Mathematical Methods for Computer Science I

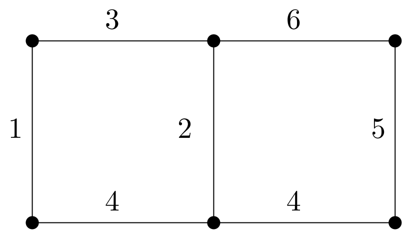
Fall 2017

Series 5 – Hand in before Monday, 23.10.2017 - 12.00

1. a) How many non-isomorphic trees on 6 vertices are there? Draw all of them.
   b) Show that if a tree has a vertex of degree d, then it has at least d leaves (vertices of degree 1).

2. a) What is the maximum number of edges that can be removed from the graph $P_{m,n}$ (see Exercise 4.5 for definition) so that the resulting graph is connected?
   b) King Uxamhwiashurh had 4 sons, 10 of his male descendants had 3 sons each, 15 had 2 sons, and all others died childless. How many male descendants did king Uxamhwiashurh have?

3. a) Let T be a spanning tree of a graph G. Show that for every edge $e \in E(G) \setminus E(T)$ there is an edge $e' \in E(T)$ such that $T + e - e'$ is a spanning tree of G again.
   b) Apply Kruskal's algorithm to find the minimum spanning tree in the graph shown below.



4. The following is the Jarník-Prim algorithm for finding the minimum spanning tree in a weighted graph.
   - Order the edges in the non-decreasing order of their weights: $(e_1, \dots, e_m)$.
   - Choose an initial vertex v.
   - Set $V_0 = \{v\}$ and $E_0 = \emptyset$.
   - If V_{i-1} and E_{i-1} are already defined, find the smallest k such that $e_k = \{x, y\}$ with $x \in V_{i-1}$ and $y \notin V_{i-1}$. Put $V_i = V_{i-1} \cup \{y\}$ and $E_i = E_{i-1} \cup \{e_k\}$.
   - The algorithm stops when no such k exists and outputs the graph (V_i, E_i).

   a) Apply Jarník-Prim's algorithm to the graph from Exercise 3, starting with the top-middle vertex.
   b) Show that the output (V_i, E_i) is a spanning tree. (In particular, one has to show that V_i coincides with the vertex set of graph G.)
   c)* Show that this spanning tree is minimal.

5. Suppose that you have a graph G with a nonnegative weight on every edge. (For example, a railroad network with the ticket price as the edge weight.) You want to find a shortest (cheapest) path between two vertices a and b, that is the path with the least possible sum of the edge weights. Edsger W. Dijkstra proposed the following algorithm.

please see overleaf

Preliminaries. Assign a preliminary distance $\mathrm{pd}(x)$ to every vertex x: zero to the vertex a and infinity to every other vertex. Mark all vertices as "unvisited" and the vertex a as "current".

Algorithm's step. For every "unvisited" neighbor x of the "current" vertex y change the value of $\mathrm{pd}(x)$ in the following way:

$$\mathrm{pd}(x) := \min\{\mathrm{pd}(x), \mathrm{pd}(y) + w(xy)\},$$

where $w(xy)$ is the weight of the edge $\{x, y\}$. After changing the value of every neighbor, mark y as "visited". The "unvisited" vertex with the least preliminary distance becomes the new "current" vertex.

The algorithm stops when every vertex is "visited".

At the end, the preliminary distance at every vertex will be equal to the cost of a shortest path from a to this vertex.

Apply this algorithm to the bottom-left vertex of the graph from Exercise 3 and find a shortest path from it to the top-right vertex. Implement the algorithm in the form of a table where the n-th row contains the preliminary distances after the n-th step.

6.* Let $(d_1, d_2, \ldots, d_n)$ be a sequence of positive integers such that $\sum_{i=1}^{n} d_i = 2n - 2$. Show that there exists a tree on n vertices with the degree sequence $(d_1, d_2, \ldots, d_n)$.