# Next Topic: Contours

## Level curves

The **level curves** or **contour curves** of a function $f$ are the set of curves of the equations $f(x, y) = k$ for varying constants $k$.

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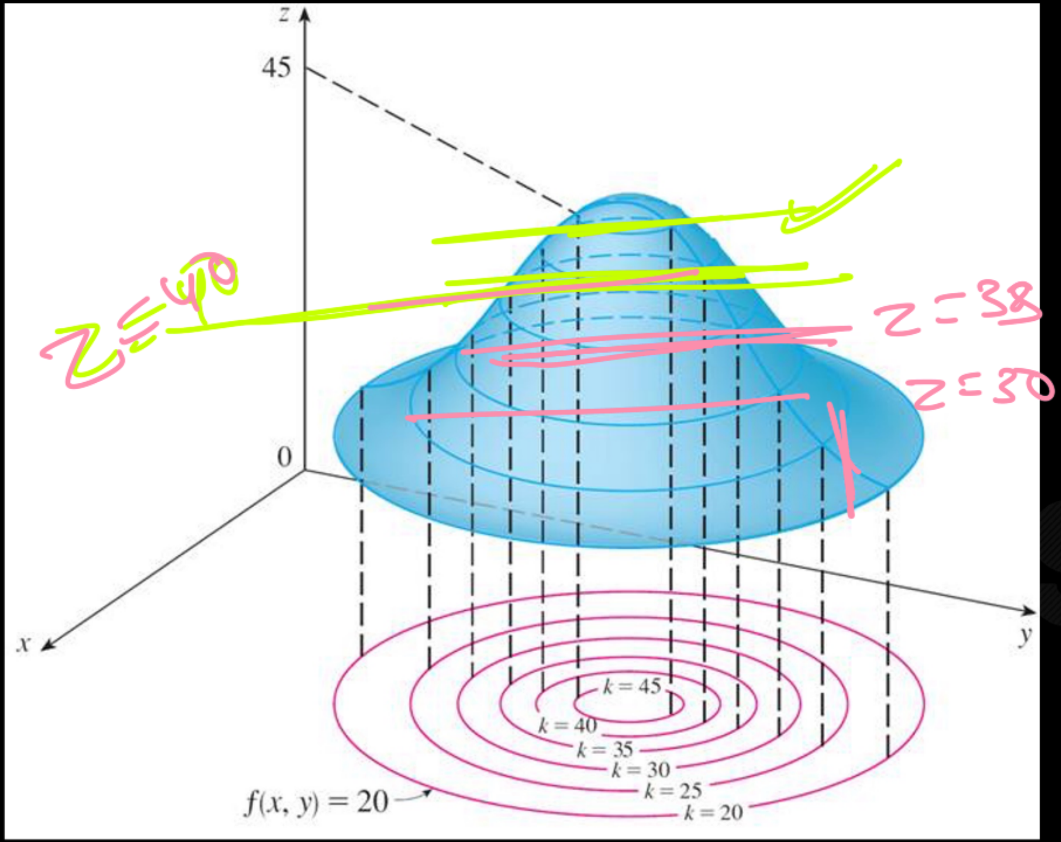
# Definition

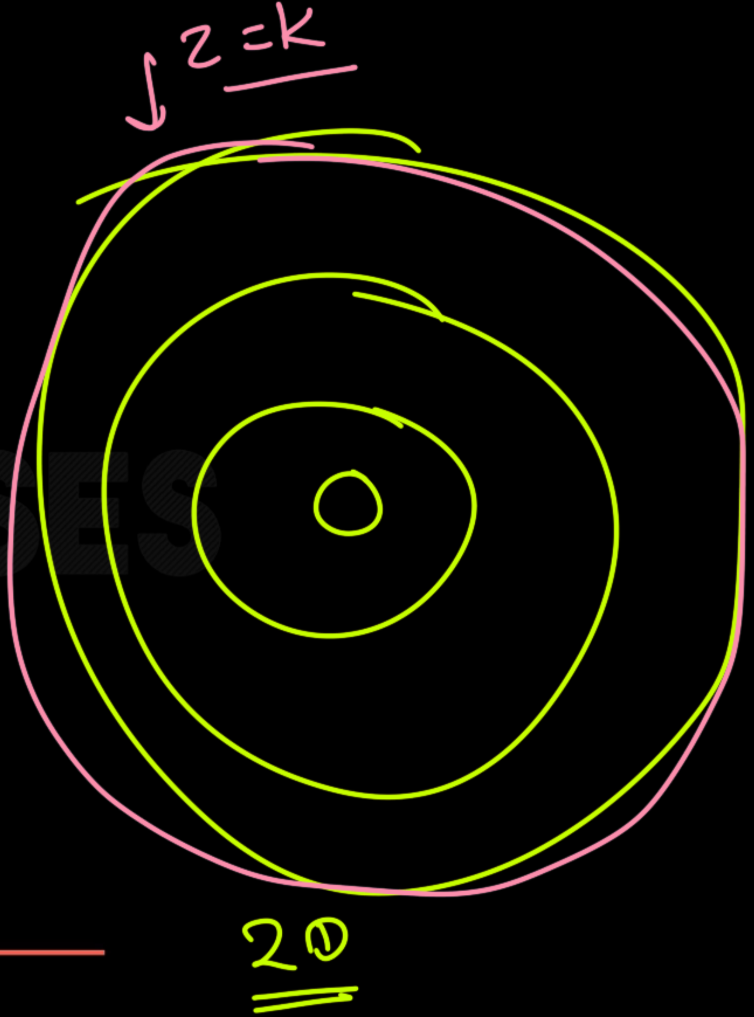
The **level curves** of a function $f$ of *two* variables are the curves $f(x, y) = k$ for $k$ a constant.

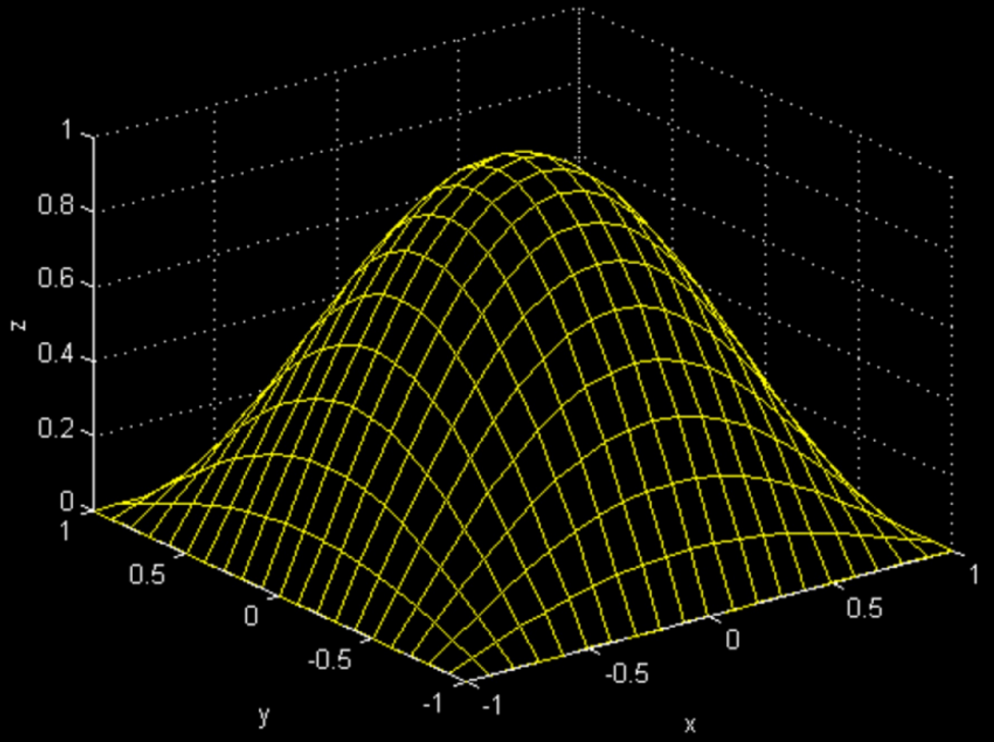
A level curve of the graph of $f$ is the same as a trace of the graph parallel to to the $xy$-plane. That is, a trace in $z = k$. You can think of such curves as analogous to the elevation lines on a topographical map.

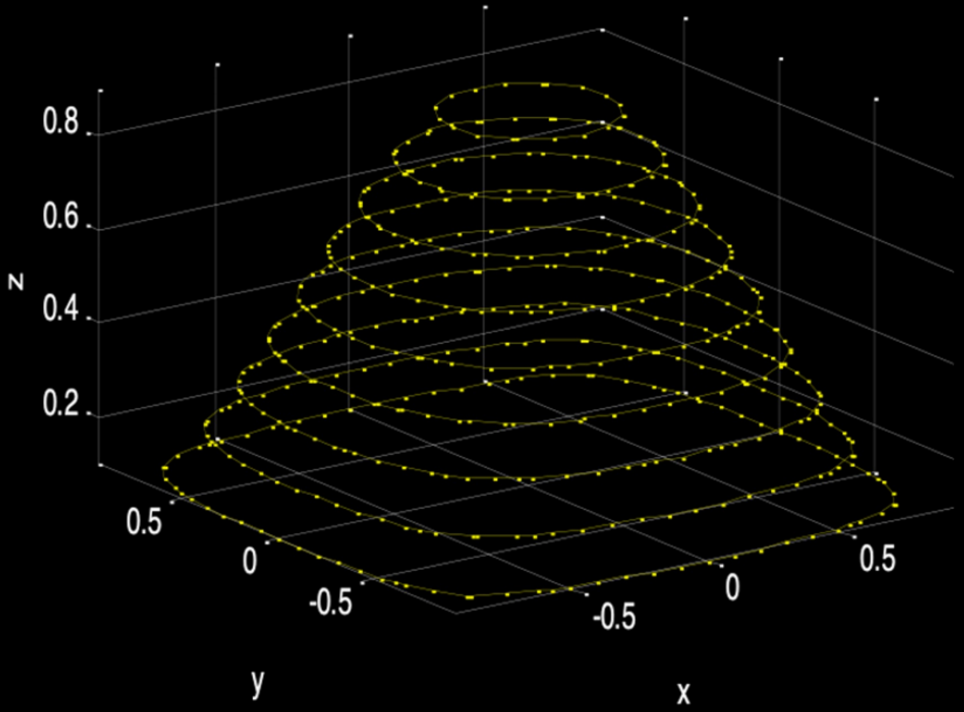
A collection of level curves charted simultaneously on the same set of axes is called a **contour map**.

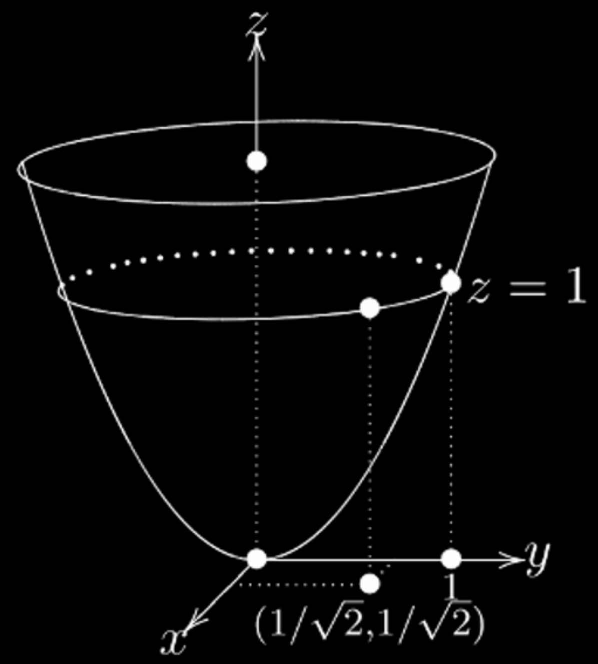
way to represent the function $f(x, y)$ graphically is to produce a 2D **plot** of the surface $z = f(x, y)$.

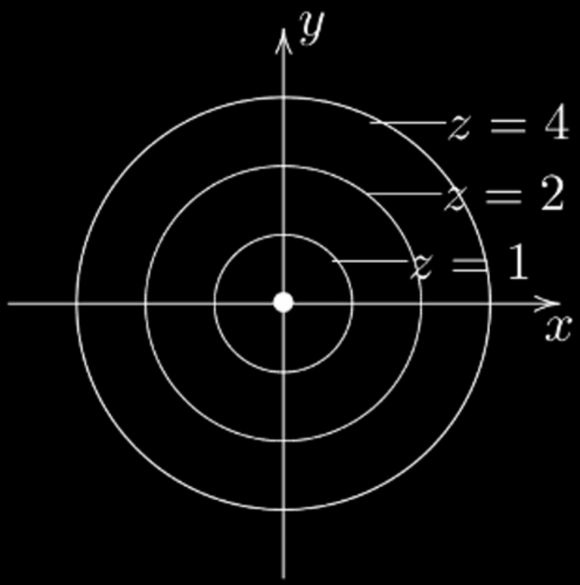

z
45
0
x
y
k = 45
k = 40
k = 35
k = 30
k = 25
k = 20
f(x, y) = 20
z=40
z=38
z=30


z=k
2D


z
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
x


z
0.8
0.6
0.4
0.2
0.5
0
-0.5
y
-0.5
0
0.5
x


z
z = 1
y
1
x
(1/√2,1/√2)


y
z = 4
z = 2
z = 1
x

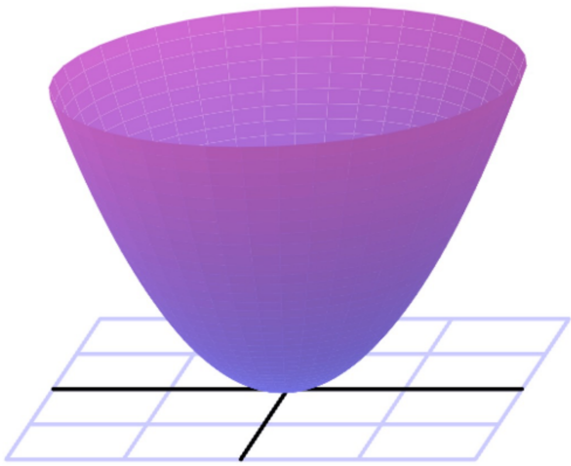
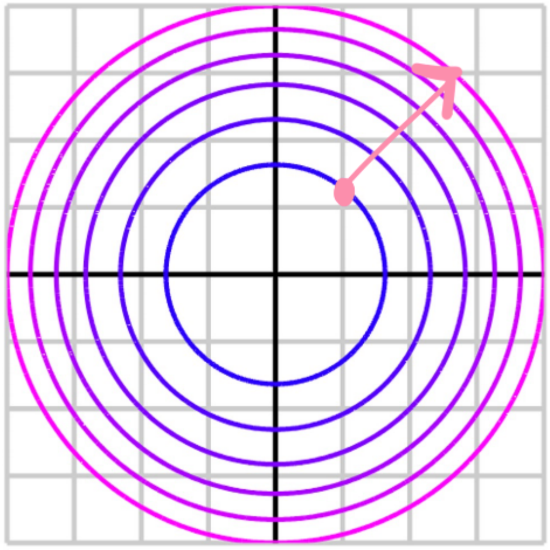
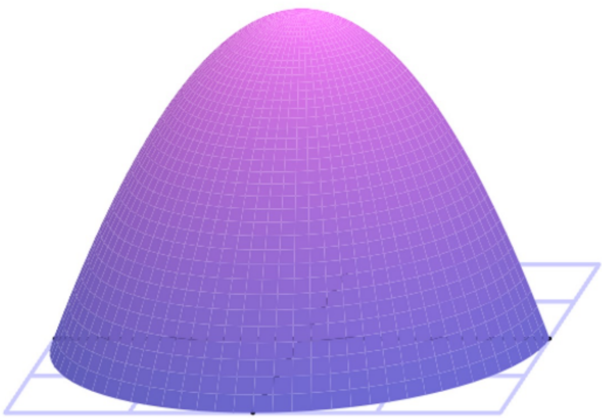


Figure 1.1.22. $z = x^2 + y^2$



Contour maps



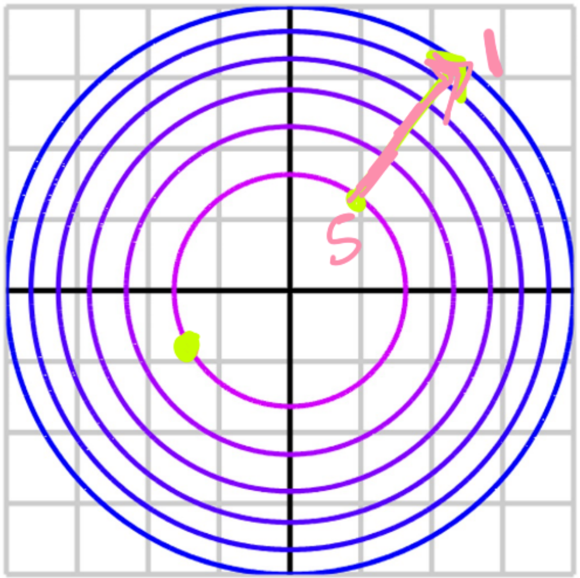



**Figure 1.1.23.** $z = 4 - (x^2 + y^2)$

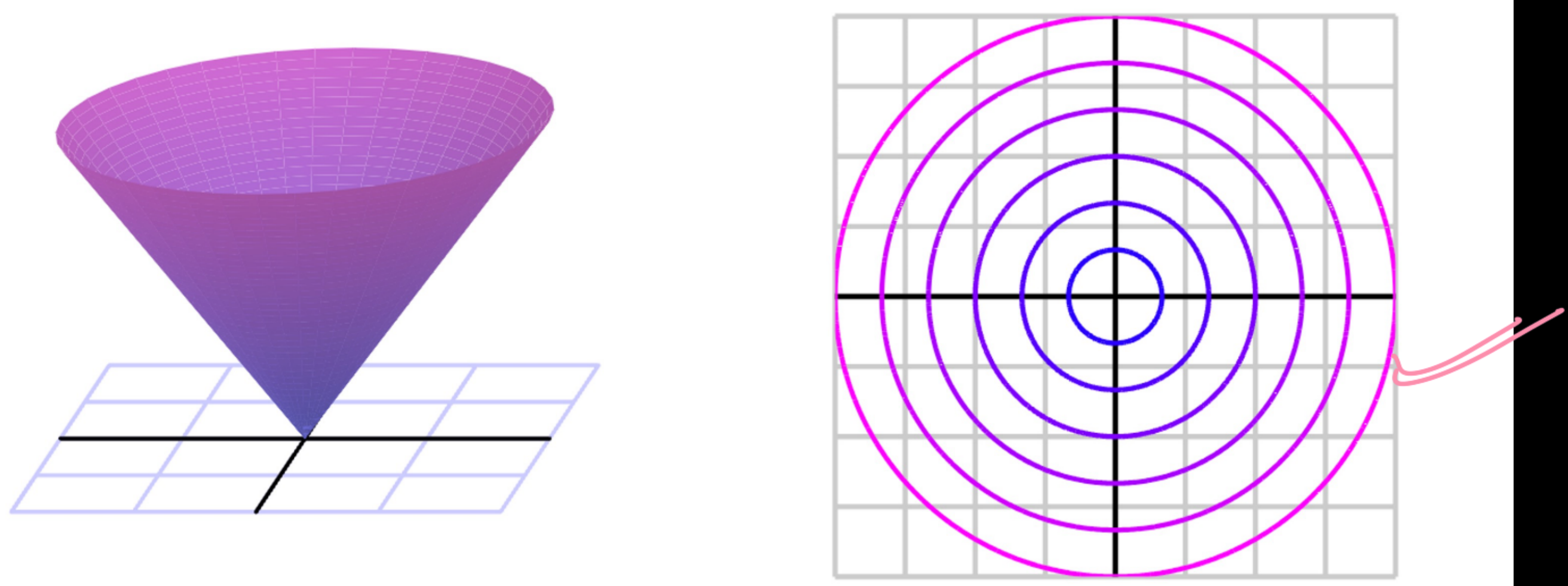
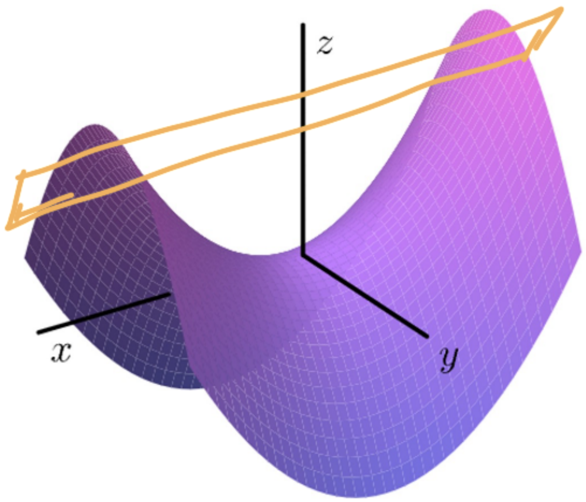


Figure 1.1.24. $z = \sqrt{x^2 + y^2}$



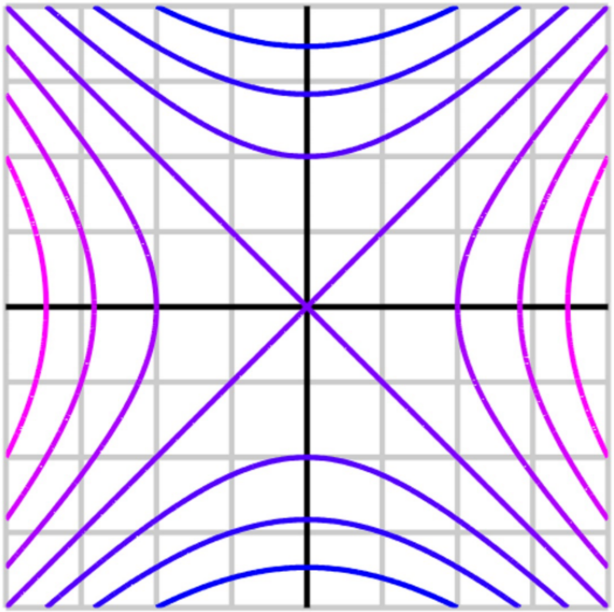
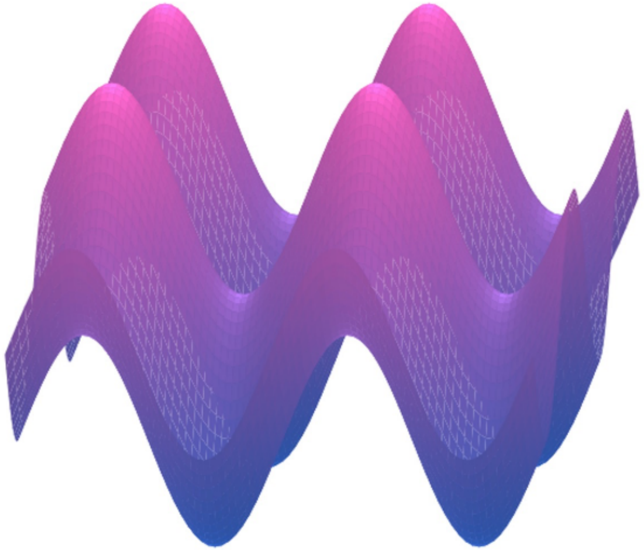



**Figure 1.1.25.** $z = x^2 - y^2$



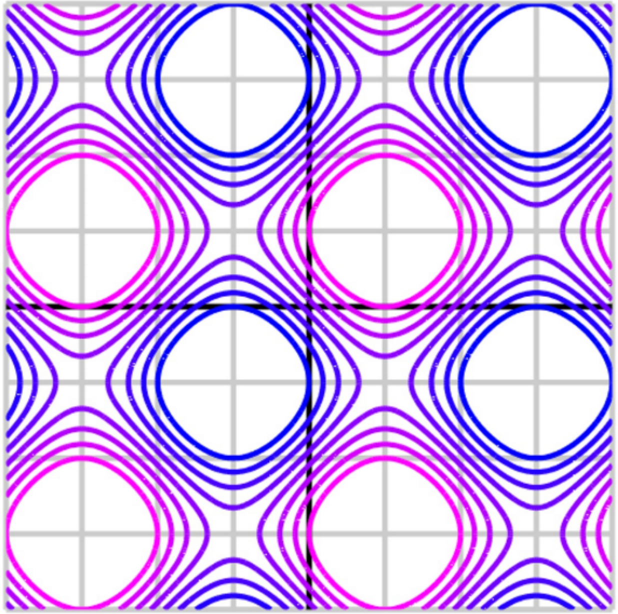


**Figure 1.1.26.** $z = \sin(x) + \sin(y)$

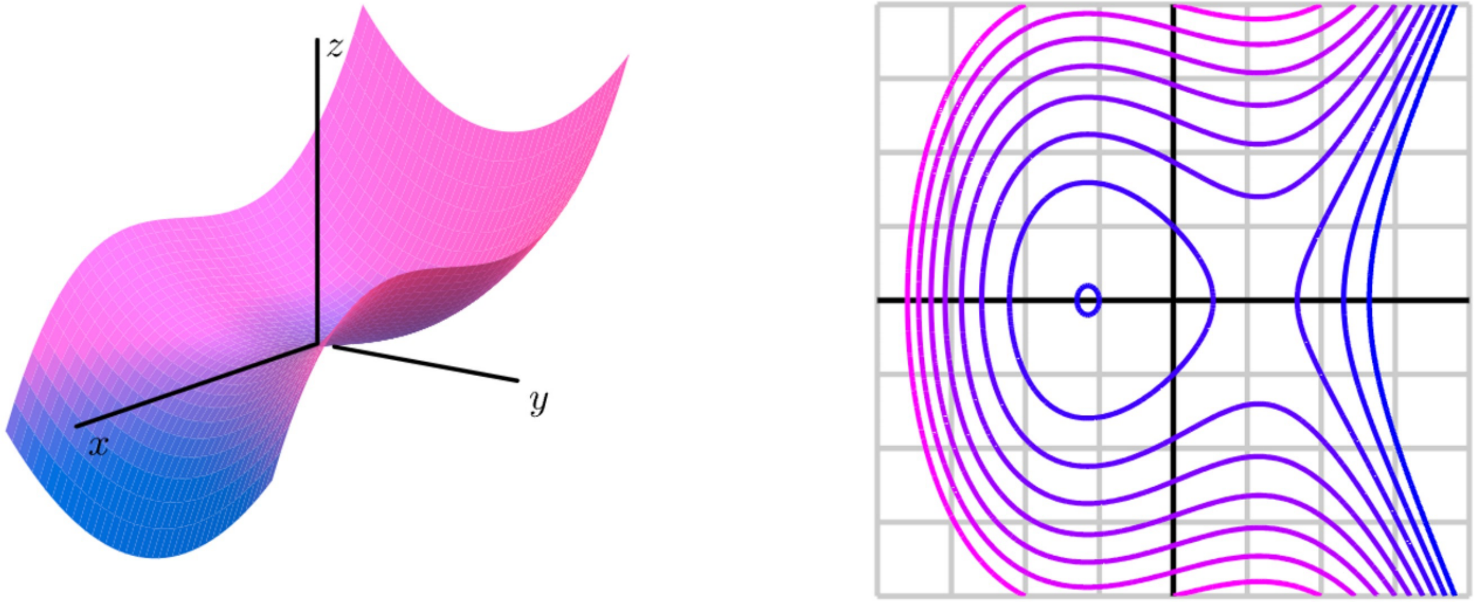



**Figure 1.1.27.** $z = y^2 - x^3 + x$

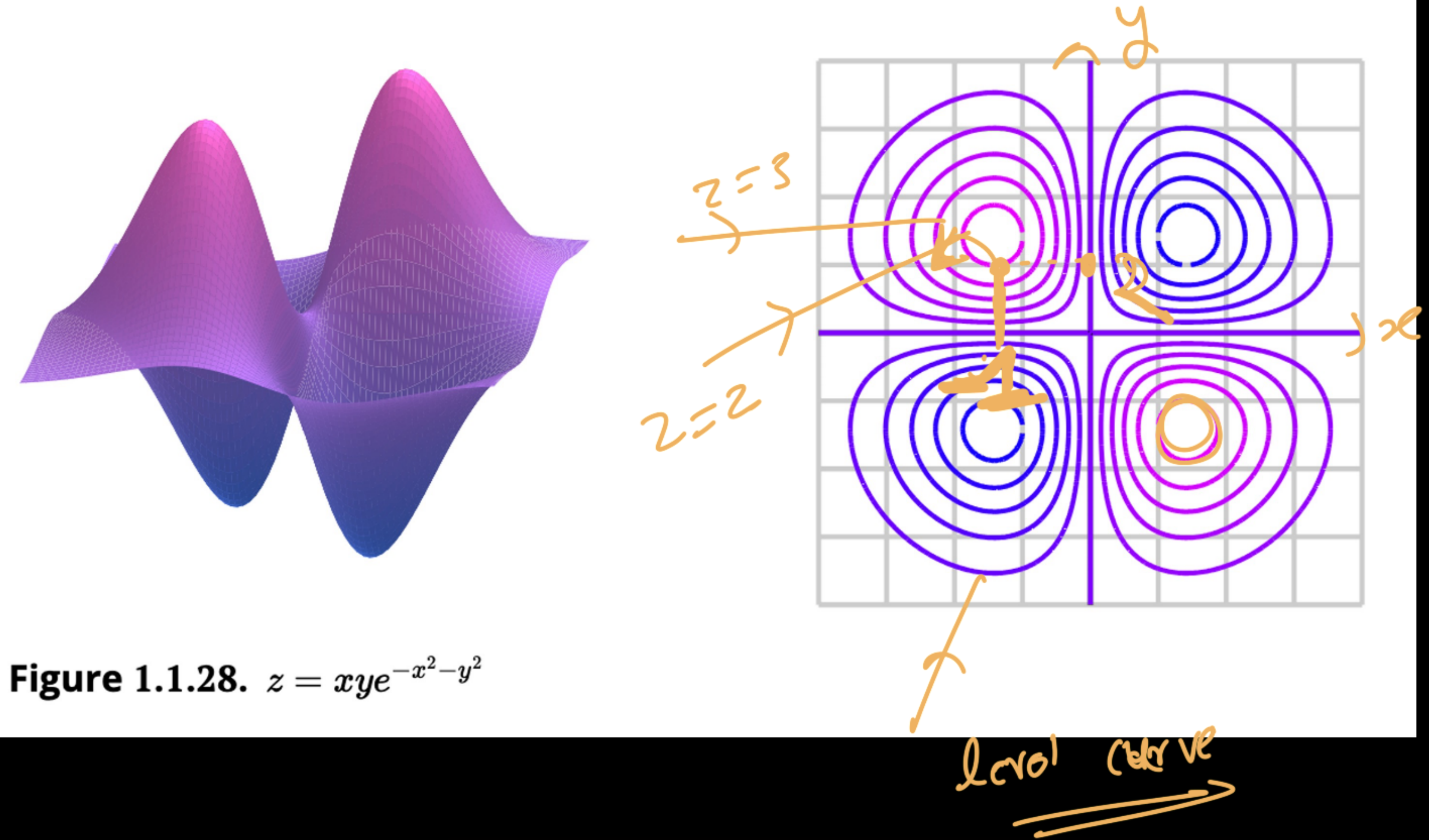



**Figure 1.1.28.** $z = xye^{-x^2-y^2}$

$x = 1$

$y = 2$

$z = 2$

## Question:

$$f(x, y) = x^2 - y,\, z = -2, -1, 0, 1, 2$$

The level curves of $z = f(x, y) = x^2 - y$ are

$z = x^2 - y$

$x^2 - y = -2$

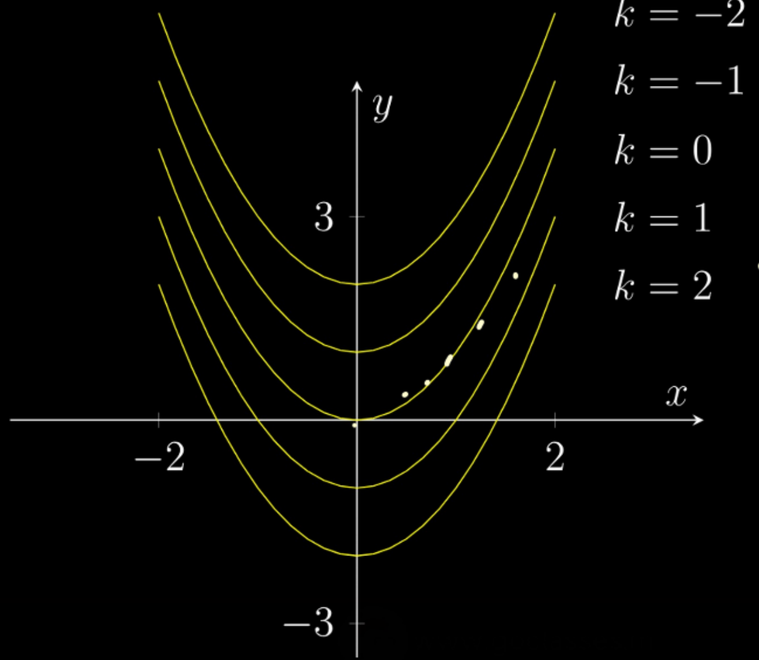
https://www.karlin.mff.cuni.cz/~kuncova/en/2021LS_FMat2/09sol.pdf

$f(x, y) = x^2 - y$, $z = -2, -1, 0, 1, 2$

The level curves of $z = f(x, y) = x^2 - y$ are

$$k = x^2 - y \implies y = x^2 - k$$

for some constant $k$. We sketch below the level curves corresponding to $k = -2, -1, 0, 1, 2$.




equally spaced

$y = x^2 - k$

$y = x^2$

$x = 0$, $y = 0$

$x = 1$, $y = 1$

$x = 2$, $y = 4$

# Important Convention

**Important Convention:** When making a contour plot, always use **equally spaced** values of $z$. For instance, we might choose $z = -2, -1.5, -1, -0.5, 0, 0.5, 1.0, 1.5, 2.0$ to obtain 9 equally spaced function values. However, the level curves themselves are **not** usually equally spaced!

# Question:

**Example 6.** Draw and label contours of the function $z = f(x, y) = y - x^2$ corresponding to function values $c = -1, 0, 1, 2$. What kind of curves are the contours?

$y - x^2 = -1$ } ← one level graph

$y - x^2 = 0$ } ← another level graph

$y - x^2 = 1$

$y - x^2 = 2$

https://www.math.uri.edu/~bkaskosz/fall21/sec-12-3.pdf

**Solution.** For each constant $c$, the curve on the $xy$-plane representing the contour along which $f(x, y) = c$ satisfies the equation:

$$y - x^2 = c.$$

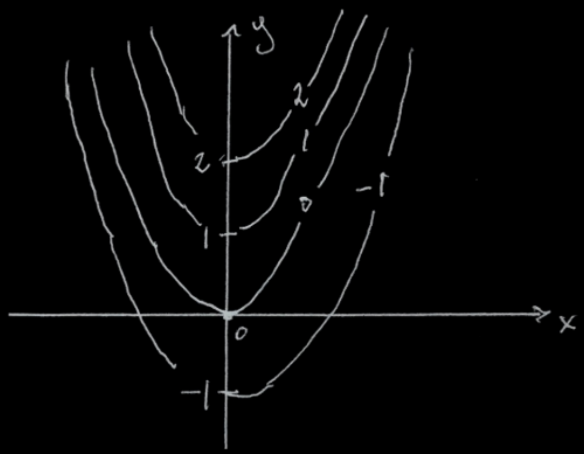
Equivalently:

$$y = x^2 + c.$$

This is the standard parabola in the $xy$-plane shifted vertically by $c$. The parabolas for the given values $c$ are:

$$y = x^2 - 1, \qquad y = x^2, \qquad y = x^2 + 1, \qquad y = x^2 + 2.$$

The contour map with $y$-intercepts and elevations marked is:



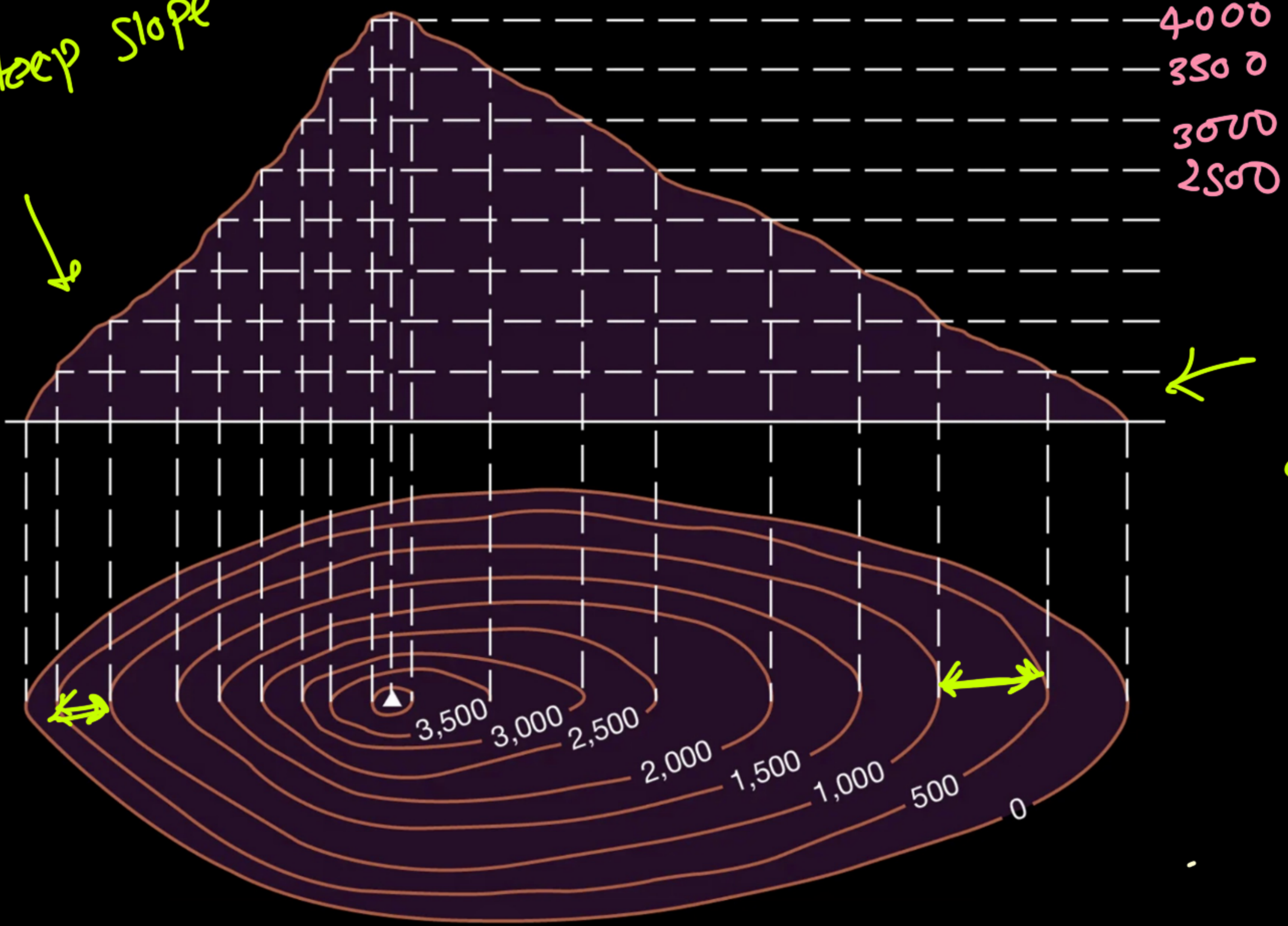
# Steepness and Level Curves distance

more distace $\Rightarrow$ less steep

change in function
is slow

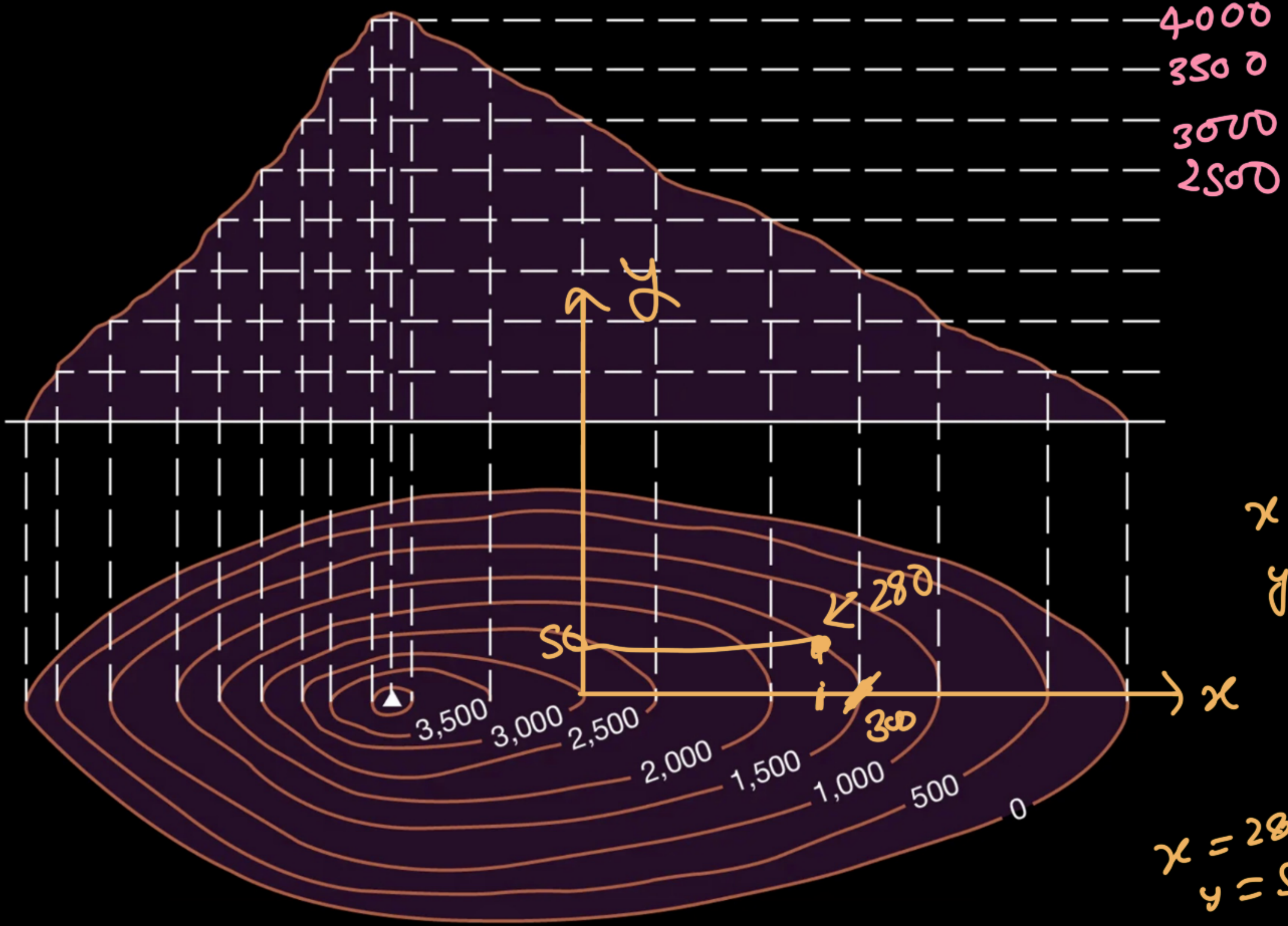
3D plot

Steep slope




Slow decrease

3D plot

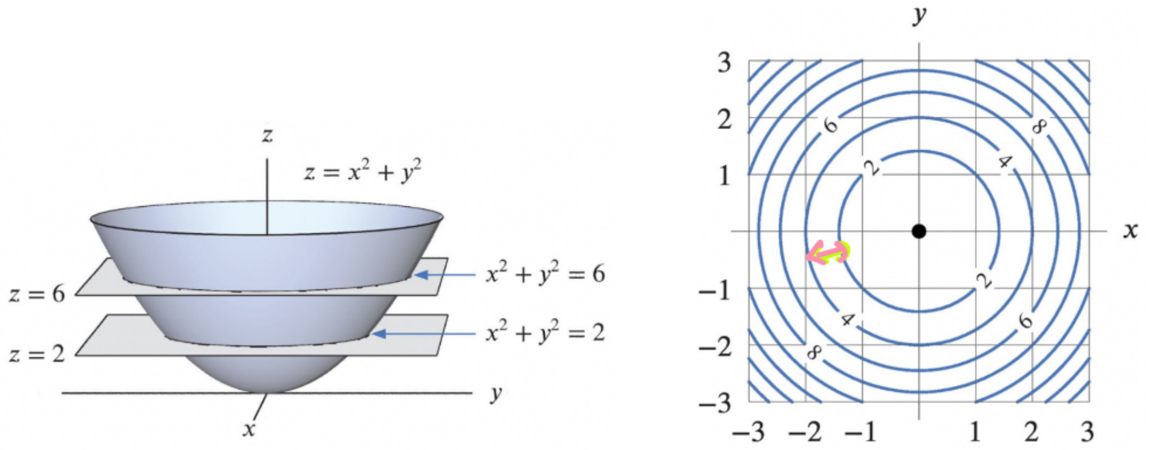



$x = 300$
$y = 0$
$\Downarrow$
$z = 1500$

$\left.\begin{array}{l} x = 280 \\ y = 50 \end{array}\right\} z = 1500$

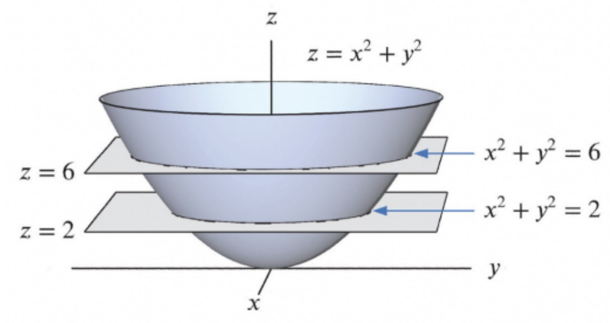
## Question:

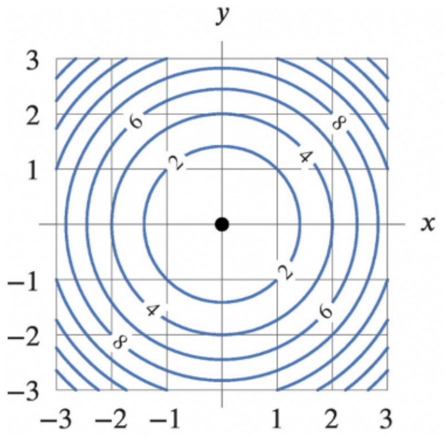
**Example 2.** Here is a paraboloid $z = x^2 + y^2$ and its contour map:




What is the shape of the level curves? How is the steepness of the paraboloid reflected in the contour map?

**Example 2.** Here is a paraboloid $z = x^2 + y^2$ and its contour map:



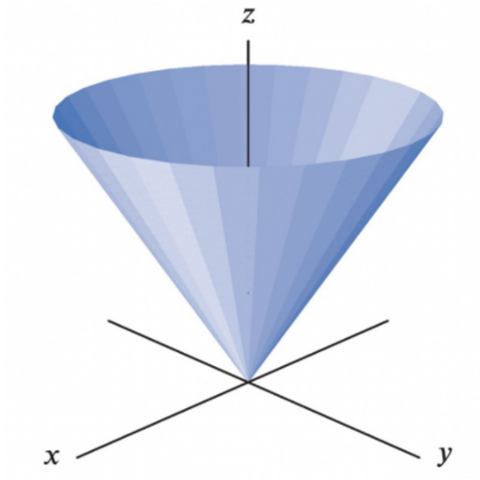



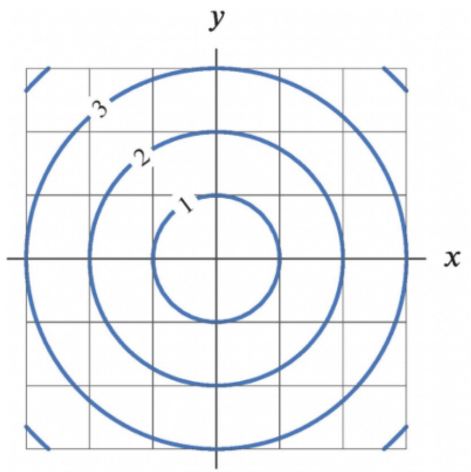

What is the shape of the level curves? How is the steepness of the paraboloid reflected in the contour map?

**Solution.** Level curves are circles as the curve $x^2 + y^2 = c$ is a cricle. The paraboloid is getting steeper and steeper so the contours are getting closer and closer together for higher and higher elevations.

# Question:

**Example 3.** Here is a cone $z = \sqrt{x^2 + y^2}$ and its contour map:




What is the shape of the level curves? How is the steepness of the cone reflected in the contour map?

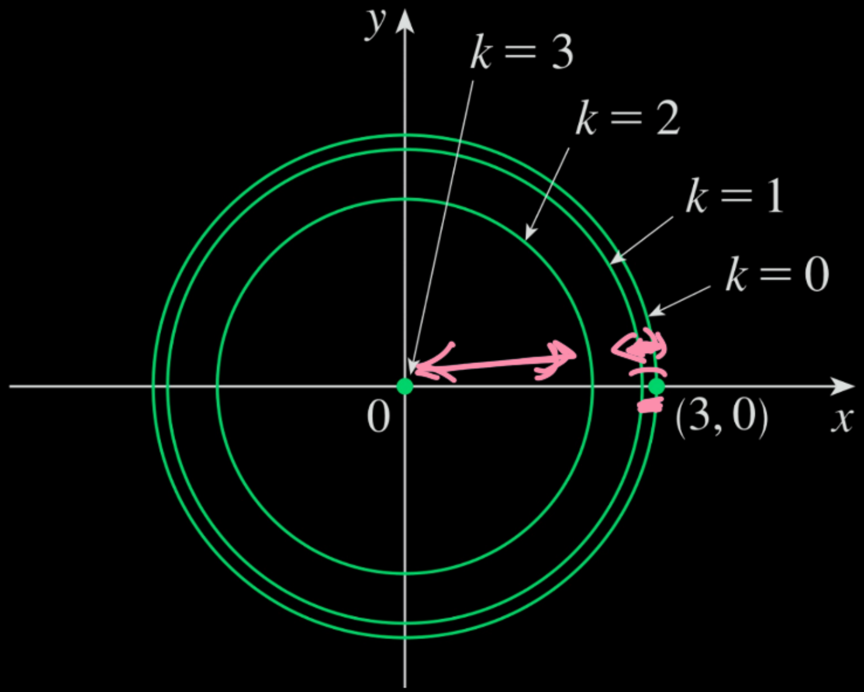
**Solution.** Level curves are circles as the curve $\sqrt{x^2 + y^2} = c$ is a circle. The cone is of constant steepness so the contours corresponding to equally spaced elevations are equally spaced.

equally spaced values of z.

### Key Points about Contour Plots

1. Every point on a particular level curve has the **same** function value.
2. Although the values chosen for a contour plot are equally spaced, this does **not** imply that the contours themselves are equally spaced.
3. Closely spaced contours indicate regions where the function is increasing (or decreasing) rapidly, while widely spaced contours correspond to locations where the function is increasing (or decreasing) slowly.
4. The set of allowable contour values is equivalent to the range of the function.

## Example, cont.



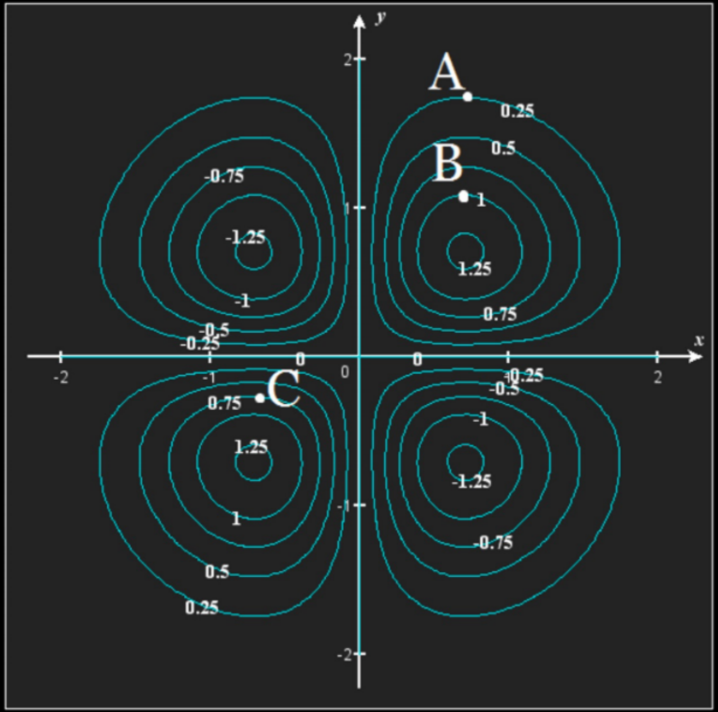

$\frac{\partial z}{\partial x}$

Four level curves of $f(x, y) = \sqrt{9 - x^2 - y^2}$ =

https://math.ou.edu/~jjackson/teaching/1401slides.pdf

Consider the contour map shown below.

# Question:




(a) If a person were walking straight from point $A$ to point $B$, would s/he be walking uphill or downhill?

(b) Is the slope steeper at point $B$ or point $C$?

(c) Starting at $C$ and moving so that $x$ remains contant and $y$ decreases, will the elevation begin to increase or decrease?

(d) Starting at $B$ and moving so that $y$ remains contant and $x$ increases, will the elevation begin to increase or decrease?

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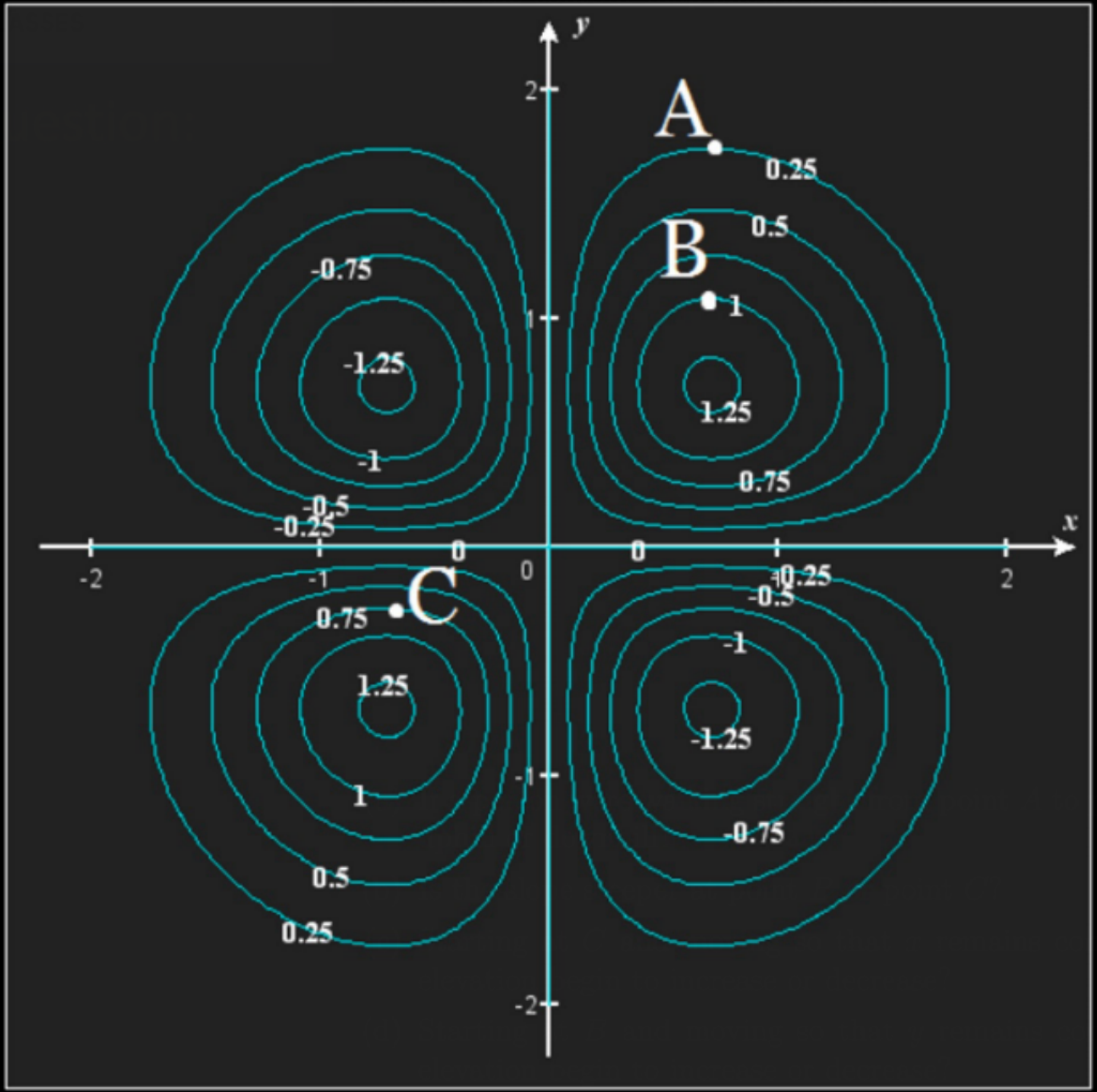



a) $A \longrightarrow B$

less height to more height

b) at C it is more steep.

c) increase

d) decrease

## Solution

(a) If a person were walking straight from point $A$ to point $B$, would s/he be walking uphill or downhill?

$\boxed{\text{Uphill}}$

(b) Is the slope steeper at point $B$ or point $C$?

$\boxed{\text{Point } C}$

(c) Starting at $C$ and moving so that $x$ remains contant and $y$ decreases, will the elevation begin to increase or decrease?

$\boxed{\text{Increase}}$

(d) Starting at $B$ and moving so that $y$ remains contant and $x$ increases, will the elevation begin to increase or decrease?

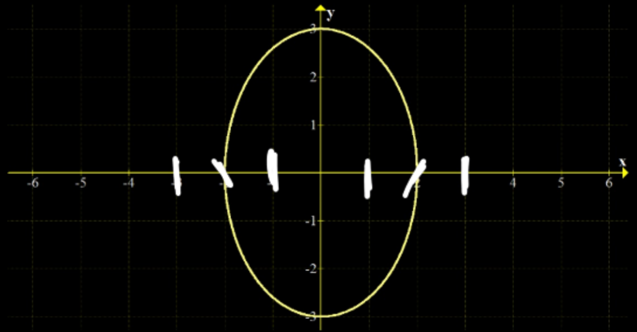
$\boxed{\text{Decrease}}$

https://www.math.drexel.edu/classes/math200/201235/resources/Chapter13/Section1/Homework13.1_Ans.pdf

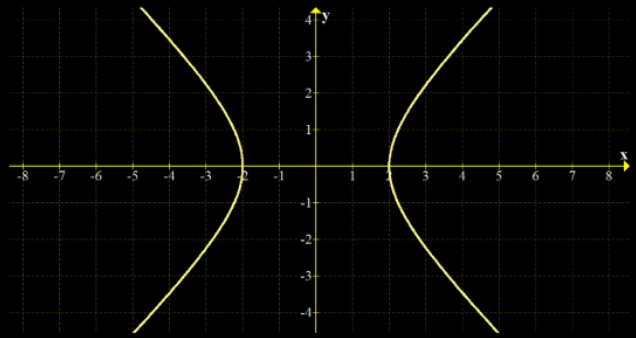
www.goclasses.in

**Question:** Which of the following graphs is the level curve of $f(x,y) = x^2 - y^2$ which passes through $P(2,0)$?

(a)



(c)



$x^2 - y^2 = c$

$4 - 0 = c$

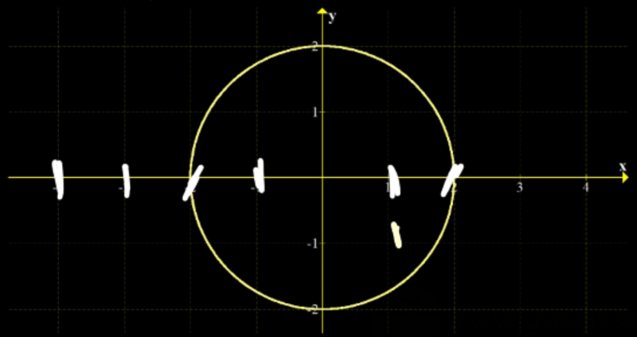
$c = 4$

$\boxed{x^2 - y^2 = 4}$

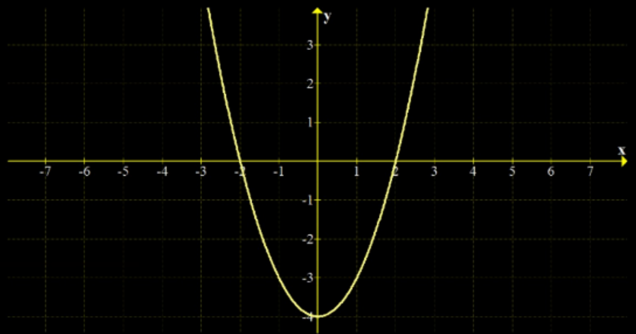
$y = 2$

$x = \pm\sqrt{8}$

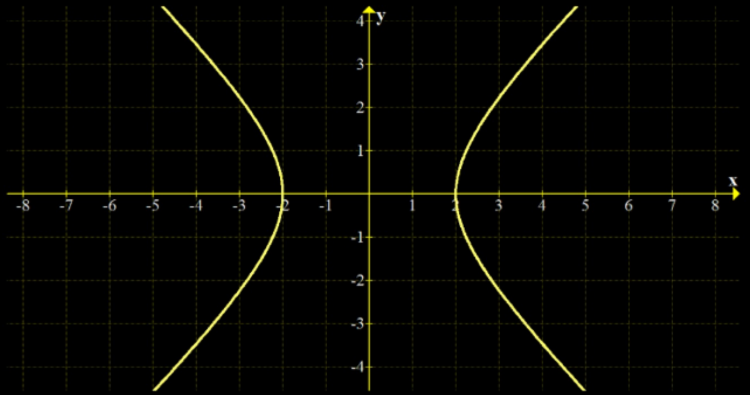
(b)



(d)



(c)




$$\boxed{x^2 - y^2 = 4}$$

$x = 3$

$9 - y^2 = 4$

$y^2 = 5$

$y = \pm\sqrt{5}$

$x = 1,$

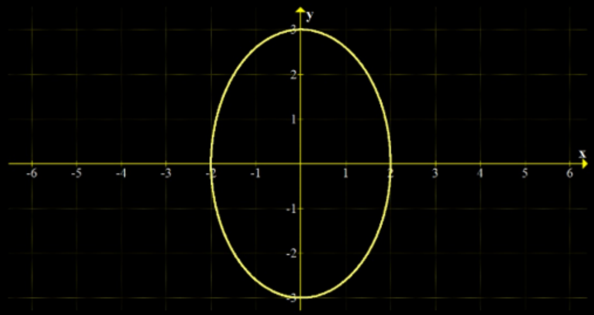
$1 - y^2 = 4$

$y^2 = -3$

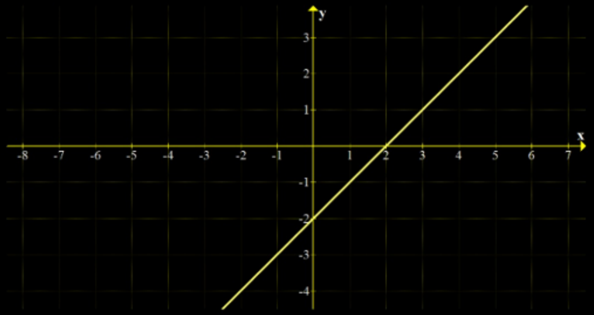
https://www.math.drexel.edu/classes/math200/201235/resources/Exam2-VersionA-Solutions.pdf

**Question:** Which of the following graphs is the level curve of $f(x,y) = x^2 - y$ which passes through $P(2,0)$?

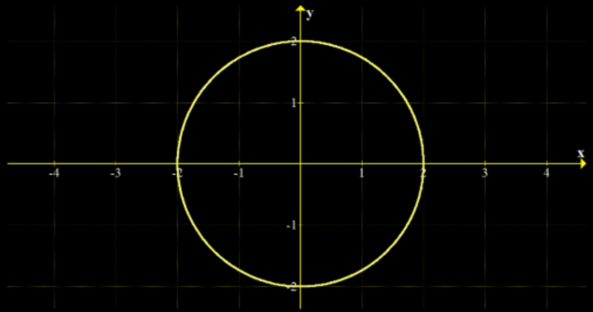
(a)



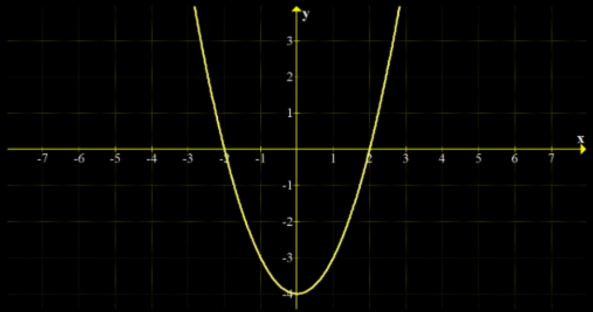
(c)



(b)



(d)



$x^2 - y = c$

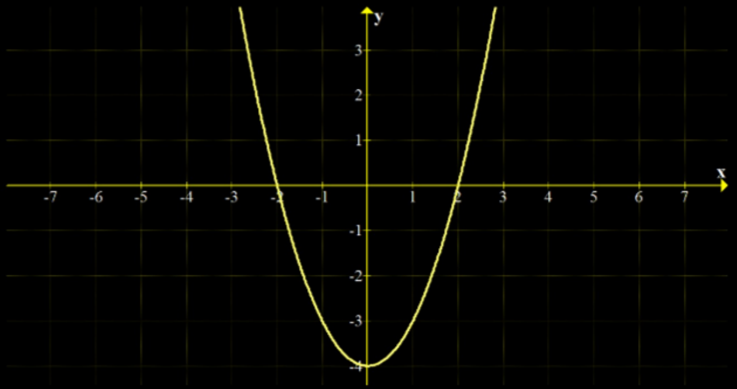
$4 - 0 = c$

$c = 4$

$x^2 - y = 4$

$y = x^2 - 4$

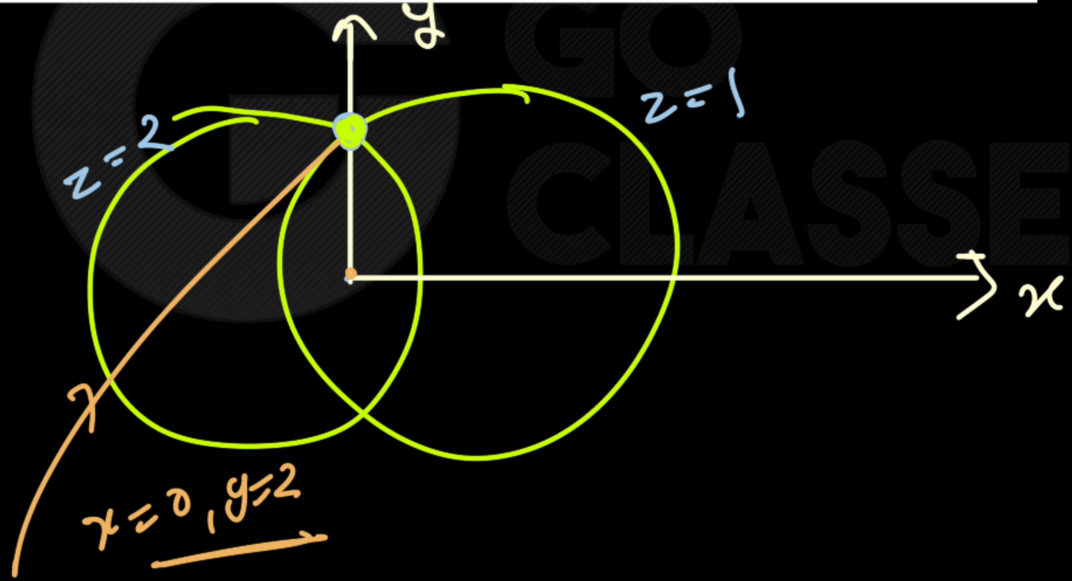
(d)




https://www.math.drexel.edu/classes/math200/201235/resources/Exam2-VersionB-Solutions.pdf

**Questions:**

1. Why is it not possible for the level curves of two different values to intersect each other?




$x = 0$
$y = 3$

$f(x) = x^2$ ✓

$f(x) = \sqrt{x}$ ✓

$f(x) = \pm\sqrt{x}$ ✗

For the given function $f(x, y) = \sqrt{16 - x^2 - y^2}$, the point $(0, 0)$ is on the level curve obtained by setting

Select one or more:

- [ ] $f(x, y) = 0.$
- [ ] $f(x, y) = 2.$
- [ ] $f(x, y) = 3.$
- [ ] $f(x, y) = 4.$
- [ ] None of the other options.

For the given function $f(x, y) = \sqrt{16 - x^2 - y^2}$, the point $(0, 0)$ is on the level curve obtained by setting

Select one or more:

- $f(x, y) = 0.$
- $f(x, y) = 2.$
- $f(x, y) = 3.$
- $f(x, y) = 4.$
- None of the other options.

The correct answer is: $f(x, y) = 4.$

Let $f(x, y) = \sqrt{3x^2 + 2y^2}$. Then the equation of the level curve passing through the point $(2, 2\sqrt{3})$ is

Select one or more:

- ☐ $\frac{x^2}{10} + \frac{y^2}{15} = 1$
- ☐ $\frac{x^2}{6} + \frac{y^2}{9} = 1$
- ☐ $\frac{x^2}{4} + \frac{y^2}{6} = 1$
- ☐ $\frac{x^2}{12} + \frac{y^2}{18} = 1$
- ☐ None of the other options.

Let $f(x, y) = \sqrt{3x^2 + 2y^2}$. Then the equation of the level curve passing through the point $(2, 2\sqrt{3})$ is

Select one or more:

- [ ] $\frac{x^2}{10} + \frac{y^2}{15} = 1$
- [ ] $\frac{x^2}{6} + \frac{y^2}{9} = 1$
- [ ] $\frac{x^2}{4} + \frac{y^2}{6} = 1$
- [ ] $\frac{x^2}{12} + \frac{y^2}{18} = 1$
- [ ] None of the other options.

$\sqrt{3 \times 4 + 2 \times (12)}$

$\sqrt{12 + 24} = \boxed{6}$

$3x^2 + 2y^2 = 36$

$\frac{x^2}{12} + \frac{y^2}{18} = 1$

Let $f(x, y) = \sqrt{3x^2 + 2y^2}$. Then the equation of the level curve passing through the point $(2, 2\sqrt{3})$ is

Select one or more:

- ☐ $\frac{x^2}{10} + \frac{y^2}{15} = 1$
- ☐ $\frac{x^2}{6} + \frac{y^2}{9} = 1$
- ☐ $\frac{x^2}{4} + \frac{y^2}{6} = 1$
- ☐ $\frac{x^2}{12} + \frac{y^2}{18} = 1$
- ☐ None of the other options.

The correct answer is: $\frac{x^2}{12} + \frac{y^2}{18} = 1$

Equation of the level surface of $f(x, y, z) = \sqrt{x - y} - \log z$ passing through the point $(6, -10, e^2)$ is

Select one or more:

- $f(x, y, z) = -1$
- $f(x, y, z) = 2$
- $f(x, y, z) = -2$
- $f(x, y, z) = 1$
- None of the other options.

Contours itself will be 3D

$\sqrt{6+10} - \log_e e^2$

$4 - 2 = \underline{\underline{2}}$

Equation of the level surface of $f(x, y, z) = \sqrt{x - y} - \log z$ passing through the point $(6, -10, e^2)$ is

Select one or more:

- [ ] $f(x, y, z) = -1$
- [ ] $f(x, y, z) = 2$
- [ ] $f(x, y, z) = -2$
- [ ] $f(x, y, z) = 1$
- [ ] None of the other options.

The correct answer is: $f(x, y, z) = 2$ ✓

# Recall

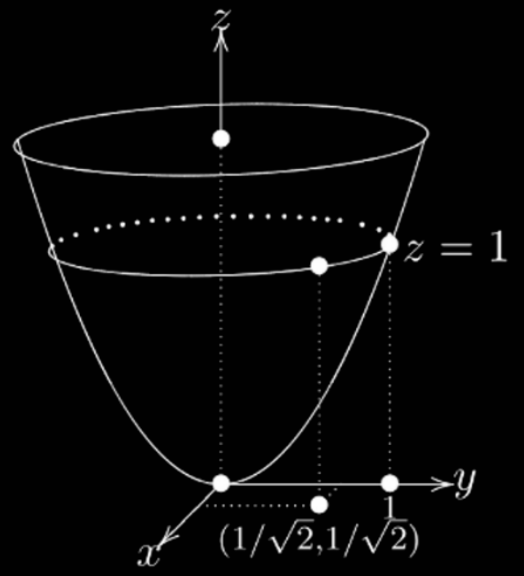
$\nabla f$

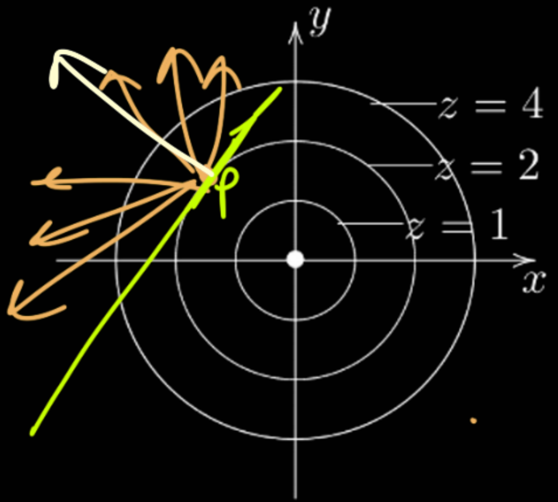
Using the dot product theorem, we deduced the following:

**Corollary (Minimum and Maximum Increase)**

*Suppose $f$ is a differentiable function with gradient $\nabla f \neq \mathbf{0}$ and $\mathbf{v}$ is a unit vector. Then the following hold:*

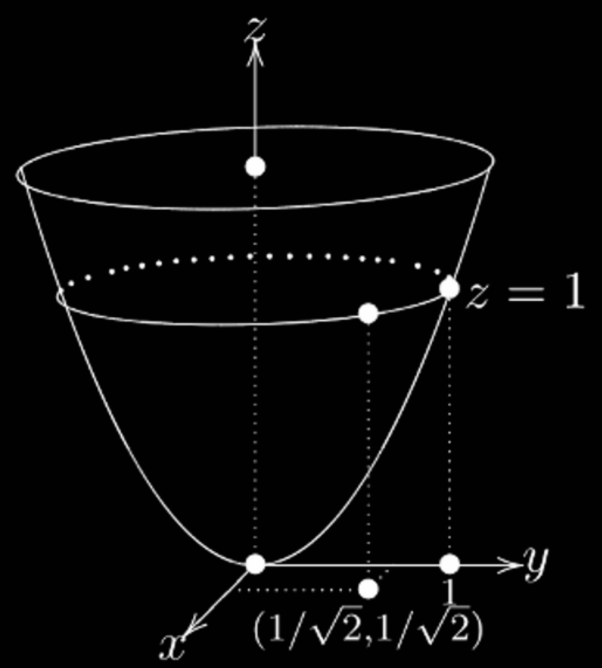
1. *The maximum value of $D_{\mathbf{v}} f$ occurs when $\mathbf{v}$ is a unit vector in the direction of $\nabla f$, and the maximum value is $\|\nabla f\|$.* $\Rightarrow \theta = 0^\circ$
2. *The minimum value of $D_{\mathbf{v}} f$ occurs when $\mathbf{v}$ is a unit vector in the opposite direction of $\nabla f$, and the minimum is $-\|\nabla f\|$.* $\Rightarrow \theta = 180$
3. *The value of $D_{\mathbf{v}} f$ is zero if and only if $\mathbf{v}$ is orthogonal to the gradient $\nabla f$.*

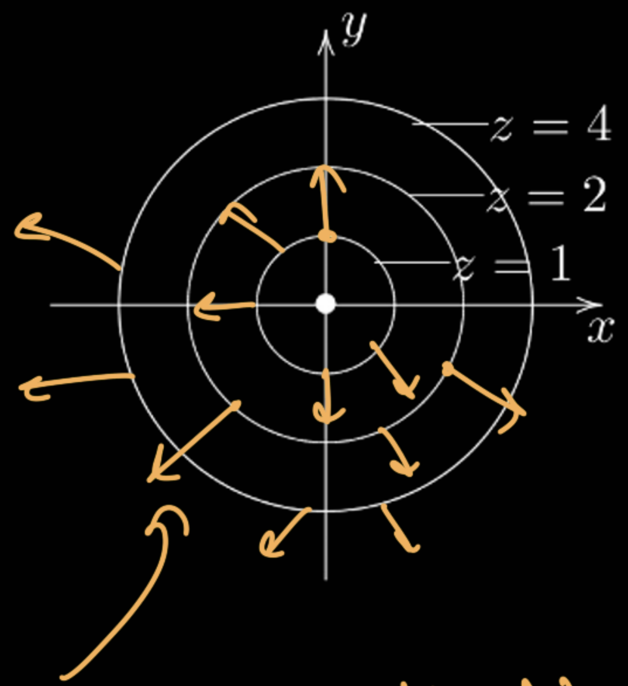


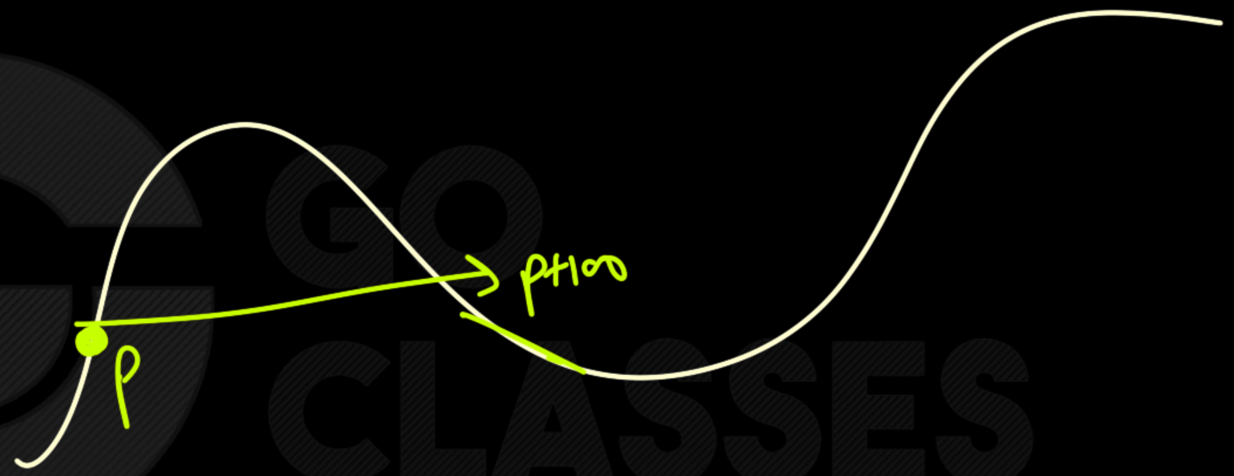

at point P, the direction of gradient should be outward. , but at outward also, there are inf. many possiblities.

⇒ function is constant in perpendicular to grad direction
⇒ function is also constant along the level curve




grad directions


p+100
p

We showed, during our discussion last time, that any direction vector orthogonal to $\nabla f$ is necessarily a direction in which the value of the function is not changing.

- In particular, if we imagine traveling along one of the level curves of $f(x, y)$, then by definition the value of $f$ is not changing.
- Therefore, by putting these two observations together, we see that $\nabla f(a, b)$ at that point will be a normal vector to the graph of the level curve at $(a, b)$.
- This is quite readily visible from plots: the gradient vector always runs perpendicular to the level curve.

This means we can use the gradient to find equations of tangent lines to implicit curves described by level sets.

in this direction function value does not change

$\nabla f$

Gradient direction is perpendicular to level curves (tangent)

# Calculus



We showed, during our discussion last time, that any direction vector orthogonal to $\nabla f$ is necessarily a direction in which the value of the function is not changing.

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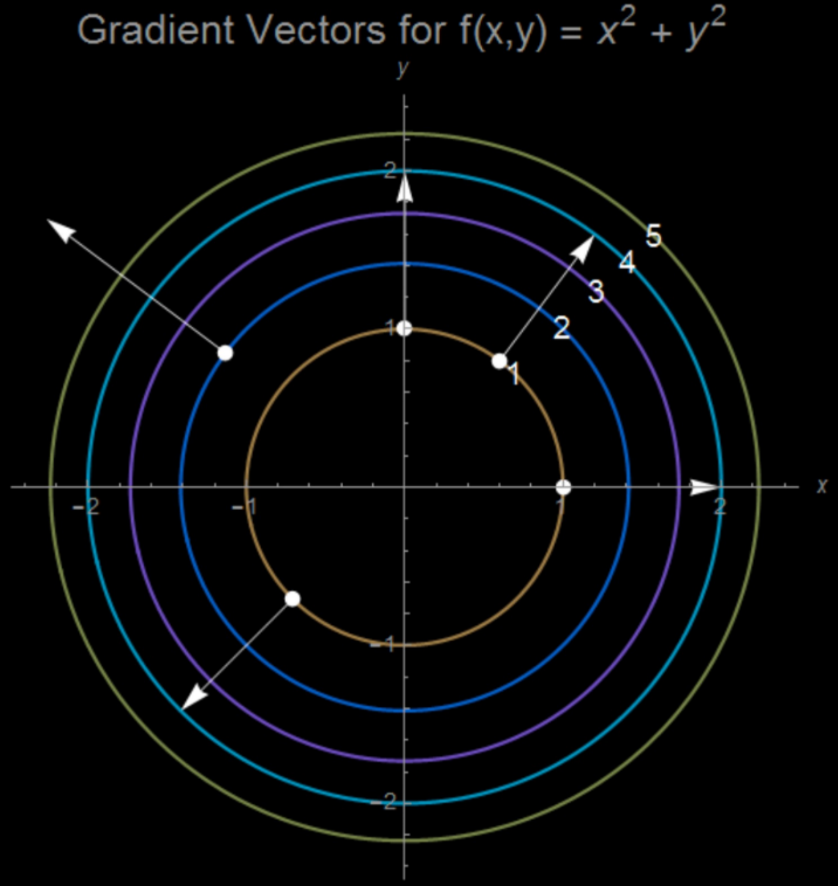
https://web.northeastern.edu/dummit/teaching_sp21_2321/2321_lecture_08_tangents_linearization.pdf

www.goclasses.in

Here are plots of some gradient vectors for $f(x, y) = x^2 + y^2$:

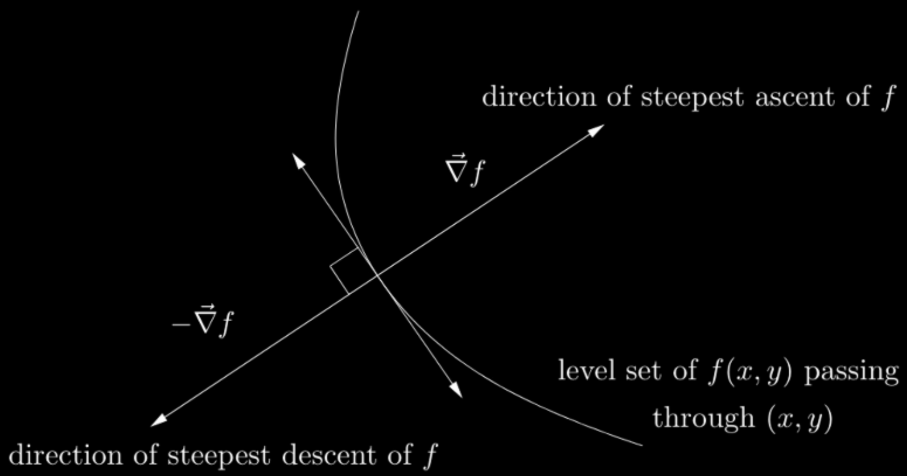


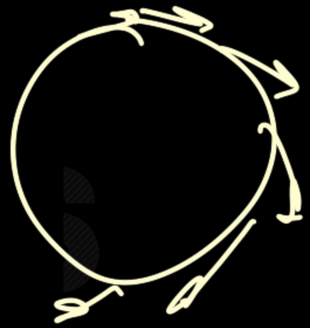

https://web.northeastern.edu/dummit/teaching_sp21_2321/2321_lecture_08_tangents_linearization.pdf

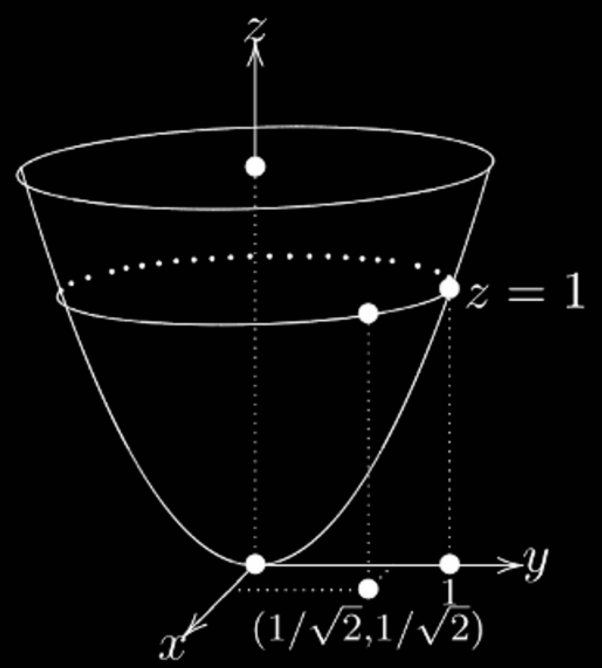
1. Directions in which the rate of change of $f$ are zero must be *tangent* to the level curve of $f$ that passes through $(a, b)$. Why? If you travel on a level curve, the value of $f$ does not change. And the instantaneous direction of motion at any point on this curve is the tangent vector to the curve at that point.

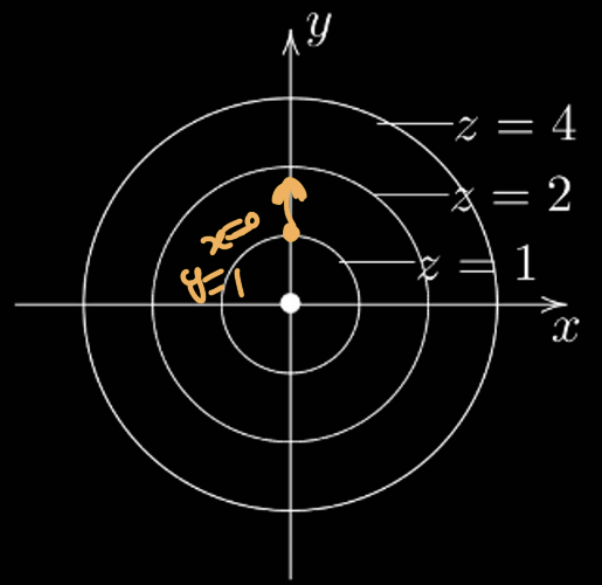
2. The gradient vector $\vec{\nabla} f(a, b)$ must be perpendicular to the level curve of $f$ that passes through $(a, b)$.

These results are sketched below.






$x = 0, \quad y =$

# Poll

$f(x, y)$

$X^{x, y}$

- Select all that are true about derivatives of a scalar function f(X) of multivariate inputs
  - At any location X, there may be many directions in which we can step, such that f(X) increases
  - The direction of the gradient is the direction in which the function increases fastest

# Poll

- Select all that are true about derivatives of a scalar function f(X) of multivariate inputs
  - At any location X, there may be many directions in which we can step, such that f(X) increases
  - The direction of the gradient is the direction in which the function increases fastest

Both are true.

# Geometric Interpretations of the Gradient Vector

The gradient vector is perpendicular to the level curves and points in the direction of largest increase.

The magnitude of the vector is equal to the largest rate of change.

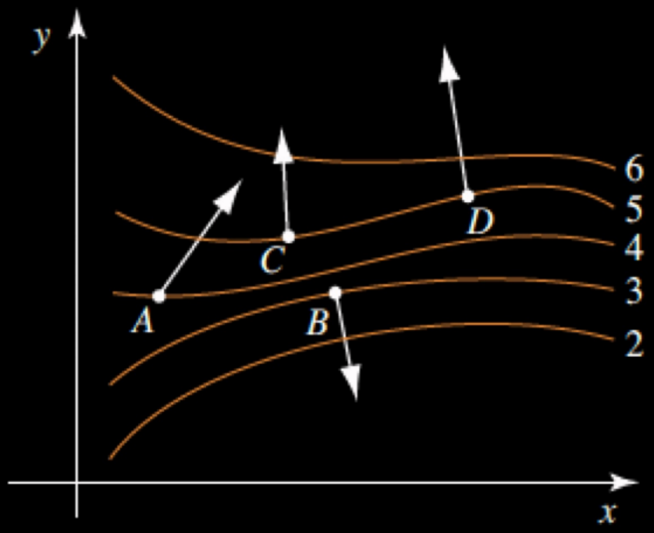
https://ms.mcmaster.ca/~clemene/1LT3/lectures/1lt3_sv_section9.pdf

Question

**Example 9.8:**

(a) Explain why the vectors at A and B cannot represent the gradient of *f*.

(b) Explain why the vectors at C and D can represent the gradient of *f*.




https://ms.mcmaster.ca/~clemene/1LT3/lectures/1lt3_sv_section9.pdf

Question

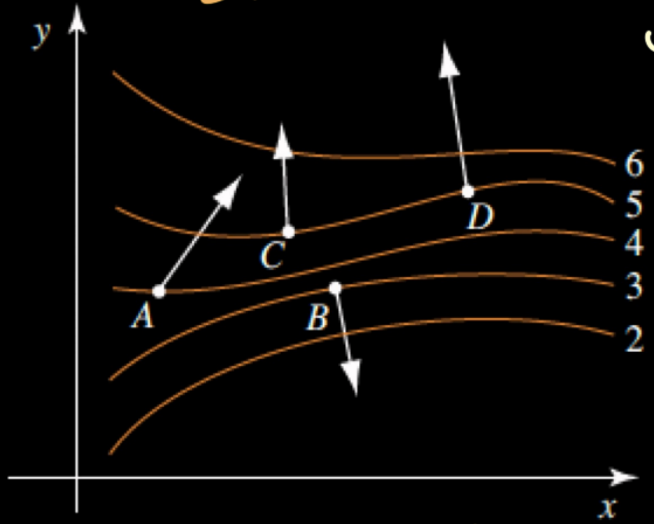
**Example 9.8:**

(a) Explain why the vectors at A and B cannot represent the gradient of *f*.

(b) Explain why the vectors at C and D can represent the gradient of *f*.

A: ⇒ Not perpendicular

B: ⇒ pointing in direction where function decrease.




↳ perpendicular & increasing direction.

https://ms.mcmaster.ca/~clemene/1LT3/lectures/1lt3_sv_section9.pdf

## Properties of the the gradient vector

Remark: If $\theta$ is the angle between $\nabla f$ and $\mathbf{u}$, then holds

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} \quad \Rightarrow \quad D_{\mathbf{u}} f = |\nabla f| \cos(\theta).$$

The formula above implies:

- The function $f$ increases the most rapidly when $\mathbf{u}$ is in the direction of $\nabla f$, that is, $\theta = 0$. The maximum increase rate of $f$ is $|\nabla f|$.
- The function $f$ decreases the most rapidly when $\mathbf{u}$ is in the direction of $-\nabla f$, that is, $\theta = \pi$. The maximum decrease rate of $f$ is $-|\nabla f|$.
- Since the function $f$ does not change along level curve or surfaces, that is, $D_{\mathbf{u}} f = 0$, then $\nabla f$ is perpendicular to the level curves or level surfaces.

## Question:

Find all points where the fastest change of the function $f(x, y) = x^2 + y^2 + x - 2y$ is in the direction of the vector $\mathbf{v} = \langle 1, 2 \rangle$.
(Note: The answer should not be a single point but rather all the points along a certain line.)

find the point(s) where the grad direction is same as $\langle 1, 2 \rangle$ direction

$$\nabla f = \begin{bmatrix} 2x+1 \\ 2y-2 \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$2x+1 = c$
$2y-2 = 2c$

$x = c - \frac{1}{2}, \quad y = \frac{2c+2}{2}$

We know that the fastest change of the function occurs when we move in the same direction as its gradient $\nabla f(x, y)$. Thus, we need $\nabla f(x, y) = c\mathbf{v}$ for any constant $c$. Note that

$$\nabla f(x, y) = \langle 2x + 1,\ 2y - 2\rangle$$

$\left(\frac{1}{2}, 2\right)$

Thus, we need

$$\begin{aligned} & \langle 2x+1,\ 2y-2\rangle = c\langle 1,\ 2\rangle \\ \Rightarrow \quad & 2x + 1 = c \quad \text{and} \quad 2y - 2 = 2c \\ \Rightarrow \quad & 2x + 1 = c \quad \text{and } y - 1 = c \end{aligned}$$

Since $2x + 1 = c$ and $y - 1 = c$, we have that $2x + 1 = y - 1$. Thus, $y = 2x + 2$. So, the points where the fastest change of the function $f(x, y)$ is in the direction of $\mathbf{v}$ is everywhere along the line $y = 2x + 2$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c - \frac{1}{2} \\ c+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$$

Question:

Suppose

$$f(x, y) = ye^{-x} + 3x.$$

Find the direction of the maximum rate of increase of $f(x, y)$ at $(0, 1)$.

(A) $\langle 2, 1 \rangle$

(B) $\langle -2, -1 \rangle$

(C) $\langle 3, 0 \rangle$

(D) $\langle -3, 0 \rangle$

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(B) $\langle -2, -1\rangle$

(C) $\langle 3, 0\rangle$

(D) $\langle -3, 0\rangle$

(E) $\langle 2e^{-1}, e\rangle$

$$\nabla f = \langle -ye^{-x} + 3, e^{-x} \rangle$$

$$\nabla f(0,1) = \langle -1e^{0} + 3, e^{0} \rangle$$

$$= \langle 2, 1 \rangle$$

# Matrix Calculus

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

$f(x, y) = x^2 + y^2$

$f : R^2 \to R$

differentiate
Scalar w.r.t. vector $\longrightarrow$ $\frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$