GATE BM 2023 Quetion 9

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Question :
Out of 1000 individuals in a town, 100 unidentified individuals are covid positive. Due to lack of adequate covid-testing kits, the health authorities of the town devised a strategy to identify these covid-positive individuals. The strategy is to:

1) Collect saliva samples from all 1000 individuals and randomly group them into sets of 5.
2) Mix the samples within each set and test the mixed sample for covid.
3) If the test done in (ii) gives a negative result, then declare all the 5 individuals to be covid negative.
4) If the test done in (ii) gives a positive result, then all the 5 individuals are separately tested for covid.

Given this strategy, no more than testing kits will be required to identify all the 100 covid positive individuals irrespective of how they are grouped

1) 700
2) 600
3) 800
4) 1000

Solution :
Given :

Parameter	Value
Number of individuals	1000
Strenght of each group	5
Number of groups	200
Number of Covid positive individuals	100

At first we ll have to examine all the groups and then test all 5 individuals of each group who tested positive.

Using binomial expansion to find the probability of a group testing positive

Parameter	Value	Expression
N	200	Number of groups to be tested initially
n	5	Number of individuals in group
p	0.1	Probability of individual testing positive
q	0.9	Probability of individual testing negative

Probability of a group with i covid negative individuals is given as
the PMF:

$$Pr(X = i) = ^n C_i(q)^i(p)^{n-i} \tag{1}$$
$$= ^5 C_i(0.9)^i(0.1)^{5-i} \tag{2}$$

Let us assume X is a random variable which denotes the number of individuals tested negative in a group.

$$Pr = \begin{cases} {}^5C_0(0.9)^0(0.1)^5 \ \{X = 0\} \\ \\ {}^5C_1(0.9)^1(0.1)^4 \ \{X = 1\} \\ \\ {}^5C_2(0.9)^2(0.1)^3 \ \{X = 2\} \\ \\ {}^5C_3(0.9)^3(0.1)^2 \ \{X = 3\} \\ \\ {}^5C_4(0.9)^4(0.1)^1 \ \{X = 4\} \\ \\ {}^5C_5(0.9)^5(0.1)^0 \ \{X = 5\} \end{cases} \tag{3}$$

Random Variable	Value	Expression	Probability
X	0	${}^5C_0(0.9)^0(0.1)^5$	0.00001
	1	${}^5C_1(0.9)^1(0.1)^4$	0.00045
	2	${}^5C_2(0.9)^2(0.1)^3$	0.0081
	3	${}^5C_3(0.9)^3(0.1)^2$	0.0729
	4	${}^5C_4(0.9)^4(0.1)^1$	0.32805
	5	${}^5C_5(0.9)^5(0.1)^0$	0.59049

Probability of group having atmost k negative tested individuals
the CDF :

$$F_X(k) = Pr(X \le k) \tag{4}$$
$$= \Sigma_{i=0}^{k} {}^nC_i(q)^i(p)^{n-i} \tag{5}$$
$$= \Sigma_{i=0}^{k} {}^5C_i(0.9)^i(0.1)^{5-i} \tag{6}$$
$$\tag{7}$$

If a group is tested positive, atleast one of the 5 individuals should be covid positive which means atmost 4 individuals are covid negative.
Finding probability of a group testing positive

$$Pr = F_X(4) \tag{8}$$
$$= \Sigma_{i=0}^{4} {}^5C_i(0.9)^i(0.1)^{5-i} \tag{9}$$
$$=^5 C_5(0.1)^5 +^5 C_4(0.1)^4(0.9) +^5 C_3(0.1)^3(0.9)^2 +^5 C_2(0.1)^2(0.9)^3 +^5 C_1(0.1)^1(0.9)^4 \tag{10}$$
$$= 1 * 0.00001 + 5 * 0.9 * 0.0001 + 10 * 0.81 * 0.001 + 10 * 0.729 * 0.01 + 5 * 0.6561 * 0.1 \tag{11}$$
$$= 0.00001 + 0.00045 + 0.0081 + 0.0729 + 0.32805 \tag{12}$$
$$= 0.4096 \tag{13}$$

Now calculating the expected number of sets: Expected number of tests for one set(a)

$$a = 1 + 5 * F_X(4) \tag{14}$$
$$= 1 + 5 * 0.4096 \tag{15}$$
$$= 3.048 \tag{16}$$

Total expected number of sets(T)

$$T = (N) * (a) \tag{17}$$
$$= 200 * 3.048 \tag{18}$$
$$= 609.6 \tag{19}$$

$\therefore$ So 609.6 is the number of tests required to detect all the covid positive individuals.
According to the options, no more than 700 tests will be required.
So, the correct answer is option A 700.