Mathematical Methods for Computer Science I

Fall 2017

Series 10 – Hand in before Monday, 04.12.2017 - 12.00

1. Construct a proof tree or a deduction tree that contains a counterexample for each of the following propositional formulas:
   a) $p \to (q \to (p \wedge q))$
   b) $(p \vee q) \to ((p \to q) \vee q)$
   c) $(p \to q) \vee (\neg p \to \neg q)$

2. Construct expanded deduction trees and use them to find a conjunctive normal form for the following formulas:
   a) $(p \to r) \to ((q \to s) \to ((p \vee q) \to r))$
   b) $(p \to q) \to ((q \to \neg r) \to \neg p)$

3. Extend the set of connectives by $\leftrightarrow$ and $\oplus$ defined as

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p), \quad p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$$

   Give Gentzen-like rules for these connectives. That is, find out what must be written above the line in the inference rules

$$\frac{}{\Gamma \vdash A \leftrightarrow B, \Delta} \quad \text{etc.}$$

4. Consider expanded deduction trees with the root $\vdash A$, where A is some propositional formula. Recall that there are different expanded deduction trees with the same root: when constructing a tree, at each vertex we have a choice from which formula to eliminate the top level logical connective.
   a) In what order should one proceed so that to minimize the number of the vertices in the expanded deduction tree?
   b) Assume that A contains m logical connectives. Show that the number of leaves in an expanded deduction tree is at most 2^m.
   c) Give an example of a formula with $2k - 1$ connectives whose expanded tree has 2^k leaves. (Hint: there is an example that contains $\wedge$ and $\vee$ only.)
   d)* Is it true that for every formula with m connectives the expanded tree has at most $2^{\lceil \frac{m}{2} \rceil}$ leaves? (Here $\lceil x \rceil$ is the smallest integer equal or greater x.)
   e)* Is the number of leaves the same in all expanded deduction trees for a given formula?

5. Assume that a proof system of Hilbert type is given.
   A set of formulas $\{A_1, \ldots, A_n\}$ is called *satisfiable* if these formulas have a common model. A set of formulas $\{A_1, \ldots, A_n\}$ is called *inconsistent* if

$$A_1, \ldots, A_n \vdash B \text{ and } A_1, \ldots, A_n \vdash \neg B$$

   for some formula B.
   a) Assume that every satisfiable set of formulas is consistent. Show that then the proof system is sound.
   b)* Assume that every consistent set of formulas is satisfiable. Does this imply that the proof system is complete?