Exercise 8 - Compute with a nonuniform mesh

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8a

We are given the problem

$$-u''(x) = 2 \tag{1}$$

for $x \in [0, 1]$ with $u(0) = 0$ and $u(1) = 1$, and we want to derive a linear system using P1 elements with a non-uniform mesh $x_0 = 0 < x_1 < ... < x_{N_n-1} = 1$.

We seek a solution $u \in V$, $V = span\{\psi_0(x), ..., \psi_{N_n-1}(x)\}$, where $\{\psi_i\}$ is a set of linearly independent basis functions for $i \in \{0, ..., N_n - 1\}$. Since $u \in V$ we can write u as

$$u(x) = \sum_{i=0}^{N_n-1} c_i\psi_i(x) \tag{2}$$

We take the innerproduct of (1) on both sides with some function $v \in V$. Letting $v = \psi_j$ gives us

$$\begin{aligned}
\langle 2, \psi_j(x)\rangle &= \langle -u''(x), \psi_j(x)\rangle \\
\int_0^1 2\psi_j(x)dx &= \int_0^1 -u''(x)\psi_j(x)dx \\
h_{j-1} + h_j &= -[u'(x)\psi_j(x)]_0^1 + \int_0^1 u'(x)\psi_j'(x)dx \\
&= \int_0^1 u'(x)\psi_j'(x)dx \\
&= \int_0^1 \sum_{i=0}^{N_n-1} c_i\psi_i'(x)\psi_j'(x)dx \\
&= \sum_{i=0}^{N_n-1} c_i \int_0^1 \psi_i'(x)\psi_j'(x)dx
\end{aligned}$$

Since ψ_j is zero everywhere but on the interval $[x_{j-1}, x_{j+1}]$, we have that the derivative is also zero everywhere except on the same interval. More precisely

$$\psi_j'(x) = \begin{cases} \frac{1}{h_{j-1}} & x \in (x_{j-1}, x_j) \\ -\frac{1}{h_j} & x \in (x_j, x_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

Put together we get that

$$h_{j-1} + h_j = c_{j-1}A_{j-1,j} + c_jA_{j,j} + c_{j+1}A_{j+1,j}$$

where

$$
\begin{aligned}
A_{j-1,j} &= \int_{x_{j-1}}^{x_j} \frac{-1}{h_{j-1}^2} dx \\
&= -\frac{1}{h_j}
\end{aligned}
$$

$$
\begin{aligned}
A_{j,j} &= \int_{x_{j-1}}^{x_j} \frac{1}{h_{j-1}^2} dx + \int_{x_j}^{x_{j+1}} \frac{1}{h_j^2} dx \\
&= \frac{1}{h_{j-1}} + \frac{1}{h_j}
\end{aligned}
$$

$$
\begin{aligned}
A_{j+1,j} &= \int_{x_j}^{x_{j+1}} \frac{-1}{h_j^2} dx \\
&= -\frac{1}{h_j}
\end{aligned}
$$

This all works for $j = 1, ..., N_n - 2$. However we now only have $N_n - 2$ equations, but we have N unknowns. This is because we don't have any equations for the boundary. We do know however that $u(0) = 0$ and that $u(1) = 1$ which gives us the last two equations $c_0 = 0$ and $c_{N_n-1} = 1$.

$$
\begin{cases}
h_{j-1} + h_j = -\frac{1}{h_j} c_{j+1} + \left(\frac{1}{h_j} + \frac{1}{h_{j-1}}\right) c_j - \frac{1}{h_{j-1}} c_{j-1} & \text{for } j = 1, ..., N_n + 2 \\
c_0 = 0 & \text{for } j = 0 \\
c_{N_n-1} = 1 & \text{for } j = N_n - 1
\end{cases}
\tag{3}
$$

8b

We want to use the finite difference method to discretize $u''(x_i) = [D_x D_x u]_i$ and comepare this to the finite element.

$$
\begin{aligned}
u''(x_i) &= [D_x(D_x u)]_i \\
&= \frac{[D_x u]_{i+\frac{1}{2}} - [D_x u]_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \\
&= \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \left(\frac{u_{i+\frac{1}{2}+\frac{1}{2}} - u_{i+\frac{1}{2}-\frac{1}{2}}}{x_{i+\frac{1}{2}+\frac{1}{2}} - x_{i+\frac{1}{2}-\frac{1}{2}}} - \frac{u_{i-\frac{1}{2}+\frac{1}{2}} - u_{i-\frac{1}{2}-\frac{1}{2}}}{x_{i-\frac{1}{2}+\frac{1}{2}} - x_{i-\frac{1}{2}-\frac{1}{2}}} \right) \\
&= \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} - \frac{u_i - u_{i-1}}{x_i - x_{i-1}} \right) \\
&= \frac{2}{x_{i+1} - x_{i-1}} \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} - \frac{u_i - u_{i-1}}{x_i - x_{i-1}} \right) \qquad (4)
\end{aligned}
$$

We have that each interval between points x_i and x_{i+1} has length h_i, so substituting $h_i = x_{i+1} - x_i$ into (1) and (4) gives

$$
\begin{aligned}
-u''(x_i) = -\frac{2}{h_i + h_{i-1}} \left(\frac{u_{i+1} - u_i}{h_i} - \frac{u_i - u_{i-1}}{h_{i-1}} \right) = 2 \\
\implies h_i + h_{i-1} = -\frac{u_{i+1} - u_i}{h_i} + \frac{u_i - u_{i-1}}{h_{i-1}} \\
= -\frac{1}{h_i} u_{i+1} + \left(\frac{1}{h_i} + \frac{1}{h_{i-1}} \right) u_i - \frac{1}{h_{i-1}} u_{i-1} \qquad (5)
\end{aligned}
$$

Comparing the end result for the discrete method (5) with the result for the finite element method (3) we can see that they are the same type of linear systems.