Mathematical Methods for Computer Science 1 Fall 2017

Series 9

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(a)

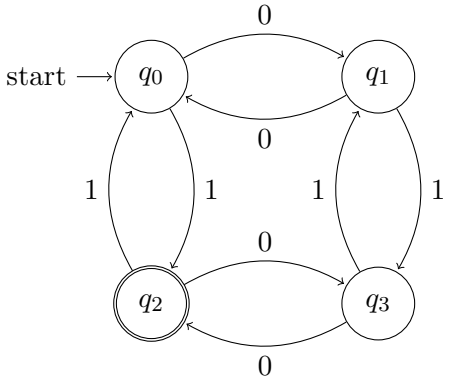
Let $w \in L$, where $L = \{\omega \in \{0,1\} * |l_0(\omega) > l_1(\omega)\}$. We assume L is regular, then there must be some decomposition $w = xyz$ with $|xy| \leq p$ and $|y| \geq 1$ such that xy^iz in L for every $i \geq 0$. We consider the string $w = 0^a1^b$ where $a > b$, $w \in L$ and $|x| \geq n$. So by the Pumping Lemma, $\exists x, y, z$ such that the Pumping Lemma hold.
Then $w = 0^a1^b = xyz$ with $|xy| \geq n$ and $|y| \geq 1$. So $x = 0^s, y = 0^t, z = 0^u1^v$ with $s + t \leq n$, $t \geq 1$, $u \geq 0$ and $s + t + u > v$. Then, for $i = 0$, we have :

$$xy^iz = xz = 0^t0^v1^u = 0^{t+v}1^u \notin L$$

So if $xyz \in L$, then $xz \notin L$, since xz has fewer 0 than xyz but the same number of 1.So the Pumping Lemma don't hold.

(b)

The following DFA accept the langage $\{\omega \in \{0,1\} * |l_0(\omega) \text{ is even}, l_1(\omega) \text{ is odd}\}$.



(c)

Let $w \in L$, where $L = \{\omega \in \{0,1\} * |l_0(\omega) \neq l_1(\omega)\}$. We assume L is regular, then there must be some decomposition $w = xyz$ with $|xy| \leq p$ and $|y| \geq 1$ such that xy^iz in L for every $i \geq 0$. We consider the string $w = 0^a1^b$ where $a < b$, $w \in L$ and $|x| \geq n$. So by the Pumping Lemma, $\exists x, y, z$ such that the Pumping Lemma hold.
Then $w = 0^a1^b = xyz$ with $|xy| \geq n$ and $|y| \geq 1$. So $x = 0^s, y = 0^t, z = 0^u1^v$ with $s + t \leq n$, $t \geq 1$, $u \geq 0$ and $s + t + u < v$.

Now, if we reapet i a certain number of time, we obtain a string where $s + t + u = v$ which is not in L. Therefore L is not regular.

2

(a)

The minimum number of states of the DFA that accepts the langage of all binary words whose tenth letter is 0, is 11, since we have to count the move inside de word in order to accept a word of L or not. Now we consider a automaton M with 10 states, then with each transition from q_i to q_{i+1}, we can count a letter in the word, and only one.
The proof is by induction on the number of states n:

Base case 1 : $n = 1$ Clearly, with only on state we can't count anything for a word, because transition only occurs between the initial state and itself.

Base case 2 : $n = 2$ With 2 states, we can count 1 element in the word, since we don't have any memory, when we go back to the initial state from the second one, we don't know how many times the transition has been taken, we can only know that it has been a transition from the initial state to the second one.

Inductive step : with m states, we can count $m - 1$ transition, each time a transition is taken from q_i to q_{i+1}. When going back from a state q_j to q_i where $j > i$, we can't make any assumption about the number of times this transition is taken. But if there are no back transition from a state q_j to a state q_i where $j > i$, we can, in a way, do some counting on constant.

(b)

No idea...

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(a)

We have $\omega = z_1z_2z_3 = z_1uvwz_3$ and the Pumping Lemma hold, since u and w can be seen as substring of z_1, respectively z_3 (we consider the Pumping Lemma for a string abc where $a = z_1u$, $b = v$ and $c = wz_3$). There are no condition on the length of u and w, therefore if both length is 0, this is just the Pumping Lemma.

(b)

In order to prove that $L = \{a^ib^jc^j | i, j \geq 0\}$ is non regular, we will show that $L' = \{b^jc^j | j \geq 0\}$ is non regular. Then, by definition, L is non regular too.
$L' = \{b^jc^j | j \geq 0\}$ is non regular :
We assume that L is regular, then the Pumping Lemma must hold. Let $w = b^nc^n$, therefore $x \in L'$ and $|x| \geq n$, so by the Pumping Lemma, $\exists u, v, w$ such that

$$x = uvw \tag{1}$$
$$|uv| \leq n \tag{2}$$
$$|v| \geq 1 \tag{3}$$
$$\forall i \geq 0 : uv^i w \in L' \tag{4}$$

We show that $\forall u, v, w$, (1) to (4) don't hold. If (1) to (3) hold, then $x = b^n c^n = uvw$ with $|uv| \leq n$ and $|v| \geq 1$. Therefore $u = b^s, v = b^t, w = b^p c^n$ with $s + t \leq n$, $t \geq 1$, $p \geq 0$, $s + t + p = n$. But here (4) don't hold for $i = 0$:

$$uv^0 w = uw = b^s b^p c^n = b^{s+p} c^n \notin L', \text{ since } s + p \neq n$$

Finally, L in non regular because L' is non regular.

4

(a)

$$\Sigma = \{a, b, c\}$$
$$\Delta = \{0, 1\}$$
$$h : \Sigma^* \mapsto \Delta^*$$
$$h(a) = 0, h(b) = 01, h(c) = 10$$
$$L = (a + b)^* + (a + c)^*$$
$$h(L) = L' = (0 + 01)^* + (0 + 10)^*$$

L' do not contains repeating 1's and don't start and end with a 1 at the same time.
In order to construct a string in L', we have to choose one of the two branch :

$L'' = (0 + 01)^*$ **:** here it's clear that a string $w \in L''$ can't start with a 1, since we have to choose either 0, 01 or ϵ to start the string with. So the second condition hold. Then, we consider a string w that's end with a 1, in order to have repeating 1, we have to pick a string $w' \in L''$ which start with a 1, which is not possible because we saw that $w \in L''$ necessarily start with a 1 or is empty.

$L''' = (0 + 10)^*$ **:** here, a string $w \in L'''$ can either start with a 1 or a 0 (or be the empty string...), but it is impossible that such a string end with a 0, since both branch of the + are ending with a 0. So the second condition hold. Then, in order to have repeating 1, we have to append 10 to a string $w' \in L'''$ that end with a 1, which is not possible because we saw that $w \in L'''$ necessarily end with a 0 or is empty.

Therefore, $L' = L'' + L'''$ fullfil the conditions.

(b)

$$\Delta = \{0, 1\}$$
$$h : \Sigma^* \mapsto \Delta^*$$
$$L = 0(01)^*$$
$$h^{-1}(L) \subset \Sigma^*$$

We can either pick

$$h(a) = 0, h(b) = 10$$

or

$$h(a) = 0, h(b) = 01$$

since $0(10)^*$ and $(01)^*0$ describe the same regular expression.

$$\Delta = \{0, 1\}$$
$$h : \Sigma^* \mapsto \Delta^*$$
$$L = 0(01)^*$$
$$h^{-1}(L) \subset \Sigma^*$$
$$h(a) = 0, h(b) = 10$$
$$h'(a) = 0, h'(b) = 01$$
$$h(L) = ab^*$$
$$h'(L) = b^*a$$
$$h'(L) = h(L)$$

$h^{-1}(L) =$ all binary word that start and end with a 0 and with alternating 0 and 1

5

L is closed under reverse.
Assume L is defined by a regular expression E, then we show that there is another regular expression $rev(E)$ such that $L(rev(E)) = rev(L(E))$, where $rev(E)$ is the reversal of the langage of E.
The proof is by induction on E :

Base case : if $E = \epsilon$, then it's clear that $rev(E) = rev(\epsilon) = \epsilon = E$.

Inductive step : there are three cases depending on E :

1. $E = F + G$, then $rev(E) = rev(F) + rev(G)$: the reverse of the union of two languages is obtain by taking the union of the reversal of these languages.

2. $E = FG$, the $rev(E) = rev(G)rev(F)$: the reverse of the concatenation of two languages is the concatenation of their reversal.

3. $E = F^*$, then $rev(E) = rev(F)^*$: any string $w \in rev(F)^*$ is of the form $w_1 w_2 \dots w_n$ where w_i is the reversal of a string in F. Therefore $rev(w) = rev(w_1)rev(w_2) \dots rev(w_n)$ and $rev(w_i) \in rev(E)$, so $rev(w) \in rev(F)^*$

We can also construct a DFA M' from the DFA M that recognize L in order to recognize $L' = rev(L)$, by

- Reversing all arcs.
- Make the initial state of M the new unique accepting state.
- Create a new initial state q_0' with $\delta(q_0', \epsilon) = F$.