# Problem 1 - Use linear/quadratic functions for vertification

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## a

We are given the equation and boundary conditions

$$u'' + \omega^2 u = f(t), \quad u(0) = 0, \quad u'(0) = V, \quad t \in [0, T] \tag{1}$$

To find $u^1$ we use the numerical representation of the double derivative.

$$u'' \sim \frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2}$$

Putting in this in the original equation gives us

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2} + \omega^2 u = f^n$$

Solving this for $u^{n+1}$ gives us

$$u^{n+1} = f^n(\Delta t)^2 - \omega^2 u^n (\Delta t)^2 + 2u^n - u^{n-1} \tag{2}$$

This equation however requires us to know $u^{-1}$ which we do not. What we do then is use the initial condition $u'(0) = V$ and approximate the derivative. We then get

$$u' \sim \frac{u^{n+1} - u^{n-1}}{2\Delta t}$$

Setting in the condition $t = 0$ we get that

$$u'(0) = \frac{u^1 - u^{-1}}{2\Delta t} = V$$

We then again solve this for $u^{-1}$ and get

$$u^{-1} = u^1 - 2V\Delta t$$

If we set $n = 0$, our expression for $u^{-1}$ and $u^0$ into equation (2), we get an expression for $u^1$

$$\begin{aligned} u^1 &= f^0(\Delta t)^2 - \omega^2 u^0 (\Delta t)^2 + 2u^0 - u^{-1} \\ &= \frac{1}{2} f^0 (\Delta t)^2 - \frac{1}{2}\omega^2 I (\Delta t)^2 + I + V\Delta t \end{aligned}$$

## b

We are given the exact solution

$$u_e(x,t) = ct + d$$

From our bourdary condisions we get that $u_e(x,0) = I$ which gives us that $d = I$ and we have that $u_e'(0) = V$ which gives us that $c = V$. So now our exact equation looks like

$$u_e(x,t) = Vt + I \tag{3}$$

From definition of the derivative we have that

$$
\begin{aligned}
[D_tD_tt]^n &= \frac{[D_tt]^{n+\frac{1}{2}} - [D_tt]^{n-\frac{1}{2}}}{\Delta t} \\
&= \frac{\frac{t^{n+\frac{1}{2}+\frac{1}{2}} - t^{n-\frac{1}{2}+\frac{1}{2}}}{\Delta t} - \frac{t^{t-\frac{1}{2}+\frac{1}{2}} - t^{n-\frac{1}{2}-\frac{1}{2}}}{\Delta t}}{\Delta t} \\
&= \frac{t^{n+1} - t^n - t^n + t^{n-1}}{(\Delta t)^2} \\
&= \frac{t^{n+1} - 2t^n + t^{n-1}}{(\Delta t)^2} \\
&= \frac{t + \Delta t - 2t + t - \Delta t}{(\Delta t)^2} \\
&= 0
\end{aligned}
$$

where I have used that $t^{n+1} = (n+1)\Delta t = t + \Delta t$.

Now using that $D_tD_t$ is a linear operator, that is,

$$
\begin{aligned}
[D_tD_t(ct+d)]^n &= c[D_tD_tt]^n + [D_tD_td]^n \\
&= 0
\end{aligned}
$$

we want to show that our exact solution is a perfect solution of the discrete equations.

We can easily see that inserted $t = 0$ into the exact solution (3) and its derivative, we get the boudary conditions. Inserting the exact solution into (1) gives us an expression for $f(t)$

$$
\omega^2(Vt + I) = f(t)
$$

Now we insert the exact solution into the descrete equation. We have already shown that $D_tD - t(Vt + I) = 0$ so we are left with

$$
\begin{aligned}
[\omega^2 u_e(x,t) = f(t)]^n \\
[\omega^2(Vt + I) = f(t)]^n
\end{aligned}
$$

which was what we found for (1) as well.