Mathematical Methods for Computer Science I

Fall 2017

Series 6 – Hand in before Monday, 30.10.2017 - 12.00

1. a) Show that for every 3-regular graph holds
$$3|V| = 2|E|.$$
   b) For an embedding of a planar graph denote by n_i the number of faces with i edges (recall that an edge counts twice if the face approaches it from both sides). Show that the following equation holds:
$$\sum_{i=3}^{\infty} i \cdot n_i = 3|V|.$$
   c) Show that for every embedding of a planar 3-regular graph the following equation holds:
$$\sum_{i=3}^{\infty} (6-i)n_i = 12.$$
   d) Show that every polytope all of whose faces is either a pentagon or a hexagon has exactly 12 pentagonal faces.
   Remark. Such polytopes are called *Buckminsterfullerenes*, see the Wikipedia article on Richard Buckminster Fuller. Try to imagine at least one such polytope before reading the article.

2. a) Show that in every triangle-free planar graph with at least three vertices holds $|E| \leq 2|V| - 4$.
   b) Deduce that $K_{3,3}$ is not planar.
   c) The graph $K_{3,3}$ is a 3-regular graph on 6 vertices. Is every 3-regular graph on 6 vertices non-planar?

3. a) Show that if a graph G has a subgraph isomorphic to a subdivision of a graph H, then H is isomorphic to a minor of G.
   b) Show that the Petersen graph has no subgraph isomorphic to a subdivision of K_5, but has K_5 as a minor.
   c) Find a subgraph of the Petersen graph that is isomorphic to a subdivision of $K_{3,3}$.

4. Consider the following greedy algorithm for graph coloring. Order the vertices of a graph arbitrarily. Then color them in this order by assigning to each vertex the smallest color available. That is, we put $c(v_1) = 1$ and proceed by
$$c(v_i) := \min\{k \mid v_i \text{ has no neighbors of color } k\}.$$
   a) Give an example of a bipartite graph and of an ordering of its vertices such that the greedy algorithm colors this graph in 3 colors.
   b) Give an example of a bipartite graph and of an ordering of its vertices such that the greedy algorithm colors this graph in 100 colors.
   c)* Give an example of a planar bipartite graph and of an ordering of its vertices such that the greedy algorithm colors this graph in 5 colors.

please see overleaf

5. Consider an embedding of a planar graph G with triangular faces. A vertex of G is called interior if it does not belong to the outer face of the embedding.
   a) Show that if the vertices of G can be colored in 3 colors, then the degree of every interior vertex is even.
   b) Show that if the degree of every interior vertex is even, then the vertices of G can be colored in 3 colors.
   c)* Show that if the degree of every interior vertex is even, then the number of edges of the outer face is divisible by 3.
   d)* Assume we have a polytope with triangular faces and the degrees of all vertices even except two. Show that the two odd vertices cannot be adjacent.