Mathematical Methods for Computer Science 1 Fall 2017

Series 11

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1

(a)

"Every ice hockey team has a goalkeeper"

$$\forall x.(\exists y.(H(x) \land G(x, y)))$$

(b)

"Nobody can be goalkeeper in two different teams"

$$\neg(\exists x.(G(x, y) \land G(x, z) \land \neg(z = y)))$$

(c)

"If Gottéron beats Berne, then Gottéron does not lose to every team"

$$B(f, b) \to \exists x.(\neg(L(f, x)))$$

(d)

"Gottéron beats some team, which beats Berne"

$$\exists x.(B(f, x) \land B(x, b))$$

2

(a)

Specification :

$$\begin{aligned} R(x) : &\quad \text{the things } x \text{ is red} \\ B(x) : &\quad \text{the things } x \text{ is in the box} \end{aligned}$$

(i)

$$\forall x.(R(x) \to B(x))$$

(ii)

$$\forall x.(B(x) \to R(x))$$

(b)

Specification :

$$\begin{aligned} f(x) : &\quad \text{function which return the father of } x \\ a : &\quad \text{constant that represent the person ``Arther''} \\ g : &\quad \text{constant that represent the person ``Gaspard''} \\ s : &\quad \text{constant that represent the person ``Ben'''} \end{aligned}$$

(i)

“Ben is a grandfather”

$$\exists x.(f(f(x)) = s)$$

(ii)

“Arthur and Gaspard have the same father”

$$f(a) = f(g)$$

(c)

Specification :

$$\begin{aligned} F(x,y) : &\quad \text{predicate that is true if } x \text{ is the father of } y \\ a : &\quad \text{constant that represent the person ``Arther''} \\ g : &\quad \text{constant that represent the person ``Gaspard''} \\ s : &\quad \text{constant that represent the person ``Ben'''} \end{aligned}$$

(i)

“Ben is a grandfather”

$$\exists x.(F(x,y) \wedge F(s,x))$$

(ii)

“Arthur and Gaspard have the same father”

$$\exists x.(F(x,a) \wedge F(x,g))$$

3

(a)

(i)

$$P(f(x,y))$$

is not a valid formula in the predicate logic, because f is a unary function symbol, and, in the formula, f is used as a binary function.

(ii)

$$Q(m, f(m))$$

is a valid formula in the predicate logic.

(iii)

$$Q(Q(m, x), y)$$

is not a valid formula in the predicate logic, because the predicate Q has a predicate in its parameters, which is not allowed. Predicate only allows terms as parameters, and Q is not a term.

(iv)

$$Q(x, y) \rightarrow \exists x.(Q(z, y))$$

is a valid formula in the predicate logic.

(b)

In the formula

$$\exists x.(P(y, z) \wedge (\forall y.(\neg Q(y, x) \wedge P(y, z))))$$

- x is a bound variable.
- y is a free variable in the scope $P(y, z) \wedge (\forall y.(\neg Q(y, x) \wedge P(y, z)))$.
- y is a bound variable in the scope $\neg Q(y, x) \wedge P(y, z)$.
- z is a free variable.

(c)

We change the free variable y and z to v and w and add universal quantifier in front of the formula for both.

$$\forall v.(\forall w.(\exists x.(P(v, w) \wedge (\forall y.(\neg Q(y, x) \wedge P(y, w))))))$$

4

(a)

The formula $\forall x.(\exists y.(P(x, y) \wedge (Q(y))))$ is satisfy by the first interpretation. For any x, we can find an y which is greater than x and greater than 0. If x is greater or equals than 0, then $x + 1$ is greater than 0 and greater than x, else, if x is less than 0, we can pick 1, which is greater than any negative number and greater than 0. So the formula is satisfiable by the first interpretation.
On the other hand, the second interpretation does not satisfy the formula, because if we assign x to a negative number, then we can't find an y which is lesser than x and greater than 0.

(b)

The formula is satisfiable under the following interpretation :

$$U = \mathbb{Z}$$
$$P = \{(x, y)|x^2 = y\}$$
$$Q = \{x|x > 0\}$$

but the formula is not satisfiable under the interpretation

$U = \{a, b, c\}$	
$P(x, y) = \{(a, b), (b, a)\}$	predicate that is true only if y is blue and x is yellow
$Q(x) = \{a, b, c\}$	predicate that is true if x is red

so the formula is not valid because we found an interpretation that falsify the formula. We can't satisfy the formula $P(x, y)$ for all y, because $P(x, y)$ can't be *true* if x or y is c.

5

The formula $\phi = \forall x.(\exists y.(P(x, y) \to Q(x, y)))$ is satisfiable under the given interpretation. In order to make the formula *true*, we found that $P(x, y)$ is false. To make that $\forall x.(\exists y.(P(x, y)))$ is $false$, we found an y for any x that falsify P :

- if x is a, then we put y is c, so P is falsify
- if x is b, then we put y is a, so P is falsify
- if x is c, then we put y is a, so P is falsify

Then, ϕ is satisfiable.
The formula ψ is not satisfiable, because we have to make it *true* for any x. For all x means that we have to check for any assignement of x with an element of U, if x is c, then whatever we pick for y, the predicate P would be satisfy :

- (c, a) is *true*, because $(c, a) \in Q$
- (c, b) is *true*, because $(c, b) \in Q$
- (c, c) is *true*, because $(c, c) \in Q$

So the first part of the implication is satisfy, but then, we can't satisfy $P(x, y)$ because $(c, a) \notin P$, $(c, b) \notin P$ and $(c, c) \notin P$.