GATE BM 2023 Quetion 9

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Question :
Out of 1000 individuals in a town, 100 unidentified individuals are covid positive. Due to lack of adequate covid-testing kits, the health authorities of the town devised a strategy to identify these covid-positive individuals. The strategy is to:

1) Collect saliva samples from all 1000 individuals and randomly group them into sets of 5.
2) Mix the samples within each set and test the mixed sample for covid.
3) If the test done in (ii) gives a negative result, then declare all the 5 individuals to be covid negative.
4) If the test done in (ii) gives a positive result, then all the 5 individuals are separately tested for covid.

Given this strategy, no more than testing kits will be required to identify all the 100 covid positive individuals irrespective of how they are grouped

1) 700
2) 600
3) 800
4) 1000

Solution :
Given :

Parameter	Value
Number of individuals	1000
Strenght of each group	5
Number of groups	200
Number of Covid positive individuals	100

At first we ll have to examine all the group(N)

$$N = \frac{1000}{5} \tag{1}$$
$$= 200 \tag{2}$$

Probability	Calculations	Value
Probability of individual testing positive	$\frac{100}{1000}$	0.1
Probability of individual testing negative	$\frac{1000-100}{1000}$	0.9

Using binomial expansion to find the probability of a group testing positive
Probability of a group with i covid negative individuals is given as
the PMF:

$$Pr(X = i) =^n C_i(q)^i(p)^{n-i} \tag{3}$$
$$=^5 C_i(0.9)^i(0.1)^{5-i} \tag{4}$$

Parameter	Value	Expression
n	5	Number of individuals in group
p	0.1	Probability of testing positive
q	0.9	Probability of testing negative

Where i is number of individuals tested positive Evaluating probability of every case in a group

$$Pr = \begin{cases} {}^5C_5(0.9)^0(0.1)^5 \ \{i = 0\} \\ \\ {}^5C_4(0.9)^1(0.1)^4 \ \{i = 1\} \\ \\ {}^5C_3(0.9)^2(0.1)^3 \ \{i = 2\} \\ \\ {}^5C_2(0.9)^3(0.1)^2 \ \{i = 3\} \\ \\ {}^5C_1(0.9)^4(0.1)^1 \ \{i = 4\} \\ \\ {}^5C_0(0.9)^5(0.1)^0 \ \{i = 5\} \end{cases} = \begin{cases} 1 * 0.00001 & = 0.00001 \\ \\ 5 * 0.9 * 0.0001 & = 0.00045 \\ \\ 10 * 0.81 * 0.001 & = 0.0081 \\ \\ 10 * 0.729 * 0.01 & = 0.0729 \\ \\ 5 * 0.6561 * 0.1 & = 0.32805 \\ \\ 1 * 0.5904 & = 0.5904 \end{cases} \tag{5}$$

Parameter	Calculation	Probability
All positive	${}^5C_5(0.9)^0(0.1)^5$	0.00001
4 positive 1 negative	${}^5C_4(0.9)^1(0.1)^4$	0.00045
3 positive 2 negative	${}^5C_3(0.9)^2(0.1)^3$	0.00810
2 positive 3 negative	${}^5C_2(0.9)^3(0.1)^2$	0.07290
1 positive 4 negative	${}^5C_1(0.9)^4(0.1)^1$	0.32805
All negaive	${}^5C_0(0.9)^5(0.1)^0$	0.5904

Probability of group having atmost k negative tested individuals
the CDF :

$$F_X(k) = Pr(X \leq k) \tag{6}$$
$$= \Sigma_{i=0}^{k} {}^nC_i(q)^i(p)^{n-i} \tag{7}$$
$$= \Sigma_{i=0}^{k} {}^5C_i(0.9)^i(0.1)^{5-i} \tag{8}$$
$$\tag{9}$$

If a group is tested positive, atleast one of the 5 individuals should be covid positive which means atmost 4 individuals are covid negative.
Finding probability of a group testing positive

$$Pr = F_X(4) \tag{10}$$
$$= \Sigma_{i=0}^{4} {}^5C_i(0.9)^i(0.1)^{5-i} \tag{11}$$
$$={}^5 C_5(0.1)^5 +{}^5 C_4(0.1)^4(0.9) +{}^5 C_3(0.1)^3(0.9)^2 +{}^5 C_2(0.1)^2(0.9)^3 +{}^5 C_1(0.1)^1(0.9)^4 \tag{12}$$
$$= 0.00001 + 0.00045 + 0.0081 + 0.0729 + 0.32805 \tag{13}$$
$$= 0.4096 \tag{14}$$

Now calculating the expected number of sets: Expected number of tests for one set(a)

$$a = 1 + 5 * F_X(4) \tag{15}$$
$$= 1 + 5 * 0.4096 \tag{16}$$
$$= 3.048 \tag{17}$$

Total expected number of sets(T)

$$T = 200 * (a) \tag{18}$$
$$= 200 * 3.048 \tag{19}$$
$$= 609.6 \tag{20}$$

$\therefore$ So 609.6 is the number of tests required to detect all the covid positive individuals.
According to the options, no more than 700 tests will be required.
So, the correct answer is option A 700.