Mathematical Methods for Computer Science 2
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Series 1

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2.1)

$$
\begin{aligned}
(1+\sqrt{5})^n &= \binom{n}{0} 1^n \sqrt{5}^0 + \binom{n}{1} 1^{n-1} \sqrt{5}^1 + \cdots + \binom{n}{n-1} 1^1 \sqrt{5}^{n-1} + \binom{n}{n} 1^0 \sqrt{5}^n \\
&= 1 + \binom{n}{1} \sqrt{5} + \binom{n}{2} \sqrt{5}^2 + \cdots + \binom{n}{n-1} \sqrt{5}^{n-1} + \binom{n}{n} \sqrt{5}^n \\
&= \sum_{i=0}^{n} \binom{n}{i} \sqrt{5}^i
\end{aligned}
$$

a.2)

$$
\begin{aligned}
(1-\sqrt{5})^n &= (1+(-\sqrt{5}))^n \\
&= \binom{n}{0} 1^n (-\sqrt{5})^0 + \binom{n}{1} 1^{n-1} (-\sqrt{5})^1 + \cdots + \binom{n}{n-1} 1^1 (-\sqrt{5})^{n-1} + \binom{n}{n} 1^0 (-\sqrt{5})^n \\
&= 1 - \binom{n}{1} \sqrt{5} + \binom{n}{2} \sqrt{5}^2 - \cdots + \binom{n}{n-1} (-\sqrt{5})^{n-1} + \binom{n}{n} (-\sqrt{5})^n \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sqrt{5}^i
\end{aligned}
$$

b)

$$
a_n = \frac{\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + \ldots}{2^{n-1}}
$$

The proof is by expanding the Binet's formula using the binomial coefficient. So we have

$$
a_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)
$$

and after expansion we obtain

$$
\begin{aligned}
a_n &= \frac{1}{2^n\sqrt{5}}\left(\sum_{i=0}^{n}\left(\binom{n}{k}(\sqrt{5}^k)\right) - \sum_{i=0}^{n}\left(\binom{n}{k}(\sqrt{5}^k)(-1)^k\right)\right) \\
&= \frac{1}{2^n\sqrt{5}}\left(\sum_{i=0}^{n}\left(\binom{n}{k}(\sqrt{5})^k(1-(-1)^k)\right)\right)
\end{aligned}
$$

When k is even, $1-(-1)^k = 0$ and $1-(-1)^k = 2$ when k is odd. So we obtain

$$
\begin{aligned}
a_n &= \frac{1}{2^n\sqrt{5}}\left(\sum_{2j+1=1}^{2j+1=n}\left(\binom{n}{2j+1}(\sqrt{5})^{2j+1}\cdot 2\right)\right) \\
&= \frac{1}{2^{n-1}\sqrt{5}}\sum_{j=0}\left(\binom{n}{2j+1}(5^j\sqrt{5})\right) \\
&= \frac{1}{2^{n-1}}\sum_{j=0}\left(5^j\binom{n}{2j+1}\right) \\
&= \frac{5^0\binom{n}{1} + 5^1\binom{n}{3} + 5^2\binom{n}{5}\ldots}{2^{n-1}}
\end{aligned}
$$

2

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 5 \\
a_n &= 5a_{n-1} - 6a_{n-2}
\end{aligned}
$$

We find that

$$
\begin{gathered}
a_2 = 5a_1 - 6a_0 = 5 \\
5 = 5*1 - 6*a_0 = 5 - 6*a_0 \\
6*a_0 = 0 \\
a_0 = 0
\end{gathered}
$$

$$X = (1, \lambda, \lambda^2)$$

$$\lambda^n = 5\lambda^{n-1} - 6\lambda^{n-2}$$

$$\lambda^2 = 5\lambda - 6$$

$$-\lambda^2 + 5\lambda - 6 = 0$$

We compute the discriminant :

$$\Delta = 25 - (4 * -1 * -6) = 25 - 24 = 1$$

$$\lambda_1 = \frac{-5 + \sqrt{1}}{2 * (-1)} = \frac{-4}{-2} = 2$$

$$\lambda_2 = \frac{-5 - \sqrt{1}}{2 * (-1)} = \frac{-6}{-2} = 3$$

And we obtain

$$A = C_1\lambda_1 + C_2\lambda_2$$

$$a_n = C_1\lambda_1^n + C_2\lambda_2^n$$

For $n = 0$

$$a_0 = C_1 + C_2$$

For $n = 1$

$$a_1 = C_1\lambda_1 + C_2\lambda_2$$

and we obtain the following system :

$$\begin{cases} C_1 + C_2 = 0 \\ C_1\lambda_1 + C_2\lambda_2 = 1 \end{cases} = \begin{cases} C_1 + C_2 = 0 \\ 2C_1 + 3C_2 = 1 \end{cases}$$

Resolution :

$$C_1 = -C_2$$

$$2(-C_2) + 3C_2 = 1$$

$$C_2 = 1$$

$$C_1 = -1$$

Finally we have

$$\begin{aligned} a_n &= C_1\lambda_1^n + C_2\lambda_2^n \\ &= (-1) * 2^n + 1 * 3^n \\ &= 3^n - 2^n \end{aligned}$$

3

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_3 &= 5 \\ a_n &= 3a_{n-2} - 2a_{n-3} \end{aligned}$$

We find that

$$a_3 = 3a_1 - 2a_0$$

$$5 = 3 * 1 - 2 * a_0$$

$$2 = -2 * a_0$$

$$a_0 = -1$$

$$\lambda^n = 3\lambda^{n-2} - 2\lambda^{n-3}$$

We pick $n = 3$

$$\begin{gathered} \lambda^3 = 3\lambda - 2 \\ -\lambda^3 + 3\lambda - 2 = 0 \end{gathered}$$

We find that 1 is a root of the equation and we obtain

$$\begin{gathered} -\lambda^3 + 3\lambda - 2 = (\lambda - 1)(-\lambda^2 - \lambda + 2) = -(\lambda - 1)^2(\lambda + 2) \\ \lambda_1 = 1 \\ \lambda_2 = -2 \end{gathered}$$

$$\begin{aligned} P(\lambda) &= (\lambda - \lambda_1)^2(\lambda - \lambda_2) \\ &= (\lambda - 1)^2(\lambda - (-2)) \end{aligned}$$

$$\begin{aligned} a_n &= C_1\lambda_1^n + C_2 n\lambda_1^n + C_3\lambda_2^n \\ &= C_1 * 1^n + C_2 * n * 1^n + C_3 * (-2)^n \\ &= C_1 + n * C_2 + (-2)^n * C_3 \end{aligned}$$

For $n = 0$

$$a_0 = C_1 + C_3 = -1$$

For $n = 1$

$$a_1 = C_1 + C_2 - 2C_+1$$

For $n = 2$

$$a_2 = C_1 + 2C_2 + 4C_3 = 2$$

And we obtain the following system :

$$\begin{cases} C_1 + C_3 = -1 \\ C_1 + C_2 - 2C_3 = 1 \\ C_1 + 2C_2 + 4C_3 = 2 \end{cases}$$

$$C_1 = -1 - C_3$$

$$\begin{cases} -C_3 - 1 + C_2 - 2C_3 = 1 \\ -C_3 - 1 + 2C_2 + 4C_3 = 2 \end{cases} = \begin{cases} C_2 - 3C_3 - 1 = 1 \\ 3C_3 - 1 + 2C_2 = 2 \end{cases}$$

$$C_2 = 3C_3 + 2$$

$$3C_3 - 1 + 2(3C_3 + 2) = 2$$

$$3C_3 - 1 + 4 + 6C_3 = 2$$

$$9C_3 + 3 = 2$$

$$9C_3 = -1$$

$$C_3 = -\frac{1}{9}$$

$$C_2 = 3 * -\frac{1}{9} + 2 = \frac{5}{3}$$

$$C_1 = -1 - \frac{1}{9} = -\frac{8}{9}$$

Finally we have

$$\begin{aligned} a_n &= C_1\lambda_1^n = C_2 n\lambda_1^n + C_3\lambda_2^n \\ &= -\frac{4}{3} * 1^n + 2 * n * 1^n + (-\frac{1}{6}) * (-2)^n \\ &= -\frac{8}{9} + n\frac{5}{3} + (-2)^n \cdot -\frac{1}{9} \end{aligned}$$

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a)

It's clear that for $n = 1$, there is only one possible way of placing the domino. So $b_1 = 1 = a_2$ which is correct. The same for b_2, we could either place both two dominos horizontally or vertically, then we obtain $b_2 = 2 = a_3$ which is also correct.
Then we use a recurence relation, we got a rectangle of $2 \times n$ denote by R_n. At the first step, we could place a domine either horizontally or vertically, which let us have two cases :

- Horizontally : if the domino is placed horizontally, then we obtain a rectangle R_{n-1} to fill, which we know how to do it.
- Vertically : if the domino is placed vertically, then we are forced to place the second one next to him (bottom or top) in order to fill the hole. Then we obtain a rectangle R_{n-2} to fill, which we know how to do it.

Now to count the number of ways to arrange an R_n rectangle with dominos, we count the number of ways to arrange R_{n-1} and R_{n-2} and sum them. Which is exactly the recurence relation of the Fibonacci's sequence.

b)

It is the same problem as before. Any edge that is in the matching represent a domino tilled on a rectangle $2 \times n$. So the number of perfect matching in $P_{2,n}$ is given by a_{n+1}.

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a)

The number of different compositions of a positive integer n with all summands equal to 1 or 2 is given by a_{n+1}. For $n = 0$, there is 1 solution (empty compositions of 1 and 2). For $n = 1$, there is 1 solution. For $n = 2$, there are 2 solutions, which is the fibonacci sequence.
Then, for any n, we use the construction of n from 1 and 2 as a recursive one. We could either pick 1 or 2 at the beginning :

- If we pick 1, we count the number of different compositions with 1 and 2 of $n - 1$.
- If we pick 2, we count the number of different compositions with 1 and 2 of $n - 2$.

Then we add the number of different compositions of $n - 1$ and $n - 2$, and sum them. Which is exactly the fibonacci recursive relation.

b)

$$a_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-k}{k} = \sum_{k \geq 0} \binom{n-k}{k}$$

The proof is by induction on n :

Base cases :

- If $n = 0$, we have 1, which is a_1.
- If $n = 1$, we have 1, which is a_2.
- If $n = 2$, we have 2, which is a_3.

Inductive step : We use the following equation for the proof :

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

We suppose $n > 1$, then we have for $n + 1$

$$\begin{aligned}
\sum_{k\geq 0} \binom{n+1-k}{k} &= \sum_{k\geq 0} \binom{n-k}{k} + \sum_{k\geq 0} \binom{n-k}{k-1} \\
&= \sum_{k\geq 0} \binom{n-k}{k} + \sum_{k\geq 0} \binom{n-1-(k-1)}{k-1} \\
&= \sum_{k\geq 0} \binom{n-k}{k} + \sum_{j\geq 0} \binom{n-1-j}{j} \\
&= a_{n+1} + a_n = a_{n+2}
\end{aligned}$$