Mathematical Methods for Computer Science 1 Fall 2017

Series 7

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1

(a)

The complete bipartite graph $K_{n,n} = (X \sqcup Y, E)$ where $X \sqcup Y = \emptyset$ has $n!$ perfect matching : we can take any vertice from X in any order and associate with one of Y. So at the first pick, it would be n choices, then $n-1$, hand so on, until 1 : $n * (n-1) * \cdots * 1 = n!$.

(b)

Because the graph is complete, this problem is the same has counting the number of ways of partitioning all the $2n$ vertices into to group.
To compute the formula, we just take 1 vertice at first, it could go with $2n-1$ other different vertices, then the second can go with $2n-3$ vertices, hand so on. So, we have $(2n-1) * (2n-3) * \cdots = (2n-1)!!$, which is the double factorial of a number.

$$(2n-1)!! = (2n-1) * (2n-3) * (2n-5) * \cdots * 1$$

2

(a)

In an Hamiltonian graph, if $|V| \equiv 0 \mod 2$, the graph holds a perfect matching. The cycle of the graph is isomorphic to $C_{|V|}$, and C_n has a perfect matching only if n is even.
In a matching, there is an even number of vertices because if a vertices appers twice, this is not a matching. So if n is odd it can't be a perfect matching.
If n is even, we just take alternatively one edge from the cycle and that leads to a perfect matching.

(b)

In the graph given in the exercice is bipartite (like a chess board), so the perfect matching will associtate any vertices from the first groupe to exactly one of the second group. Becase a group has more vertices than the other, it can't be a perfect matching.

3

(a)

$|X| = |Y|$, because there is exactly $k|X|$ edges than are going out of X and $k|Y|$ are going inside Y, so we have $k|X| = k|Y| \implies |X| = |Y|$.

(b)

We know that $|X| = |Y|$ and that G is k-regular, if $k = 1$, this is already a perfect matching, because every vertices of X is adjacent to at least one vertice of Y. If $k > 1$, then we can rhmwk/hm4.pdfemove edges in order to obtain the case of G is 1-regular.
If we can't find a way such that we would obtain a 1-regular from a k-regular graph, it means that $|X| \neq |Y|$ so there would be a contradiction.

4

We consider a bipartite graph $G = (X \sqcup Y, E)$, X represent the set of all the 13 possible piles and Y the set of the possible 13 possible rank. There is an edge $\{x, y\}$ such that $x \in X$ and $y \in Y$.
If there is a perfect matching in G, then we would show that it is always possible to select exactly one card from each pile in such a way that among the 13 selected cards there is exactly one card of each rank.
G has a perfect matching, because if we take any k piles, there would be at least k possible rank. Finally, according to hall's theorem, there exist a perfect matching.

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Given the graph $G = (V, E)$, where $V = X \sqcup Y = \{\underbrace{x_1, x_2, x_3, x_4, x_5}_{X}, \underbrace{y_1, y_2, y_3, y_4, y_5}_{Y}\}$ and $M = \{\{x_2, y_2\}, \{x_3, y_3\}, \{x_5, y_5\}\}$.

Step 1

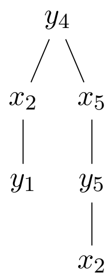
$N(S) = N(x_1) = \{y_2, y_3\}$
$T = \emptyset$
$N(S) \neq T$, we choose $y \in N(S) \setminus T \rightarrow y = y_2$. y is matched, so $\{y_2, x_2\} \in M$.
$S := S \cup \{x_2\} \rightarrow S = \{x_1, x_2\}$
$T := T \cup \{y_2\} \rightarrow T = \{y_2\}$

Step 2

$N(S) = N(\{x_1, x_2\}) = \{y_1, y_2, y_3, y_4, y_5\}$
$T = \{y_2\}$
$N(S) \neq T$, we choose $y \in N(S) \setminus T \rightarrow y = y_5$. y is matched, so $\{y_5, x_5\} \in M$.
$S := S \cup \{x_5\} \rightarrow S = \{x_1, x_2, x_5\}$
$T := T \cup \{y_5\} \rightarrow T = \{y_2, y_5\}$

Step 3

$N(S) = N(\{x_1, x_2, x_5\}) = \{y_1, y_2, y_3, y_4, y_5\}$
$T = \{y_2, y_5\}$
$N(S) \neq T$, we choose $y \in N(S) \setminus T \to y = y_4$. y is not matched, so we need to find an augmenting path.



Problem, x_2 appears in both branch, there is a contradiction, G does not contains a maximum matching.

6

(a)

Because there is a perfect matching M, the player 2 just have to pick the edges which is adjacent to the vertices picked by player 1 in M. Because M is a perfect matching, player 2 would never loose. If player 2 can't choose an edges, it means that M is not a perfect matching.

(b)

Player 1 need to found a maximal matching M and pick an edge v_1 that is not matched. Then, player 2 has two kinds of move :

- Pick v_2 an edge matched by M, then player 1 pick the adjacent to v_2 in M.
- Pick v not in M, player 2 pick an edge that is not in M too. If player 2 can't found such an edge, it means that $\{v_1, v_2\}$ would have been in M.