Mathematical Methods for Computer Science 1 Fall 2017

Series 12

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(a)

The formula

$$\phi = (\forall x.(P(x) \wedge (\exists y.(\forall z.(Q(y,z)))))) \to (\forall x.(P(x) \vee (\exists y.(\forall z.(Q(y,z))))))$$

is a special case of the formula $(q \wedge p) \to (q \vee p)$, so if $(q \wedge p) \to (q \vee p)$ is valid in our proof system, then ϕ is valid too.
So we have to prove $\vdash (q \wedge p) \to (q \vee p)$, by deduction lemma, we are going to prove $q \wedge p \vdash q \vee p$.

Proof

(1)	$q \wedge p$	premise
(2)	$(q \wedge p) \to q$	Axiom 3a
(3)	q	Modus Ponens of 1 and 2
(4)	$q \to (q \vee p)$	Axiom 5b
(5)	$q \vee p$	Modus Ponens of 3 and 4

(b)

The formula

$$\phi = Q(x,y) \to (\forall z.(P(z)) \to (Q(x,y) \to \forall z.(P(z))))$$

is a special case of the formula $p \to (q \to (p \wedge q))$ and $p \to (q \to (p \wedge q))$ is an axiom in our proof system, then ϕ is a valid formula.

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(a)

$$\phi = (\forall x.(\exists y.(P(x,y)))) \vee (\forall x.(\exists y.(\neg P(x,y))))$$

$$P(x,y) = \{(x,y) | x = y\}$$
$$U = \{0,1\}$$

In order to falsify ϕ, we have to falsify both side of the $\lor$ connective.
To falsify $\forall x.(\exists y.(P(x,y)))$, for any x, we can found an y that is not equal to x, and then falsify the formula. For $x = 1$, we pick $y = 0$ and for $x = 0$ we pick $y = 1$.
To falsify $\forall x.(\exists y.(\neg P(x,y)))$, for any x, we can found an y that is equal to x, and then falsify the formula. For $x = 1$, we pick $y = 1$ and for $x = 0$, we pick $y = 0$.
We can't make U smaller because :

- $|U| = 0$ is not possible, because $U \neq \emptyset$
- $|U| = 1$ is not possible, because the predicate have to "return" $false$ or $true$, and it is not possible to construct a predicate that is non-deterministic. For example, $P(1,1)$ can't "return" $true$ one time and $false$ on another time.

(b)

$$\phi = \forall x.(\exists y.(P(x,y))) \to \exists x.(\forall y.(P(x,y)))$$

$$P(x,y) = \{(x,y)|x = y\}$$
$$U = \{0,1\}$$

In order to falsify ϕ, we have to satisfy the left hand side and falsify the right hand side of the $\to$ connective.
To satisfy $\forall x.(\exists y.(P(x,y)))$, we have to find, for any x, and y which ois equal to x. For $x = 1$, we pick $y = 1$ and for $x = 0$, we pick $y = 0$.
To falsify $\exists x.(\forall y.(P(x,y)))$, we have to find an x, that for all y, x is not always equal to y. We pick $x = 1$, then if $y = 1$, the formula is satisfy, but for $y = 0$, the formula is not.
We can't make U smaller because :

- $|U| = 0$ is not possible, because $U \neq \emptyset$
- $|U| = 1$ is not possible, we use the same argument as before. The predicate $P(x,y)$ is deterministic. So if we have only one choice to fill P, with only one single value, then $P(\lambda,\lambda)$ would always "return" the same value, either $true$ or $false$. Then both left and right hand side of the $\to$ connective have the same value and $true \to true$ and $false \to false$ are valid.

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(a)

We will prove $\vdash \neg\forall x.(\neg\phi) \to \exists x.(\phi)$, by proving the contrapositive : $\vdash \neg\exists x.(\phi) \to \forall x.(\neg\phi)$ and by deduction lemma, $\neg\exists x.(\phi) \vdash \forall x.(\neg\phi)$.

Proof

(1)	$\neg\exists x.(\phi)$	Premise
(2)	$\phi \to \exists x.(\phi)$	Axiom 11
(3)	$\neg\exists x.(\phi) \to (\phi \to \neg\exists x.(\phi))$	Axiom 1
(4)	$\phi \to \neg\exists x.(\phi)$	Modus Ponens of 1 and 3

(5)	$(\phi \to \exists x.(\phi)) \to ((\phi \to \neg\exists x.(\phi)) \to \neg\phi)$	Axiom 8
(6)	$(\phi \to \neg\exists x.(\phi)) \to \neg\phi$	Modus Ponens of 2 and 5
(7)	$\neg\phi$	Modus Ponens of 5 and 6
(8)	$\neg\phi \to \forall x.(\neg\phi)$	Generalisation rule
(9)	$\forall x.(\neg\phi)$	Modus Ponens of 7 and 8

(b)

We will prove $\vdash \neg\exists x.(\neg\phi) \to \forall x.(\phi)$, and by deduction lemma, we prove $\neg\exists x.(\neg\phi) \vdash \forall x.(\phi)$

Proof

(1)	$\neg\exists x.(\neg\phi)$	Premise
(2)	$\neg\phi \to \exists x.(\neg\phi)$	Axiom 11
(3)	$\neg\exists x.(\neg\phi) \to (\neg\phi \to \neg\exists x.(\neg\phi))$	Axiom 1
(4)	$\neg\phi \to \neg\exists x.(\neg\phi)$	Modus Ponens of 1 and 3
(5)	$\neg\phi \to \exists x.(\neg\phi) \to ((\neg\phi \to \neg\exists x.(\neg\phi)) \to \neg\neg\phi)$	Axiom 8
(6)	$(\neg\phi \to \neg\exists x.(\neg\phi)) \to \neg\neg\phi$	Modus Ponens of 2 and 5
(7)	$\neg\neg\phi$	Modus Ponens of 4 and 5
(8)	ϕ	Double negation elimination
(9)	$\phi \to \forall x.(\phi)$	Generalisation rule
(10)	$\forall x.(\phi)$	Modus Ponens of 8 and 9

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$$
\begin{aligned}
\phi = &\forall x.(\forall y.(P(x,y) \land P(y,x) \to x = y)) \land && R \\
&\forall x.(\forall y.(P(x,y) \lor P(y,x))) \land && S \\
&\forall x.(\forall y.(\forall z.(P(x,y) \land P(y,z) \to P(x,z)))) \to && T \\
&\exists x.(\forall y.(P(x,y)))
\end{aligned}
$$

(a)

In order to falsify ϕ, we have to satisfy $R \land S \land T$ and falsify $\exists x.(\forall y.(P(x,y)))$. So we have to satisfy R, satisfy S and satisfy T :

$$
\begin{aligned}
&U = \mathbb{Z} \\
&P(x,y) \leftrightarrow \{(x,y) | x >= y\}
\end{aligned}
$$

R is satisfy, because, for any x and y, we have $x >= y$ and $y >= x$, and the only way satisfy this is to give the same value to x and y.
S is satisfy, because for any x and y from $\mathbb{Z}$, either one is greater than the other or the invers.

T is satisfy, because the relation $>=$ is transitive, for any x, y and z from $\mathbb{Z}$, if $x >= y$ and $y >= z$, then $x >= z$.
Finally, we falsify $\exists x.(\forall y.(P(x,y)))$, we pick $x = 3$ (for example), then for any y from $\mathbb{Z}$, we can't make $P(x,y)$ true, because if $x = 3$, for the value of $y = 4$, then $P(x,y)$ is not true, because $(3,4) \notin P(x,y)$.
Then we find a counter-example to ϕ.

(b)

This formula is true for a finite universe, because we can put $U = \{1\}$, then R, S and T are *true* and $\exists x.(\forall y.(P(x,y)))$ is also *true*, because we just have to check if $P(1,1)$, which is *true*.

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In order to satisfy S and R but falsify T. To do this, we use a modified version of the rock, paper, scissors game. The predicate $P(x,y)$ is *true* is x beats y, we consider that $P(x,y)$ is *true* if $x = y$. Then, we have

$$
\begin{aligned}
&U = \{rock, paper, scissors\} \\
&P = \{ \\
&(rock, rock), \\
&(scissors, scissors), \\
&(paper, paper), \\
&(rock, scissors), \\
&(scissors, paper), \\
&(paper, rock) \\
&\}
\end{aligned}
$$

Then, T is *true* because $P(x,y)$ is *true* if $x = y$. R is also *true*, because either x beats y or y beats x. But T is $false$, because the relation P is not transitive, for any x, y and z, we would have $x = rock$, $y = scissors$ and $z = paper$ at one point. Then, x beats y is *true*, y beats z is *true* but x beats z is $false$.