Wave Project - Finite difference simulation of 2D waves

Marte Fossum

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Discretization

We are give the equation

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t) \tag{1}$$

with the boundary condition

$$\frac{\partial u}{\partial n} = 0$$

on the domain $\Omega = [0, L_x] \times [0, L_y]$, with initial conditions

$$u(x,y,0) = I(x,y) \tag{2}$$

$$u_t(x,y,0) = V(x,y) \tag{3}$$

We want to discretize equation (1) in order to simulate it on the computer. By letting $u(x_i, y_j, t_n) = u^n_{i,j}$ we can rewrite each term as

$$\frac{\partial^2 u}{\partial t^2} = \frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2}$$

$$b\frac{\partial u}{\partial t} = b\frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2\Delta t}$$

$$\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) = \frac{\frac{1}{2}(q_{i,j} + q_{i+1,j})(u^n_{i+1,j} - u^n_{i,j}) - \frac{1}{2}(q_{i,j} + q_{i-1,j})(u^n_{i,j} - u^n_{i-1,j})}{\Delta x^2}$$

$$\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) = \frac{\frac{1}{2}(q_{i,j} + q_{i,j+1})(u^n_{i,j+1} - u^n_{i,j}) - \frac{1}{2}(q_{i,j} + q_{i,j-1})(u^n_{i,j} - u^n_{i,j-1})}{\Delta y^2}$$

$$f(x,y,t) = f(x_i, y_j, t_n)$$

Inserting this into (1) gives us

$$\begin{aligned}\frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2} + b\frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2\Delta t} =& \frac{\frac{1}{2}(q_{i,j} + q_{i+1,j})(u^n_{i+1,j} - u^n_{i,j}) - \frac{1}{2}(q_{i,j} + q_{i-1,j})(u^n_{i,j} - u^n_{i-1,j})}{\Delta x^2} \\ &+ \frac{\frac{1}{2}(q_{i,j} + q_{i,j+1})(u^n_{i,j+1} - u^n_{i,j}) - \frac{1}{2}(q_{i,j} + q_{i,j-1})(u^n_{i,j} - u^n_{i,j-1})}{\Delta y^2} \\ &+ f(x_i, y_j, t_n)\end{aligned} \tag{4}$$

Now we solve for $u^{n+1}_{i,j}$.

$$u_{i,j}^{n+1} = \left(\frac{\Delta t^2}{1+\frac{\Delta tb}{2}}\right)\left(\frac{\frac{1}{2}(q_{i,j}+q_{i+1,j})(u_{i+1,j}^n - u_{i,j}^n) - \frac{1}{2}(q_{i,j}+q_{i-1,j})(u_{i,j}^n - u_{i-1,j}^n)}{\Delta x^2}\right)$$
$$+ \left(\frac{\Delta t^2}{1+\frac{\Delta tb}{2}}\right)\left(\frac{\frac{1}{2}(q_{i,j}+q_{i,j+1})(u_{i,j+1}^n - u_{i,j}^n) - \frac{1}{2}(q_{i,j}+q_{i,j-1})(u_{i,j}^n - u_{i,j-1}^n)}{\Delta y^2}\right) \tag{5}$$
$$+ \left(\frac{1}{1+\frac{\Delta tb}{2}}\right)\left(\Delta t^2 f(x_i, y_j, t_n) + 2u_{i,j}^n + u_{i,j}^{n-1}\left(\frac{\Delta tb}{2} - 1\right)\right)$$

In order for us to use this equation, we have to know the first time step. To get u^1 we use the initial conditions (2) and (3). From (2) we get that $u^0 = I(x,y)$ and by discretizing (3) we get that

$$\frac{u^1 - u^{-1}}{2\Delta t} = V \implies u^{-1} = u^1 - 2V\Delta t$$

We set $n = 0$ and set in for u^{-1} and solve for u^1. This gives us

Vertification

We now assume that $u(x,y,t) = C$ where C is a constant. We want to show that the constant solution is a solution of the descrete equation also.

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = \frac{C - 2C + C}{\Delta t^2} = 0$$

$$b\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} = b\frac{C - C}{2\Delta t} = 0$$

$$\frac{q_{1+\frac{1}{2}}(u_{i+1,j}^n - u_{i,j}^n) - q_{1-\frac{1}{2}}(u_{i,j}^n - u_{i-1,j}^n)}{\Delta x^2} = \frac{q_{1+\frac{1}{2}}(C-C) - q_{1-\frac{1}{2}}(C-C)}{\Delta x^2} = 0$$

$$\frac{q_{1+\frac{1}{2}}(u_{i+1,j}^n - u_{i,j}^n) - q_{1-\frac{1}{2}}(u_{i,j}^n - u_{i-1,j}^n)}{\Delta y^2} = \frac{q_{1+\frac{1}{2}}(C-C) - q_{1-\frac{1}{2}}(C-C)}{\Delta y^2} = 0$$

So the constant solution is a solution for the discrete equations when $f(x,y,t) = 0$ and the initial conditions are $u(x,y,0) = C$ and $u_t(x,y,t) = 0$.

5 possible bugs

- 1
- 2
- 3
- 4
- 5