Mathematical Methods for Computer Science 2
Spring 2018

Series 3

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1

a)

By extending the brackets of

$$(1-x)(1-x^2)(1-x^4)(1-x^8)\cdot\ldots$$

We can see that any x^n is obtained only one time. For example, in order to obtain x^7, we have to multiply x, x^2, x^4 and pick the 1 in all the other factors. If we don't pick a 1, we obtain a n which is greater that 7. By the way, it is the encoding of the integer in the binary format.

b)

The result obtained is the sum from a) and the sign in front of x^n is plus only when the number we encode into the binary format is encoded with an even number of bits.
If the number of bits is even, it means we have take an even number of x^{2^k} from the product and then an even number of negative x^{2^k} which leads to a positive x^n.

2

a)

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n = (1+x+x^2+x^3+\ldots)^k &= (\frac{1}{(1-x)})^k \\ &= \frac{1}{(1-x)^k} \\ &= (1-x)^{-k} \\ &= \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n\end{aligned}$$

b)

$$
\begin{aligned}
\frac{1}{1-x-x^2} &= \frac{1}{1-(x+x^2)} \\
&= 1+(x+x^2)x+(x+x^2)x^2+(x+x^2)x^3+(x+x^2)x^4+\dots \\
&= 1+x+2x^2+3x^3+5x^4+8x^5+13x^6+\dots
\end{aligned}
$$

The coefficient in front of each x^n follow the fibonnaci sequence. The summands are represented by the factors $(x+x^2)$, which means the summands 1 and 2. Then, by expanding we see that the number of compositions with summands 1 and 2.

3

a)

b)

4

a)

The generating function for $p_{n,k}$ is :

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_n x^n &= (1+x+x^2+x^3+\dots)(1+x^2+x^4+x+\dots)(1+x^3+x^6+x^9+\dots)\dots \\
&= a_0+a_1x+a_2x^2+\dots \\
&= \prod_{k=1}^{\infty}(\frac{1}{1-x^k})
\end{aligned}
$$

The x^n term in the series $a_0+a_1x+a_2x^2+\dots$ count the number of ways to obtain n where $n=a_1+2a_2+3a_3+\dots$, which is the number of partition of n.

b)

If we need to have the number of partitions of n into exactly k parts, we multiply the factor by x^k :

$$
\prod_{k=1}^{\infty}(\frac{x^k}{1-x^k})
$$

c)

$$
p_{n-k,\geq k} = p_{n,k}
$$

Algebraic proof :

$$p_{n-k,\geq k} = \prod_{k=1}^{\infty}(\frac{1}{1-x^k})$$
$$p_{n,k} = \prod_{k=1}^{\infty}(\frac{x^k}{1-x^k})$$

5

a)

$$p_{n-k,\geq k} = p_{n,k}$$

Proof by bijection : we consider two sets P_1 and P_2 where P_1 is the set of partitions of $n-k$ with no parts greater than k and P_2 is the set of partitions of n with parts into exactly k parts.
Clearly, we have $P_1 \cap P_2 = \emptyset$
We can construct the set P_1 from the set P_2 by using the following construction for each parts :

- create the Ferrers diagram of the part
- separate k into $K = \underbrace{\{1, 1, \ldots, 1\}}_{k}$
- for each $k_i \in K$ add 1 square to the first row of the diagram, then one to the second row, one to the third, hand so on.
- We add one square to the row only if the row index is smaller or equals to k.
- If we are at the last row, we get back to the first one and repeat the process until we have no k_i left.

This way, each element of P_1 is associated with an unique element in P_2.

b)

Let

$$f(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\cdot \ldots$$
$$g(x) = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^4+x^8\ldots)\cdot \ldots$$

Then we have

$$
\begin{aligned}
g(x) &= (1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^4+x^8\dots)\cdot\dots \\
&= (\frac{1}{1-x})(\frac{1}{1-x})(\frac{1}{1-x^4})\cdots \\
&= \frac{(\frac{1}{1-x^3})}{(\frac{1}{1-x^3})(\frac{1}{1-x})(\frac{1}{1-x^2})(\frac{1}{1-x^4})\cdots} \\
&= \frac{\prod(1-x^{3i})}{\prod(1-x^i)} \\
&= (\frac{1-x^3}{1-x})(\frac{1-x^6}{1-x^2})(\frac{1-x^9}{1-x^3})(\frac{1-x^{12}}{1-x^5})(\frac{1-x^{15}}{1-x^6})\cdots \\
&= (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\cdot\dots = f(x)
\end{aligned}
$$