Series 1

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1

2.1)

$$(1+\sqrt{5})^n = \binom{n}{0} 1^n \sqrt{5}^0 + \binom{n}{1} 1^{n-1}\sqrt{5}^1 + \cdots + \binom{n}{n-1} 1^1 \sqrt{5}^{n-1} + \binom{n}{n} 1^0 \sqrt{5}^n$$

$$= 1 + \binom{n}{1}\sqrt{5} + \binom{n}{2}\sqrt{5}^2 + \cdots + \binom{n}{n-1}\sqrt{5}^{n-1} + \binom{n}{n}\sqrt{5}^n$$

$$= \sum_{i=0}^n \binom{n}{i}\sqrt{5}^i$$

a.2)

$$(1-\sqrt{5})^n = (1+(-\sqrt{5}))^n$$

$$= \binom{n}{0} 1^n (-\sqrt{5})^0 + \binom{n}{1} 1^{n-1}(-\sqrt{5})^1 + \cdots + \binom{n}{n-1} 1^1 (-\sqrt{5})^{n-1} + \binom{n}{n} 1^0 (-\sqrt{5})^n$$

$$= 1 - \binom{n}{1}\sqrt{5} + \binom{n}{2}\sqrt{5}^2 - \cdots + \binom{n}{n-1}(-\sqrt{5})^{n-1} + \binom{n}{n}(-\sqrt{5})^n$$

$$= \sum_{i=0}^n (-1)^i \binom{n}{i}\sqrt{5}^i$$

b)

$$a_n = \frac{\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + \ldots}{2^{n-1}}$$

The proof is by expanding the Binet's formula using the binomial coefficient. So we have

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

and after expansion we obtain

$$a_n = \frac{1}{2^n\sqrt{5}} \left(\sum_{i=0}^{n} \left(\binom{n}{k} (\sqrt{5}^k) \right) - \sum_{i=0}^{n} \left(\binom{n}{k} (\sqrt{5}^k)(-1)^k \right) \right)$$

$$= \frac{1}{2^n\sqrt{5}} \left(\sum_{i=0}^{n} \left(\binom{n}{k} (\sqrt{5})^k (1-(-1)^k)) \right) \right)$$

When k is even, $1-(-1)^k = 0$ and $1-(-1)^k = 2$ when k is odd. So we obtain

$$a_n = \frac{1}{2^n\sqrt{5}} \left(\sum_{2j+1=1}^{2j+1=n} \left(\binom{n}{2j+1} (\sqrt{5})^{2j+1} \cdot 2 \right) \right)$$

$$= \frac{1}{2^{n-1}\sqrt{5}} \sum_{j=0} \left(\binom{n}{2j+1} (5^j\sqrt{5}) \right)$$

$$= \frac{1}{2^{n-1}} \sum_{j=0} \left(5^j \binom{n}{2j+1} \right)$$

$$= \frac{5^0 \binom{n}{1} + 5^1 \binom{n}{3} + 5^2 \binom{n}{5} \dots}{2^{n-1}}$$

2

$$a_1 = 1$$
$$a_2 = 5$$
$$a_n = 5a_{n-1} - 6a_{n-2}$$

We find that

$$a_2 = 5a_1 - 6a_0 = 5$$
$$5 = 5*1 - 6*a_0 = 5 - 6*a_0$$
$$6*a_0 = 0$$
$$a_0 = 0$$

$$X = (1, \lambda, \lambda^2)$$

$$\lambda^n = 5\lambda^{n-1} - 6\lambda^{n-2}$$

$$\lambda^2 = 5\lambda - 6$$

$$-\lambda^2 + 5\lambda - 6 = 0$$

We compute the discriminant :

$$\Delta = 25 - (4 * -1 * -6) = 25 - 24 = 1$$

$$\lambda_1 = \frac{-5 + \sqrt{1}}{2 * (-1)} = \frac{-4}{-2} = 2$$

$$\lambda_2 = \frac{-5 - \sqrt{1}}{2 * (-1)} = \frac{-6}{-2} = 3$$

And we obtain

$$A = C_1 \lambda_1 + C_2 \lambda_2$$

$$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n$$

For $n = 0$

$$a_0 = C_1 + C_2$$

For $n = 1$

$$a_1 = C_1 \lambda_1 + C_2 \lambda_2$$

and we obtain the following system :

$$\begin{cases} C_1 + C_2 = 0 \\ C_1\lambda_1 + C_2\lambda_2 = 1 \end{cases} = \begin{cases} C_1 + C_2 = 0 \\ 2C_1 + 3C_2 = 1 \end{cases}$$

Resolution :

$$C_1 = -C_2$$

$$2(-C_2) + 3C_2 = 1$$

$$C_2 = 1$$

$$C_1 = -1$$

Finally we have

$$\begin{aligned} a_n &= C_1 \lambda_1^n + C_2 \lambda_2^n \\ &= (-1) * 2^n + 1 * 3^n \\ &= 3^n - 2^n \end{aligned}$$

3

$$a_1 = 1$$
$$a_2 = 2$$
$$a_3 = 5$$
$$a_n = 3a_{n-2} - 2a_{n-3}$$

We find that

$$a_3 = 3a_1 - 2a_0$$
$$5 = 3 * 1 - 2 * a_0$$
$$2 = -2 * a_0$$
$$a_0 = -1$$

$$\lambda^n = 3\lambda^{n-2} - 2\lambda^{n-3}$$

We pick $n = 3$

$$\lambda^3 = 3\lambda - 2$$
$$-\lambda^3 + 3\lambda - 2 = 0$$

We find that 1 is a root of the equation and we obtain

$$-\lambda^3 + 3\lambda - 2 = (\lambda - 1)(-\lambda^2 - \lambda + 2) = -(\lambda - 1)^2(\lambda + 2)$$
$$\lambda_1 = 1$$
$$\lambda_2 = -2$$

$$P(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$$
$$= (\lambda - 1)^2(\lambda - (-2))$$

$$a_n = C_1\lambda_1^n + C_2 n\lambda_1^n + C_3\lambda_2^n$$
$$= C_1 * 1^n + C_2 * n * 1^n + C_3 * (-2)^n$$
$$= C_1 + n * C_2 + (-2)^n * C_3$$

For $n = 0$

$$a_0 = C_1 + C_3 = -1$$

For $n = 1$

$$a_1 = C_1 + C_2 - 2C_+1$$

For $n = 2$

$$a_2 = C_1 + 2C_2 + 4C_3 = 2$$

And we obtain the following system :

$$\begin{cases} C_1 + C_3 = -1 \\ C_1 + C_2 - 2C_3 = 1 \\ C_1 + 2C_2 + 4C_3 = 2 \end{cases}$$

$$C_1 = -1 - C_3$$

$$\begin{cases} -C_3 - 1 + C_2 - 2C_3 = 1 \\ -C_3 - 1 + 2C_2 + 4C_3 = 2 \end{cases} = \begin{cases} C_2 - 3C_3 - 1 = 1 \\ 3C_3 - 1 + 2C_2 = 2 \end{cases}$$

$$C_2 = 3C_3 + 2$$

$$3C_3 - 1 + 2(3C_3 + 2) = 2$$

$$3C_3 - 1 + 4 + 6C_3 = 2$$

$$9C_3 + 3 = 2$$

$$9C_3 = -1$$

$$C_3 = -\frac{1}{9}$$

$$C_2 = 3 * -\frac{1}{9} + 2 = \frac{5}{3}$$

$$C_1 = -1 - \frac{1}{9} = -\frac{8}{9}$$

Finally we have

$$a_n = C_1 \lambda_1^n = C_2 n \lambda_1^n + C_3 \lambda_2^n$$
$$= -\frac{4}{3} * 1^n + 2 * n * 1^n + \left(-\frac{1}{6}\right) * (-2)^n$$
$$= -\frac{8}{9} + n\frac{5}{3} + (-2)^n \cdot -\frac{1}{9}$$

4

a)

b)

5

a)

b)

6

a)

b)