NUMERICAL MODELING AND SIMULATION FOR ACOUSTICS
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LECTURER: Prof. I. Mazzieri

HOMEWORK 1

Consider the following wave propagation problem:

$$\begin{cases} rho(x)u_{tt}(x,t) - (\mu(x)u_x)_x(x,t) = f(x,t) & (x,t) \in (0,L) \times (0,T], \\ u(x,0) = u_0(x) & x \in (0,L), \\ u_t(x,0) = v_0(x) & x \in (0,L), \\ u(0,t) = g_D(t) & t \in (0,T], \\ \mu(1)u_x(1,t) = g_N(t) & t \in (0,T], \end{cases} \tag{1}$$

where u_0, v_0, g_D and g_N are given regular functions, and μ and ρ represent the (variable) stiffness and mass density of the medium.

1. Write the weak formulation of problem (1) and the corresponding finite element discretization. Consider linear basis functions. Show that the Galerkin discretization leads to the following system
$$M\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \mathbf{F}(t), \tag{2}$$
together with initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}(0) = \mathbf{v}_0$. Define precisely the entries of the matrices M and A and of the right hand side $\mathbf{F}$.

2. Implement in Matlab a finite element solver for problem (1), by considering the Newmark-β scheme as time integrator for (2). Consider the values $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$. Verify your implementation on a test problem for which you know the exact solution u. Report the behaviour of the norm $\|u - u_h\|_{L^2(0,L)}$ at the final observation time for different choiches of the discretization parameters, h and Δt. Consider both constant and variable (piece-wice constant/linear) model parameters μ and ρ. Comment on the results.

3. Consider the data: $L = 1.5$, $h = L/200$, $T = 5$, $\Delta t = 0.01$, $f = u_0 = v_0 = g_N = 0$, while
$$g_D(t) = 4\pi\sqrt{e}(t - 0.3)e^{-8\pi^2(t-0.3)^2}, \tag{3}$$
and the parameters μ and ρ as:

   a)
$$\rho = \begin{cases} 1 & x \in [0, 0.75], \\ 4 & x \in [0.75, 1.5], \end{cases} \quad \text{and} \quad \mu = \begin{cases} 4 & x \in [0, 0.75], \\ 1 & x \in [0.75, 1.5]. \end{cases}$$

   b)
$$\rho = \begin{cases} 1 & x \in [0, 0.5], \\ 1 + 6(x - 0.5) & x \in [0.5, 1], \\ 4 & x \in [1, 1.5], \end{cases} \quad \text{and} \quad \mu = \begin{cases} 4 & x \in [0, 0.5], \\ 4 + 3(x - 0.5) & x \in [0.5, 1], \\ 1 & x \in [1, 1.5]. \end{cases}$$

c)

$$\rho = 1 + 2x, \quad \text{and} \quad \mu = 4 - 2x.$$

Report the results and comment on them.