

Mathematical Methods for Computer Science 2  
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Series 3

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1

a)

By extending the brackets of

$$(1-x)(1-x^2)(1-x^4)(1-x^8)\dots$$

We can see that any  $x^n$  is obtained only one time. For example, in order to obtain  $x^7$ , we have to multiply  $x, x^2, x^4$  and pick the 1 in all the other factors. If we don't pick a 1, we obtain a  $n$  which is greater than 7. By the way, it is the encoding of the integer in the binary format.

b)

The result obtained is the sum from a) and the sign in front of  $x^n$  is plus only when the number we encode into the binary format is encoded with an even number of bits.

If the number of bits is even, it means we have taken an even number of  $x^{2^k}$  from the product and then an even number of negative  $x^{2^k}$  which leads to a positive  $x^n$ .

2

a)

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= (1 + x + x^2 + x^3 + \dots)^k = \left(\frac{1}{1-x}\right)^k \\ &= \frac{1}{(1-x)^k} \\ &= (1-x)^{-k} \\ &= \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n\end{aligned}$$

b)

$$\begin{aligned}\frac{1}{1-x-x^2} &= \frac{1}{1-(x+x^2)} \\ &= 1 + (x+x^2)x + (x+x^2)x^2 + (x+x^2)x^3 + (x+x^2)x^4 + \dots \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \dots\end{aligned}$$

The coefficient in front of each  $x^n$  follow the fibonnaci sequence. The summands are represented by the factors  $(x+x^2)$ , which means the summands 1 and 2. Then, by expanding we see that the number of compositions with summands 1 and 2.

**3**

a)

b)

**4**

a)

The generating function for  $p_{n,k}$  is :

$$\begin{aligned}\sum_{n=0}^{\infty} p_n x^n &= (1+x+x^2+x^3+\dots)(1+x^2+x^4+x+\dots)(1+x^3+x^6+x^9+\dots)\dots \\ &= a_0 + a_1x + a_2x^2 + \dots \\ &= \prod_{k=1}^{\infty} \left( \frac{1}{1-x^k} \right)\end{aligned}$$

The  $x^n$  term in the series  $a_0 + a_1x + a_2x^2 + \dots$  count the number of ways to obtain  $n$  where  $n = a_1 + 2a_2 + 3a_3 + \dots$ , which is the number of partition of  $n$ .

b)

If we need to have the number of partitions of  $n$  into exactly  $k$  parts, we multiply the factor by  $x^k$  :

$$\prod_{k=1}^{\infty} \left( \frac{x}{1-x^k} \right)$$

c)

$$p_{n-k, \geq k} = p_{n,k}$$

**Algebraic proof :** If we have

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a)

$$p_{n-k,\geq k} = p_{n,k}$$

b)