Mathematical Methods for Computer Science 1
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Series 2

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1

a)

Picking n object out of r possible choice when repetition is allowed and order not allowed is computed by using the formula $f(n,r) = \binom{n+r-1}{r-1}$. So to choose 8 out of 10 postcards : $f(8,10) = \binom{8+10-1}{10-1} = \binom{17}{9} = 24'310$.

b)

In order to pick 15 postcards out of 10 choice, such that we have at least one postcards of every kind, we separate the problem into two parts :

1. Pick the 10 different postcards for the 10 first choice, because there is no choice and repetition is alowed, we have $f(10,10) = 1$ total different possible choice.
2. Pick the last 5 out of 10 using the formula in $1.a$: $f(5,10) = \binom{5+10-1}{5-1} = \binom{14}{4} = 1001$.

Finally, we multiply the two to obtain the total number of ways of obtaining 15 postcards is $1 * 1001 = 1001$.

c)

The number of ways to buy 8 different postcards out of 10 possible choice is given by $\frac{10!}{2!}$. Because at the first choice we have 10 possible choice, at the second we have $10-1$, hand so on.

d)

The number of ways of splitting 8 postcards into 5 different letter-boxes is dividing into 2 problems :

1. We separate the first 5 postcards into the 5 boxes so every friends is getting a postcard. The number of possible ways to do this is given by : $\frac{n}{n-k} = \frac{8!}{3!} = 6720$ where n is the total number of postcards and k the number of different letter-boxes.

2. Then we separate the rest of the $8-5=3$ letters into the 5 letter-boxes, it's the same as choosing for each postcard the letter-box in which to put the card, so the total number of ways of doing this is : $5*5*5=5^3=125$.

Finally we multiply the two numbers 6720 and 125 in order to obtain the total number of ways of splitting 8 postcards into 5 different letter-boxes : $6720*125=840'000$.

2

a)

The number of ways to pick 6 balls out of 49 is given by $\binom{49}{6}=13'983'816$.
The number of 6-balls-pick that are guessing exactly 3 numbers is the same as picking 3 out of 6 good choices and 3 out of $49-6$ bad choices : $\binom{6}{3}*\binom{43}{3}=8'815$.
We can formulate the function $f(n,k,r)$ which compute the total numbers of ways of picking k good numbers and r bad numbers out of n possible numbers where $k+r=6$ (in this case) : $f(n,k,r)=\binom{r+k}{k}*\binom{n-(r+k)}{r}$.
The number of 6-balls-pick that are guessing at least 3 numbers is the sum of the number of 6-balls-pick that are guessing exactly 3,4,5 and 6 numbers. So we take the formula above and we compute the total number of 6-balls-pick that are guessing at least 3 numbers :

$$f(49,3,3)+f(49,4,2)+f(49,5,1)+f(49,6,0)=$$
$$\binom{6}{3}*\binom{43}{3}+\binom{6}{4}*\binom{43}{2}+\binom{6}{5}*\binom{43}{1}+\binom{6}{6}*\binom{43}{0}=$$
$$246'820+13'545+258+1=260'624$$

b)

Prove $\binom{m+n}{k}=\binom{m}{0}*\binom{n}{k}+\binom{m}{1}*\binom{n}{k-1}+\cdots+\binom{m}{k}*\binom{n}{0}$.
The idea is to think about picking k balls out of a bag that contains m black balls and n white balls. At each try we can get i black balls and $j=k-i$ white balls. The total number of possibility $\binom{m+n}{k}$ is the same as summing k times :

- Picking 0 black balls and k white balls
- Picking 1 black balls and $k-1$ white balls
- $\cdots$
- Picking k black balls and 0 white balls

So

$$\binom{m+n}{k}=\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$$

3

a)

The number of latice which are going from A to B through C is the product of the number of latice which are going from A to C by the number of latice which are going from C to B.
Number of (A, C) latice : $\binom{5}{2} = \binom{5}{3} = 10$.
Number of (C, B) latice : $\binom{5}{2} = \binom{5}{3} = 10$.
Total number of latice from A to B through C : $10 * 10 = 100$.

b)

Prove the identity $I =$

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

In order to prove I, we use the formula $f(k, l) = \binom{k+l}{k}$ which compute the number of monotone path between $(0, 0)$ and (k, l). We know that the number of white points n_w in the big diagonals, perpendicular to the big diagonals AB, is $n_w = k = l$. So we can compute the number of monotone path between A and B using a similar methods as in $3.a$.
The number of monotone path between A and B is given by the sum of the whole number of path between A and C_i and between C_i and B where C_i represents the points in the middle of the path. Then we compute each case (in this case we use $k = l = 6$ to simplify the reasoning) :

- From $A = (0, 0)$ to $B = (k, l)$ passing by $C_i = (0, l)$: number of monotone path is $\binom{k}{0} * \binom{l}{l}$
- From $A = (0, 0)$ to $B = (k, l)$ passing by $C_i = (1, l-1)$: number of monotone path is $\binom{k}{1} * \binom{l}{l-1}$
- $\cdots$
- From $A = (0, 0)$ to $B = (k, l)$ passing by $C_i = (k, 0)$: number of monotone path is $\binom{k}{k} * \binom{l}{0}$

And because $k = l$ and $\binom{m}{n} = \binom{m}{m-n}$, we have prove I

c)

Prove the identity $I =$

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

using $I' =$

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

I' is the same as I where picking k balls from a bag of $m+n$ total balls where the number of white n and black m balls is the same : $m = n$. In this case we know that we have to pick k balls out of $2k = m+n$ total balls. So we pick the formula I' and replace m by n and k by n, because $m = n$ and $k = n$:

$$\binom{n+n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i}$$

And because $\binom{n}{i} = \binom{n}{n-i}$ we finally have

$$\binom{n+n}{n} = \binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

4

The total number of composition of n is given by 2^{n-1}, having a sum of n 1 : $1_0+1_1+1_2+...+1_n$ we have to place between the 1 either a + or a ,, so we can obtain a composition of n.
Example with $n = 5$:

- $Comp_n = \{1, 1, 1, 1, 1\}$
- $Comp_n = \{1+1, 1, 1+1\} = \{2, 1, 2\}$
- hand so on.

We have to place 2 different kind of "operator" and we have $n-1$ place to put them so we have 2^{n-1} possible composition for n.

5

a)

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\dots & & & & & & & & & &
\end{array}$$

Comment : $1+3+6 = 10$

b)

Using the Pascal trianble, $\binom{n}{0}$ is always starting in the left side of the triangle at 1. As we know $\binom{n+1}{1}$, we can compute $\binom{n+2}{1}$ because there is only 1 on the left side of the triangle. Then, as we know $\binom{n+2}{2}$, we can compute $\binom{n+3}{2}$ by using $\binom{n}{0}$,$\binom{n+1}{1}$ and $\binom{n+2}{2}$. So, by induction, if we know the entire "diagonals" $\binom{n}{0}, \binom{n+1}{1}, \cdots, \binom{n+k-1}{k-1}$ we can compute $\binom{n+l}{k-1}$.