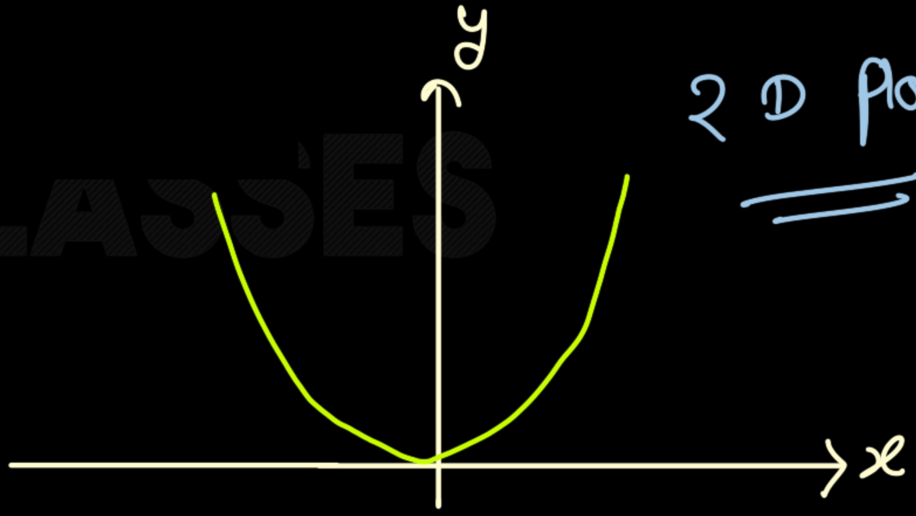
$y = x^2$

For functions of two variables can write

$$z = f(x, y).$$

2 D Plot





$$f(x, y) = x^2 + 2y^2$$

$$f(2, 1) = ?$$

$$f(1, 2) = ?$$

$f(2,1) = 2^2 + 2 = 4+2 = 6$

$f(1,2) = 1 + 2 \cdot 2^2 = 1+8 = 9$

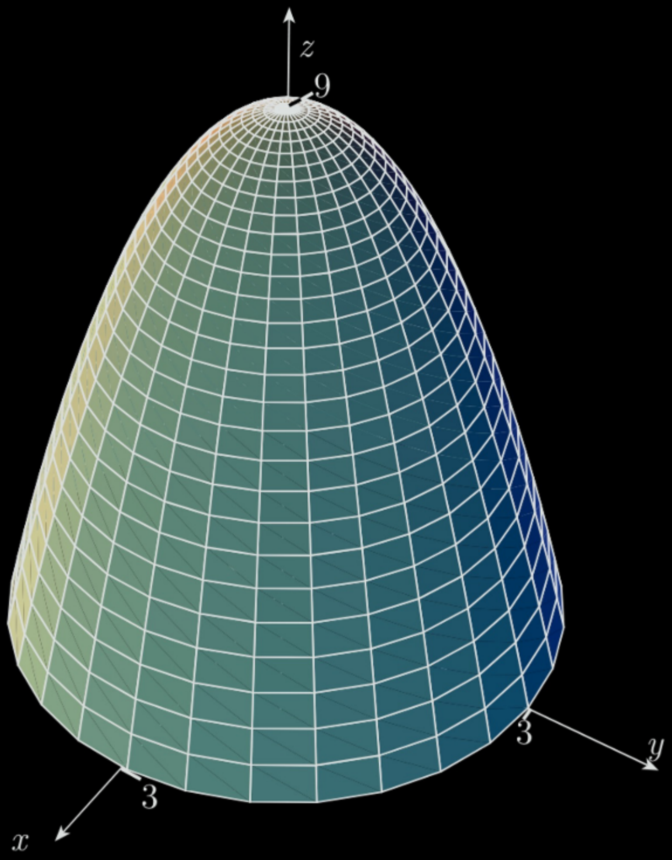


Figure 1.1: The paraboloid $z = 9 - x^2 - y^2$.

$x = 0, \; y = 0$

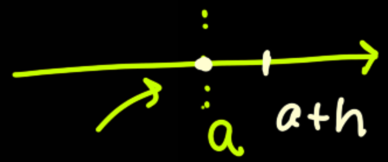
$\Rightarrow z = 9$

How to differentiate Multivariate function ?

Single-Variable Functions

Recall that the derivative of a single-variable function $f(x)$ at $x = a$ is defined as follows:

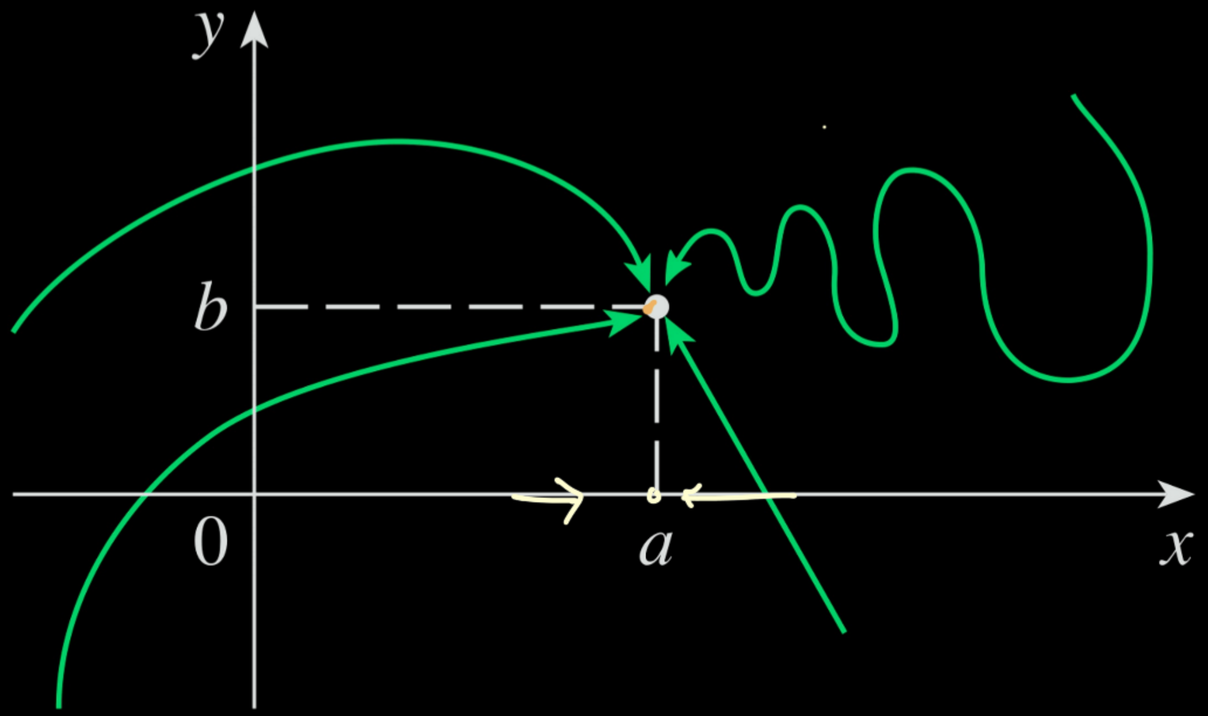
$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

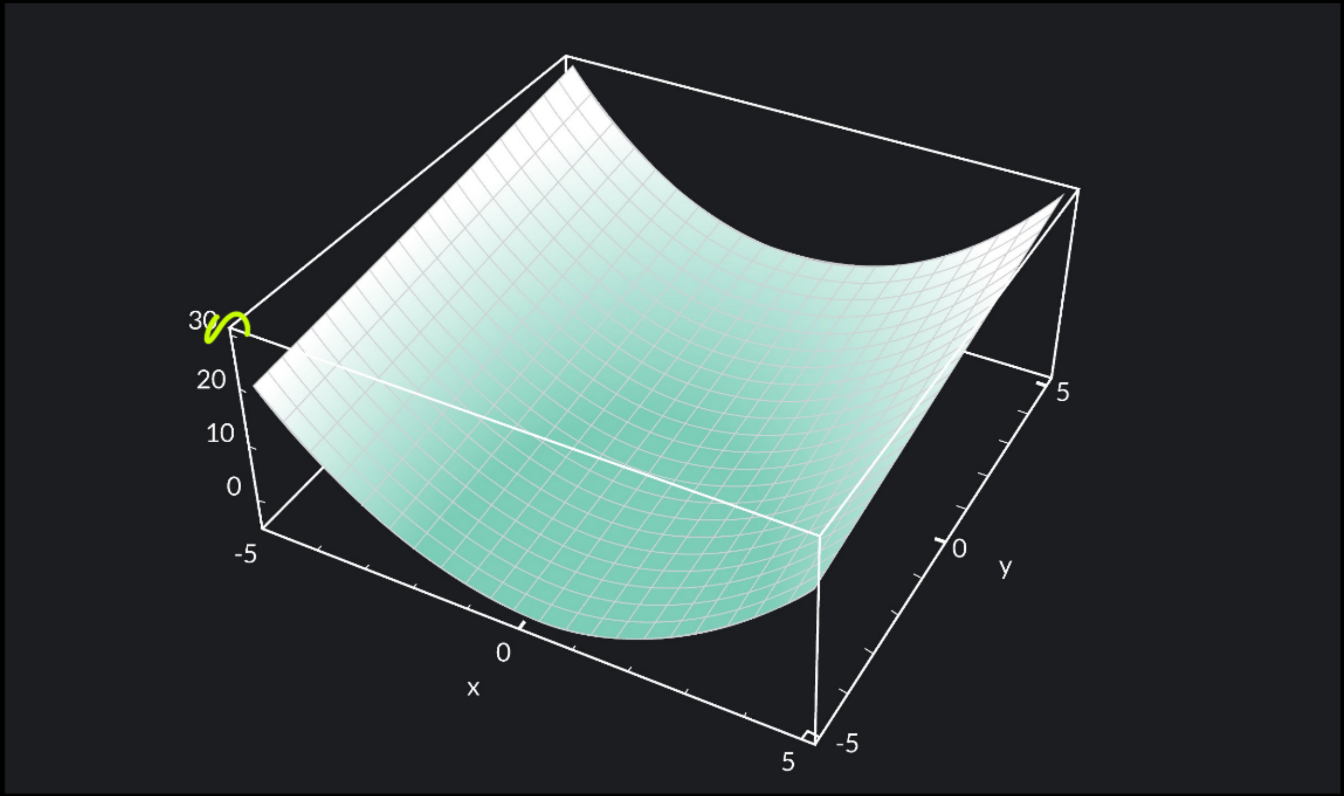


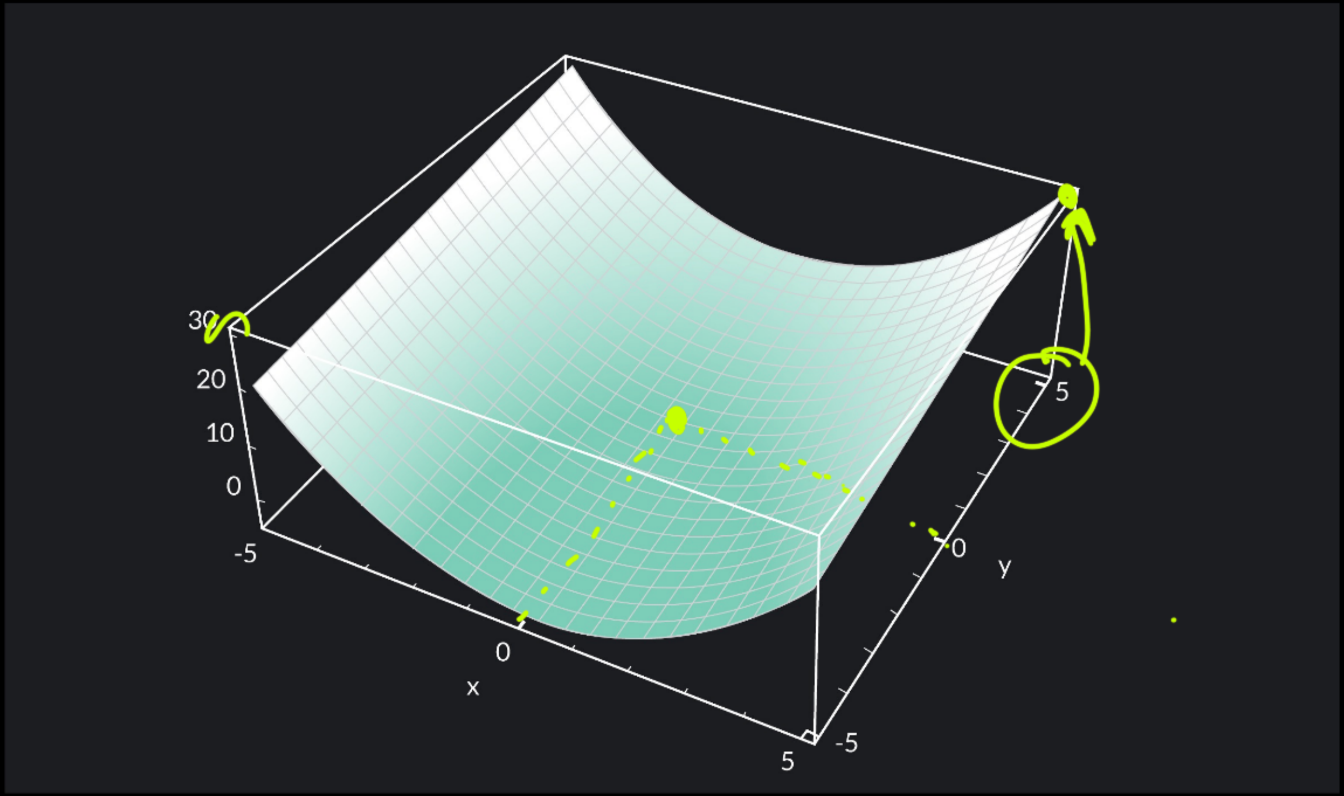
$h \to 0$

This definition is designed specifically to tell us the instantaneous rate of change of f at $x = a$.

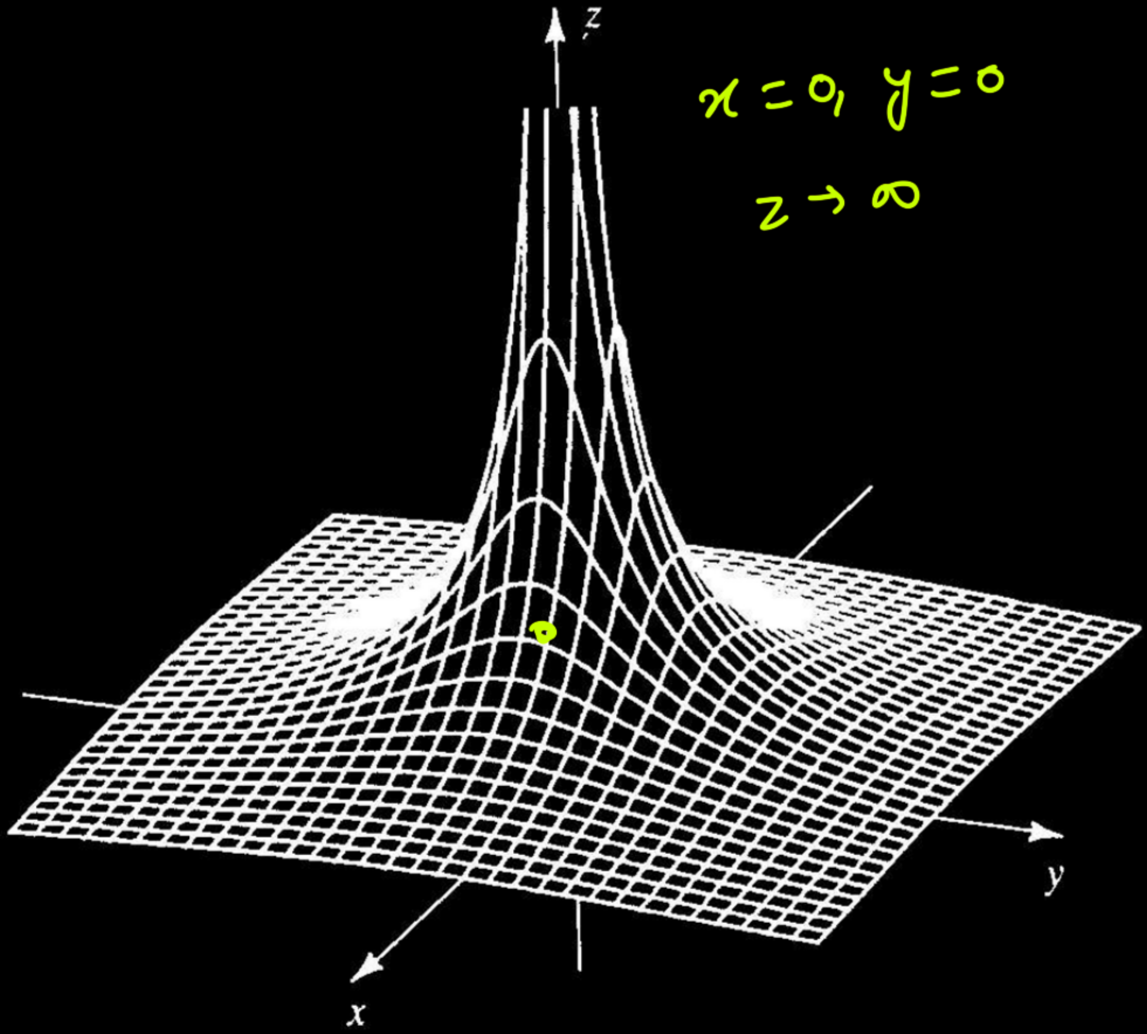
With this motivation in mind, how might we define the derivative of, say, a two-variable function $f(x, y)$? Could we design it to give us the instantaneous rate of change of $f(x, y)$ at a point, (a, b)?


y
b
0
a
x


30
20
10
0
-5
0
x
5
-5
0
y
5


30
20
10
0
-5
0
x
5
-5
0
y
5

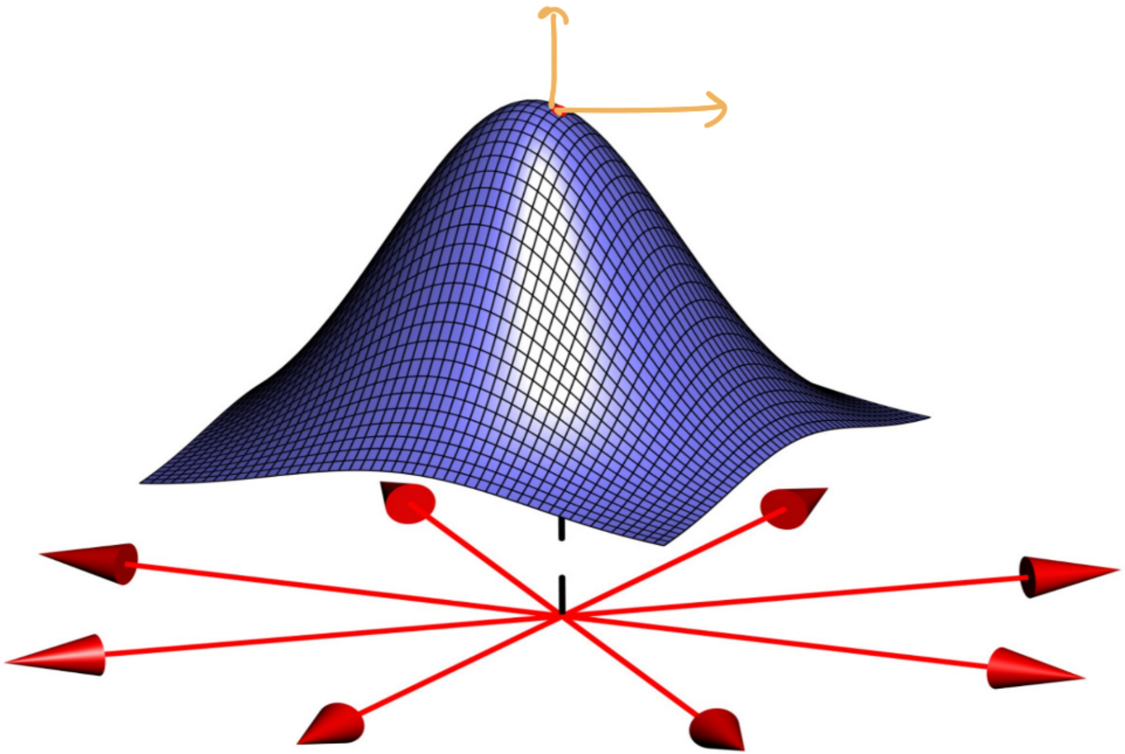
Calculus



$x = 0, \ y = 0$

$z \to \infty$

Multiple Possible Directions for $f : \mathbb{R}^2 \to \mathbb{R}$



The Derivative?

Now suppose we wanted to calculate the instantaneous rate of change of f at the point (a, b), something we would like to call f'(a, b). Unfortunately, this is inherently ambiguous.

Every possible direction we might head away from (a,b) in gives a different instantaneous rate of change of f.

Thus, there is no way to define "the" derivative of a two-variable function at a point!

https://math.ou.edu/~jjackson/teaching/1403slides.pdf

Not “the” Derivative, but Derivatives!

Instead, to give an unambiguous derivative (i.e. the instantaneous rate of change of a function) we need to specify two things: the point at which we wish to take such a derivative, and the direction in which we wish to take it. Today — for two-variable functions, anyway — we will only talk about two directions: parallel to the x-axis, and parallel to the y-axis. These are the so-called **partial derivatives**.

instead of finding derivative, we will find partial derivatives.

Limit Definition of Partial Derivatives

A partial derivative measures the rate of change of a multivariable function as one variable changes, but the others remain constant.

Definition

The **partial derivatives** of a two-variable function $f(x, y)$ are the functions

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

and

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

https://www.math.emory.edu/~mpcarr/math211/print/ch14.pdf

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,\, b) - f(a,b)}{h}$$

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{(a,b)} = \lim_{h\to 0} \frac{f(a,\, b+h) - f(a,b)}{h}$$

Definition for Two-Variable Functions

The **partial derivative of f with respect to x at** (a, b) is:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

This tells us the instantaneous rate at which f is changing at (a, b) when we move parallel to the x-axis in the direction of increasing x, with y held fixed.

Similarly the **partial derivative of f with respect to y at** (a, b) is:

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

This tells us the instantaneous rate at which f is changing at (a, b) when we move parallel to the y-axis in the direction of increasing y, with x held fixed.

Notation

The partial derivative of a function can be denoted a variety of ways. Here are some equivalent notations

- f_x
- $\frac{\partial f}{\partial x}$
- $\frac{\partial z}{\partial x}$
- $\frac{\partial}{\partial x} f$
- $D_x f$

$$\frac{\partial f}{\partial x} = f_x$$

Question:

Evaluate f_x when $f(x, y) = x^2y + y^2$.

$$\frac{\partial f}{\partial x} = 2xy + 0 = 2xy$$

Solution: To do so, we write

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x^2y + y^2\right) = \frac{\partial}{\partial x}x^2y + \frac{\partial}{\partial x}y^2$$

and then we evaluate the derivative as if y is a constant. In particular,

$$f_x(x,y) = y\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial x}y^2 = y \cdot 2x + 0$$

That is, y factors to the front since it is considered constant with respect to x. Likewise, y^2 is considered constant with respect to x, so that its derivative with respect to x is 0. Thus, $f_x(x,y) = 2xy$.

Question:

Find f_x and f_y when $f(x, y) = y \sin(xy)$

Solution: To find f_x, we use the chain rule

$$f_x = y\frac{\partial}{\partial x}\sin(xy) = y\cos(xy)\frac{\partial}{\partial x}xy = y^2\cos(xy)$$

However, to find f_y, we begin with the product rule:

$$f_y = \frac{\partial}{\partial y}[y\sin(xy)] = \left(\frac{\partial}{\partial y}y\right)\sin(xy) + y\frac{\partial}{\partial y}\sin(xy)$$

We then use the chain rule to evaluate $\partial_y \sin(xy)$:

$$\begin{aligned} f_y &= \sin(xy) + y\cos(xy)\frac{\partial}{\partial y}(xy) \\ &= \sin(xy) + xy\cos(xy) \end{aligned}$$

Question:

Which of the following options is correct regarding the first partial derivatives of the function $f(x, y) = \ln(x^4 + 9y^2)$?

A. $\frac{\partial f}{\partial x} = \frac{4x^3}{x^4+9y^2}, \quad \frac{\partial f}{\partial y} = \frac{18y}{x^4+9y^2}$

B. $\frac{\partial f}{\partial x} = \frac{4x^3}{x^4+9y^2}, \quad \frac{\partial f}{\partial y} = \frac{18y}{x^4+4y^2}$

C. $\frac{\partial f}{\partial x} = \frac{4x^3}{x^4+4y^2}, \quad \frac{\partial f}{\partial y} = \frac{18y}{x^4+4y^2}$

D. $\frac{\partial f}{\partial x} = \frac{4x^2}{x^4+9y^2}, \quad \frac{\partial f}{\partial y} = \frac{9y}{x^4+4y^2}$

Question:

Which of the following options is correct regarding the first partial derivatives of the function $f(x, y) = \ln(x^4 + 9y^2)$?

A. $\frac{\partial f}{\partial x} = \frac{4x^3}{x^4+9y^2}, \quad \frac{\partial f}{\partial y} = \frac{18y}{x^4+9y^2}$

B. $\frac{\partial f}{\partial x} = \frac{4x^3}{x^4+9y^2}, \quad \frac{\partial f}{\partial y} = \frac{18y}{x^4+4y^2}$

C. $\frac{\partial f}{\partial x} = \frac{4x^3}{x^4+4y^2}, \quad \frac{\partial f}{\partial y} = \frac{18y}{x^4+4y^2}$

D. $\frac{\partial f}{\partial x} = \frac{4x^2}{x^4+9y^2}, \quad \frac{\partial f}{\partial y} = \frac{9y}{x^4+4y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{x^4+9y^2} \cdot (4x^3)$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^4+9y^2} \cdot 18y$$

Question:

Given the function $f(x, y) = x^2y + x^3y^3 + 1$, find the second partial derivative of f with respect to x, $\frac{\partial^2 f}{\partial x^2}$.

A) $2y + 6xy^3$

B) $2y + 9x^2y^2$

C) $4x + 9x^2y^3$

D) $2x + 3x^2y^2$

Question:

Given the function $f(x, y) = x^2y + x^3y^3 + 1$, find the second partial derivative of f with respect to x, $\frac{\partial^2 f}{\partial x^2}$.

A) $2y + 6xy^3$

B) $2y + 9x^2y^2$

C) $4x + 9x^2y^3$

D) $2x + 3x^2y^2$

$$\frac{\partial f}{\partial x} = 2xy + 3x^2y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 2y + 6x \cdot y^3$$

Solution:

Given the function $f(x, y) = x^2y + x^3y^3 + 1$, we need to find $\frac{\partial^2 f}{\partial x^2}$.

First, find the first partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y + x^3y^3 + 1) = 2xy + 3x^2y^3$$

Next, find the second partial derivative with respect to x:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2xy + 3x^2y^3) = 2y + 6xy^3$$

So, $\frac{\partial^2 f}{\partial x^2} = 2y + 6xy^3$.

Correct answer: A) $2y + 6xy^3$

Second Derivatives

Second Derivatives

The second partial derivative of f with respect to x is denoted f_{xx} and is defined

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y)$$

$$\frac{\partial^2 f}{\partial x^2} \qquad\qquad \frac{\partial}{\partial x}$$

Second Derivatives

The second partial derivative of f with respect to x is denoted f_{xx} and is defined

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y)$$

That is, f_{xx} is the derivative of the first partial derivative f_x. Likewise, the second partial derivative of f with respect to y is denoted f_{yy} and is defined

$$f_{yy}(x, y) = \frac{\partial}{\partial y} f_y(x, y)$$

Second Derivatives

The second partial derivative of f with respect to x is denoted f_{xx} and is defined

$$f_{xx}(x,y) = \frac{\partial}{\partial x} f_x(x,y)$$

That is, f_{xx} is the derivative of the first partial derivative f_x. Likewise, the second partial derivative of f with respect to y is denoted f_{yy} and is defined

$$f_{yy}(x,y) = \frac{\partial}{\partial y} f_y(x,y)$$

Finally, the *mixed* partial derivatives are denoted f_{xy} and f_{yx}, respectively, and are defined

$$f_{xy}(x,y) = \frac{\partial}{\partial y} f_x(x,y) \qquad \text{and} \qquad f_{yx} = \frac{\partial}{\partial x} f_y(x,y)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \qquad f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

most of the time:

$$\underbrace{\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}}$$

↓

(for continious functions)

Question:

Find the second partial derivatives of

$$f(x, y) = x^3 + 3x^2y^2$$

$f_x = 3x^2 + 6xy^2$

$f_{xx} = 6x + 6y^2$

$f_{xy} = 12xy$

$f_y = 6yx^2$

$f_{yx} = 12xy$

$f_{yy} = 6x^2$

Solution: The first partial derivatives are $f_x = 3x^2 + 6xy^2$ and $f_y = 6x^2y$. As a result, we have

$$\begin{aligned}
f_{xx}(x,y) &= \frac{\partial}{\partial x} f_x(x,y) = \frac{\partial}{\partial x}\left(3x^2 + 6xy^2\right) = 6x + 6y^2 \\
f_{yy}(x,y) &= \frac{\partial}{\partial y} f_y(x,y) = \frac{\partial}{\partial y}\left(6x^2y\right) = 6x^2 \\
f_{xy}(x,y) &= \frac{\partial}{\partial y} f_x(x,y) = \frac{\partial}{\partial y}\left(3x^2 + 6xy^2\right) = 12xy \\
f_{yx}(x,y) &= \frac{\partial}{\partial x} f_y(x,y) = \frac{\partial}{\partial x}\left(6x^2y\right) = 12xy
\end{aligned}$$

Question:

H.W.

Find $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ for $f(x, y) = x^3 + x^2y^2 + 2y^3 + 2x + y$.

Solution

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2 + 0 + 2 + 0 = 3x^2 + 2xy^2 + 2$$

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x + 2y^2 + 0 = 6x + 2y^2.$$

$$\frac{\partial f}{\partial y} = 0 + x^2 \times 2y + 6y^2 + 0 + 1 = 2x^2y + 6y^2 + 1$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2x^2 + 12y.$$

Question:

$$f(x,y) = \frac{x^3 - 3xy^2}{x^2 + y^2}$$

Compute $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$

$$\frac{d\left(\frac{f}{g}\right)}{dx} = \frac{f'g - g'f}{g^2}$$

$$\frac{\partial f}{\partial x} = \frac{(3x^2 - 3y^2)(x^2+y^2) - (2x)(x^3 - 3xy^2)}{(x^2+y^2)^2}$$

Solution: By the quotient rule

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{(x^2+y^2)(3x^2-3y^2)-(x^3-3xy^2)(2x)}{(x^2+y^2)^2} \\ &= \frac{x^4+6x^2y^2-3y^4}{(x^2+y^2)^2}.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{(x^2+y^2)(-6xy)-(x^3-3xy^2)(2y)}{(x^2+y^2)^2} \\ &= \frac{-8x^3y}{(x^2+y^2)^2}.\end{aligned}$$

Question:

H.W.

Consider the function $f(x, y) = e^{\frac{y}{3x}}$.

(a) Find all the first partial derivatives of f.

(b) Find all the second partial derivatives of f.

(a) (2 points) Find all the first partial derivatives of f.

Solution: We have

$$f_x = -e^{\frac{y}{3x}} \frac{y}{3x^2};$$
$$f_y = e^{\frac{y}{3x}} \frac{1}{3x}.$$

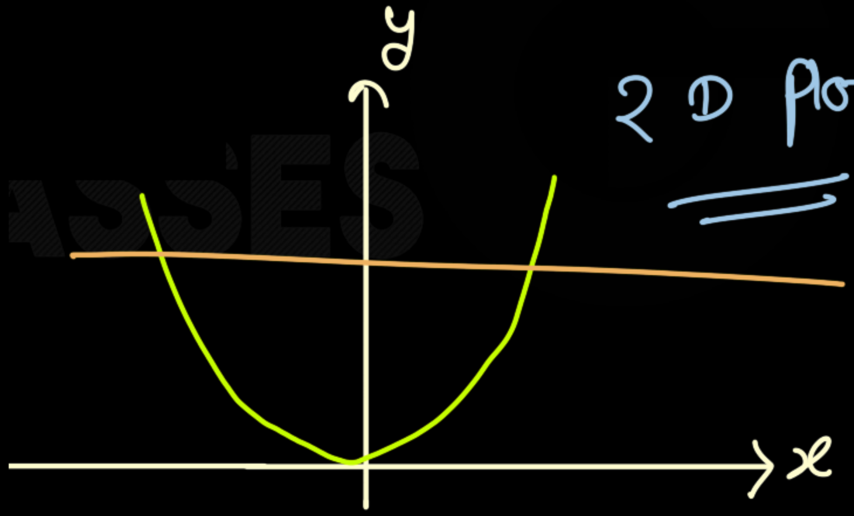
(b) (2 points) Find all the second partial derivatives of f.

Solution: We have

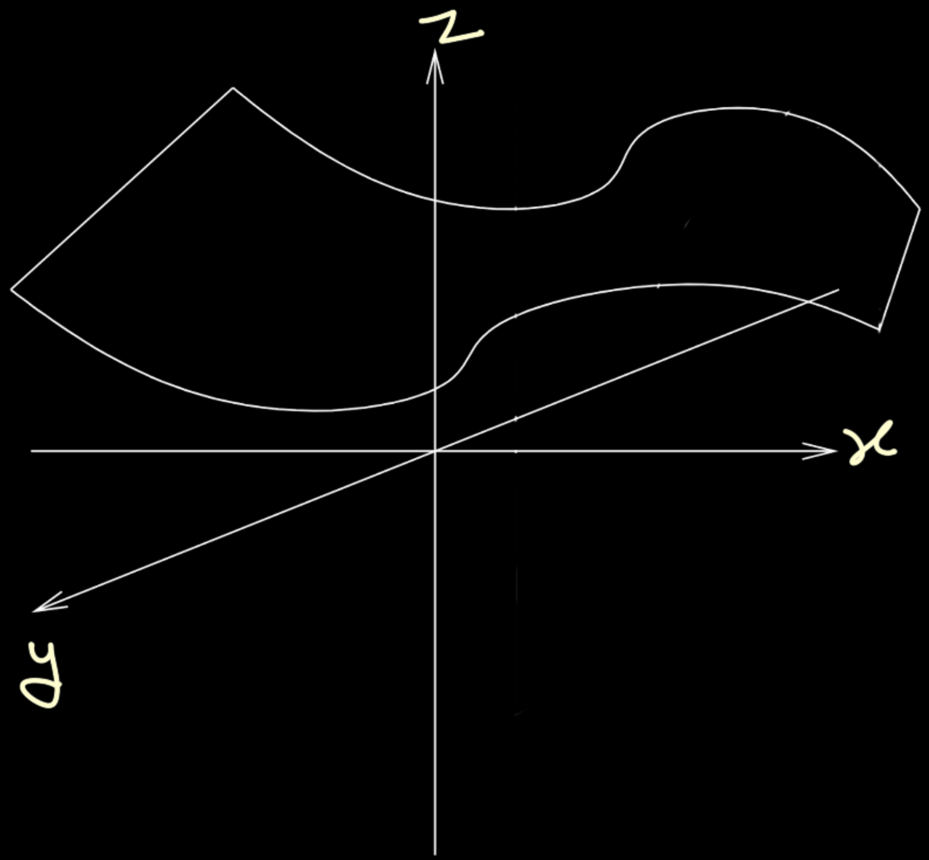
$$\begin{aligned}
f_{xx} &= e^{\frac{y}{3x}} \left(-\frac{y}{3x^2}\right)^2 + e^{\frac{y}{3x}} \frac{2y}{3x^3} \\
&= e^{\frac{y}{3x}} \left(\frac{y^2}{9x^4} + \frac{2y}{3x^3}\right) \\
f_{xy} &= f_{yx} = -e^{\frac{y}{3x}} \frac{y}{3x^2} \frac{1}{3x} + e^{\frac{y}{3x}} \left(-\frac{1}{3x^2}\right); \\
&= -e^{\frac{y}{3x}} \left(\frac{y}{9x^3} + \frac{1}{3x^2}\right); \\
f_{yy} &= e^{\frac{y}{3x}} \left(\frac{1}{3x}\right)^2 \\
&= e^{\frac{y}{3x}} \frac{1}{9x^2}.
\end{aligned}$$

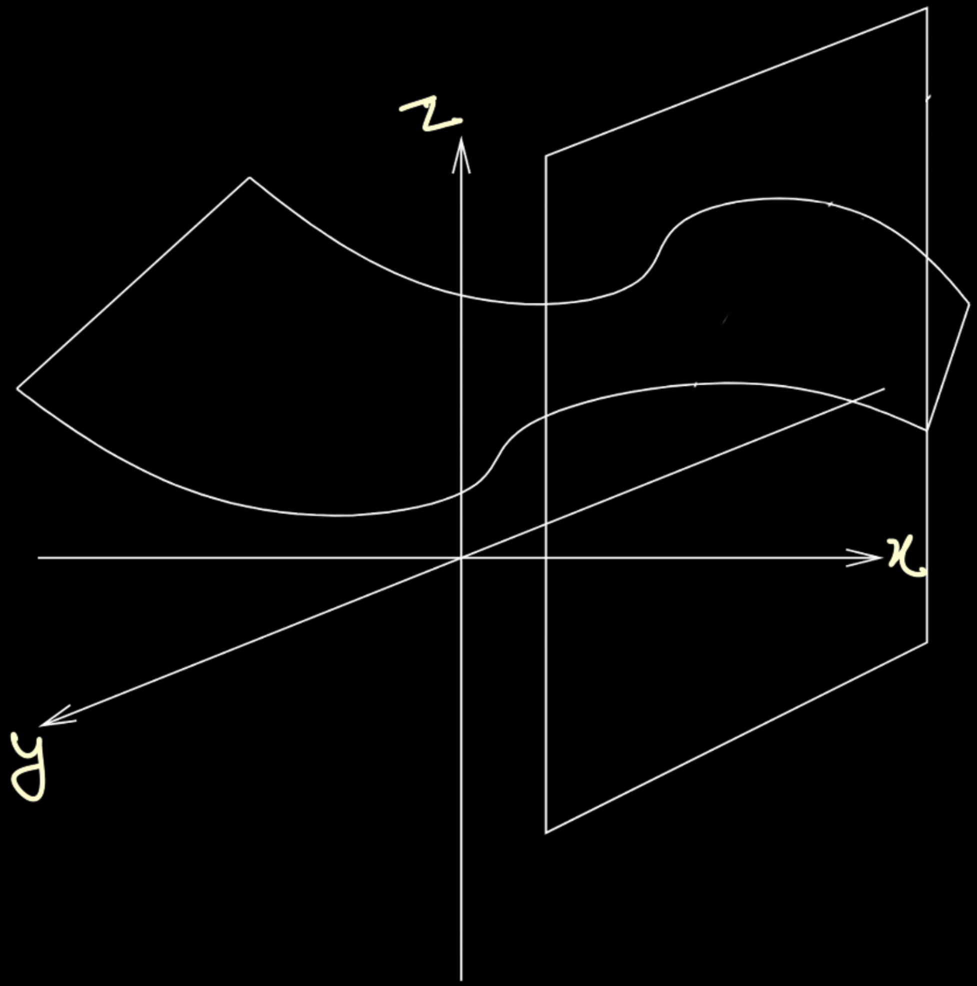
Graphical Interpretation partial derivatives

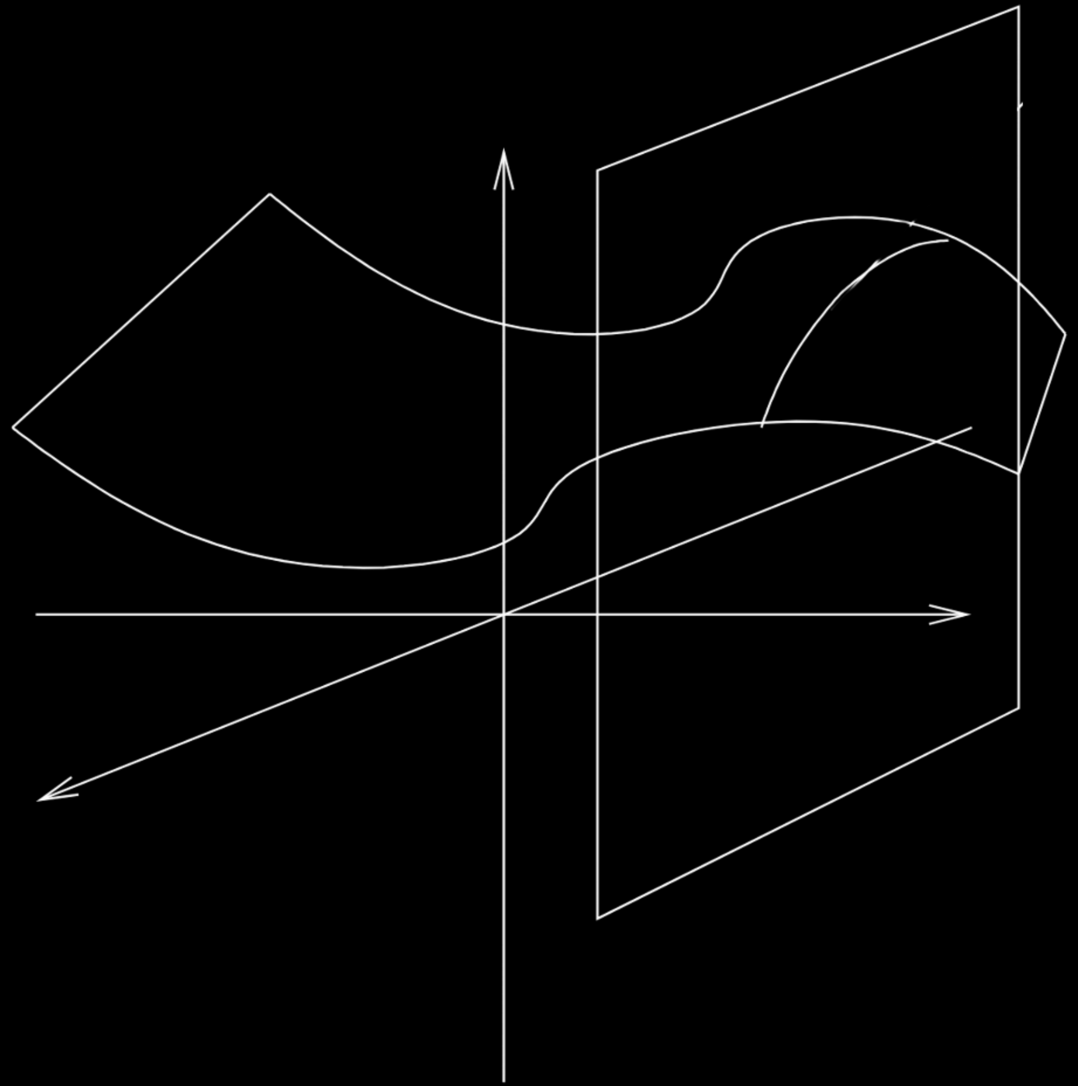
$y = x^2$

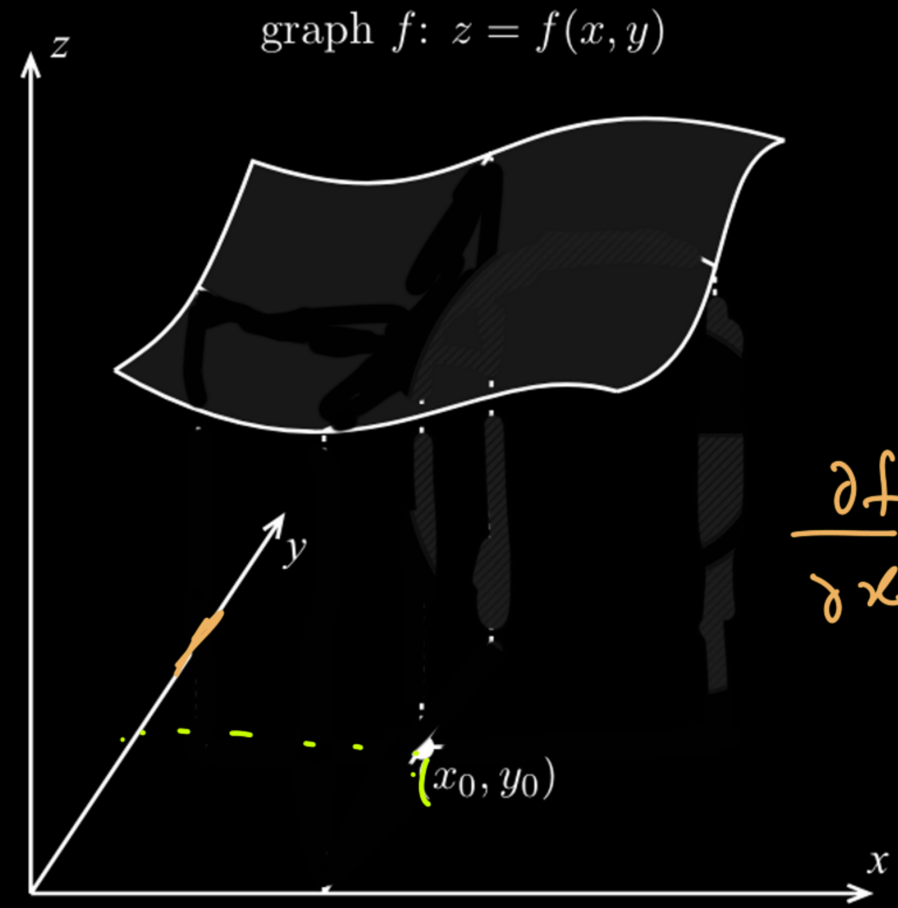


2D Plot

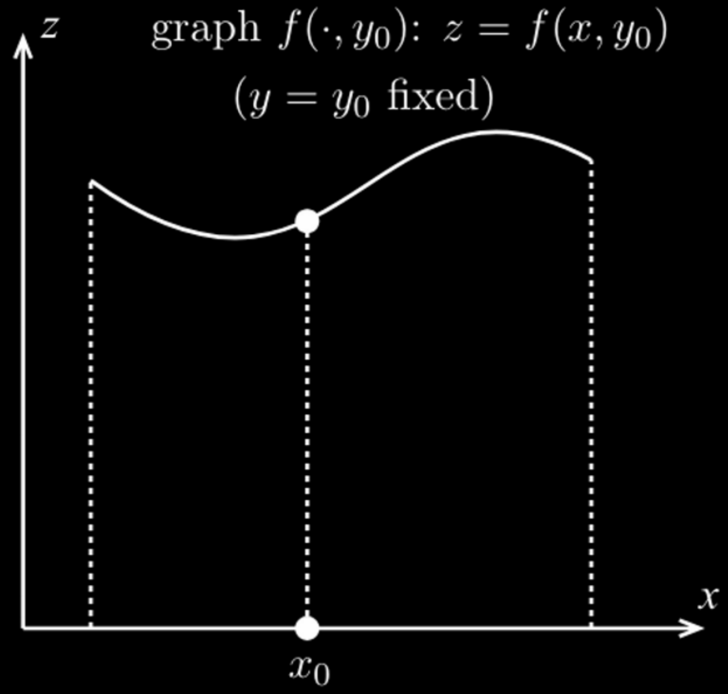

z
x
y


z
x
y

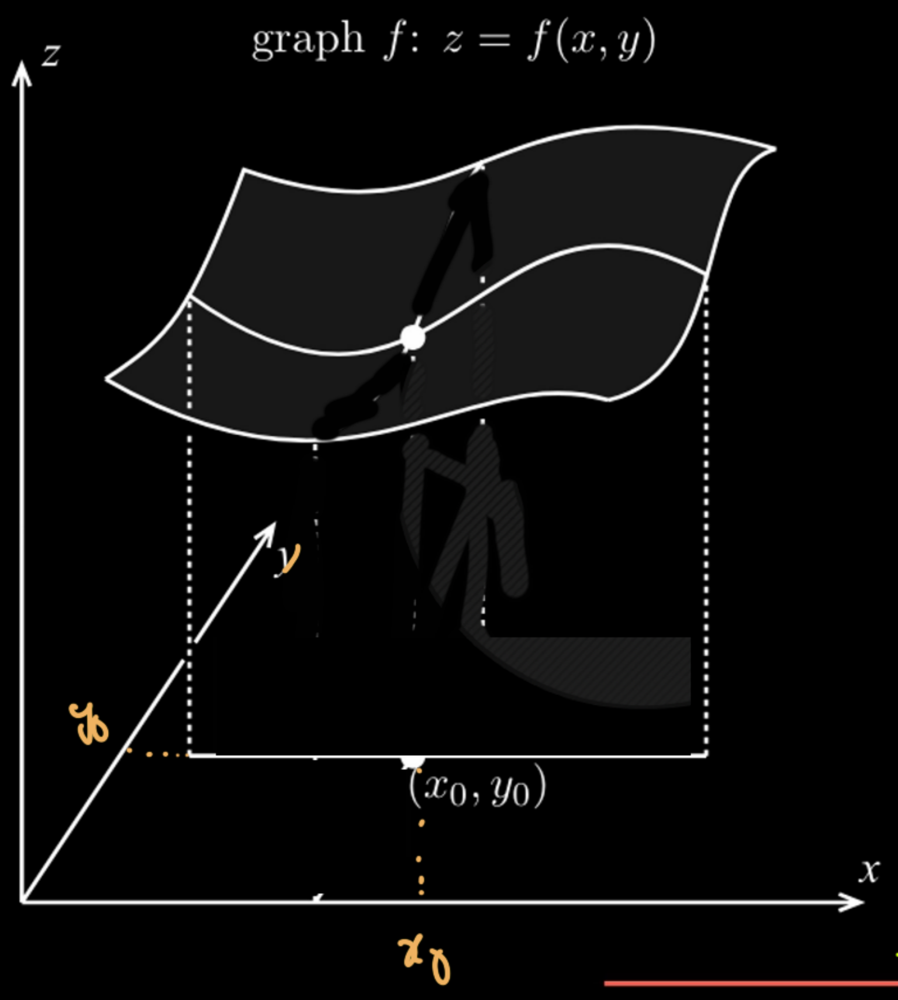


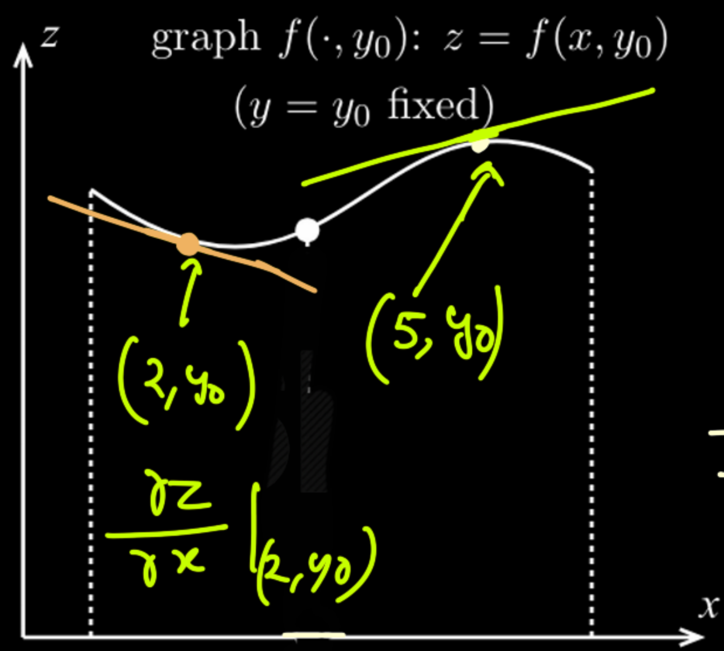


$$\frac{\partial f}{\partial x}$$



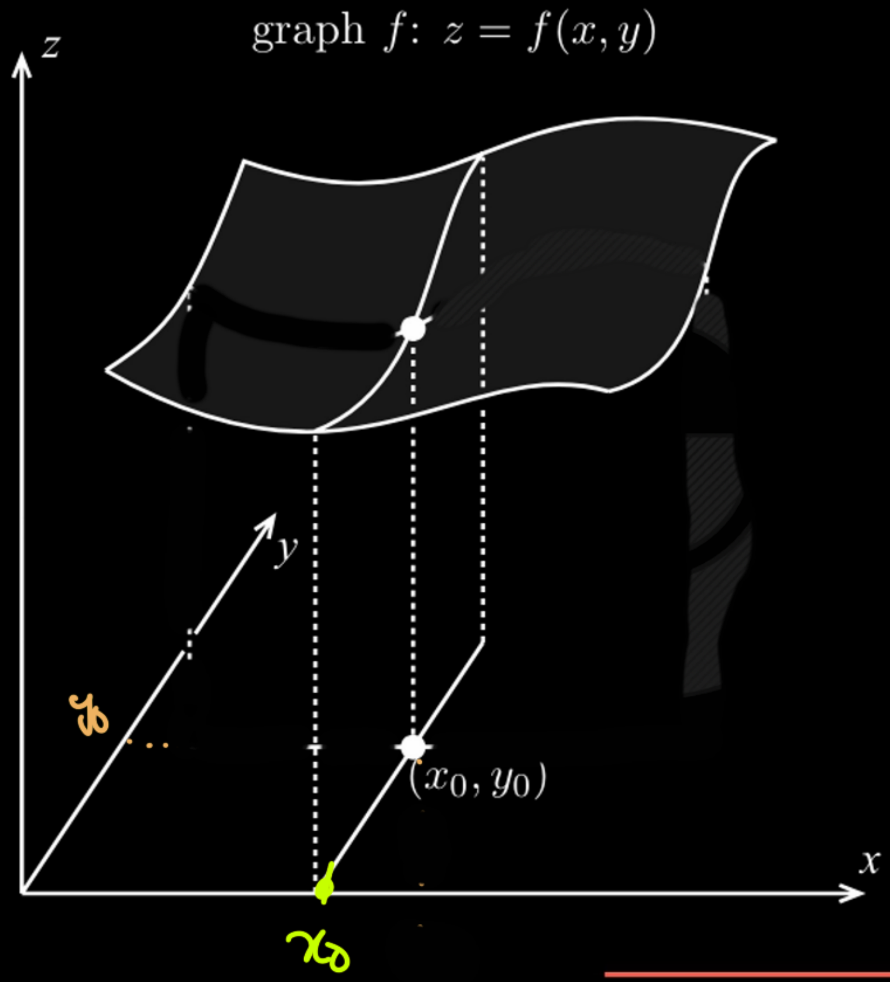
Calculus

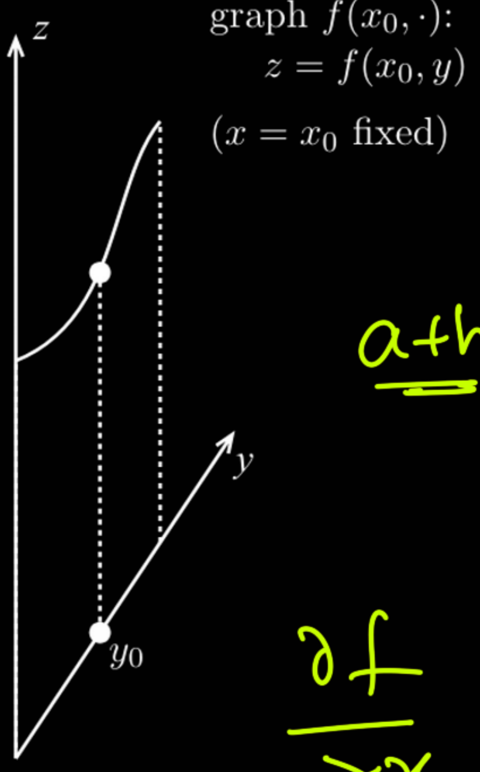




$$\left.\frac{\partial z}{\partial x}\right|_{x_0, y_0}$$

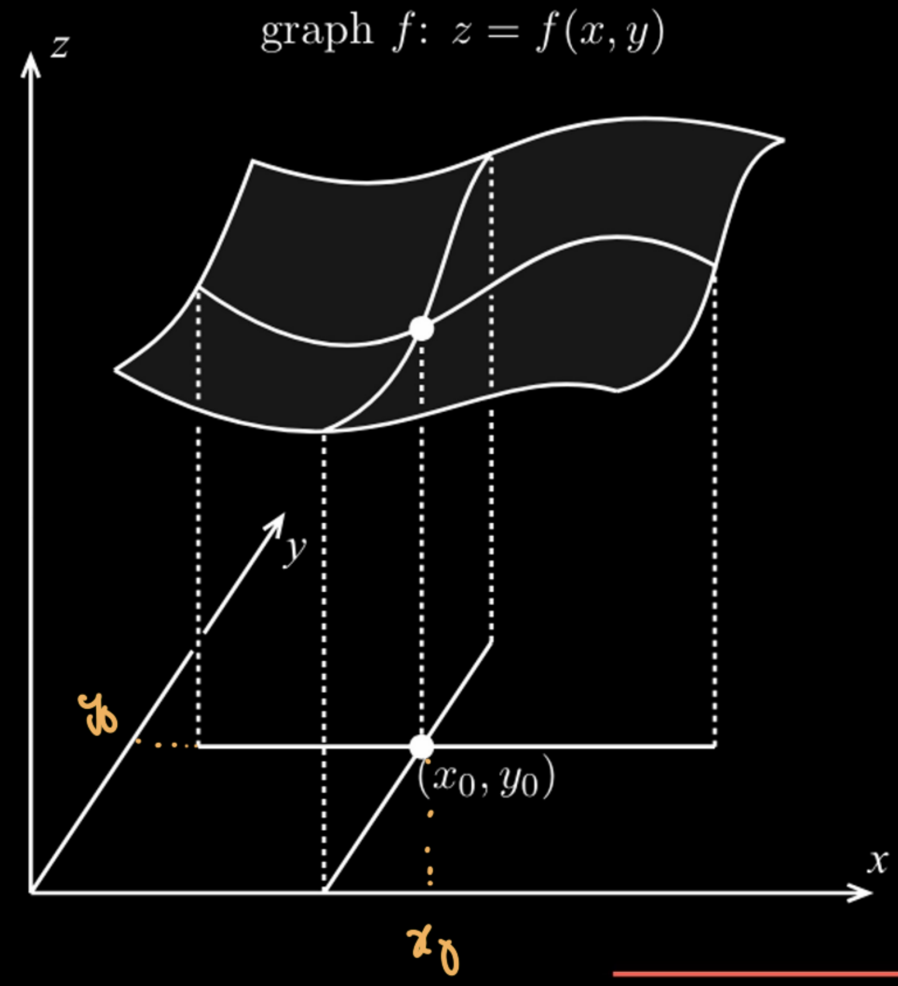
z as a function of just one variable

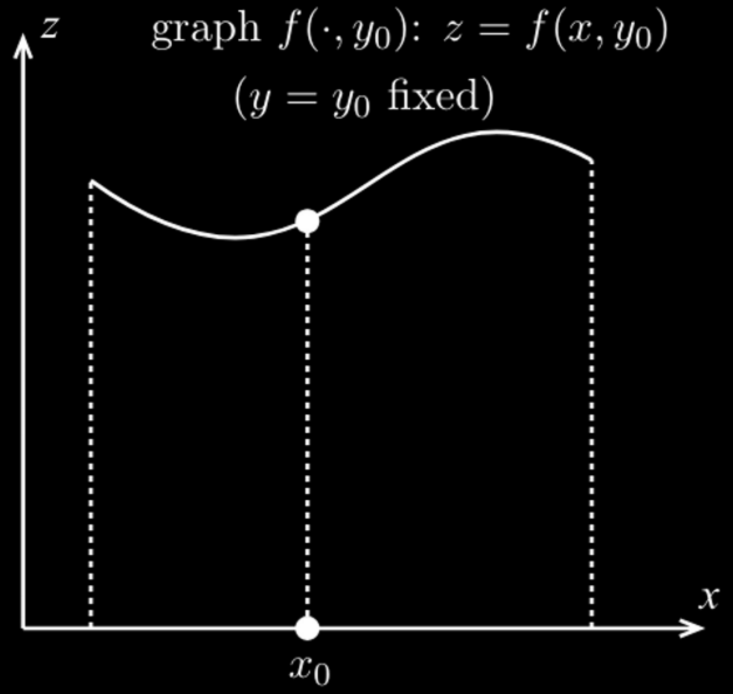


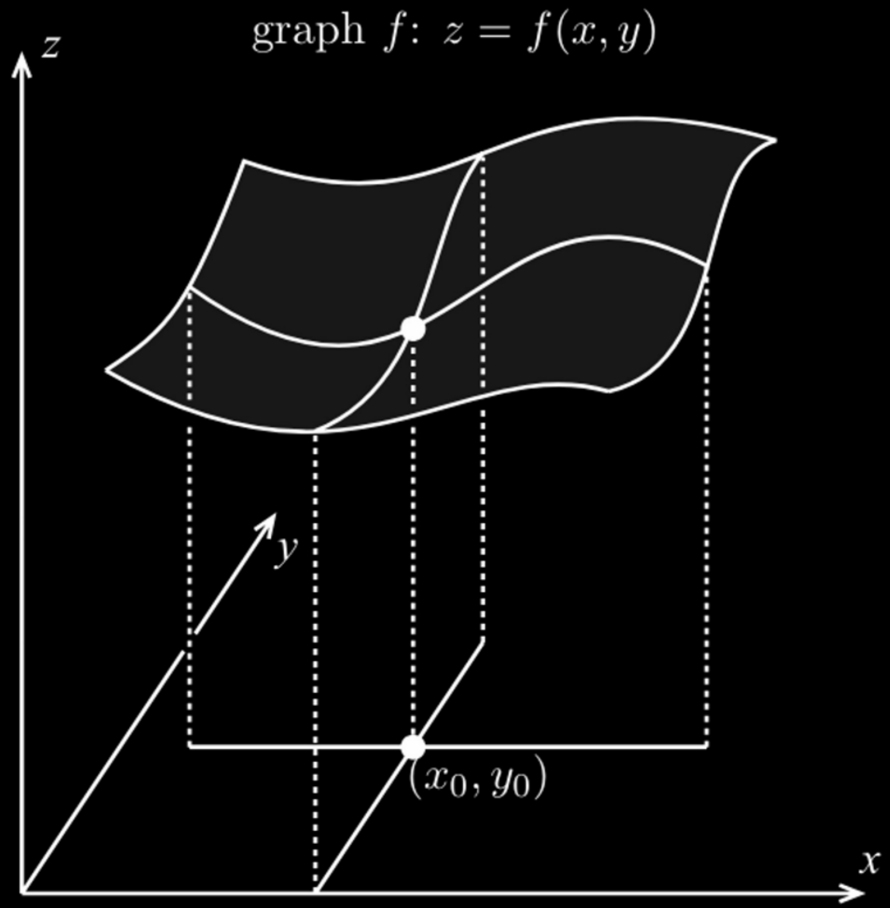


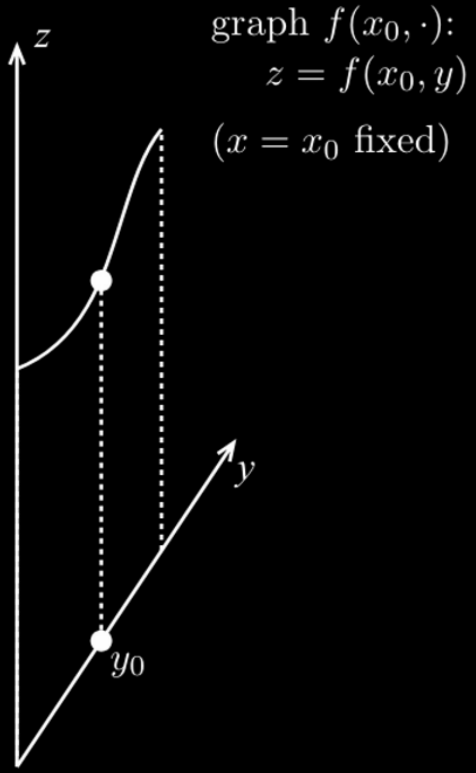
$a+h, b$

$\frac{\partial f}{\partial x}\Big|_{(\ \ )}$

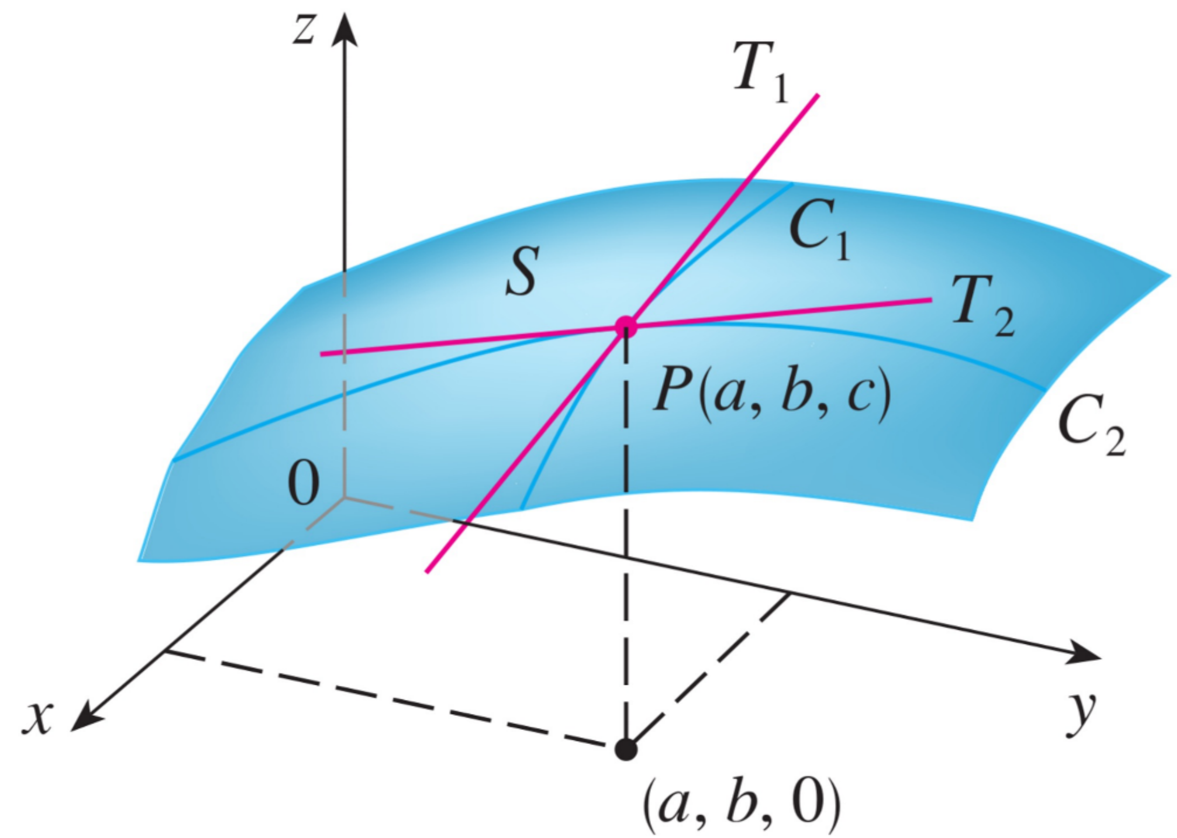

graph f: $z = f(x, y)$
z
y
y_0
(x_0, y_0)
x
x_0


z
graph $f(\cdot, y_0)$: $z = f(x, y_0)$
($y = y_0$ fixed)
x
x_0

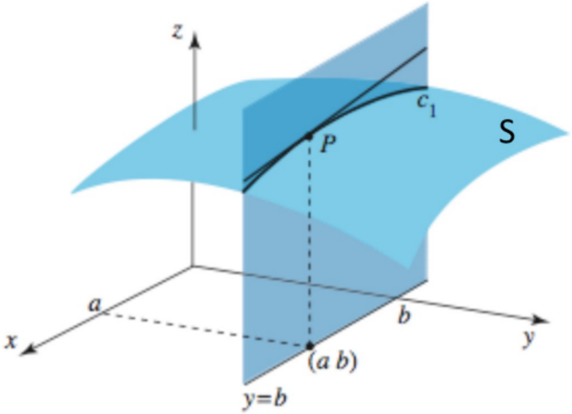

graph f: $z = f(x, y)$
z
y
(x_0, y_0)
x


z
graph $f(x_0, \cdot)$:
$z = f(x_0, y)$
$(x = x_0$ fixed$)$
y
y_0

Graphical Interpretation, cont.



Geometric Interpretation of the Partial Derivatives of z=f(x,y)



Let $z=f(x,y)$ be a function of two variables whose graph is the surface S.

Fix $y=b$ (constant) and let x vary.

The curve c_1 on the surface S is defined by $z=f(x,b)$. (Note: this is now only a function of the variable x)

https://ms.mcmaster.ca/~clemene/1LT3/lectures/1lt3_sv_section4.pdf

Question:

Which of the following partial derivatives gives the slope of the tangent line at $y = 1$ to the $x = 2$ cross-section of $f(x, y)$? (Circle your answer.)

$\frac{\partial f}{\partial x}(1, 2)$ $\frac{\partial f}{\partial x}(2, 1)$ $\frac{\partial f}{\partial y}(1, 2)$ $\frac{\partial f}{\partial y}(2, 1)$

keeping x fix

$$\frac{\partial f}{\partial y}(2, 1)$$

The $x = 2$ cross-section involves keeping x constant. The slope of this cross-section is therefore related to $\dfrac{\partial f}{\partial y}$ since it tells you how f changes as y changes (and x is kept constant). Since it is the $x = 2$ cross-section, and the slope is taken at $y = 1$, this means the slope is equal to the partial derivative evaluated at $(2, 1)$. So the correct answer is the last option.

Chain rule multivariate

$f(x)$ $\quad$ $f(g(x))$

$$\frac{\partial f(g(x))}{\partial x} = f'(g(x)) \cdot g'(x)$$

Chain rule multivariate

Question:

Let $z = f(x, y) = x^2 + y$
where $x = 2t + 1$ and $y = 3t - 1$. Compute dz/dt.

$$z = (2t+1)^2 + (3t-1)$$

$$\frac{dz}{dt} = 2(2t+1)\cdot 2 + 3 = 8t+4+3 = 8t+7$$

Let $z = f(x, y) = x^2 + y$

where $x = 2t + 1$ and $y = 3t - 1$. Compute dz/dt.

Explicitly compute z as a function of t.
Plug x and y into z, in terms of t:

$$\begin{aligned} z = x^2 + y &= (2t+1)^2 + (3t-1) \\ &= 4t^2 + 4t + 1 + 3t - 1 \\ &= 4t^2 + 7t \end{aligned}$$

Then compute dz/dt:

$$\frac{dz}{dt} = 8t + 7$$

method 1

Second method: Chain rule

Let $z = f(x, y) = x^2 + y$

where $x = 2t + 1$ and $y = 3t - 1.$

$$
\begin{tikzcd}
& z \arrow[dl, "\frac{\partial f}{\partial x}"'] \arrow[dr, "\frac{\partial f}{\partial y}"] & \\
x \arrow[d, "\frac{dx}{dt}"'] & & y \arrow[d, "\frac{dy}{dt}"] \\
t & & t
\end{tikzcd}
$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$2x \cdot 2 + 1 \cdot 3 = 4x + 3$$

$$= 4(2t+1) + 3 = 8t + 7$$

Let $z = f(x,y) = x^2 + y$
where $x = 2t+1$ and $y = 3t-1$.

- Chain rule formula:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= 2x \cdot 2 + 1 \cdot 3 = 4x + 3$$

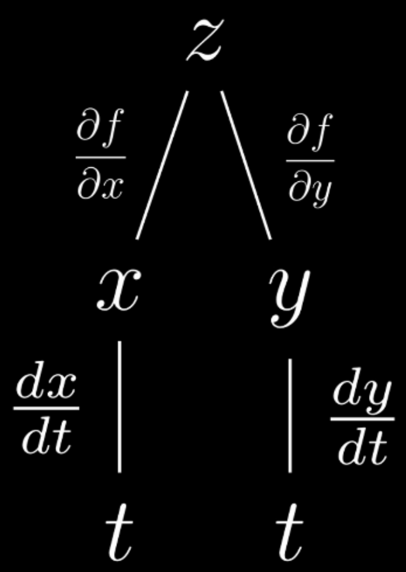
- Plug in x, y in terms of t:

$$= 4(2t+1) + 3 = 8t + 4 + 3 = \boxed{\mathbf{8t+7}}$$

- This agrees with the first method.

Chain rule multivariate

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$



Remark: We write $\frac{dz}{dt}$ rather than $\frac{\partial z}{\partial t}$ because z ultimately depends on only the single variable t, so we are actually computing a single-variable derivative and not a partial derivative.

$$
\begin{tikzcd}[column sep=small]
 & & z \arrow[dll, dash, "\frac{\partial z}{\partial x}"'] \arrow[drr, dash, "\frac{\partial z}{\partial y}"] & & \\
x \arrow[d, dash, "\frac{dx}{dt}"'] & & & & y \arrow[d, dash, "\frac{dy}{dt}"] \\
t & & & & t
\end{tikzcd}
$$

$z = f(x, y)$ depends on two variables.
Use partial derivatives.

x and y each depend on one variable, t.
Use ordinary derivative.

To compute $\frac{dz}{dt}$:

- There are two paths from z at the top to t's at the bottom.
- Along each path, multiply the derivatives.
- Add the products over all paths.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Question:

For $f(x, y) = x^2 + y^2$, with $x = t^2$ and $y = t^4$, find $\dfrac{df}{dt}$, both directly and via the chain rule.

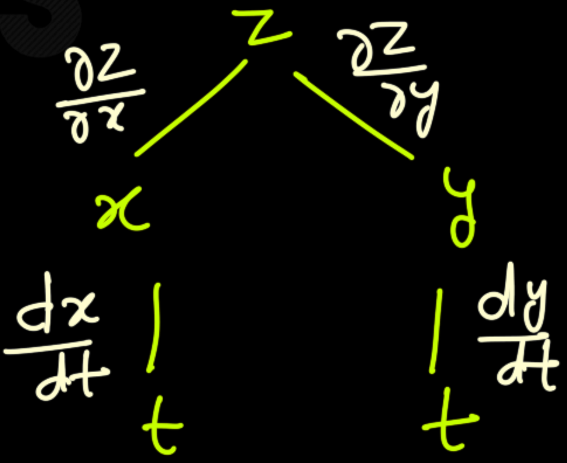
Question:

For $f(x, y) = x^2 + y^2$, with $x = t^2$ and $y = t^4$, find $\frac{df}{dt}$, both directly and via the chain rule.

$$z = x^2 + y^2 = t^4 + t^8 = 4t^3 + 8t^7$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

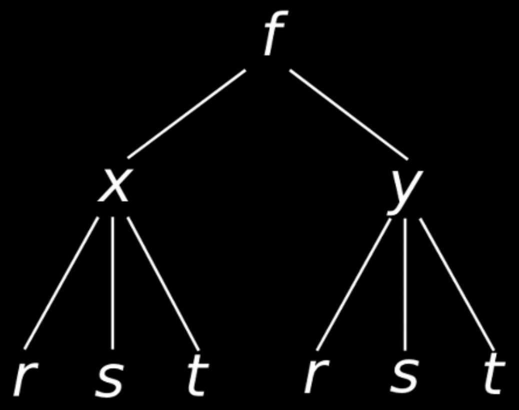
$$= 2x \cdot 2t + 2y \cdot 4t^3 = 4t^3 + 8t^7$$

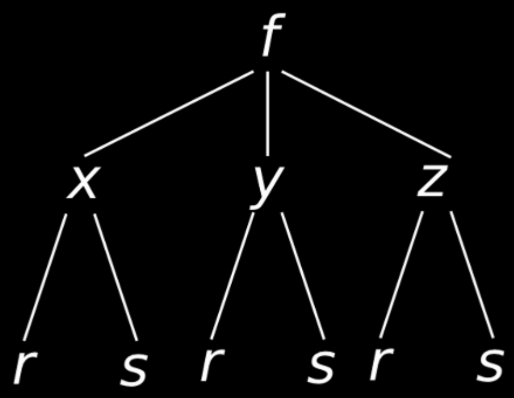


Example: For $f(x, y) = x^2 + y^2$, with $x = t^2$ and $y = t^4$, find $\frac{df}{dt}$, both directly and via the chain rule.

- In this instance, the multivariable chain rule says that $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$.
- Computing the derivatives shows $\frac{df}{dt} = (2x) \cdot (2t) + (2y) \cdot (4t^3)$.
- Plugging in $x = t^2$ and $y = t^4$ yields $\frac{df}{dt} = (2t^2) \cdot (2t) + (2t^4) \cdot (4t^3) = 4t^3 + 8t^7$. } chain rule
- To do this directly, we would plug in $x = t^2$ and $y = t^4$: this gives $f(x, y) = t^4 + t^8$, so that $\frac{df}{dt} = \underline{4t^3 + 8t^7}$. } direct substitute
- Of course, we obtain the same answer either way!

For instance, if f is a function of x and y, and we make x and y both functions of r, s and t, we get the tree on the left.

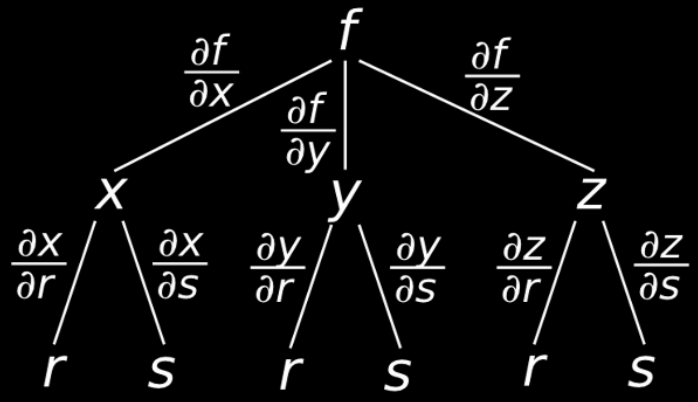




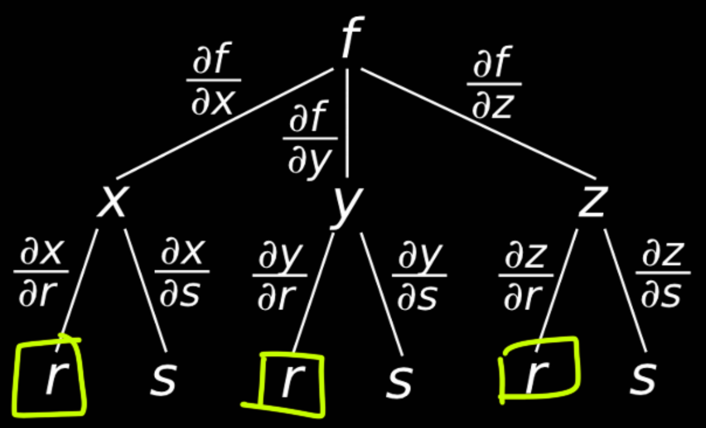
If f is a function of x, y and z, and we make x, y and z all functions of r and s, we get the tree on the right.

We label each branch in a given tree diagram with the partial derivative of the vertex above with respect to the vertex below.

So in the second example above we get the labelling:

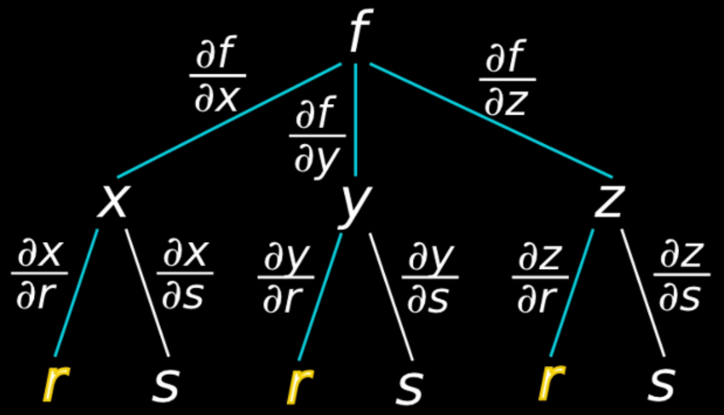


To compute the derivative of f with respect to a variable in the bottom row, we follow every path to that variable, multiplying as we go, and add the results.



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

Continuing with our example, there are three paths (in red) to the variable r (in blue):

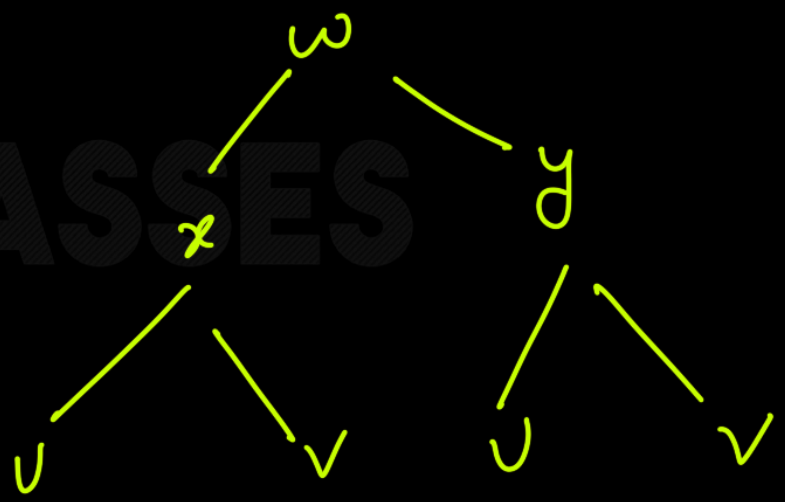


"Multiplying down and adding across" gives the result

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}.$$

Question:

Example: Given $w = x^2y + y^2 + x, \quad x = u^2v, \quad y = uv^2 \quad$ find $\dfrac{\partial w}{\partial u}$.



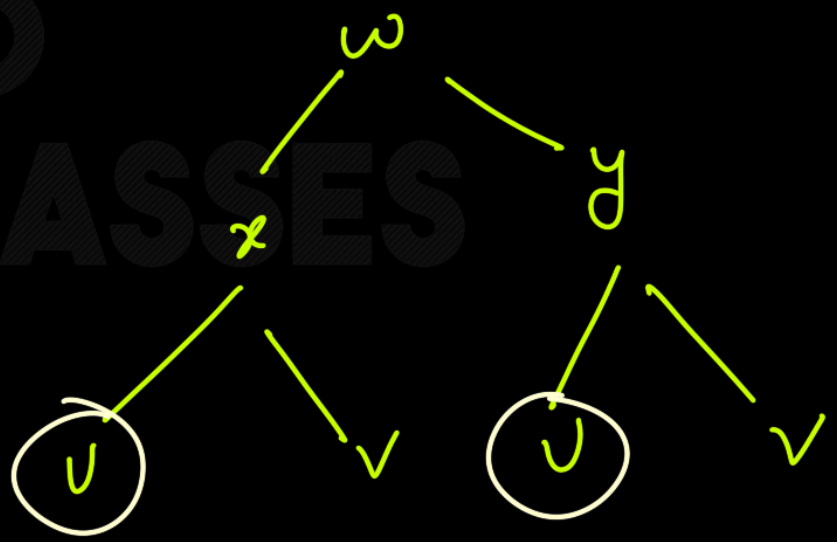
https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/95e0a67b5970599a87a98dd4b47200b3_MIT18_02SC_notes_17.pdf

Question:

$$\frac{\partial w}{\partial x} = 2xy+1 \qquad \frac{\partial w}{\partial y} = x^2+2y$$

Example: Given $w = x^2y + y^2 + x, \quad x = u^2v, \quad y = uv^2$ find $\dfrac{\partial w}{\partial u}$.

$$\frac{\partial w}{\partial u} = (2xy+1)\cdot 2uv + (x^2+2y)\cdot v^2$$



https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/95e0a67b5970599a87a98dd4b47200b3_MIT18_02SC_notes_17.pdf

Answer: First we compute

$$\frac{\partial w}{\partial x} = 2xy + 1, \quad \frac{\partial w}{\partial y} = x^2 + 2y, \quad \frac{\partial x}{\partial u} = 2uv, \quad \frac{\partial y}{\partial u} = v^2, \quad \frac{\partial x}{\partial v} = u^2, \quad \frac{\partial y}{\partial v} = 2uv.$$

The chain rule then implies

$(u, v) = (1, 2)$

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \\ &= (2xy+1)2uv + (x^2+2y)v^2 \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \\ &= (2xy+1)u^2 + (x^2+2y)2uv.\end{aligned}$$

answer

Often, it is okay to leave the variables mixed together. If, for example, you wanted to compute $\frac{\partial w}{\partial u}$ when $(u, v) = (1, 2)$ all you have to do is compute x and y and use these values, along with u, v, in the formula for $\dfrac{\partial w}{\partial u}$.

$$x = 2,\ y = 4 \ \Rightarrow\ \frac{\partial w}{\partial u} = (5)(4) + (12)(4) = 68.$$

If you actually need the derivatives expressed in just the variables u and v then you would have to substitute for x, y and z.

Question:

HW.

Example: Let $f(x,y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

1. Find $\dfrac{\partial f}{\partial s}$.
2. Find $\dfrac{\partial f}{\partial t}$.

Example: Let $f(x,y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

1. Find $\dfrac{\partial f}{\partial s}$.

- By the chain rule we have $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial s} = (2x)\cdot(2s) + (2y)\cdot(3s^2)$.
- Plugging in $x = s^2 + t^2$ and $y = s^3 + t^4$ yields

$$\begin{aligned}\frac{\partial f}{\partial s} &= (2s^2 + 2t^2)\cdot(2s) + (2s^3 + 2t^4)\cdot(3s^2) \\ &= 4s^3 + 4st^2 + 6s^5 + 6s^2t^4.\end{aligned}$$

Example: Let $f(x, y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

2. Find $\dfrac{\partial f}{\partial t}$.

- By the chain rule we have
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = (2x) \cdot (2t) + (2y) \cdot (4t^3).$$
- Plugging in $x = s^2 + t^2$ and $y = s^3 + t^4$ yields

$$\begin{aligned}\frac{\partial f}{\partial s} &= (2s^2 + 2t^2) \cdot (2t) + (2s^3 + 2t^4) \cdot (4t^3) \\ &= 4s^2t + 4t^3 + 8s^3t^3 + 8t^7.\end{aligned}$$

Question:

H.W.

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x, y) = 3x^2 - 2xy + y^2,\ x = x(u, v) = 3u + 2v,\ y = y(u, v) = 4u - v.$$

Solution

To implement the chain rule for two variables, we need six partial derivatives—$\partial z / \partial x$, $\partial z / \partial y$, $\partial x / \partial u$, $\partial x / \partial v$, $\partial y / \partial u$,and $\partial y / \partial v$:

$$\begin{aligned} \frac{\partial z}{\partial x} &= 6x - 2y & \frac{\partial z}{\partial y} &= -2x + 2y \\ \frac{\partial x}{\partial u} &= 3 & \frac{\partial x}{\partial v} &= 2 \\ \frac{\partial y}{\partial u} &= 4 & \frac{\partial y}{\partial v} &= -1. \end{aligned}$$

To find $\partial z / \partial u$, we use Equation 13.4.2:

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ &= 3(6x - 2y) + 4(-2x + 2y) \\ &= 10x + 2y. \end{aligned}$$

Next, we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial u} &= 10x + 2y \\ &= 10(3u + 2v) + 2(4u - v) \\ &= 38u + 18v.\end{aligned}$$

To find $\partial z/\partial v$, we use Equation 13.4.3:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= 2(6x - 2y) + (-1)(-2x + 2y) \\ &= 14x - 6y.\end{aligned}$$

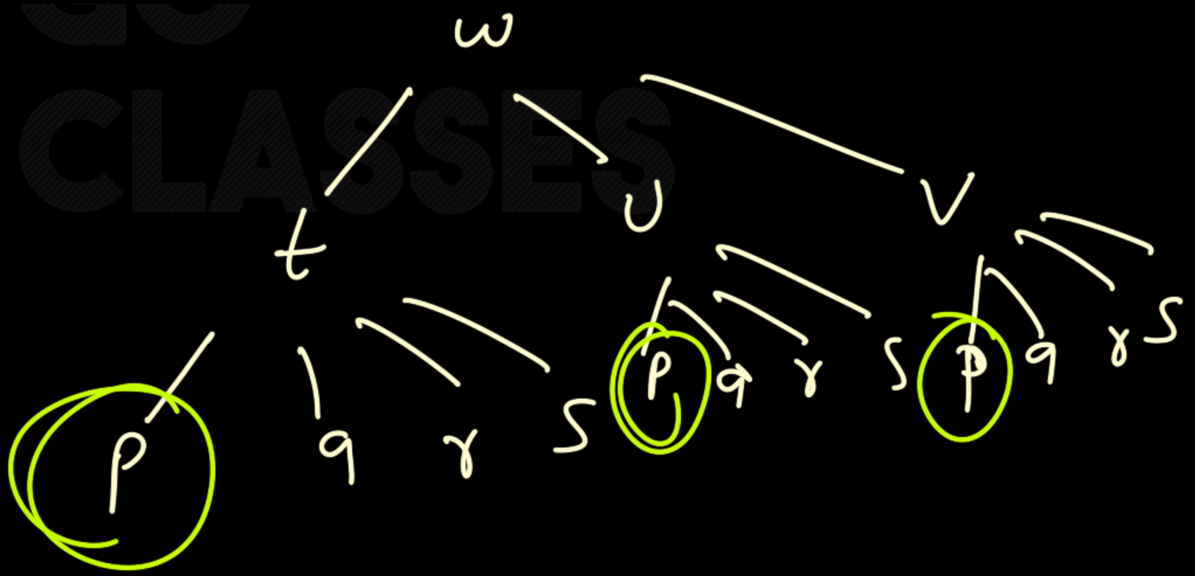
Then we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= 14x - 6y \\ &= 14(3u + 2v) - 6(4u - v) \\ &= 18u + 34v\end{aligned}$$

Question:

Use a tree diagram to write out the chain rule for the case where $w = f(t, u, v)$, $t = t(p, q, r, s)$, $u = u(p, q, r, s)$, and $v = v(p, q, r, s)$ are all differentiable functions.

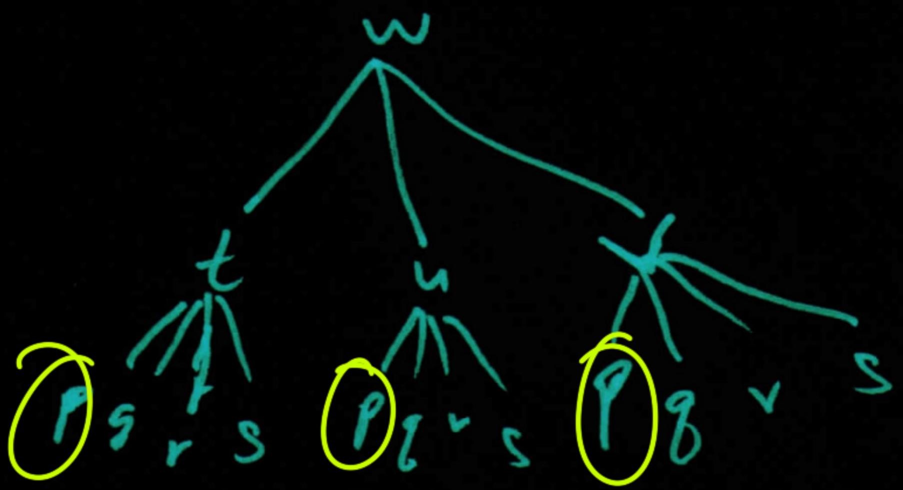
$$\frac{\partial w}{\partial p} =$$



$$w = f(t, u, v), \quad t = t(p, q, r, s)$$

$$u = u(p, q, r, s)$$

$$v = v(p, q, r, s).$$



$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}.$$

Question:

H.W.

Let $z = 2x^2 - xy + y^2$, with $x = s + t$, $y = st$.

Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

https://people.math.sc.edu/lu/teaching/2003spring_10C/exam2_1_sol.pdf

Solution: We have

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= (4x - y) + (2y - x)t;\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= (4x - y) + (2y - x)s.\end{aligned}$$

Question:

H.W.

Example: State the chain rule that computes $\frac{df}{dt}$ for the function $f(x, y, z)$, where each of x, y, and z is a function of the variable t.

- First, we draw the tree diagram: $\begin{array}{ccccc} & & f & & \\ & \swarrow & \downarrow & \searrow & \\ x & & y & & z \\ \downarrow & & \downarrow & & \downarrow \\ t & & t & & t \end{array}$.
- In the tree diagram, there are 3 paths from f to t: they are $f \to x \to t$, $f \to y \to t$, and $f \to z \to t$.
- The path $f \to x \to t$ gives $\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t}$, while the path $f \to y \to t$ gives $\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$, and the path $f \to z \to t$ gives $\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$.
- So, the statement is $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$.

- You can interpret this statement as saying that the total change in f is the sum of three components:
  1. The change $\frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$ in f resulting from the change in x.
  2. The change $\frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$ in f resulting from the change in y.
  3. The change $\frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$ in f resulting from the change in z.

Question:

H.W.

Given $w = yz + zx + xy$, $x = s^2 - t^2$, $y = s^2 + t^2$ and $z = s^2t^2$, find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

Solution: This is a partial derivative problem and so we apply Chain Rule (2).

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = (z+y)(2s) + (z+x)2s + (x+y)2st^2$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = -(z+y)(2t) + (z+x)2t + (x+y)2s^2t$$

Question:

H.W.

Example : Given $p = f(x, y, z)$, $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$, write the chain rule formulas giving the partial derivatives of the dependent variable p with respect to each independent variable.

Solution: This is a partial derivative problem and so we apply Chain Rule.

$$\frac{\partial p}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial p}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

Question:

H.W.

Let $f(x, y) = x^4 + xy^2$. Let $x(s, t) = s + t^2$, $y(s, t) = st$, and let $h(s, t) = f(x(s, t), y(s, t))$. Compute $\frac{\partial h}{\partial t}$ at $(s, t) = (-2, -1)$.

(a) 40

(b) 2

(c) 8

(d) I don't know.

Next Topic:

Gradients
and
Directional Derivatives