Mathematical Methods for Computer Science 1 Fall 2017

Series 3

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1

a)

Show that for $n = k + l + m$, we have

$$\binom{n}{k, l, m} = \binom{n-1}{k-1, l, m} + \binom{n-1}{k, l-1, m} + \binom{n-1}{k, l, m-1}$$

The idea is to pick k black balls, l red balls and m white balls out of a bag of $m = k + l + m$ balls. We fix the first picked balls and that leads us to three differents posibility :

1. We have pick a black balls, so there is $m - 1 = (k - 1) + l + m$ balls left in the bags, and the number of ways of picking the $m - 1$ last balls is $\binom{m-1}{k-1, m, l}$.
2. We have pick a red balls, so there is $m - 1 = k + (l - 1) + m$ balls left in the bags, and the number of ways of picking the $m - 1$ last balls is $\binom{m-1}{k, l-1, m}$.
3. We have pick a white balls, so there is $m - 1 = k + l + (m - 1)$ balls left in the bags, and the number of ways of picking the $m - 1$ last balls is $\binom{m-1}{k, l, m-1}$.

By summing the three differents possibilities we obtain the total number of ways $\binom{n}{k, l, m}$:

$$\binom{n-1}{k-1, l, m} + \binom{n-1}{k, l-1, m} + \binom{n-1}{k, l, m-1} = \binom{n}{k, l, m}$$

b)

Show that for every n we have

$$\sum_{k+l+m=n \vee k,l,m \geq 0} \binom{n}{k, l, m} = 3^n$$

Using the multinomial theorem

$$(x_1 + x_2 + \cdots + x_n)^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \cdots, k_r} a_1^{k_1} * a_2^{k_2} * \cdots * a_r^{k_r}$$

If we replace x_i by 1, we obtain

$$\underbrace{(1+1+\overset{n}{\cdots}+1)}_{r} = r^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \cdots, k_r} * \underbrace{1*1*\cdots*1}_{r}$$

And when we replace $k_1, k_2, \cdots, k_r$ by k, l, m, we obtain $r = 3$ and

$$r^n = 3^n = \sum_{k+l+m=n} \binom{n}{k, l, m}$$

2

Prove

$$\binom{n}{k_1, \cdots, k_r} = \binom{n}{k_r}\binom{n-k_r}{k_1, \cdots, k_{r-1}}$$

a)

With the multinomial coefficients formula, we have

$$\binom{n}{k_1, \cdots, k_m} = \frac{n!}{\prod_{i=1}^{n} k_i!}$$

then

$$\begin{aligned}
\binom{n}{k_m}\binom{n-k_m}{k_1, \cdots, k_{m-1}} &= \frac{n!}{k_m!(n-k_m)!}\frac{(n-k_m)!}{k_1!k_2!\cdots k_{m-1}!} \\
&= \frac{n!}{k_m!k_1!k_2!\cdots k_{m-1}!} \\
&= \binom{n}{k_1, k_2, \cdots, k_m}
\end{aligned}$$

b)

With a combinatorial argument : the idea is to cunt how many differents words are possible with k_1 letters α, k_2 letters β, ..., k_m letters γ.
The first ways of doing this by using the multinomial coefficients : $\binom{n}{k_1, \cdots, k_m}$ with $n = \sum_{i=1}^{m} k_i$.
The second ways is to pick, at first, the k_m γ letters : $\binom{n}{k_m}$, then we pick the lefting letters without the k_m γ : $\binom{n-k_m}{k_1, k_2, \cdots, k_{m-1}}$.
Because we count the same set of objetc using two differents ways, the combinatorial proof is done.

3

a)

This is the same as putting 7 objects into 3 sets of capacity 4,2 and 1 : $\binom{7}{4,2,1} = \frac{7!}{4!2!1!} = \frac{7*6*5}{2} = 105$

b)

This is the same as putting 10 people into 5 rooms where each room as a capacity of 2 : $\binom{10}{2,2,2,2,2} = \frac{10!}{2!2!2!2!2!} = \frac{10!}{32} = 113400$.

4

The set of student who ski : A, the set of the student who climb : B, the set of the student who do ico hockey : C. We have to following informations :

$$\begin{aligned} |A| &= 20 \\ |B| &= 15 \\ |C| &= 8 \\ |A \cup B| &= 6 \\ |A \cup C| &= 2 \\ |B \cup C| &= 3 \end{aligned}$$

Using the inclusion-exclusion formula, we have :

$$\begin{aligned} |A \cup B \cup C| = 33 \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 20 + 15 + 8 - 6 - 2 - 3 + |A \cap B \cap C| \\ &= 32 + |A \cap B \cap C| \end{aligned}$$

Using simple algebra we have

$$33 = 32 + |A \cap B \cap C|$$

so

$$|A \cap B \cap C| = 1$$

The is 1 student who do all three sport

5

The number of monotone maps $f : \{1, \cdots, n\} \to \{1, \cdots, n\}$ is given by $\binom{2n}{n}$.
The idea is that a map is a sequence from 1 to n where the entries are from 1 to n. The number k of increases between the entries is specifying by throwing k balls into n boxes, so we have

$$\sum_{k=0}^{n} \binom{n+k-1}{k} = \binom{2n}{n}$$

6

First, we consider that the couple are a unique entity. The number of ways to put them around the table is $k!$. Because the table is a ring, we have to divide the result by m because we can rotate a pick, example : $\{1,2,3\}$,$\{3,1,2\}$ and $\{2,3,1\}$ are the same because the table is cyclic. So we have $\frac{k!}{m}$ number of ways of putting the couples around the table.
Because we can swap the people inside the couples, we have to compute the number of permutation inside the couples : 2^n.
Finally, the number of differents ways can k married couples sit at this table so that every couple sits next to each other is given by $2^k \frac{k!}{m}$

7

This is similar to the ménage problem proposed by Lucas, except that the people of the same sex can sit next to each other. The idea to solve this problem is to use the inclusion-exclusion principle on the sets of all the possible ways of putting the $2n$ individuals person : $2n!$

$$m_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha_k$$

α_k is computed using the following formula :

$$\alpha_k = 2n * (2n - k - 1)! * 2^k$$

Finally we have

$$m_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} * 2n * (2n - k - 1)! * 2^k$$

About the answer : I did not found the answer alone, I have search a bit in the literature to find out the answer[1].

[1] https://math.dartmouth.edu/ doyle/docs/menage/menage/menage.html