12 - 3 - 24

Ejercicios

3

$(4-\lambda)^2 - 16 = 0$

$$16 - 8\lambda + \lambda^2 - 16 = \lambda^2 + 8\lambda$$
$$= \lambda(\lambda + 8) = 0$$

Así: $\lambda_1 = -8$ $\quad\lambda_2 = 0$

Como $\lambda_i \leq 0$ entonces no es positiva definida

1 Si tomamos $\hat{\Sigma}_k = \hat{\Sigma}$

$$\hat{\Sigma} = \frac{\sum_{k=0}^{k-1} (x_i - \mu_k)(x_i - \mu_k)^T}{n-k} = \frac{\sum_{k=0}^{k-1} (n_k - 1)\hat{\Sigma}_k}{n-k}$$

Al incluir las k en una sumatoria para det $\hat{\Sigma}$

3 $\log\left(\frac{f_1(x)\pi_1}{f_k(x)\pi_k}\right) = \log(f_1(x)\pi_1) - \log(f_k(x)\pi_k)$

Con δ_k del 1

$$\delta_k = -\frac{1}{2}\log(\Sigma_1) - \frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1) + \log(\pi_1) + \frac{1}{2}\log(\Sigma_k)$$
$$+ \frac{1}{2}(x-\mu_k)^T (x-\mu_k) - \log(\pi_k)$$

$$= -\frac{1}{2}\log\left(\frac{\Sigma_1}{\Sigma_k}\right) - \frac{1}{2}\left((x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1) - (x-\mu_k)^T \Sigma_k^{-1}(x-\mu_k)\right) + \log\left(\frac{\pi_1}{\pi_k}\right)$$

$$= -\frac{1}{2}\left((x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1) - (x-\mu_k)^T \Sigma_k^{-1}(x-\mu_k)\right)$$

4) Multnormal $\bar{x} = 0$

$$P(\bar{x}) = 4x_1^2 + x_2^2 + 4x_3^2 + 2x_1x_2$$

$x = (x_1, x_2, x_3)$ $\qquad A = \Sigma^{-1}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Asi $a_{11} = 4$, $a_{12} = a_{21}$, $a_{13} = 0$, $a_{22} = 1$, $a_{23} = 0$, $a_{31} = 0$, $a_{32} = 0$, $a_{33} = 4$

$$\Sigma^{-1} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \begin{aligned} 2a_{21} &= 2 \\ a_{12} &= 1 \end{aligned}$$

$$\Sigma = \begin{pmatrix} \frac{1}{4-1} & \frac{-1}{4-1} & 0 \\ -\frac{1}{4-1} & \frac{4}{4-1} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \quad \therefore \quad X \sim \text{Norm}(0, \Sigma)$$

5) a) $\text{Det}(\Sigma_1 - \lambda I) = (7-\lambda)(1-\lambda) - 4 = 0$

$$= 7 - \lambda - 7\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 8\lambda + 3 = 0$$

$\lambda_1 = 7.6$ $\qquad \lambda_2 = .39$

Como λ_1 y $\lambda_2 > 0 \Rightarrow$ es positivo definida

$\therefore$ Si existe un vector gaussiano y matriz de cov.

b) Sea $x = (x_1, x_2)$

$$L(x) = \log\left(\frac{f_1(x)\pi_1}{f_2(x)\pi_2}\right) = b_0 + b^T x = \frac{5}{3} + \frac{7}{3}x_1 - \frac{20}{3}x_2$$

$$\Sigma_1^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} 1/2 & -2/3 \\ -2/3 & 7/3 \end{pmatrix}$$

$$\Sigma_2^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}$$

$$b = \Sigma_1^{-1}(\mu_1 - \mu_2) = \begin{pmatrix} 1/3 & -\frac{2}{3} \\ -\frac{2}{3} & 7/3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 + 4/3 \\ -2 - \frac{14}{3} \end{pmatrix} = \begin{pmatrix} 7/3 \\ -20/3 \end{pmatrix}$$

$$b_0 = -\frac{1}{2}\left(\mu_1^T \Sigma_1^{-1} \mu_1 - \mu_2^T \mu_2\right) + \log\left(\frac{\pi_2}{\pi_1}\right)$$

$$= -\frac{1}{2}\left((1 \quad 0)\begin{pmatrix} 1/3 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{7}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - (-2 \quad 2)\begin{pmatrix} 1/3 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{7}{3} \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix}\right) + \log(1)$$

$$= \frac{47}{6}$$

c) $l(x_1 = 2, x_2\ 1) = \frac{47}{6} + \frac{7}{3}(2) - \frac{20}{3}(1)$

$= \frac{35}{6} > 0 \quad \Rightarrow (2,1)$ se asigna a π_1

d) $Q(x) = \beta_0 + \beta^T x + x^T \Omega x$

$$= \frac{5}{2} + \left(\frac{7}{3} \quad -\frac{4}{3}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1 \quad x_2)\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Omega = -\frac{1}{2}\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right) = -\frac{1}{2}\begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 \end{pmatrix}$$

$$\beta = \Sigma_1^{-1}\mu_1 - \Sigma_2^{-1}\mu_2 = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{7}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ -\frac{4}{3} \end{pmatrix}$$

$$\beta_0 = \frac{1}{2}\left(\log(1) + (10) + 10\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{7}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - (-2 \quad 2)\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix}\right) - \log(1)$$

$$= \frac{5}{2}$$

e) $Q(x_1 = 2, x_2 = 1)$

$$= \frac{5}{2} + \left(\frac{7}{3} \quad -\frac{4}{3}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + (2 \quad 1)\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{15}{2} > 0$$

$\therefore$ Se asigna $(2,1)$ a π_1