NUMERICAL MODELING AND SIMULATION FOR ACOUSTICS
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HOMEWORK 3

Consider the following one dimensional Burger's equation:

$$u_t(x,t) + \left(\frac{u^2}{2}(x,t)\right)_x = 0 \quad (x,t) \in (0,L) \times (0,T], \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x) \quad x \in (0,L), \tag{2}$$

and, the Dirichlet boundary conditions

$$u(0,t) = u(L,t) = 0 \quad t \in (0,T], \tag{3}$$

Consider a cartesian mesh for the domain $[0,L] \times [0,T]$ by dividing the interval $[0,T]$ into N steps $0 = t_0 \leq t_1 \leq ... \leq t_N = T$ with constant time step $\Delta t = T/N$ and $t_n = n\Delta t$ for $n = 1,2,...,N$ and the space interval $[0,L]$ into M nodes $0 \leq x_0 \leq x_1 \leq ... \leq x_M = L$ with constant spacing step $h = ih$ for $i = 1,2,...,M$.

1. Write the finite volume approximation of (1) by considering the Gudunov method with either constant (firt order scheme) and linear riconstruction (second order scheme) of the numerical solution. For the latter consider the following options:

    1. **Upwind**
    $$\begin{cases} U^-_{i+\frac{1}{2}} = U_i + \frac{1}{2}(U_i - U_{i-1}), \\ U^+_{i+\frac{1}{2}} = U_i - \frac{1}{2}(U_i - U_{i-1}). \end{cases}$$

    2. **Lax-Wendroff**
    $$\begin{cases} U^-_{i+\frac{1}{2}} = U_i + \frac{1}{2}(U_{i+1} - U_i), \\ U^+_{i+\frac{1}{2}} = U_i - \frac{1}{2}(U_{i+1} - U_i). \end{cases}$$

    3. **Fromm**
    $$\begin{cases} U^-_{i+\frac{1}{2}} = U_i + \frac{1}{4}(U_{i+1} - U_{i-1}), \\ U^+_{i+\frac{1}{2}} = U_i - \frac{1}{4}(U_{i+1} - U_{i-1}). \end{cases}$$

2. Consider the following data: $L = 1$, $T = 1$ and $u_0(x) = \frac{1}{2\pi}\sin(2\pi x)$. Compare the solutions obtained by using methods at the previous point. Comment on the results.

3. Finally, to achieve second order space and time approximation consider the following predictor-corrector algortihm based on two steps:

    - **predictor**

$$U_i^{n+\frac{1}{2}} = U_i^n - \frac{\Delta t}{2h}\left[F_{i+\frac{1}{2}}(U_i^n, U_{i+1}^n) - F_{i-\frac{1}{2}}(U_i^n, U_{i+1}^n)\right],$$

– **corrector**

$$U_i^{n+\frac{1}{2}} = U_i^n - \frac{\Delta t}{h}\left[F_{i+\frac{1}{2}}(U_{i+\frac{1}{2}}^{n+\frac{1}{2},-}, U_{i+\frac{1}{2}}^{n+\frac{1}{2},+}) - F_{i-\frac{1}{2}}(U_{i+\frac{1}{2}}^{n+\frac{1}{2},-}, U_{i+\frac{1}{2}}^{n+\frac{1}{2},+})\right],$$

begin $U_{i+\frac{1}{2}}^{n+\frac{1}{2},-}$ and $U_{i+\frac{1}{2}}^{n+\frac{1}{2},-}$ the reconstruction value at time $t^{n+\frac{1}{2}}$ on the cell-interface with one of the methods given at point 1. Solve the problem at point 2., report the results and comment on them.

Extra. Like Godunov's method, Glimm's method is also based on solving a collection of Riemann problems on the interval $[t^n, t^{n+1}]$. But instead of averaging procedure for getting the numerical solution on the next layer a random choice procedure is used. Glimm's time step proceeds in two stages

$$\begin{aligned} U_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= \xi(U_j^n, U_{j+1}^n), \\ U_j^{n+1} &= \xi(U_{j-\frac{1}{2}}^{n+\frac{1}{2}}, U_{j+\frac{1}{2}}^{n+\frac{1}{2}}). \end{aligned}$$

$\xi(u_L, u_R)$ is a random variable taking values either u_L or u_R with probabilities proportional to lenghts of the corresponding intervals:

$$\xi(u_L, u_R) = \begin{cases} u_L, & p = \frac{\frac{h}{2} + s\frac{\Delta t}{2}}{h}, \\ u_R, & p = \frac{\frac{h}{2} - s\frac{\Delta t}{2}}{h}, \end{cases}$$

where $s = (u_R + u_L)/2$ is the shock speed. Alternatively, by setting $\lambda = \frac{\Delta t}{h}$, we can write ξ as follows

$$\xi(u_L, u_R) = \begin{cases} u_L, & p = \frac{1}{2}(1 + s\lambda), \\ u_R, & p = \frac{1}{2}(1 - s\lambda). \end{cases}$$

Implement the Glimm's method for the solution of the problem at point 2. Compare the results with the one obtained at the previous points.