Positive Definite Matrices

- Every real symmetric matrix has real eigenvalues
- A positive definite matrix is a real symmetric matrix with all positive eigenvalues.

- Note that as it’s a real symmetric matrix all the eigenvalues are real, so it makes sense to talk about them being positive or negative.

Now, it's not always easy to tell if a matrix is positive definite. Quick, is this matrix?

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Hard to tell just by looking at it.[1]

Now, it's not always easy to tell if a matrix is positive definite. Quick, is this matrix?

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

← to tell if this matrix is positive definate or not

⇓

we need to find out eigen values

Hard to tell just by looking at it.[1]

One way to determine if a matrix A is positive definite is to calculate all the eigenvalues λ_i and check if $\lambda_i > 0$ for all i.

The problem with this approach is that calculating eigenvalues can be a real pain, especially for large matrices.

Therefore, today we will learn some easier methods to determine if a matrix A is positive definite.

Positive Definite Matrices

What Are They, and What Do They Want?

I've already told you what a positive definite matrix is.

A matrix is positive definite if it's symmetric and all its eigenvalues are positive.

we are just interested to know sign of eigenvalues.

not the eigenvalues by itself.

Motivation : why we are studying positive definate matrices ?

minima or maxima

- we need to optimise multivariate functions there we need Positive Definate matrices concept

Property of Symmetric matrices

Sign of Pivots = Signs of eigenvalues

Property of Symmetric matrices

Sign of Pivots = Signs of eigenvalues

- Number of +ve eigenvalues = Number of +ve pivots
- Number of -ve eigenvalues = Number of -ve pivots
- Number of 0 eigenvalues = Number of 0 pivots

this is true for symmetric matrices.

A

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$$

$A - \lambda I$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

Can we say eigenvalues of this matrix is 1, −3 ? ⇒ No

we can apply row operations here.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$$

this is a symmetric matrix

$\Rightarrow$ one eigen value is +ve

other eigen value is -ve

The thing is, there are a lot of other equivalent ways to define a positive definite matrix. One equivalent definition can be derived using the fact that for a symmetric matrix the signs of the pivots are the signs of the eigenvalues. So, for example, if a 4×4 matrix has three positive pivots and one negative pivot, it will have three positive eigenvalues and one negative eigenvalue.

A matrix is positive definite if it's symmetric and all its pivots are positive.

Pivots are, in general, *way* easier to calculate than eigenvalues.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

If we perform elimination (subtract $2\times$ row 1 from row 2) we get

$$\begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{-3} \end{pmatrix} \Rightarrow$$

this matrix is NOT positive definite

The pivots are 1 and -3. In particular, one of the pivots is -3, and so the matrix is not positive definite.

A symmetric matrix **A** is to be **positive definite** if

1. all the eigenvalues are positive
2. or all the pivots are positive
3. or all the n upper left determinants are positive — Principle determinants
4. or $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0\ \forall \mathbf{x}$ except $\mathbf{x} = 0$.

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Question:

Example - Is the following matrix positive definite?

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$2$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8 - 2 - 2 = 4$$

$2, 3, 4 > 0 \Rightarrow$ Positive definite.

Question:

Find whether $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ is positive definite or not.

Consider $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

1. $\lambda = 2, 4.$ ✓

2. $\begin{pmatrix} \boxed{3} & 1 \\ 1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 \\ 0 & \boxed{\frac{8}{3}} \end{pmatrix}$ ✓

3. $|3| = 3, \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8$ ✓

4. $(x_1 \quad x_2) \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1^2 + 3x_2^2 + 2x_1x_2$
$= (x_1 + x_2)^2 + 2(x_1^2 + x_2^2) > 0$ for any $\mathbf{x}$.

$$(x_1 + x_2)^2 + 2(x_1^2 + x_2^2) > 0$$

$x^T A x$

$\therefore$ It is a positive definite matrix.

($\boxed{\text{Any one}}$ of the above tests is sufficient to check whether a matrix is positive definite or not)

Question: T/F

If $\mathbf{A}$ is positive definite, $\mathbf{A}^{-1}$ will also be positive definite.

$\lambda_i > 0 \quad \Rightarrow \quad \frac{1}{\lambda_i} > 0$

If **A** is positive definite, $\mathbf{A}^{-1}$ will also be positive definite.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of **A**.

The eigenvalues of $\mathbf{A}^{-1}$ are

$$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$$

As **A** is positive definite, $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$.

$$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n} > 0$$

Hence $\mathbf{A}^{-1}$ is positive definite.

TRUE

A^{-1} is also Symm.

Ajay Sreenivas to Everyone 9:44 AM

AS

(A-1)T = (AT)-1 = A-1

1

Question: T/F

If $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices, $\mathbf{A} + \mathbf{B}$ will also be a positive definite matrix.

$$Ax = \lambda x$$

$$B\boxed{x} = \mu x$$

$$(A+B)x = (\lambda+\mu)x$$

eigen vector of B need not to be x

Question: T/F

If $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices, $\mathbf{A}+\mathbf{B}$ will also be a positive definite matrix.

$$x^T (A+B) x > 0$$

$$\Downarrow$$

$$\underbrace{x^T A x}_{>0 \text{ for } \forall x} + \underbrace{x^T B x}_{>0 \ \forall x}$$

If $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices, $\mathbf{A} + \mathbf{B}$ will also be a positive definite matrix.

$$\mathbf{x}^T(\mathbf{A} + \mathbf{B})\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{B}\mathbf{x}$$

Since $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ and $\mathbf{x}^T\mathbf{B}\mathbf{x} > 0$,

$$\therefore \mathbf{x}^T(\mathbf{A} + \mathbf{B})\mathbf{x} > 0$$

Hence $\mathbf{A} + \mathbf{B}$ is also positive definite.

Positive Definite

A symmetric matrix **A** is to be **positive definite** if

1. all the eigenvalues are positive >0
2. all the pivots are positive >0
3. all the n upper left determinants are positive >0
4. $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \ \forall \mathbf{x}$ except $\mathbf{x} = 0$.

Positive Semi Definite (PSD)

A symmetric matrix $\mathbf{A}$ is to be *positive semidefinite* if

1. all the eigenvalues are *nonnegative* ≥ 0
2. all the pivots are *nonnegative* ≥ 0
3. all the n upper left determinants are *nonnegative* ≥ 0
4. $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$.

Question:

Find whether $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is positive semidefinite or not.

http://www.cse.iitm.ac.in/~vplab/courses/LARP_2018/chap6_PSD.pdf

Solution

1. The eigenvalues are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 3$ (a zero eigenvalue). } $\Rightarrow$ Not positive Def. but positive semi def.

2. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (missing pivot).

PSD

3. $\det A = 0$ and smaller determinants are positive.

4. $x^T A x = (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 \geq 0$ (zero if $x_1 = x_2 = x_3$).

$\therefore$ It is a positive *semi definite* matrix.

(Any one of the above tests is sufficient to check whether a matrix is positive *semi definite* or not)

Question:

Example - For what numbers b is the following matrix positive semidefinite?

$$\begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$$

https://www.math.utah.edu/~zwick/Classes/Fall2012_2270/Lectures/Lecture33_with_Examples.pdf

$$\begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$$

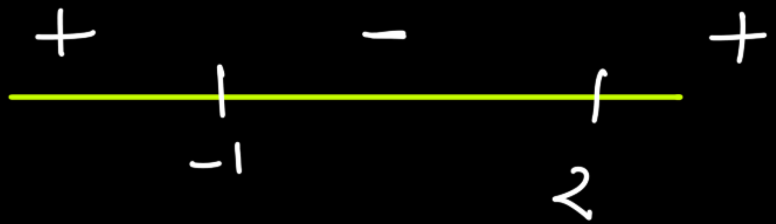
$$\begin{vmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{vmatrix} = 8 + b + b - 2 - 2 - 2b^2$$

$$= -2b^2 + 2b + 4 \geq 0$$

$$\Rightarrow \quad b^2 - b - 2 \leq 0$$

$$(b-2)(b+1) \leq 0$$

$$\boxed{-1 \leq b \leq 2}$$



Negitave Definite

① all $\lambda_i < 0$

or ② all pivots < 0

or ③ all principle det < 0

④ $x^T A x < 0 \quad \forall x$ (except $x = 0$)

Negitive Semi definate

① all $\lambda_i \leq 0$

or ② all pivots ≤ 0

or ③ all principle det ≤ 0

④ $x^T A x \leq 0 \quad \forall x$ (excep $x = 0$

Q. it is not neccessary for a matrix be either of PD, PSD, Negitive definite or Negative semi def.

eg = $\begin{bmatrix} 2 & 5 \\ 0 & -3 \end{bmatrix}$ ← this matrix is neither of any above.

Quadratic Form

GATE DA Syllabus:

Linear Algebra: Vector space, subspaces, linear dependence and independence of vectors, matrices, projection matrix, orthogonal matrix, idempotent matrix, partition matrix and their properties, quadratic forms, systems of linear equations and solutions; Gaussian elimination, eigenvalues and eigenvectors, determinant, rank, nullity, projections, LU decomposition, singular value decomposition.

- **Linear functions:** sum of terms of the form $c_i x_i$ where the c_i are parameters and x_i are variables. General form:

$$c_1 x_1 + \cdots + c_n x_n = c^T x$$

- **Linear functions:** sum of terms of the form $c_i x_i$ where the c_i are parameters and x_i are variables. General form:

$$c_1 x_1 + \cdots + c_n x_n = c^T x$$

- **Quadratic functions:** sum of terms of the form $a_{ij} x_i x_j$ where a_{ij} are parameters and x_i are variables. General form:

$$a_{11} x_1^2 + a_{12} x_1 x_2 + \cdots + a_{nn} x_n^2$$

Examples 7.9. The following are quadratic forms:

1. $f(x_1) = x_1^2$ $\rightarrow x^T A x = x_1^T I x_1$
2. $f(x_1, x_2) = 2x_1^2 + 3x_2^2 - x_1x_2$
3. $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

$$[x_1 \ x_2] \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[x_1 \ x_2] \begin{bmatrix} 2 & -1/2 \\ -1/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A function $f(x_1, x_2, \ldots, x_n)$ *is a quadratic if and only if it can be written as*

$$f(x_1, x_2, \ldots, x_n) = \mathbf{x}^T \underline{A}\, \mathbf{x}, \qquad \textit{where } \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$x^T \underline{A} x$

for a symmetric $n \times n$ *matrix* A.

The matrix A is uniquely determined by the quadratic form, and is called the symmetric matrix associated with the quadratic form.

$\boxed{x^T A x}$

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$f(x_1, x_2) = 2x_1^2 + 3x_2^2 - x_1x_2$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \underline{\begin{bmatrix} 2 & -1/2 \\ -1/2 & 3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Downarrow$

Symm. ✓

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Downarrow$

Symm ✗

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \underline{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Downarrow$

Symm ✗

-1

$-1/2 \quad -1/2$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

-1

$-1/2 \quad -1/2$

Example: find the quadratic polynomial for the following symmetric matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

- $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} = [x_1 \quad x_2] \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = {x_1}^2 + 2{x_2}^2$
- $Q(\boldsymbol{x}) = \boldsymbol{x}^T B \boldsymbol{x} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = {x_1}^2 + 2{x_2}^2 - {x_3}^2 - 2x_1x_2 + 2x_2x_3$

$x_2 x_3$

$x_1 x_2$

Question:

Compute the product $x^T A\,\mathbf{x}$ *when* A *is the symmetric* 2×2 *matrix* $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

https://www.dr-eriksen.no/teaching/GRA6035/2010/lecture4.pdf

Question:

Compute the product $x^T A\mathbf{x}$ when A is the symmetric 2×2 matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - x_2^2 + 4x_1x_2$$

Ajay Sreenivas to Everyone 10:15 AM

AS

x1^2 - X2^2 +4x1x2

Tejash More to Everyone 10:15 AM

x1^2 - x2^2 + 4X1x2

https://www.dr-eriksen.no/teaching/GRA6035/2010/lecture4.pdf

Solution

We compute the matrix product

$$\mathbf{x}^T A\,\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 & 2x_1 - x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= x_1^2 + 2x_2x_1 + 2x_1x_2 - x_2^2 = x_1^2 + 4x_1x_2 - x_2^2$$

Question:

Let $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute $\mathrm{x}^T A \mathrm{x}$ for the following matrices:

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ b. $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

$4x_1^2 + 3x_2^2$

$3x_1^2 + 7x_2^2 - 4x_1x_2$

Solution:

a.

$$\mathbf{x}^T A \mathrm{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2$$

b. There are two -2 entries in A. Watch how they enter the calculations. The $(1, 2)$-entry in A is in boldface type.

$$\begin{aligned} \mathbf{x}^T A \mathbf{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix} \\ &= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2) \\ &= 3x_1^2 - 2x_1x_2 - 2x_2x_1 + 7x_2^2 \\ &= 3x_1^2 - 4x_1x_2 + 7x_2^2 \end{aligned}$$

The presence of $-4x_1x_2$ in the quadratic form in Example $1(b)$ is due to the -2 entries off the diagonal in the matrix A. In contrast, the quadratic form associated with the diagonal matrix A in Example 1 (a) has no x_1x_2 cross-product term.

Question:

Let $Q(\mathbf{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$.

Compute $Q(\mathbf{x})$ for $\mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Solution:

$$Q(-3,1) = (-3)^2 - 8(-3)(1) - 5(1)^2 = 9 + 24 - 5 = 28$$
$$Q(2,-2) = (2)^2 - 8(2)(-2) - 5(-2)^2 = 4 + 32 - 20 = 16$$
$$Q(1,-3) = (1)^2 - 8(1)(-3) - 5(-3)^2 = 1 + 24 - 45 = -20$$

Question:

For $\mathrm{x} \in \mathbb{R}^3$, let $Q(\mathrm{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$.

Write this quadratic form as $\mathbf{x}^T A\mathbf{x}$.

https://www.math.chalmers.se/Math/Grundutb/GU/LGMA40/H18-1/kvadratiska_former_stencil.pdf

Question:

For x $\in \mathbb{R}^3$, let $Q(\mathrm{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$.

Write this quadratic form as $\mathbf{x}^T A\mathbf{x}$.

$$\begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

Ajay Sreenivas to Everyone 10:18 AM

[5 -1/2 0
-1/2 3 4
0 4 2]

Tejash More to Everyone 10:18 AM

5 -1/2 0
-1/2 3 4
0 4 2

Aditya to Everyone 10:18 AM

5 -1/2 0
-1/2 3 4
0 2 2

044_sumaya to Everyone 10:18 AM

4 4

https://www.math.chalmers.se/Math/Grundutb/GU/LGMA40/H18-1/kvadratiska_former_stencil.pdf

Solution:

The coefficients of x_1^2, x_2^2, and x_3^2 go on the diagonal of A. To make A symmetric, the coefficient of $x_i x_j$ for $i \neq j$ must be split evenly between the (i, j) and (j, i) entries in A. The coefficient of $x_1 x_3$ is 0. It is readily checked that:

$$Q(\mathrm{x}) = \mathrm{x}^T A \mathrm{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example

$$2x^2 + 6xy - 7y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$4x^2 - 5y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3 =$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example

$$2x^2 + 6xy - 7y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$4x^2 - 5y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3 =$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 3 \\ -1 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Answer

Symmetric Matrix

- Symmetric matrices are useful, but not essential, for representing quadratic forms.
- For example, the quadratic form $2x^2+6xy-7y^2$ can be written as

$$2x^2 + 6xy - 7y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where the coefficient 6 of the cross-product term has been split as 5+1 rather than 3+3, as in the symmetric representation.

Symmetric Matrix

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  where the coefficient 6 of the cross-product term has been split as 5+1 rather than 3+3, as in the symmetric representation.

$x^T A x$

- However, symmetric matrices are usually more convenient to work with, so it will always be understood that A is symmetric when we write a quadratic form as $\mathbf{x}^T A \mathbf{x}$, even if not stated explicitly.

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Question:

Consider the quadratic form $Q(x_1, x_2) = 5x_1^2 - 10x_1x_2 + x_2^2$.

Find the symmetric matrix A such that $Q(x) = x^T Ax$, where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

A. $\begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$

C. $\begin{pmatrix} 5 & -10 \\ -10 & 1 \end{pmatrix}$

B. $\begin{pmatrix} 5 & -5 \\ -5 & 0 \end{pmatrix}$

D. $\begin{pmatrix} 5 & -10 \\ -10 & 0 \end{pmatrix}$

https://rmi.tsu.ge/~kade/LecturesT.Kadeishvili/MathEconomics/Term3/Week3QuadraticLEC.pdf

Solution:

Example. The quadratic form $Q(x_1, x_2, x_3) = 5x_1^2 - 10x_1x_2 + x_2^2$ whose symmetric matrix is $A = \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$ is the product of three matrices

$$(x_1, x_2, x_3) \cdot \boxed{\begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

1.1.1 Symmetrization of matrix

The quadratic form $Q(x_1, x_2, x_3) = 5x_1^2 - 10x_1x_2 + x_2^2$ can be represented, for example, by the following 2×2 matrices

-10: -8, -2

-10: -7, -3

-10: -5, -5

$$\begin{pmatrix} 5 & -2 \\ -8 & 1 \end{pmatrix}, \begin{pmatrix} 5 & -3 \\ -7 & 1 \end{pmatrix}, \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$$

the last one is *symmetric*: $a_{ij} = a_{ji}$.

Question:

Example

Find the symmetric matrix associated with the quadratic form

$$Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + x_2^2 + 4x_2x_3 - x_3^2.$$

https://www.dr-eriksen.no/teaching/GRA6035/2010/lecture4.pdf

Solution

We note that the diagonal entry a_{ii} *in the matrix* A *is the coefficient in front of the term* x_i^2*, and that the sum* $a_{ij} + a_{ji}$ *of corresponding entries off the diagonal in* A *is the coefficient in front of the term* $x_i x_j$*. Since* A *is symmetric,* $a_{ij} = a_{ji}$ *is half the coefficient in front of* $x_i x_j$*. Therefore, the quadratic form* $Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + x_2^2 + 4x_2x_3 - x_3^2$ *has symmetric matrix*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

In some cases, quadratic forms are easier to use when they have no cross-product terms—that is, when the matrix of the quadratic form is a diagonal matrix. Fortunately, the cross-product term can be eliminated by making a suitable change of variable.

if we don't want any cross product term then corresponding matrix should be diagonal

$$Q(x) = x^T \underset{PDP^T}{A} x = y^T D y$$

$$Ax = \lambda x = x\lambda$$

$$AU = U\Lambda$$

$$A = U\Lambda U^{-1} \underset{\text{Symm } A}{=} U\Lambda U^T$$

$$Q(x) = x^T \underset{PDP^T}{A} x = y^T D y$$

$$x^T P D \underbrace{P^T x}_{y}$$

$$y = P^T x$$

$$\Rightarrow y^T = x^T P$$

$$Q(x) = x^T A x = y^T D y$$

Step 1: put $A = PDP^T$

$$Q(x) = x^T P D P^T x$$

Step 2: put $y = P^T x$

$$Q(x) = (P^T x)^T \underset{\sim}{D} (P^T x)$$

$$= y^T \underset{\sim}{D} y$$

$$Q(x) = x^T A x = y^T D y$$

y is just x only

but in the basis of P.

Step 1: put $A = PDP^T$

$$Q(x) = x^T P D P^T x$$

Step 2: put $y = P^T x$

$$Q(x) = (P^T x)^T \underset{\sim}{D} (P^T x)$$
$$= y^T \underset{\sim}{D} y$$

Change of Basis :

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} [x]_E = \Big[\quad P \quad\Big] [x]_P$$

Standard basis

$$[x]_P = P^{-1} [x]_E$$

$$= P^T [x]_E$$

Change of Variable in a Quadratic Form

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Change of Variable in a Quadratic Form

If $\mathbf{x}$ represents a variable vector in $\mathbb{R}^n$, then a **change of variable** is an equation of the form

$$\mathbf{x} = P\mathbf{y}, \qquad \text{or equivalently,} \qquad \mathbf{y} = P^{-1}\mathbf{x} = P^T x \tag{1}$$

where P is an invertible matrix and $\mathbf{y}$ is a new variable vector in $\mathbb{R}^n$. Here $\mathbf{y}$ is the coordinate vector of $\mathbf{x}$ relative to the basis of $\mathbb{R}^n$ determined by the columns of P. (See Section 4.4.)

If the change of variable (1) is made in a quadratic form $\mathbf{x}^T A\mathbf{x}$, then

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T(P^T AP)\mathbf{y} \tag{2}$$

and the new matrix of the quadratic form is $P^T AP$. Since A is symmetric, Theorem 2 guarantees that there is an *orthogonal* matrix P such that $P^T AP$ is a diagonal matrix D, and the quadratic form in (2) becomes $\mathbf{y}^T D\mathbf{y}$. This is the strategy of the next example.

Question:

Make a change of variable that transforms the quadratic form

$$Q(\mathbf{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$$

into a quadratic form with no cross-product term.

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$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

first eigenvector

2nd eigen vector

$$P = \begin{bmatrix} & \end{bmatrix}$$

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SOLUTION The matrix of the quadratic form in Example 3 is

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

The first step is to orthogonally diagonalize A. Its eigenvalues turn out to be $\lambda = 3$ and $\lambda = -7$. Associated unit eigenvectors are

$$\lambda = 3: \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}; \qquad \lambda = -7: \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

These vectors are automatically orthogonal (because they correspond to distinct eigenvalues) and so provide an orthonormal basis for $\mathbb{R}^2$. Let

$$P = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

Then $A = PDP^{-1}$ and $D = P^{-1}AP = P^TAP$, as pointed out earlier. A suitable change of variable is

$$\mathbf{x} = P\mathbf{y}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Then

$$
\begin{aligned}
x_1^2 - 8x_1x_2 - 5x_2^2 &= \mathbf{x}^T A \mathbf{x} = (P\mathbf{y})^T A (P\mathbf{y}) \\
&= \mathbf{y}^T P^T A P \mathbf{y} = \mathbf{y}^T D \mathbf{y} \\
&= 3y_1^2 - 7y_2^2
\end{aligned}
$$

Then

$$\begin{aligned} x_1^2 - 8x_1x_2 - 5x_2^2 &= \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) \\ &= \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} \\ &= 3y_1^2 - 7y_2^2 \end{aligned}$$

To illustrate the meaning of the equality of quadratic forms in Example 4, we can compute $Q(\mathbf{x})$ for $\mathbf{x} = (2, -2)$ using the new quadratic form. First, since $\mathbf{x} = P\mathbf{y}$,

$$\mathbf{y} = P^{-1}\mathbf{x} = P^T\mathbf{x}$$

so

$$\mathbf{y} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

Hence

$$\begin{aligned} 3y_1^2 - 7y_2^2 &= 3(6/\sqrt{5})^2 - 7(-2/\sqrt{5})^2 = 3(36/5) - 7(4/5) \\ &= 80/5 = 16 \end{aligned}$$

$Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$

find out $Q(x)$ for $(2, -2)$ but using the diagonal form.

$$\boxed{[x]_E = P\,[x]_P}$$

$x_P = P^T x$

Convert $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ in Basis of P

$$P = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

P^T ✓

$$[x]_P = P^T x = \begin{bmatrix} 6/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

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$[x]_P = P^T x = \begin{bmatrix} 6/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$

$x^T A x$

$Q(x) = y^T \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} y = \underline{3y_1^2 - 7y_2^2}$

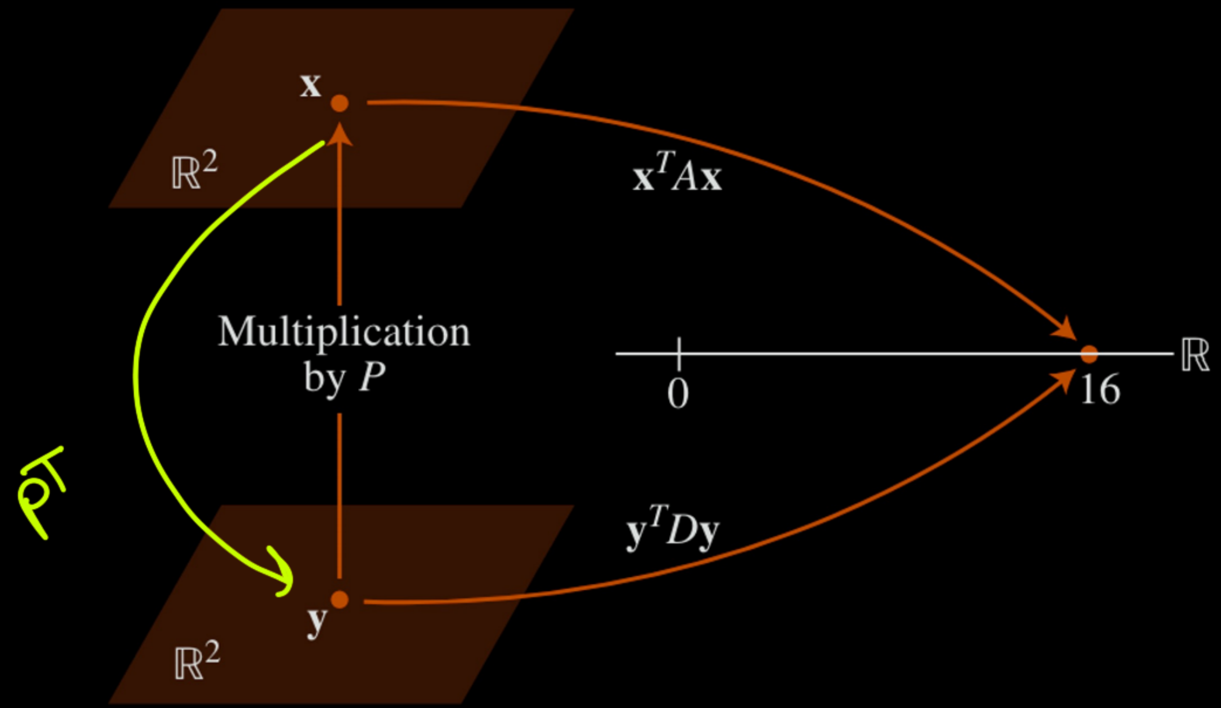

FIGURE 1 Change of variable in $\mathbf{x}^T A\mathbf{x}$.

Question:

Find an orthogonal change of variable of the quadratic form
$Q = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$.

$$Q = \boldsymbol{x}^T A \boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- The characteristic equation of the matrix A is

$$\begin{vmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{vmatrix} = \lambda^3 - 9\lambda = \lambda(\lambda+3)(\lambda-3) = 0$$

- The eigenvalues are 0, -3, 3. The orthonormal bases for the three eigenspace are

$$\lambda = 0 : \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \qquad \lambda = -3 : \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \qquad \lambda = 3 : \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

- A substitution **x**=*P***y** that eliminates the cross product terms is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

- This produces the new quadratic form

$$Q = \boldsymbol{y}^T(P^TAP)\boldsymbol{y} = \underbrace{\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}$$

- A substitution **x**=*P***y** that eliminates the cross product terms is

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- This produces the new quadratic form

$$Q = \boldsymbol{y}^T(P^TAP)\boldsymbol{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -3y_2^2 + 2y_3^2$$