Constrained Optimization

Today's Topics

- Unconstrained Optimization **(Revision)**
- Equality Constrained Optimization
- Equality/Inequality Constrained Optimization

today

tommorow

Unconstrained Optimization

Problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to:} \quad & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

First Order Necessary Conditions

If $\mathbf{x}^*$ is a local minimizer of $f(\mathbf{x})$ and $f(\mathbf{x})$ is continuously differentiable in an open neighbourhood of $\mathbf{x}^*$, then

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

That is, $f(\mathbf{x})$ is stationary at $\mathbf{x}^*$

$f'(x) = 0 \Rightarrow$ minima

$\Rightarrow$ maxima

$\Rightarrow$ neither minima nor maxima

$f'(x)$ should be zero for minima or maxima

$f'(x) = 0$ is neccessary for minima or maxima.

Unconstrained Optimization

Second Order Necessary Conditions

If $\mathbf{x}^*$ is a local minimizer of $f(\mathbf{x})$ and $\nabla^2 f(\mathbf{x})$ is continuously differentiable in an open neighbourhood of $\mathbf{x}^*$, then

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

$\nabla^2 f(\mathbf{x}^*)$ is positive semi definite $H(x) \geq 0$

PSD

Second Order Sufficient Conditions

Suppose that $\nabla^2 f(\mathbf{x})$ is continuously differentiable in an open neighbourhood of $\mathbf{x}^*$. If the following two conditions are satisfied, then $\mathbf{x}^*$ is a local minimum of $f(\mathbf{x})$.

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

$\nabla^2 f(\mathbf{x}^*)$ is positive definite

minima $\Rightarrow$ $\nabla f = 0$

$\not\Rightarrow$ PSD

$$\left.\begin{array}{l}\nabla f = 0 \\ \text{Hessian is PSD}\end{array}\right\} \overset{?}{\Longrightarrow} \text{it is minima}$$

NO

$$\left.\begin{array}{l}\nabla f = 0 \\ \text{Hessian is PD}\end{array}\right\} \overset{?}{\Longrightarrow} \text{it is minima}$$

YES

Constrained Optimization

min $\qquad x_1^2 + x_2^2$

without any constraint min. $= 0$

$$\begin{cases} \min & x_1^2 + x_2^2 \\ \text{s.t.} & x_1 + x_2 = 2 \end{cases}$$

How to solve these kind of questions

$\min \quad x_1^2 + x_2^2$

$\text{s.t.} \quad x_1 + x_2 = 2$

✓ Equality Constrained Optimization

$\min \quad x_1^2 + x_2^2$

$\text{s.t.} \quad x_1 + x_2 \leq 2$

Inequality Constrained Optimization

Direct Substitution

- When the constraint(s) are **equalities**, we can convert the problem from a constrained optimization to an unconstrained optimization problem by substituting for some of the variables.

Question:

Minimize: $f(\mathbf{x}) = 4x_1^2 + 5x_2^2$

Subject to: $2x_1 + 3x_2 = 6$

https://www.ou.edu/class/che-design/che5480-13/Optimization%20of%20Chemical%20Processes-Chapter%208.pdf

Minimize: $f(\mathbf{x}) = 4x_1^2 + 5x_2^2$

Subject to: $2x_1 + 3x_2 = 6$

$$x_1 = \frac{6 - 3x_2}{2}$$

$$f(x_2) = 4\left(\frac{6-3x_2}{2}\right)^2 + 5x_2^2$$

unconstraint optimisation problem

$$= 14x_2^2 - 36x_2 + 36 \Rightarrow \boxed{x_2 = \frac{9}{7}}$$

https://www.ou.edu/class/che-design/che5480-13/Optimization%20of%20Chemical%20Processes-Chapter%208.pdf

Solution:

$$\text{Minimize:} \quad f(\mathbf{x}) = 4x_1^2 + 5x_2^2 \tag{8.2a}$$

$$\text{Subject to:} \quad 2x_1 + 3x_2 = 6 \tag{8.2b}$$

Either x_1 or x_2 can be eliminated without difficulty. Solving for x_1,

$$x_1 = \frac{6 - 3x_2}{2} \tag{8.3}$$

we can substitute for x_1 in Equation (8.2a). The new equivalent objective function in terms of a single variable x_2 is

$$f(x_2) = 14x_2^2 - 36x_2 + 36 \tag{8.4}$$

The constraint in the original problem has now been eliminated, and $f(x_2)$ is an unconstrained function with 1 degree of freedom (one independent variable). Using constraints to eliminate variables is the main idea of the generalized reduced gradient method, as discussed in Section 8.7.

We can now minimize the objective function (8.4), by setting the first derivative of f equal to zero, and solving for the optimal value of x_2:

$$\frac{df(x_2)}{dx_2} = 28x_2 - 36 = 0 \quad x_2^* = \frac{9}{7}$$

Once x_2^* is obtained, then, x_1^* can be directly obtained via the constraint (8.2b):

$$x_1^* = \frac{6 - 3x_2^*}{2} = \frac{15}{14}$$

Limitations of Direct Substitution

In some cases, we cannot use substitution easily: for instance, suppose the constraint is $x^4 + 5x^3y + y^2x + x^6 + 5 = 0$. Here, it is not possible to solve this equation to get x as a function of y or vice versa.

$x_1 + x_2 = 6$

Limitations of Direct Substitution (Contd.)

Moreover, if there are inequality constraints instead of equality constraints, such as $g\left(x_1, \ldots, x_n\right) \leq 0$ or $g\left(x_1, \ldots, x_n\right) \geq 0$, then the direct substitution method cannot be applied.

$x_1 + x_2 \leq 2$

- Two reasons for an alternative approach:

1. In some cases, we cannot use substitution easily: for instance, suppose the constraint is $x^4 + 5x^3y + y^2x + x^6 + 5 = 0$. Here, it is not possible to solve this equation to get x as a function of y or vice versa.

today

2. In many cases, the economic constraints are written in the form $g(x_1, \ldots, x_n) \leq 0$ or $g(x_1, \ldots, x_n) \geq 0$. While the Lagrangian technique can be modified to take care of such cases, the substitution technique cannot be modified, or can be modified only with some difficulty.

tommorow

The Lagrangian Approach

Joseph Louis Lagrange (1736-1813)

Joseph-Louis Lagrange (25 January 1736 – 10 April 1813) was an Italian Enlightenment Era mathematician and astronomer. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

https://bpb-us-w2.wpmucdn.com/u.osu.edu/dist/c/91883/files/2021/01/04-OC-KKT-and-beyond.pdf

General problem

Find the minimum and maximum of

$$f(x, y, z, \ldots)$$

subject to

$$g(x, y, z, \ldots) = c$$

CLASSES

n variable

max or min $f(x)$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

s.t. $g(x) = 0$

$n+1$ variables

$$L(x, \lambda) = f(x) + \lambda g(x)$$

$$\boxed{\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial \lambda} = 0}$$

max or min $f(x)$ ← n variable $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

s.t. $g(x) = 0$

n+1 variables

$$\left.\begin{array}{l} L(x, \lambda) = f(x) + \lambda g(x) \end{array}\right\} \text{Step 1}$$

$$\left.\boxed{\frac{\partial L}{\partial x_i} = 0 \quad \frac{\partial L}{\partial \lambda} = 0}\right\} \text{Step 2}$$

x_i's is answer,

The Lagrangian Approach

- To find the extreme points of a function $f(x)$ subject to the constraint $g(x) = 0$, we utilize the Lagrangian method:

$$L(x, \lambda) = f(x) - \lambda g(x)$$

- Solve $\nabla L(x, \lambda) = 0$:

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0$$

The Lagrangian Approach

- To find the extreme points of a function $f(x)$ subject to the constraint $g(x) = 0$, we utilize the Lagrangian method:

$$L(x, \lambda) = f(x) - \lambda g(x)$$

- Solve $\nabla L(x, \lambda) = 0$:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

∇f is parallel to ∇g

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow g(x) = 0$$

orignal constraint

Question:

Apply Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 1.$ $\}\, g(x,y)$

$$\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$
$$= x+y+\lambda(x^2+y^2-1)$$

$$\frac{\partial L}{\partial x} = 1+2\lambda x = 0 \quad \therefore x = -1/2\lambda$$

$$\partial L/\partial y = 1+2\lambda y = 0 \Rightarrow y = -1/2\lambda$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2-1 = 0$$

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} - 1 = 0$$

$$\therefore \frac{1}{2\lambda^2} = 1$$

$$\therefore \lambda = \sqrt{\frac{1}{2}}$$

$$\therefore x = -\frac{1}{2}\cdot\sqrt{2} = -\frac{1}{\sqrt{2}}$$

https://mast.queensu.ca/~math121/AssignmentSolutions/sol_s15_03.pdf

Apply Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 1$.

$$L(x, y, \lambda) = f(x) + \lambda g(x)$$

$$L(x, y, \lambda) = (x+y) + \lambda(x^2 + y^2 - 1)$$

$$\frac{\partial L}{\partial x} = 1 + 2\lambda x = 0 \qquad \frac{\partial L}{\partial y} = 1 + 2\lambda y = 0 \qquad \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0$$

$$2\lambda x = -1 \quad — ① \qquad 2\lambda y = -1 \quad — ②$$

divide ①/② $\frac{x}{y} = 1 \Rightarrow x = y$

$$x^2 + x^2 - 1 = 0$$

$$2x^2 = 1$$

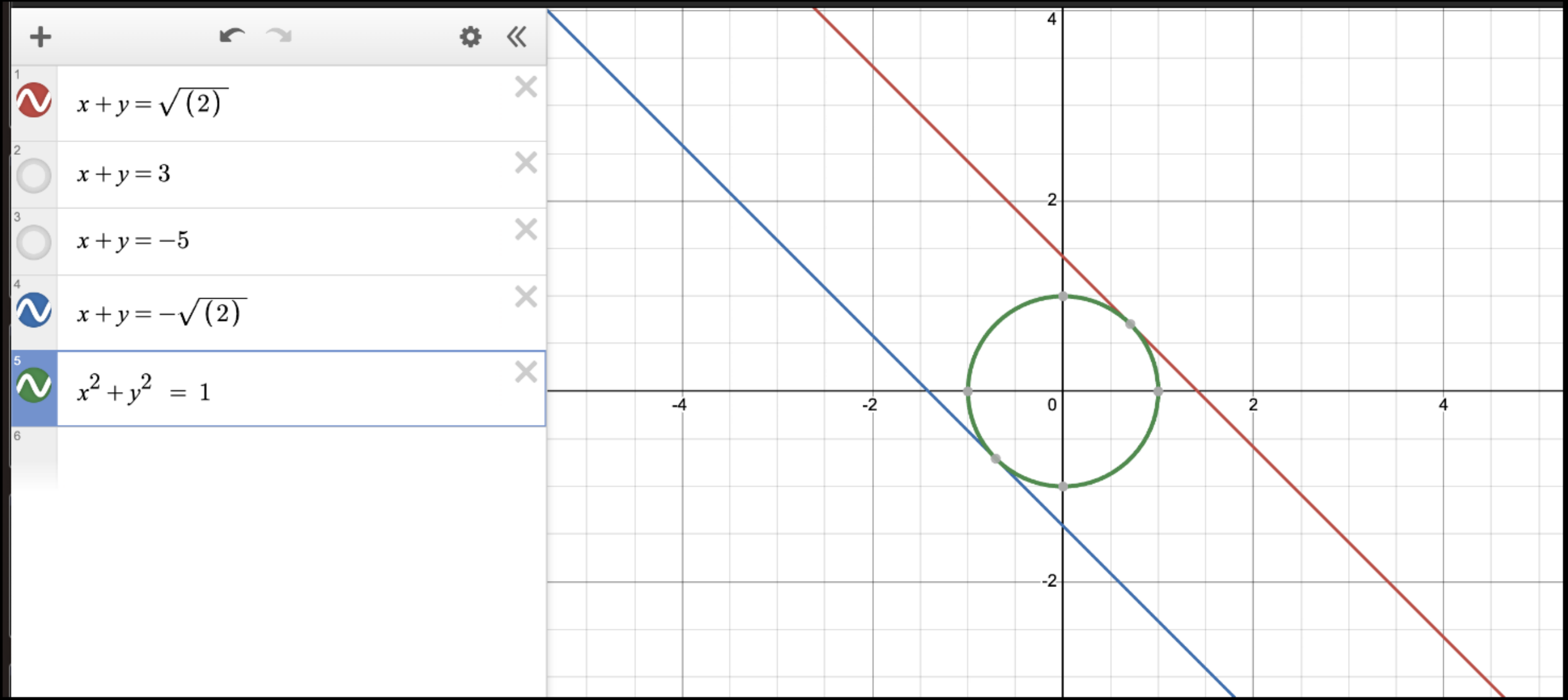
$$x = \pm\sqrt{1/2}$$

$$(x, y) = \left.\begin{array}{l} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ \\ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \end{array}\right\} \begin{array}{l} \frac{2}{\sqrt{2}} = \sqrt{2} \leftarrow \text{maxima} \\ \\ x+y = -\sqrt{2} \leftarrow \text{minima} \end{array}$$

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \times$$

because $x = y$

$\underline{x+y}$



1 $x+y=\sqrt{(2)}$
2 $x+y=3$
3 $x+y=-5$
4 $x+y=-\sqrt{(2)}$
5 $x^2+y^2=1$
6
4
2
-4
-2
0
2
4
-2

Set up the Lagrange multiplier equations:

$$\begin{aligned} f_x = \lambda g_x &\quad \Rightarrow \quad 1 = \lambda 2x \\ f_y = \lambda g_y &\quad \Rightarrow \quad 1 = \lambda 2y \\ \text{constraint:} &\quad \Rightarrow \quad x^2 + y^2 = 1 \end{aligned}$$

Taking (1) / (2), (assuming $\lambda \neq 0$)

$$\frac{1}{1} = \frac{\lambda 2x}{\lambda 2y} = \frac{x}{y}$$

so $y = x$

Sub into (3) to find

$$2x^2 = 1 \quad \Rightarrow \quad x = \pm\sqrt{1/2}$$

Combining with $y = x$, we get the solutions $(x, y) = (\sqrt{1/2}, \sqrt{1/2})$ and $(-\sqrt{1/2}, -\sqrt{1/2})$.
Since our constraint is closed and bounded, we can simply compare the value of f at these two points to determine the maximum and minimum values of f subject to the constraint.

$$f(\sqrt{1/2}, \sqrt{1/2}) = 2\sqrt{1/2}$$
$$f(-\sqrt{1/2}, -\sqrt{1/2}) = -2\sqrt{1/2}$$

From this, the maximum of f on $x^2 + y^2 = 1$ is at $(\sqrt{1/2}, \sqrt{1/2})$ and the minimum is at $(-\sqrt{1/2}, -\sqrt{1/2})$

- Given a problem

$$\max f(x_1, \ldots, x_n) \text{ subject to } g(x_1, \ldots, x_n) = 0$$

- Write down the Lagrangian function

$$\mathcal{L}(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n)$$

- Note that the Lagrangian is a function of $n + 1$ variables: $(x_1, \ldots, x_n, \lambda)$. We then look for the *critical points* of the Lagrangian, that is, points where all the partial derivatives of the Lagrangian are zero.
- *Note that we are not trying to maximize or minimize the Lagrangian function.*

n+1 variables

n+1 equations

$$\frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = 0$$

$$\vdots$$

$$\frac{\partial L}{\partial x_n} = 0 \quad \frac{\partial L}{\partial \lambda} = 0$$

- Using a Lagrangian, we get $n+1$ first order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0, (i = 1, \ldots, n)$$

$$\text{and } \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

- Solving these equations will give us candidate solutions for the constrained optimization problem.
- Candidate solutions have the same status as the unconstrained case. That is, they need to be checked using the second-order conditions.

Question:

(10 points) Using Lagrange multipliers find the maximum and minimum values of the function $f(x, y) = 4x^2 + 9y^2$ subject to the constraint $xy = 4$

$$L(x, y, \lambda) = 4x^2 + 9y^2 + \lambda(xy - 4)$$

$$\frac{\partial L}{\partial x} = 8x + \lambda y = 0 \qquad \frac{\partial L}{\partial y} = 18y + \lambda x = 0$$

$$\frac{\partial L}{\partial \lambda} = xy - 4 = 0$$

$$\left.\begin{aligned} 8x &= -\lambda y \\ 18y &= -\lambda x \end{aligned}\right\} \begin{aligned} \lambda^2 &= 18 \times 8 \\ \lambda &= \pm 12 \end{aligned}$$

https://sites.pitt.edu/~evt3/a/0240/2012_Fall/Exams/e1(0240)_9am_slns_fall_2012.pdf

$$8x = -\lambda y$$
$$18y = -\lambda x$$

$\lambda = 12$

$$8x = -12y$$
$$18y = -12x$$
$$xy = 4$$
$$y = 4/x$$

$$8x = -12\left(\frac{4}{x}\right) \Rightarrow 8x^2 = -48$$

No soln

$\lambda = -12$

$$8x = 12y$$
$$xy = 4 \Rightarrow y = \frac{4}{x}$$
$$8x = 12\left(\frac{4}{x}\right) \Rightarrow x^2 = 6$$
$$x = \pm\sqrt{6}$$

$\lambda = 12$

No soln

$\lambda = -12$

$8x = 12y$

$xy = 4 \Rightarrow y = \frac{4}{x}$

$8x = 12\left(\frac{4}{x}\right) \Rightarrow x^2 = 6$

$x = \pm\sqrt{6}$

$x = \sqrt{6}$

$y = \frac{2}{3}\sqrt{6}$

$x = -\sqrt{6}$

$y = -\frac{2}{3}\sqrt{6}$

$\lambda = -12$

$8x = 12y$

$xy = 4 \Rightarrow y = \frac{4}{x}$

$8x = 12\left(\frac{4}{x}\right) \Rightarrow x^2 = 6$

$x = \pm\sqrt{6}$

$x = \sqrt{6}$ $\qquad$ $x = -\sqrt{6}$

$y = \frac{2}{3}\sqrt{6}$ $\qquad$ $y = -\frac{2}{3}\sqrt{6}$

$4x^2 + 9y^2 \quad = \quad 48 \qquad 48$

Solution: Under the constraint $xy = 4$ x can be made arbitrary large (while y is small). Therefore, there is no upper bound for the function $f(x, y)$ and its maximum never attains. Let's find minimum using Lagrange multipliers. Denote $g(x, y) = xy$.

Then using $\bar{\nabla} f(x, y) = \lambda \bar{\nabla} g(x, y)$, $g(x, y) = 4$ we get the system of equations

$$\begin{aligned} 8x &= \lambda y \\ 18y &= \lambda x \\ xy &= 4 \end{aligned}$$

$L(x,y,\lambda) = f(x) - \lambda g(x)$

Multiplying the first line by the second we obtain $8 \cdot 18xy = \lambda^2 xy$ or $8 \cdot 18 = \lambda^2$. (Note: $xy = 4$ and hence $x \neq 0$ and $y \neq 0$). Then $\lambda^2 = 8 \cdot 18 = 16 \cdot 9 = (4 \cdot 3)^2$ and $\lambda = -12$ or $\lambda = 12$. From the first line $y = \frac{8}{\lambda} x$.

$L(x, y, \lambda) = f(x) + \lambda g(x)$

Case $\lambda = -12$. Then $y = -\frac{2}{3}x$ and the last line gives $-\frac{2}{3}x^2 = 4$, no solutions.

Case $\lambda = 12$. Then $y = \frac{2}{3}x$ and the last line gives $\frac{2}{3}x^2 = 4$, $x^2 = 6$, $x = -\sqrt{6}$ or $x = \sqrt{6}$. There are two solutions of the system $(x, y) = (-\sqrt{6}, -\frac{2}{3}\sqrt{6})$ and $(x, y) = (\sqrt{6}, \frac{2}{3}\sqrt{6})$.

$f(\sqrt{6}, \frac{2}{3}\sqrt{6}) = f(-\sqrt{6}, -\frac{2}{3}\sqrt{6}) = 4 \cdot 6 + 9 \cdot \frac{4}{9} \cdot 6 = 48$ is the minimum value of $f(x, y)$.

$$L(x, \lambda) = f(x) - \lambda g(x)$$

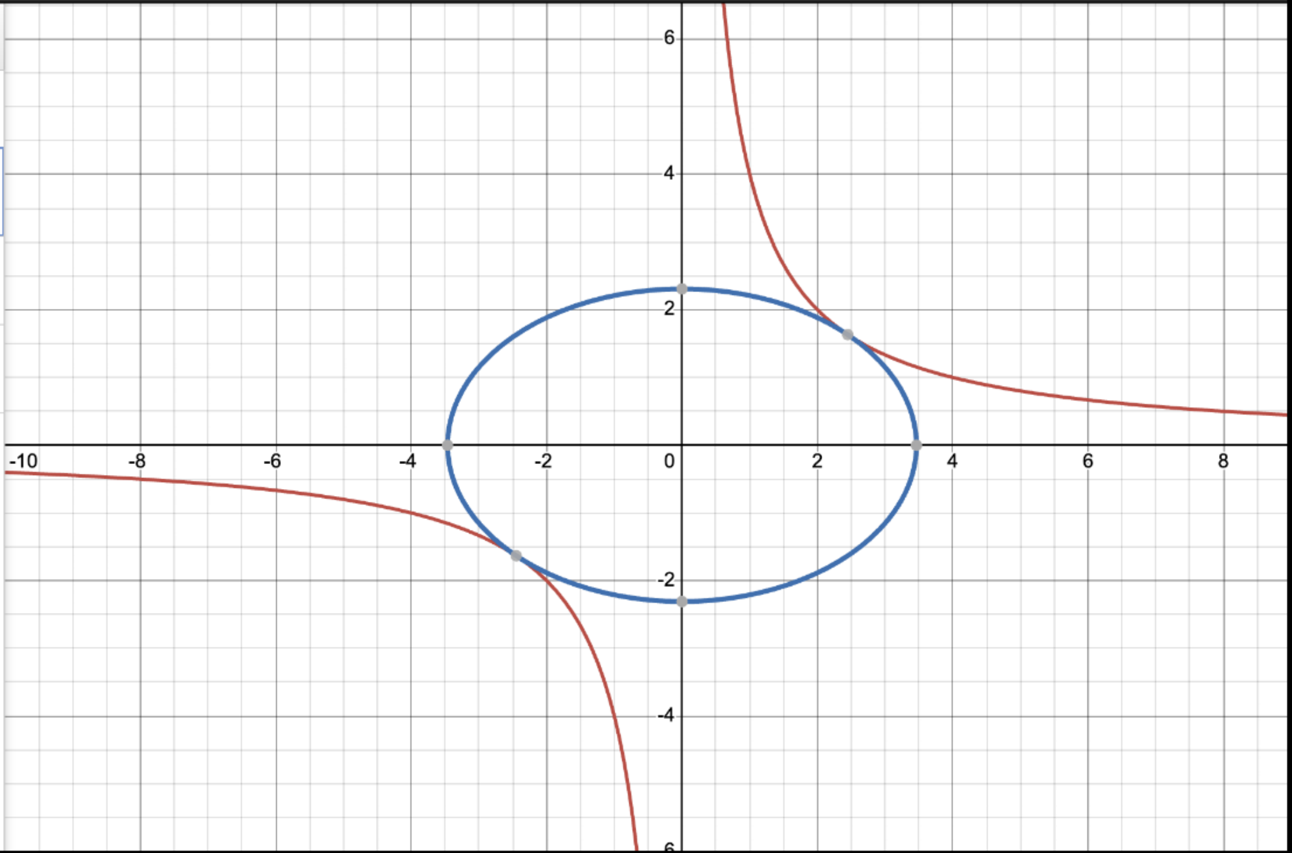
$$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$xy = 4$

$4x^2 + 9y^2 = 48$

$4x^2 + 9y^2 = 40$

$4x^2 + 9y^2 = 60$



18 5A9 to Everyone 9:29 PM

Are concepts in mtech in IISc has similar concepts like this or these are just prerequisite for mtech?

exact same concepts.

Question:

Find the extreme values of $f(x, y) = xe^y$ subject to the constraint $x^2 + y^2 = 2$.

Which pair (x, y) below satisfy the maxima and minima constraints?

A. Maxima will be obtained at $(1, 1)$, Minima will be obtained at $(-1, 1)$

B. Maxima will be obtained at $(-1, 1)$, Minima will be obtained at $(1, 1)$

C. Maxima will be obtained at $(1, -1)$, Minima will be obtained at $(-1, 1)$

D. Maxima will be obtained at $(-1, 1)$, Minima will be obtained at $(1, -1)$

$L(x, y, \lambda) = f(x) - \lambda g(x)$

$\frac{\partial L}{\partial x} = e^y - \lambda 2x = 0$

$\frac{\partial L}{\partial y} = xe^y - \lambda 2y = 0$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^2 + y^2 = 2$

$\underset{e}{(1, 1)} \quad \underset{-e}{(-1, 1)}$

https://www.ms.uky.edu/~ma213/213-s19/_assets/old-exams/MA213_F18_Exam_2_KEY.pdf

$$L(x, y, \lambda) = f(x) - \lambda g(x)$$

$$\frac{\partial L}{\partial x} = e^y - \lambda 2x = 0 \Rightarrow e^y = \lambda 2x$$

$$\frac{\partial L}{\partial y} = x e^y - \lambda 2y = 0 \longrightarrow x(\cancel{\lambda}\cancel{2}x) = \cancel{\lambda}\cancel{2}y$$

$$x^2 = y$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^2 + y^2 = 2 \longrightarrow y^2 + y = 2$$

$$y = 1, -2$$

$x^2 = y$

$y^2 + y = 2$

$y = 1, -2$

$y = 1$

$x^2 = 1$

$x = \pm 1$

$y = -2$

$x^2 = y$

$x^2 = -2$ ✗

$\Rightarrow (1, 1), (-1, 1)$

Solution: The objective function is $f(x,y) = xe^y$ and the constraint function is $g(x,y) = x^2 + y^2$. (2 points) Computing the gradients and setting $\nabla f = \lambda \nabla g$ we get

$$e^y = 2\lambda x \qquad \text{(2 points)} \qquad (1)$$
$$xe^y = 2\lambda y \qquad \text{(2 points)} \qquad (2)$$
$$x^2 + y^2 = 2 \qquad \text{(1 points)} \qquad (3)$$

Note that e^y is never zero so $\lambda \neq 0$. Substituting (1) into (2) to eliminate e^y, we get

$$2\lambda x^2 = 2\lambda y$$

or $y = x^2$. Using this relation in (3) we get $y + y^2 = 2$ or $y^2 + y - 2 = 0$. The two roots are $y = 1$ and $y = -2$, but $y = -2$ is not admissible since $y = x^2$. Thus we need to test $(1,1)$ and $(-1,1)$. (8 points)

(x,y)	$f(x,y) = xe^y$
$(1,1)$	e
$(-1,1)$	$-e$

From these computations we conclude that the maximum value of $f(x,y)$ with constraint $x^2 + y^2 = 2$ is e, and the minimum value is $-e$. (3 points)

Question:

Find all the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

$$x^2+2y^2 - \lambda(x^2+y^2-1)$$

$$L(x, y, \lambda) = f(x,y) - \lambda\, g(x, y)$$

$$\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial y} = 0 \quad \frac{\partial L}{\partial \lambda} = 0$$

$$\Rightarrow \lambda = 1$$

$$2x - 2\lambda x = 0$$

$$4y - 2\lambda y = 0 \Rightarrow \lambda = 2$$

$$x^2+y^2 = 1$$

Example, cont.

$$\nabla f(x, y) = \langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle = \lambda \nabla g(x, y)$$

and

$$x^2 + y^2 = 1$$

By comparing the components of the vectors, we arrive at the more useful set of equations:

$$2x = \lambda 2x \tag{1}$$

$$4y = \lambda 2y \tag{2}$$

$$x^2 + y^2 = 1 \tag{3}$$

Thus, our goal, according to the method of Lagrange multipliers, is to find all constants λ and ordered pairs (x, y) satisfying all three of these equations at once.

Example, cont.

There are many ways to solve this system, but here's one that stands out to me: First, note that:

$$
\begin{aligned}
(1) \Rightarrow \ & 2x(1-\lambda) = 0 \\
\Rightarrow \ & x = 0 \text{ or } \lambda = 1
\end{aligned}
$$

Therefore, for (1) to be true, we must have either $x = 0$ or $\lambda = 1$. We now investigate each case separately.

$x=0$ or $\lambda=1$ $\Leftarrow$ $2x - 2\lambda x = 0$ $\Rightarrow \lambda = 1$

$4y - 2\lambda y = 0 \Rightarrow \lambda = 2$

$x^2 + y^2 = 1$

Example, cont.

First, suppose that $x = 0$. Note that:

$$\begin{aligned}(3) \text{ and } x = 0 &\Rightarrow y^2 = 1 \\ &\Rightarrow y = \pm 1\end{aligned}$$

Therefore, if $x = 0$, (1) and (3) can only be true at the same time if $y = \pm 1$. But then note that:

$$\begin{aligned}(2),\ x = 0, \text{ and } y = 1 &\Rightarrow 4 = 2\lambda \\ &\Rightarrow \lambda = 2\end{aligned}$$

and

$$\begin{aligned}(2),\ x = 0, \text{ and } y = -1 &\Rightarrow -4 = -2\lambda \\ &\Rightarrow \lambda = 2\end{aligned}$$

$x = 0$ | $\lambda = 1$

$y = \pm 1$ | $y = 0$

$(0, 1)$ | $x = \pm 1$

$(0, -1)$ | $(1, 0)$

$(-1, 0)$

Example, cont.

Therefore, if $x = 0$, then all three equations are true at the same time:

at the point $(0, 1)$ with $\lambda = 2$

or

at the point $(0, -1)$ with $\lambda = 2$

We will hang onto the points $(0, 1)$ and $(0, -1)$ for later.

Example, cont.

Now we investigate the case where $\lambda = 1$. Note that:

$$\begin{aligned}
(2) \text{ and } \lambda = 1 &\Rightarrow 4y = 2y \\
&\Rightarrow 2y = 0 \\
&\Rightarrow y = 0
\end{aligned}$$

Therefore, if $\lambda = 1$, (1) and (2) can only be true at the same time if $y = 0$. But then note that:

$$\begin{aligned}
(3),\ \lambda = 1, \text{ and } y = 0 &\Rightarrow x^2 = 1 \\
&\Rightarrow x = \pm 1
\end{aligned}$$

Example, cont.

Therefore, if $\lambda = 1$, then all three equations are true at the same time:

at the point $(1, 0)$ with $\lambda = 1$

or

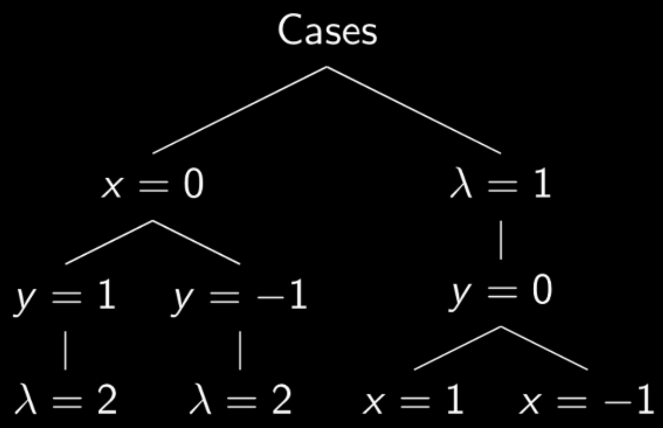
at the point $(-1, 0)$ with $\lambda = 1$

Example, cont.

At this point, we've found all possible solutions to this system of equations, as we've exhausted all the cases we discovered. Indeed, we found that for the first equation to be true, it must be that either $x = 0$ or $\lambda = 1$, and then we investigated which possible values of y and λ; and x and y, respectively, can satisfy the rest of the system in each case.

You may find it somewhat challenging to keep track of which case you're working on at a given time, especially if the number of cases grows large; and you may find it especially challenging to know when you've finished investigating all of the cases you found. To that end, you may find a tree like the one on the following slide helpful.

Example, cont.



$\overset{2}{(0, 1)}$, $\overset{2}{(0, -1)}$, $\overset{1}{(1, 0)}$, and $\overset{1}{(-1, 0)}$

This is a chart that you can construct as you go through solving the system of equations. For example, in this problem we would proceed as follows: the first equation tells us that for all equations in the system to be true simultaneously, we must have that either $x = 0$ or $\lambda = 1$. So, we add a node below "Cases" for each of these. Then, when we investigate the $x = 0$ case, we see that y must either be 1 or -1 when $x = 0$, so we add these nodes below $x = 0$, etc. When all nodes terminate with a value of x, y, and λ, you're finished!

Example, cont.

Let's finish up the problem. In solving the system of equations above, we obtained the following points:

$$\underbrace{\overset{2}{(0,1)}, \ \overset{2}{(0,-1)}}, \ \underbrace{\overset{1}{(1,0)}, \text{ and } \overset{1}{(-1,0)}}$$

x^2+2y^2

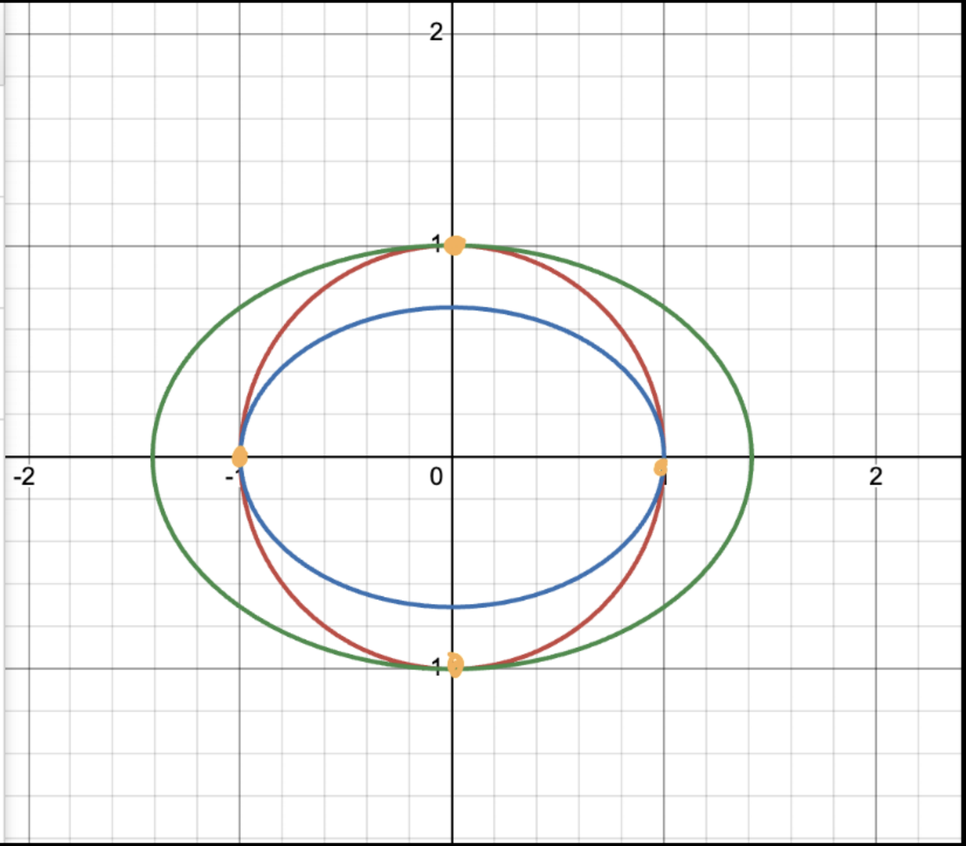
We now plug all of these into $f(x,y)$. The method of Lagrange multipliers tells us that the largest value we get is the absolute maximum value of $f(x,y)$ on the circle $x^2+y^2=1$; and the smallest value we get is the absolute minimum value of $f(x,y)$ on the circle $x^2+y^2=1$.

GO CLASSES

$x^2 + y^2 = 1$ ← constraint

$x^2 + 2y^2 = 1$ level set

$x^2 + 2y^2 = 2$ level set



Example, cont.

We have:

$$\begin{aligned} f(0,1) &= 0^2 + 2 \cdot 1^2 = 2 \\ f(0,-1) &= 2 \\ f(1,0) &= 1 \\ f(-1,0) &= 1 \end{aligned}$$

Therefore, the absolute maximum value of $f(x, y)$ on the unit circle is $f(0, 1) = f(0, -1) = 2$, and the absolute minimum value of $f(x, y)$ on the unit circle is $f(1, 0) = f(-1, 0) = 1$.

Geometric proof for Lagrange

https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/resources/mit18_02sc_notes_22/

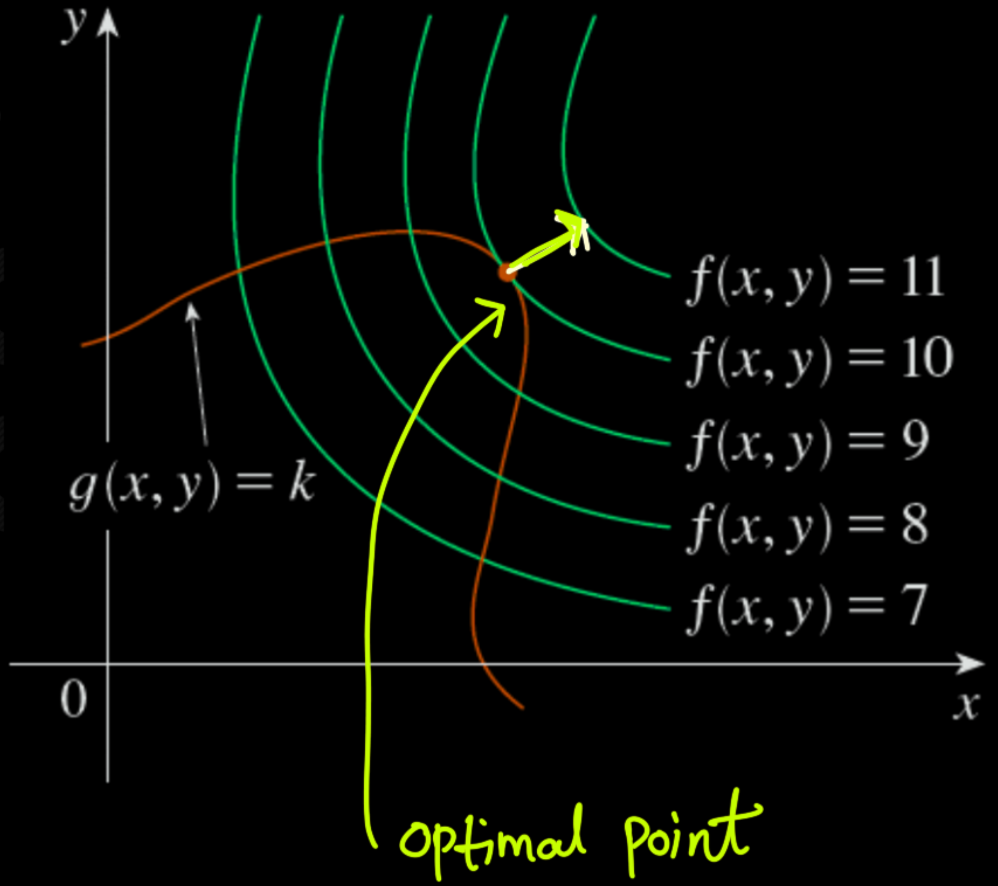
$\nabla f, \nabla g$

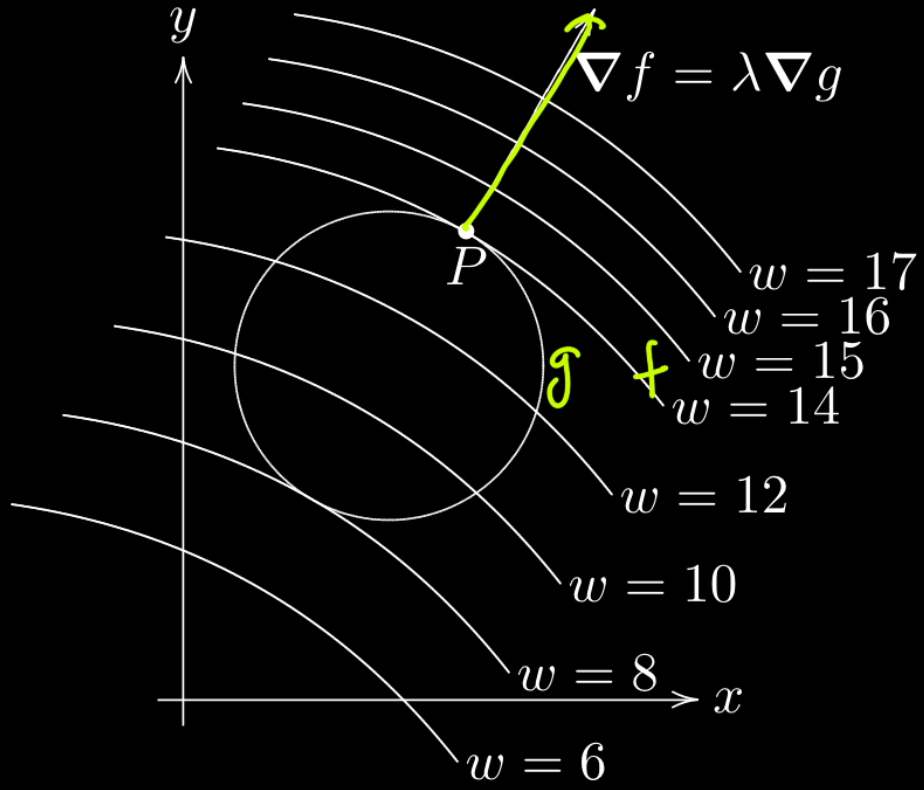
either in same direction

or in opposite direction

$\nabla f = +\nabla g$

$\nabla f = -\nabla g$

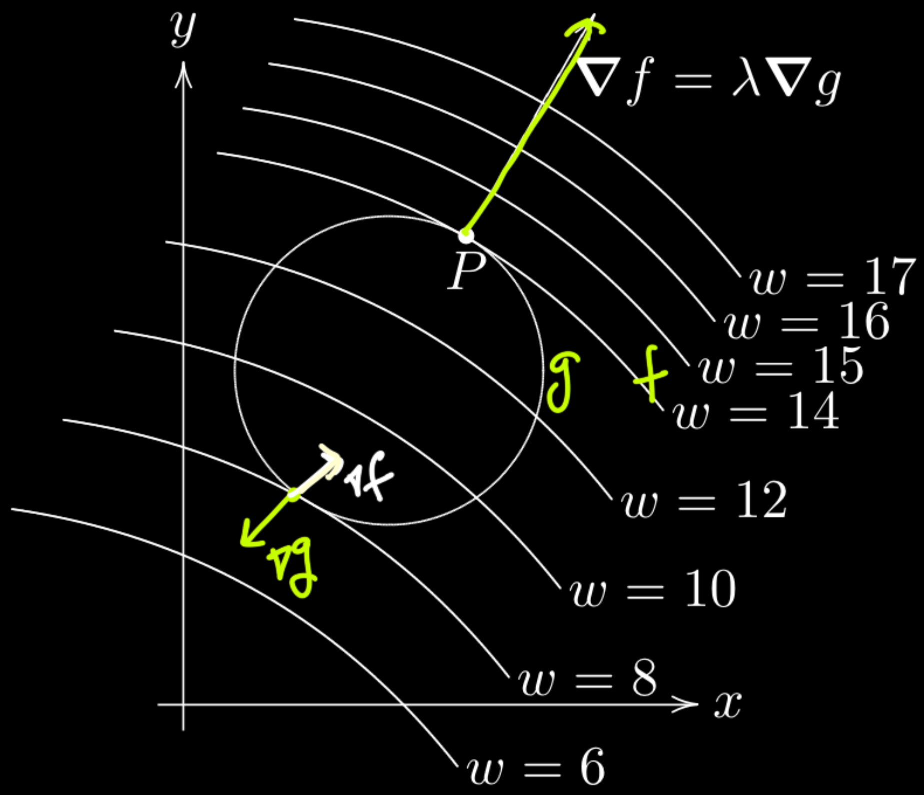




$\nabla f = +ve \ \nabla g$

$(x-a)^2 + (y-b)^2 = 10$

$(x-a)^2 + (y-b)^2 = 20$



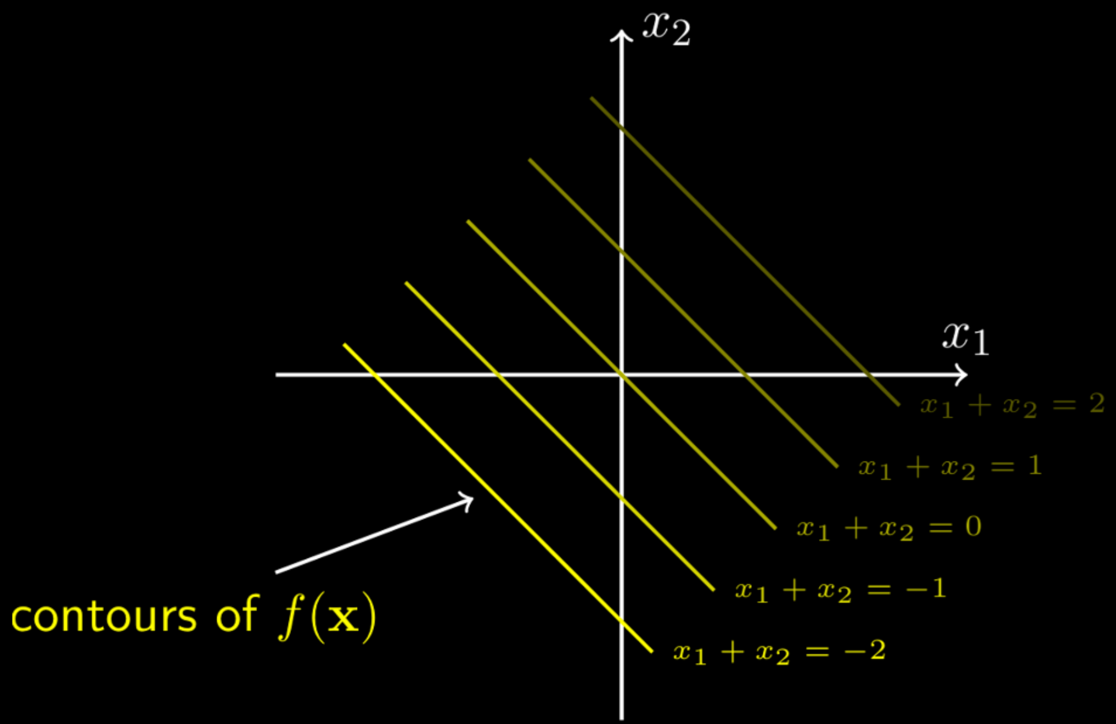
$\nabla f = -ve \ \nabla g$

$(x-a)^2 + (y-b)^2 = 10$

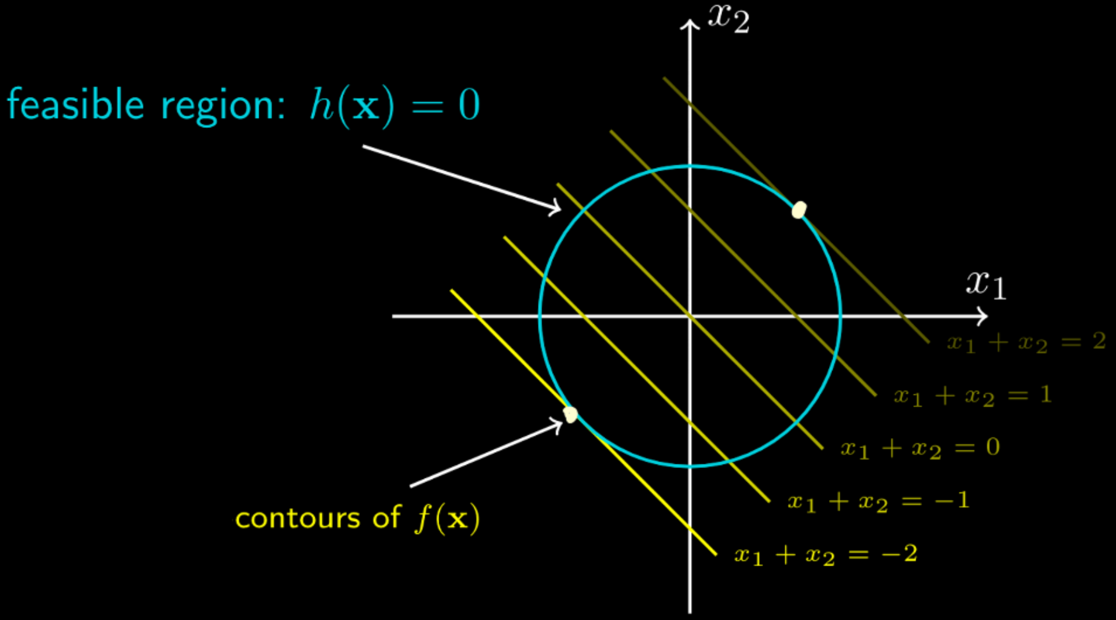
$(x-a)^2 + (y-b)^2 = 20$

https://ocw.mit.edu/courses/18-02sc-multivariable-calculus-fall-2010/ebbeb8e61827a8058d2c45b674d003b3_MIT18_02SC_notes_22.pdf

$$\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) = x_1 + x_2 \text{ and subject to } x_1^2 + x_2^2 = 2$$



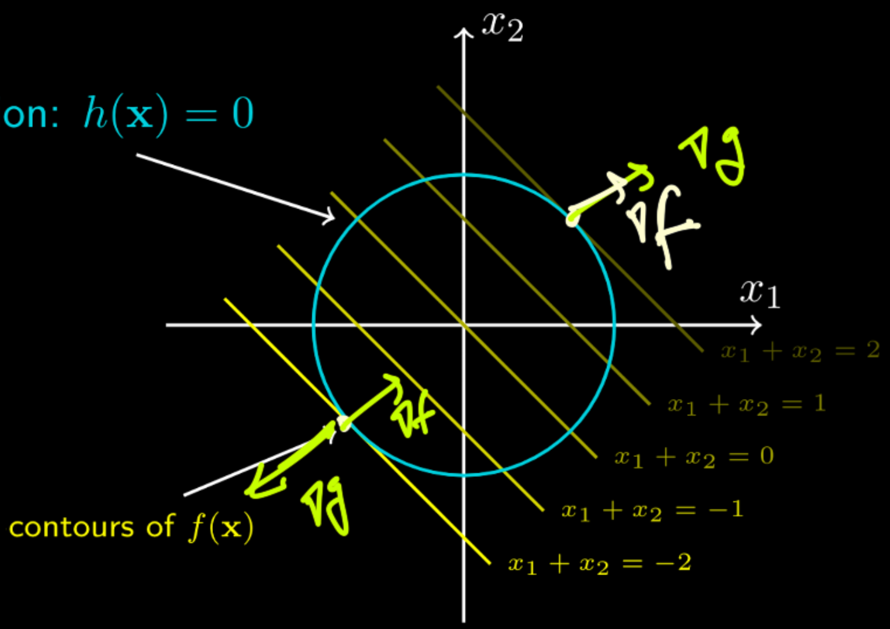
$$\boxed{f(\mathbf{x}) = x_1 + x_2}$$



$$\boxed{f(\mathbf{x}) = x_1 + x_2}$$

$$\boxed{h(\mathbf{x}) = x_1^2 + x_2^2 - 2}$$

min = −2 , max = 2



$$f(\mathbf{x}) = x_1 + x_2$$

$$h(\mathbf{x}) = x_1^2 + x_2^2 - 2$$

min $= -2$, max $= 2$

Optional

- At the minimum point, $\nabla f(x)$ and $\nabla g(x)$ are parallel and opposite to each other.
- At the maximum point, $\nabla f(x)$ and $\nabla g(x)$ are parallel in the same direction

The Lagrangian Approach

- To find the extreme points of a function $f(x)$ subject to the constraint $g(x) = 0$, we utilize the Lagrangian method:

$$L(x, \lambda) = f(x) - \lambda g(x)$$

- Solve $\nabla L(x, \lambda) = 0$:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow g(x) = 0$$

Question:

$$\max_{\mathbf{x}} \quad \mathbf{x}^T A \mathbf{x}$$
$$\text{s.t.} \quad \mathbf{x}^T \mathbf{x} = 1.$$

(Consider A as symmetric matrix)

$Ax + A^T x$

$2Ax$

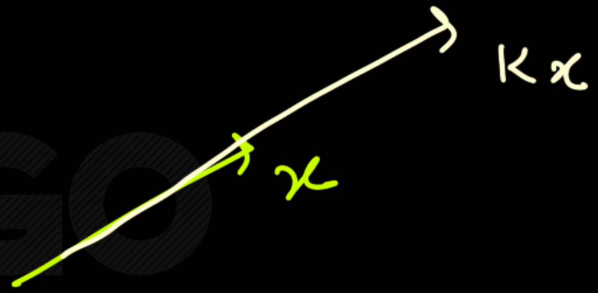
$\boxed{\frac{1}{2} x^T A x} \Rightarrow Ax$

$L(x, \lambda) = x^T A x - \lambda(x^T x - 1)$

$\frac{\partial L}{\partial x} = 2Ax - \lambda 2x = 0 \Rightarrow \boxed{Ax = \lambda x}$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^T x = 1 \Rightarrow \|x\| = 1$

if x is eigenvector $\Rightarrow$ Kx is also eigenvector



$$x^T A x = x^T \lambda x = \lambda x^T x = \lambda$$

max. value will be at maximum eigenvalue.

$$\min \quad x^T A x$$
$$\text{s.t.} \quad \|x\| = 1$$

min value of $x^T A x = \lambda_{min}$ of min. $\underline{\underline{\lambda}}$

To maximize $\mathbf{x}^T A\mathbf{x}$ subject to $\mathbf{x}^T\mathbf{x} = 1$:

- Formulate the Lagrangian $\mathcal{L}(\mathbf{x}, \lambda) = \mathbf{x}^T A\mathbf{x} + \lambda(1 - \mathbf{x}^T\mathbf{x})$, where λ is the Lagrange multiplier.
- Compute the gradient of $\mathcal{L}$ with respect to $\mathbf{x}$: $\nabla_{\mathbf{x}}\mathcal{L} = 2A\mathbf{x} - 2\lambda\mathbf{x} = 0$.
- Simplify to $(A - \lambda I)\mathbf{x} = 0$, indicating $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ.
- Maximize $\mathbf{x}^T A\mathbf{x}$ by choosing $\mathbf{x}$ as the eigenvector corresponding to the largest eigenvalue $\lambda_{\max}$ of A, given $\mathbf{x}^T\mathbf{x} = 1$.

Therefore, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to $\mathbf{x}^T\mathbf{x} = 1$ is $\lambda_{\max}$, where $\lambda_{\max}$ is the largest eigenvalue of A, and $\mathbf{x}$ is the corresponding eigenvector.

GATE IT 2007 | Question: 2

62

Let A be the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. What is the maximum value of $x^T Ax$ where the maximum is taken over all x that are the unit vectors x?

A. 5

B. $\frac{(5+\sqrt{5})}{2}$

C. 3

D. $\frac{(5-\sqrt{5})}{2}$

$\max \quad x^T A x$

s.t. $\quad x^T x = 1$

$\Rightarrow \lambda_{max}$

$\lambda_{max} = \frac{5+\sqrt{5}}{2}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 5\lambda + 5 = 0$ (on solving this quadratic equation)

$\lambda = \frac{5 \pm \sqrt{5}}{2}$

on-2

GATE IT 2007 | Question: 2

62

Let A be the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. What is the maximum value of $x^T A x$ where the maximum is taken over all x that are the unit eigenvectors of A?

A. 5
B. $\frac{(5+\sqrt{5})}{2}$
C. 3
D. $\frac{(5-\sqrt{5})}{2}$

great hint

max. $x^T A x$

s.t. $x^T x = 1$

also $Ax = \lambda x$

$$\boxed{x^T A x \Rightarrow x^T \lambda x \Rightarrow \lambda}$$

λ_{max}

λ_{min}

https://gateoverflow.in/3433/gate-it-2007-question-2

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 5\lambda + 5 = 0$ (on solving this quadratic equation)

$$\lambda = \frac{5 \pm \sqrt{5}}{2}$$

in PCA

$x^T A x$

covariance matrix

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & & \cdot & \\ x_2 & \cdot & & \\ x_3 & & & \end{array}$$

(A is symm)

$\max/\min \; x^T A x$

s.t. $x^T x = 1$

$\Downarrow$

eigenvalues of A

$A \downarrow$

$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$

$\min \Rightarrow \lambda_4$

$\max \Rightarrow \lambda_1$

SVM Support vector Machine

Constrained optimisation