Mathematical Methods for Computer Science 2
Spring 2018

Series 2

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a)

We have

$$(1 - x + 2x^3) = (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)$$

we use the following equation to find a_0, a_1, a_2 and a_3

$$(1 - x + 2x^3)(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) = 1$$

$$\begin{aligned}
&(1 - x + 2x^3)(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) = \\
&a_0 - a_0x + 2a_0x^3 + \\
&a_1x - a_1x^2 + 2a_1x^4 + \\
&a_2x^2 - a_2x^3 + 2a_2x^5 + \\
&a_3x^3 - a_3x^4 + 2a_3x^6 + \\
&a_4x^4 - a_4x^5 + 2a_4x^7 + \dots = 1
\end{aligned}$$

So we can extract the following system :

$$\begin{cases}
a_0 = 1 \\
-a_0x + a_1x = 0 \\
-a_1x^2 + a_2x^2 = 0 \\
2a_0x^3 - a_2x^3 + a_3x^3 = 0
\end{cases}$$

which leads to the following :

$$a_0 = 1$$
$$-1x + a_1x = 0 \longrightarrow a_1 = 1$$
$$-x^2 + a_2x^2 = 0 \longrightarrow a_2 = 1$$
$$2x^3 - x^3 + a_3x^3 = 0 \longrightarrow a_3 = -1$$

2

a)

$$
\begin{aligned}
a_0 &= 1 \qquad a_{n+1} = 2a_n + 3 \\
a_1 &= 5 \\
a_2 &= 13 \\
a_3 &= 29 \\
&\dots \\
a_{n+1} - 2a_n &= 3
\end{aligned}
$$

$$
\begin{aligned}
F(x) &= a_0 + a_1 x + a_2 x^2 + \dots \\
xF(x) &= a_0 x + a_1 x^2 + \dots \\
F(x) - 2xF(x) &= (a_0 + a_1 x + a_2 x^2 + \dots) - (2a_0 x + 2a_1 x^2 + \dots) \\
&= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + \dots \\
&= 1 + \underbrace{(5-2)}_{3} x + \underbrace{(13-10)}_{3} x^2 + \dots \\
&= 1 + 3x + 3x^2 + \dots \\
&= 1 + 3x(1 + x + x^2 + x^3 + \dots) \\
&= 1 + \frac{3x}{1-x} = \frac{(1-x)+3x}{1-x} = \frac{2x+1}{1-x}
\end{aligned}
$$

$$
\begin{aligned}
F(x) &= \frac{2x+1}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x} \\
2x+1 = A(1-2x) + B(1-x) &= A - 2Ax + B - Bx = 2x+1
\end{aligned}
$$

We extract the following system :

$$
\begin{cases}
A + B = 1 \\
-2A - B = 2
\end{cases}
$$

$$
\begin{gathered}
A = 1 - B \\
(-2)(1-B) - B = 2 \\
-2 + 2B - B = 2 \\
B = 4 \\
A = 1 - 4 = -3
\end{gathered}
$$

$$
F(x) = \frac{-3}{1-x} + \frac{4}{2x+1} =
$$

b)

$$
\begin{aligned}
F(x) &= \frac{-3}{1-x} + \frac{4}{2x+1} = -3\frac{1}{1-x} + 4\frac{1}{2x+1} \\
&= -3\frac{1}{1-x} + 4\frac{1}{1-(-2x)} \\
&= -3\sum_{k=0}^{\infty} x^k + 4\sum_{k=0}^{\infty}(-1)^k\ 2^k x^k \\
&= 4\sum_{k=0}^{\infty}(-1)^k\ 2^k x^k - 3\sum_{k=0}^{\infty} x^k \\
&= \sum_{k=0}^{\infty}(-1)^k \cdot 4 \cdot 2^k x^k - \sum_{k=0}^{\infty} 3 \cdot 1^k \cdot x^k \\
&= \sum_{k=0}^{\infty} 4(2^k - 1) - 3x^k
\end{aligned}
$$

$$a_n = 4(2^n - 1) + 1$$

3

$$\frac{x^2+x}{(x-1)(x+2)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x-1)}$$

$$
\begin{aligned}
x^2 + x &= A(x-1)(x+2) + B(x-1) + C(x+2)^2 \\
&= A(x^2 - 2 + x) + Bx + B + C(x^2 + 4 + 4x) \\
&= Ax^2 + Ax - 2A + Bx - B + Cx^2 + 4C + 4Cx
\end{aligned}
$$

We extract the following system :

$$
\begin{cases}
-2A - B + 4C = 0 \\
A + B + 4C = 1 \\
A + C = 1
\end{cases}
$$

$$A + C = 1 \longrightarrow A = 1 - C$$

$$\longrightarrow$$

$$
\begin{cases}
-2 + 2C - B + 4C = 0 \\
1 + 3C + B = 1
\end{cases}
$$

$$\longrightarrow$$

$$B = -3C$$

$$-2 + 2C + 3C + 4C = 0$$
$$9C = 2$$
$$C = \frac{2}{9}$$

$$B = -3\frac{2}{9} = -\frac{6}{9} = -\frac{2}{3}$$
$$A = 1 - C$$
$$A = 1 - \frac{2}{9} = \frac{7}{9}$$

$$\frac{x^2 + x}{(x-1)(x+2)} = \frac{7}{9(x+2)} - \frac{2}{3(x+2)^2} + \frac{2}{9(x-1)}$$

4

a)

$$(2n-1)!! = \frac{(2n)!}{2^n n!}$$
$$\longrightarrow$$
$$(2n-1)!!2^n n! = (2n!)$$

$$(2n-1)!!2^n n! = ((2n-1)(2n-3)\dots(1)) \cdot 2^n \cdot n!$$
$$\longrightarrow$$
$$2^n = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot \dots \cdot 2}_{n}$$
$$n! = \underbrace{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n}_{n}$$
$$2^n \cdot n! = \underbrace{2 \cdot 4 \cdot 6 \cdot 8 \cdot \dots \cdot 2n}_{n} = (2n)(2n-2)(2n-4)\dots(2)$$
$$(2n-1)!!2^n n! = (2n-1)(2n-2)(2n-3)(2n-4)\dots(2)(1) = (2n)!$$

Therefore

$$(2n-1)!! = \frac{(2n)!}{2^n n!}$$

b)

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$$

$$\binom{\alpha}{k} = \binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-3)\dots((\frac{1}{2}-k+1))}{k!}$$

We expand the sum :

$$1 - \frac{(\frac{1}{2})(\frac{1}{2})}{1!} + \frac{(\frac{1}{2})(\frac{1}{2})(\frac{3}{2})}{2!} - \frac{(\frac{1}{2})(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{3!} + \dots =$$
$$1 - \frac{(-1)!!}{1! * 2^1} + \frac{1!!}{2! * 2^2} - \frac{3!!}{3!2^3} + \dots = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-3)!!}{k!2^k}$$

c)

$$a_0 = 1$$
$$a_1 x = -\frac{1}{2}x$$
$$a_2 x^2 = \frac{1}{8}x^2$$

5

a)

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

So we have

$$\begin{aligned}
\frac{d}{dx}(\frac{1}{1-x}) &= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots \\
\frac{d}{dx}(\frac{1}{1+x}) &= \frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - \dots \\
\frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} &= 4x + 8x^3 + 12x^5 + 16x^7 + \dots \\
&= 4x\underbrace{(1 + 2x^2 + 3x^4 + 4x^6)}_{F(x)}
\end{aligned}$$

$$
\begin{aligned}
F(x) &= \frac{1}{4x}(\frac{1}{(1-x)^2} + \frac{1}{(1+x)^2}) \\
&= \frac{1}{4x}(\frac{(1+x)^2-(1-x)^2}{((1-x)^2)((1+x)^2)}) \\
&= \frac{1}{4x}(\frac{1+x^2+2x-1-x^2+2x}{((1-x)^2)((1+x)^2)}) \\
&= \frac{1}{4x}(\frac{4x}{((1-x)^2)((1+x)^2)}) \\
&= \frac{1}{((1-x)^2)((1+x)^2)} \\
&= \frac{1}{(1+x^2-2x)(1+x^2+2x)} \\
&= \frac{1}{1+x^2+2x+x^2+x^4+2x^3-2x-2x^3-4x^2} \\
&= \frac{1}{1+-2x^2+x^4} \\
&= \frac{1}{(1-x^2)^2}
\end{aligned}
$$

$$1-2x+3x^2-4x^3+5x^4-\dots = \frac{d}{dx}(\frac{1}{1+x}) = -\frac{1}{(1+x)^2}$$

b)

$$\underbrace{1+2x+3x+4x^3+\dots}_{\frac{1}{(1-x)^2}} \cdot \underbrace{1-2x+3x^2-4x^3+\dots}_{\frac{1}{(1+x)^2}} = \underbrace{1+2x^2+3x^4+4x^6}_{\frac{1}{(1-x^2)^2}}$$

Hence

$$
\begin{aligned}
(1-x)^2 \cdot (1+x)^2 &= (1+x^2+2x) \cdot (1+x^2-2x) \\
&= (1-x^2)^2
\end{aligned}
$$