Mathematical Methods for Computer Science I

Fall 2017

Series 8 – Hand in before Monday, 20.11.2017 - 12.00

1. a) Show that the set of connectives $\{\neg, \rightarrow\}$ is complete (that is, every Boolean function can be expressed by a formula that contains only these connectives).
   b) The connective $\uparrow$ is defined as $p \uparrow q = \neg(p \wedge q)$. (Whence its other name NAND.) Show that this single connective suffices to express every Boolean function. In other words, show that the set of connectives $\{\uparrow\}$ is complete.

2. Write the formula $(p \rightarrow q) \wedge ((q \vee r) \rightarrow p)$ in a disjunctive normal form
   a) by transforming it with the help of the distributive, de Morgan and other laws of logic;
   b) by filling in a truth table and reading its rows.

3. a) Let $A = (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$. Write a conjunctive normal form for $\neg A$.
   b) Let A be a propositional formula that contains a subformula F. Let B be the formula obtained from A by replacing F with a formula G. It turned out that B is equivalent to A. Does this imply that G is equivalent to F?

4. Which of the following propositional formulas are tautologies? Explain your answer.
   a) $A = (p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$
   b) $B = \neg(p \rightarrow q) \vee (\neg p \vee q)$
   c) $C = (p \vee q \vee r) \wedge (p \vee q \vee \neg s)$

5. The recipe for constructing a formula in DNF for a given boolean function in n variables produces conjunctive clauses of length n. On the other hand, a formula in DNF may contain shorter conjunctive clauses. For example, $p \vee q$ is a formula in DNF with two clauses of length 1 each. (At the same time, this is CNF with a single disjunctive clause of length 2.)
   a) Transform the formula
   $$(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge r)$$
   to a simpler formula in DNF.
   b) Transform the formula
   $$(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee r)$$
   to a simpler formula in CNF.
   c) Give an example of a boolean function φ of n variables such that there is no formula in DNF for φ that contains a conjunctive clause of length less than n. (*Hint:* What if the function changes its value always when you change the value of one of its arguments?)

please see overleaf

6.* Define the connectives $\cdot$ and $+$ by the truth table below.

p	q	$p \cdot q$	$p+q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

(These are the multiplication and addition in the field $\mathbb{Z}_2$; note that $p \cdot q = p \wedge q$.) Show that every boolean function has a unique representation as a $\mathbb{Z}_2$-polynomial. Here a $\mathbb{Z}_2$-polynomial in the variables $p_1, \dots, p_n$ is a sum of monomials. A monomial is either an expression of the form

$$p_{i_1} \cdot \dots \cdot p_{i_k}, \quad 1 \leq i_1 < \dots < i_k \leq n$$

or a constant 1.