Mathematical Methods for Computer Science 1 Fall 2017

Series 5

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1

(a)

We know that the graph $G = (V, E)$ is 3-regular. So, $\forall v \in V, deg(v) = 3$. We also know that $\sum_{v \in V} deg(v) = 2|E|$. Finally, we have $\sum_{v \in V} deg(v) = 3|V| = 2|E|$.

(b)

In the computation $\sum_{i=3}^{\infty} i * n_i$, every edges is counted twice because a segment can only separate the plane into two faces. So, we have $\sum_{i=3}^{\infty} i * n_i = 2|E|$, because the graph is 3-regular from the previous exercice, we know that $2|E| = 3|V|$.

(c)

From previous exercice, we know that $\sum_{i=3}^{\infty} i * n_i = 3|V|$ and we assume that $\sum_{i_3}^{\infty}(6-i)n_i = 12$:

$$(\sum_{i=3}^{\infty} i * n_i) + (\sum_{i_3}^{\infty}(6-i)n_i) = \sum_{i=3}^{\infty} 6n_i - in_i + in_i = 12 + 3|V|$$
$$\sum_{i=3}^{\infty} 6n_i = 12 + 3|V|$$
$$\sum_{i=3}^{\infty} n_i = 2 + \frac{|V|}{2}$$

$\sum_{i=3}^{\infty} n_i$ is the total number of faces f, so we have to prove that $f = \frac{|V|}{2} + 2$ for any 3-regular planar graph :
For a 3-regular graph, we know that $2|E| = 3|V|$. Using the Euler's formula, we could deduce the formula for the number of faces :

$$|V| - |E| + f = 2$$
$$2|V| - 2|E| + 2f = 4$$
$$2|V| - 3|V| + 2f = 4$$
$$2f - |V| = 4$$
$$2f = |V| + 4$$
$$f = \frac{|V|}{2} + 2$$

(d)

We know that the graph is 3-regular and its faces are only polygon or pentagon. So we can transform $\sum_{i=3}^{\infty}(6-i)n_i = (6-5)n_5 + \underbrace{(6-6)}_{0}n_6 = n_5 = 12$. The total number of pentagonal faces n_5 is 12.

2

(a)

We know that $|V| - |E| + f = 2 \implies |E| = |V| + f - 2$ and every faces have at least 4 edges (triangle-free graph) : $4f \leq 2|E| \implies f \leq \frac{|E|}{2}$. Finally, we have

$$
\begin{aligned}
|E| &\leq |V| + \frac{|E|}{2} - 2 \\
\frac{|E|}{2} &\leq |V| - 2 \\
|E| &\leq 2|V| - 4
\end{aligned}
$$

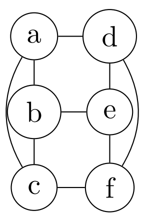
(b)

$K_{3,3}$ is triangle-free because it is bipartite, so his chromatic number is 2 so there is no triangle. $K_{3,3} = (V, E)$, $|V| = 6$, $|E| = 9$ (count by hand) and we finally have :

$$9 \not\leq 2 * 6 - 4 \implies 9 \not\leq 8$$

(c)

No, for example the following graph is planar, 3-regular and on 6 vertices :



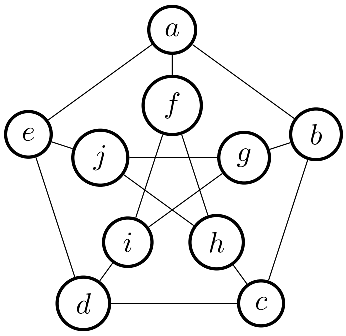
3

(a)

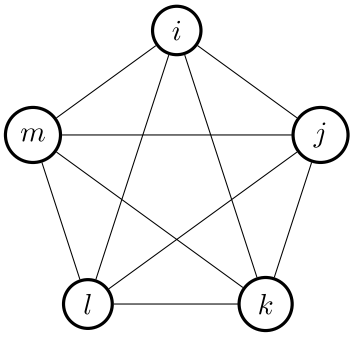
For any graph $G = (V, E)$, if we do a subdivision of $G$to obtain G', we could contract the added vertices and corresponding edges in order to recover G from G'. So, if G holds a subgraph G_{sub} isomorphic to a subdivision of H, denote H', we could delete all the vertices and edges which are not part of G_{sub}, then we use the contraction on edges, edges deletion and vertices deletion to obtain H.

(b)

We use the following Petersen graph :

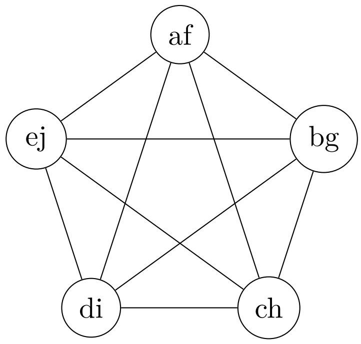


The K_5 graph :



The Petersen graph is not a subdivision of $K_{3,3}$, because the subdivision could never delete edges. It only add a vertices v on an edges $e = \{v_1, v_2\}$ such that $E' = E \setminus \{v_1, v_2\} \cup \{\{v_1, v\}, \{v, v_2\}\}$ and $V' = V \cup \{v\}$.
So, we could never obtain the Petersen graph by successive subdivision because we can't produce a vertices of degree 3. We can't modifiy the degree of existing vertices and the vertices obtained by subdivision are always of degrees 2.
The Petersen graph has K_5 has a minor. We use the following contraction in order to obtain k_5 :

- contraction of a with f
- contraction of b with g
- contraction of c with h
- contraction of d with i
- contraction of e with j

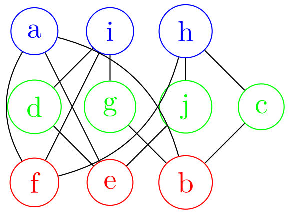


(c)

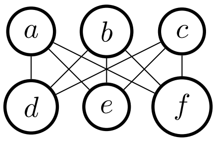
The subgraph G', of the Petersen graph :

$$V = \{a, b, c, d, e, f, g, h, i, j\}$$
$$E = \{\{a, f\}, \{a, c\}, \{a, b\}, \{i, d\}, \{d, e\}, \{i, g\}, \{g, b\}, \{i, f\}, \{h, f\}, \{h, c\}, \{c, b\}, \{h, j\}, \{j, e\}, \}$$

We can "see" the subdivision of the $K_{3,3}$ graph, the blue nodes are from the top groups, the red nodes are from the bottom groups and the gree nodes are the from the subdivision of $K_{3,3}$.



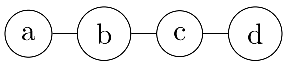
is isomorphic to the following subdivision of $K_{3,3}$:
The graph $K_{3,3}$:



4

(a)

The bipartite graph we use :

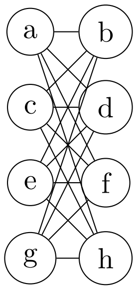


Vertices ordering for the algorithm : (a, d, b, c) :

1. Vertices a : color 1.
2. Vertices d : color 1.
3. Vertices b : color 2.
4. Vertices c : color 3.

(b)

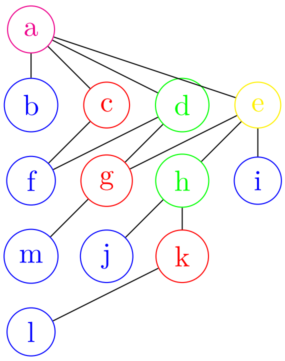
For this, we use a Crown graph on 200 vertices, the Crown graph is the complete bipartite graph from which the edges of a perfect mathching has been removed, for example, in the following $K_{4,4}$ graph :



We have to remove the edges $\{a, b\}, \{c, d\}, \{e, f\}, \{g, h\}$ (the edges from vertices at the same height) to obtain a crown graph. Then we use the ordering $(a, b, c, d, e, f, \cdots)$ (in alphabetical order) so, on 200 vertices, the chromatic number of the graph would be 100.

(c)

The following graph is bipartite and 5-coloring



under the ordering $(b, f, m, j, l, i, k, g, c, d, h, e, a)$.

5

(a)

We consider only the graph which has 1 interior vertex, because the property need to holds for every interior vertex, we can generalise this for any number of interior vertex. We also consider n exterior vertices $\{v_0, \cdots, v_n\}$ which are adjacent to the interior vertex v. The n exterior vertices are adjacent and it would be 2 cases :

- n is even : we have the subgraph $G' = (V', E')$ where $V' = V \setminus v$ and $E' = \{e | e = \{x, y\} \in E \wedge (x \neq v \wedge y \neq v)\}$. G' is 2-coloring, so G is 3-coloring because v is adjacent to vertices with color number 1 and 2 and n is even.

- n is odd : we have the subgraph $G' = (V', E')$ where $V' = V \setminus v$ and $E\prime = \{e | e = \{x, y\} \in E \wedge (x \neq v \wedge y \neq v)\}$. G' is 3-coloring, so G is 4-coloring and n is odd.

So we avec to show that C_n is 2-coloring for n even and 3-coloring for n odd and $n \geq 3$. If n is even, the graph is bipartite so 2-coloring. If n is odd, we show that C_n is 3-coloring because we use C_{n-1} and we add a vertex between two already present one. We can only add a vertex between vertices of color 1 and 2, so C_n is 3-coloring if n is odd.

(b)

From previous exercice, if v has an odd number of adjacent vertices, it means that the C_n graph that is representing thos adjacent vertices is 3-coloring (or 2-coloring if $deg(v) \equiv 0 \mod 2$). So if $deg(v)$ is even, G is 3-coloring.