Mathematical Methods for Computer Science 1 Fall 2017

Series 5

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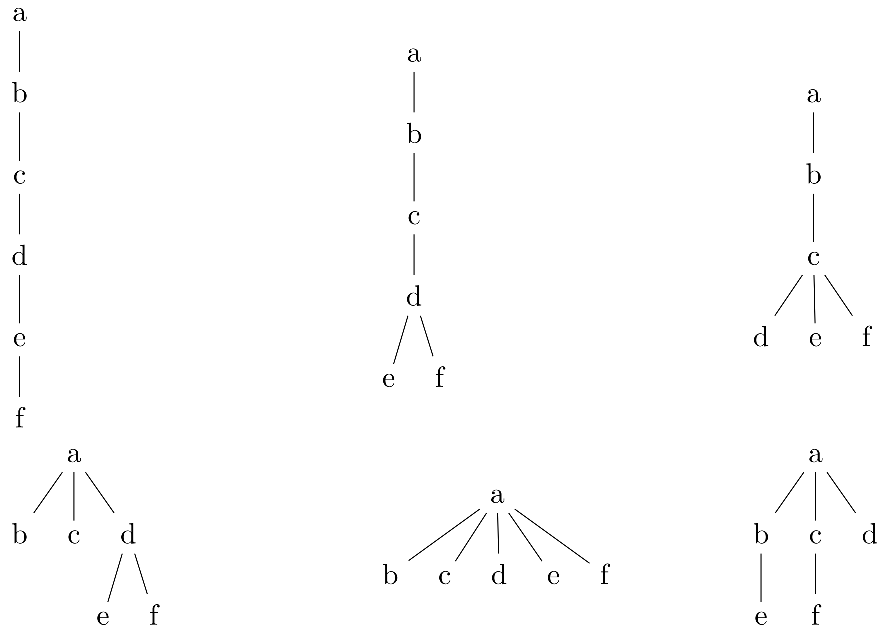
1

a)

Note : the labels are here for visual purpose, the trees are unlabelled.



b)

We use the following lemma : in every tree, there is at least two leaf (proven during the course). We denote v the vertices with degree d. We consider the sub-tree of d which are all tree (of at least one element). If there is only one element in the sub-tree, there is one leaf. If there is multiple element in the sub-tree, there is at least 2 leaves in the sub-tree and we remove 1 because the leaves could be connected to v. So every sub-tree adjacent to v hold at least 1 leaves, because there is d subtree, there is at least d leaves in total in the tree.

2

a)

In the graph $P_{m,n} = (V, E)$, there is $|V| = m * n$. In order to obtain the maximal number of removable edges, such that the graph remains connected, we could contruct a tree $T = (V', E')$ where $V' = V$ and $E' \subset E$. We know that $|V'| = |E'| + 1$, so the number of removable edges from E is $|E| - (|E'| + 1)$, where $|E| = n(m-1) + m(n-1) = mn - n + mn - m = 2mn - m - n$ and $|E'| = |V'| - 1 = |V| - 1 = mn - 1$. Finally we have the number of removable vertices : $2mn - m - n - (mn - 1 + 1) = mn - m - n$.

b)

We denote the vertex for the king K, and $deg(K) = 4$. From "10 of his male descendants had 3 sons each", we know that there is 10 vertices of degree 4 each, and from "15 had 2 sons", we know that there is 15 vertices of degree 3 each.
We also know, from theorem, that $|V| = |E| + 1$ and $\sum_{v \in V} deg(v) = 2|E|$.
Then, we have $4 + (10 * 4) + (15 * 3) + x = 2|E|$. Where x is the number of childless sons of degree 1. Then, the number of vertices $|V|$ is $|V| = 1 + 10 + 15 + x$, 1 for the king and x for the number of childless sons. Finally, using the formula $|V| = |E| + 1$, we have

$$\begin{aligned}
|V| &= |E| + 1 \\
|V| - 1 &= |E| \\
1 + 10 + 15 - 1 + x &= \frac{4 + (10 * 4) + (15 * 3) + x}{2} \\
25 + x &= \frac{89 + x}{2} \\
x - \frac{x}{2} &= \frac{89}{2} - 25 \\
\frac{x}{2} &= \frac{89}{2} - 25 \\
x &= 2(\frac{89}{2} - 25) \\
x &= 89 - 50 \\
x &= 39
\end{aligned}$$

So the number of male descendants is $39 + 10 + 15 = 64$.

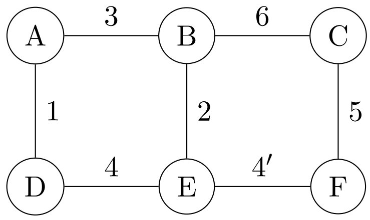
3

a)

We assume that $T + e - e'$ also remove and add the corresponding tree otherwise $T + e - e'$ would not be a tree but a forest.
$T + e - e'$ is a spanning tree, because if $E(G) \setminus E(T) \neq \emptyset$, that means there is a cycle in G so there is multiple path to go to e. If $E(G) \setminus E(T) = \emptyset$, that means there is no cycle in G so the graph is already a tree.

b)

We denote AB the edges which is going from A to B.



- Order the edges by weight : $e_{sort} = (1, 2, 3, 4, 4', 5, 6) = (AD, BE, AB, DE, EF, CF, BC)$
- $E_0 = \emptyset$
- $E_1 = E_0 \cup \{AD\} = \{AD\}$
- $E_2 = E_1 \cup \{BE\} = \{AD, BE\}$
- $E_3 = E_2 \cup \{AB\} = \{AD, BE, AB\}$
- $E_4 = E_3 = \{AD, BE, AB\}$ because $E = \{AB, BE, DE, AD\}$ has a cycle
- $E_5 = E_4 \cup \{EF\} = \{AD, BE, AB, EF\}$
- $E_6 = E_5 \cup \{FC\} = \{AD, BE, AB, EF, CF\}$
- Stop, because $|E_6| = 5 = |V| - 1$

So the minimal spanning tree is $T = \{AD, BE, AB, EF, CF\} = \{1, 2, 3, 4', 5\}$.

4

a)

We denote AB the edges which is going from A to B.

- Order the edges in the non-decreasing order pf their weight : $e_{sort} = (6, 5, 4, 4', 3, 2, 1) = (BC, CF, DE, EF, AB, BE, AD)$
- Choose an initial vertex v : $v = B$
- $V_0 = \{v\}, E_0 = \emptyset$
- $e_k = \{B, E\}, k = 2, V_1 = V_0 \cup \{E\}, E_1 = E_0 \cup \{BE\} = \{BE\}$
- $e_k = \{B, A\}, k = 3, V_2 = V_1 \cup \{A\} = \{E, A\}, E_2 = E_1 \cup \{BA\} = \{BE, AB\}$
- $e_k = \{A, D\}, k = 1, V_3 = V_2 \cup \{D\} = \{E, A, D\}, E_3 = E_2 \cup \{AD\} = \{BE, AB, AD\}$
- $e_k = \{E, F\}, k = 4, V_4 = V_3 \cup \{F\} = \{E, A, D, F\}, E_4 = E_3 \cup \{EF\} = \{BE, AB, AD, EF\}$
- $e_k = \{F, C\}, k = 5, V_5 = V_4 \cup \{C\} = \{E, A, D, F, C\}, E_5 = E_4 \cup \{FC\} = \{BE, AB, AD, EF, CF\}$
- The algorithm stop because we can't find a k, all the vertices of E are in E_5.

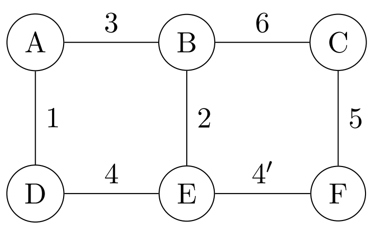
b)

$T = (V_i, E_i)$ is a spanning of $G = (V, E)$, because each vertices has been choose from V, so we have $V_i = V$. If every vertices from V are not in V_i, it means that there exist a k and the algorithm should not have to stop. T does not have any cycle, because we only add vertices to V_i (and corresponding edges to E_i) that are not in V_i, so having a cycle is impossible.

c)

Let T be the spanning tree of G construct with the Jarnik-Prism algorithm, and T' the known minimal spanning tree of G. If $T = T'$, the spanning tree is minimal. If $T \neq T'$, it means that $w(T) > w(T')$, so, at a certain moment in the algorithm, $w(T_i)$ would have increase wrongly with an edge that has been choosen wrongly by the algorithm. Because the algorithm choose the smallest possible weight at each iteration, this is impossible.

5



We start with D.

Table 1: Dijkstra algorithm application

n	$pd(D)$	$pd(A)$	$pd(B)$	$pd(C)$	$pd(E)$	$pd(F)$	current	visited set
0	0	∞	∞	∞	∞	∞	D	$\{\}$
1	0	1	∞	∞	4	∞	D	$\{\}$
2	0	1	4	∞	4	∞	A	$\{D\}$
3	0	1	4	∞	4	8	E	$\{D, A\}$
4	0	1	4	10	4	8	B	$\{D, A, E\}$
5	0	1	4	10	4	8	F	$\{D, A, E, B\}$
6	0	1	4	10	4	8	C	$\{D, A, E, B, F\}$
								$\{D, A, E, B, F, C\}$

The shortest path from D to C is $D \to A \to B \to C$ with a total weight of $1 + 3 + 6 = 10$.

6

We use the fact that in every tree $T = (V, E)$, $|V| = |E| + 1$, that implies $|V| - 1 = |E|$, and in every graph $G = (V, E)$, $\sum_{v \in V} deg(v) = 2 * |E|$. So we have :

$$
\begin{aligned}
&|E| = |V| - 1 \\
&2 * |E| = 2(|V| - 1) \\
&|V| = n \\
&\sum_{v \in V} deg(v) = 2 * |E|
\end{aligned}
$$

Finally we conclude that

$$
\begin{aligned}
\sum_{v \in V} deg(v) &= 2 * |E| \\
\sum_{v \in V} deg(v) &= 2 * (|V| - 1) \\
\sum_{v \in V} deg(v) &= 2 * (n - 1) \\
\sum_{v \in V} deg(v) &= 2n - 2 \\
\sum_{i=0}^{n} d_i &= 2n - 2
\end{aligned}
$$