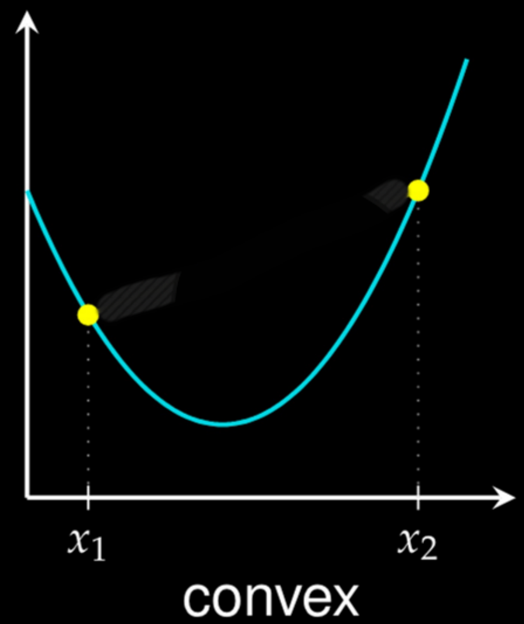
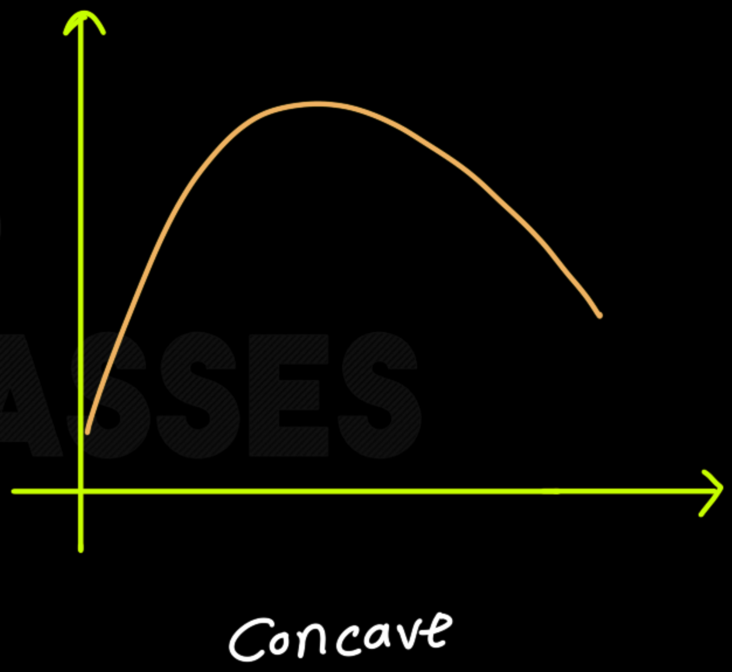
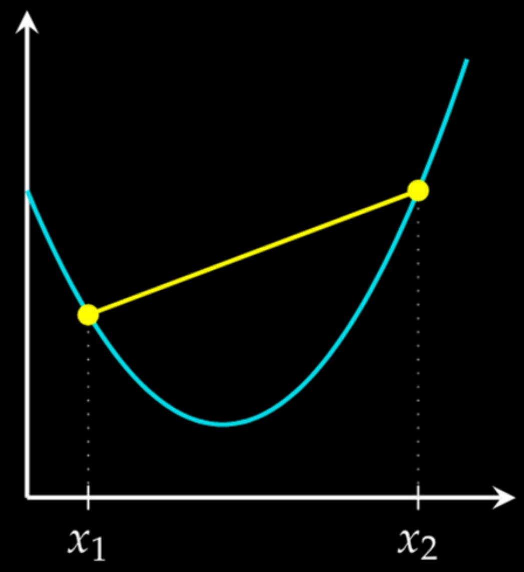
Convex Functions

convex

concave


x_1
x_2
convex


Concave



Secant line should be above the function.

at this point
it is $\Rightarrow$ concave

at this point $\Rightarrow$ convex

Sometimes convex, Sometimes concave.

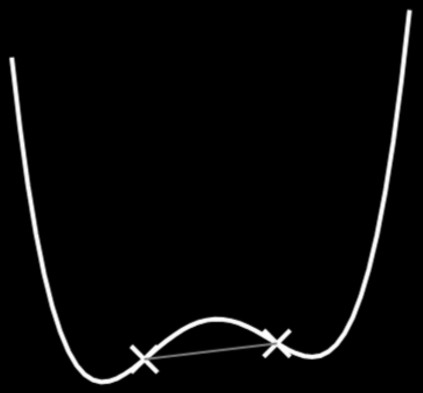
Hence, this function is neither convex nor concave is this line always above function? ⇒ NO.

at this point it is ⇒ concave

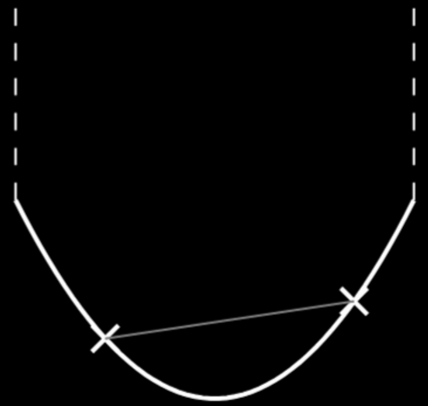
at this point ⇒ convex

Sometimes convex, sometimes concave.

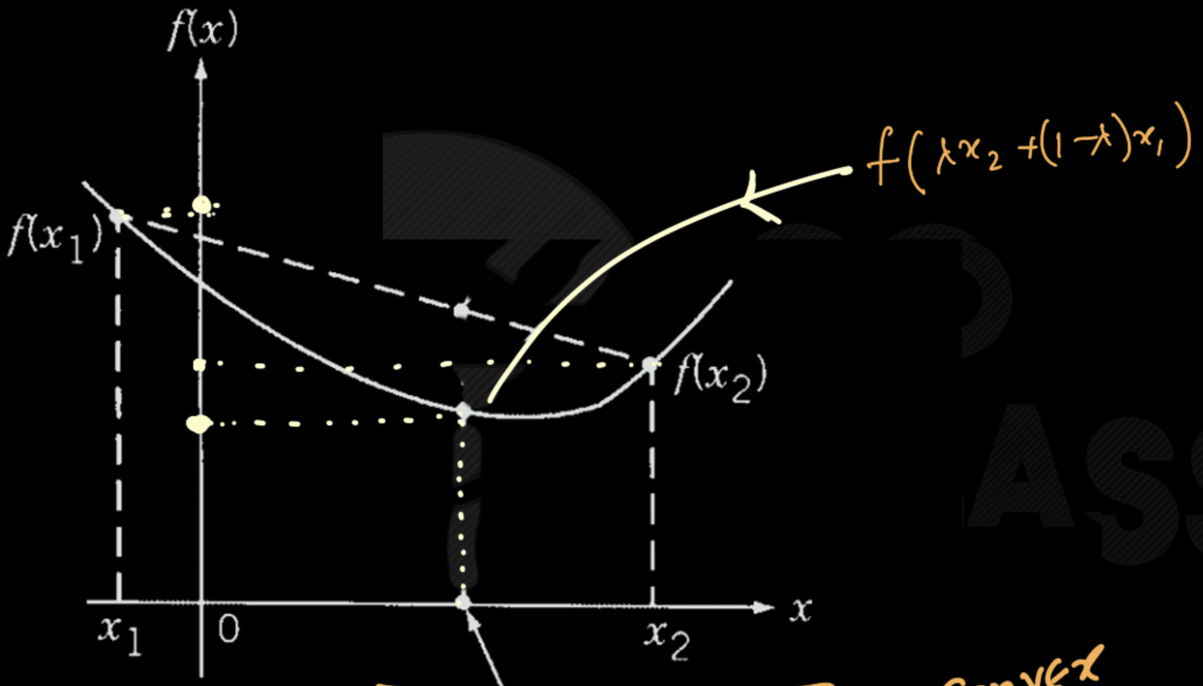
Hence, this function is neither convex nor concave



nonconvex function



convex function



$\boxed{\lambda x_2 + (1-\lambda)x_1}$ ← Convex combination

a) $f(\lambda x_2 + (1-\lambda)x_1)$

b) $f(\lambda x_1 + (1-\lambda)x_2)$

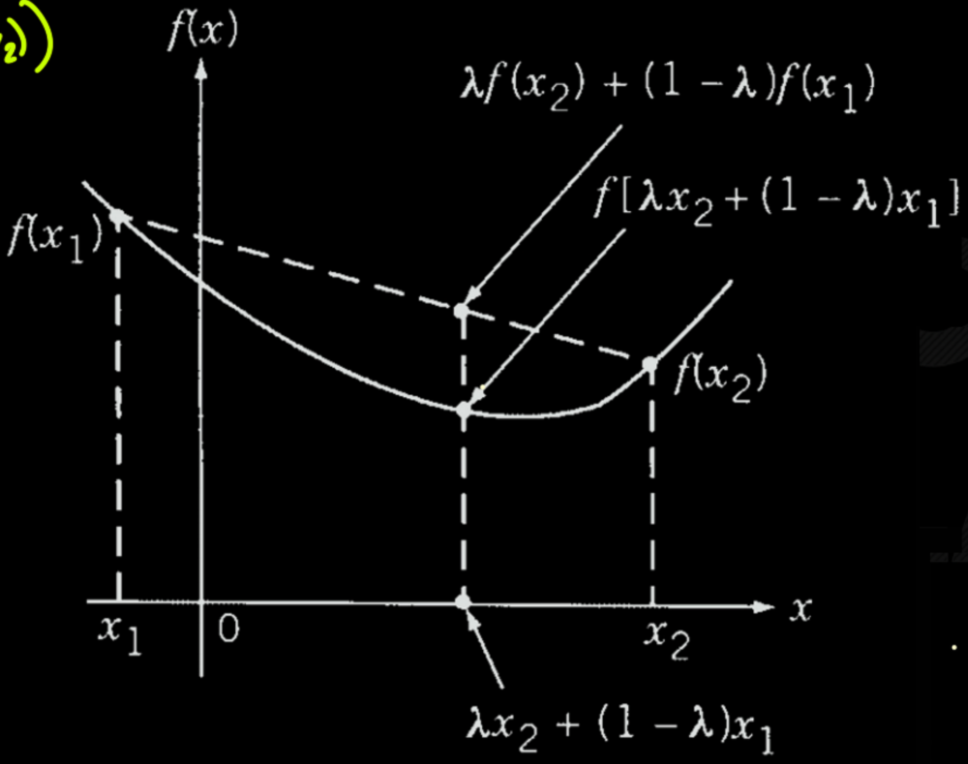
c) $f\left(\frac{x_1+x_2}{2}\right)$

d) $f(x_1 + \lambda x_2)$

$\lambda \in (0, 1)$

$(x_1, f(x_1))$

$(x_2, f(x_2))$

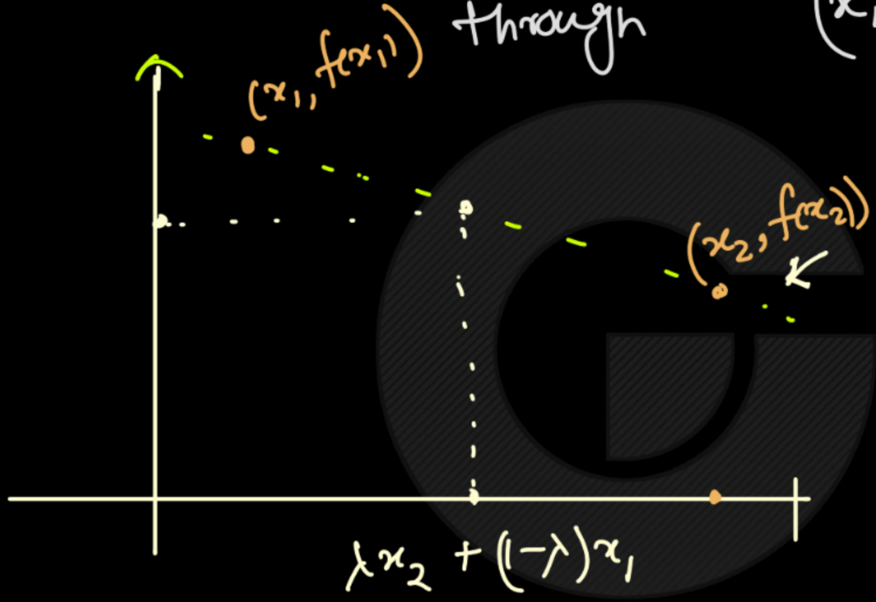


$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781119454816

Equation of a line passing through (x_1, y_1) (x_2, y_2)



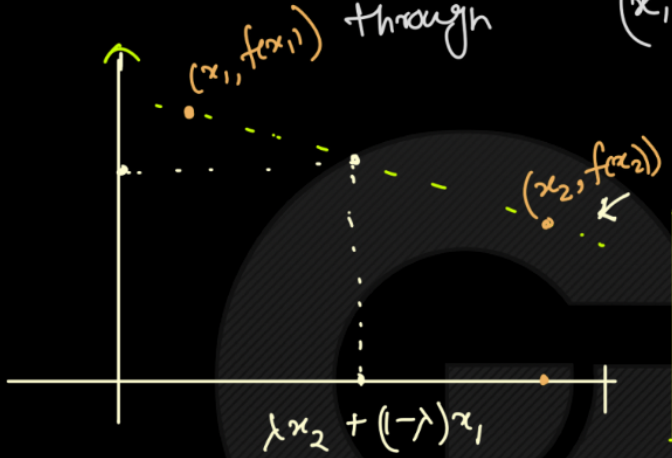
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\boxed{y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)}$$

put $x = \lambda x_2 + (1-\lambda)x_1$

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}\lambda(x_2 - x_1)$$

Equation of a line passing through (x_1, y_1) (x_2, y_2)



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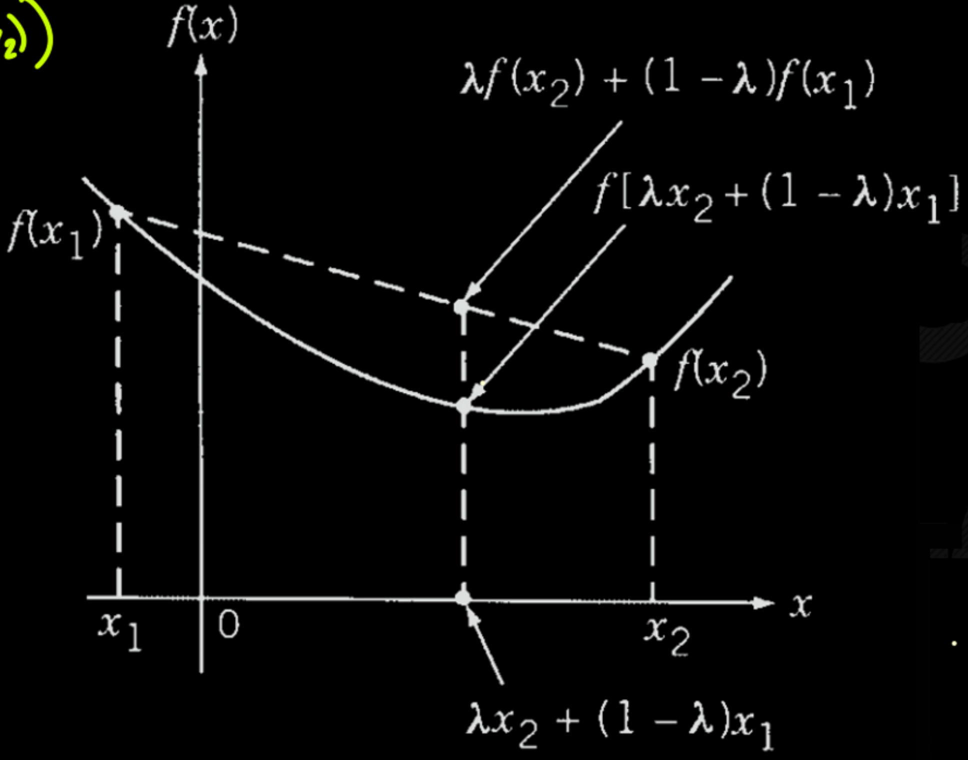
put $x = \lambda x_2 + (1-\lambda)x_1$

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{\cancel{x_2 - x_1}} \lambda \cancel{(x_2 - x_1)}$$

$$y = \lambda f(x_2) + (1-\lambda) f(x_1)$$

$(x_1, f(x_1))$

$(x_2, f(x_2))$



$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

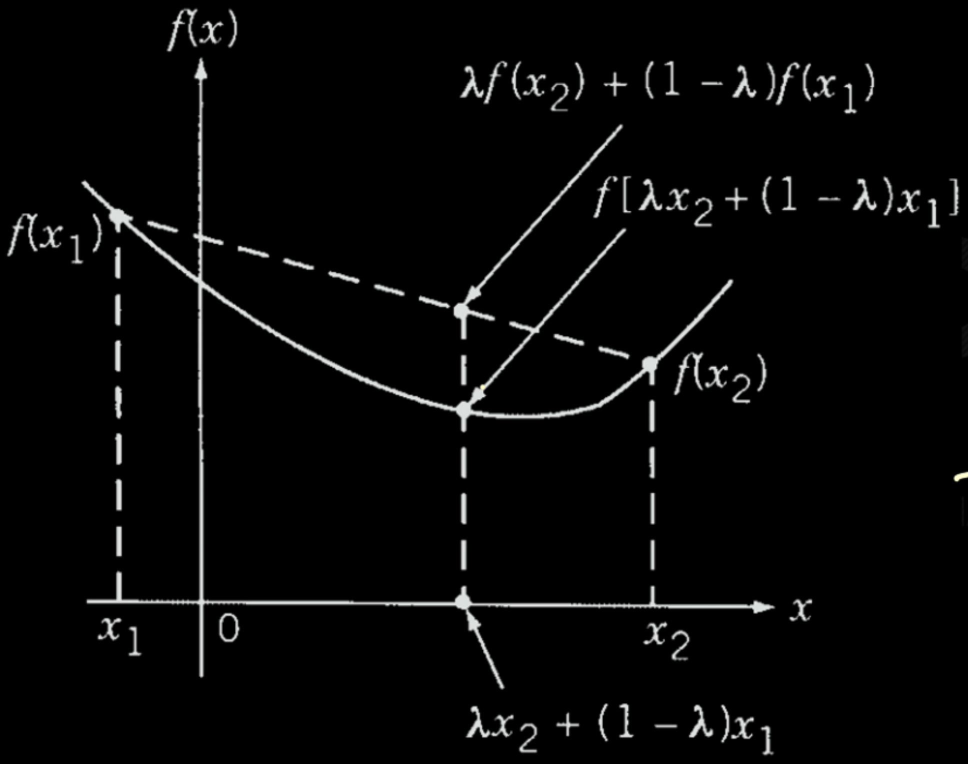
$$x = \lambda x_2 + (1-\lambda)x_1$$

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{\cancel{x_2 - x_1}}\lambda\cancel{(x_2 - x_1)}$$

$$f(\lambda x_2 + (1-\lambda)x_1) \leq \underline{\lambda f(x_2) + (1-\lambda)f(x_1)}$$

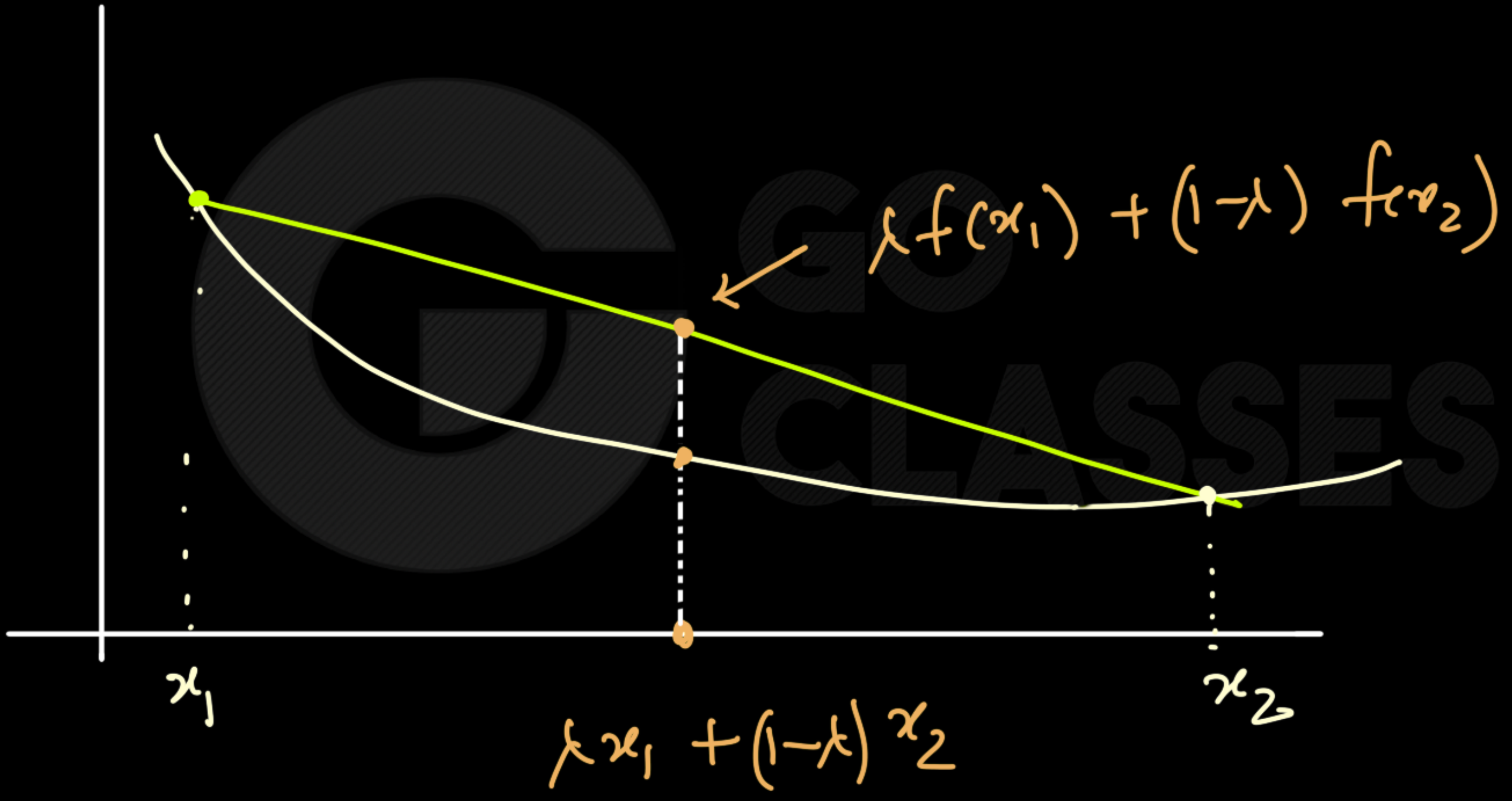
$\forall x_1, x_2$

https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781119454816

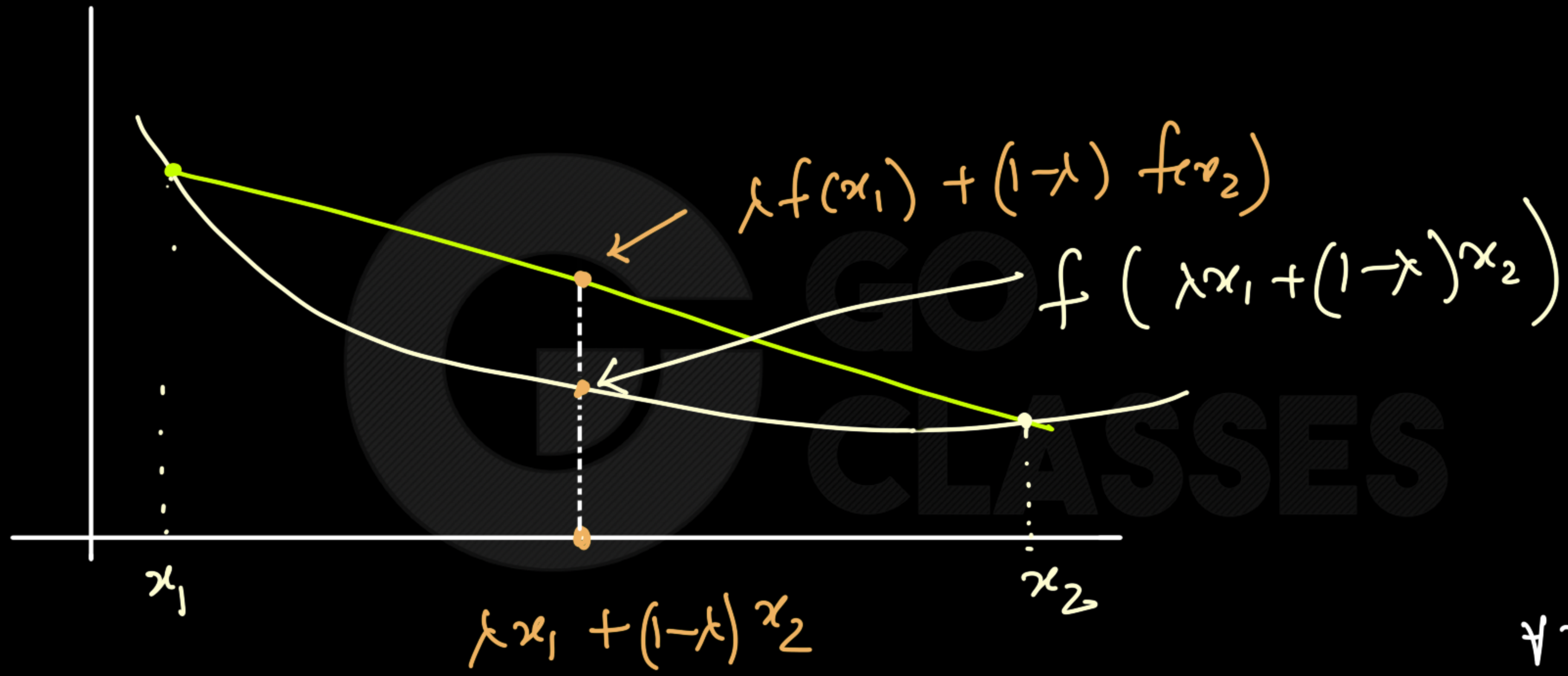


$$f(\lambda x_2 + (1-\lambda)x_1) \leq \lambda f(x_2) + (1-\lambda) f(x_1)$$

$\forall x_1, x_2$



$\lambda f(x_1) + (1-\lambda) f(x_2)$
x_1
x_2
$\lambda x_1 + (1-\lambda) x_2$



$\forall x_1, x_2$

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \quad \lambda \in [0,1]$$

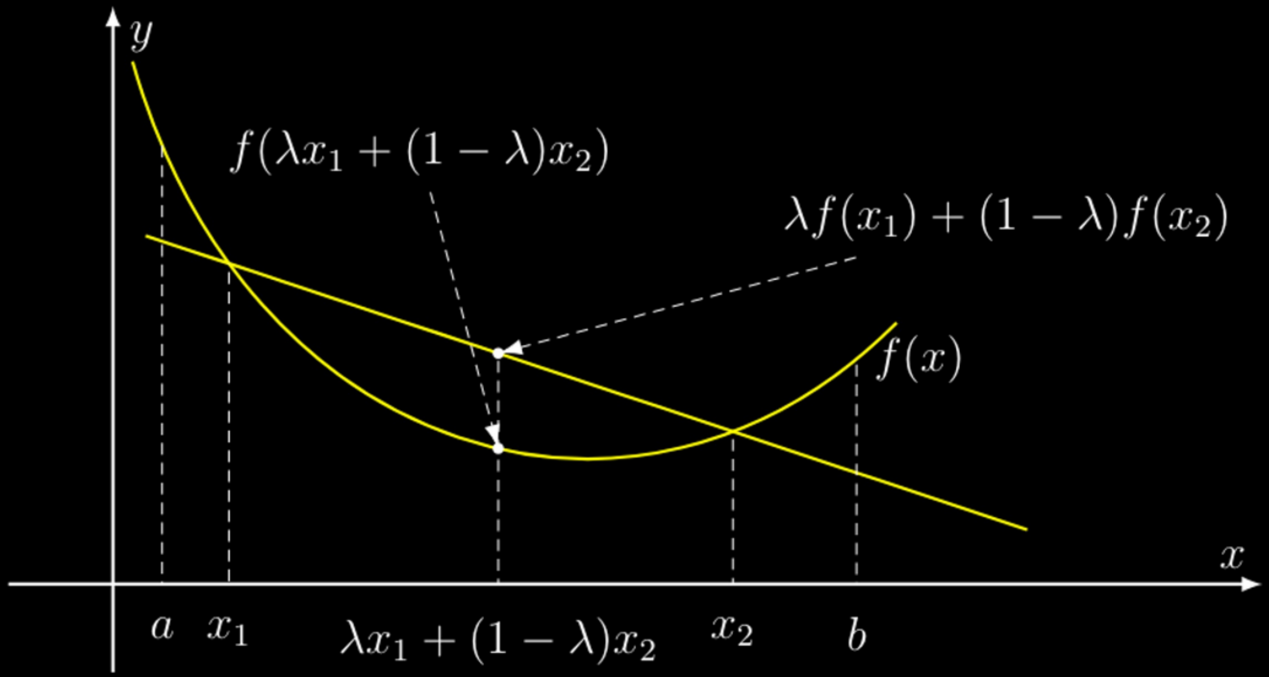
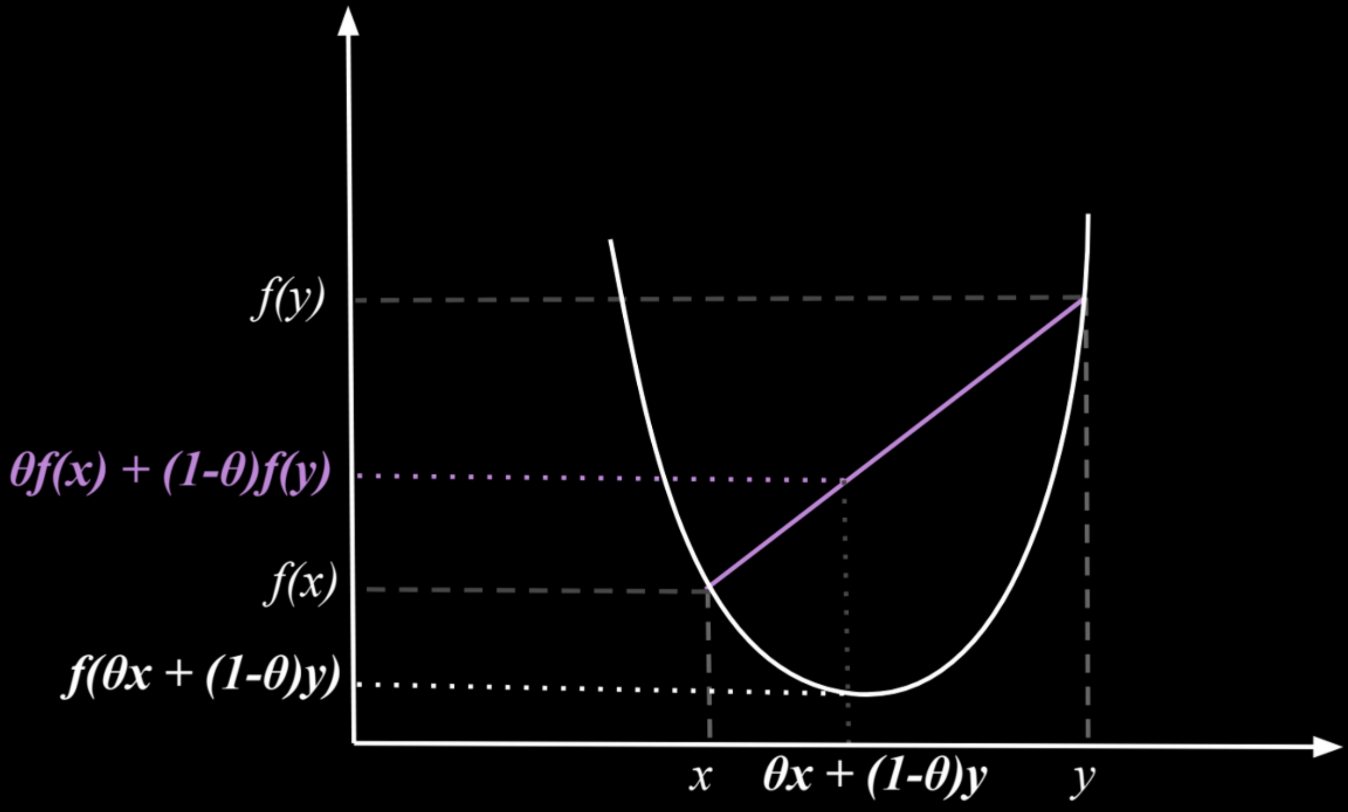


Figure 2: A convex function $f(x)$

Function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

https://www.control.lth.se/fileadmin/control/Education/EngineeringProgram/FRTN50/Lectures/cvxfcn.pdf


f(y)
θf(x) + (1-θ)f(y)
f(x)
f(θx + (1-θ)y)
x
θx + (1-θ)y
y

Convexity: Zero-order condition

A real-valued function is **convex** *if*

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y),$$

for all $x,\ y \in \mathbb{R}^n$ *and all* $0 \leq \theta \leq 1$.

- Function is *below the chord* from x to y.

is this a convex function

is this a convex function

"yes"

but it is not strict convex.

1.2 Examples of univariate convex functions

This is a selection from [1]; see this reference for more examples.

- e^{ax}
- $-\log(x)$
- x^a (defined on $\mathbb{R}_{++}$), $a \geq 1$ or $a \leq 0$
- $-x^a$ (defined on $\mathbb{R}_{++}$), $0 \leq a \leq 1$
- $|x|^a$, $a \geq 1$
- $x \log(x)$ (defined on $\mathbb{R}_{++}$)

$x^2 + y^2$ convex

$x^2 - y^2$ × (neither)

$-x^2 - y^2$ is concave

f is convex $\Longleftrightarrow$ $-f$ is concave

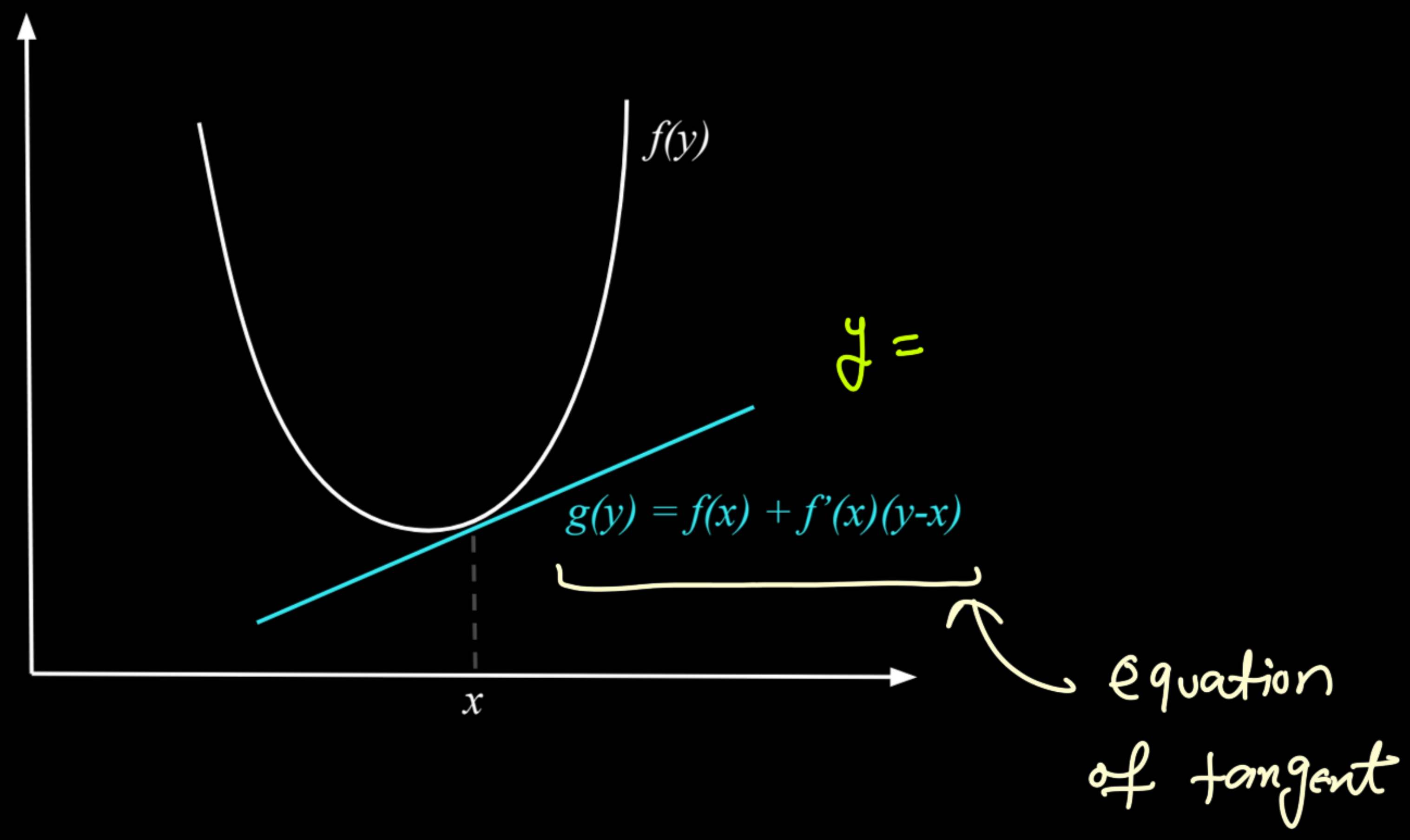
Convexity: First-order condition

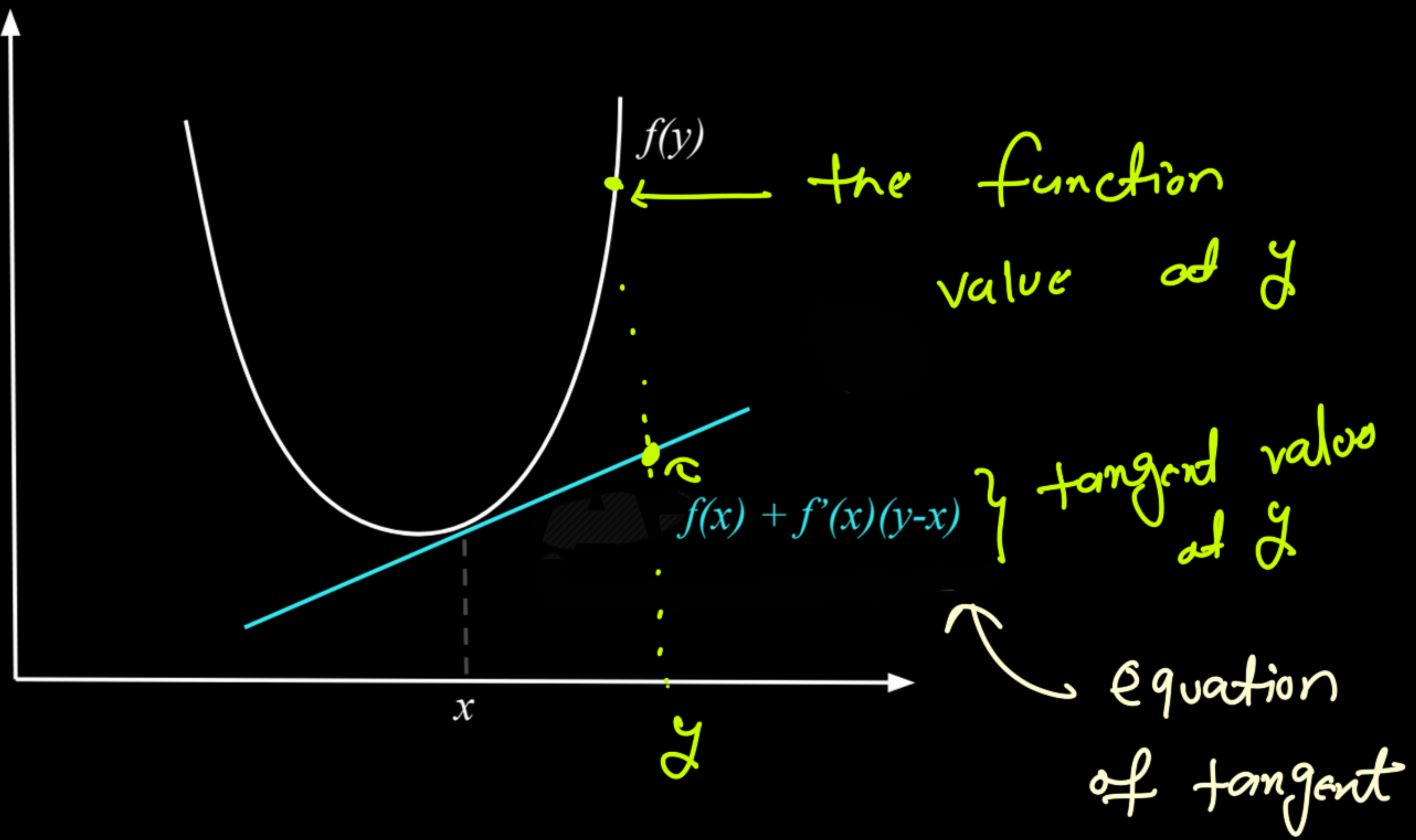
A real-valued *differentiable* *function is* **convex** *iff*

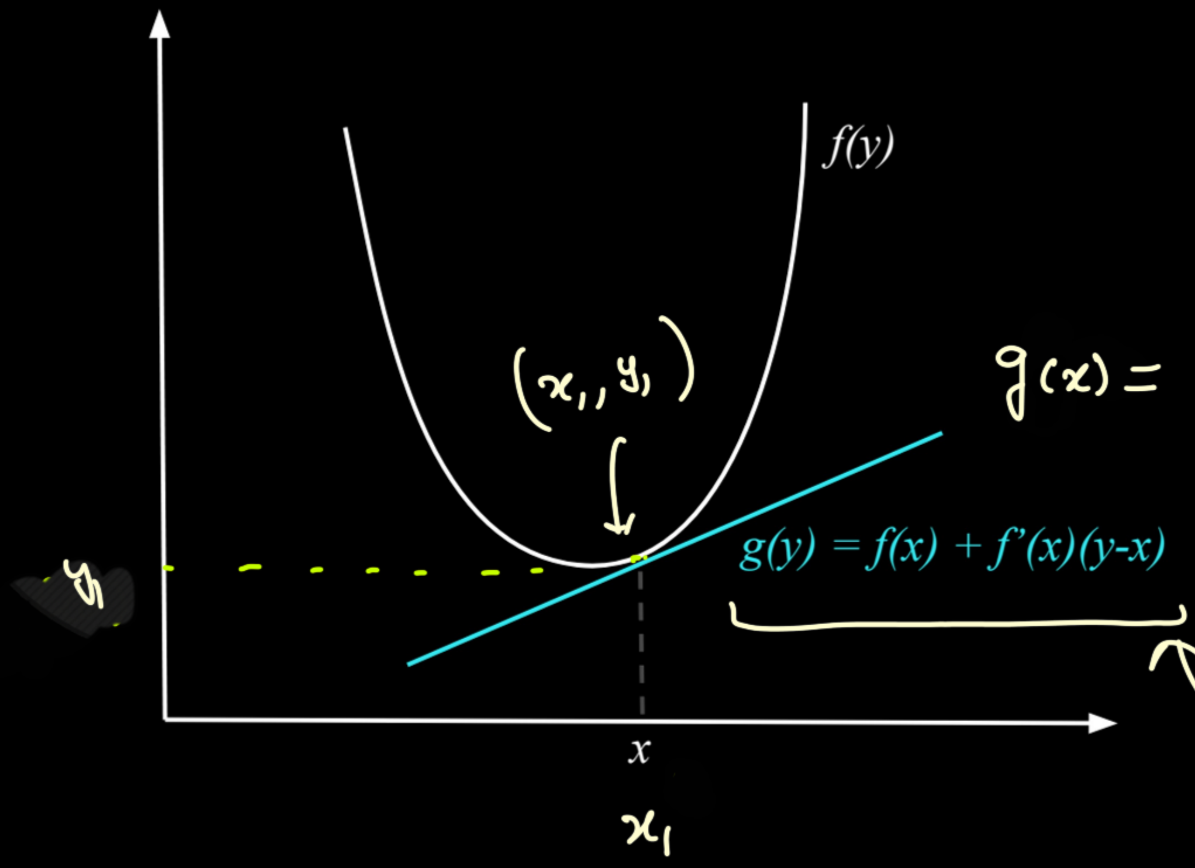
$$f(y) \geq f(x) + \nabla f(x)^T (y - x),$$

for all $x,\ y \in \mathbb{R}^n$.

- The function is globally *above the tangent* at y.
- Show that any stationary point is a global minimum.


f(y)
y =
g(y) = f(x) + f'(x)(y-x)
x
equation of tangent


f(y)
the function value at y
f(x) + f'(x)(y-x)
tangent value at y
equation of tangent
x
y



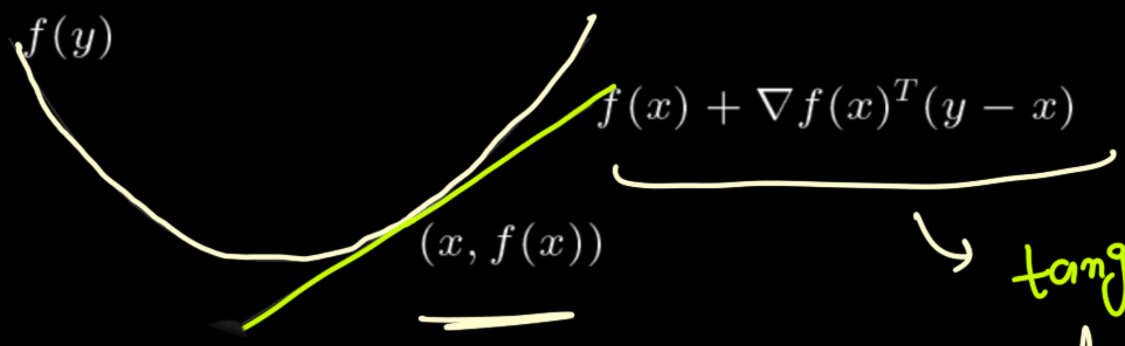
$$g(x) = y_1 + f'(x_1)(x - x_1)$$

equation of tangent

$$\underbrace{f(y)}_{\text{function value at } y} \geq \underbrace{f(x) + \nabla f(x)^T (y - x)}_{\text{tangent value at } y},$$

- **First-order condition**: a differentiable f with convex domain is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \forall x, y \in \mathbf{dom}\, f$$



tangent value at any point y

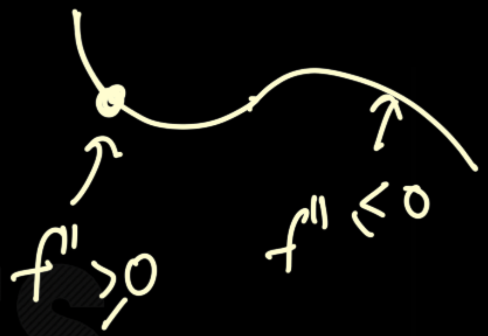
Intuitively, this says that for a convex function, a tangent of its graph at any point must lie below the graph.

Testing for convexity of a single variable function

$f''(x) \geq 0 \implies$ Concave up

$\downarrow$

Convex



$f''(x) \leq 0 \implies$ Concave down (concave)

Testing for convexity of a single variable function

- A function is convex if its slope is non decreasing or $\partial^2 f / \partial x^2 \geq 0$. It is strictly convex if its slope is continually increasing or $\partial^2 f / \partial x^2 > 0$ throughout the function.

Convexity: Second-order condition

A real-valued twice-differentiable function is **convex** *iff*

$$\underbrace{\nabla^2 f(x)}_{\text{Hessian}} \succeq 0$$

for all $x \in \mathbb{R}^n$.

- The function is *flat or curved upwards* in every direction.

Convexity: Second-order condition

A real-valued twice-differentiable function is **convex** *iff*

$$\underbrace{\nabla^2 f(x)}_{\text{Hessian}} \succeq 0 \quad \text{Symbol of PSD}$$

for all $x \in \mathbb{R}^n$.

- The function is *flat or curved upwards* in every direction.

$x_i \geq 0$

$d^T \nabla^2 f(x)\, d \geq 0$

Meaning of Eigenvalues

If the Hessian at a given point has all positive eigenvalues, it is said to be a positive-definite matrix. This is the multivariable equivalent of “concave up”.

If all of the eigenvalues are negative, it is said to be a negative-definite matrix. This is like “concave down”.

https://web.stanford.edu/group/sisl/k12/optimization/MO-unit4-pdfs/4.10applicationsofhessians.pdf

Meaning of Eigenvalues

If either eigenvalue is 0, then you will need more information (possibly a graph or table) to see what is going on.

And, if the eigenvalues are mixed (one positive, one negative), you have a saddle point:

Here, the graph is concave up in one direction and concave down in the other.

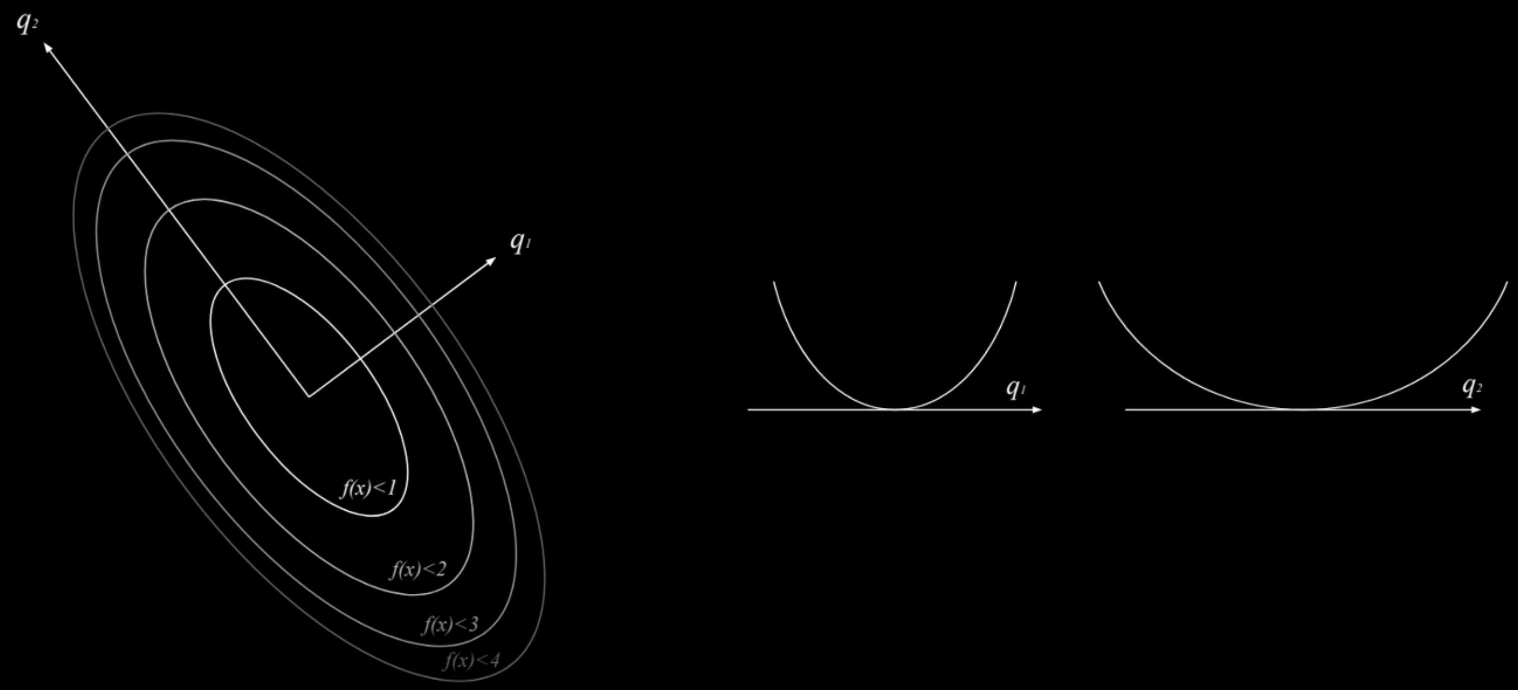
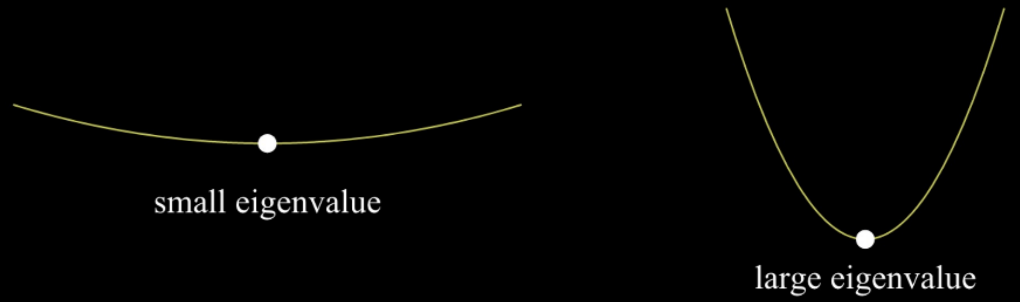


Figure 7: **Left:** Looking along the principle directions of the quadratic function $f(x) = \frac{1}{2}x^\top Hx$, we see that the curve changes faster along q_1 than q_2. Here[4], $\lambda_1 > \lambda_2$. **Right:** Cross-sections along q_1 and q_2, showing that f has higher curvature along q_1 than q_2.

https://mitliagkas.github.io/ift6085-2020/ift-6085-lecture-2-notes.pdf

Calculus

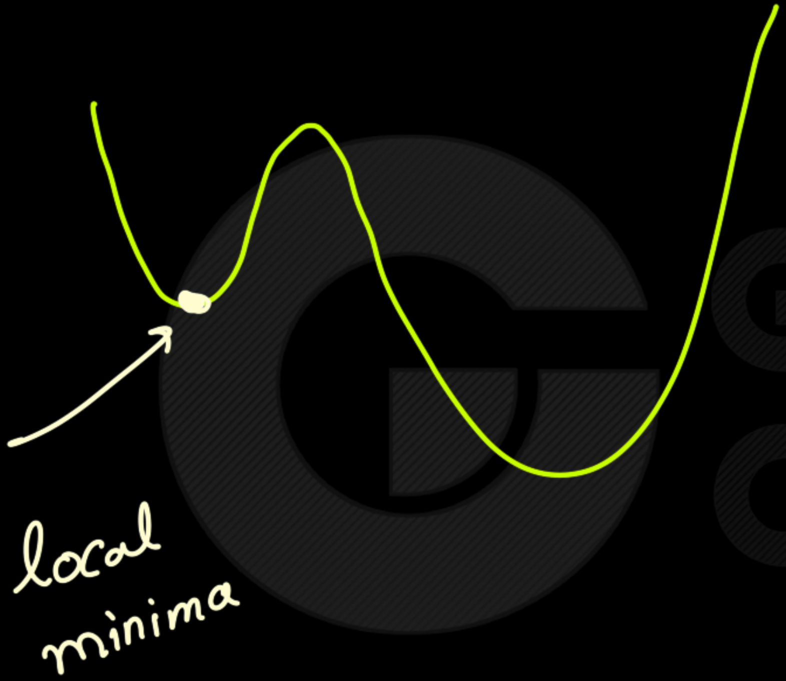
- The function $f(x)$ has a minimum at x_* (the minimizer) if the Hessian index is zero:



https://trygvehelgaker.no/Presentations/ESQC_Optimization_2009.pdf

Key property of Convex Functions:

- All local minima are global minima.



local minima + convex $\Rightarrow$ global minima

Question:

Consider the function $f(\mathbf{x}) = f(x_1, x_2) = (x_1 + x_2^2)^2$.

Derive the Hessian of $f(x)$.

At the point $\mathbf{x}_0 = (0, 1)^T$, we consider the search direction $\mathbf{p} = (1, -1)^T$. Show that p is a descent direction.

https://www.cs.cmu.edu/~ggordon/10725-F12/hws/midterm_sol.pdf

$$\nabla f(x) = \begin{pmatrix} 2x_1 + 2x_2^2 \\ 4x_1x_2 + 4x_2^3 \end{pmatrix}.$$

$$f(\mathbf{x}) = f(x_1, x_2) = (x_1 + x_2^2)^2.$$

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 4x_2 \\ 4x_2 & 4x_1 + 12x_2^2 \end{pmatrix}.$$

$8x_1 + 24x_2^2 - 16x_2^2$

$8x_1 + 8x_2^2$

$x_0 = (0, 1)$

$\nabla f(\mathbf{x}_0) = (2, 4)^T.$
$\nabla f(\mathbf{x}_0)^T\mathbf{p} = 2 - 4 = -2 < 0.$
Therefore, $\mathbf{p}$ is a descent direction at $\mathbf{x}_0$.

$(0, 1) \to (1, -1)$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Question:

Problem 1 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^4y^2 + x^4 - 2x^3y - 2x^2y - x^2 + 2x + 2$$

Determine whether the function f is convex or not.
(5 points)

https://wiki.math.ntnu.no/_media/tma4180/2018v/

Solution: For, example, one can argue like this. A twice differentiable function is convex iff its Hessian matrix is positive semidefinite. For our function, however, we have:

$$\nabla^2 f(0,0) = \nabla^2[-x^2 + 2x + 2]|_{(x,y)=(0,0)} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix},$$

where we ignore all polynomial terms of degree 3 or higher since their second derivatives are 0 at $(0,0)$. Thus $\nabla^2 f(0,0)$ is negative semidefinite (the eigenvalues of a diagonal matrix are elements on the diagonal), and the function is not convex.

at $(0,0)$ the function is concave but we can not say at all points

Question:

Show that the function $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$,

$$F(x,y) = x^4 + y^4 - 4xy + x^2 + y^2$$

is non-convex.

$$\nabla f_{(0,0)} \begin{bmatrix} -4y + 2x \\ -4x + 2y \end{bmatrix} \quad H_{(0,0)} = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$$

$\downarrow$

Not PSD at $(0,0)$

https://wiki.math.ntnu.no/_media/tma4180/2016v/exam_summer16_lf.pdf

Solution

$$\nabla^2 F(x,y) = \begin{pmatrix} 12x^2+2 & -4 \\ -4 & 12y^2+2 \end{pmatrix},$$

which is not positive semi-definite for $(x,y) = (0,0)$, as $\det \nabla^2 F(0,0) = -12 < 0$. Thus F is non-convex.

Question:

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = y^4 + 3y^2 - 4xy - 2y + x^2.$$

H.W.

We start by computing the gradient of f, which is

$$\nabla f(x,y) = \begin{pmatrix} -4y + 2x \\ 4y^3 + 6y - 4x - 2 \end{pmatrix},$$

In order to check whether the function f is convex, we compute its Hessian as

$$\nabla^2 f(x,y) = \begin{pmatrix} 2 & -4 \\ -4 & 12y^2 + 6 \end{pmatrix}.$$

At the point $(x,y) = (0,0)$ we obtain

$$\nabla^2 f(0,0) = \begin{pmatrix} 2 & -4 \\ -4 & 6 \end{pmatrix},$$

the determinant of which is negative. Thus $\nabla^2 f(0,0)$ is not positive semi-definite, and thus f is non-convex.

Question:

H.W.

the product $f(u,v) = uv$, with $\mathbf{dom}\, f = \mathbf{R}^2$

☐ convex ☐ concave ☐ neither

https://sharif.edu/~mtefagh/cvx/solutions/midterm_18s_sol.pdf

Neither. The Hessian $\nabla f^2(u, v)$ has a positive and negative eigenvalue, thus this function is neither convex nor concave $\cdot$

Question:

For each of the following functions determine whether it is convex or concave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

H.W.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.
Solution. Strictly convex

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.
Solution. The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is neither positive semidefinite nor negative semidefinite. Therefore, f is neither convex nor concave.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.
Solution. The Hessian of f is

$$\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} 2/(x_1^2) & 1/(x_1 x_2) \\ 1/(x_1 x_2) & 2/x_2^2 \end{bmatrix} \succeq 0$$

Therefore, f is convex

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.
Solution. The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$

which is not positive or negative semidefinite. Therefore, f is not convex or concave.

Question:

Consider the loss function $L : \Re^2 \Rightarrow \Re$ where

$$L(x) = a(x_1 + x_2)^2 + b(x_1 - x_2)^2$$

where $a > 0$ and $b > 0$ are scalar parameters and $x = [x_1, x_2]$.

Q5.1 Miminum
(5 Points)
Prove that L convex.
Hint: You can use the "linear transform" property of convexity from the notes.

Q5.2
(5 Points)
Does $L(x)$ take on a global minimum?
Justify your answer, and if the answer is "yes", then give its value?

https://engineering.purdue.edu/DeepLearn/exams/solutions/2021-exam1-solution.pdf

Proof:
Let $s : \Re \Rightarrow \Re$ denote the square function where $s(y) = y^2$ for $y \in \Re$, and $f : \Re^2 \Rightarrow \Re$ be

$$f(x) = as(x_1) + bs(x_2) = ax_1^2 + bx_2^2$$

$a, b > 0$

Given $a > 0, b > 0$, $f(x)$ is a convex function since s is convex and f is a linear combination of the convex function.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, the linear transform of the convex function $f(x)$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ A

$$\underbrace{a g_1 + b g_2}$$

$$L(x) = f(Ax) = f(\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}) = f(\begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}) = a(x_1 + x_2)^2 + b(x_1 - x_2)^2$$

$\downarrow$ convex

is convex by "linear transformation" property of convexity.

Q5.2

$L(x)$ takes on a global minimum because it's continuously differentiable and convex on $\Re^2$. By solving $\Delta_x L(x) = 0$, we know the function achieves global minimum value 0 at $[x_1, x_2] = [0, 0]$.

$f(x)$ $f(Ax)$

$a g_1(x) + b g_2(x)$ convex if

$a, b > 0$ g_1, g_2 are convex

$f(x) \Rightarrow f(Ax)$

convex convex

Question:

Let $f : \Re^N \Rightarrow \Re$ be a convex function with local minimum $x^* \in \Re^N$.

Q4.1 Miminum

(5 Points)

Is x^* a global minimum?

1 — yes

Q4.2

(5 Points)

Is x^* a unique local minimum?

2 — NO

$f(\ ) = c$

Solution:

Q4.1

Yes, x^* is a global minimum.

Q4.2

No, x^* is not necessarily a unique local minimum.

Counter example:
The local minimum of a convex function is not necessarily unique.

For example, considering the constant function

$$\boxed{f(x) = c}$$

where $x \in \Re^N$ and $c \in \Re$ is a constant, the local minimum exists but not unique because every point is a local minimum.

Question: True/False

Problem 3 (Convex Function): Given function $f(x, y) = x^T Ax + 2x^T By + y^T Cy$, where matrices $A, C \in S^n$ and $x, y \in R^n$, then $f(x, y)$ is concave if and only if the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ is negative semidefinite.

Problem 3 (Convex Function): Given function $f(x, y) = x^T Ax + 2x^T By + y^T Cy$, where matrices $A, C \in S^n$ and $x, y \in R^n$, then $f(x, y)$ is concave if and only if the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ is negative semidefinite.

T/F: **T**. We can prove the statement via conversion to matrix formulation.

Question:

Consider the following function and determine if the function is convex.

$$f(x) := \frac{1}{2}\|Ax - b\|_2^2, \text{ where } A \in \mathbb{R}^{m \times n} \text{ and } b \in \mathbb{R}^m.$$

$$\nabla f(x) = A^T(Ax - b).$$

$$\nabla^2 f(x) = A^T A \text{ for all } x,\ f \text{ is convex.}$$

$$\begin{aligned}\mathbf{x}^T A^T A\mathbf{x} &= (A\mathbf{x})^T A\mathbf{x},\\ &= (A\mathbf{x}) \cdot (A\mathbf{x}),\\ &= \|A\mathbf{x}\|_2^2 \geq 0.\end{aligned}$$

Question:

Show that the functions

$$f(x_1, x_2) = x_1^2 + x_2^3, \quad \text{and} \quad g(x_1, x_2) = x_1^2 + x_2^4$$

both have a critical point at $(x_1, x_2) = (0, 0)$ and that their associated Hessians are positive semi-definite. Then show that $(0, 0)$ is a local (global) minimizer for g but is not a local minimizer for f.

https://sites.math.washington.edu//~burke/crs/408/samples/feguide-s14-sol.pdf

Solution

Both f and g are completely separable. The origin is the unique critical point for both functions. However,

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & 0 \\ 0 & 6x_2 \end{bmatrix} \qquad \nabla^2 g(x_1, x_2) = \begin{bmatrix} 2 & 0 \\ 0 & 12x_2^2 \end{bmatrix}.$$

Clearly, $\nabla^2 f$ is not positive semi-definite for $x_2 < 0$, so f is not convex, while $\nabla^2 g$ is everywhere positive semi-definite and so is convex. Thus, f has no local (global) optima, while the origin is a global minimizer of g.

Question:

Let λ be a constant, $x \in \mathbb{R}^2$ and define function

$$f(x) = x_1^2 - 4x_1x_2 + 3x_2^2 + \lambda||x||_2^2.$$

What is the minimum λ to make $f(x)$ a convex function?

(A) $2 - \sqrt{5}$

(B) $2 + \sqrt{5}$

(C) $-2 + \sqrt{5}$

(D) $-2 - \sqrt{5}$

https://courses.cs.washington.edu/courses/cse446/24sp/sections/05/practice_midterm_solutions.pdf

Let λ be a constant, $x \in \mathbb{R}^2$ and define function

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(B) $2 + \sqrt{5}$

(C) $-2 + \sqrt{5}$

(D) $-2 - \sqrt{5}$

Solution:

The solution is (C). $\nabla_x^2(x_1^2 - 4x_1x_2 + 3x_2^2) = \begin{bmatrix} 2 & -4 \\ -4 & 6 \end{bmatrix}$, so its eigenvalues $\lambda_{1,2}$ are $4 \pm 2\sqrt{5}$. To make f convex, we need $2\lambda + \min\{\lambda_1, \lambda_2\} \geq 0$.

Question:

Assume that f is a convex function on $\mathbb{R}^n$, and that $L : \mathbb{R}^m \to \mathbb{R}^n$ is affine, which means that it has the form

$$L(x) = Bx + c$$

where B is an $n \times m$ matrix and $c \in \mathbb{R}^n$.

Let $g(x) = f(L(x))$, and prove that g is a convex function on $\mathbb{R}^m$.

https://www.math.utoronto.ca/adkins/APM462Summer2016/Midterm_APM462_S2016.pdf

Solution. Let $\alpha, \beta \geq 0$, and $\alpha + \beta = 1$. Note that $L(\alpha x + \beta y) = B(\alpha x + \beta y) + c = \alpha Bx + \beta By + c = \alpha(Bx + c) + \beta(By + c) = \alpha L(x) + \beta L(y)$.

By convexity of f we have, $g(\alpha x+\beta y) = f(L(\alpha x+\beta y)) = f(\alpha L(x)+\beta L(y)) \leq \alpha f(L(x))+\beta f(L(y)) = \alpha g(x) + \beta g(y)$.

Question:

Consider the function f defined on $\mathbb{R}^3$, given by

$$f(x, y, z) = -e^{x+y+z}$$

Which statement is true?

(a) f is a convex function but not a concave function
(b) f is a convex function and a concave function
(c) f is not a convex function but a concave function
(d) f is neither a convex nor a concave function

https://www.dr-eriksen.no/teaching/GRA6035/Exams/midterm-2012-02.pdf

We compute the Hessian matrix of $f(x, y, z) = -e^{x+y+z}$: First, we compute the first order partial derivatives

$$f'_x = f'_y = f'_z = -e^{x+y+z}$$

and then we compute the second order partial derivatives and form the Hessian matrix

$$H(f) = \begin{pmatrix} -e^{x+y+z} & -e^{x+y+z} & -e^{x+y+z} \\ -e^{x+y+z} & -e^{x+y+z} & -e^{x+y+z} \\ -e^{x+y+z} & -e^{x+y+z} & -e^{x+y+z} \end{pmatrix}$$

The principal minors are $\Delta_1 = -e^{x+y+z}, -e^{x+y+z}, -e^{x+y+z}$, $\Delta_2 = 0, 0, 0$ and $\Delta_3 = 0$. Since $-e^{x+y+z} < 0$, it follows that f is concave but not convex. The correct answer is alternative **C**.

Question:

Determine whether the function $f(x, y, z) = x^4 + 2x^2 + 3y^2 - 6xz + 6z^2$ is convex or concave.

https://www.dr-eriksen.no/teaching/GRA6035/Exams/midterm-2023-01.pdf

The Hessian matrix $H(f)$ has leading principal minors $D_1 = 12x^2 + 4 > 0$, $D_2 = 6(12x^2 + 4) > 0$ and $D_3 = 6(12(12x^2 + 4) - (-6)^2) = 6(144x^2 + 12) > 0$ since $H(f)$ is given by

$$H(f) = \begin{pmatrix} 12x^2 + 4 & 0 & -6 \\ 0 & 6 & 0 \\ -6 & 0 & 12 \end{pmatrix}$$

The Hessian $H(f)$ is therefore positive definite for all (x, y, z), hence f is convex (but not concave).

Question:

Consider the function $f(x, y, z) = x^3 + 3xy^2 - z^4 - 3x + 4z$. **Which statement is true?**

(a) f has a local maximum point but no local minimum
(b) f has a local minimum point but no local maximum
(c) f has stationary points, but all are saddle points
(d) f has both a local maximum and a local minimum point

We compute the first order derivatives to find stationary points, and find

$$3x^2 + 3y^2 - 3 = 0, \quad 6xy = 0, \quad -4z^3 + 4 = 0$$

This gives four stationary points $(\pm 1, 0, 1)$ and $(0, \pm 1, 1)$. We compute the Hessian matrix of f and find

$$H(f) = \begin{pmatrix} 6x & 6y & 0 \\ 6y & 6x & 0 \\ 0 & 0 & -12z^2 \end{pmatrix} = \begin{pmatrix} \pm 6 & 0 & 0 \\ 0 & \pm 6 & 0 \\ 0 & 0 & -12 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \pm 6 & 0 \\ \pm 6 & 0 & 0 \\ 0 & 0 & -12 \end{pmatrix}$$

Since $D_2 = -36$ at the last two stationary points, they are saddle points. At $(1, 0, 1)$ we have $D_1 = 6$, $D_2 = 36$ and $D_3 = -12 \cdot 36 < 0$ so this is also a saddle point. At $(-1, 0, 1)$ we have $D_1 = -6$, $D_2 = 36$ and $D_3 = -12 \cdot 36 < 0$ so this is a local maximum. It follows that there are local max but not local min for f. The correct answer is alternative **A**.

Question:

Consider the function $f(x, y, z) = x^4 + y^4 + z^4 - 4xy$. **Which statement is true?**

a) All stationary points of f are saddle points

b) The function f has both a saddle point and a local minimum point.

c) The function f has a local minimum point, and f is convex.

d) The function f has a global minimum point, and f is convex.

https://www.dr-eriksen.no/teaching/GRA6035/Exams/midterm-2020-01-sol.pdf

The function $f(x,y,z) = x^4 + y^4 + z^4 - 4xy$ has first order partial derivatives and first order conditions given by

$$f'_x = 4x^3 - 4y = 0, \quad f'_y = 4y^3 - 4x = 0, \quad f'_z = 4z^3 = 0$$

The stationary points are given by $z = 0$, $x^3 = y$, and $x = y^3$. This gives $y = x^3 = (y^3)^3 = y^9$, or $y(1 - y^8) = 0$. Since $y^8 = 1$ gives $y = \pm 1$, there are three stationary point given by $y = 0$, $y = 1$ and $y = -1$; the stationary points are $(0,0,0)$, $(1,1,0)$, and $(-1,-1,0)$. The Hessian matrix is

$$H(f) = \begin{pmatrix} 12x^2 & -4 & 0 \\ -4 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{pmatrix}$$

which gives

$$H(f)(0,0,0) = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H(f)(\pm 1, \pm 1, 0) = \begin{pmatrix} 12 & -4 & 0 \\ -4 & 12 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have that $H(f)(0,0,0)$ is indefinite, since $D_2 = -16 < 0$, hence $(0,0,0)$ is a saddle point. We also have that $H(f)(\pm 1, \pm 1, 0)$ is positive semi-definite (but not positive definite), since the principal minors are given by $D_1 = 12$, $D_2 = 144 - 16 = 128 > 0$ and $D_3 = 0$ and we can apply the reduced rank criterion since the matrix has rank two. Hence the second derivative test is inconclusive. The stationary points $(1,1,0)$ and $(-1,-1,0)$ are in fact local minimum points: To see this, note that $f(x,y) = x^4 + y^4 - 4xy$ has local minimum points $(1,1)$, $(-1,-1)$ by the second derivative test and the results above. Moreover, $f(1,1,0) = -2$ and $f(1,1,z) = -2 + z^4 \geq -2$. The correct answer is alternative **B**.

Question:

Suppose a function $f(x)$ is everywhere differentiable and is locally maximized when $x = x^*$. Which of the following statements is correct?

A. $f(x)$ is a convex function
B. $f''(x^*) < 0$
C. $f'(x^*) > 0$
D. None of the above

https://cramton.umd.edu/econ300/2nd-midterm-w-spr-13-answers.pdf

Suppose a function $f(x)$ is everywhere differentiable and is locally maximized when $x = x^*$. Which of the following statements is correct?

A. $f(x)$ is a convex function
B. $f''(x^*) < 0$
C. $f'(x^*) > 0$
D. None of the above

Question: True/False

(Convex Function): Given two convex functions $f(x)$ and $g(x)$, where $x \in R^n$, its product $f(x) \times g(x)$ is a convex function.
T/F:

https://cseweb.ucsd.edu/classes/wi24/cse203B-a/exam/Midterm_Sols.pdf

I.4 (Convex Function): Given two convex functions $f(x)$ and $g(x)$, where $x \in R^n$, its product $f(x) \times g(x)$ is a convex function.
T/F:

False: Consider the counter example $f(x) = x(x-1)$ and $g(x) = x-2$. $f(x)$ and $g(x)$ are convex but $f(x) \times g(x)$ is not convex.

Question: True/False

(b) (3 points) **True or False**: If $f(x)$ is a convex function, then for all $c \in \mathbb{R}$, $c \cdot f(x)$ must be a convex function.

Solution: False (this only applies if $c \geq 0$; if $c \leq 0$, then $c \cdot f(x)$ becomes concave).

Question:

A family of concave utility functions. For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha} \qquad \text{with } \mathbf{dom}\, u_\alpha = \mathbf{R}_+.$$

Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

By inspection we have

$$u_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = 0.$$

The derivative is given by

$$u'_\alpha(x) = x^{\alpha-1},$$

which is positive for all x (since $0 < \alpha < 1$), so these functions are increasing. To show concavity, we examine the second derivative:

$$u''_\alpha(x) = (\alpha - 1)x^{\alpha-2}.$$

Since this is negative for all x, we conclude that u_α is strictly concave.

- A function f is *concave* if $-f$ is convex

→ zero order

→ first order

→ second order

Hessian PSD Convex

$\Downarrow$ ✗

minima / maxima

PD $\Rightarrow$ minima