Mathematical Methods for Computer Science 1 Fall 2017

Series 4

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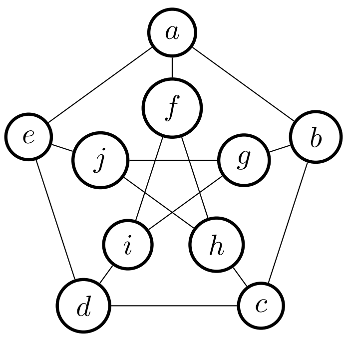
1

a)

To show that the graph $G = (V = \{\text{ all two-element subsests of } \{1,2,3,4,5\}\}, E = \{\{A,B\} | A \cap B = \emptyset\}\})$, we could use the following bijection:

$$
\begin{aligned}
f : \{f &\mapsto \{1,2\}, \\
j &\mapsto \{2,4\}, \\
i &\mapsto \{3,4\}, \\
h &\mapsto \{3,5\}, \\
g &\mapsto \{1,5\}, \\
a &\mapsto \{4,5\}, \\
b &\mapsto \{2,3\}, \\
c &\mapsto \{1,4\}, \\
d &\mapsto \{2,5\}, \\
e &\mapsto \{1,3\}\}
\end{aligned}
$$

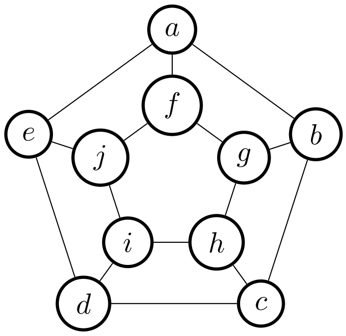
With the following graph



We ca see that the intersection of the sets of two adjacents nodes are empty.

b)

The following graph



is not isomorphic to the petersen graph, because this graph has an Hamiltonian cycle and the Petersen graph does not, so this graph is Hamiltonian and the Petersen graph is not Hamiltonian.

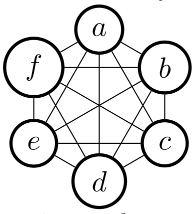
2

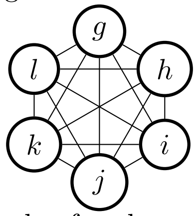
a)

In a graph with n vertices, if a vertice v has a degree $n-1$, it means that he is connected to every other vertices of the graph except itself. It implies that from any vertices other than v, we can go to v and then to an another vertice of the graph, so every vertices is accessible from any other vertices with 2 jump. Therefore, the graph is connected.

b)

We imagine we separate the graph G into two group of 7 vertices each. In each groupe we fully connected the nodes so every nodes of the two groups has a degree 6 :





In order to have a 7-regular graph, we have to connect every node of each graph to an another ones that is not part of the opposing graph. Due to $7+7=14$, we have just 1 node left to connect each nodes of each subgraph. So, each vertices of each subgraph is connected to the one left, so the graph is connected according to the previous point in exercice 2.

3

a)

From the two following degrees sequences : $A=(5,4,3,2,2,1)$ and $B=(5,4,3,2,1,1)$, only B could be a degree sequences in a graph with 6 vertices because $5+4+3+2+1+1=16$ is even and $5+4+3+2+2+1=17$ is odd, and the sum of all the degrees of a Graph is even.

b)

No, we represent the segment as vertices in a graph G and the intersections as connections between two vertices. So if a segment s_1 intersect with a segment s_2, s_1 and s_2 are adjacent in G.

We know that the segments set we want to draw is 3-regular, so there is 9 vertices with a degree 3 for each, then the sum of all the degrees is : $\sum_{i=1}^{9} 3 = 3 * 9 = 27$, 27 is odd so it is not possible because the sum of the degree of all vertices in a graph is even.

4

a)

We take the smallest adjacency matrix A_G that could contains a triangle :

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

If A_G contains a triangle, it means that we have a fully connected adjacency matrix :

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

If A_G^3 contains a 0 in his diagonals, it means that there would be a 0 in a row and in a columns (because the matrix diagonals is a mirror, if a is connected to b, then b is connected to h) which result in a 0 in the diagonals of A_G^3. Example with a triangle:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$$

And without a triangle

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

Because a_{12} and a_{21} are 0, a_{22} would be 0 anyway after multiple power of the matrix.
We have show this for a graph of 3 vertices, but we can extends it to any graph, because a triangle in any graph is a subgraph which has 3 vertices and his fully connected.

b)

In a graph, a triangle is a walks of length 3, by using the matrix adjacency A_G, the number of walks between vertex i and j of length n is given by the element (i, j) of the matrix A_G^n. Because a triangle is a walks of length 3, we compute the trace of A_G^3, which gives the total number of triangle in G. But, we have to divide $trace(A_G^3)$ by 6, because the adjacency matrix is symetric and because a triangle has 3 vertices, we counted a walks 3 times for 3 vertices. Then, we have the number of triangle in G is given by

$$trace(A_G^3)\frac{1}{\underbrace{3}_{\text{a triangle has 3 vertices}}\underbrace{2}_{A_G \text{ is symetric}}}$$

5

a)

The is $\frac{n!}{2n}$ distinct Hamiltonian cycle in the complete graph K_n. From any vertices of the graph, we can choose $n-1$ different path, then $n-2$, hand so on... so we have $(n-1)!$ differents cycle, we divide by $2n$ because the cycle $(1,2,3,4)$ is the same as the cycle $(4,1,2,3)$ and $(3,4,1,2)$, hand so on. Therefore we have $\frac{(n-1)!}{2} = \frac{n!}{2n}$.

b)

In any graph $P_{m,n}$, there is always a Hamiltonian path that simply traverse the rows in alternate direction, if m is even (or n, we could juste transpose the grid graph to inverse m and n), so there would be a path to come back to the start of the cycle.
The idea is to start from the top left and traverse the first row, and the second except for the first element of the second row, the we go to the third and so on. In order to go and comeback, we need to have 2 row, so an even number of row gives us the go and comeback strategy.

c)

If n and m are odd, it means that we can't comeback after the "go" from the previous exercice.