Formal Methods
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S03

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Exercise 1 :

We denote V the variable P from the SAT-algorithm given in the course. Note : we assume that the algorithm is smart enough to simplify expressions like $\top \wedge P = P$ or $\bot \vee Q = Q$, in order to decrease the size of the following resolution :

$$SAT((P \vee \neg Q) \leftrightarrow (\neg P \wedge Q))$$

$$V = P$$

$$SAT((\top \vee \neg Q) \leftrightarrow (\neg \top \wedge Q)) \vee SAT((\bot \vee \neg Q) \leftrightarrow (\neg \bot \wedge Q))$$

$$\begin{array}{cc}
SAT((\top \vee \neg Q) \leftrightarrow (\neg \top \wedge Q)) & SAT((\bot \vee \neg Q) \leftrightarrow (\neg \bot \wedge Q)) \\
SAT((\top) \leftrightarrow (\bot \wedge Q)) & SAT((\neg Q) \leftrightarrow (\top \wedge Q)) \\
SAT((\top) \leftrightarrow (\bot)) & SAT(\neg Q \leftrightarrow Q) \\
SAT(\bot) & V = Q \\
false & SAT(\neg \top \leftrightarrow \top) \vee SAT(\neg \bot \leftrightarrow \bot)
\end{array}$$

$$\begin{array}{cc}
SAT(\neg \top \leftrightarrow \top) & SAT(\neg \bot \leftrightarrow \bot) \\
SAT(\bot \leftrightarrow \top) & SAT(\top \leftrightarrow \bot) \\
SAT(\bot) & SAT(\bot) \\
false & false
\end{array}$$

So we have $false \vee false \vee false$ is $false$.

Exercice 2

(a)

Two formulae are equivalent if they have the same models, equisatisfiability is weaker than equivalence. If two formulae A and B are equisatisfiable, it means that if A is satisfiable, B is satisfiable too and if B is satisfiable, A is satisfiable. Two formulae can be satisfiable but not equivalent.
For example, A and $\top$ are equisatisfiable, if $\top$ is satisfiable (which is always the case), so A is satisfiable, but A is not equivalent to $\top$.
The two formulae $(P \vee \neg P)$ and $(P \implies P)$ are equivalent, both are tautology. Because they are equivalent, they are also equisatisfiable. So the equivalence implies the equisatisfiability, but the equisatisfiability don't imply the equivalence.
Another example of equisatisfiability : $A \vee B$ and $(A \vee n) \wedge (B \vee \neg n)$, but they are not equivalent.

(b)

The transformation of a formula ϕ into an equivalent formula ϕ' in CNF has an exponential complexity (in the worst case). Turning ϕ into an equisatisfiability formula ϕ_{eq} has a linear complexity. Because we just have to check for the satisfiability of a formula, we can use a equisatisfiable one to reduce the space and time complexity.

Exercise 3

$$F \equiv \neg P \wedge (Q \to R)$$

Transformation into a equisatisfiability formula F'

$$\begin{aligned}
rep(F) &= rep(\neg P \wedge (Q \to R)) = P_1 \\
rep(\neg P) &= P_2 \\
rep(Q \to R) &= P_3 \\
rep(Q) &= Q \\
rep(P) &= P \\
rep(R) &= R
\end{aligned}$$

$$\begin{aligned}
F' = P1 &\wedge (\neg P_1 \vee P_2) \wedge (\neg P_1 \vee P_3) \wedge (P_1 \vee \neg P_2 \vee \neg P_3) && (enc(\neg P \wedge (Q \to R))) \\
&\wedge (\neg P_2 \vee \neg P) \wedge (P_2 \vee P) && (enc(\neg P)) \\
&\wedge (P_3 \vee Q) \wedge (P_3 \vee \neg R) \wedge (\neg P_3 \vee \neg Q \vee R) && (enc(Q \to R))
\end{aligned}$$

Applying the resolution

Using P_1

$$\begin{gathered}
P1 \wedge (\neg P_1 \vee P_2) \wedge (\neg P_1 \vee P_3) \wedge (P_1 \vee \neg P_2 \vee \neg P_3) \wedge \cdots = \\
P_3 \wedge P_2 \wedge \underbrace{(P_2 \vee \neg P_2 \vee \neg P_3)}_{\top} \wedge \underbrace{(P_3 \vee \neg P_2 \vee \neg P_3)}_{\top} \wedge \cdots = \\
P_3 \wedge P_2 \wedge \cdots
\end{gathered}$$

Using P_2

$$P_3 \wedge P_2 \wedge (\neg P_2 \vee \neg P) \wedge (P_2 \vee P) \wedge \cdots =$$
$$P_3 \wedge \neg P \wedge \underbrace{(\neg P \vee P)}_{\top} \wedge \cdots =$$
$$P_3 \wedge \neg P \wedge \cdots$$

Using P_3

$$P_3 \wedge \neg P \wedge (P_3 \vee Q) \wedge (P_3 \vee \neg R) \wedge (\neg P_3 \vee \neg Q \vee R) =$$
$$\neg P \wedge (\neg Q \vee R) \wedge \underbrace{Q \vee \neg Q \vee R}_{\top} \wedge \underbrace{\neg R \vee \neg Q \vee R}_{\top} =$$
$$\neg P \wedge (\neg Q \vee R)$$

F is satisfiable with the interpretation $I : \{P \mapsto false, Q \mapsto false, R \mapsto true\}$