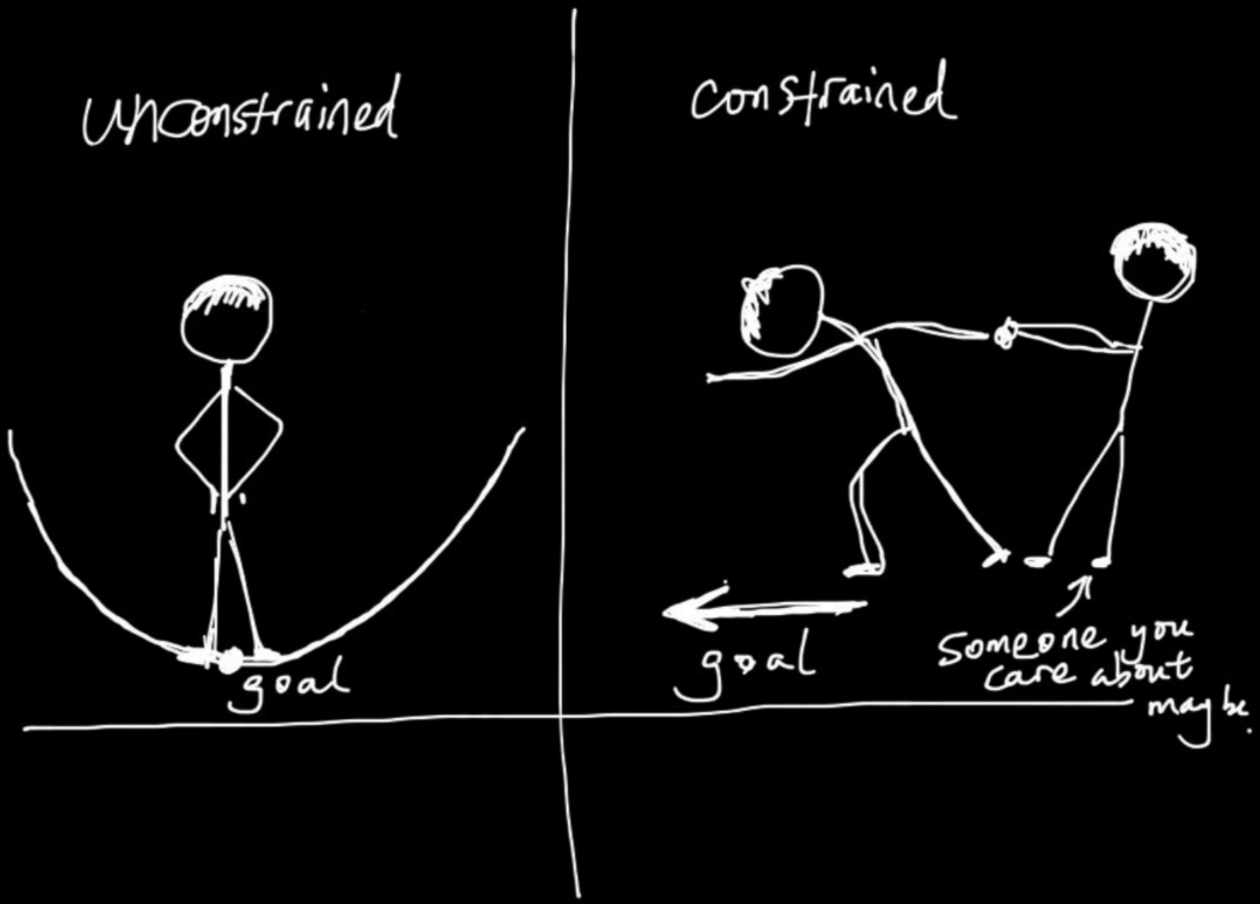
Constrained Optimization

- Inequality Constrained Optimization
- KKT Conditions

Today's Topics

- Equality/Inequality Constrained Optimization


unconstrained
constrained
goal
goal
someone you care about
maybe.

Equality vs. Inequality Constraints

Equality

$$\text{minimize} \ \ x_1 + x_2$$

$$10x_1 + 4x_2 = 20$$

$$5x_1 + 5x_2 = 20$$

$$2x_1 + 12x_2 = 12$$

Inequality

$$\text{minimize} \ \ x_1 + x_2$$

$$10x_1 + 4x_2 \geq 20$$

$$5x_1 + 5x_2 \geq 20$$

$$2x_1 + 12x_2 \geq 12$$

$$x_1, \ x_2 \geq 0$$

$$\begin{array}{lllr} \text{minimize} & f(x) & & \\ \text{subject to} & g_i(x) & \leq 0 & i = 1 \ldots m \end{array}$$

$h_j(x) = 0 \quad j = 1 \ldots n$

$g_1(x) \leq 0$

$g_2(x) \leq 0$

$t(x) \geq 0$

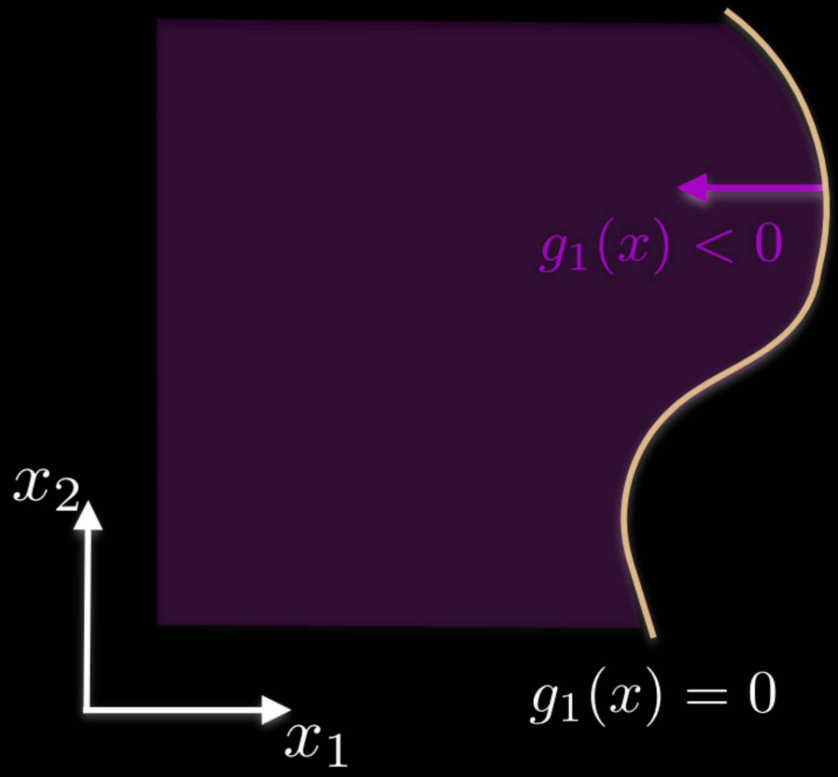
$\boxed{-t(x) \leq 0}$

GO CLASSES

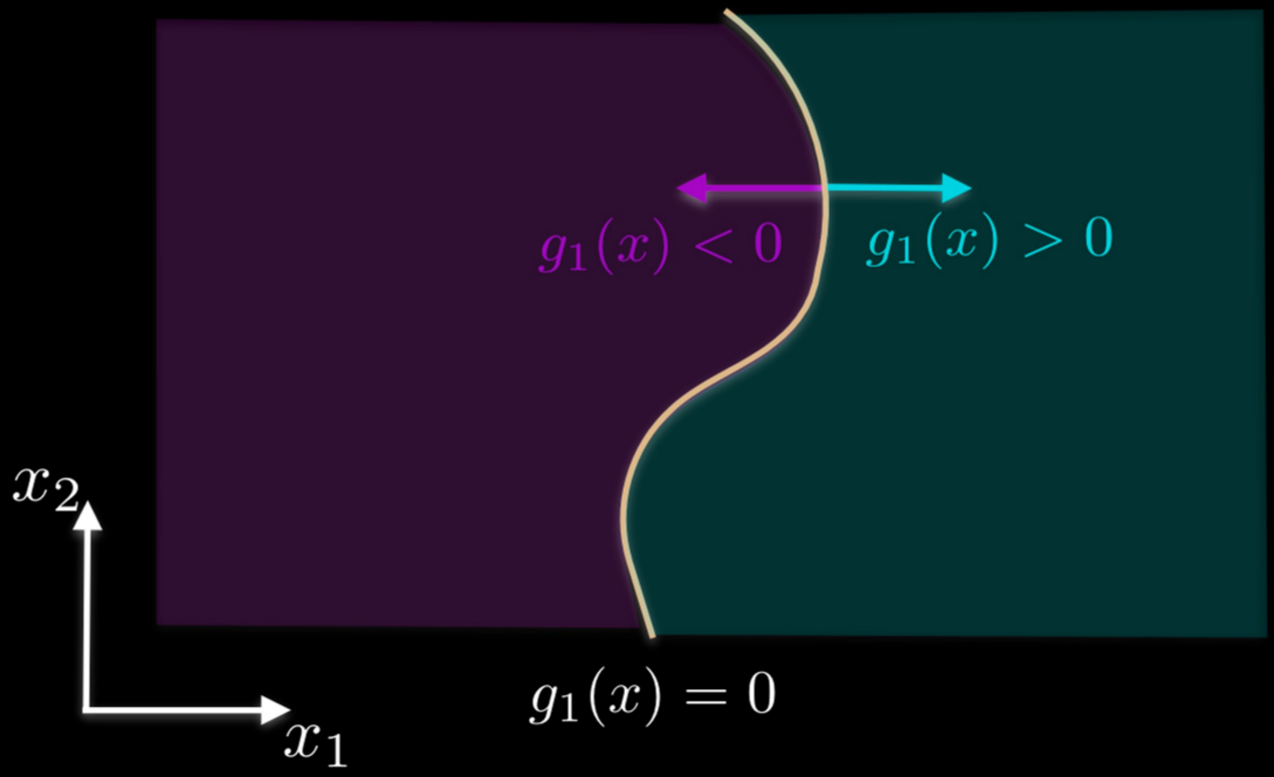
$$g_1(x) = 0$$

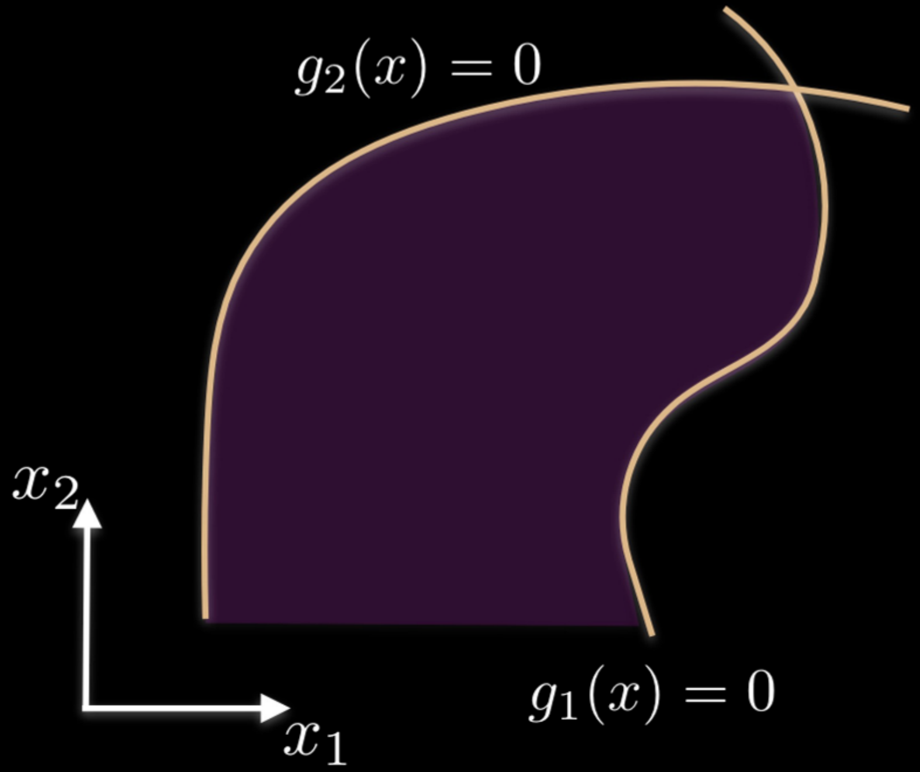
SSES

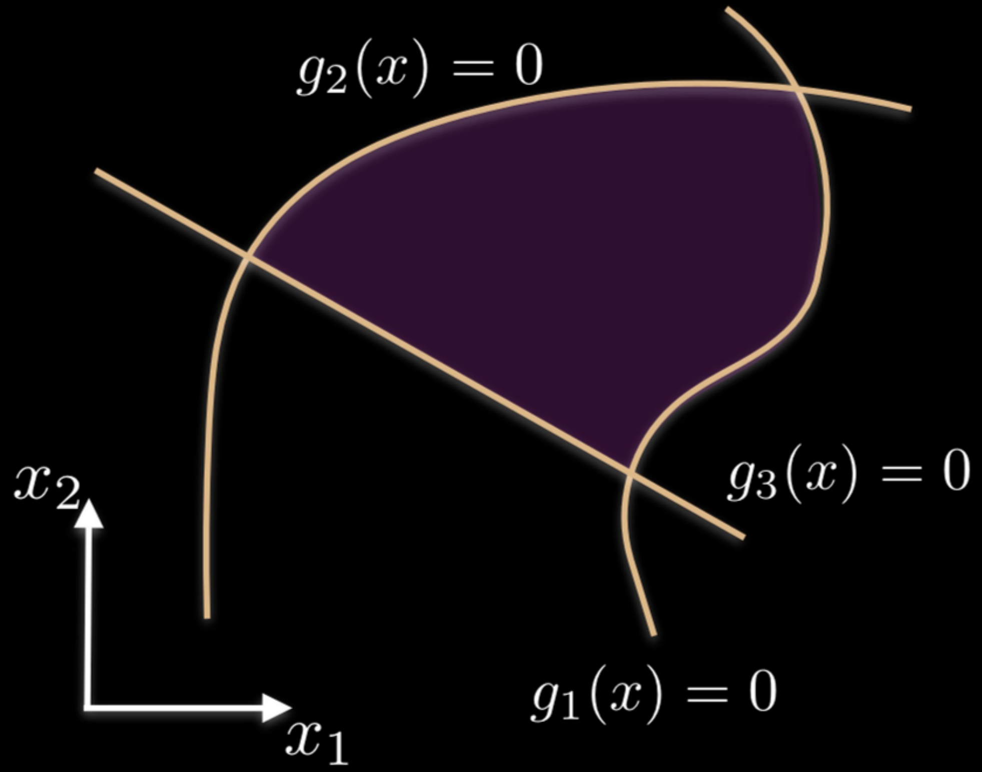
http://igs2016.mi.fu-berlin.de/summerschool/talks/course3_part1.pdf

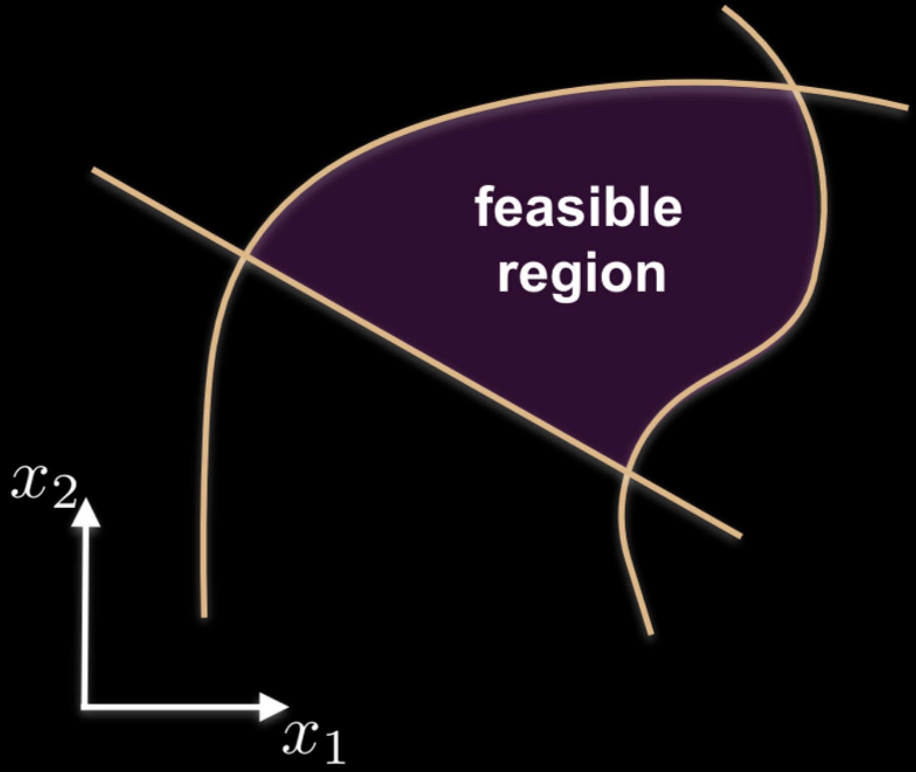


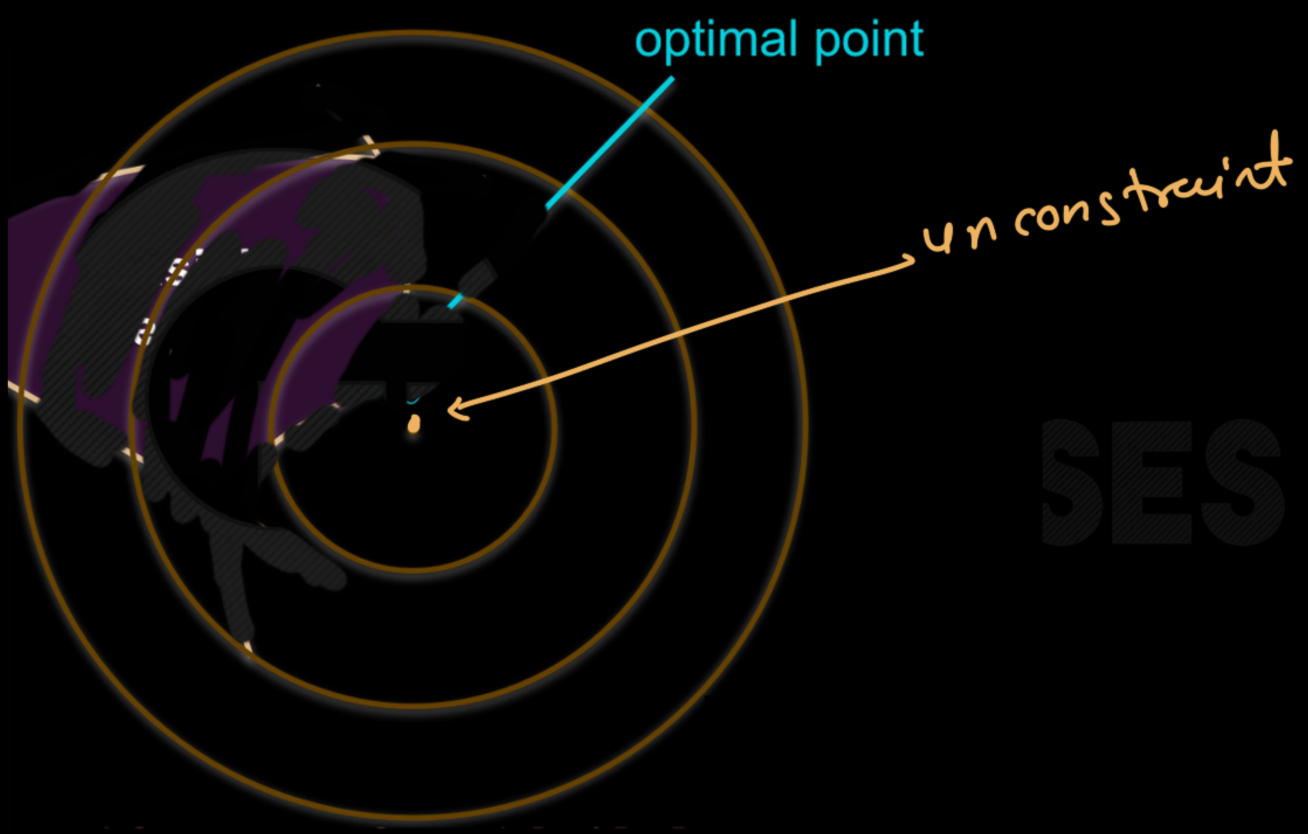
GO CLASSES

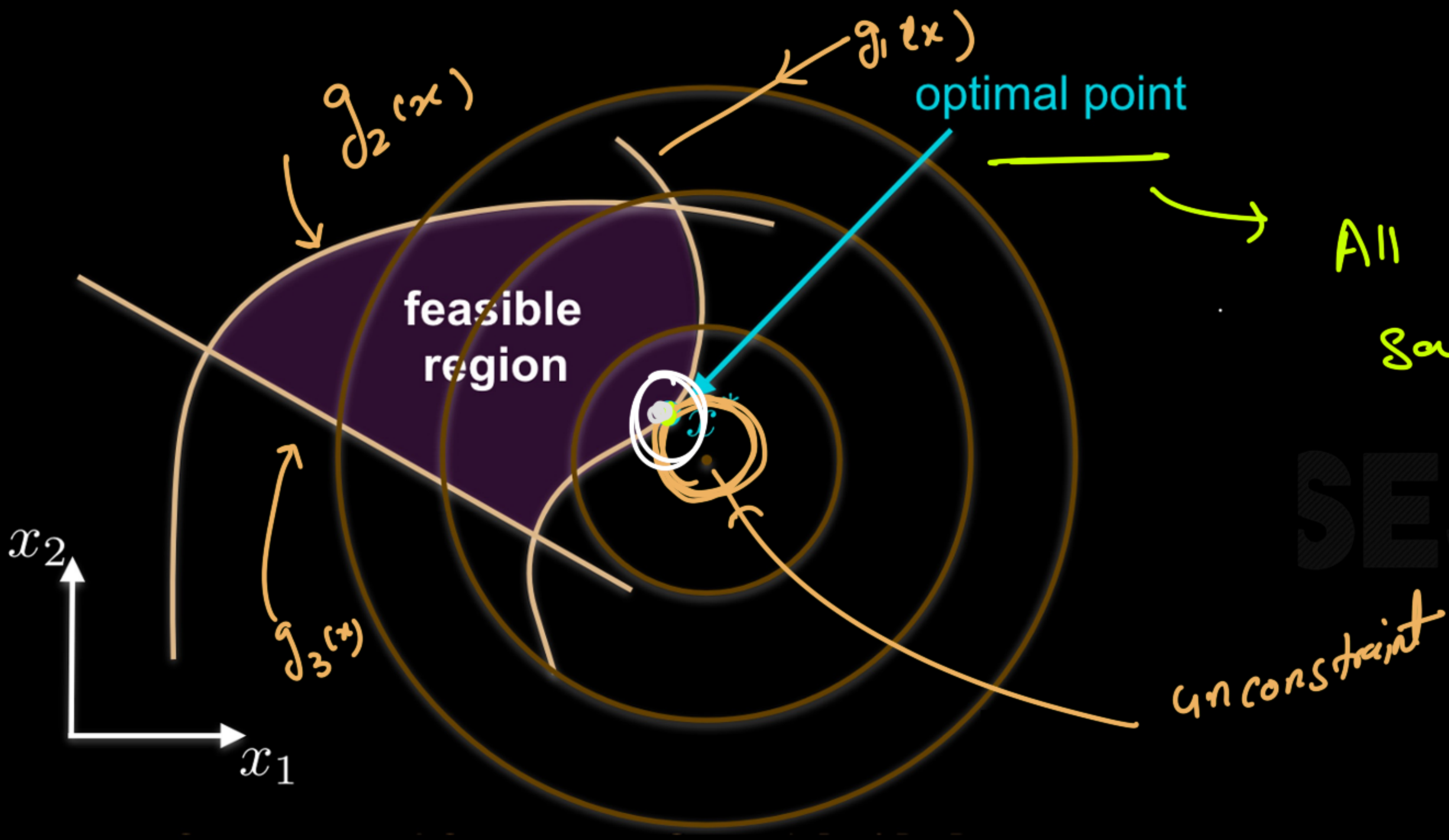

$g_1(x) < 0$
$g_1(x) > 0$
x_2
x_1
$g_1(x) = 0$


$g_2(x) = 0$
x_2
x_1
$g_1(x) = 0$


$g_2(x) = 0$
$g_3(x) = 0$
$g_1(x) = 0$
x_2
x_1


feasible
region
x_2
x_1


optimal point
un constraint



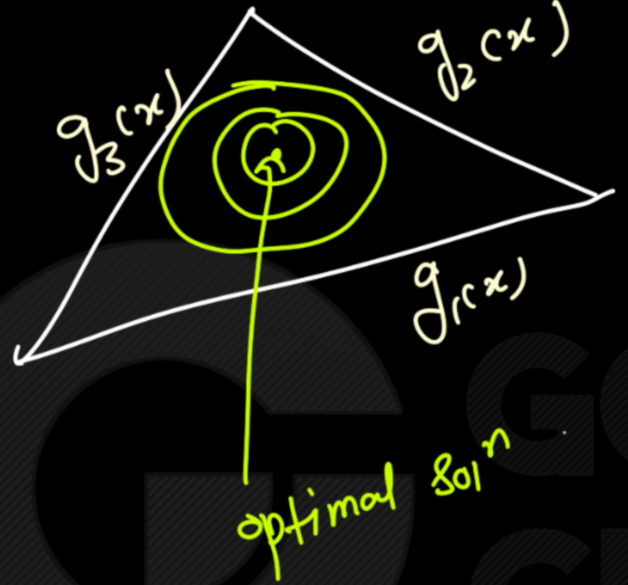
All constraints are satisfied

At optimal point

$g_1(x^*) = 0$ Active

$g_2(x^*) < 0$ Not active

$g_3(x^*) < 0$



at optimal :-

all constraints satisfied ✓

None of the constraint is active

$$\min x_1^2 + x_2^2$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

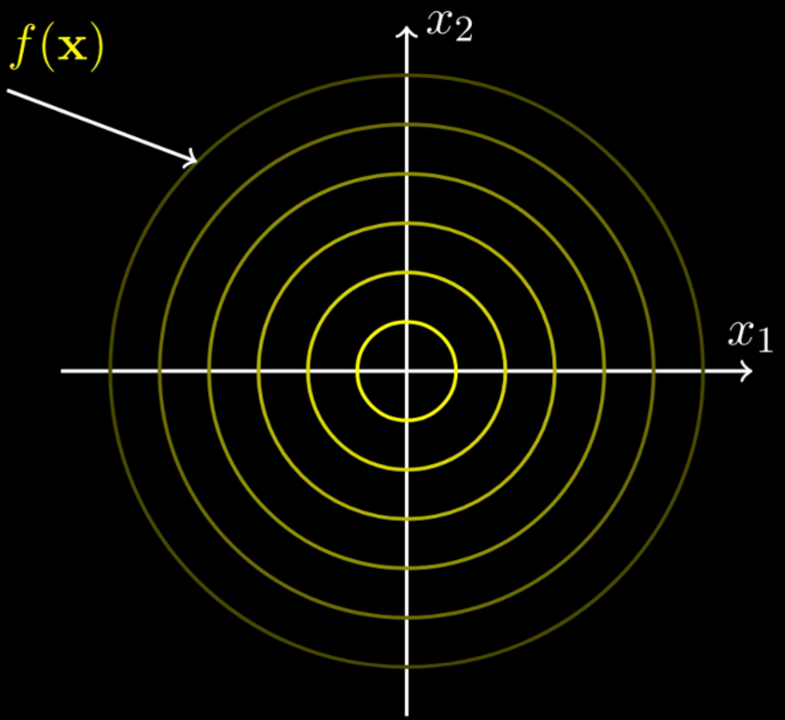
Constraint and unconstraint sol^n both are same in this example.

Want to solve this constrained optimization problem

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) = \min_{\mathbf{x}\in\mathbb{R}^2} .4\left(x_1^2 + x_2^2\right)}$$

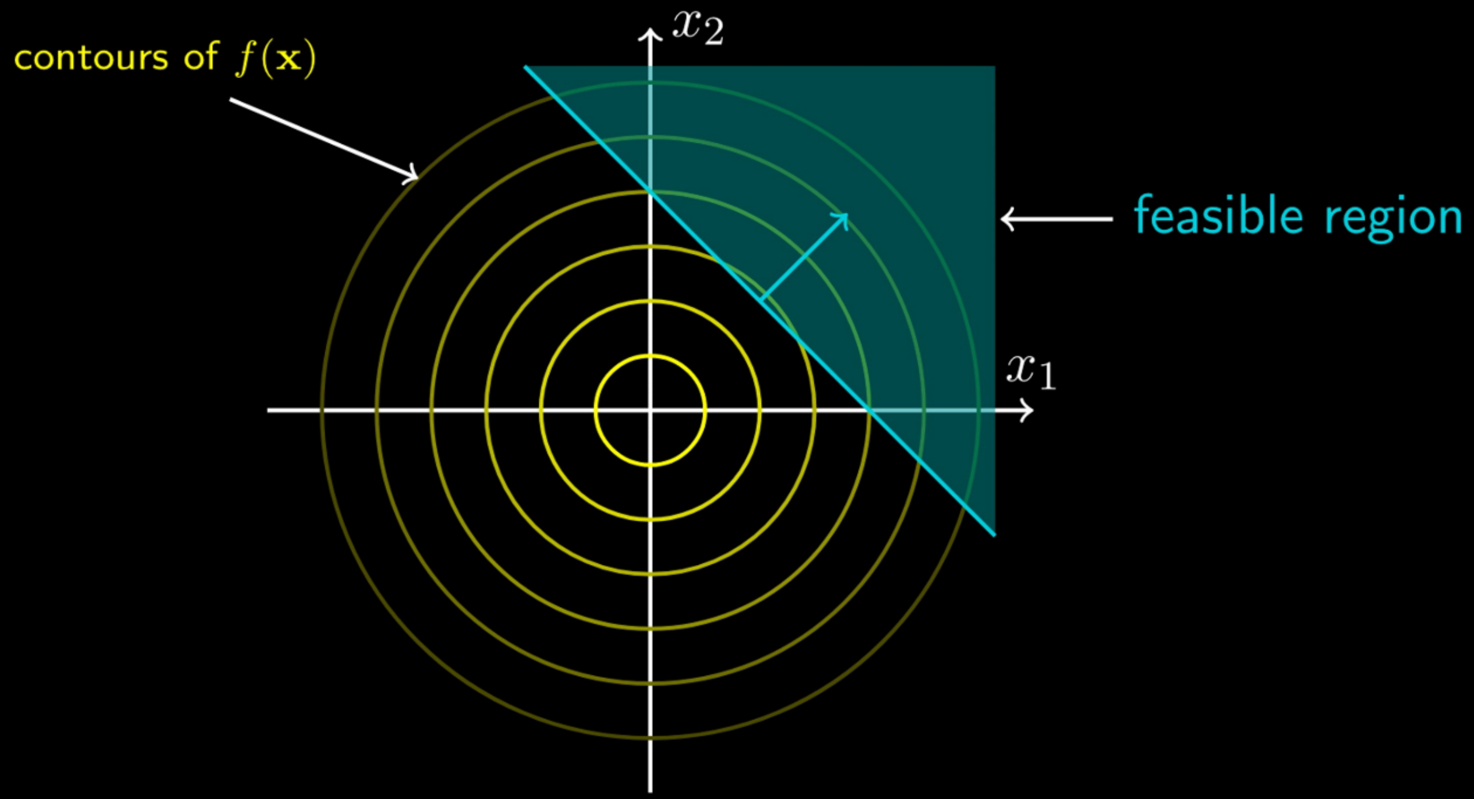
subject to

$$\boxed{g(\mathbf{x}) = 2 - x_1 - x_2 \le 0}$$



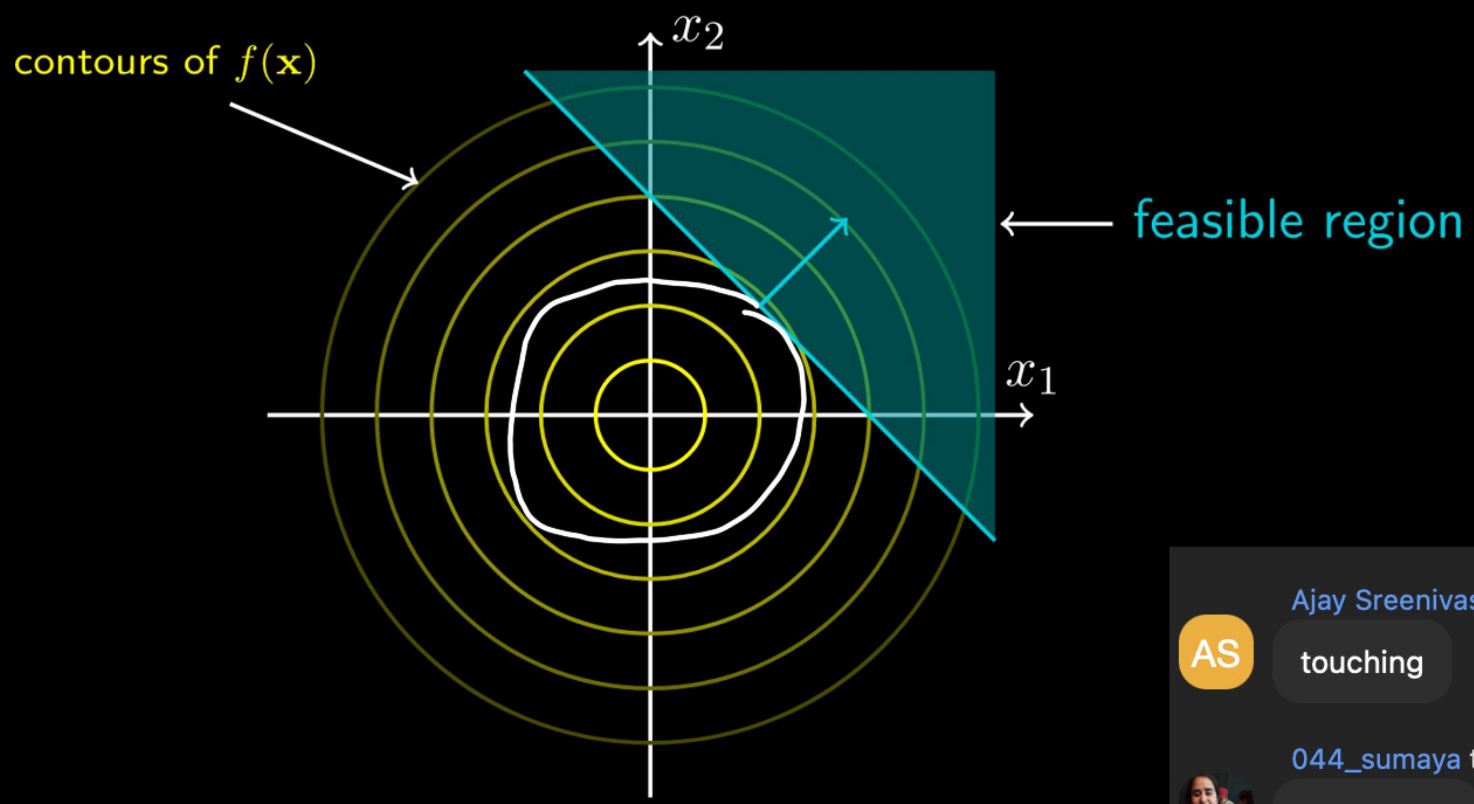
$.4\left(x_1^2 + x_2^2\right)$

$$\boxed{f(\mathbf{x}) = .4\left(x_1^2 + x_2^2\right)}$$



$$.4\left(x_1^2 + x_2^2\right)$$

$$\boxed{g(\mathbf{x}) = 2 - x_1 - x_2 \leq 0}$$



$.4\left(x_1^2 + x_2^2\right)$

$$\boxed{g(\mathbf{x}) = 2 - x_1 - x_2 \leq 0}$$

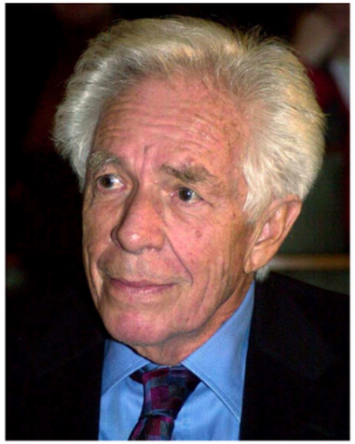
Ajay Sreenivas to Everyone 9:47 AM
AS
touching

044_sumaya to Everyone 9:47 AM
First point

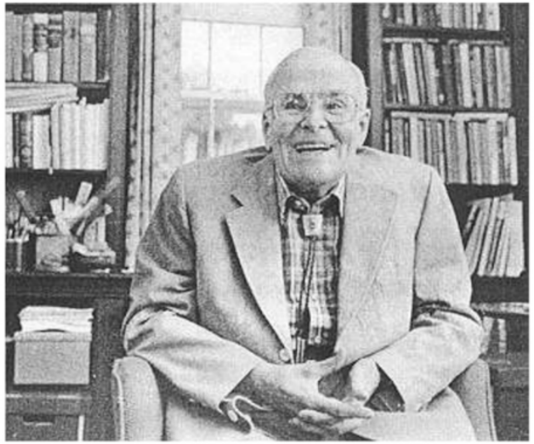
The Karush-Kuhn-Tucker Theorem

KKT

William Karush



Harold W. Kuhn



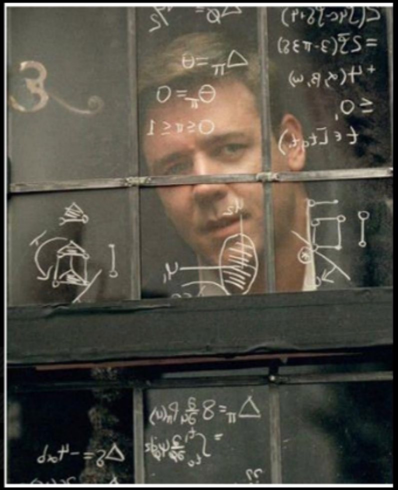
Albert W. Tucker

- The expression `Kuhn-Tucker` has 5,100,000 hits on Google.
- Needless to say, it is a cornerstone of Optimization.
- Proved in 1939 in the Master Thesis of Karush, rediscovered in 1951 by Kuhn and Tucker.

https://people.math.ethz.ch/~patrickc/co/CO18L06.pdf

Calculus

- Albert William Tucker (1905-1995)
  (Necessary and sufficient conditions
  for the optimal solution of programming
  problems, nonlinear programming, game
  theory: his PhD student
  was John Nash)



John Nash from
A Beautiful Mind

https://www.cs.rit.edu/~ago/courses/633/lec03_climbing.pdf

Constrained optimization problems are formulated as

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g(x) \leq 0 \\ & h(x) = 0 \end{aligned}$$

General Constrained Optimization Problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s. to} \quad & g(x) \leq 0 \\ & h(x) = 0 \end{aligned}$$

$\mu, \lambda \Rightarrow$ KKT cofficients

If x^* is a local minimum, then the following necessary conditions hold:

$$L(x, \lambda, \mu) = f(x) + \mu g(x) + \lambda h(x).$$

KKT Conditions

$\nabla_x L = 0$

$$\nabla f(x^*) + \mu \nabla g(x^*) + \lambda \nabla h(x^*) = 0 \quad \text{Stationarity} \quad (1)$$

$$g(x^*) \leq 0 \quad \text{Feasibility} \quad (2)$$

$$h(x^*) = 0 \quad \text{Feasibility} \quad (3)$$

$$\mu \geq 0 \quad \text{Non-negativity} \quad (4)$$

$$\mu g(x^*) = 0 \quad \text{Complementary slackness} \quad (5)$$

$$\begin{array}{ll} \min & f(x) \\ \text{s. to} & g(x) \leq 0 \\ & h(x) = 0 \end{array}$$

$$\mathcal{L} = f(x) + \lambda\, h(x) + \mu\, g(x)$$

KKT conditions

1) $\nabla_x L = 0 \Rightarrow \nabla_x f(x) + \lambda \nabla_x h + \mu \nabla g = 0$

2) $g(x) \leq 0$, $h(x) = 0$

3) $\mu \geq 0$

4) $\mu\, g(x) = 0$

$$\min_x \quad f(x)$$

s. t.

$$h(x) = 0$$

$$g(x) \leq 0$$

$$\mathcal{L} = f(x) + \lambda\, h(x) + \mu\, g(x)$$

KKT conditions are:

$$\nabla_x f(x) + \lambda\, \nabla_x h(x) + \mu\, \nabla_x g(x) = 0 \tag{1a}$$

$$h(x) = 0 \tag{1b}$$

$$g(x) \leq 0 \tag{1c}$$

$$\mu\, g(x) = 0 \tag{1d}$$

$$\mu \geq 0 \tag{1e}$$

$\Leftarrow$ x is optimal

x is optimal sol^n $\Rightarrow$ KKT conditions will be satisfied

KKT conditions are satisfied $\Rightarrow$ x may or may not be optimal

$$\nabla_x f(x) + \lambda \nabla_x h(x) + \mu \nabla_x g(x) = 0 \tag{1a}$$

$$h(x) = 0 \tag{1b}$$

$$g(x) \leq 0 \tag{1c}$$

$$\mu \, g(x) = 0 \tag{1d}$$

$$\mu \geq 0 \tag{1e}$$

Constraint (1.a) states that the gradient of the Lagrangian at an optimal solution x should be zero.

Constraint (1.b) states the equality constraints.

Constraint (1.c) enforces the inequality constraints.

Constraint (1.d) states that the product of the multiplier vector of the inequality constraints and the inequality constraint vector is zero.

Constraint (1.e) states that the multiplier vector of the inequality constraints is nonnegative.

Question:

The following optimization problem is given:

$$\begin{aligned} &\text{maximize} \quad x_1^2 + 4x_2^2 \\ &\text{subject to} \quad x_1^2 + 2x_2^2 \leq 2 \end{aligned}$$

$\longrightarrow$ $\min \; -x_1^2 - 4x_2^2$

s.t. $x_1^2 + 2x_2^2 - 2 \leq 0$

Derive the KKT conditions and find the set of points satisfying these conditions

$$L(x, \mu) = f(x) + \mu g(x)$$

$$L(x, \mu) = -x_1^2 - 4x_2^2 + \mu(x_1^2 + 2x_2^2 - 2)$$

$$L(x,\mu) = -x_1^2 - 4x_2^2 + \mu(x_1^2 + 2x_2^2 - 2)$$

KKT conditions:

1) $\nabla_x L = 0$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow -2x_1 + \mu 2x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow -8x_2 + \mu 4x_2 = 0$$

2) $x_1^2 + 2x_2^2 - 2 \leq 0$

3) $\mu \geq 0$

4) $\mu(x_1^2 + 2x_2^2 - 2) = 0$

$$L(x,\mu) = -x_1^2 - 4x_2^2 + \mu(x_1^2+2x_2^2-2)$$

1) $\nabla_x L = 0$ $\quad \dfrac{\partial L}{\partial x_1} = -2x_1 + \mu 2x_2 = 0$

$$\frac{\partial L}{\partial x_2} = -8x_2 + \mu 4 x_2 = 0$$

2) $x_1^2 + 2x_2^2 \leq 2$

3) $\mu \geq 0$

4) $\mu \cdot (x_1^2 + 2x_2^2 - 2) = 0$

$$L(x,\mu) = -x_1^2 - 4x_2^2 + \mu(x_1^2+2x_2^2-2)$$

1) $\nabla_x L = 0$ $\quad \frac{\partial L}{\partial x_1} = -2x_1 + \mu 2x_2 = 0$ Stationary cond^n

$$\frac{\partial L}{\partial x_2} = -8x_2 + \mu 4x_2 = 0$$

2) $x_1^2 + 2x_2^2 \leq 2$ feasibility cond^n

3) $\mu \geq 0$ Non-Negitivity cond^n

4) $\mu \cdot (x_1^2 + 2x_2^2 - 2) = 0$ Complementary Slackness cond^n

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$$L(x, \mu) = -x_1^2 - 4x_2^2 + \mu\left(x_1^2 + 2x_2^2 - 2\right)$$

1) $\nabla_x L = 0 \qquad \dfrac{\partial L}{\partial x_1} = -2x_1 + \mu 2x_1 = 0$

$$\frac{\partial L}{\partial x_2} = -8x_2 + \mu 4x_2 = 0$$

2) $x_1^2 + 2x_2^2 \leq 2$

3) $\mu \geq 0$

4) $\mu \cdot \left(x_1^2 + 2x_2^2 - 2\right) = 0$

$\mu = 0$

$$\left\{\begin{array}{l} -2x_1 + \mu 2x_1 = 0 \\ -8x_2 + \mu 4x_2 = 0 \end{array}\right.$$

$x_1 = 0$

$x_2 = 0$

$x_1^2 + 2x_2^2 - 2 = 0$

$-2x_1 + \mu 2x_1 = 0$

$-8x_2 + \mu 4x_2 = 0$

$x_1(1 - \mu) = 0$

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(x_1, x_2, μ)

$0, 0, 0$

$\pm\sqrt{2}, 0, 1$

$0, \pm 1, 2$

$\mu = 0$

$-2x_1 + \mu 2x_1 = 0$

$-8x_2 + \mu 4x_2 = 0$

$x_1^2 + 2x_2^2 < 2$

$x_1 = 0$

$x_2 = 0$

$x_1^2 + 2x_2^2 - 2 = 0$

$-2x_1 + \mu 2x_1 = 0$

$-8x_2 + \mu 4x_2 = 0$

$x_1(1-\mu) = 0$

$x_1 = 0$

$x_2 = \pm 1$

$\mu = 2$

$1-\mu = 0 \Rightarrow \mu = 1$

$x_2 = 0$

$x_1 = \pm\sqrt{2}$

(x_1, x_2, μ)

$0, 0, 0$

$\pm\sqrt{2}, 0, 1$

$0, \pm 1, 2$

$\left.(0, 0, 0)\right\}\ \mu = 0$

$$\left.\begin{matrix}(\sqrt{2}, 0, 1) \\ (-\sqrt{2}, 0, 1)\end{matrix}\right\}\ \mu = 1$$

$$\left.\begin{matrix}(0, 1, 2) \\ (0, -1, 2)\end{matrix}\right\}\ \mu = 2$$

Solutions: The Lagrangian can be written as:

$$\mathcal{L}(x,\mu) = -(x_1^2 + 4x_2^2) + \mu(x_1^2 + 2x_2^2 - 2).$$

Applying the KKT conditions, we obtain the following candidates:

- $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mu = 0$
- $x = \pm \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}, \mu = 1$
- $x = \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mu = 2$

$(0, 0, 0) \Big\} \mu = 0$

$\left.\begin{matrix} (\sqrt{2}, 0, 1) \\ (-\sqrt{2}, 0, 1) \end{matrix}\right\} \mu = 1$

$\left.\begin{matrix} (0, 1, 2) \\ (0, -1, 2) \end{matrix}\right\} \mu = 2$

Question:

Minimize $f = x_1^2 + 2x_2^2 + 3x_3^2$

subject to

$$g_1 = x_1 - x_2 - 2x_3 \le 12$$

$$g_2 = x_1 + 2x_2 - 3x_3 \le 8$$

min $x_1^2 + 2x_2^2 + 3x_3^2$

s.t.

$x_1 - x_2 - 2x_3 - 12 \le 0$

$x_1 + 2x_2 - 3x_3 - 8 \le 0$

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0 \Rightarrow$$

$$2x_1 + \lambda_1 + \lambda_2 = 0 \qquad (2)$$

$$4x_2 - \lambda_1 + 2\lambda_2 = 0 \qquad (3)$$

$$6x_3 - 2\lambda_1 - 3\lambda_2 = 0 \qquad (4)$$

$$\lambda_j g_j = 0 \Rightarrow$$

$$\lambda_1(x_1 - x_2 - 2x_3 - 12) = 0 \qquad (5)$$

$$\lambda_2(x_1 + 2x_2 - 3x_3 - 8) = 0 \qquad (6)$$

$$g_j \le 0 \Rightarrow$$

$$x_1 - x_2 - 2x_3 - 12 \le 0 \qquad (7)$$

$$x_1 + 2x_2 - 3x_3 - 8 \le 0 \qquad (8)$$

$$\lambda_j \ge 0 \Rightarrow$$

$$\lambda_1 \ge 0 \qquad (9)$$

$$\lambda_2 \ge 0 \qquad (10)$$

From (5) either $\lambda_1 = 0$ or $x_1 - x_2 - 2x_3 - 12 = 0$

Case 1

- From *(2), (3) and (4)* we have $x_1 = x_2 = -\lambda_2 / 2$ and $x_3 = \lambda_2 / 2$
- Using these in *(6)* we get $\lambda_2^2 + 8\lambda_2 = 0, \therefore \lambda_2 = 0 \; or \; -8$
- From *(10)*, $\lambda_2 \geq 0$, therefore, $\lambda_2 = 0$,
- Therefore, $\mathbf{X}^* = [\, 0, 0, 0 \,]$

This solution set satisfies all of *(6)* to *(9)*

<u>Case 2:</u> $x_1 - x_2 - 2x_3 - 12 = 0$

- Using *(2)*, *(3) and (4),* we have $\dfrac{-\lambda_1 - \lambda_2}{2} - \dfrac{\lambda_1 - 2\lambda_2}{4} - \dfrac{2\lambda_1 + 3\lambda_2}{3} - 12 = 0$

  or $17\lambda_1 + 12\lambda_2 = -144$

- But conditions *(9)* and *(10)* give us $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ simultaneously,

  which cannot be possible with $17\lambda_1 + 12\lambda_2 = -144$

Hence the solution set for this optimization problem is $X^* = [\,0\ 0\ 0\,]$

Question:

Minimize $f = x_1^2 + x_2^2 + 60x_1$

subject to

$$g_1 = x_1 - 80 \geq 0$$

$$g_2 = x_1 + x_2 - 120 \geq 0$$

Case 1 : $\mu_1 = 0$, $\mu_2 = 0$

$\nabla_x L = 0$ ✓

2 equations

Case 2 : $\mu_1 = 0$ $\mu_2 > 0$ → complementry slackness

Case 3 : $\mu_2 = 0$ $\mu_1 > 0$

x_1

x_2

μ

Case 4 $\mu_1 \neq 0$ $\mu_2 \neq 0$

$\mu_1 > 0$ $\mu_2 > 0$

Case 1 : $\mu_1 = 0, \quad \mu_2 = 0$ } 2 equations, 2 unknowns

Case 2 : $\mu_1 = 0 \quad \mu_2 > 0$

Case 3 : $\mu_2 = 0 \quad \mu_1 > 0$

} 3 equations, 3 unknown

Case 4 : $\mu_1 \neq 0 \quad \mu_2 \neq 0$

$\mu_1 > 0 \quad \mu_2 > 0$

} 4 equations, 4 unknowns

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0 \quad \Rightarrow \quad \begin{aligned} &2x_1 + 60 + \lambda_1 + \lambda_2 = 0 && (11) \\ &2x_2 + \lambda_2 = 0 && (12) \end{aligned}$$

$$\lambda_j g_j = 0 \quad \Rightarrow \quad \begin{aligned} &\lambda_1 (x_1 - 80) = 0 && (13) \\ &\lambda_2 (x_1 + x_2 - 120) = 0 && (14) \end{aligned}$$

$$g_j \le 0 \quad \Rightarrow \quad \begin{aligned} &x_1 - 80 \ge 0 && (15) \\ &x_1 + x_2 + 120 \ge 0 && (16) \end{aligned}$$

$$\lambda_j \ge 0 \quad \Rightarrow \quad \begin{aligned} &\lambda_1 \le 0 && (17) \\ &\lambda_2 \le 0 && (18) \end{aligned}$$

From (13) either $\lambda_1 = 0$ or $(x_1 - 80) = 0$,

Case 1

- From *(11) and (12)* we have $x_1 = -\lambda_2/2 - 30$ and $x_2 = -\lambda_2/2$
- Using these in *(14)* we get $\lambda_2\left(\lambda_2 - 150\right) = 0$

  $\therefore \lambda_2 = 0 \; or \; -150$
- Considering $\lambda_2 = 0$, $\mathbf{X}^* = [\,30, 0]$. But this solution set violates *(15)* and *(16)*
- For $\lambda_2 = -150$, $\mathbf{X}^* = [\,45, 75]$. But this solution set violates *(15)*

Case 2: $(x_1 - 80) = 0$

- Using $x_1 = 80$ in *(11) and (12)*, we have

$$\lambda_2 = -2x_2$$
$$\lambda_1 = 2x_2 - 220 \qquad (19)$$

- Substitute (19) in (14), we have $-2x_2\left(x_2 - 40\right) = 0$
- For this to be true, either $x_2 = 0$ *or* $x_2 - 40 = 0$

- For $x_2 = 0,\ \lambda_1 = -220$
- This solution set violates *(15)* and *(16)*
- For $x_2 - 40 = 0,\ \lambda_1 = -140\ \ and\ \ \lambda_2 = -80$
- This solution set is satisfying all equations from (15) to (19) and hence the desired
- Thus, the solution set for this optimization problem is $\mathbf{X}^* = [\ 80\ 40\ ]$.

Question:

Using the KKT conditions discussed in class, obtain all the candidate strict local minima for the following nonlinear optimization problem:

$$\begin{aligned} \max \quad & -x_1^2 - 2x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \geq 3 \\ & x_2 - x_1^2 \geq 1 \end{aligned}$$

There are many cases to consider. Make sure that you don't miss any.

Solutions:

First, rewrite the optimization problem in standard form:

$$\begin{aligned} \text{min} \quad & x_1^2 + 2x_2^2 \\ \text{subject to} \quad & -x_1 - x_2 + 3 \le 0 \\ & -x_2 + x_1^2 + 1 \le 0 \end{aligned}$$

Solutions:

First, rewrite the optimization problem in standard form:

$$\begin{array}{rl} \min & x_1^2 + 2x_2^2 \\ \text{subject to} & -x_1 - x_2 + 3 \leq 0 \\ & -x_2 + x_1^2 + 1 \leq 0 \end{array}$$

Then, construct the Lagrangian:

$$L(x_1, x_2, \mu_1, \mu_2) = x_1^2 + 2x_2^2 + \mu_1(-x_1 - x_2 + 3) + \mu_2(-x_2 + x_1^2 + 1).$$

KKT conditions

1 $\nabla_x L = 0$

2 $-x_1 - x_2 + 3 \leq 0$

$-x_2 + x_1^2 + 1 \leq 0$

3 $\mu_1(-x_1 - x_2 + 3) = 0$

$\mu_2(-x_2 + x_1^2 + 1) = 0$

4 $\mu_1 \geq 0, \quad \mu_2 \geq 0$

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Solutions:

GO Classes

First, rewrite the optimization problem in standard form:

$$\begin{aligned} \min \quad & x_1^2 + 2x_2^2 \\ \text{subject to} \quad & -x_1 - x_2 + 3 \leq 0 \\ & -x_2 + x_1^2 + 1 \leq 0 \end{aligned}$$

Then, construct the Lagrangian:

$$L(x_1, x_2, \mu_1, \mu_2) = x_1^2 + 2x_2^2 + \mu_1(-x_1 - x_2 + 3) + \mu_2(-x_2 + x_1^2 + 1).$$

The KKT conditions are:

1. $\nabla_{x_1} L(x_1, x_2, \mu_1, \mu_2) = 2x_1 - \mu_1 + 2\mu_2 x_1 = 0$
2. $\nabla_{x_2} L(x_1, x_2, \mu_1, \mu_2) = 4x_2 - \mu_1 - \mu_2 = 0$
3. $\mu_1(x_1 + x_2 - 3) = 0$
4. $\mu_2(x_2 - x_1^2 - 1) = 0$
5. $\mu_1, \mu_2 \geq 0$
6. $-x_1 - x_2 + 3 \leq 0$
7. $-x_2 + x_1^2 + 1 \leq 0$

$\mu_1 = 0 \ , \mu_2 = 0$

$\Downarrow$

$x_1 = 0, \ x_2 = 0$

There are few cases to consider:

Case 1 : $\mu_1 = 0$, $\mu_2 = 0$

Case 2 : $\mu_1 = 0$ $\underline{\mu_2 > 0}$ $\rightarrow$ complementary slackness

Case 3 : $\mu_2 = 0$ $\underline{\mu_1 > 0}$

Case 4 $\mu_1 \neq 0$ $\mu_2 \neq 0$

$\underline{\mu_1 > 0}$ $\underline{\mu_2 > 0}$

There are few cases to consider:

Case 1— $\mu_1 = \mu_2 = 0 \Rightarrow x_1 = x_2 = 0$. However, condition 6 would be violated. Thus, this point doesn't satisfy the KKT condition and is not a candidate for a minimizer.

Case 2— $\mu_1 \geq 0, \mu_2 = 0$. Given this assumption, and solving

$$x_2 - x_1^2 - 1 = 0, 2x_1 - \mu_1 = 0, 4x_2 - \mu_1 = 0,$$

we obtain a unique solution: $x_1 = 2, x_2 = 1$. However, this solution violates condition 7. Thus, this point doesn't satisfy the KKT condition and is not a candidate for a minimizer.

Case 3— Similar to Case 2, we choose $\mu_2 \geq 0, \mu_1 = 0$. The solution obtained is $x_1 = 0$ and $x_2 = 1$, which violates condition 6. Thus, this point doesn't satisfy the KKT condition and is not a candidate for a minimizer.

Case 4— Next Page

Case 2— $\mu_1 \geq 0, \mu_2 = 0$. Given this assumption, and solving

$$x_2 - x_1^2 - 1 = 0, 2x_1 - \mu_1 = 0, 4x_2 - \mu_1 = 0,$$

we obtain a unique solution: $x_1 = 2, x_2 = 1$. However, this solution violates condition 7. Thus, this point doesn't satisfy the KKT condition and is not a candidate for a minimizer.

The KKT conditions are:

1. $\nabla_{x_1} L(x_1, x_2, \mu_1, \mu_2) = 2x_1 - \mu_1 + 2\mu_2 x_1 = 0$
2. $\nabla_{x_2} L(x_1, x_2, \mu_1, \mu_2) = 4x_2 - \mu_1 - \mu_2 = 0$
3. $\mu_1(x_1 + x_2 - 3) = 0$
4. $\mu_2(x_2 - x_1^2 - 1) = 0$
5. $\mu_1, \mu_2 \geq 0$
6. $-x_1 - x_2 + 3 \leq 0$
7. $-x_2 + x_1^2 + 1 \leq 0$

There are few cases to consider:

$\mu_1 > 0$, $\mu_2 = 0$

$x_2 - x_1^2 - 1 = 0$

$x_2 - x_1^2 - 1 \neq 0$

$$2x_1 - \mu_1 = 0,$$

$$4x_2 - \mu_1 = 0,$$

$x_1 + x_2 - 3 = 0$

There are few cases to consider:

Case 1— $\mu_1 = \mu_2 = 0 \Rightarrow x_1 = x_2 = 0$. However, condition 6 would be violated. Thus, this point doesn't satisfy the KKT condition and is not a candidate for a minimizer.

Case 2— $\mu_1 \geq 0, \mu_2 = 0$. Given this assumption, and solving

$$x_2 - x_1^2 - 1 = 0, 2x_1 - \mu_1 = 0, 4x_2 - \mu_1 = 0,$$

we obtain a unique solution: $x_1 = 2, x_2 = 1$. However, this solution violates condition 7. Thus, this point doesn't satisfy the KKT condition and is not a candidate for a minimizer.

Case 3— Similar to Case 2, we choose $\mu_2 \geq 0, \mu_1 = 0$. The solution obtained is $x_1 = 0$ and $x_2 = 1$, which violates condition 6. Thus, this point doesn't satisfy the KKT condition and is not a candidate for a minimizer.

Case 4— Next Page

Case 4— We now consider the case when $\mu_1, \mu_2 > 0$. This case implies that

$$\begin{aligned} -x_1 - x_2 + 3 &= 0 \\ -x_2 + x_1^2 + 1 &= 0, \end{aligned}$$

which implies that $(3 - x_1) - x_1^2 - 1 = 0$ or $x_1^2 + x_1 - 2 = 0$. This quadratic polynomial has two solutions: $x_1^{(1)} = -2, x_2^{(2)} = 1$. By substitution, the two solutions generate $x_2^{(1)} = 5$ and $x_2^{(2)} = 2$.

The first candidate point, $x_1^{(1)} = -2, x_2^{(2)} = 5$ implies that $-4 - \mu_1 - 4\mu_2$ (from Condition 1), which is impossible for two positive variables μ_1 and μ_2.

The second candidate point, $x_1^{(2)} = 1, x_2^{(2)} = 2$ implies that $\mu_1 = 6, \mu_2 = 2$ (from Conditions 1 and 2)—satisfying all the KKT conditions. Therefore, $x_1^* = 1, x_2^* = 2$.

(x_1, x_2, μ_1, μ_2)

$1, 2, 6, 2$

$$\begin{array}{ll} \min & f(x) \\ \text{s. to} & g(x) \le 0 \\ & h(x) = 0 \end{array}$$

if all constrains and objective function is conve

then we call it convex optimisation problem.

Theorem

If $f(\mathbf{x})$ is convex and $g_i(\mathbf{x})$, $h(x)$ are convex functions then a feasible KKT point is optimal

KKT point and convex problem $\rightarrow$ global optimality at x.

Question:

Example 2.1 *Verify that the point* $(x_1, x_2) = (\frac{4}{5}, \frac{8}{5})$ *is a local/global solution of the problem*

$$\begin{aligned} \min\ & x_1^2 + x_2^2, \\ s.t.\ & x_1^2 + x_2^2 \le 5, \\ & x_1 + 2x_2 = 4, \\ & x_1, x_2 \ge 0. \end{aligned}$$

https://www.karlin.mff.cuni.cz/~branda/download/01_02_NLP.pdf

Solution: Write the Lagrange function

$$L(x_1, x_2, u_1, u_2) = x_1^2 + x_2^2 + u_1(x_1^2 + x_2^2 - 5) - u_2x_1 - u_3x_2 + v(x_1 + 2x_2 - 4),\ u_1, u_2, u_3 \geq 0.$$

Derive the KKT conditions

$$\begin{aligned} &\text{i) feasibility,} \\ &\text{ii) } u_1(x_1^2 + x_2^2 - 5) = 0,\ u_1 \geq 0, \\ &\quad u_2x_1 = 0,\ u_2 \geq 0, \\ &\quad u_3x_2 = 0,\ u_3 \geq 0, \\ &\text{iii) } \frac{\partial L}{\partial x_1} = 2x_1 + 2u_1x_2 - u_2 + v = 0, \\ &\quad \frac{\partial L}{\partial x_2} = 2x_2 + 2u_1x_2 - u_3 + 2v = 0. \end{aligned} \tag{4}$$

For point $(x_1, x_2) = (\frac{4}{5}, \frac{8}{5})$, we have that $u_{1,2,3} = 0$ (from complementarity conditions, i.e. none of the inequality constraints is active) and $v = -\frac{8}{5}$ which is feasible value for Lagrange multiplier corresponding to equality constraint. So we have obtained KKT point $(\frac{4}{5}, \frac{8}{5}, 0, 0, 0, -\frac{8}{5})$.

Since the objective function and inequality constraints are convex, and the equality constraint is linear (affine), $(x_1, x_2) = (\frac{4}{5}, \frac{8}{5})$ is a global solution.

Question:

Problem: Locate all of the KKT points for the following problem.
Are these points local solutions?
Are they global solutions?

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 - 4x_1 - 4x_2 \\ \text{subject to} & x_1^2 \le x_2 \\ & x_1 + x_2 \le 2 \end{array}$$

https://sites.math.washington.edu//~burke/crs/408/samples/kkt_example.pdf

The Lagrangian for this problem is

$$L((x_1, x_2), (u_1, u_2)) = (x_1 - 2)^2 + (x_2 - 2)^2 + u_1(x_1^2 - x_2) + u_2(x_1 + x_2 - 2)\ .$$

Let us now write the KKT conditions for this problem.

1. (Primal Feasibility) $x_1^2 \leq x_2$ and $x_1 + x_2 \leq 2$
2. (Dual Feasibility) $0 \leq u_1$ and $0 \leq u_2$
3. (Complementarity) $u_1(x_1^2 - x_2) = 0$ and $u_2(x_1 + x_2 - 2) = 0$
4. (Stationarity of the Lagrangian)

$$0 = \nabla_x L((x_1, x_2), (u_1, u_2)) = \begin{pmatrix} 2(x_1 - 2) + 2u_1 x_1 + u_2 \\ 2(x_2 - 2) - u_1 + u_2 \end{pmatrix} \begin{matrix} = 0 \\ = 0 \end{matrix},$$

or equivalently

$$\begin{aligned} 4 &= 2x_1 + 2u_1 x_1 + u_2 \\ 4 &= 2x_2 - u_1 + u_2. \end{aligned}$$

Next observe that the global minimizer for the objective function is $(x_1, x_2) = (2, 2)$. Thus, if this point are feasible, it would be the global solution and the multipliers would both be zero. But it is not feasible. Indeed, both constraints are violated by this point. Hence, we conjecture that both constraints are active at the solution. In this case, the KKT pair $((x_1, x_2), (u_1, u_2))$ must satisfy the following 4 key equations

$$\begin{aligned} x_2 &= x_2^2 \\ 2 &= x_1 + x_2 \\ 4 &= 2x_1 + 2u_1x_1 + u_2 \\ 4 &= 2x_2 - u_1 + u_2. \end{aligned}$$

GO CLASSES

This is 4 equations in 4 unknowns that we can try to solve by elimination. Using the first equation to eliminate x_2 from the second equation, we see that x_1 must satisfy

$$0 = x_1^2 + x_1 - 2 = (x_1 + 2)(x_1 - 1),$$

so $x_1 = -2$ or $x_1 = 1$. Thus, either $(x_1, x_2) = (-2, 4)$ or $(x_1, x_2) = (1, 1)$. Since $(1, 1)$ is closer the global minimizer of the objective f_0, let us first investigate $(x_1, x_2) = (1, 1)$ to see if it is a KKT point. For this we must find the KKT multipliers (u_1, u_2).
By plugging $(x_1, x_2) = (1, 1)$ into the second of the key equations given above, we get

$$2 = 2u_1 + u_2 \text{ and } 2 = -u_1 + u_2.$$

By subtracting these two equations, we get $0 = 3u_1$ so $u_1 = 0$ and $u_2 = 2$. Since both of these values are non-negative, we have found a KKT pair for the original problem. Hence, by convexity we know that $(x_1, x_2) = (1, 1)$ is the global solution to the problem.

GO CLASSES

KKT conditions are sufficient

if the problem is convex

objective function is convex

& all constraints are convex

Question:

Calculate all points (with according Lagrange-multipliers) which satisfy the Karush-Kuhn-Tucker-conditions for the following optimization problem:

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \quad & -x_1 - x_2 \\ \text{subject to} \quad & x_2 \geq x_1^2, \\ & x_2 \leq x_1 + 2. \end{aligned}$$

https://wiki.math.ntnu.no/_media/tma4180/2021v/completeexamsolutions.pdf

Solution: The Lagrange function reads $\mathcal{L}(x,\lambda) = -x_1 - x_2 + \lambda_1(x_1^2 - x_2) + \lambda_2(x_1 - 2 - x_2)$. The KKT conditions are

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{pmatrix} -1 + 2\lambda_1 x_1 - \lambda_2 \\ -1 - \lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$\lambda_1(x_1^2 - x_2) = 0 \tag{2}$$

$$\lambda_2(-x_1 - 2 + x_2) = 0 \tag{3}$$

$$\lambda_1, \lambda_2 \geq 0 \tag{4}$$

$$x_2 \geq x_1^2$$

$$x_2 \leq x_1 + 2.$$

- $\lambda_1 = \lambda_2 = 0$: This contradicts (1).

- $\lambda_1 = 0, \lambda_2 > 0$: By (1), we get $\lambda_2 = -1$, a contradiction.

- $\lambda_2 = 0, \lambda_1 > 0$: By the second line in (1), we get $\lambda_1 = -1$, a contradiction.

- $\lambda_1 > 0, \lambda_2 > 0$: (3) yields $-x_1 - 2 + x_2 = 0$, thus $x_1 = x_2 - 2$. Together with (2), we get $x_2 = x_1^2 = (x_2 - 2)^2$ and thus $x_2 = 4$ and $x_2 = 1$. If $x_2 = 1$, (2) implies that $x_1 = 1$, but this contradicts (3). If $x_2 = 4$, we get from (3) that $x_1 = \pm 2$. We get a contradiction for $x_1 = -2$, as λ_1 would be calculated to $-\frac{2}{5}$. $x_2 = 4$ and $x_1 = 2$ yield $\lambda_1 = \frac{2}{3}$ and $\lambda_2 = \frac{5}{3}$. So, the pair (x, λ) with $x = (2, 4)^\top$ and $\lambda = (\frac{2}{3}, \frac{5}{3})^\top$ is the only point that satisfies the KKT conditions.

Question:

Consider the optimization problem

$$\begin{array}{rl} \min & -x_1 - 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 1 \\ & x_2 = 2x_1 + 1 \\ & x_1 \leq 2 \end{array}$$

Perform the following steps of the Karush-Kuhn-Tucker (KKT) method towards obtaining the solution.

https://et.engr.iupui.edu/~skoskie/ECE580/ECE580_exams/ECE580_efsoln_f15.pdf

(a) Set up the problem in standard form.

Solution:

$$\begin{array}{rl} \min & -x_1 - 3x_2 \\ \text{subject to} & h(x) = x_2 - 2x_1 - 1 = 0 \\ & g_1(x) = 1 - x_1 - x_2 \leq 0 \\ & g_1(x) = x_1 - 2 \leq 0 \end{array}$$

(b) State the KKT conditions that the solution will have to satisfy. Make sure that you have as many conditions as variables.

Solution: We have five unknowns, $x_1^*, x_2^*, \lambda^*, \mu_1^*$, and μ_2^*. With

$$f(x) = \begin{bmatrix} -1 & -3 \end{bmatrix} x,$$

the conditions are

$$\mu^* \geq 0,$$

$$Df(x^*) + \lambda^{*T} Dh(x^*) + \mu^* Dg(x^*) = 0,$$

$$\mu_1^* g_1(x^*) = \mu^*(1 - x_1 - x_2) = 0, \tag{1}$$

$$\mu_2^* g_2(x^*) = \mu_2^* x_1 = 2, \tag{2}$$

$$h(x^*) = x_2 - 2x_1 - 1 = 0, \tag{3}$$

$$g_1(x^*) = 1 - x_1 - x_2 \leq 0,$$

and

$$g_2(x^*) = x_1 - 2 \leq 0.$$

The transpose of the condition on the derivatives is

$$\begin{bmatrix} -1 \\ -3 \end{bmatrix} + \lambda^* \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \mu_1^* \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \mu_2^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

which can be rewritten as the two scalar equations

$$\begin{aligned} -1 - 2\lambda^* - \mu_1^* + \mu_2^* &= 0 \qquad (4) \\ -3 + \lambda^* - \mu_1^* &= 0. \qquad (5) \end{aligned}$$

Thus we have five equations in five unknowns.

(c) Determine the candidate points that should be tested for optimality.

Solution: Because the equations are linear, the set of options will correspond to two pairs of options. From (1) we have either

$$\mu_1^* = 0 \quad \text{or} \quad x_1 + x_2 = 1$$

and from (2) either

$$\mu_2^* = 0 \quad \text{or} \quad x_1 = 2.$$

Case 1: If $\mu_1^* = 0$ then from the derivative constraints (4) and (5),

$$\begin{aligned} -1 - 2\lambda^* + \mu_2^* &= 0 \\ -3 + \lambda^* &= 0. \end{aligned}$$

Thus $\lambda^* = 3$ and $\mu_2^* = 7$, so by (2) $7(x_1 - 2) = 0$, which implies $x_2^* = 5$. The candidate augmented vector is then

$$\left[\begin{array}{lllll} \lambda & \mu_1 & \mu_2 & x_1 & x_2 \end{array}\right]_1 = \left[\begin{array}{lllll} 3 & 0 & 7 & 2 & 5 \end{array}\right].$$

Case 2: If $\mu_1^* \neq 0$ then $x_1^* + x_2^* = 1$, and if $\mu_2^* = 0$ then from the derivative constraints (4) and (5),

$$\begin{aligned} -1 - 2\lambda^* - \mu_1^* &= 0 \\ -3 + \lambda^* - \mu_1^* &= 0. \end{aligned}$$

Subtracting the second equation from the first and solving for λ^* yields $\lambda^* = 2/3$. Then $\mu_1^* = 7/3$ and (1) and (3) together imply $x_1^* = 0$, in which case $x_2 = 1$. The resulting candidate augmented vector is

$$\left[\begin{array}{lllll} \lambda & \mu_1 & \mu_2 & x_1 & x_2 \end{array}\right]_2 = \left[\begin{array}{lllll} 2/3 & 7/3 & 0 & 0 & 1 \end{array}\right].$$

Case 3: If $\mu_1^* \neq 0$ then $x_1^* + x_2^* = 1$, and if $\mu_2^* \neq 0$ then $x_1 = 2$. Then from (1) $x_2 = -1$. However, the equality constraint (3) is not satisfied, so this case does not occur.

Finally,

Case 4: $\mu_1^* = \mu_2^* = 0$. Then (4) and (5) are inconsistent, so this case cannot occur.

Thus we have two candidate solutions:

$$\begin{bmatrix} 3 \\ 0 \\ 7 \\ 2 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2/3 \\ 7/3 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

Question:

7. (10 points) Consider the following nonlinear program

$$\begin{aligned} \max \quad & x_1 \\ \text{s.t.} \quad & x_1 + x_2 \leq 0 \\ & x_1^2 + x_2^2 \geq 18. \end{aligned}$$

(a) (2 points) Graphically solve the nonlinear program.

(b) (2 points) Determine whether the nonlinear program is a convex program. Explain why.

(c) (3 points) Does $(-3, 3)$ satisfy the KKT condition? Explain why.

(d) (3 points) Suppose the second constraint becomes $x_1^2 + x_2^2 \leq 18$, does $(-3, 3)$ satisfy the KKT condition for the new NLP? Explain why.

7. (a) The nonlinear program can be graphed as shown below

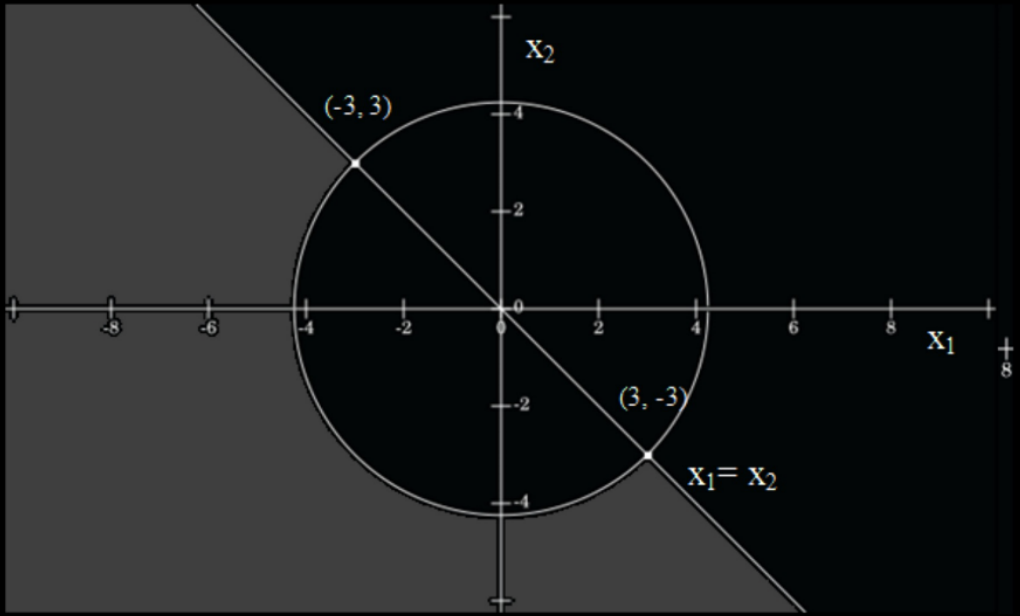


Figure 2: Graph for Problem 7

To maximize x_1 over this unbounded feasible region, the problem is unbounded.

(b) No. $(-3,3)$ and $(3,-3)$ are in the feasible region, but a combination of these two points $\frac{1}{2}(-3,3)+\frac{1}{2}(3,-3)=(0,0)$ is not in the feasible region. Hence, it is not convex program.

(c) Yes. Let $f(x) = x_1, g_1(x) = x_1 + x_2$, and $g_2(x) = -x_1^2 - x_2^2$

i. Primal feasibility: $g_1(-3,3) = -3+3 = 0 \leq 0$ and $g_2(-3,3) = -9-9 \leq -18$.

ii. Dual feasibility: Given that $\nabla f(-3,3) = (1,0)$, $\nabla g_1(-3,3) = (1,1)$, and $\nabla g_2(-3,3) = (6,-6)$, we need to find $\lambda \geq 0$ such that $\nabla f(-3,3) = \lambda_1 \nabla g_1(-3,3) + \lambda_2 \nabla g_2(-3,3)$. It turns out that $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{12}$ work.

iii. Complementary slackness: $\lambda_1[0 - g_1(-3,3)] = \frac{1}{2}[0-0] = 0$ and $\lambda_2[-18 - g_2(-3,3)] = \frac{1}{12}[-18+18] = 0$.

(d) No. Let $f(x) = x_1, g_1(x) = x_1 + x_2$, and $g_2(x) = x_1^2 + x_2^2$. For dual feasibility, given that $\nabla f(-3,3) = (1,0)$, $\nabla g_1(-3,3) = (1,1)$, and $\nabla g_2(-3,3) = (-6,6)$, we need to find $\lambda \geq 0$ such that $\nabla f(-3,3) = \lambda_1 \nabla g_1(-3,3) + \lambda_2 \nabla g_2(-3,3)$. As the unique solution has $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{12}$, dual feasibility does not hold.

GO CLASSES

- KKT conditions ✓
- How to solve for the points satisfying KKT conditions ✓

Optional Discussion

$$\min_x f(x)$$
$$\text{s.t. } g(x) \le 0$$

Lagrangian: $\mathcal{L}(x, \lambda) = f(x) + \lambda g(x).$

Dual functions

https://www.cs.princeton.edu/courses/archive/spring16/cos495/slides/ML_basics_lecture5_SVM_II.pdf

$$\min_x f(x)$$
$$\text{s.t. } g(x) \leq 0$$

Lagrangian: $\mathcal{L}(x, \lambda) = f(x) + \lambda g(x).$

- Consider the quantity:

$,\lambda \geq 0$

$$p(x) := \max_{\lambda} \mathcal{L}(x, \lambda)$$

$$p(x) = \begin{cases} f(x) & \lambda = 0 \quad \text{if constraints are satisfied} \\ \infty & \text{for } \lambda = \infty \quad \text{Constraint is NOT satisfied} \end{cases}$$

$$\min_x f(x)$$
$$\text{s.t. } g(x) \leq 0$$

Lagrangian: $\mathcal{L}(x, \lambda) = f(x) + \lambda g(x)$.

- Consider the quantity:

$$p(x) := \max_{\lambda, \lambda \geq 0} \mathcal{L}(x, \lambda)$$

$$p(x) = \begin{cases} f(x), & \text{if } x \text{ satisfies all the constraints} \\ +\infty, & \text{if } x \text{ does not satisfy the constraints} \end{cases}$$

$$\min_x p(x) = \min_x \max_\lambda L(x, \lambda)$$

$$= \max_\lambda \min_x L(x, \lambda)$$

$$\min_x f(x)$$
$$\text{s.t. } g(x) \leq 0$$

Lagrangian: $\mathcal{L}(x, \lambda) = f(x) + \lambda g(x).$

- Consider the quantity:

$$p(x) := \max_{\lambda, \lambda \geq 0} \mathcal{L}(x, \lambda)$$

$$p(x) = \begin{cases} f(x), & \text{if } x \text{ satisfies all the constraints} \\ +\infty, & \text{if } x \text{ does not satisfy the constraints} \end{cases}$$

- So minimizing $f(x)$ is the same as minimizing $p(x)$:

$$\min_x f(x) = \min_x p(x) = \min_x \max_{\lambda : \lambda \geq 0} \mathcal{L}(x, \lambda)$$
$$= \max_{\lambda : \lambda \geq 0} \min_x \mathcal{L}(x, \lambda).$$

https://www.cs.princeton.edu/courses/archive/spring16/cos495/slides/ML_basics_lecture5_SVM_II.pdf

$$\underbrace{\boxed{\begin{array}{lll} \min\limits_{x} & f(x) & \\ \text{s.t.} & g_i(x) \leq 0 & i = 1, \ldots, m \\ & h_j(x) = 0 & j = 1, \ldots, p \end{array}}}_{\text{Primal Problem}}$$

$$\underbrace{\boxed{\begin{array}{ll} \max\limits_{\lambda,\mu} & F(\lambda, \mu) \\ \text{s.t.} & \lambda \geq 0 \end{array}}}_{\text{Dual Problem}}$$

Dual function: $F(\lambda, \mu) := \min L(x, \lambda, \mu)$

convex

$$\boxed{\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad i = 1, \ldots, m \\ & h_j(x) = 0 \quad j = 1, \ldots, p \end{aligned}}$$

Primal Problem

$=$

$$\boxed{\begin{aligned} \max_{\lambda,\mu} \quad & \min L(x, \lambda, \mu) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}}$$

Dual Problem

Dual function: $F(\lambda, \mu) := \min L(x, \lambda, \mu)$

Where do these KKT conditions come from ?

Where do these conditions come from? Think about *maximizing* Lagrangian $\mathcal{L}(\lambda, x)$ with respect to x and *minimizing* it with respect to λ, unconstrained, except for $x \geq 0$ and $\lambda \geq 0$. This is the saddle-point. K-T conditions (2) are FOCs for such max-min (or saddle-point) problem.

$$= \max_{\lambda:\lambda\geq 0} \min_{x} \mathcal{L}(x, \lambda).$$

http://users.econ.umn.edu/~jwerner/K-Ttheory.pdf

Question:

Want to solve this constrained optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) = \min_{\mathbf{x}\in\mathbb{R}^2} .4\left(x_1^2 + x_2^2\right)$$

subject to

$$g(\mathbf{x}) = 2 - x_1 - x_2 \leq 0$$

https://www.ee.iitm.ac.in/~uday/2023b-EE5121/Duality.pdf

Alternate solution:

Construct the *Lagrangian dual function*

$$q(\lambda) = \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \lambda) = \min_{\mathbf{x}} \ (f(\mathbf{x}) + \lambda g(\mathbf{x}))$$

Find optimal value of $\mathbf{x}$ wrt $\mathcal{L}(\mathbf{x}, \lambda)$ in terms of the Lagrange multiplier:

$$x_1^* = \frac{5}{4}\lambda, \qquad x_2^* = \frac{5}{4}\lambda$$

Substitute back into the expression of $\mathcal{L}(\mathbf{x}, \lambda)$ to get

$$= \max_{\lambda:\lambda\geq 0} \min_{x} \mathcal{L}(x, \lambda).$$

$$q(\lambda) = \frac{5}{4}\lambda^2 + \lambda\,(2 - \frac{5}{4}\lambda - \frac{5}{4}\lambda)$$

$$\mathcal{L}(\mathbf{x}, \lambda) = .4\, x_1^2 + .4\, x_2^2 + \lambda\,(2 - x_1 - x_2)$$

Find $\lambda \geq 0$ which maximizes $q(\lambda)$. Luckily in this case the global optimum of $q(\lambda)$ corresponds to the constrained optimum

$$\frac{\partial q(\lambda)}{\partial \lambda} = -\frac{5}{2}\lambda + 2 = 0 \implies \lambda^* = \frac{4}{5} \implies x_1^* = x_2^* = 1$$

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_1} = .8\, x_1^* - \lambda^* = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_2} = .8\, x_2^* - \lambda^* = 0$$

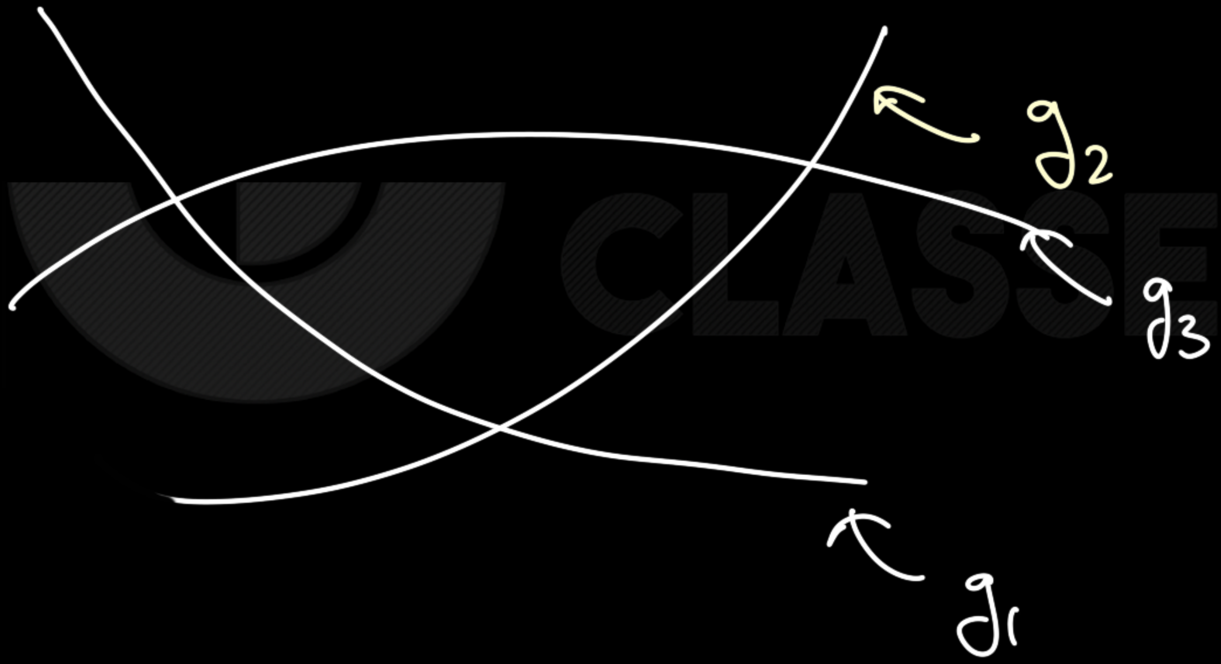
Geometric proof for KKT Conditions

Lagrangian: $\mathcal{L}(x, \lambda) = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$

$\nabla_x L = 0 \quad \Rightarrow \quad \nabla_x f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x) = 0$

Lagrangian: $\mathcal{L}(x, \lambda) = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x) + \lambda_3 g_3(x) = 0$

$$\nabla_x L = 0 \quad \Rightarrow \quad \nabla_x f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x) + \lambda_3 \nabla g_3(x) = 0$$



Lagrangian: $\mathcal{L}(x, \lambda) = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x) + \lambda_3 g_3(x) = 0$

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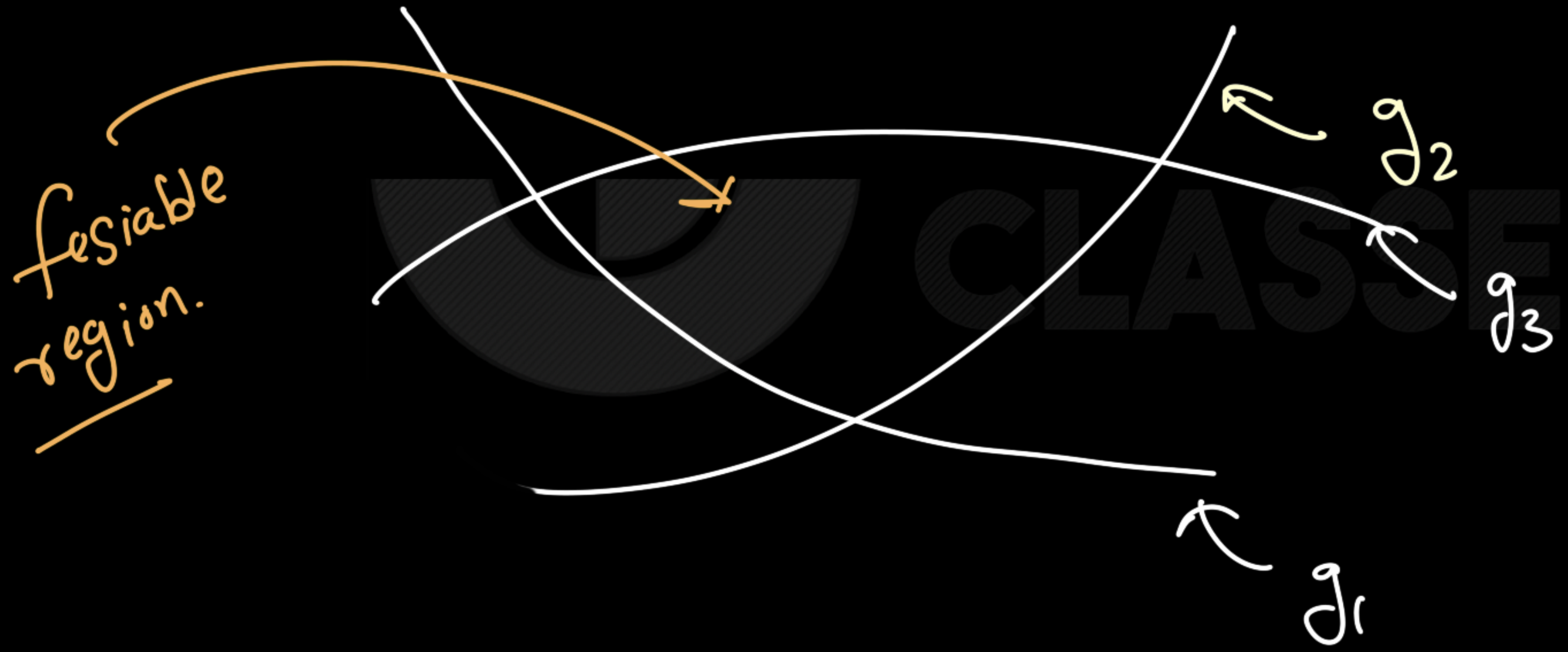
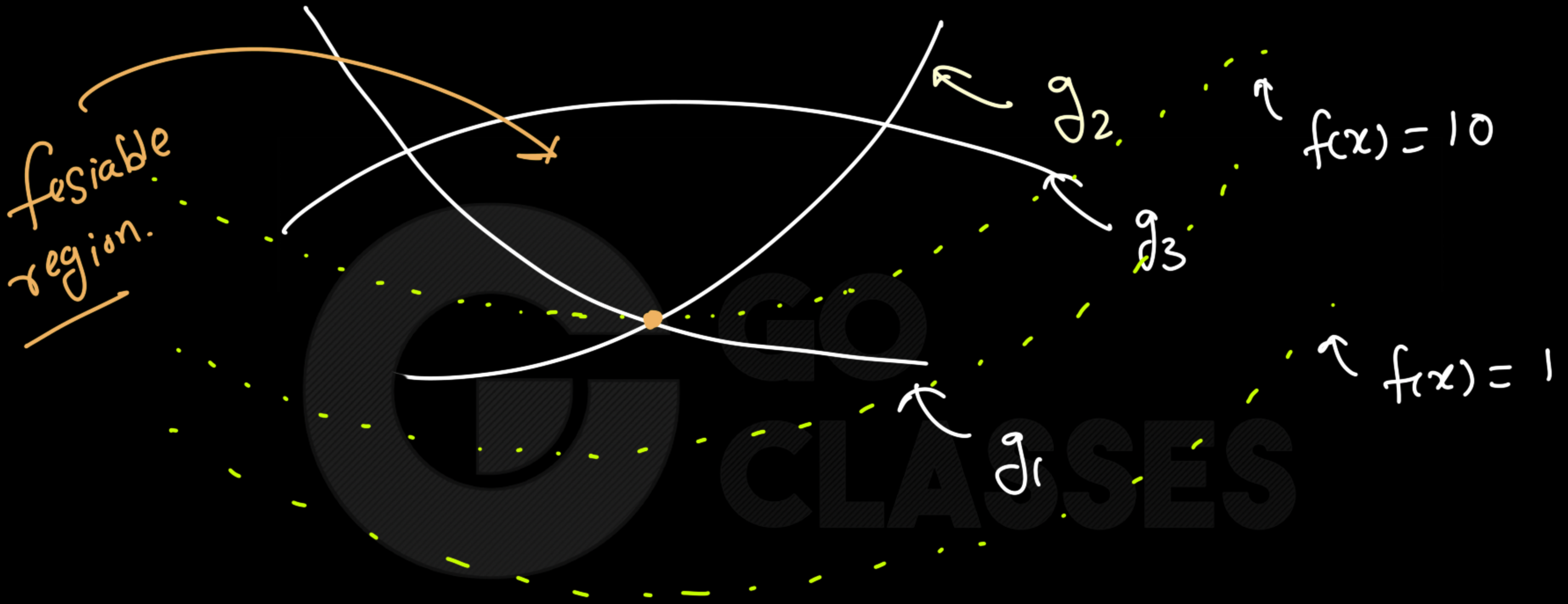
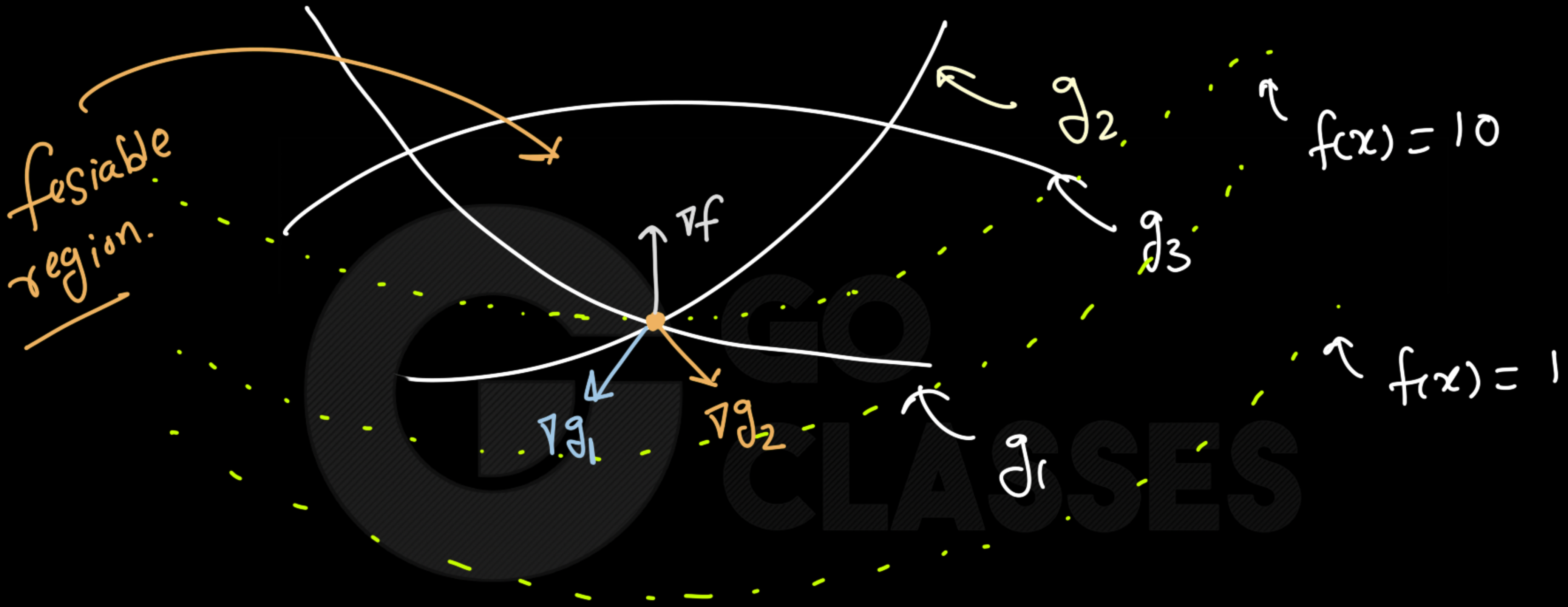
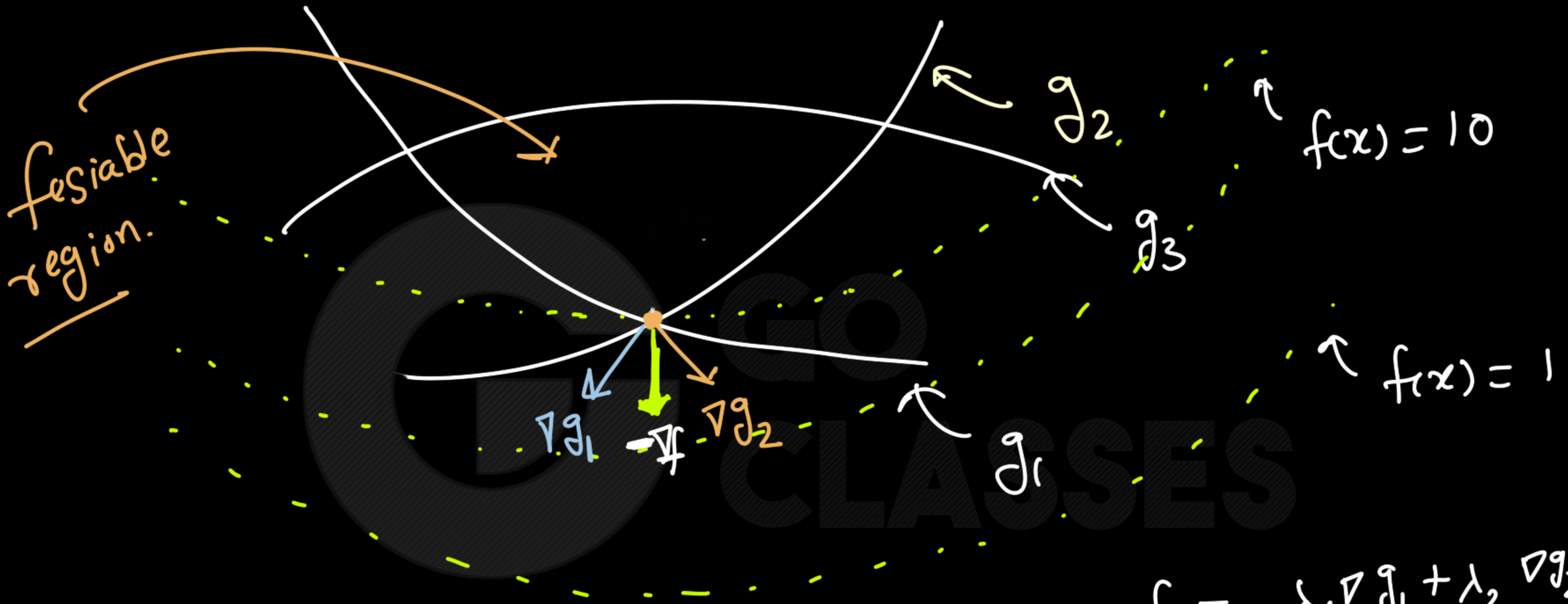




fesiable region.
g_2
g_3
g_1
$f(x)=10$
$f(x)=1$



GO CLASSES

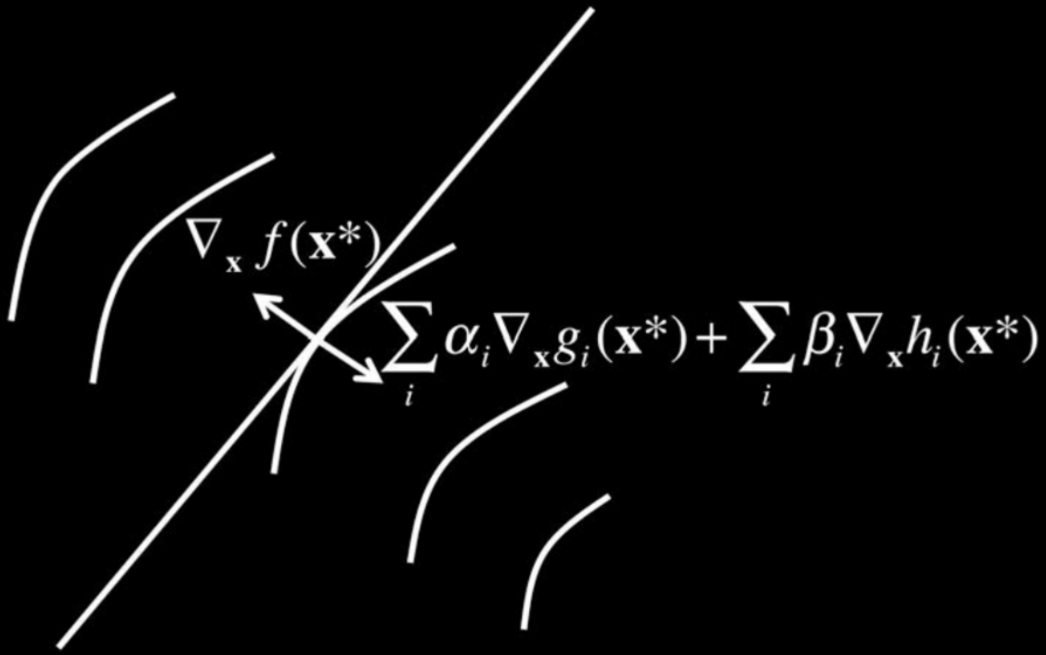


$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\Rightarrow \nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0$$

$-\nabla f$ = linear combination of ∇g_1, ∇g_2 } → cofficients of linear combination will be positive

Calculus



https://ocw.mit.edu/courses/15-097-prediction-machine-learning-and-statistics-spring-2012/3c580801bf99c95efc0559294eb9d2e2_MIT15_097S12_lec11.pdf

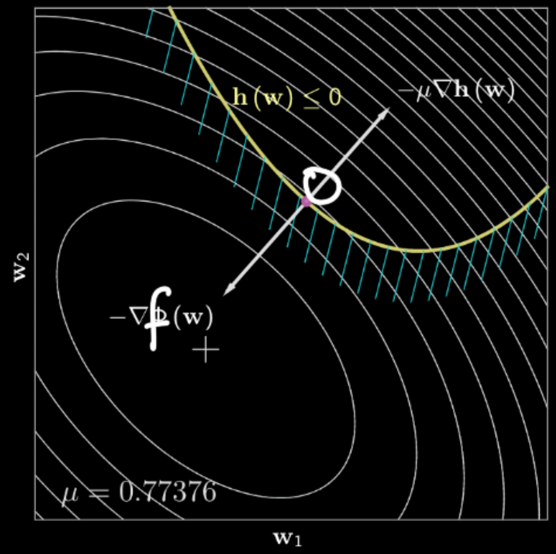
Some intuitions on the KKT conditions

$$\min_{\mathbf{w}} \ \Phi(\mathbf{x})$$

$$\text{s.t. } h(\mathbf{w}) \leq \mathbf{0}$$

Ball rolling down a valley blocked by a fence

- $-\nabla\Phi$ is the gravity
- $-\mu\nabla h$ is the force of the fence. Sign $\mu \geq 0$ means the fence can only "push" the ball.



$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

KKT

How to write KKT conditions

How to solve for KKT conditions