Welcome to Calculus Module -2

GATE DA Calculus Syllabus

Calculus and Optimization: Functions of a single variable, limit, continuity and differentiability, Taylor series, maxima and minima, optimization involving a single variable.

Calculus Module -2

- Taylor series (1D)
- Multivariate Calculus
- Matrix Calculus (Differentiation of a scalar with respect to a vector)
- Gradient
- Hessian
- Positive symmetric matrices
- Taylor series (multivariate)
- First-order conditions for optimality
- Second-order conditions for optimality
- Gradient descent
- Contours
- Convexity and global minima
- Constraint optimization
- Lagrangian and KKT conditions

$x =$ [ ] $x^T =$ [ ]

$x^T x$

SVM

optimisation in multivariate system

$$L = \left(\underline{\underline{w^T x}} \right)^2$$

$$\frac{\partial L}{\partial w}$$

scalar

vector

Taylor series (1 D)

Recall that for a function f, its derivative at a is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Recall that for a function f, its derivative at a is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$ (h is very small)

slope of the line passing through $(a, f(a))$ & $(a+h, f(a+h))$

This means essentially that $\frac{f(a+h)-f(a)}{h}$ gets close to $f'(a)$ as h becomes smaller and smaller. Or in other words, for a small h value,

$$f'(a) \approx \frac{f(a+h)-f(a)}{h}$$

Or if we solve for $f(a+h)$,

$$f(a+h) \approx f(a) + hf'(a)$$

The above is called the local linear approximation centered at $\mathbf{a}$ for the function f. It allows us to estimate values of f.

"a" $\Rightarrow$ $f(a)$ ✓

$f(a+h) \approx f(a) + hf'(a)$

$f(1) = 3 \qquad f'(1) = 5$

$f(1.00002) = ?$

$f(1.00002) = f(1) + 0.00002 * f'(1)$

Or if we solve for $f(a+h)$,

$$f(\overset{x}{a+h}) \approx f(a) + \overset{x-a}{h} f'(a)$$

The above is called the local linear approximation centered at $\mathbf{a}$ for the function f. It allows us to estimate values of f.

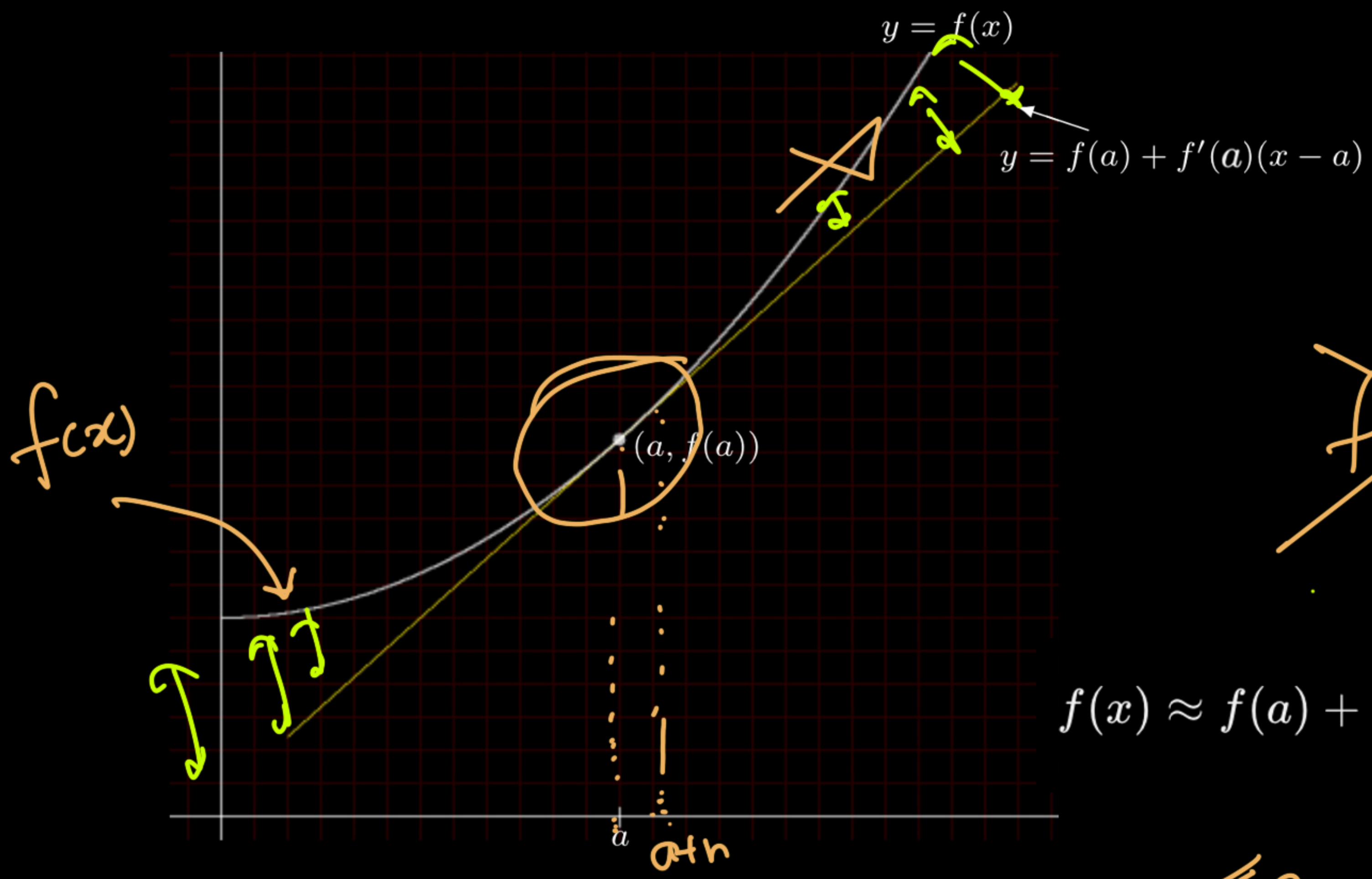
We could phrase this a slightly different way. Instead, say that for values of h close to the center a, we can make the approximation

(Putting $x = a + h$):

$$f(x) \approx f(a) + f'(a) \cdot (x - a)$$

if x is very close to a :

$$f(x) \approx f(a) + f'(a) \cdot (x - a)$$



$f(a+h)$

$\cancel{f(x)}$

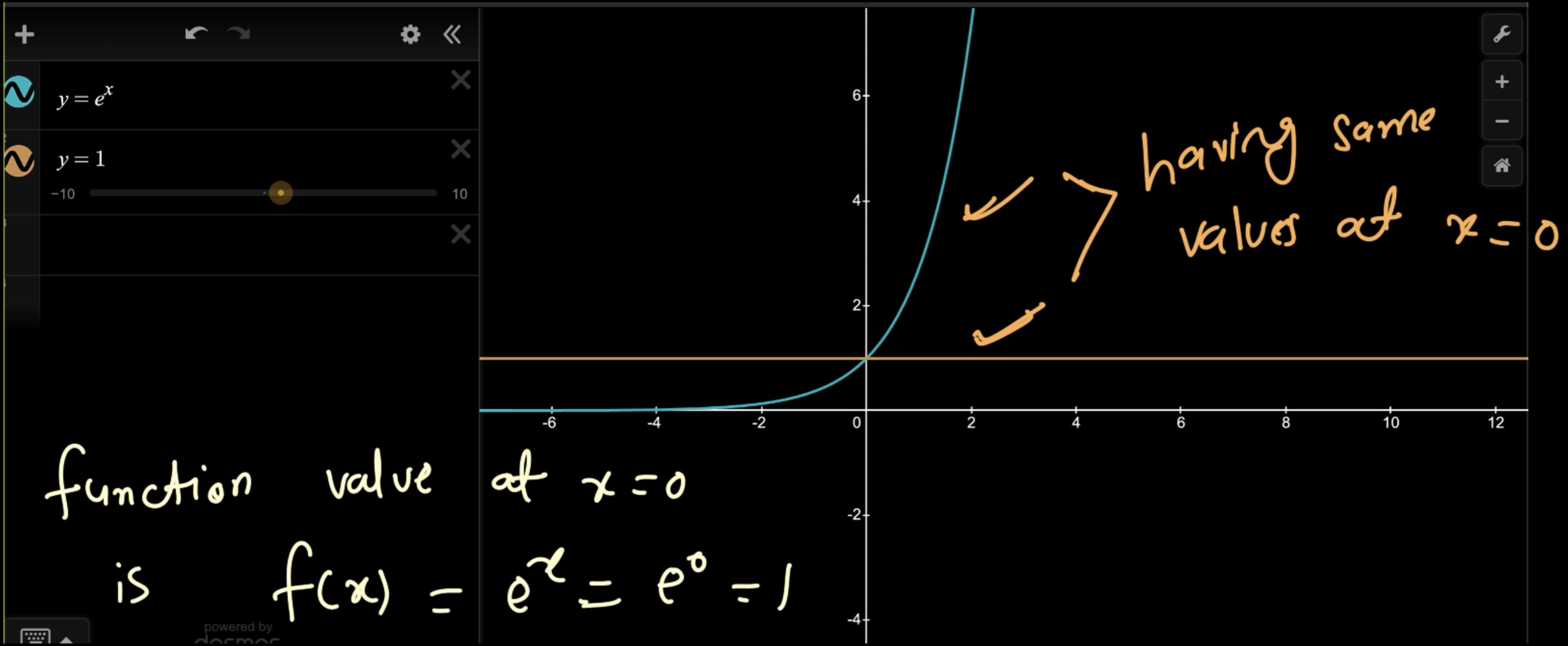
$$f(x) \approx f(a) + f'(a) \cdot (x - a)$$

$a = 2.8$

$x =$

Linear approximation

function can be approximated by tangent for very near values to tangent point.



function value at $x=0$

is $f(x) = e^x = e^0 = 1$

is same as $y=1$ (at $x=0$)

$y = e^x$ $\qquad\qquad$ $y = c_0 + c_1 x$

1) functions values must be same.

2) their derivatives must also be same

$\underline{\underline{1+x}}$

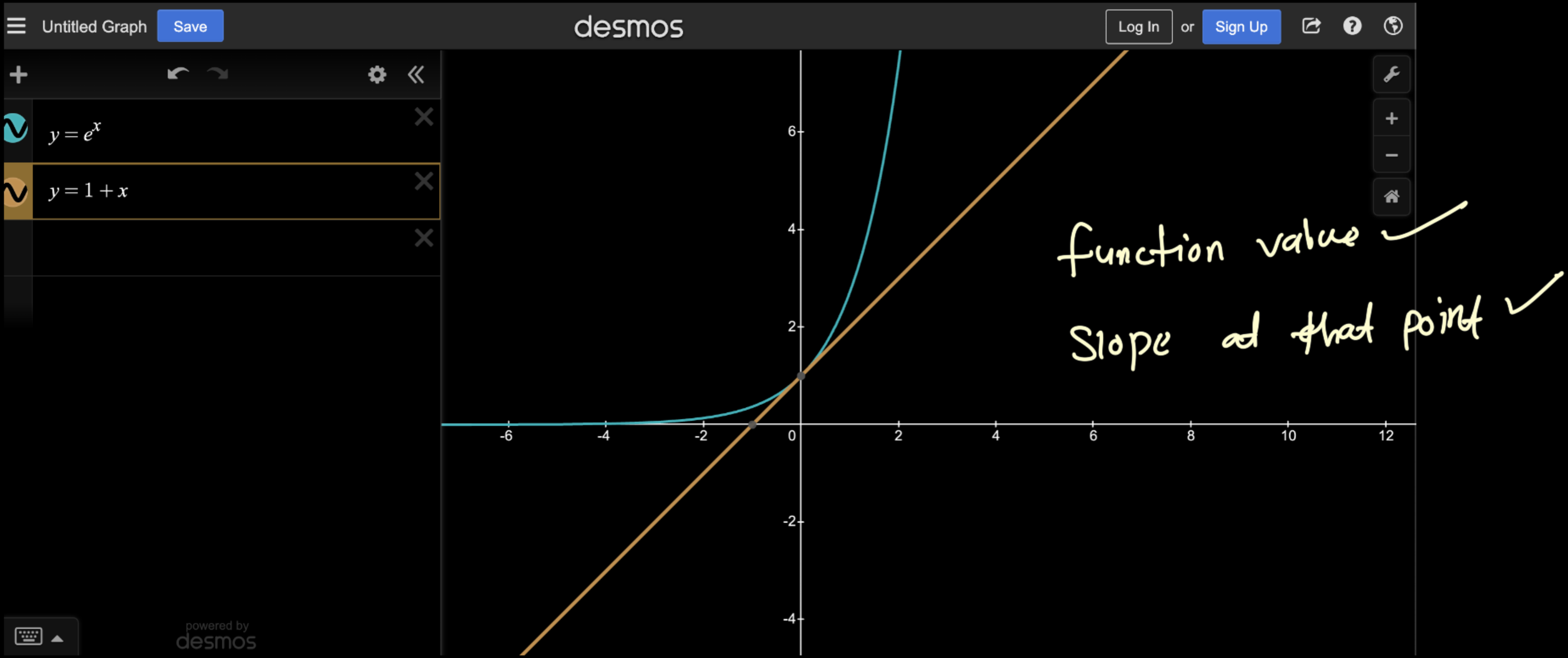
$y = e^x$ $\qquad$ $y = c_0 + c_1 x$

1) functions values must be same. $\qquad e^0 = c_0$

2) their derivatives must also be same $\qquad f'(x) = e^x \quad f'(x) = c_1$

$e^0 = c_1$

$$\boxed{c_0 = 1, \quad c_1 = 1}$$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$y = e^x$
$y = 1 + x$
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function value
Slope at that point

Can we improve our approximation ?

$f(x)$

$y = e^x$

$p(x)$

$y = c_0 + c_1 x + c_2 x^2$

$x = 0$ $\quad e^0 = c_0$ $\quad f(0) = p(0)$

$x = 0$ $\quad e^0 = c_1$ $\quad f'(0) = p'(0)$

$x = 0$ $\quad e^0 = 2c_2$ $\quad f''(0) = p''(0)$

$f(x)$

$y = e^x$

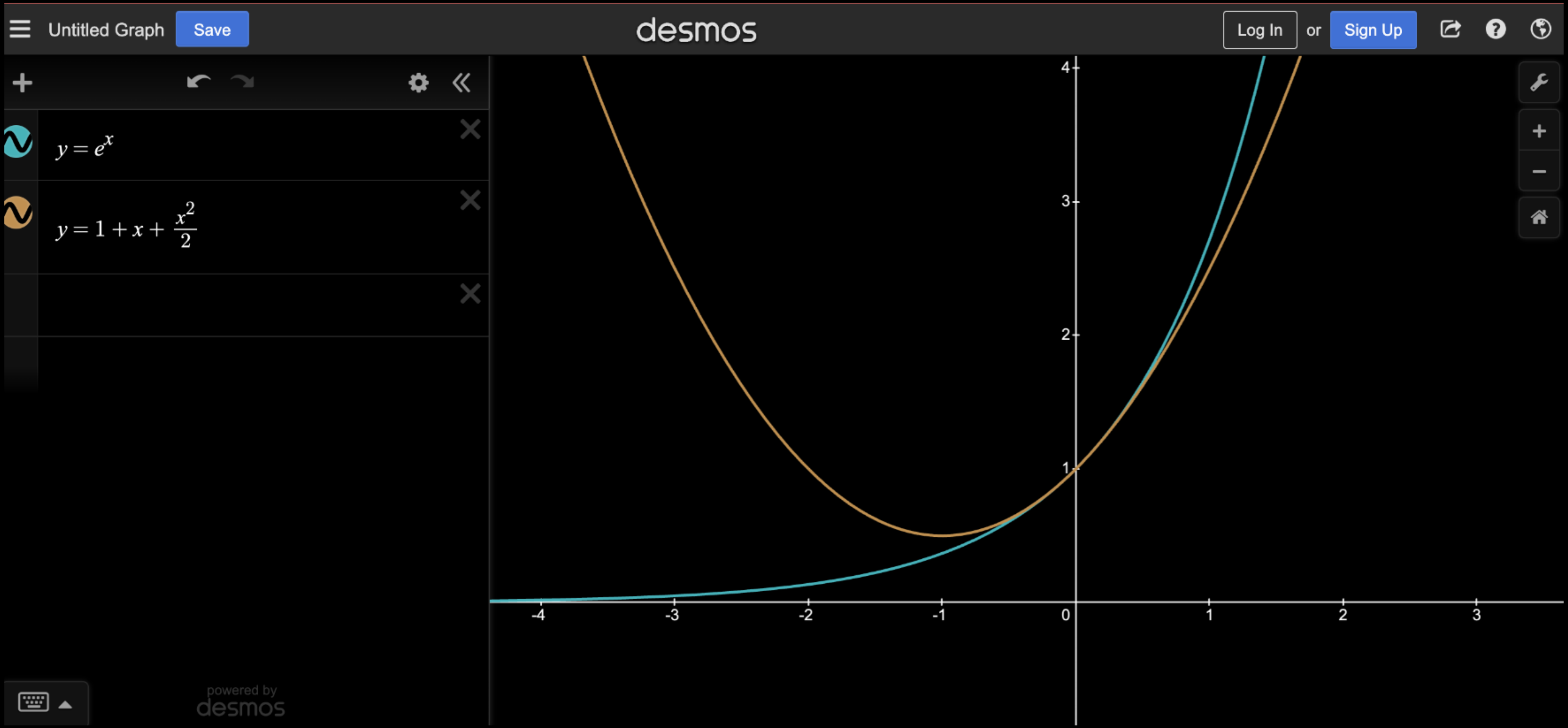
$p(x)$

$y = c_0 + c_1 x + c_2 x^2 \to 1 + x + \frac{1}{2}x^2$

$x = 0 \qquad e^0 = c_0 \qquad f(0) = p(0)$

$x = 0 \qquad e^0 = c_1 \qquad f'(0) = p'(0)$

$x = 0 \qquad e^0 = 2c_2 \qquad f''(0) = p''(0)$



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$y = e^x$
$y = 1 + x + \frac{x^2}{2}$
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$$f(x) = \quad c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \ldots$$

Suppose we want to approximate $f(x)$

using the polynomial $c_0 + c_1 x + c_3 x^3 + \ldots c_n x^n + \ldots$

Actual function

Polynomial by which i want to approximate

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots n c_n x^{n-1}$$

$$f''(x) = 2c_2 + 6c_3 x + \cdots n(n-1) c_n x^{n-2}$$

$$f'''(x) = 6c_3 + \cdots n(n-1)(n-2) c_n \cdot x^{n-3}$$

$$f^n(x) = n! \cdot c_n$$

Actual function

Polynomial by which i want to approximate

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

Use $x = 0$ as an example

$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots n c_n x^{n-1}$

$f''(x) = 2c_2 + 6c_3 x + \cdots n(n-1) c_n x^{n-2}$

$f'''(x) = 6c_3 + \cdots n(n-1)(n-2) c_n \cdot x^{n-3}$

$f^n(x) = n! \cdot c_n$

Actual function

Polynomial by which i want to approximate

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n \Rightarrow c_0 = f(0)$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots n c_n x^{n-1} \qquad c_1 = f'(0)$$

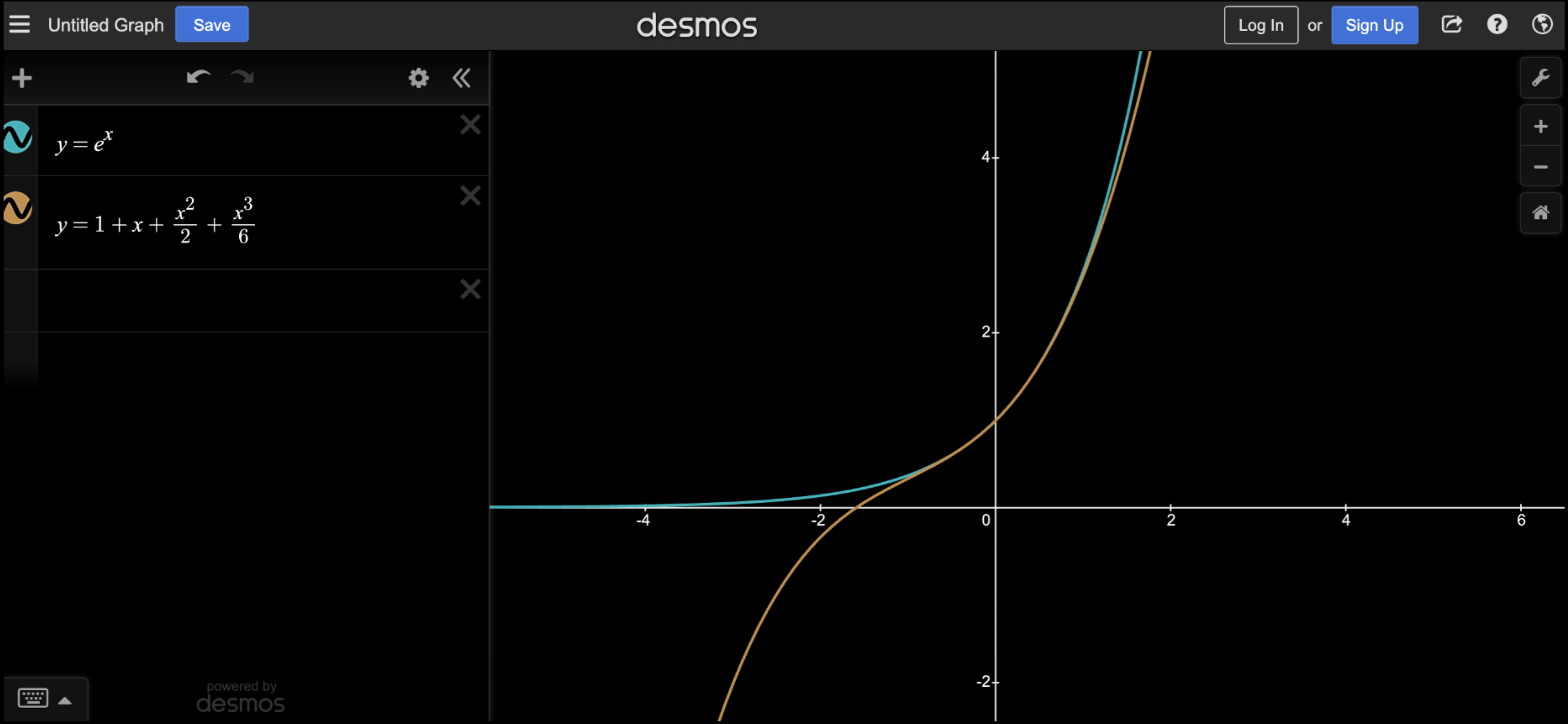
$$f''(x) = 2c_2 + 6c_3 x + \cdots n(n-1) c_n x^{n-2} \qquad c_2 = \frac{f''(0)}{2}$$

$$f'''(x) = 6c_3 + \cdots n(n-1)(n-2) c_n \cdot x^{n-3}$$

$$f^n(x) = n! \cdot c_n \Rightarrow c_n = \frac{f^n(0)}{n!}$$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots c_n x^n$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \cdots \frac{f^n(0)}{n!}$$



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$y = e^x$
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4
2
-2

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots \frac{1}{n!}f^{(n)}(a)(x-a)^n + \cdots$$

Question:

Definition: Let $f^{(k)}(0)$ exist for $k = 0, 1, 2, ..., n$. Then the *nth Taylor polynomial* of f about 0 is defined as the polynomial $p_n(x)$ of degree at most n, given by

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

4. (a) Show that p_0 has the same value at 0 as f does.

   (b) Show that p_1 has the same value at 0 and the same slope at 0 as f does.

   (c) Show that p_2 has the same value at 0, the same slope at 0, and the same concavity near (0, 0) as f does.

Question:
$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots \frac{1}{n!}f^{(n)}(a)(x-a)^n + \cdots$$

1. Write down the Taylor series for the following functions centered at a.

(a) $f(x) = e^{(x^2)}$ centered at $a = 0$.

(b) $f(x) = x^3$ centered at $a = 1$.

https://www2.math.utah.edu/~schwede/MichiganClasses/math186/Worksheet1Sols.pdf

Question:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots \frac{1}{n!}f^{(n)}(a)(x-a)^n + \cdots$$

1. Write down the Taylor series for the following functions centered at a.

(a) $f(x) = e^{(x^2)}$ centered at $a = 0$.

(b) $f(x) = x^3$ centered at $a = 1$.

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f(x) = f(1) + f'(1)\cdot(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}\cdot f'''(1)(x-1)^3$

https://www2.math.utah.edu/~schwede/MichiganClasses/math186/Worksheet1Sols.pdf

(b) $f(x) = x^3$ centered at $a = 1$.

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$f'''(x) = 6$$

$$f(x) = f(1) + f'(1)\cdot(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}\cdot f'''(1)(x-1)^3$$

$$f(x) = 1 + 3\cdot(x-1) + \frac{1}{2}6(x-1)^2 + \frac{1}{6}\cdot 6(x-1)^3$$

$$= 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$$

$(x-1)^2$

$$f(x) = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$$

$$= 1 + x-1\left[\cancel{3} + 3x - \cancel{3} + x^2 - 2x + 1\right]$$

$$1 + x-1)\left(x^2 + x + 1\right)$$

$$= \cancel{1} + x^3 + \cancel{x^2} + \cancel{x} - \cancel{x^2} - \cancel{x} - \cancel{1} = x^3$$

$e^{x^2} \cdot 4x^2$

$f(x) = e^{x^2}$

$f'(x) = e^{x^2} \cdot 2x \rightarrow f'(0) = 0$

$f''(x) = e^{x^2} \cdot 2 + e^{x^2} \cdot 2x \cdot 2x$ $\quad f''(0) = 2$

$f'''(x) = 2 \cdot e^{x^2} \cdot 2x + e^{x^2} \cdot 8x + e^{x^2} \cdot 2x \cdot 4x^2$

$f'''(0) = 0$

$f^{iv}(x) =$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{f(x)} = 1 + f(x) + \frac{(f(x))^2}{2!} + \frac{(f(x))^3}{3!}$$

1. Write down the Taylor series for the following functions centered at a.

(a) $f(x) = e^{(x^2)}$ centered at $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$\sum \frac{x^n}{n!}$

Note the uniqueness of power series (see the last page of problems) implies that this must be the Taylor series.

(b) $f(x) = x^3$ centered at $a = 1$.

$$1+\frac{3(x-1)^1}{1!}+\frac{(6)(x-1)^2}{2!}+\frac{(6)(x-1)^3}{3!} = 1+3(x-1)+3(x-1)^2+(x-1)^3+0(x-1)^4+0(x-1)^5+\ldots$$

$= x^3$

Question:

Compute the Taylor series for $f(x) = \ln(x)$ at $a = 10$.

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = 2/x^3$

$f^{iv}(x) = -6/x^4$

$f^{v}(x) = \frac{4!}{x^5}$

https://egunawan.github.io/fall18/exams/exam3practice/exam3practice_f18key.pdf

8. Compute the Taylor series for $f(x) = \ln(x)$ at $a = 10$.

Solution: *Thinking about the problem:* We will differentiate $\ln x$ enough times to see a pattern. The pattern will give us the coefficients in the Taylor series.

Doing the problem:

The first several higher derivatives of $f(x) = \ln x$ are in the table below.

n	0	1	2	3	4	5	6	7
$f^{(n)}(x)$	$\ln x$	$1/x$	$-1/x^2$	$2/x^3$	$-6/x^4$	$24/x^5$	$-120/x^6$	$720/x^7$

The pattern for $n \geq 1$ is $f^{(n)}(x) = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$, so the Taylor series of $\ln x$ at $a = 10$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(10)}{n!}(x-10)^n &= f(10) + \sum_{n=1}^{\infty} \frac{f^{(n)}(10)}{n!}(x-10)^n \\
&= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{10^n n!}(x-10)^n \\
&= \ln 10 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-10)^n}{10^n n} \\
&= \ln 10 + \frac{x-10}{10} - \frac{(x-10)^2}{200} + \frac{(x-10)^3}{3000} - \frac{(x-10)^4}{40000} + \cdots
\end{aligned}$$

Question:

Find the second order Taylor polynomial of the function $f(x) = \sqrt{4+x}$ for x near 0. *You must show the calculations that lead to your answer.*

https://dhsp.math.lsa.umich.edu/exams/116/f04s2.pdf

5. (10 points)

(a) Find the second order Taylor polynomial of the function $f(x) = \sqrt{4+x}$ for x near 0. *You must show the calculations that lead to your answer.*

We will need two derivatives of $f(x)$. It's not hard to compute $f'(x) = \frac{1}{2}(x+4)^{-1/2}$, and $f''(x) = -\frac{1}{4}(x+4)^{-3/2}$. Hence $f'(0) = 1/4$ and $f''(0) = -1/32$. Since in addition $f(0) = 2$, we obtain the second-order Taylor polynomial for $f(x)$ near $x = 0$:

$$P_2(x) \;=\; f(0) + f'(0)\,x + \frac{f''(0)}{2}\,x^2 \;=\; \mathbf{2} + \frac{\mathbf{x}}{\mathbf{4}} - \frac{\mathbf{x^2}}{\mathbf{64}}\,.$$

Question:

5.(a). Write the definition for the nth degree Taylor polynomial of a function $f(x)$ centered at $x = a$.

https://math.colorado.edu/math2300/exams/exam3/practiceExam/practicemidterm3sols.pdf

5.(a). Write the definition for the nth degree Taylor polynomial of a function $f(x)$ centered at $x = a$.

SOLUTION

(a) This is just writing the formula

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots}$$

Question:

Write down the 6-th order Taylor polynomial (i.e., through the x^6 term) for $f(x) = e^{-4x^3}$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

b) Write down the 6-th order Taylor polynomial (i.e., through the x^6 term) for $f(x) = e^{-4x^3}$.

$f(x) = e^{-4x^3} = 1 + (-4x^3) + (-4x^3)^2/2 + \cdots = 1 - 4x^3 + 8x^6 + \cdots$, so the answer is $1 - 4x^3 + 8x^6$.

Question:

The Taylor series of the polynomial $f(x) = 1 - 3x + x^2$ at $a = 0$ is

(1) $1 - 3x + x^2$ (2) $1 - 3x + x^2 + x^3 + x^4$ (3) $1 - 3x + x^2 - 3x^3 + x^4$

(d) (4 points) The Taylor series of the polynomial $f(x) = 1 - 3x + x^2$ at $a = 0$ is

(1) $1 - 3x + x^2$ (2) $1 - 3x + x^2 + x^3 + x^4$ (3) $1 - 3x + x^2 - 3x^3 + x^4$

The Taylor series of a polynomial is itself (because the best polynomial approximating a function which is a polynomial is itself). This was discussed in lecture and see also textbook. The correct answer is $1 - 3x + x^2$, that is, (1).

Answer

Question:

The Taylor expansion of x^2e^{-x} at $a = 0$ of order 3 is

(1) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$ (2) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ (3) $x^2 - x^3$

(e) (3 points) The Taylor expansion of x^2e^{-x} at $a = 0$ of order 3 is

(1) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$ (2) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ (3) $x^2 - x^3$

The Taylor expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

Thus the Taylor expansion of e^{-x} is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

So the Taylor expansion of x^2e^{-x} is

$$x^2 \cdot (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots) = x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \frac{x^6}{4!} + \dots$$

The truncation at order 3 is thus $x^2 - x^3$. The correct answer is (3).

Alternatively, it is clear that x^2e^{-x} evaluated at $a = 0$ is 0, so the constant term in the Taylor series must be 0, that discards (1) and (2) directly, thus the correct answer must be (3).

Question:

Consider the function $f(x) = e^{2x}$.

Find the degree 3 Taylor polynomial for $f(x) = e^{2x}$ centered at 0.

https://math.colorado.edu/~casa/teaching/12spring/2300/hw/midIII/2300midIIIsols.pdf

4. Consider the function $f(x) = e^{2x}$.

4.(a). [10 points] Find the degree 3 Taylor polynomial for $f(x) = e^{2x}$ centered at 0.

SOLUTION:

$$1 + 2x + 2x^2 + \frac{4}{3}x^3$$

Recall that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, so that

$$f(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Question: $$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots \frac{1}{n!}f^{(n)}(a)(x-a)^n + \cdots$$

3. (20 points) Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = -2$ is given by

$$T_3(x) = 2 - \frac{3}{8}(x+2)^2 - \frac{1}{12}(x+2)^3.$$

$T_3(-2) = f(-2$

(a) Find $f(-2)$, $f'(-2)$, and $f''(-2)$.

$-\frac{3}{4}$

(b) Determine whether f has a local minimum, a local maximum, or neither at $x = -2$. Justify your answer.

$f(-2) = 2$

$f'(-2) = 0$

$f''(-2) = -\frac{3}{4}$

$f'''(-2) = -\frac{1}{2}$

Solution.

(a) $f(-2) = 2$, $f'(-2) = 0$, and $f''(-2) = -3/4$ (since $f''(-2)/2! = -3/8$).

(b) Since $f'(-2) = 0$, and $f''(-2) = -3/4$, f has a ~~local~~ a local maximum at $x = -2$ by the Second Derivative Test.

Question:

Compute the quadratic $(n = 2)$ Taylor polynomial $P_2(x)$ for the function $x^{2/3}$ at the value $a = 8$.

https://www.math.upenn.edu/sites/www.math.upenn.edu/files/solutionsII171026.pdf

3. (a) Compute the quadratic ($n = 2$) Taylor polynomial $P_2(x)$ for the function $x^{2/3}$ at the value $a = 8$.

Solution: Since $f(x) = x^{2/3}$ has

$$f'(x) = \frac{2x^{-1/3}}{3} \text{ and } f''(x) = -\frac{2x^{-4/3}}{9},$$

one computes

$$\begin{aligned} P_2(x) &= f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 4 + \frac{1}{3}(x-8) - \frac{1}{144}(x-8)^2. \end{aligned}$$

Next Topic: Multivariate Calculus