Lagrangian Duality

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Lagrangian Duality

- Given a nonlinear programming problem, known as the *primal problem*, there exists another nonlinear programming problem, closely related to it, that receives the name of the *Lagrangian dual problem*.

- Under certain convexity assumptions and suitable constraint qualifications, the primal and dual problems have equal optimal objective values.

The Primal Problem

Consider the following nonlinear programming problem:

Primal Problem P

$$
\begin{aligned}
&\text{minimise } f(x), \qquad (1)\\
&\text{subject to:}\\
&g_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m,\\
&h_i(x) = 0 \quad \text{for } i = 1, \ldots, \ell,\\
&x \in X.
\end{aligned}
$$

The Dual Problem

Then the *Lagrangian dual problem* is defined as the following nonlinear programming problem.

Lagrangian Dual Problem D

$$\text{maximise } \theta(u, v), \tag{2}$$

$$\text{subject to:}$$

$$u \geq 0,$$

where,

$$\theta(u, v) = \inf\{f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{i=1}^{\ell} v_i h_i(x) : x \in X\}, \tag{3}$$

is the *Lagrangian dual function*.

The Dual Problem

- In the dual problem (2)–(3), the vectors u and v have as their components the Lagrange multipliers u_i for $i = 1, \ldots, m$, and v_i for $i = 1, \ldots, \ell$.

- Note that the Lagrange multipliers u_i, corresponding to the inequality constraints $g_i(x) \leq 0$, are restricted to be nonnegative, whereas the Lagrange multipliers v_i, corresponding to the equality constraints $h_i(x) = 0$, are unrestricted in sign.

- Given the primal problem P (1), several Lagrangian dual problems D of the form of (2)–(3) can be devised, depending on which constraints are handled as $g_i(x) \leq 0$ and $h_i(x) = 0$, and which constraints are handled by the set X. (An appropriate selection of the set X must be made, depending on the nature of the problem.)

Geometric Interpretation

Consider the following primal problem P:

Primal Problem P

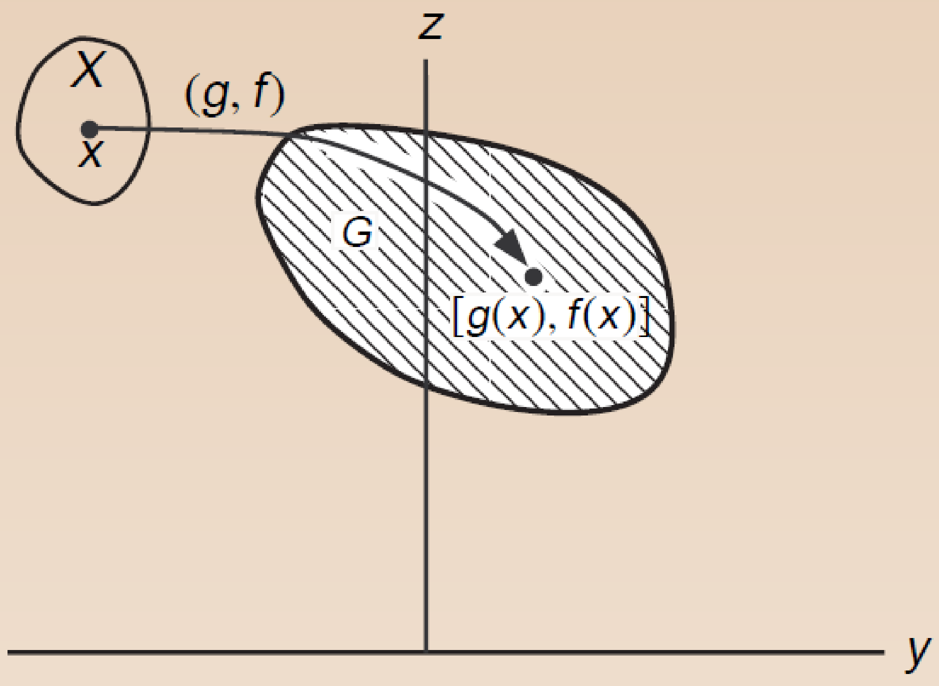
minimise $f(x)$,

subject to:

$g(x) \leq 0$,

$x \in X$,

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$.



Define the following set in $\mathbb{R}^2$:

$$G = \{(y, z) : y = g(x), z = f(x) \text{ for some } x \in X\},$$

that is, G is the image of X under the (g, f) map.

Geometric Interpretation

$$G = \{(y, z) : y = g(x), z = f(x) \text{ for some } x \in X\},$$

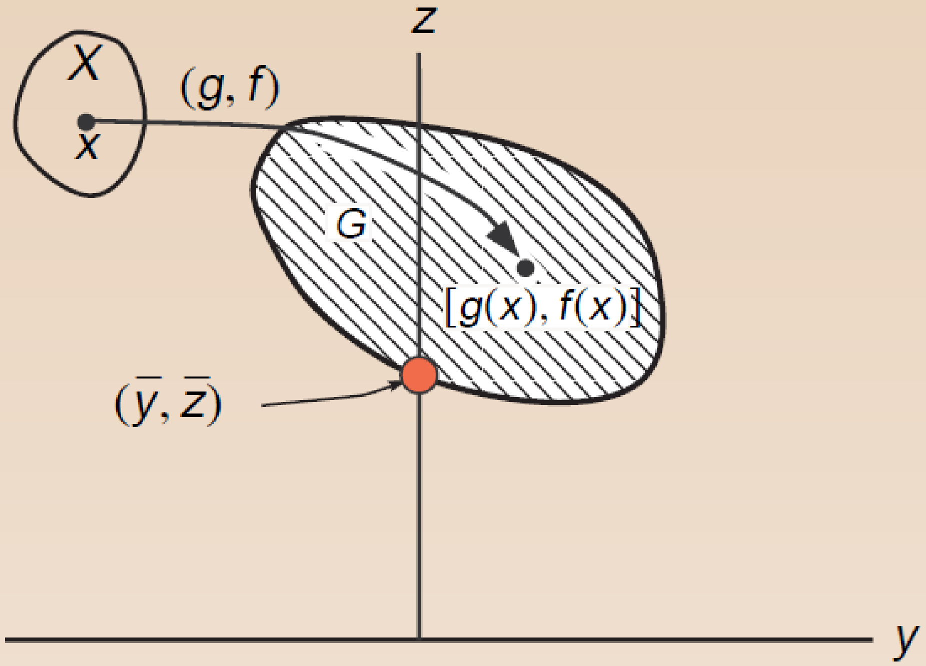
Primal Problem P

minimise $f(x)$,

subject to:

$g(x) \le 0$,

$x \in X$.



Then, the primal problem consists in finding a point in G with $y \le 0$ that has minimum ordinate z.

Obviously this point is $(\bar{y}, \bar{z})$.

Geometric Interpretation

Lagrangian Dual Problem D

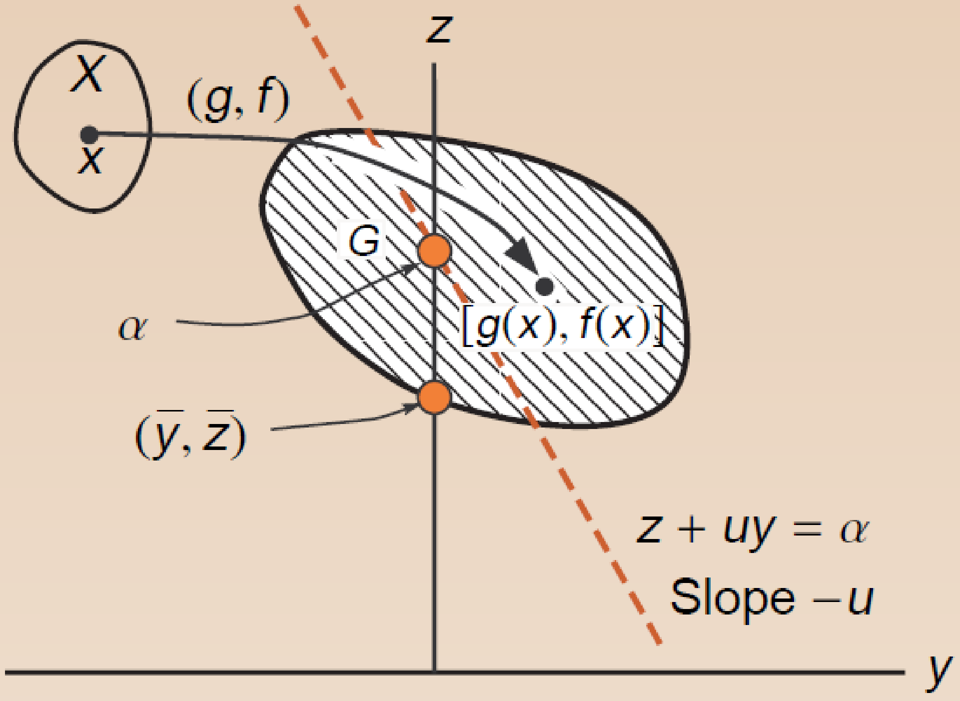
maximise $\theta(u)$,

subject to:

$u \geq 0$,

where (*Lagrangian dual subproblem*):

$\theta(u) = \inf\{f(x) + ug(x) : x \in X\}$.



Given $u \geq 0$, the Lagrangian dual subproblem is equivalent to minimise $z + uy$ over points (y, z) in G. Note that $z + uy = \alpha$ is the equation of a straight line with slope $-u$ that intercepts the z-axis at α.

Geometric Interpretation

Lagrangian Dual Problem D

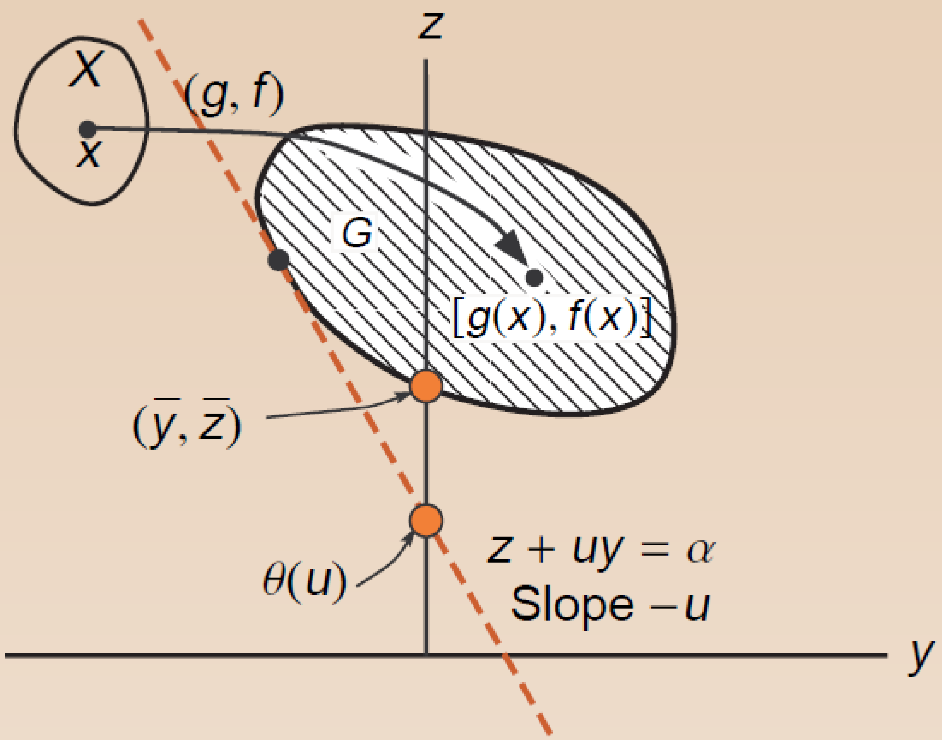
maximise $\theta(u)$,

subject to:

$u \geq 0$,

where (*Lagrangian dual subproblem*):

$\theta(u) = \inf\{f(x)+ug(x) : x \in X\}$.



In order to minimise $z + uy$ over G we need to move the line $z + uy = \alpha$ parallel to itself as far down as possible, whilst it remains in contact with G. The last intercept on the z-axis thus obtained is the value of $\theta(u)$ corresponding to the given $u \geq 0$.

Geometric Interpretation

Lagrangian Dual Problem D

maximise $\theta(u)$,

subject to:

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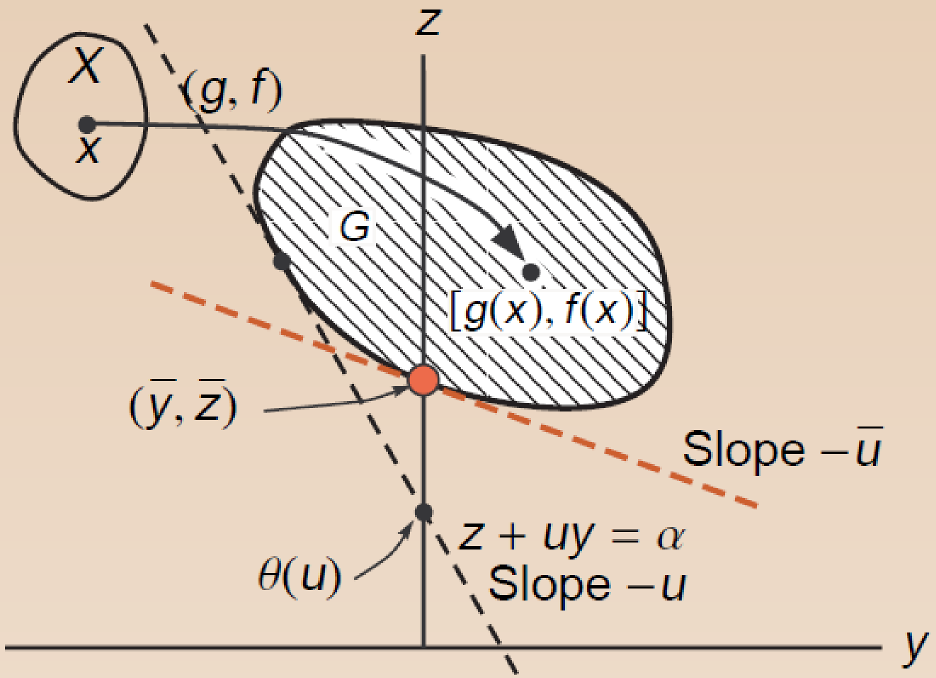
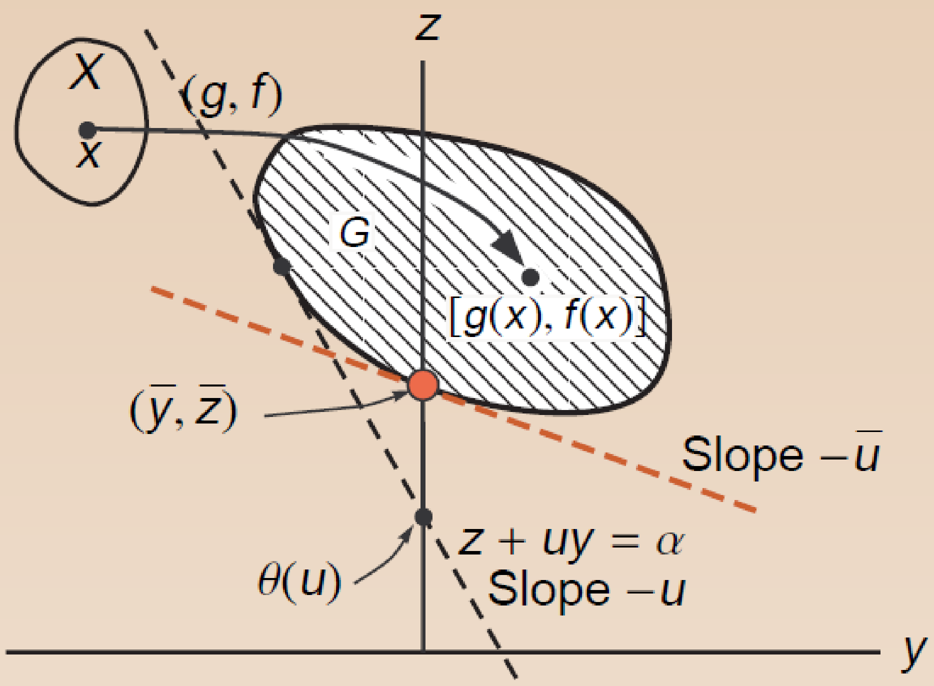
where (*Lagrangian dual subproblem*):

$\theta(u) = \inf\{f(x)+ug(x) : x \in X\}$.



Finally, to solve the dual problem, we have to find the line with slope $-u$ ($u \geq 0$) such that the last intercept on the z-axis, $\theta(u)$, is maximal. Such a line has slope $-\bar{u}$ and supports the set G at the point $(\bar{y}, \bar{z})$. Thus, the solution to the dual problem is $\bar{u}$, and the optimal dual objective value is $\bar{z}$.

Geometric Interpretation



- The solution of the Primal problem is $\bar{z}$, and the solution of the Dual problem is also $\bar{z}$.
- It can be seen that, in the example illustrated, the optimal primal and dual objective values are equal. In such cases, it is said that there is no *duality gap* (strong duality).

Weak Duality

The following result shows that the objective value of any feasible solution to the dual problem constitutes a lower bound for the objective value of any feasible solution to the primal problem.

Theorem (Weak Duality Theorem)

Consider the primal problem P *given by* (1) *and its Lagrangian dual problem* D *given by* (2). *Let* x *be a feasible solution to* P*; that is,* $x \in X$, $g(x) \leq 0$, *and* $h(x) = 0$. *Also, let* (u, v) *be a feasible solution to* D*; that is,* $u \geq 0$. *Then:*

$$f(x) \geq \theta(u, v).$$

$\circ$

Weak Duality

Proof.

We use the definition of θ given in (3), and the facts that $x \in X$, $u \geq 0$, $g(x) \leq 0$ and $h(x) = 0$. We then have

$$\begin{aligned}\theta(u, v) &= \inf\{f(\tilde{x}) + u^\top g(\tilde{x}) + v^\top h(\tilde{x}) : \tilde{x} \in X\} \\ &\leq f(x) + u^\top g(x) + v^\top h(x) \leq f(x),\end{aligned}$$

and the result follows. $\square$

Weak Duality

We then have, as a corollary of the previous theorem, the following result.

Corollary

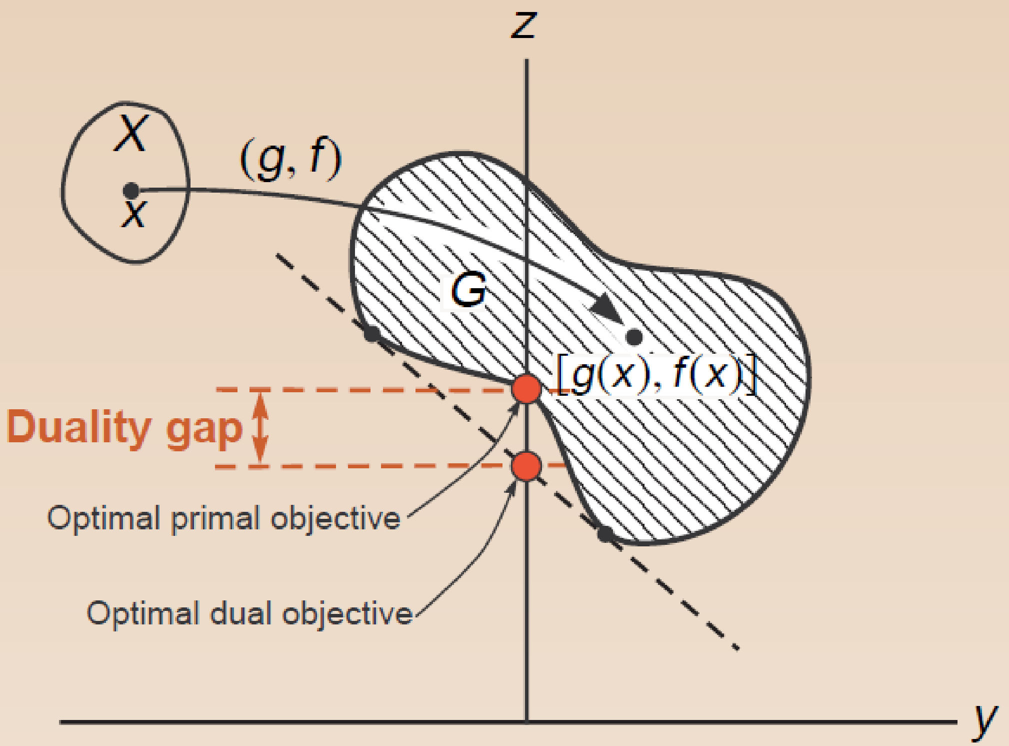
$$\inf\{f(x) : x \in X, g(x) \leq 0, h(x) = 0\} \geq \sup\{\theta(u, v) : u \geq 0\}.$$

Note from the corollary that the optimal objective value of the primal problem is greater than or equal to the optimal objective value of the dual problem.

If the inequality holds as a *strict* inequality, then it is said that there exists a *duality gap*.

Weak Duality

The figure shows an example of the geometric interpretation of the primal and dual problems.



Notice that, in the case shown in the figure, there exists a duality gap due to the nonconvexity of the set G.

We will see, in the **Strong Duality Theorem**, that if some suitable convexity conditions are satisfied, then there is no duality gap between the primal and dual optimisation problems.