Mathematical Methods for Computer Science 1 Fall 2017

Series 9

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(a)

Proof

(1)	$A \to B$	premise
(2)	A	premise
(3)	B	Modus Ponens of 1 and 2
(4)	$B \to C$	premise
(5)	C	Modus Ponens of 3 and 4

(b)

By deduction lemma, we transform $(A \to B) \to ((B \to C) \to (A \to C))$ to $(A \to B), (B \to C), A \vdash C$:

$$\cfrac{\cfrac{\cfrac{(A \to B) \to ((B \to C) \to (A \to C))}{(A \to B) \vdash ((B \to C) \to (A \to C))}}{(A \to B), (B \to C) \vdash (A \to C)}}{(A \to B), (B \to C), A \vdash C}$$

Then we prove C :

Proof

(1)	$A \to B$	premise
(2)	A	premise
(3)	B	Modus Ponens of 1 and 2
(4)	$B \to C$	premise
(5)	C	Modus Ponens of 3 and 4

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We know that Γ_1 can prove A and Γ_2, A can prove B. In order to show that $\Gamma_1 \cup \Gamma_2, A \vdash B$, we would construct Γ_3 by pick either a formulas from Γ_1 or Γ_2 (with respect in wich they appear in the sequence) until both are empty. Clearly, Γ_3 is $\Gamma_1 \cup \Gamma_2$. Then, A will appear in the sequence and A is provable by Γ_1. Then, B is provable because Γ_2 and A appears in Γ_3.

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(a)

By deduction lemma, we transform $((A \wedge B) \to C) \to (A \to (B \to C))$ to $((A \wedge B) \to C), A, B \vdash C$:

$$\cfrac{\cfrac{\cfrac{((A \wedge B) \to C) \to (A \to (B \to C))}{((A \wedge B) \to C) \vdash (A \to (B \to C))}}{((A \wedge B) \to C), A \vdash (B \to C)}}{((A \wedge B) \to C), A, B \vdash C}$$

Then we prove C :

Proof

(1)	A	premise
(2)	B	premise
(3)	$(A \to (B \to (A \wedge B)))$	Axiom 4
(4)	$(B \to (A \wedge B))$	Modus Ponens of 1 and 3
(5)	$(A \wedge B)$	Modus Ponens of 1 and 4
(6)	$((A \wedge B) \to C)$	premise
(7)	C	Modus Ponens of 5 and 6

(b)

By deduction lemma, we transform $(A \wedge B) \to (B \wedge A)$ to $(A \wedge B) \vdash (B \wedge A)$:

$$\cfrac{(A \wedge B) \to (B \wedge A)}{(A \wedge B) \vdash (B \wedge A)}$$

Proof

(1)	$(A \wedge B)$	premise
(2)	$(A \wedge B) \to A$	Axiom 3a
(3)	A	Modus Ponens of 1 and 2
(4)	$(A \wedge B) \to B$	Axiom 3b
(5)	B	Modus Ponens of 2 and 4

(6)	$(B \to (A \to (B \wedge A)))$	Axiom 4
(7)	$(A \to (B \wedge A))$	Modus Ponens of 6 and 5
(8)	$(B \wedge A)$	Modus Ponens of 7 and 3

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In order to show that

$$((A \wedge C) \vee (B \wedge C)) \to ((A \vee B) \wedge C)$$

is provable in our proof system, we will use the case distinction lemma :

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, (A \vee B) \vdash C}$$

By using the deduction lemma, we transform $((A \wedge C) \vee (B \wedge C)) \to ((A \vee B) \wedge C)$ into $((A \wedge C) \vee (B \wedge C)) \vdash ((A \vee B) \wedge C)$:

$$\frac{((A \wedge C) \vee (B \wedge C)) \to ((A \vee B) \wedge C)}{((A \wedge C) \vee (B \wedge C)) \vdash ((A \vee B) \wedge C)}$$

Then, if we show that $\Gamma, (A \wedge C) \vdash ((A \vee B) \wedge C)$ and $\Gamma, (B \wedge C) \vdash ((A \vee B) \wedge C$, we could conclude by the case distinction lemma that

$$\frac{\Gamma, (A \wedge C) \vdash ((A \wedge B) \vee C) \qquad \Gamma, (B \wedge C) \vdash ((A \wedge B) \vee C)}{\Gamma, (A \wedge B) \vee (B \wedge C) \vdash ((A \wedge B) \vee C)}$$

Proof of $\Gamma, (A \wedge C) \vdash ((A \wedge B) \vee C)$

(1)	$A \wedge C$	premise
(2)	$(A \wedge C) \to A$	Axiom 3a
(3)	$(A \wedge C) \to C$	Axiom 3b
(4)	A	Modus Ponens of 1 and 3
(5)	C	Modus Ponens of 1 and 2
(6)	$A \to (A \vee B)$	Axiom 5a
(7)	$A \vee B$	Modus Ponens of 4 and 6
(8)	$(A \vee B) \to (C \to ((A \vee B) \wedge C))$	Axiom 4
(9)	$C \to ((A \vee B) \wedge C)$	Modus Ponens of 7 and 8
(10)	$(A \vee B) \wedge C$	Modus Ponens of 5 and 9

Proof of $\Gamma, (B \wedge C) \vdash ((A \wedge B) \vee C)$

(1)	$B \wedge C$	premise
(2)	$(B \wedge C) \to B$	Axiom 3a
(3)	$(B \wedge C) \to C$	Axiom 3b

(4)	B	Modus Ponens of 1 and 2
(5)	C	Modus Ponens of 1 and 3
(6)	$B \to (A \vee B)$	Axiom 5b
(7)	$A \vee B$	Modus Ponens of 4 and 6
(8)	$(A \vee B) \to (C \to ((A \vee B) \wedge C))$	Axiom 4
(9)	$C \to ((A \vee B) \wedge C)$	Modus Ponens of 7 and 8
(10)	$(A \vee B) \wedge C$	Modus Ponens of 5 and 9

Finally, by case distinction lemma, we have

$$\frac{\Gamma, (A \wedge C) \vdash ((A \wedge B) \vee C) \qquad \Gamma, (B \wedge C) \vdash ((A \wedge B) \vee C)}{\Gamma, (A \wedge B) \vee (B \wedge C) \vdash ((A \wedge B) \vee C)}$$

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We use the following lemma (Lemma 1) in order to prove $\Gamma \vdash \neg\neg A \to A$:

$$\frac{A \qquad A}{A}$$

By deduction lemma, $\vdash A \to A$ is provable if and only if $\Gamma, A \vdash A$. Because A is a premise, A holds.
Then, we prove $\vdash \neg\neg A \to A$:

Proof

(1)	$(\neg\neg A \to (A \to A)) \to ((\neg\neg A \to A) \to (\neg\neg A \to A))$	Axiom 2
(2)	$(\neg\neg A \to A) \to ((\neg\neg A \to A) \to (\neg\neg A \to A))$	By Lemma 1
(3)	$(\neg\neg A \to A) \to (\neg\neg A \to A)$	By Lemma 1
(4)	$\neg\neg A \to A$	By Lemma 1

So we have that $\Gamma \vdash \neg\neg A \to A$ holds in our proof system. Then, we have to show that the axiom set without the axiom 9 $(A \vee \neg A)$ and with the axiom 9′ $(\neg\neg A \to A)$ can prove $A \vee \neg A$. So we prove $\Gamma \setminus \{A \vee \neg A\} \cup \{\neg\neg A \to A\} \vdash A \vee \neg A$.

Proof

(1)	$\neg(A \vee \neg A)$	Assumption
(2)	A	Assumption
(3)	$A \to (A \vee \neg A)$	Axiom 5a
(4)	$A \vee \neg A$	Modus Ponens of 2 and 3
(5)	$\perp$	Because 1 and 2 are in contrary, one is false
(6)	$\neg A$	Proof by contradiction

(7)	$\neg A \rightarrow (A \vee \neg A)$	Axiom 5b
(8)	$A \vee \neg A$	Modus Ponens of 6 and 7
(9)	$\perp$	Because 1 and 8 are in contrary, one is false
(10)	$\neg\neg(A \vee \neg A)$	Proof by contradiction
(11)	$A \vee \neg A$	Double Negation elimination (see before)