**Definition 0.1.** *Fix an algebraic structure $\mathbb{R}$ and a set of polynomials $\mathcal{F} = \{F_i\}$. Let $d$ be a non-negative integer. A $d$-design for $\mathbb{F}$ is a mapping $D$ from $\mathbb{R}$-polynomials of degree $\leq d$ to $\mathbb{R}$ such that:*

1. *$D(1) = 1$*
2. *$D$ is linear, which means $D(X + Y) = D(X) + D(Y)$.*
3. *$D(Q \cdot F_i) = 0$ for any polynomial $Q$ and $F_i \in \mathbb{F}$ such that $deg(Q) + deg(F_i) \leq d$.*
4. *$D(x_i^2 Q) = D(x_i Q)$, where $x$ is a variable in $R[x_1, x_2, ..., x_n]$ and $deg(Q) < d - 1$.*

Notice that the mapping $D$ is actually determined by the values it assigns to monomials of degree less than $d$, since it is a linear mapping and any polynomial can be decomposed into some monomials. Furthermore, due to property 4, it suffices to consider only multilinear monomials when determining the value of $D$. Alternatively, we can assume that $\mathcal{F}$ includes $x_i(x_i - 1)$ for all $i$, in which case property 3 implies property 4.

Naturally, for multilinear monomials, we can view them as conjunctions of the atomic statements represented by the propositional variables in the monomial. Furthermore, we found there is a connection between $d$-designs and Nullstellensatz refutation of degree $d$. Next, two theorems describe this connection.

**Theorem 0.1.** *If $\mathcal{F}$ exists a $d$-design, then there cannot be a Nullstellensatz refutation of degree $\leq d$ to refutate $\mathcal{F}$.*

*Proof.* If there exists a Nullstellensatz refutation of degree $\leq d$, then

$$1 = \sum_i P_i \cdot F_i + \sum_j Q_j \cdot (x_j^2 - x_j)$$

. Using the $d$-design for $\mathcal{F}$,

$$D(1) = 1 \neq 0 = \sum_i D(P_i \cdot F_i) + \sum_j D(Q_j x_j^2) - D(Q_j x_j)$$

. ◻

A converse of the above theorem, to some extent, also holds:

**Theorem 0.2.** *Suppose the algebraic structure $\mathbb{R}$ is a field. If $\mathcal{F}$ does not have a Nullsltellensatz refutation of degree $d$, then there is a $d$-design for $\mathcal{F}$.*

*Proof.* Suppose there are $n$ variables used in our proposition. Let $\delta$ be the number of monomials of degree $\leq d$. Then we can calculate $\delta = \sum_{i=0}^{d} \binom{n+i-1}{i}$. Recall that $\binom{n+i-1}{i} = \binom{n+i-1}{n-1}$. It means $\delta$ is not infinite. Then for any polynomial $H$ of degree $\leq d$, it can be viewed as a vector $\mathbf{v_H}$ where each element $a_Q$ is the coefficient of a monomial $Q$. Therefore, $\mathbf{v_H}$ has dimension $\delta$.

Let $\mathcal{G}$ be a set of polynomials in the form $Q \cdot F$, where $Q$ is a monomial and $F \in \mathcal{F}$, and $\deg(Q \cdot F) \leq d$. For simplicity, we consider $\mathcal{F}$ to include all polynomials of the form $x_i(x_i - 1)$. Each $g_i \in \mathcal{G}$ can be viewed as a vector $\mathbf{v_{g_i}}$, analogous to what was done for $H$.

Define a vector $\mathbf{v_1}$ with only one nonzero element on the constant term. A Nullstellensatz refutation of degree at most $d$ exists if and only if there is an $R$-linear combination of the vectors $\{\mathbf{v_{g_i}}\}$. Let $\mathbf{M}$ be the $\delta \times |\mathcal{G}|$ matrix with columns $\mathbf{v_{g_i}}$,, where $\delta = \sum_{i=0}^{d} \binom{n+i-1}{i}$. This is equivalent to finding a solution $\mathbf{w}$ to the linear equation $\mathbf{M}\mathbf{w} = \mathbf{v_1}$.

It's equivalent to checking whether the rank of coefficient matrix $\mathbf{M}$ equals the rank of the augmented matrix $[\mathbf{M}, \mathbf{v_1}]$. Since there is only one nonzero element in $\mathbf{v_1}$, that element cannot be a linear combination of zeros. It's therefore equivalent to checking whether the last row of $\mathbf{M}$ is a linear combination of other rows in $\mathbf{M}$.

Thus, there is a Nullstellensatz refutation of degree $\leq d$ if and only if the last row of $\mathbf{M}$ is linearly independent of the rest of the row vectors. If the last row is a linear combination, we can indeed find a possible $d$-design. Denote each row as $\mathbf{u_Q}$ where $Q$ corresponds to that row. Suppose the combination is

$$\sum_{\deg(Q)\le d} \alpha_Q \mathbf{u_Q} = 0. \tag{1}$$

Then we can actually use $\alpha_Q$ as $D(Q)$ for all monomials $Q$, and $D(1) = 1$.

The rest is to check if $D$ is valid. Since it's sufficient that all properties hold for monomials, we only need to consider the case for monomials. For any monomial $Q$ and $F \in \mathcal{F}$, $Q \cdot F$ must be in $\mathcal{G}$. Because this is a column in matrix $\mathbf{M}$, equation (1) holds for the column $Q \cdot F$. Then $D(Q \cdot F) = D(\sum_{\deg(Q)\le d} \alpha_Q u_Q^{Q\cdot F}) = 0$. It's not difficult to check that $D$ is linear. $\square$