Resume : Mathematical Methods for Computer Science

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Chapter 1

Combinatorics

We are counting elements of set :

$$A \cap B \neq \emptyset \to |A \cup B| = |A| + |B| \tag{1.1}$$
$$|A \times B| = |A| * |B| \tag{1.2}$$

Making two choices independently out of m and n possibilites leads to $m*n$ different possibilities. For example, throwing a dice twice leads to $6 * 6 = 36$ possibilities consecutevely or not.

$$\underbrace{|A_1 \times \cdots \times A_n|}_{|\{(a_1,\ldots,a_n),a_i \in A_i\}|} = |A_1| * \cdots * |A_n| \tag{1.3}$$

Example : Throwing n coins leads to 2^n possibilities.
Choosing 2 out of n people is $n(n-1)$.

$$B \subset A \to |A \setminus B| = |A| - |B| \tag{1.4}$$

Example : Getting at least one 6 from two dice : $\frac{6*2-1}{6*6} \frac{11}{36}$

1.1 Quotient rate

The number of sheep in a herd is equals to the number of legs divided by 4.
Given a set A with equivalence relation $\sim$ such that every equivalence class contains n element, we have

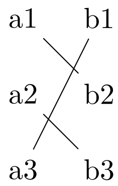
$$|\frac{A}{\sim}| = \frac{|A|}{\sim} \tag{1.5}$$

Have a map $f : x \mapsto y$ such that for every elements of y there are n corresponding elements of x, then

$$|y| = \frac{|x|}{\sim} \tag{1.6}$$

1.2 Permutation

A permutation of a set A is a bijective map $f : A \mapsto A$, where usually $A = \{1, 2, \dots, n\}$. There are three ways to represent a permutation :

1. Using a graph-like draw 

2. $$\begin{pmatrix} 1 & 2 & 3 \\ f(1) & f(2) & f(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

3. $$\begin{pmatrix} f(1) & f(2) & f(3) \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

There are $n!$ different permutation of the set $\{1, 2, \dots, n\}$.

1.3 Ordered choice

Take k out of n element and remember the order, there are

$$n * (n-1) * \cdots * (n-k+1) = \frac{n!}{(n-k)!} \tag{1.7}$$

number of k-permutation in a set of n elements.

1.4 Unordered choice

The number of ways to pick k out of n elements without order is

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

Theorem 1.

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof.

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{(n-k)! \cdot k!} \\ \binom{n}{n-k} &= \frac{n!}{(n-(n-k))! \cdot (n-k)!} \\ &= \frac{n!}{k! \cdot (n-k)!} \end{aligned}$$

□

Number of subset of a set of n element is 2^n. Either we pick an element or not, so its a sequence of 1 and 0 where 1 means that we pick the element and 0 not. Example with $n = 5 :$ 10010.

1.5 Unordered choices of subsets

$\binom{n}{k}$ is the number of k-elements subsets of $\{1, 2, \ldots, n\}$.

Theorem 2. $\binom{n}{k}$ *is the number of binary words of length n with k digits "1".*

Proof. Encode a subset $A \subset \{1, 2, \ldots, n\}$ by a binary word : ith digit is "1" if $i \in A$ and "0" if $i \notin A$. So binary words with A-digit "1" $\leftrightarrow$ k-elements subsets. $\square$

Theorem 3. *For every positive integer n we have*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Proof. Both sides of the equation counter the number of binary words of length n. On the LHS, the words are split into groups with the same number of "1". $\square$

1.6 Monotone path

We encode a monotone path in the plane with "0" and "1", "0" indicates that we go vertically and "1" horizontally. So we would have a sequence $(0, 1, 1, 1, 0, 0, \ldots)$.

Theorem 4. *The number of monotone paths from $(0, 0)$ to (k, l) is*

$$\binom{k+l}{k}$$

Proof. Encode a monotone path into a binary word like before, then path from $(0, 0)$ to (k, l) of k "1" and l "0" have a length of $k + l$ with k digit "1". Therefore, the number of monotone paths is

$$\binom{k+l}{k}$$

$\square$

1.7 Pascal's Triangle

$n=0$							1						
$n=1$						1		1					
$n=2$					1		2		1				
$n=3$				1		3		3		1			
$n=4$			1		4		6		4		1		
$n=5$		1		5		10		10		5		1	
$n=6$	1		6		15		20		15		6		1

Theorem 5. *The k-th number in the n-th row of the Pascal's triangle is* $\binom{n}{k}$.

Lemma 1. *For any $0 < k < n$ we have*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

due to the Pascal's triangle.

Proof. $\binom{n}{k}$ is the number of binary words of length n with k digits "1". There are 2 kinds of words :

- Start with "0" : $\binom{n-1}{k-1}$
- Start with "1" : $\binom{n-1}{k}$

Therefore

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

□

1.8 Binomial theorem

Recall :

$$(a+b)^2 = a^2 + b^2 + 2ab$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{k} a^{n-k}b^k + \cdots + \binom{n}{n} b^n \quad \text{for } n \geq 0 \text{ and } n \in \mathbb{N}$$

Thus one take the nth row of the Pascal's triangle :

$$(a+b)^5 = \underbrace{a^5}_{1} + \underbrace{5a^4b}_{5} + \underbrace{10a^3b^2}_{10} + \underbrace{10a^2b^3}_{10} + \underbrace{5ab^4}_{5} + \underbrace{b^5}_{1}$$

Proof.

$$(a+b)^2 = (a+b)(a+b) = aa + ab + ab + bb = a^2 + 2ab + b^2$$

$$(a+b)^n = \prod_{i=1}^{n}(a+b) \rightarrow \text{ sum of all binary words } (a,b) \text{ of length } n\text{, the order doesn't matter.}$$

Every word with $n-k$ letters "a" and k letter "b" gives the term $a^{n-k}b^k$. The coefficient at $a^{n-k}b^k$ is the number of words with $n-k$ "a" and k "b". There are $\binom{n}{k}$, so

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

□

1.9 Multinomial Coefficient

Theorem 6. *Let n balls of r differents colors be given, with k_i balls of colors i. These balls can be arranged in a row in*

$$\binom{n}{k_1, k_2, \dots, k_n} = \frac{n!}{\prod_{i=1}^{n}(k_i)!}$$

Proof. Use the quotient rule and make the balls of the same color distinguishable. There are $n!$ ways of putting these balls in a row.

$$X = \{\text{all arrangements with distinguishable balls}\}$$
$$Y = \{\text{all arrangements with undistinguishable ballsdistinguishable}\}$$

Balls of color i can be distinguished in $k_i!$ ways

$$\Rightarrow$$

The map has multiplicity $(k_1!, \dots, k_2!)$

$$\Rightarrow$$

$$|Y| = \frac{|X|}{k_i! \cdot \dots \cdot k_i!} = \frac{n!}{\prod_{i=1}^{n}(k_i)!}$$

□

Example : The number of words of length $n = k + l + m$ with k "a", l "b" and m "c" is

$$\frac{n!}{k! \cdot l! \cdot m!} = \binom{n}{k, l, m}$$

Theorem 7 (Mutltinomial Theorem)**.**

$$(a_1 + \dots + a_r)^n = \sum_{k_1, k_2, \dots, k_r = n} \binom{n}{k_1, \dots, k_n} a_1^{k_1} \cdot \dots \cdot a_r^{k_r}$$

for $k_1 + \dots + k_r = n$ *and* $n \geq 0$

Proof. The sum of all words of length n with letters $a_1, \dots, a_v$ get the term $a_1^{k_1}, \dots, a_r^{k_r}$ with coefficient equal to the number of words containing k_i letters a_i with $i = 1, \dots, n$. □

Chapter 2

Graph Theory

Chapter 3

Logic