Mathematical Methods for Computer Science 1 Fall 2017

Series 8

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(a)

In order to show that $\{\rightarrow, \neg\}$ is complete, we have to found the equivalent formula of $a \vee b$ and $a \wedge b$ ($\neg a$ and $a \rightarrow b$ are trivial...).

$a \vee b$

$$(\neg a) \rightarrow b = (\neg\neg a) \vee b = a \vee b$$

and the truth table to validate the result :

a	b	¬	a	→	b
T	T	F	T	T	T
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	F	F	F

$a \wedge b$

$$\begin{aligned}\neg(a \rightarrow \neg b) &= \neg(\neg a \vee \neg b) \\ &= \neg\neg(a \wedge b) \\ &= (a \wedge b)\end{aligned}$$

and the truth table to validate the result :

a	b	¬	(	a	→	¬	b	)
T	T	T		T	F	F	T	
T	F	F		T	T	T	F	
F	T	F		F	T	F	T	
F	F	F		F	T	T	F	

(b)

In order to show that $\{\uparrow\}$ is complete, we have to found the equivalent formula of $a \rightarrow b$ and $\neg a$, because we just showed that $\{\neg, \rightarrow\}$ is complete.
We know that $a \uparrow b \leftrightarrow \neg(a \wedge b)$.

$\neg a$

$$a \uparrow a = \neg(a \wedge a) = \neg a$$

and the truth table to validate the result :

a	¬	(	a	&	a	)
T	F		T	T	T	
F	T		F	F	F	

$a \to b$

$$\begin{aligned} a \to b &= \neg a \vee b \\ &= \neg\neg(\neg a \vee b) \\ &= \neg(a \wedge \neg b) \\ &= \neg(a \wedge \underbrace{\neg(b \wedge b)}_{b \uparrow b}) \\ &= \underbrace{\neg(a \wedge (b \uparrow b))}_{a \uparrow (b \uparrow b)} \\ &= a \uparrow (b \uparrow b) \end{aligned}$$

and the truth table to validate the result :

a	b	a	↑	(	b	↑	b	)
T	T	T	T		T	F	T	
T	F	T	F		F	T	F	
F	T	F	T		T	F	T	
F	F	F	T		F	T	F	

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Note : we denote $\top$ the formula which represent *true* and $\bot$ the formula which represent *false* using the following equivalence :

$$\begin{aligned} \top &= (\neg A \vee A) \\ \bot &= (\neg A \wedge A) \end{aligned}$$

We can also use the following equivalence :

$$(A \wedge B) \vee A \leftrightarrow A$$

We can also use the following equivalence :

$$(A \wedge B) \vee A \leftrightarrow A$$

and use thr truth table for proving the equivalence :

a	b	((	a	&	b	)	∨	a	)	↔	a
1	1		1	1	1		1	1		1	1
1	0		1	0	0		1	1		1	1
0	1		0	0	1		0	0		1	0
0	0		0	0	0		0	0		1	0

(a)

$$
\begin{aligned}
(p \to q) \land ((q \lor r) \to p) &= (\neg p \lor q) \land (\neg(q \lor r) \lor p)\\
&= (\neg p \lor q) \land ((\neg q \land \neg r) \lor p)\\
&= (\neg p \lor q) \land (p \lor \neg q) \land (p \lor \neg r)\\
&= ((p \land (\neg p \lor q)) \lor (\neg q \land (\neg p \lor q))) \land (p \lor \neg r)\\
&= (\underbrace{(p \land \neg p)}_{\perp} \lor (p \land q) \lor (\neg q \land \neg p) \lor \underbrace{(\neg q \land q)}_{\perp}) \land (p \lor \neg r)\\
&= ((p \land q) \lor (\neg q \land \neg p)) \land (p \lor \neg r)\\
&= ((p \lor \neg r) \land (\neg q \land \neg p)) \lor ((p \lor \neg r) \land (p \land q))\\
&= \underbrace{(\neg q \land \neg p \land p)}_{\perp} \lor (\neg q \land \neg r \land \neg p) \lor \underbrace{(p \land q \land p)}_{p \land q} \lor (p \land q \land \neg r)\\
&= (\neg q \land \neg r \land \neg p) \lor (p \land q) \lor (p \land q \land \neg r)\\
&= (\neg q \land \neg r \land \neg p) \lor (p \land q)
\end{aligned}
$$

(b)

Truth table of $\phi = (p \to q) \land ((q \lor r) \to p)$:

p	q	r	(p	→	q)	&	((q	∨	r)	→	p)
1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1	0	1	1
1	0	1	1	0	0	0	0	1	1	1	1
1	0	0	1	0	0	0	0	0	0	1	1
0	1	1	0	1	1	0	1	1	1	0	0
0	1	0	0	1	1	0	1	1	0	0	0
0	0	1	0	1	0	0	0	1	1	0	0
0	0	0	0	1	0	1	0	0	0	1	0

We pick the row where $\phi = 1$ and we construct the DNF form :

$$
\begin{gathered}
(p \land q \land r) \lor (\neg p \land \neg q \land \neg r) \lor (p \land q)\\
=\\
(\neg p \land \neg q \land \neg r) \lor (p \land q)
\end{gathered}
$$

Because

$$(p \land q \land r) \lor (p \land r) \leftrightarrow (p \land r)$$

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Using De Morgan's law, $\neg(DNF) = CNF$ and A is already in DNF.

(a)

$$A = (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

$$\begin{aligned}
\neg A &= \neg((p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)) \\
&= (\neg(p \wedge q \wedge \neg r) \wedge \neg(p \wedge \neg q \wedge r) \wedge \neg(\neg p \wedge \neg q \wedge \neg r)) \\
&= (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee q \vee r)
\end{aligned}$$

(b)

We can turn out the formulation into the following formula :

$(A \leftrightarrow B)$	A is equivalent to B
$\wedge(A \leftrightarrow (C \vee F))$	A contains a sub-formula F
$\wedge(A \leftrightarrow (C \vee G))$	B is the sub-formula A with F replace by B
$\rightarrow(G \leftrightarrow F)$	implies that F is equivalent to G

Using the following truth table, we can see that the formula is not a tautology, because F and G can be different in the formula but A and B still remains equivalent.

A	B	C	F	G	((A	$\leftrightarrow$	B)	$\wedge$	((A	$\leftrightarrow$	(C	$\vee$	F))	$\wedge$	(B	$\leftrightarrow$	(C	$\vee$	G))))	$\rightarrow$	(G	$\leftrightarrow$	F)
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1
1	1	1	0	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1	0	0
1	1	1	0	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	0	1	0
1	1	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1
1	1	0	1	0	1	1	1	0	1	1	0	1	1	0	1	0	0	0	0	1	0	0	1
1	1	0	0	1	1	1	1	0	1	0	0	0	0	0	1	1	0	1	1	1	1	0	0
1	1	0	0	0	1	1	1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0
1	0	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1
1	0	1	1	0	1	0	0	0	1	1	1	1	1	0	0	0	1	1	0	1	0	0	1
1	0	1	0	1	1	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	0	0
1	0	1	0	0	1	0	0	0	1	1	1	1	0	0	0	0	1	1	0	1	0	1	0
1	0	0	1	1	1	0	0	0	1	1	0	1	1	0	0	0	0	1	1	1	1	1	1
1	0	0	1	0	1	0	0	0	1	1	0	1	1	1	0	1	0	0	0	1	0	0	1
1	0	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0
1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	1	0
0	1	1	1	1	0	0	1	0	0	0	1	1	1	0	1	1	1	1	1	1	1	1	1
0	1	1	1	0	0	0	1	0	0	0	1	1	1	0	1	1	1	1	0	1	0	0	1
0	1	1	0	1	0	0	1	0	0	0	1	1	0	0	1	1	1	1	1	1	1	0	0
0	1	1	0	0	0	0	1	0	0	0	1	1	0	0	1	1	1	1	0	1	0	1	0
0	1	0	1	1	0	0	1	0	0	0	0	1	1	0	1	1	0	1	1	1	1	1	1
0	1	0	1	0	0	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1	0	0	1
0	1	0	0	1	0	0	1	0	0	1	0	0	0	1	1	1	0	1	1	1	1	0	0
0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	1	0	1	0
0	0	1	1	1	0	1	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1	1
0	0	1	1	0	0	1	0	0	0	0	1	1	1	0	0	0	1	1	0	1	0	0	1
0	0	1	0	1	0	1	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	0	0
0	0	1	0	0	0	1	0	0	0	0	1	1	0	0	0	0	1	1	0	1	0	1	0
0	0	0	1	1	0	1	0	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	1
0	0	0	1	0	0	1	0	0	0	0	0	1	1	0	0	1	0	0	0	1	0	0	1
0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	0	0
0	0	0	0	0	0	1	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0

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(a)

$(p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$ is a tautology :

$$
\begin{aligned}
(p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p) &= (\neg p \vee q) \rightarrow ((\neg p \vee \neg q) \rightarrow \neg p) \\
&= (\neg p \vee q) \rightarrow (\neg(\neg p \vee \neg q) \vee \neg p) \\
&= (\neg p \vee q) \rightarrow ((p \wedge q) \vee \neg p) \\
&= (\neg p \vee q) \rightarrow (\underbrace{(\neg p \vee p)}_{\top} \wedge (\neg p \vee q)) \\
&= (\neg p \vee q) \rightarrow (\neg p \vee q) \\
&= \top
\end{aligned}
$$

$A \to A$ is a tautology :

$$A \to A = \neg A \lor A = \top$$

So $(\neg p \lor q)$ can be subsitute by A, then we would have $A \to A$.

(b)

$\neg(p \to q) \lor (\neg p \lor q)$ is a tautology :

$$\begin{aligned}\neg(p \to q) \lor (\neg p \lor q) &= \neg(\neg p \lor q) \lor (\neg p \lor q)\\ &= (\neg\neg p \land \neg q) \lor (\neg p \lor q)\\ &= (p \land \neg q) \lor (\neg p \lor q)\end{aligned}$$

Here, we put $A = (p \land \neg q)$, so $\neg A = \neg(p \land \neg q) = (\neg p \lor \neg\neg q) = (\neg p \lor q)$. So we have $A \lor \neg A$ (by subsitution) and this is a tautology.

(c)

$\phi = (p \lor q \lor r) \land (p \lor q \lor \neg s)$ is not a tautology. For example, the following interpretation I does not satisfies the formula :

$$I = \{p \mapsto \bot, q \mapsto \bot, r \mapsto \bot, s \mapsto \top\}$$

$$\phi^I = \underbrace{(\bot \lor \bot \lor \bot)}_{\bot} \land \underbrace{(\bot \lor \bot \lor \neg\top)}_{\bot} = \bot \land \bot = \bot$$

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(a)

$$(\neg p \land \neg q \land \neg r) \lor (\neg p \land q \land r) \lor (p \land q \land r)$$

We can transform $(\neg p \land q \land r) \lor (p \land q \land r)$ into $(q \land r)$, because if p is $true$, we don't look at $(\neg p \land q \land r)$ because it would be $false$ anyway and if p is $false$, we don't look at $(p \land q \land r)$ because it would be false anyway.

$$(\neg p \land \neg q \land \neg r) \lor (q \land r)$$

We can't simplify this anymore.

(b)

$$(\neg p \lor \neg q \lor \neg r) \land (\neg p \lor q \lor r) \land (p \lor q \lor r)$$

We can use the same principle as before, if p is $true$, $(p \lor q \lor r)$ would be $true$ and $(\neg p \lor q \lor r)$ would be transform to $(q \lor r)$, else if p is $false$, $(\neg p \lor q \lor r)$ would be $true$ and $(p \lor q \lor r)$ would be equal to $(q \lor r)$.

$$(\neg p \lor \neg q \lor \neg r) \land (p \lor q)$$

(c)

We can use the following function ϕ, we denote the arguments of ϕ by $\lambda = \{\underbrace{a, b, \ldots}_{|\lambda|}\})$:

$$\phi(\lambda) = \bigwedge_{\alpha \in \lambda} \alpha = a \wedge b \wedge c \wedge \cdots$$

The DNF form of such a formula can't be less than $|\lambda|$, because each variable needs to appear in the DNF form. If one of the variable would be false, the whole formula is going to be false.